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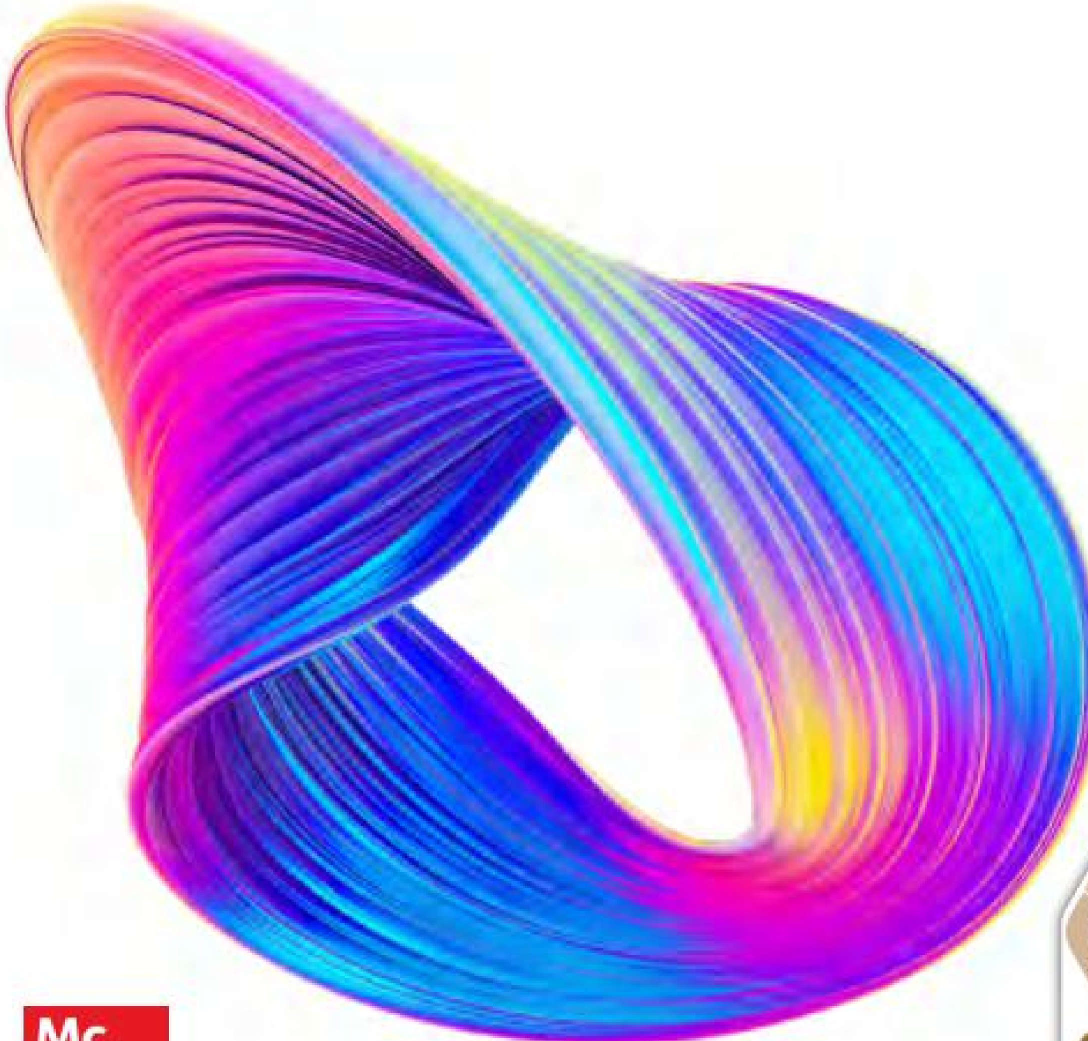

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Printed in the United Arab Emirates.

ISBN: 978-1-39-891505-3 *(Student Edition)*
MHID: 1-39-891505-X *(Student Edition)*

1 2 3 4 5 6 7 8 9 XXX 22 21 20 19 18 17

ePub Edition

ISBN: 978-1-44-709234-6 *(Student Edition)*
MHID: 1-44-709234-1 *(Student Edition)*
ISBN: 978-1-44-709235-3 *(Teacher Edition)*
MHID: 1-44-709235-X *(Teacher Edition)*

Contents in Brief

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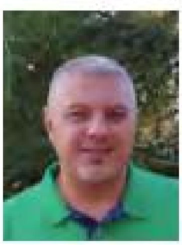
In addition to *Calculus: Early Transcendental Functions*, Professors Smith and Minton are also co-authors of *Calculus: Concepts and Connections* © 2006, and three earlier books for McGraw-Hill Higher Education. Earlier editions of *Calculus* have been translated into Spanish, Chinese and Korean and are in use around the world.



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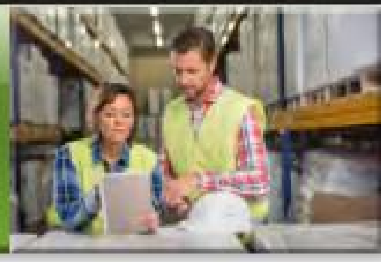
CHAPTER 1 Integration



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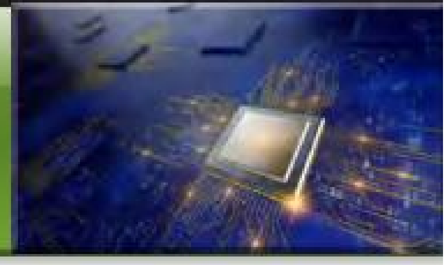
CHAPTER

3 Integration Techniques



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CHAPTER 5 Infinite Series

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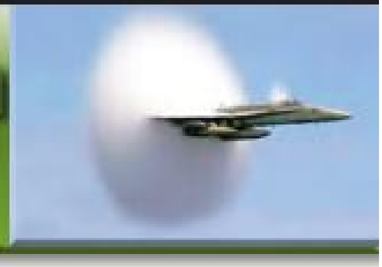
CHAPTER 6 Parametric Equations and Polar Coordinates



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Appendices

Appendix A: Answers to Odd-Numbered Exercises

Student Handbook: Symbols, Formulas, and Key Concepts

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Infinite Series

CHAPTER

5

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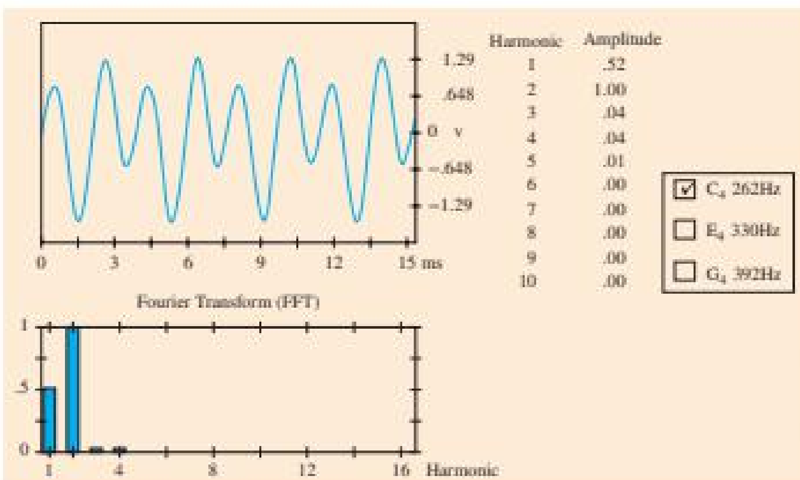
Chapter Topics

Everywhere we look today, we are surrounded by digital technologies. For instance, music and video are now routinely delivered digitally, while digital still and video cameras are now the standard. An essential ingredient in this digital revolution has been the use of Fourier analysis, a mathematical idea that is introduced in this chapter.

The key to these digital technologies is the ability to transform various kinds of information into a digital format. For instance, the music made by a saxophone might be initially represented as a series of notes on sheet music, but the musician brings her own special interpretation to the music. Such an individual performance can then be recorded, to be copied and replayed later. While this is easily accomplished with conventional analog technology, the advent of digital technology has allowed recordings with a previously unknown fidelity. To do this, the music is broken down into its component parts, which are individually recorded and then reassembled on demand to recreate the original sound. So, the complex rhythms and intonations generated by the saxophone reed and body are converted into a stream of digital bits (zeros and ones), which are then turned back into music.



In this chapter, we learn how calculators can quickly approximate a quantity like sin 1.234567, but we'll also see how music synthesizers work. The mathematics of these two modern marvels is surprisingly similar. Quite significantly, we see how to express a wide range of functions in terms of much simpler functions, opening up a new world of important applications.

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5.1 SEQUENCES OF REAL NUMBERS

The mathematical notion of sequence is not much different from the common English usage of the word. For instance, to describe the sequence of events that led up to a traffic accident, you'd not only need to list the events, but you'd need to do so in the correct *order*. In mathematics, the term *sequence* refers to an infinite collection of real numbers, written in a specific order.

We have already seen sequences several times now. For instance, to find approximate solutions to nonlinear equations such as $\tan x - x = 0$, we began by first making an initial guess, x_0, and then using Newton's method to compute a sequence of successively improved approximations, $x_1, x_2, \ldots, x_n, \ldots$.

DEFINITION OF SEQUENCE

A **sequence** is any function whose domain is the set of integers starting with some integer n_0 (often 0 or 1). For instance, the function $a(n) = \frac{1}{n}$, for $n = 1, 2, 3, \ldots$, defines the sequence

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

Here, $\frac{1}{1}$ is called the first **term**, $\frac{1}{2}$ is the second term and so on. We call $a(n) = \frac{1}{n}$ the **general term**, since it gives a (general) formula for computing all the terms of the sequence. Further, we use subscript notation instead of function notation and write a_n instead of $a(n)$.

EXAMPLE 1.1 The Terms of a Sequence

Write out the first four terms of the sequence whose general term is given by $a_n = \frac{n+1}{n}$, for $n = 1, 2, 3, \ldots$.

Solution We have the sequence

$$a_1 = \frac{1+1}{1} = \frac{2}{1}, \quad a_2 = \frac{2+1}{2} = \frac{3}{2}, \quad a_3 = \frac{4}{3}, \quad a_4 = \frac{5}{4}, \ldots. \blacksquare$$

We often use set notation to denote a sequence. For instance, the sequence with general term $a_n = \frac{1}{n^2}$, for $n = 1, 2, 3, \ldots$, is denoted by

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n^2}\right\}_{n=1}^{\infty},$$

or equivalently, by listing the terms of the sequence:

$$\left\{\frac{1}{1}, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{n^2}, \ldots\right\}.$$

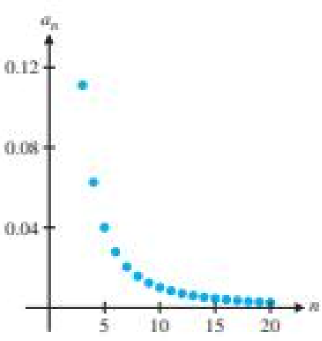


FIGURE 5.1
$a_n = \frac{1}{n^2}$

To graph this sequence, we plot a number of discrete points, since a sequence is a function defined only on the integers. (See Figure 5.1.) Note that as n gets larger and larger, the terms of the sequence, $a_n = \frac{1}{n^2}$, get closer and closer to zero. In this case, we say that the sequence **converges** to 0 and write

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0.$$

In general, we say that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L (i.e., $\lim_{n\to\infty} a_n = L$) if we can make a_n as close to L as desired, simply by making n sufficiently large. Notice that this language parallels that used in the definition of the limit $\lim_{x\to\infty} f(x) = L$ for a function of a real variable x (given in section 1.6). The only difference is that n can take on only integer values, while x can take on any real value (integer, rational or irrational).

When we say that we can make a_n as close to L as desired (i.e., arbitrarily close), just how close must we be able to make a_n to L? Well, if you pick any (small) real number, $\varepsilon > 0$, you must be able to make a_n within a distance ε of L, simply by making n sufficiently large. That is, we need $|a_n - L| < \varepsilon$.

We summarize this in Definition 1.1.

DEFINITION 1.1

The sequence $\{a_n\}_{n=n_0}^{\infty}$ **converges** to L if and only if given any number $\varepsilon > 0$, there is an integer N for which

$$|a_n - L| < \varepsilon, \quad \text{for every } n > N.$$

If there is no such number L, then we say that the sequence **diverges**.

We illustrate Definition 1.1 in Figure 5.2. Notice that the definition says that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L if, given any number $\varepsilon > 0$, we can find an integer N so that the terms of the sequence stay between $L - \varepsilon$ and $L + \varepsilon$ for all values of $n > N$.

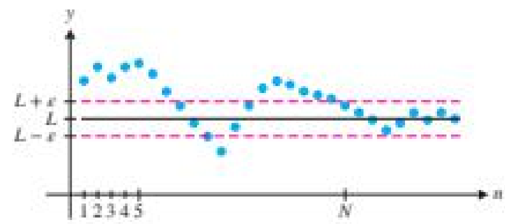


FIGURE 5.2
Convergence of a sequence

In example 1.2, we show how to use Definition 1.1 to prove that a sequence converges.

EXAMPLE 1.2 Proving That a Sequence Converges

Use Definition 1.1 to show that the sequence $\left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$ converges to 0.

Solution Here, we must show that we can make $\frac{1}{n^2}$ as close to 0 as desired, just by making n sufficiently large. So, given any $\varepsilon > 0$, we must find N sufficiently large so that for every $n > N$,

$$\left|\frac{1}{n^2} - 0\right| < \varepsilon \quad \text{or} \quad \frac{1}{n^2} < \varepsilon. \tag{1.1}$$

Since n^2 and ε are positive, we can divide both sides of (1.1) by ε and multiply by n^2, to obtain

$$\frac{1}{\varepsilon} < n^2.$$

Taking square roots gives us

$$\sqrt{\frac{1}{\varepsilon}} < n.$$

Working backwards now, observe that if we choose N to be an integer with $N \geq \sqrt{\frac{1}{\varepsilon}}$, then $n > N$ implies that $\frac{1}{n^2} < \varepsilon$, as desired. ■

Most of the usual rules for computing limits of functions of a real variable also apply to computing the limit of a sequence, as we see in Theorem 1.1.

THEOREM 1.1

Suppose that $\{a_n\}_{n=n_0}^{\infty}$ and $\{b_n\}_{n=n_0}^{\infty}$ both converge. Then

(i) $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$,

(ii) $\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$,

(iii) $\lim\limits_{n\to\infty}(a_n b_n) = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n)$ and

(iv) $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ (assuming $\lim\limits_{n\to\infty} b_n \neq 0$).

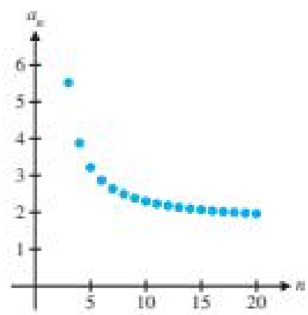


FIGURE 5.3
$a_n = \dfrac{5n+7}{3n-5}$

The proof of Theorem 1.1 is virtually identical to the proof of the corresponding theorem about limits of a function of a real variable (see Theorem 3.1 in section 1.3 and the appendix) and is omitted.

To find the limit of a sequence, you should work largely the same as when computing the limit of a function of a real variable, but keep in mind that sequences are defined *only* for integer values of the variable.

REMARK 1.1

If you (incorrectly) apply l'Hôpital's Rule in example 1.3, you get the right answer. (Go ahead and try it; nobody's looking.) Unfortunately, you will not always be so lucky. It's a lot like trying to cross a busy highway: while there are times when you can successfully cross with your eyes closed, it's not generally recommended. Theorem 1.2 describes how you can safely use l'Hôpital's Rule.

EXAMPLE 1.3 Finding the Limit of a Sequence

Evaluate $\lim\limits_{n\to\infty}\dfrac{5n+7}{3n-5}$.

Solution This has the indeterminate form $\frac{\infty}{\infty}$. The graph in Figure 5.3 suggests that the sequence tends to some limit around 2. Note that we cannot apply l'Hôpital's Rule here, since the functions in the numerator and the denominator are defined only for integer values of n and, hence, are not differentiable. Instead, simply divide numerator and denominator by the highest power of n in the denominator. We have

$$\lim_{n\to\infty}\frac{5n+7}{3n-5} = \lim_{n\to\infty}\frac{(5n+7)\left(\frac{1}{n}\right)}{(3n-5)\left(\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{5+\frac{7}{n}}{3-\frac{5}{n}} = \frac{5}{3}.$$

In example 1.4, we see a sequence that diverges by virtue of its terms tending to $+\infty$.

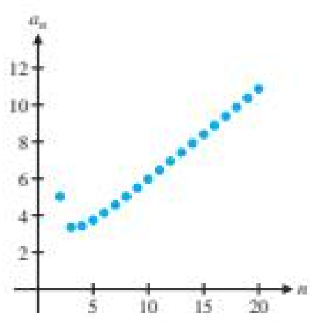


FIGURE 5.4
$a_n = \dfrac{n^2+1}{2n-3}$

EXAMPLE 1.4 A Divergent Sequence

Evaluate $\lim\limits_{n\to\infty}\dfrac{n^2+1}{2n-3}$.

Solution Again, this has the indeterminate form $\frac{\infty}{\infty}$, but from the graph in Figure 5.4, the sequence appears to be increasing without bound. Dividing top and bottom by n (the highest power of n in the denominator), we have

$$\lim_{n\to\infty}\frac{n^2+1}{2n-3} = \lim_{n\to\infty}\frac{(n^2+1)\left(\frac{1}{n}\right)}{(2n-3)\left(\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{n+\frac{1}{n}}{2-\frac{3}{n}} = \infty$$

and so, the sequence $\left\{\dfrac{n^2+1}{2n-3}\right\}_{n=1}^{\infty}$ diverges.

In example 1.5, we see that a sequence doesn't need to tend to $\pm\infty$ in order to diverge.

EXAMPLE 1.5 A Divergent Sequence Whose Terms Do Not Tend to ∞

Determine the convergence or divergence of the sequence $\{(-1^n)\}_{n=1}^{\infty}$.

Solution If we write out the terms of the sequence, we have

$$\{-1, 1, -1, 1, -1, 1, \ldots\}.$$

That is, the terms of the sequence alternate back and forth between -1 and 1 and so the sequence diverges. To see this graphically, we plot the first few terms of the sequence in Figure 5.5. Notice that the points do not approach any limit (a horizontal line). ■

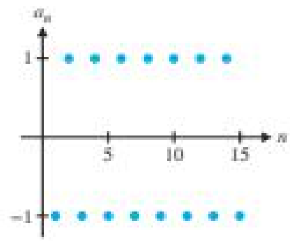


FIGURE 5.5
$a_n = (-1)^n$

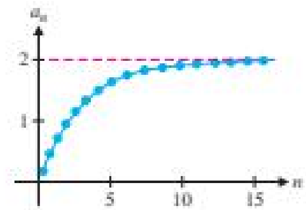


FIGURE 5.6
$a_n = f(n)$, where $f(x) \to 2$, as $x \to \infty$

You can use an advanced tool like l'Hôpital's Rule to find the limit of a sequence, but you must be careful. Theorem 1.2 says that if $f(x) \to L$ as $x \to \infty$ through all real values, then $f(n)$ must approach L, too, as $n \to \infty$ through integer values. (See Figure 5.6 for a graphical representation of this.)

THEOREM 1.2

Suppose that $\lim_{x\to\infty} f(x) = L$. Then, $\lim_{n\to\infty} f(n) = L$, also.

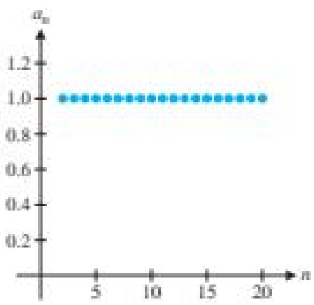


FIGURE 5.7a
$a_n = \cos(2\pi n)$

REMARK 1.2

The converse of Theorem 1.2 is false. That is, if $\lim_{n\to\infty} f(n) = L$, it need *not* be true that $\lim_{x\to\infty} f(x) = L$. This is clear from the following observation. Note that

$$\lim_{n\to\infty} \cos(2\pi n) = 1,$$

since $\cos(2\pi n) = 1$ for every *integer* n. (See Figure 5.7a.) However,

$$\lim_{x\to\infty} \cos(2\pi x) \text{ does not exist,}$$

since as $x \to \infty$, $\cos(2\pi x)$ oscillates between -1 and 1. (See Figure 5.7b.)

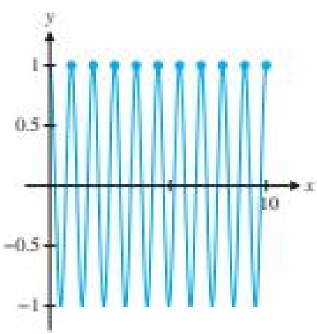


FIGURE 5.7b
$y = \cos(2\pi x)$

EXAMPLE 1.6 Applying l'Hôpital's Rule to a Related Function

Evaluate $\lim_{n\to\infty} \frac{n+1}{e^n}$.

Solution This has the indeterminate form $\frac{\infty}{\infty}$, but the graph in Figure 5.8 suggests that the sequence converges to 0. However, there is no obvious way to resolve this,

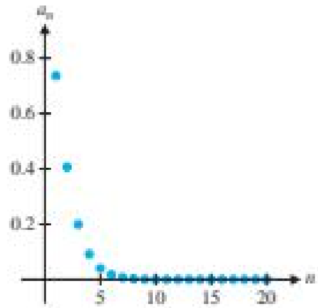


FIGURE 5.8
$a_n = \frac{n+1}{e^n}$

except by l'Hôpital's Rule (which does *not* apply to limits of sequences). So, we instead consider the limit of the corresponding function of a real variable to which we may apply l'Hôpital's Rule. (Be sure you check the hypotheses.) We have

$$\lim_{x\to\infty} \frac{x+1}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(x+1)}{\frac{d}{dx}(e^x)} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

From Theorem 1.2, we now have

$$\lim_{n\to\infty} \frac{n+1}{e^n} = 0, \text{ also.}$$ ■

For many sequences (including infinite series, which we study in the remainder of this chapter), we don't even have an explicit formula for the general term. In such cases, we must test for convergence in some indirect way. Our first indirect tool corresponds to the result (of the same name) for limits of functions of a real variable presented in section 1.3.

THEOREM 1.3 (Squeeze Theorem)

Suppose $\{a_n\}_{n=n_0}^{\infty}$ and $\{b_n\}_{n=n_0}^{\infty}$ are convergent sequences, both converging to the limit L. If there is an integer $n_1 \geq n_0$ such that for all $n \geq n_1$, $a_n \leq c_n \leq b_n$, then $\{c_n\}_{n=n_0}^{\infty}$ converges to L, too.

In example 1.7, we demonstrate how to apply the Squeeze Theorem to a sequence. Observe that the trick here is to find two sequences, one on each side of the given sequence (i.e. one larger and one smaller) that have the same limit.

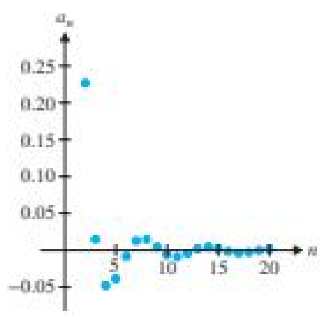


FIGURE 5.9
$a_n = \frac{\sin n}{n^2}$

EXAMPLE 1.7 Applying the Squeeze Theorem to a Sequence

Determine the convergence or divergence of $\left\{\frac{\sin n}{n^2}\right\}_{n=1}^{\infty}$.

Solution From the graph in Figure 5.9, the sequence appears to converge to 0, despite the oscillation. Further, note that you cannot compute this limit using the rules we have established so far. (Try it!) However, since

$$-1 \leq \sin n \leq 1, \text{ for all } n,$$

dividing through by n^2 gives us

$$\frac{-1}{n^2} \leq \frac{\sin n}{n^2} \leq \frac{1}{n^2}, \text{ for all } n \geq 1.$$

Finally, since
$$\lim_{n\to\infty} \frac{-1}{n^2} = 0 = \lim_{n\to\infty} \frac{1}{n^2},$$

the Squeeze Theorem also gives us that
$$\lim_{n\to\infty} \frac{\sin n}{n^2} = 0.$$ ■

The following useful result follows immediately from Theorem 1.3.

COROLLARY 1.1

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$, also.

PROOF

Notice that for all n, $$-|a_n| \le a_n \le |a_n|.$$

Further, $$\lim_{n\to\infty} |a_n| = 0 \quad \text{and} \quad \lim_{n\to\infty}(-|a_n|) = -\lim_{n\to\infty}|a_n| = 0.$$

So, from the Squeeze Theorem, $\lim_{n\to\infty} a_n = 0$, too. ■

Corollary 1.1 is particularly useful for sequences with both positive and negative terms, as in example 1.8.

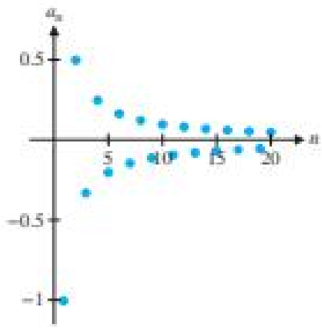


FIGURE 5.10

$a_n = \frac{(-1)^n}{n}$

EXAMPLE 1.8 A Sequence with Terms of Alternating Signs

Determine the convergence or divergence of $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$.

Solution The graph of the sequence in Figure 5.10 suggests that although the sequence oscillates, it still may be converging to 0. Since $(-1)^n$ oscillates back and forth between -1 and 1, we cannot compute $\lim_{n\to\infty} \frac{(-1)^n}{n}$ directly. However, notice that

$$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$$

and $$\lim_{n\to\infty} \frac{1}{n} = 0.$$

From Corollary 1.1, we get that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$, too. ■

We remind you of the following definition, which we use throughout the chapter.

DEFINITION 1.2

For any integer $n \ge 1$, the **factorial** $n!$ is defined as the product of the first n positive integers,

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n.$$

We define $0! = 1$.

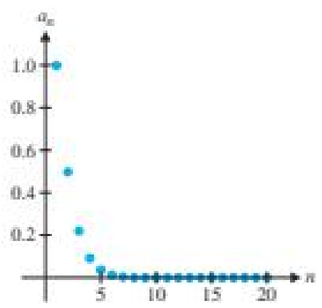


FIGURE 5.11

$a_n = \frac{n!}{n^n}$

Example 1.9 shows a sequence whose limit would be extremely difficult to find without the Squeeze Theorem.

EXAMPLE 1.9 A Proof of Convergence by the Squeeze Theorem

Investigate the convergence of $\left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty}$.

Solution First, notice that we have no means of computing $\lim_{n\to\infty} \frac{n!}{n^n}$ directly. (Try this!) From the graph of the sequence in Figure 5.11, it appears that the sequence is converging to 0. Notice that the general term of the sequence satisfies

$$0 < \frac{n!}{n^n} = \underbrace{\frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n}}_{n \text{ factors}} = \left(\frac{1}{n}\right)\underbrace{\frac{2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n}}_{n-1 \text{ factors}} \le \left(\frac{1}{n}\right)(1) = \frac{1}{n}. \quad (1.2)$$

From the Squeeze Theorem and (1.2), we have that since

$$\lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} 0 = 0,$$

then $$\lim_{n\to\infty} \frac{n!}{n^n} = 0, \text{ also.}$$ ■

Just as we did with functions of a real variable, we need to distinguish between sequences that are increasing and those that are decreasing. The definitions are straightforward.

DEFINITION 1.3

(i) The sequence $\{a_n\}_{n=1}^{\infty}$ is **increasing** if

$$a_1 \le a_2 \le \cdots \le a_n \le a_{n+1} \le \cdots.$$

(ii) The sequence $\{a_n\}_{n=1}^{\infty}$ is **decreasing** if

$$a_1 \ge a_2 \ge \cdots \ge a_n \ge a_{n+1} \ge \cdots.$$

If a sequence is either increasing or decreasing, it is called **monotonic**.

There are several ways to show that a sequence is monotonic. Regardless of which method you use, you will need to show that either $a_n \le a_{n+1}$ for all n (increasing) or $a_{n+1} \le a_n$ for all n (decreasing). We illustrate two very useful methods in examples 1.10 and 1.11.

EXAMPLE 1.10 An Increasing Sequence

Investigate whether the sequence $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ is increasing, decreasing or neither.

Solution From the graph in Figure 5.12, it appears that the sequence is increasing. However, you should not be deceived by looking at the first few terms of a sequence. More generally, we look at the ratio of two successive terms. Defining $a_n = \frac{n}{n+1}$, we have $a_{n+1} = \frac{n+1}{n+2}$ and so,

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{n+1}{n+2}\right)}{\left(\frac{n}{n+1}\right)} = \left(\frac{n+1}{n+2}\right)\left(\frac{n+1}{n}\right) = \frac{n^2+2n+1}{n^2+2n} = 1 + \frac{1}{n^2+2n} > 1. \quad (1.3)$$

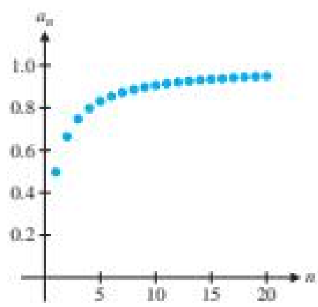


FIGURE 5.12
$a_n = \frac{n}{n+1}$

Multiplying both sides of (1.3) by $a_n > 0$, we obtain

$$a_{n+1} > a_n,$$

for all n and so, the sequence is increasing. Alternatively, consider the function $f(x) = \frac{x}{x+1}$ (of the real variable x) corresponding to the sequence. Observe that

$$f'(x) = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0,$$

which says that the function $f(x)$ is increasing. From this, it follows that the corresponding sequence $a_n = \frac{n}{n+1}$ is also increasing. ■

EXAMPLE 1.11 A Sequence That Is Increasing for $n \geq 2$

Investigate whether the sequence $\left\{\frac{n!}{e^n}\right\}_{n=1}^{\infty}$ is increasing, decreasing or neither.

Solution From the graph of the sequence in Figure 5.13, it appears that the sequence is increasing (and rather rapidly, at that). Here, for $a_n = \frac{n!}{e^n}$, we have $a_{n+1} = \frac{(n+1)!}{e^{n+1}}$, so that

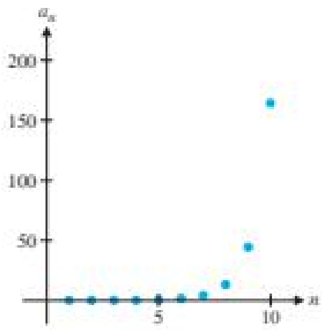


FIGURE 5.13

$a_n = \frac{n!}{e^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\left[\frac{(n+1)!}{e^{n+1}}\right]}{\left(\frac{n!}{e^n}\right)} = \frac{(n+1)!}{e^{n+1}}\frac{e^n}{n!}$$

$$= \frac{(n+1)n!e^n}{e(e^n)n!} = \frac{n+1}{e} > 1, \text{ for } n \geq 2. \tag{1.4}$$

Since $(n+1)! = (n+1) \cdot n!$ and $e^{n+1} = e \cdot e^n$.

Multiplying both sides of (1.4) by $a_n > 0$, we get

$$a_{n+1} > a_n, \text{ for } n \geq 2.$$

Notice that in this case, although the sequence is not increasing for all n, it is increasing for $n \geq 2$. Keep in mind that it doesn't really matter what the first few terms do, anyway. We are only concerned with the behavior of a sequence as $n \to \infty$. ■

We need to define one additional property of sequences.

DEFINITION 1.4

We say that the sequence $\{a_n\}_{n=n_0}^{\infty}$ is **bounded** if there is a number $M > 0$ (called a **bound**) for which $|a_n| \leq M$, for all n.

It is important to realize that a given sequence may have any number of bounds (for instance, if $|a_n| \leq 10$ for all n, then $|a_n| \leq 20$, for all n, too).

EXAMPLE 1.12 A Bounded Sequence

Show that the sequence $\left\{\frac{3-4n^2}{n^2+1}\right\}_{n=1}^{\infty}$ is bounded.

Solution We use the fact that $4n^2 - 3 > 0$, for all $n \geq 1$, to get

$$|a_n| = \left|\frac{3-4n^2}{n^2+1}\right| = \frac{4n^2-3}{n^2+1} < \frac{4n^2}{n^2+1} < \frac{4n^2}{n^2} = 4.$$

So, this sequence is bounded by 4. (We might also say in this case that the sequence is bounded between −4 and 4.) Further, note that we could also use any number greater than 4 as a bound. ■

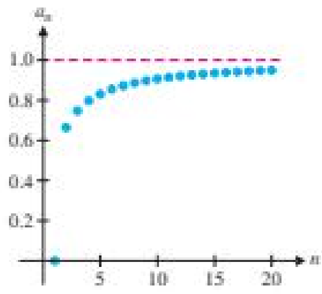


FIGURE 5.14a
A bounded and increasing sequence

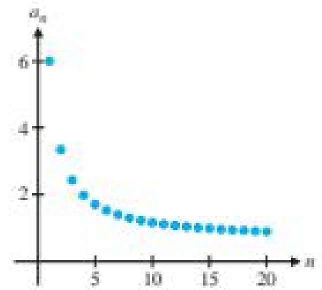


FIGURE 5.14b
A bounded and decreasing sequence

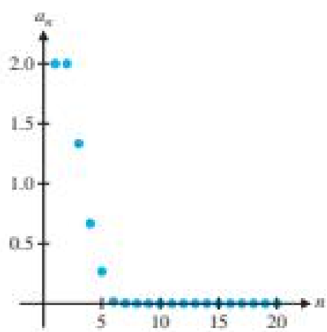


FIGURE 5.15
$a_n = \dfrac{2^n}{n!}$

n	$a_n = \dfrac{2^n}{n!}$
2	2
4	0.666667
6	0.088889
8	0.006349
10	0.000282
12	0.0000086
14	1.88×10^{-7}
16	3.13×10^{-9}
18	4.09×10^{-11}
20	4.31×10^{-13}

Theorem 1.4 provides a powerful tool for the investigation of sequences.

THEOREM 1.4

Every bounded, monotonic sequence converges.

A typical bounded and increasing sequence is illustrated in Figure 5.14a, while a bounded and decreasing sequence is illustrated in Figure 5.14b. In both figures, notice that a bounded and monotonic sequence can't oscillate or increase/decrease without a bound and, consequently, must converge. The proof of Theorem 1.4 is rather involved and we leave it to the end of the section.

Theorem 1.4 says that if we can show that a sequence is bounded and monotonic, then it must also be convergent, although we may have little idea of what its limit might be. Still, once we establish that a sequence converges, we can approximate its limit by computing a sufficient number of terms, as in example 1.13.

EXAMPLE 1.13 A Bounded, Monotonic Sequence

Investigate the convergence of the sequence $\left\{\dfrac{2^n}{n!}\right\}_{n=1}^{\infty}$.

Solution First, note that we do not know how to compute $\lim\limits_{n\to\infty} \dfrac{2^n}{n!}$. This has the indeterminate form $\dfrac{\infty}{\infty}$, but we cannot use l'Hôpital's Rule here directly or indirectly. (Why not?) The graph in Figure 5.15 suggests that the sequence converges to 0. To confirm this suspicion, we first show that the sequence is monotonic. We have

$$\frac{a_{n+1}}{a_n} = \frac{\left[\dfrac{2^{n+1}}{(n+1)!}\right]}{\left(\dfrac{2^n}{n!}\right)} = \frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n}$$

$$= \frac{2(2^n)n!}{(n+1)n!2^n} = \frac{2}{n+1} \leq 1, \text{ for } n \geq 1. \quad (1.5)$$

Since $2^{n+1} = 2 \cdot 2^n$ and $(n+1)! = (n+1)n!$.

Multiplying both sides of (1.5) by $a_n > 0$ gives us $a_{n+1} \leq a_n$ for all n and so, the sequence is decreasing. Next, since the sequence is decreasing, we have that

$$|a_n| = \frac{2^n}{n!} \leq \frac{2^1}{1!} = 2,$$

for $n \geq 1$ (i.e., the sequence is bounded by 2). Since the sequence is both bounded and monotonic, it must be convergent, by Theorem 1.4. We display a number of terms of the sequence in the table in the margin, from which it appears that the sequence is converging to approximately 0. We can make a slightly stronger statement, though. Since we have established that the sequence is *decreasing* and convergent, we have from our computations that

$$0 \leq a_n \leq a_{20} \approx 4.31 \times 10^{-13}, \quad \text{for } n \geq 20.$$

Further, the limit L must also satisfy the inequality

$$0 \leq L \leq 4.31 \times 10^{-13}.$$

We can confirm that the limit is 0, as follows. From (1.5),

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(\frac{2}{n+1}\right)a_n,$$

so that

$$L = \left(\lim_{n\to\infty}\frac{2}{n+1}\right)\left(\lim_{n\to\infty} a_n\right) = 0 \cdot L = 0.$$

REMARK 1.3

Do not underestimate the importance of Theorem 1.4. This indirect way of testing a sequence for convergence takes on additional significance when we study infinite series (a special type of sequence that is the topic of the remainder of this chapter).

Before we can prove Theorem 1.4, we need to state one of the properties of the real number system.

THE COMPLETENESS AXIOM

If a nonempty set S of real numbers has a lower bound, then it has a *greatest lower bound.* Equivalently, if such a set has an upper bound, it has a *least upper bound.*

This axiom says that if a nonempty set S has an upper bound, that is, a number M such that

$$x \le M, \text{ for all } x \in S,$$

then there is an upper bound L, for which

$$L \le M \text{ for } all \text{ upper bounds, } M,$$

with a corresponding statement holding for lower bounds.

The Completeness Axiom enables us to prove Theorem 1.4.

PROOF

(Increasing sequence) Suppose that $\{a_n\}_{n=n_0}^{\infty}$ is increasing and bounded. This is illustrated in Figure 5.16, where you can see an increasing sequence bounded by 1. We have

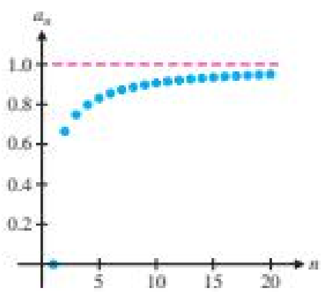


FIGURE 5.16
Bounded and increasing

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le a_{n+1} \le \cdots$$

and for some number $M > 0$, $|a_n| \le M$ for all n. This is the same as saying that

$$-M \le a_n \le M, \quad \text{for all } n.$$

Now, let S be the set containing all of the terms of the sequence, $S = \{a_1, a_2, \ldots, a_n, \ldots\}$. Notice that M is an upper bound for the set S. From the Completeness Axiom, S must have a least upper bound, L. That is, L is the *smallest* number for which

$$a_n \le L, \quad \text{for all } n. \tag{1.6}$$

Notice that for any number $\varepsilon > 0$, $L - \varepsilon < L$ and so, $L - \varepsilon$ is *not* an upper bound, since L is the *least* upper bound. Since $L - \varepsilon$ is not an upper bound for S, there is some element, a_N, of S for which

$$L - \varepsilon < a_N.$$

Since $\{a_n\}$ is increasing, we have that for $n \ge N$, $a_N \le a_n$. Finally, from (1.6) and the fact that L is an upper bound for S and since $\varepsilon > 0$, we have

$$L - \varepsilon < a_N \le a_n \le L < L + \varepsilon,$$

or more simply

$$L - \varepsilon < a_n < L + \varepsilon,$$

for $n \ge N$. This is equivalent to

$$|a_n - L| < \varepsilon, \quad \text{for } n \ge N,$$

which says that $\{a_n\}$ converges to L. The proof for the case of a decreasing sequence is similar and is left as an exercise. ■

BEYOND FORMULAS

The essential logic behind sequences is the same as that behind much of calculus. When evaluating limits (including limits of sequences and those that define derivatives and integrals), we are frequently able to compute an exact answer directly, as in example 1.3. However, some limits are more difficult to determine and can be found only by using an indirect method, as in example 1.13. Such indirect methods prove to be extremely important (and increasingly common) as we expand our study of sequences to those defining infinite series in the rest of this chapter.

EXERCISES 5.1

WRITING EXERCISES

1. Compare and contrast $\lim_{x\to\infty} \sin \pi x$ and $\lim_{n\to\infty} \sin \pi n$. Indicate the domains of the two functions and how they affect the limits.

2. Explain why Theorem 1.2 should be true, taking into account the respective domains and their effect on the limits.

3. In words, explain why a decreasing bounded sequence must converge.

4. A sequence is said to diverge if it does not converge. The word "diverge" is well chosen for sequences that diverge to ∞, but is less descriptive of sequences such as {1, 2, 1, 2, 1, 2, . . .} and {1, 2, 3, 1, 2, 3, . . .}. Briefly describe the limiting behavior of these sequences and discuss other possible limiting behaviors of divergent sequences.

In exercises 1–4, write out the terms $a_1, a_2, \ldots, a_6$ of the given sequence.

1. $a_n = \dfrac{2n-1}{n^2}$
2. $a_n = \dfrac{3}{n+4}$
3. $a_n = \dfrac{4}{n!}$
4. $a_n = (-1)^n \dfrac{n}{n+1}$

In exercises 5–8, (a) find the limit of each sequence, (b) use the definition to show that the sequence converges and (c) plot the sequence on a calculator or CAS.

5. $a_n = \dfrac{1}{n^3}$
6. $a_n = \dfrac{2}{\sqrt{n}}$
7. $a_n = \dfrac{n}{n+1}$
8. $a_n = \dfrac{2n+1}{n}$

9. Plot each sequence in exercises 5–8 and illustrate the convergence.

10. Plot the sequence $a_n = \left(\sin \dfrac{n\pi}{2} + \cos \dfrac{n\pi}{2}\right)\dfrac{n}{n+1}$ and describe the behavior of the sequence.

In exercises 11–24, determine whether the sequence converges or diverges.

11. $a_n = \dfrac{3n^2+1}{2n^2-1}$
12. $a_n = \dfrac{5n^3-1}{2n^3+1}$
13. $a_n = \dfrac{n^2+1}{n+1}$
14. $a_n = \dfrac{n^2+1}{n^3+1}$
15. $a_n = (-1)^n \dfrac{n+2}{3n-1}$
16. $a_n = (-1)^n \dfrac{n+4}{n+1}$
17. $a_n = (-1)^n \dfrac{n+2}{n^2+4}$
18. $a_n = \cos \pi n$
19. $a_n = ne^{-n}$
20. $a_n = \dfrac{\cos n}{e^n}$
21. $a_n = \dfrac{e^n+2}{e^{2n}-1}$
22. $a_n = \dfrac{3^n}{e^n+1}$
23. $a_n = \dfrac{n2^n}{3^n}$
24. $a_n = \dfrac{n!}{2^n}$

In exercises 25–30, evaluate each limit.

25. $\lim_{n\to\infty} n \sin \dfrac{1}{n}$
26. $\lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right)$
27. $\lim_{n\to\infty} [\ln(2n+1) - \ln(n)]$
28. $\lim_{n\to\infty} \left|\cos \dfrac{n\pi}{2}\right| \dfrac{2n-1}{n+2}$
29. $\lim_{n\to\infty} \dfrac{n^3+1}{e^n}$
30. $\lim_{n\to\infty} \dfrac{\ln(n)}{\sqrt{n+1}}$

In exercises 31–34, use the Squeeze Theorem and Corollary 1.1 to prove that the sequence converges to 0 (given that $\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^2} = 0$).

31. $a_n = \dfrac{\cos n}{n^2}$
32. $a_n = \dfrac{\cos n\pi}{n^2}$
33. $a_n = (-1)^n \dfrac{e^{-n}}{n}$
34. $a_n = (-1)^n \dfrac{\ln n}{n^2}$

In exercises 35–38, determine whether the sequence is increasing, decreasing or neither.

35. $a_n = \dfrac{n+3}{n+2}$ **36.** $a_n = \dfrac{n-1}{n+1}$

37. $a_n = \dfrac{e^n}{n}$ **38.** $a_n = \dfrac{3^n}{(n+2)!}$

In exercises 39–42, show that the sequence is bounded.

39. $a_n = \dfrac{3n^2-2}{n^2+1}$ **40.** $a_n = \dfrac{6n-1}{n+3}$

41. $a_n = \dfrac{\sin(n^2)}{n+1}$ **42.** $a_n = e^{1/n}$

In exercises 43–46, write a formula that produces the given terms of the sequence.

43. $a_1 = \frac{1}{8}, a_2 = \frac{-1}{4}, a_3 = \frac{1}{2}, a_4 = -1, a_5 = 2$

44. $a_1 = 1, a_2 = \frac{1}{3}, a_3 = \frac{1}{5}, a_4 = \frac{1}{7}, a_5 = \frac{1}{9}$

45. $a_1 = 1, a_2 = \frac{3}{4}, a_3 = \frac{5}{9}, a_4 = \frac{7}{16}, a_5 = \frac{9}{25}$

46. $a_1 = \frac{1}{4}, a_2 = \frac{-2}{9}, a_3 = \frac{3}{16}, a_4 = \frac{-4}{25}, a_5 = \frac{5}{36}$

47. Prove Theorem 1.4 for a decreasing sequence.

48. Define the sequence a_n with $a_1 = \sqrt{3}$ and $a_n = \sqrt{3 + 2a_{n-1}}$ for $n \geq 2$. Show that $\{a_n\}$ converges and estimate the limit of the sequence.

49. (a) Numerically estimate the limits of the sequences $a_n = \left(1 + \frac{2}{n}\right)^n$ and $b_n = \left(1 - \frac{2}{n}\right)^n$. Compare the answers to e^2 and e^{-2}.

(b) Given that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$, show that $\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n = e^r$ for any constant r. (Hint: Make the substitution $n = m/r$.)

50. (a) Suppose that $a_1 = 1$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{4}{a_n}\right)$. Show numerically that the sequence converges to 2. To find this limit analytically, let $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$ and solve the equation $L = \frac{1}{2}\left(L + \frac{4}{L}\right)$.

(b) Determine the limit of the sequence defined by $a_1 = 1$ $a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$ for $c > 0$ and $a_n > 0$.

51. Define the sequence a_n with $a_1 = \sqrt{2}$ and $a_n = \sqrt{2 + \sqrt{a_{n-1}}}$ for $n \geq 2$. Show that $\{a_n\}$ is increasing and bounded by 2. Evaluate the limit of the sequence by estimating the appropriate solution of $x = \sqrt{2 + \sqrt{x}}$.

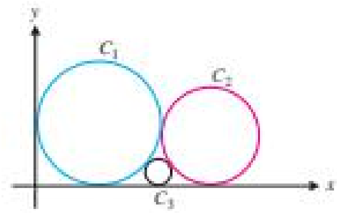
52. (a) Find all values of p such that the sequence $a_n = \frac{1}{p^n}$ converges.

(b) Find all values of p such that the sequence $a_n = \frac{1}{n^p}$ converges.

53. Define $a_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}$. Evaluate the sum using a formula from section 4.2 and show that the sequence converges. By thinking of a_n as a Riemann sum, identify the definite integral to which the sequence converges.

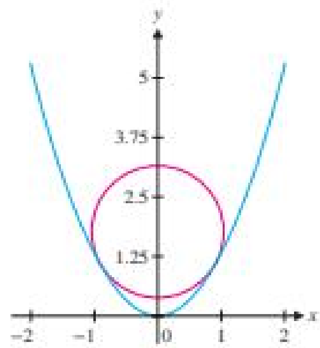
54. Define $a_n = \sum_{k=1}^{n} \dfrac{1}{n+k}$. By thinking of a_n as a Riemann sum, identify the definite integral to which the sequence converges.

55. Start with two circles C_1 and C_2 of radius r_1 and r_2, respectively, that are tangent to each other and each tangent to the x-axis. Construct the circle C_3 that is tangent to C_1, C_2 and the x-axis. (See the accompanying figure.) (a) If the centers of C_1 and C_2 are (c_1, r_1) and (c_2, r_2), respectively, show that $(c_2 - c_1)^2 + (r_2 - r_1)^2 = (r_1 + r_2)^2$ and then $|c_2 - c_1| = 2\sqrt{r_1 r_2}$. (b) Find similar relationships for circles C_1 and C_3 and for circles C_2 and C_3. (c) Show that the radius r_3 of C_3 is given by $\sqrt{r_3} = \dfrac{\sqrt{r_1 r_2}}{\sqrt{r_1} + \sqrt{r_2}}$.



(d) Construct a sequence of circles where C_4 is tangent to C_2, C_3 and the x-axis; then C_5 is tangent to C_3, C_4 and the x-axis. If you start with unit circles $r_1 = r_2 = 1$, find a formula for the radius r_n in terms of F_n, the nth term in the Fibonacci sequence of exercises 61 and 62. (Suggested by James Albrecht.)

56. (a) Let C be the circle of radius r inscribed in the parabola $y = x^2$. (See the figure.) Show that the y-coordinate c of the center of the circle equals $c = \frac{1}{4} + r^2$.



(b) Let C_1 be the circle of radius $r_1 = 1$ inscribed in $y = x^2$. Construct a sequence of circles C_2, C_3 and so on, where each circle C_n rests on top of the previous circle C_{n-1} (that is, C_n is tangent to C_{n-1}) and is inscribed in the parabola. If r_n is the radius of circle C_n, find a (simple) formula for r_n. (Suggested by Gregory Minton.)

57. Archimedes showed that if S_n is the length of an n-gon inscribed in a circle, then $S_{2n} = \sqrt{2 - \sqrt{4 - S_n^2}}$. If $S_6 = 1$, find S_{48} and show that $S_{48} \approx \dfrac{\pi}{24}$.

58. Show that $\dfrac{n!}{n^n} < \dfrac{1}{n}$ and prove that $\lim_{n\to\infty} \dfrac{n!}{n^n} = 0$. Nevertheless, show numerically that $\lim_{n\to\infty} \dfrac{\ln(n!)}{\ln(n^n)} = 1$. What graphical property of $\ln x$ explains this?

APPLICATIONS

1. A jewelry company works with 12 cm square boxes. Show that for $n = 1, 2, 3, \ldots$, a total of n^2 circular disks of diameter $\frac{12\text{ cm}}{n}$ fit into the bottom of a box. Let a_n be the wasted area in the bottom of a box with n^2 disks. Compute a_n.

2. The pattern of a sequence can't always be determined from the first few terms. Start with a circle, pick two points on the circle and connect them with a line segment. The circle is divided into $a_1 = 2$ regions. Add a third point, connect all points and show that there are now $a_2 = 4$ regions. Add a fourth point, connect all points and show that there are $a_3 = 8$ regions. Is the pattern clear? Show that $a_4 = 16$ and then compute a_5 for a surprise!

3. A different population model was studied by Fibonacci, an Italian mathematician of the thirteenth century. He imagined a population of rabbits starting with a pair of newborns. For one month, they grow and mature. The second month, they have a pair of newborn baby rabbits. We count the number of pairs of rabbits. Thus far, $a_0 = 1$, $a_1 = 1$ and $a_2 = 2$. The rules are: adult rabbit pairs give birth to a pair of newborns every month, newborns take one month to mature and no rabbits die. Show that $a_3 = 3$, $a_4 = 5$ and in general $a_n = a_{n-1} + a_{n-2}$. This sequence of numbers, known as the **Fibonacci sequence**, occurs in an amazing number of applications.

4. In this exercise, we visualize the Fibonacci sequence (see exercise 3). Start with two squares of side 1 placed next to each other (see Figure A). Place a square on the long side of the resulting rectangle (see Figure B); this square has side 2. Continue placing squares on the long sides of the rectangles: a square of side 3 is added in Figure C, then a square of side 5 is added to the bottom of Figure C, and so on.

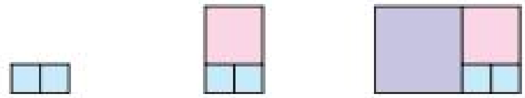


FIGURE A FIGURE B FIGURE C

Argue that the sides of the squares are determined by the Fibonacci sequence of exercise 3.

EXPLORATORY EXERCISES

1. (a) If $a_1 = 3$ and $a_{n+1} = a_n + \sin a_n$ for $n \geq 2$, show numerically that $\{a_n\}$ converges to π. With the same relation $a_{n+1} = a_n + \sin a_n$, try other starting values a_1 (Hint: Try $a_1 = -3$, $a_1 = \pm 6$, $a_1 = \pm 9$, $a_1 = \pm 12$ and other values.) and state a general rule for the limit of the sequence as a function of the starting value. (b) If $a_1 = 6$ and $a_{n+1} = a_n - \sin a_n$ for $n \geq 2$, numerically estimate the limit of $\{a_n\}$ in terms of π. Then try other starting values and state a general rule for the limit of the sequence as a function of the starting value. (c) State a general rule for the limit of the sequence with $a_{n+1} = a_n + \cos a_n$ as a function of the starting value a_1.

2. Let $a_1 = 0$, $a_2 = 1$ and $a_n = \dfrac{a_{n-1} + a_{n-2}}{2}$ for $n \geq 3$. Find a general formula for $|a_n - a_{n-1}|$ and prove that the sequence converges. Show that $a_n = \frac{2}{3} + \frac{1}{3}\left(-\frac{1}{2}\right)^{n-2}$ and find the limit of the sequence. Generalize to $a_1 = a$, $a_2 = b$.

5.2 INFINITE SERIES

Recall that we write the decimal expansion of $\frac{1}{3}$ as the repeating decimal $\frac{1}{3} = 0.3333333\overline{3}$, where we understand that the 3s in this expansion go on forever. Alternatively, we can think of this as

$$\begin{aligned} \frac{1}{3} &= 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots \\ &= 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + 3(0.1)^4 + \cdots + 3(0.1)^k + \cdots. \end{aligned} \tag{2.1}$$

For convenience, we write (2.1) using summation notation as

$$\frac{1}{3} = \sum_{k=1}^{\infty} 3(0.1)^k. \tag{2.2}$$

Since we can't add together infinitely many terms, we need to carefully define the *infinite sum* indicated in (2.2). Equation (2.2) means that as you add together more and more terms, the sum gets closer and closer to $\frac{1}{3}$.

In general, for any sequence $\{a_k\}_{k=1}^{\infty}$, suppose we start adding the terms together. We define the **partial sums** $S_1, S_2, \ldots, S_n, \ldots$ by

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2 = S_1 + a_2, \end{aligned}$$

$$S_3 = \underbrace{a_1 + a_2}_{S_2} + a_3 = S_2 + a_3,$$

$$S_4 = \underbrace{a_1 + a_2 + a_3}_{S_3} + a_4 = S_3 + a_4,$$

$$\vdots$$

$$S_n = \underbrace{a_1 + a_2 + \cdots + a_{n-1}}_{S_{n-1}} + a_n = S_{n-1} + a_n, \tag{2.3}$$

and so on. Note that each partial sum S_n is the sum of two numbers: the nth term, a_n, and the previous partial sum, S_{n-1}, as indicated in (2.3).

For instance, for the sequence $\left\{\frac{1}{2^k}\right\}_{k=1}^{\infty}$, consider the partial sums

$$S_1 = \frac{1}{2}, \qquad S_2 = \frac{1}{2} + \frac{1}{2^2} = \frac{3}{4},$$

$$S_3 = \frac{3}{4} + \frac{1}{2^3} = \frac{7}{8}, \qquad S_4 = \frac{7}{8} + \frac{1}{2^4} = \frac{15}{16}$$

and so on. Look at these carefully and you might notice that $S_2 = \frac{3}{4} = 1 - \frac{1}{2^2}$, $S_3 = \frac{7}{8} = 1 - \frac{1}{2^3}$, $S_4 = \frac{15}{16} = 1 - \frac{1}{2^4}$ and so on, so that $S_n = 1 - \frac{1}{2^n}$, for each $n = 1, 2, \ldots$. Observe that the sequence $\{S_n\}_{n=1}^{\infty}$ of partial sums converges, since

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1.$$

This says that as we add together more and more terms of the sequence $\left\{\frac{1}{2^k}\right\}_{k=1}^{\infty}$, the partial sums are drawing closer and closer to 1. In view of this, we write

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1. \tag{2.4}$$

It's very important to understand what's going on here. This new mathematical object, $\sum_{k=1}^{\infty} \frac{1}{2^k}$, is called a **series** (or **infinite series**). It is *not a sum* in the usual sense of the word, but rather, the *limit* of the sequence of partial sums. Equation (2.4) says that as we add together more and more terms, the sums are approaching the limit of 1.

In general, for any sequence, $\{a_k\}_{k=1}^{\infty}$, we can write down the series

$$a_1 + a_2 + \cdots + a_k + \cdots = \sum_{k=1}^{\infty} a_k.$$

DEFINITION 2.1

If the sequence of partial sums $S_n = \sum_{k=1}^{n} a_k$ converges (to some number S), then we say that the series $\sum_{k=1}^{\infty} a_k$ **converges** (to S) and write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = S. \tag{2.5}$$

In this case, we call S the **sum** of the series. Alternatively, if the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ diverges (i.e., $\lim_{n\to\infty} S_n$ does not exist), then we say that the series **diverges**.

EXAMPLE 2.1 A Convergent Series

Determine whether the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges or diverges.

Solution From our work on the introductory example, observe that

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1.$$

In this case, we say that the series converges to 1. ■

In example 2.2, we examine a simple divergent series.

EXAMPLE 2.2 A Divergent Series

Investigate the convergence or divergence of the series $\sum_{k=1}^{\infty} k^2$.

Solution Here, the nth partial sum is

$$S_n = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2$$

and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (1^2 + 2^2 + \cdots + n^2) = \infty.$$

Since the sequence of partial sums diverges, the series diverges also. ■

Determining the convergence or divergence of a series is only rarely as simple as it was in examples 2.1 and 2.2.

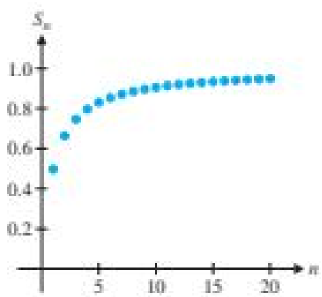


FIGURE 5.17
$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$

n	$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$
10	0.90909091
100	0.99009901
1000	0.999001
10,000	0.99990001
100,000	0.99999
1×10^6	0.999999
1×10^7	0.9999999

EXAMPLE 2.3 A Series with a Simple Expression for the Partial Sums

Investigate the convergence or divergence of the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.

Solution In Figure 5.17, we have plotted the first 20 partial sums. In the accompanying table, we list a number of partial sums of the series.

From both the graph and the table, it appears that the partial sums are approaching 1, as $n \to \infty$. However, we must urge caution. It is extremely difficult to look at a graph or a table of any partial sums and decide whether a given series converges or diverges. In the present case, we can find a simple expression for the partial sums. The partial fractions decomposition of the general term of the series is

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}. \tag{2.6}$$

Now, consider the nth partial sum. From (2.6), we have

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right). \end{aligned}$$

Notice how nearly every term in the partial sum is canceled by another term in the sum (the next term). For this reason, such a sum is referred to as a **telescoping sum** (or **collapsing sum**). We now have

$$S_n = 1 - \frac{1}{n+1}$$

and so,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1.$$

This says that the series converges to 1, as suggested by the graph and the table. ■

It is rare that we can find the sum of a convergent series exactly. Usually, we must test a series for convergence using some indirect method and then approximate the sum by calculating some partial sums. The series we considered in example 2.1, $\sum_{k=1}^{\infty}\frac{1}{2^k}$, is an example of a *geometric series,* whose sum is known exactly. We have the following result.

NOTES

A geometric series is any series that can be written in the form $\sum_{k=0}^{\infty} ar^k$ for non-zero constants a and r. In this case, each term in the series equals the constant r times the previous term.

THEOREM 2.1

For $a \neq 0$, the **geometric series** $\sum_{k=0}^{\infty} ar^k$ converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$. (Here, r is referred to as the **ratio**.)

PROOF

The proof relies on a clever observation. Since the first term of the series corresponds to $k = 0$, the nth partial sum (the sum of the first n terms) is

$$S_n = a + ar^1 + ar^2 + \cdots + ar^{n-1}. \tag{2.7}$$

Multiplying (2.7) by r, we get

$$rS_n = ar^1 + ar^2 + ar^3 + \cdots + ar^n. \tag{2.8}$$

Subtracting (2.8) from (2.7), we get

$$\begin{aligned}(1-r)S_n &= (a + ar^1 + ar^2 + \cdots + ar^{n-1}) - (ar^1 + ar^2 + ar^3 + \cdots + ar^n)\\ &= a - ar^n = a(1-r^n).\end{aligned}$$

Dividing both sides by $(1 - r)$ gives us

$$S_n = \frac{a(1-r^n)}{1-r}.$$

If $|r| < 1$, notice that $r^n \to 0$ as $n \to \infty$ and so,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}.$$

We leave it as an exercise to show that if $|r| \geq 1$, $\lim_{n\to\infty} S_n$ does not exist. ■

EXAMPLE 2.4 A Convergent Geometric Series

Investigate the convergence or divergence of the series $\sum_{k=2}^{\infty} 5\left(\frac{1}{3}\right)^k$.

Solution The first 20 partial sums are plotted in Figure 5.18. It appears from the graph that the sequence of partial sums is converging to some number around 0.8. Further evidence is found in the adjacent table of partial sums.

n	$S_n = \sum_{k=2}^{n+1} 5\left(\frac{1}{3}\right)^k$
6	0.83219021
8	0.83320632
10	0.83331922
12	0.83333177
14	0.83333316
16	0.83333331
18	0.83333333
20	0.83333333

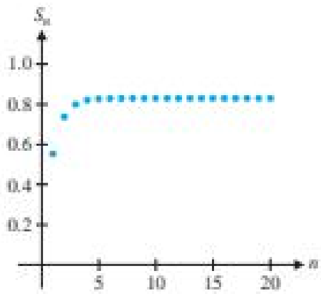


FIGURE 5.18
$S_n = \sum_{k=2}^{n+1} 5\left(\frac{1}{3}\right)^k$

While the table suggests that the series converges to approximately 0.83333333, we must urge caution. Some sequences and series converge (or diverge) far too slowly to observe graphically or numerically. You must *always* confirm your suspicions with careful mathematical analysis. In the present case, note that the series is geometric, as follows:

$$\begin{aligned}\sum_{k=2}^{\infty} 5\left(\frac{1}{3}\right)^k &= 5\left(\frac{1}{3}\right)^2 + 5\left(\frac{1}{3}\right)^3 + 5\left(\frac{1}{3}\right)^4 + \cdots + 5\left(\frac{1}{3}\right)^n + \cdots \\ &= 5\left(\frac{1}{3}\right)^2\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots\right] \\ &= \sum_{k=0}^{\infty}\left\{5\left(\frac{1}{3}\right)^2\left(\frac{1}{3}\right)^k\right\}.\end{aligned}$$

You can now see that this is a geometric series with ratio $r = \frac{1}{3}$ and $a = 5\left(\frac{1}{3}\right)^2$. Further, since

$$|r| = \frac{1}{3} < 1,$$

we have from Theorem 2.1 that the series converges to

$$\frac{a}{1-r} = \frac{5\left(\frac{1}{3}\right)^2}{1-\left(\frac{1}{3}\right)} = \frac{\left(\frac{5}{9}\right)}{\left(\frac{2}{3}\right)} = \frac{5}{6} = 0.8333333\bar{3},$$

which is consistent with the graph and the table of partial sums. ■

EXAMPLE 2.5 A Divergent Geometric Series

Investigate the convergence or divergence of the series $\sum_{k=0}^{\infty} 6\left(-\frac{7}{2}\right)^k$.

Solution A graph showing the first 20 partial sums (see Figure 5.19) is not particularly helpful, until you look at the vertical scale. The following table showing a number of partial sums is more revealing.

n	$S_n = \sum_{k=0}^{n-1} 6\left(-\frac{7}{2}\right)^k$
11	1.29×10^6
12	-4.5×10^6
13	1.6×10^7
14	-5.5×10^7
15	1.9×10^8
16	-6.8×10^8
17	2.4×10^9
18	-8.3×10^9
19	2.9×10^{10}
20	-1×10^{11}

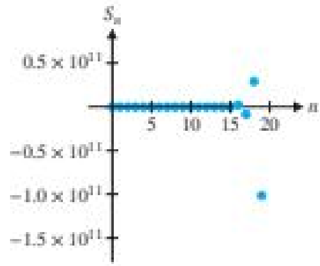


FIGURE 5.19

$$S_n = \sum_{k=0}^{n-1} 6\left(-\frac{7}{2}\right)^k$$

Note that while the partial sums are oscillating back and forth between positive and negative values, they are growing in absolute value. We can confirm our suspicions by observing that this is a geometric series with ratio $r = -\frac{7}{2}$. Since

$$|r| = \left|-\frac{7}{2}\right| = \frac{7}{2} \geq 1,$$

the series is divergent, as we suspected. ■

The following simple observation provides us with a very useful test.

THEOREM 2.2

If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

PROOF

Suppose that $\sum_{k=1}^{\infty} a_k$ converges to some number L. This means that the sequence of partial sums defined by $S_n = \sum_{k=1}^{n} a_k$ also converges to L. Notice that

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n-1} a_k + a_n = S_{n-1} + a_n.$$

Subtracting S_{n-1} from both sides, we have

$$a_n = S_n - S_{n-1}.$$

This gives us

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0,$$

as desired. ■

REMARK 2.1

The converse of Theorem 2.2 is *false*. That is, having $\lim_{k\to\infty} a_k = 0$ does *not* guarantee that the series $\sum_{k=1}^{\infty} a_k$ converges. *Be very clear about this point. This is a very common misconception.*

The following very useful test follows directly from Theorem 2.2.

kth-Term Test for Divergence

If $\lim_{k\to\infty} a_k \neq 0$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

The kth-term test is so simple, you should use it to test every series you run into. It says that if the terms don't tend to zero, the series is divergent and there's nothing more to do. However, as we'll soon see, if the terms *do* tend to zero, the series may or may not converge and additional testing is needed.

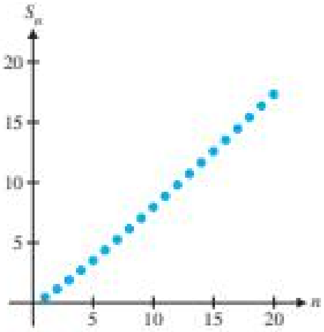


FIGURE 5.20

$S_n = \sum_{k=1}^{n} \frac{k}{k+1}$

EXAMPLE 2.6 A Series Whose Terms Do Not Tend to Zero

Investigate the convergence or divergence of the series $\sum_{k=1}^{\infty} \frac{k}{k+1}$.

Solution A graph showing the first 20 partial sums is shown in Figure 5.20. The partial sums appear to be increasing without bound as n increases. Further,

$$\lim_{k\to\infty} \frac{k}{k+1} = 1 \neq 0.$$

So, by the kth-term test for divergence, the series must diverge. ■

Example 2.7 shows an important series whose terms tend to 0 as $k \to \infty$, but that diverges, nonetheless.

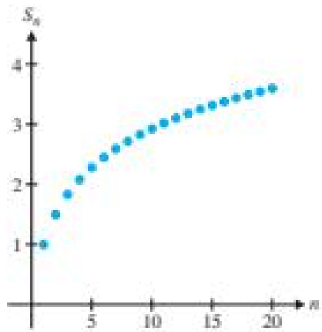


FIGURE 5.21

$S_n = \sum_{k=1}^{n} \frac{1}{k}$

EXAMPLE 2.7 The Harmonic Series

Investigate the convergence or divergence of the **harmonic series**: $\sum_{k=1}^{\infty} \frac{1}{k}$.

Solution In Figure 5.21, we see the first 20 partial sums of the series. In the table, we display several partial sums. The table and the graph suggest that the series might converge to a number around 3.6. As always with sequences and series, we need to confirm this suspicion. First, note that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0.$$

Be careful: once again, this does *not* say that the series converges. If the limit had been nonzero, we would have concluded that the series diverges. In the present case, where the limit is 0, we can conclude only that the series *may* converge, but we will need to investigate further.

The following clever proof provides a preview of things to come. Consider the nth partial sum

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Note that S_n corresponds to the sum of the areas of the n rectangles superimposed on the graph of $y = \frac{1}{x}$, as shown in Figure 5.22 for the case where $n = 7$.

Since each of the indicated rectangles lies partly above the curve, we have

$$\begin{aligned} S_n &= \text{Sum of areas of } n \text{ rectangles} \\ &\geq \text{Area under the curve} = dx \int_1^{n+1} \frac{1}{x} \\ &= \ln |x| \Big|_1^{n+1} = \ln(n+1). \end{aligned} \quad (2.9)$$

However, the sequence $\{\ln(n+1)\}_{n=1}^{\infty}$ diverges, since

$$\lim_{n\to\infty} \ln(n+1) = \infty .$$

Since $S_n \geq \ln(n+1)$, for all n [from (2.9)], we must also have that $\lim_{n\to\infty} S_n = \infty$, from which it follows that the series, $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, too, even though $\lim_{k\to\infty} \frac{1}{k} = 0$. ■

n	$S_n = \sum_{k=1}^{n} \frac{1}{k}$
11	3.01988
12	3.10321
13	3.18013
14	3.25156
15	3.31823
16	3.38073
17	3.43955
18	3.49511
19	3.54774
20	3.59774

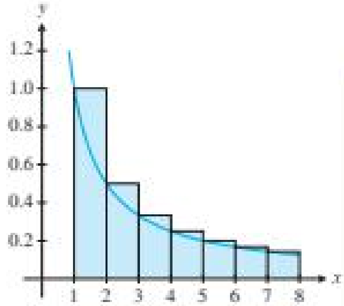


FIGURE 5.22
$y = \frac{1}{x}$

We conclude this section with several unsurprising results.

THEOREM 2.3

(i) If $\sum_{k=1}^{\infty} a_k$ converges to A and $\sum_{k=1}^{\infty} b_k$ converges to B, then the series $\sum_{k=1}^{\infty} (a_k \pm b_k)$ converges to $A \pm B$ and $\sum_{k=1}^{\infty} (c\,a_k)$ converges to cA, for any constant c.

(ii) If $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} (a_k \pm b_k)$ diverges.

The proof of this theorem is left as an exercise.

BEYOND FORMULAS

The harmonic series illustrates one of the most counterintuitive facts in calculus. A full understanding of this particular infinite series will help you recognize many of the subtle issues that arise in later mathematics courses. The general result may be stated this way: in the case where $\lim_{k\to\infty} a_k = 0$, the series $\sum_{k=1}^{\infty} a_k$ might diverge or might converge, depending on *how fast* the sequence a_k approaches zero. Keep thinking about why the harmonic series diverges and you will develop a deeper understanding of how infinite series in particular and calculus in general work.

EXERCISES 5.2

WRITING EXERCISES

1. Suppose that your friend is confused about the difference between the convergence of a sequence and the convergence of a series. Carefully explain the difference between convergence or divergence of the sequence $a_k = \frac{k}{k+1}$ and the series $\sum_{k=1}^{\infty} \frac{k}{k+1}$.

2. Explain in words why the kth-term test for divergence is valid. Explain why it is *not* true that if $\lim_{k\to\infty} a_k = 0$ then $\sum_{k=1}^{\infty} a_k$ necessarily converges. In your explanation, include an important example that proves that this is not true and comment on the fact that the convergence of a_k to 0 can be slow or fast.

3. In Theorems 2.2 and 2.3, the series start at $k = 1$, as in $\sum_{k=1}^{\infty} a_k$. Explain why the conclusions of the theorems hold if the series start at $k = 2$, $k = 3$ or at any positive integer.

4. We emphasized in the text that numerical and graphical evidence for the convergence of a series can be misleading. Suppose your calculator carries 14 digits in its calculations. Explain why for large enough values of n, the term $\frac{1}{n}$ will be too small to change the partial sum $\sum_{k=1}^{n} \frac{1}{k}$. Thus, the calculator would incorrectly indicate that the harmonic series converges.

In exercises 1–24, determine whether the series converges or diverges. For convergent series, find the sum of the series.

1. $\sum_{k=0}^{\infty} 3\left(\frac{1}{5}\right)^k$

2. $\sum_{k=0}^{\infty} \frac{1}{3}(5)^k$

3. $\sum_{k=0}^{\infty} \frac{1}{2}\left(-\frac{1}{3}\right)^k$

4. $\sum_{k=0}^{\infty} 4\left(\frac{1}{2}\right)^k$

5. $\sum_{k=0}^{\infty} \frac{1}{2}(3)^k$

6. $\sum_{k=3}^{\infty} (-1)^k \frac{3}{2^k}$

7. $\sum_{k=1}^{\infty} \frac{4}{k(k+2)}$

8. $\sum_{k=1}^{\infty} \frac{4k}{k+2}$

9. $\sum_{k=1}^{\infty} \frac{3k}{k+4}$

10. $\sum_{k=1}^{\infty} \frac{9}{k(k+3)}$

11. $\sum_{k=1}^{\infty} \frac{2}{k}$

12. $\sum_{k=0}^{\infty} \frac{4}{k+1}$

13. $\sum_{k=1}^{\infty} \frac{2k+1}{k^2(k+1)^2}$

14. $\sum_{k=1}^{\infty} \frac{4}{k(k+1)(k+3)(k+4)}$

15. $\sum_{k=2}^{\infty} 2e^{-k}$

16. $\sum_{k=1}^{\infty} \sqrt[k]{3}$

17. $\sum_{k=0}^{\infty}\left(\frac{1}{2^k} - \frac{1}{k+1}\right)$

18. $\sum_{k=0}^{\infty}\left(\frac{1}{2^k} - \frac{1}{3^k}\right)$

19. $\sum_{k=2}^{\infty}\left(\frac{2}{3^k} + \frac{1}{2^k}\right)$

20. $\sum_{k=2}^{\infty}\left(\frac{1}{k} - \frac{1}{4^k}\right)$

21. $\sum_{k=0}^{\infty}(-1)^{k+1}\frac{3k}{k+1}$

22. $\sum_{k=0}^{\infty}(-1)^k\frac{k^3}{k^2+1}$

23. $\sum_{k=1}^{\infty}\sin\left(\frac{k}{5}\right)$

24. $\sum_{k=1}^{\infty}\tan^{-1}k$

In exercises 25–28, determine all values of c such that the series converges.

25. $\sum_{k=0}^{\infty} 3(2c+1)^k$

26. $\sum_{k=0}^{\infty}\frac{2}{(c-3)^k}$

27. $\sum_{k=0}^{\infty}\frac{c}{k+1}$

28. $\sum_{k=0}^{\infty}\frac{2}{ck+1}$

In exercises 29–32, use graphical and numerical evidence to conjecture the convergence or divergence of the series.

29. $\sum_{k=1}^{\infty}\frac{1}{k^2}$

30. $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$

31. $\sum_{k=1}^{\infty}\frac{3}{k!}$

32. $\sum_{k=1}^{\infty}\frac{2^k}{k!}$

33. (a) Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=m}^{\infty} a_k$ converges for any positive integer m. In particular, if $\sum_{k=1}^{\infty} a_k$ converges to L, what does $\sum_{k=m}^{\infty} a_k$ converge to? (b) Prove that if $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=m}^{\infty} a_k$ diverges for any positive integer m.

34. Explain why the partial fractions technique of example 2.3 does not work for $\sum_{k=1}^{\infty}\frac{1}{k(k+1/2)}$.

35. Prove Theorem 2.3 (i).

36. Prove Theorem 2.3 (ii).

37. Let $S_n = \sum_{k=1}^{n}\frac{1}{k}$. Show that $S_1 = 1$ and $S_2 = \frac{3}{2}$. Since $\frac{1}{3} > \frac{1}{4}$, we have $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Therefore, $S_4 > \frac{3}{2} + \frac{1}{2} = 2$. Similarly, $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$, so $S_8 > \frac{5}{2}$. Show that $S_{16} > 3$ and $S_{32} > \frac{7}{2}$. For which n can you guarantee that $S_n > 4$? $S_n > 5$? For any positive integer m, determine n such that $S_n > m$. Conclude that the harmonic series diverges.

38. Compute several partial sums S_n of the series $1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$. Prove that the series diverges. Find the **Cesaro sum** of this series: $\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} S_k\right)$.

39. (a) Write $0.9999\overline{9} = 0.9 + 0.09 + 0.009 + \cdots$ and sum the geometric series to prove that $0.9999\overline{9} = 1$. (b) Prove that $0.19999\overline{9} = 0.2$.

40. (a) Write $0.1818\overline{18}$ as a geometric series and then write the sum of the geometric series as a fraction. (b) Write $2.134\overline{134}$ as a fraction.

41. Give an example where $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both diverge but $\sum_{k=1}^{\infty}(a_k + b_k)$ converges.

42. If $\sum_{k=0}^{\infty} a_k$ converges and $\sum_{k=0}^{\infty} b_k$ diverges, is it necessarily true that $\sum_{k=0}^{\infty}(a_k + b_k)$ diverges?

43. Prove that the sum of a convergent geometric series $1 + r + r^2 + \cdots$ must be greater than $\frac{1}{2}$.

44. Prove that if the series $\sum_{k=0}^{\infty} a_k$ converges, then the series $\sum_{k=0}^{\infty}\frac{1}{a_k}$ diverges.

45. Show that the partial sum $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ does not equal an integer for any prime $n < 100$. Is the statement true for all integers $n > 1$?

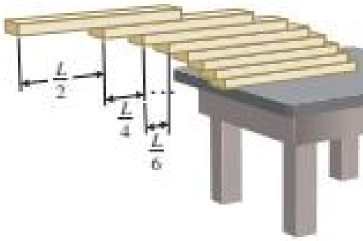
46. The **Cantor set** is one of the most famous sets in mathematics. To construct the Cantor set, start with the interval $[0, 1]$. Then remove the middle third, $\left(\frac{1}{3}, \frac{2}{3}\right)$. This leaves the set $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. For each of the two subintervals, remove the middle third; in this case, remove the intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$. Continue in this way, removing the middle thirds of each remaining interval. The Cantor set is all points in $[0, 1]$ that are *not* removed. Argue that 0, 1, $\frac{1}{3}$ and $\frac{2}{3}$ are in the Cantor set, and identify four more points in the set. It can be shown that there are an infinite number of points in the Cantor set. On the other hand, the total length of the subintervals removed is $\frac{1}{3} + 2\left(\frac{1}{9}\right) + \cdots$. Find the third term in this series, identify the series as a convergent geometric series and find the sum of the series. Given that you started with an interval of length 1, how much "length" does the Cantor set have?

47. For $0 < x < 1$, show that $1 + x + x^2 + \cdots + x^n < \frac{1}{1-x}$. Does this inequality hold for $-1 < x < 0$?

48. For any positive integer n, show that $2 > 1 + \frac{1}{2} + \cdots + \frac{1}{2^n}$, $\frac{3}{2} > 1 + \frac{1}{3} + \cdots + \frac{1}{3^n}$ and $\frac{5}{4} > 1 + \frac{1}{5} + \cdots + \frac{1}{5^n}$. Use these facts to show that $2 \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \cdots \cdot \frac{p}{p-1} > 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ where p is the largest prime that is less than n. Conclude that there are an infinite number of primes.

APPLICATIONS

1. Suppose you have n boards of length L. Place the first board with length $\frac{L}{2n}$ hanging over the edge of the table. Place the next board with length $\frac{L}{2(n-1)}$ hanging over the edge of the first board. The next board should hang $\frac{L}{2(n-2)}$ over the edge of the second board. Continue on until the last board hangs $\frac{L}{2}$ over the edge of the $(n-1)$st board. Theoretically, this stack will balance (in practice, don't use quite as much overhang). With $n = 8$, compute the total overhang of the stack. Determine the number of boards n such that the total overhang is greater than L. This means that the last board is entirely beyond the edge of the table. What is the limit of the total overhang as $n \to \infty$?



2. Have you ever felt that the line you're standing in moves more slowly than the other lines? In *An Introduction to Probability Theory and Its Applications,* William Feller proved just how bad your luck is. Let N be the number of people who get in line until someone waits longer than you do (you're the first, so $N \geq 2$). The probability that $N = k$ is given by $p(k) = \frac{1}{k(k-1)}$. Prove that the total probability equals 1; that is, $\sum_{k=2}^{\infty} \frac{1}{k(k-1)} = 1$. From probability theory, the average (mean) number of people who must get in line before someone has waited longer than you is given by $\sum_{k=2}^{\infty} k \frac{1}{k(k-1)}$. Prove that this diverges to ∞. Talk about bad luck!

3. To win a deuce tennis game, one player or the other must win the next two points. If each player wins one point, the deuce starts over. If you win each point with probability p, the probability that you win the next two points is p^2. The probability that you win one of the next two points is $2p(1-p)$. The probability that you win a deuce game is then $p^2 + 2p(1-p)p^2 + [2p(1-p)]^2p^2 + [2p(1-p)]^3p^2 + \cdots$. Explain what each term represents, explain why the geometric series converges and find the sum of the series. If $p = 0.6$, you're a better player than your opponent. Show that you are more likely to win a deuce game than you are a single point. The slightly strange scoring rules in tennis make it more likely that the better player wins.

4. On an analog clock, at 1:00, the minute hand points to 12 and the hour hand points to 1. When the minute hand reaches 1, the hour hand has progressed to $1 + \frac{1}{12}$. When the minute hand reaches $1 + \frac{1}{12}$, the hour hand has moved to $1 + \frac{1}{12} + \frac{1}{12^2}$. Find the sum of a geometric series to determine the time at which the minute hand and hour hand are in the same location.

5. A dosage d of a drug is given at times $t = 0, 1, 2, \ldots$. The drug decays exponentially with rate r in the bloodstream. The amount in the bloodstream after $n + 1$ doses is $d + de^{-r} + de^{-2r} + \cdots + de^{-nr}$. Show that the eventual level of the drug (after an "infinite" number of doses) is $\frac{d}{1 - e^{-r}}$. If $r = 0.1$, find the dosage needed to maintain a drug level of 2.

6. Two bicyclists are 40 km apart, riding toward each other at 20 km/h (each). A fly starts at one bicyclist and flies toward the other bicyclist at 60 km/h. When it reaches the bike, it turns around and flies back to the first bike. It continues flying back and forth until the bikes meet. Determine the distance flown on each leg of the fly's journey and find the sum of the geometric series to get the total distance flown. Verify that this is the right answer by solving the problem the easy way.

7. Suppose $100,000 of counterfeit money is introduced into the economy. Each time the money is used, 25% of the remaining money is identified as counterfeit and removed from circulation. Determine the total amount of counterfeit money successfully used in transactions. This is an example of the **multiplier effect** in economics. Suppose that a new marking scheme on dollar bills helps raise the detection rate to 40%. Determine the reduction in the total amount of counterfeit money successfully spent.

8. In this exercise, we will find the **present value** of a plot of farmland. Assume that a crop of value $c will be planted in years 1, 2, 3 and so on, and the yearly inflation rate is r. The present value is given by

$$P = ce^{-r} + ce^{-2r} + ce^{-3r} + \cdots.$$

9. Find the sum of the geometric series to compute the present value.

10. Suppose you repeat a game at which you have a probability p of winning each time you play. The probability that your first win comes in your nth game is $p(1-p)^{n-1}$. Compute $\sum_{n=1}^{\infty} p(1-p)^{n-1}$ and state in terms of probability why the result makes sense.

11. In general, the total time it takes for a ball to complete its bounces is $\frac{2v}{g} \sum_{k=0}^{\infty} r^k$ and the total distance the ball moves is $\frac{v^2}{g} \sum_{k=0}^{\infty} r^{2k}$, where r is the coefficient of restitution of the ball. Assuming $0 < r < 1$, find the sums of these geometric series.

12. (a) Here is a magic trick (from Art Benjamin). Pick any positive integer less than 1000. Divide it by 7, then divide the answer by 11, then divide the answer by 13. Look at the first six digits after the decimal of the answer and call out any five of them in any order and the magician will tell what the other digit is. The "secret" knowledge used by the magician is that the sum of the six digits will equal 27. Try this! (b) Let x be a positive integer less than 1000, and let $c = \frac{x}{1000} + \frac{999 - x}{1{,}000{,}000}$. Show that $c + \frac{c}{1{,}000{,}000} + \frac{c}{1{,}000{,}000^2} + \frac{c}{1{,}000{,}000^3} + \cdots$ converges to $\frac{x+1}{1001}$. (c) Explain why part (b) implies that the decimal expansion of the fraction in part (a) will repeat every six digits, with the first three digits being one less than the original number and the remaining three digits being the 9's-complement of the first three digits.

EXPLORATORY EXERCISES

1. **Infinite products** are of great interest to mathematicians. Numerically explore the convergence or divergence of the infinite product $\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{25}\right)\left(1 - \frac{1}{49}\right) \cdots = \prod_{p = \text{prime}} \left(1 - \frac{1}{p^2}\right)$. Note that the product is taken over the prime numbers, not all integers. Compare your results to the number $\frac{6}{\pi^2}$.

2. In example 2.7, we showed that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n + 1)$. Superimpose the graph of $f(x) = \frac{1}{x-1}$ onto Figure 5.22 and show that $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln(n)$. Conclude that $\ln(n + 1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln(n)$. **Euler's constant** is defined by

$$\gamma = \lim_{n\to\infty}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n)\right].$$

Look up the value of γ. (Hint: Use your CAS.) Use γ to estimate $\sum_{i=1}^{n}\frac{1}{i}$ for $n = 10{,}000$ and $n = 100{,}000$. Investigate whether the sequence $a_n = \sum_{k=n}^{2n}\frac{1}{k}$ converges or diverges.

3. Define $a_1 = 1$, $a_2 = 1 - \frac{1}{2} - \frac{1}{3}$ and $a_n = a_{n-1} + (-1)^{n-1}\sum_{k=2^{n-1}}^{2^n-1}\frac{1}{k}$ for $n \geq 2$. Illustrate numerically that the sequence diverges, alternating between two values with $\lim_{n\to\infty}|a_n - a_{n-1}| = \ln 2$. Explore the sequence with $b_1 = 1$ and $b_n = b_{n-1} + (-1)^{n-1}\sum_{k=3^{n-1}}^{3^n-1}\frac{1}{k}$. (Chris Davis and David Taylor have extended these results from base 2 and base 3 to any base $b > 1$.*)

* http://www.airitilibrary.com/Publication/alDetailedMesh?docid=21600368-201305-201601290028-201601290028-309-316

5.3 THE INTEGRAL TEST AND COMPARISON TESTS

We need to test most series for convergence in some indirect way that does not result in finding the sum of the series. In this section, we will develop several such tests for convergence of series. The first of these is a generalization of the method we used in section 5.2 to show that the harmonic series is divergent.

For a given series $\sum_{k=1}^{\infty} a_k$, suppose that there is a function f for which

$$f(k) = a_k, \quad \text{for } k = 1, 2, \ldots,$$

where f is continuous and decreasing and $f(x) \geq 0$ for all $x \geq 1$. We consider the nth partial sum

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

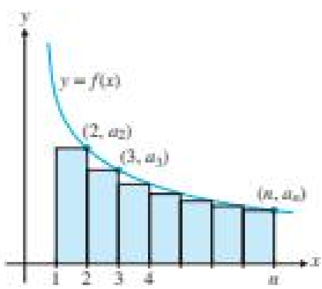


FIGURE 5.23a
$(n - 1)$ rectangles, lying beneath the curve

In Figure 5.23a, we show $(n - 1)$ rectangles constructed on the interval $[1, n]$, each of width 1 and with height equal to the value of the function at the right-hand endpoint of the subinterval on which it is constructed. Notice that since each rectangle lies completely beneath the curve, the sum of the areas of the $(n - 1)$ rectangles shown is less than the area under the curve from $x = 1$ to $x = n$. That is,

$$0 \leq \text{Sum of areas of } (n - 1) \text{ rectangles} \leq \text{Area under the curve} = \int_1^n f(x)\,dx. \tag{3.1}$$

Note that the area of the first rectangle is length × width = $(1)(a_2)$, the area of the second rectangle is $(1)(a_3)$ and so on. We get that the sum of the areas of the $(n - 1)$ rectangles shown is

$$a_2 + a_3 + a_4 + \cdots + a_n = S_n - a_1.$$

Together with (3.1), this gives us

$$\begin{aligned} 0 &\leq \text{Sum of areas of } (n - 1) \text{ rectangles} \\ &= S_n - a_1 \leq \text{Area under the curve} = \int_1^n f(x)\,dx. \end{aligned} \tag{3.2}$$

Now, suppose that the improper integral $\int_1^\infty f(x)\,dx$ converges. Then, from (3.2), we have

$$0 \leq S_n - a_1 \leq \int_1^n f(x)\,dx \leq \int_1^\infty f(x)\,dx.$$

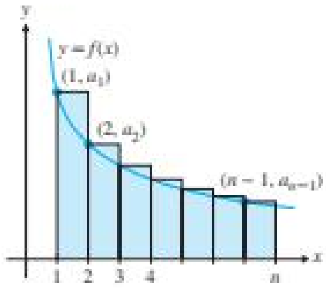


FIGURE 5.23b
$(n-1)$ rectangles, partially above the curve

Adding a_1 to all the terms gives us

$$a_1 \le S_n \le a_1 + \int_1^{\infty} f(x)\,dx,$$

so that the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ is bounded. Since $\{S_n\}_{n=1}^{\infty}$ is also monotonic (why is that?), $\{S_n\}_{n=1}^{\infty}$ is convergent by Theorem 1.4 and so, the series $\sum_{k=1}^{\infty} a_k$ is also convergent.

In Figure 5.23b, we show $(n-1)$ rectangles constructed on the interval $[1, n]$, each of width 1, but with height equal to the value of the function at the left-hand endpoint of the subinterval on which it is constructed. In this case, the sum of the areas of the $(n-1)$ rectangles shown is greater than the area under the curve. That is,

$$\begin{aligned} 0 &\le \text{Area under the curve} = \int_1^{n} f(x)\,dx \\ &\le \text{Sum of areas of } (n-1) \text{ rectangles.} \end{aligned} \tag{3.3}$$

Further, note that the area of the first rectangle is length × width = $(1)(a_1)$, the area of the second rectangle is $(1)(a_2)$ and so on. We get that the sum of the areas of the $(n-1)$ rectangles indicated in Figure 5.23b is

$$a_1 + a_2 + \cdots + a_{n-1} = S_{n-1}.$$

Together with (3.3), this gives us

$$\begin{aligned} 0 &\le \text{Area under the curve} = \int_1^{n} f(x)\,dx \\ &\le \text{Sum of areas of } (n-1) \text{ rectangles} = S_{n-1}. \end{aligned} \tag{3.4}$$

Now, suppose that the improper integral $\int_1^{\infty} f(x)\,dx$ diverges. Since $f(x) \ge 0$, this says that $\lim_{n\to\infty} \int_1^{n} f(x)\,dx = \infty$. From (3.4), we have that

$$\int_1^{n} f(x)\,dx \le S_{n-1}.$$

This says that
$$\lim_{n\to\infty} S_{n-1} = \infty,$$

also. So, the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ diverges and hence, the series $\sum_{k=1}^{\infty} a_k$ diverges, too.

We summarize the results of this analysis in Theorem 3.1.

HISTORICAL NOTES

Colin Maclaurin (1698–1746) Scottish mathematician who discovered the Integral Test. Maclaurin was one of the founders of the Royal Society of Edinburgh and was a pioneer in the mathematics of actuarial studies. The Integral Test was introduced in a highly influential book that also included a new treatment of an important method for finding series of functions. Maclaurin series, as we now call them, are developed in section 5.7.

THEOREM 3.1 (Integral Test)

If $f(k) = a_k$ for all $k = 1, 2, \ldots$, f is continuous and decreasing, and $f(x) \ge 0$ for $x \ge 1$, then $\int_1^{\infty} f(x)\,dx$ and $\sum_{k=1}^{\infty} a_k$ either *both* converge or *both* diverge.

Note that while the Integral Test might say that a given series and improper integral both converge, it does *not* say that they will converge to the same value. In fact, this is generally not the case, as we see in example 3.1.

EXAMPLE 3.1 Using the Integral Test

Investigate the convergence or divergence of the series $\sum_{k=0}^{\infty} \frac{1}{k^2+1}$.

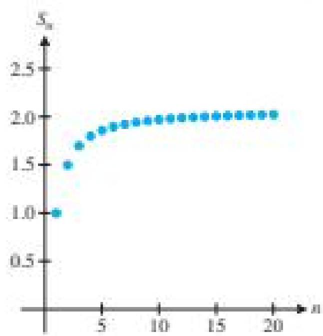


FIGURE 5.24
$S_n = \sum_{k=0}^{n-1} \frac{1}{k^2+1}$

Solution The graph of the first 20 partial sums shown in Figure 5.24 suggests that the series converges to some value around 2. In the accompanying table, we show some selected partial sums. Based on this, it is difficult to say whether the series is

n	$S_n = \sum_{k=0}^{n-1} \frac{1}{k^2+1}$
10	1.97189
50	2.05648
100	2.06662
200	2.07166
500	2.07467
1000	2.07567
2000	2.07617

converging very slowly to a limit around 2.076 or whether the series is instead diverging very slowly. To determine which is the case, we must test the series further. Define $f(x) = \frac{1}{x^2+1}$. Note that f is continuous and positive everywhere and $f(k) = \frac{1}{k^2+1} = a_k$, for all $k \geq 1$. Further,

$$f'(x) = (-1)(x^2+1)^{-2}(2x) < 0,$$

for $x \in (0, \infty)$, so that f is decreasing. This says that the Integral Test applies to this series. So, we consider the improper integral

$$\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{R\to\infty}\int_0^R \frac{1}{x^2+1}\,dx = \lim_{R\to\infty} \tan^{-1} x \Big|_0^R$$

$$= \lim_{R\to\infty}(\tan^{-1} R - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

The Integral Test says that since the improper integral converges, the series must converge, also. Now that we have established that the series is convergent, our earlier calculations give us the estimated sum 2.076. Notice that this is *not* the same as the value of the corresponding improper integral, which is $\frac{\pi}{2} \approx 1.5708$. ■

In example 3.2, we discuss an important type of series.

EXAMPLE 3.2 The p-Series

Determine for which values of p the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ (a **p-series**) converges.

Solution First, notice that for $p = 1$, this is the harmonic series, which diverges. For $p > 1$, define $f(x) = \frac{1}{x^p} = x^{-p}$. Notice that for $x \geq 1$, f is continuous and positive. Further,

$$f'(x) = -px^{-p-1} < 0,$$

so that f is decreasing. This says that the Integral Test applies. We now consider

$$\int_1^\infty x^{-p}\,dx = \lim_{R\to\infty}\int_1^R x^{-p}\,dx = \lim_{R\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg|_1^R$$

$$= \lim_{R\to\infty}\left(\frac{R^{-p+1}}{-p+1} - \frac{1}{-p+1}\right) = \frac{-1}{-p+1}.$$

Since $p > 1$ implies that $-p + 1 < 0$.

In this case, the improper integral converges and so too must the series. We leave it as an exercise to show that the series diverges when $p < 1$. ■

We summarize the result of example 3.2 as follows.

p-SERIES

The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Notice that in each of examples 3.1 and 3.2, we were able to use the Integral Test to establish the convergence of a series. While you can use the partial sums of a convergent series to estimate its sum, it remains to be seen how precise a given estimate is. First, if

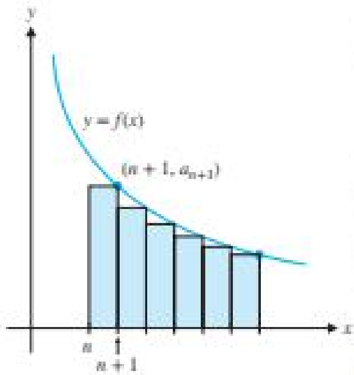


FIGURE 5.25
Estimate of the remainder

we estimate the sum s of the series $\sum_{k=1}^{\infty} a_k$ by the nth partial sum $S_n = \sum_{k=1}^{n} a_k$, we define the **remainder** R_n to be

$$R_n = s - S_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{\infty} a_k.$$

Notice that this says that the remainder R_n is the error in approximating s by S_n. For any series shown to be convergent by the Integral Test, we can estimate the size of the remainder, as follows. From Figure 5.25, observe that R_n corresponds to the sum of the areas of the indicated rectangles. Further, under the conditions of the Integral Test, this is less than the area under the curve $y = f(x)$ on the interval $[n, \infty)$. (Recall that this area is finite, since $\int_1^{\infty} f(x)dx$ converges.) This gives us the following result.

THEOREM 3.2 (Error Estimate for the Integral Test)

Suppose that $f(k) = a_k$ for all $k = 1, 2, \ldots$, where f is continuous and decreasing, and $f(x) \geq 0$ for all $x \geq 1$. Further, suppose that $\int_1^{\infty} f(x)\,dx$ converges. Then, the remainder R_n satisfies

$$0 \leq R_n = \sum_{k=n+1}^{\infty} a_k \leq \int_n^{\infty} f(x)\,dx.$$

Whenever the Integral Test applies, we can use Theorem 3.2 to estimate the error in using a partial sum to approximate the sum of a series.

A more interesting and far more practical question related to example 3.3 is to determine the number of terms of the series necessary to obtain a given accuracy.

EXAMPLE 3.3 Estimating the Error in a Partial Sum

Estimate the error in using the partial sum S_{100} to approximate the sum of the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$.

Solution First, recall that in example 3.2, we used the Integral Test to show that this series—a p-series, with $p = 3$—is convergent. From Theorem 3.2, the remainder satisfies

$$0 \leq R_{100} \leq \int_{100}^{\infty} \frac{1}{x^3}\,dx = \lim_{R\to\infty} \int_{100}^{R} \frac{1}{x^3}\,dx = \lim_{R\to\infty} \left(-\frac{1}{2x^2}\right)\Bigg|_{100}^{R}$$

$$= \lim_{R\to\infty} \left(\frac{-1}{2R^2} + \frac{1}{2(100)^2}\right) = 5 \times 10^{-5}.$$

EXAMPLE 3.4 Finding the Number of Terms Needed for a Given Accuracy

Determine the number of terms needed to obtain an approximation to the sum of the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ correct to within 10^{-5}.

Solution Again, we already used the Integral Test to show that the series in question converges. Then, by Theorem 3.2, we have that the remainder satisfies

$$0 \leq R_n \leq \int_n^{\infty} \frac{1}{x^3}\,dx = \lim_{R\to\infty} \int_n^{R} \frac{1}{x^3}\,dx = \lim_{R\to\infty} \left(-\frac{1}{2x^2}\right)\Bigg|_n^{R}$$

$$= \lim_{R\to\infty} \left(\frac{-1}{2R^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}.$$

So, to ensure that the remainder is less than 10^{-5}, we require that

$$0 \le R_n \le \frac{1}{2n^2} \le 10^{-5}.$$

Solving this last inequality for n yields

$$n^2 \ge \frac{10^5}{2} \quad \text{or} \quad n \ge \sqrt{\frac{10^5}{2}} = 100\sqrt{5} \approx 223.6.$$

So, taking $n \ge 224$ will guarantee the required accuracy and, consequently, we have $\sum_{k=1}^{\infty} \frac{1}{k^3} \approx \sum_{k=1}^{224} \frac{1}{k^3} \approx 1.202047$, which is correct to within 10^{-5}, as desired. ■

Comparison Tests

We next present two results that allow us to compare a given series with one that is already known to be convergent or divergent, much as we did with improper integrals in section 6.6.

THEOREM 3.3 (Comparison Test)

Suppose that $0 \le a_k \le b_k$, for all k.

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges, too.

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges, too.

Intuitively, this theorem should make abundant sense: if the "larger" series converges, then the "smaller" one must also converge. Likewise, if the "smaller" series diverges, then the "larger" one must diverge, too.

PROOF

Given that $0 \le a_k \le b_k$ for all k, observe that the nth partial sums of the two series satisfy

$$0 \le S_n = a_1 + a_2 + \cdots + a_n \le b_1 + b_2 + \cdots + b_n.$$

(i) If $\sum_{k=1}^{\infty} b_k$ converges (say, to B), this says that

$$0 \le S_n = a_1 + a_2 + \cdots + a_n \le b_1 + b_2 + \cdots + b_n \le \sum_{k=1}^{\infty} b_k = B, \tag{3.5}$$

for all $n \ge 1$. From (3.5), the sequence $\{S_n\}_{n=1}^{\infty}$ of partial sums of $\sum_{k=1}^{\infty} a_k$ is bounded. Notice that $\{S_n\}_{n=1}^{\infty}$ is also increasing. (Why?) Since every bounded, monotonic sequence is convergent (see Theorem 1.4), we get that $\sum_{k=1}^{\infty} a_k$ is convergent, too.

(ii) If $\sum_{k=1}^{\infty} a_k$ is divergent, we have (since all of the terms of the series are non-negative) that

$$\lim_{n\to\infty} (b_1 + b_2 + \cdots + b_n) \ge \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n) = \infty.$$

Thus, $\sum_{k=1}^{\infty} a_k$ must be divergent, also. ■

You can use the Comparison Test to test the convergence of series that look similar to series that you already know are convergent or divergent (notably, geometric series or p-series).

EXAMPLE 3.5 Using the Comparison Test for a Convergent Series

Investigate the convergence or divergence of $\sum_{k=1}^{\infty} \frac{1}{k^3+5k}$.

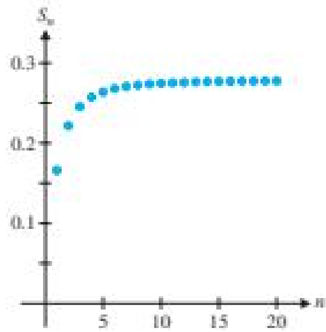


FIGURE 5.26
$S_n = \sum_{k=1}^{n} \frac{1}{k^3+5k}$

Solution The graph of the first 20 partial sums shown in Figure 5.26 suggests that the series converges to some value near 0.3. To confirm such a conjecture, we must carefully test the series. Note that for large values of k, the general term of the series looks like $\frac{1}{k^3}$, since when k is large, k^3 is much larger than $5k$. This observation is significant, since we already know that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series ($p = 3 > 1$). Further, observe that

$$0 \le \frac{1}{k^3+5k} \le \frac{1}{k^3},$$

for all $k \ge 1$. Since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges, the Comparison Test says that $\sum_{k=1}^{\infty} \frac{1}{k^3+5k}$ converges, too. As with the Integral Test, although the Comparison Test tells us that both series converge, the two series *need not* converge to the same sum. A quick calculation of a few partial sums should convince you that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges to approximately 1.202, while $\sum_{k=1}^{\infty} \frac{1}{k^3+5k}$ converges to approximately 0.2798. (Note that this is consistent with what we saw in Figure 5.26.) ■

EXAMPLE 3.6 Using the Comparison Test for a Divergent Series

Investigate the convergence or divergence of $\sum_{k=1}^{\infty} \frac{5^k+1}{2^k-1}$.

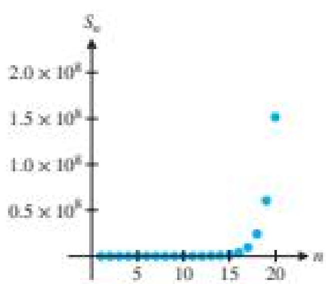


FIGURE 5.27
$S_n = \sum_{k=1}^{n} \frac{5^k+1}{2^k-1}$

Solution From the graph of the first 20 partial sums seen in Figure 5.27, it appears that the partial sums are growing very rapidly. On this basis, we would conjecture that the series diverges. Of course, to verify this, we need further testing. Notice that for k large, the general term looks like $\frac{5^k}{2^k} = \left(\frac{5}{2}\right)^k$ and we know that $\sum_{k=1}^{\infty} \left(\frac{5}{2}\right)^k$ is a divergent geometric series $\left(|r| = \frac{5}{2} > 1\right)$. Further,

$$\frac{5^k+1}{2^k-1} \ge \frac{5^k}{2^k-1} \ge \frac{5^k}{2^k} = \left(\frac{5}{2}\right)^k \ge 0.$$

By the Comparison Test, $\sum_{k=1}^{\infty} \frac{5^k+1}{2^k-1}$ diverges, too. ■

There are plenty of series whose general term looks like the general term of a familiar series, but for which it is unclear how to get the inequality required for the Comparison Test to go in the right direction.

EXAMPLE 3.7 A Comparison That Does Not Work

Investigate the convergence or divergence of the series $\sum_{k=3}^{\infty} \frac{1}{k^3-5k}$.

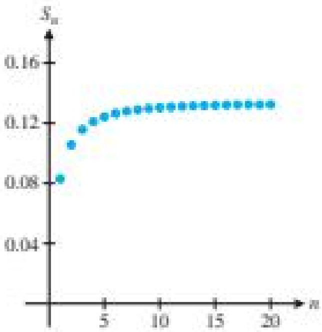


FIGURE 5.28
$S_n = \sum_{k=3}^{n+2} \frac{1}{k^3 - 5k}$

Solution Note that this is nearly identical to example 3.5, except that there is a "−" sign in the denominator instead of a "+" sign. The graph of the first 20 partial sums seen in Figure 5.28 looks somewhat similar to the graph in Figure 5.26, except that the series appears to be converging to about 0.12. In this case, however, we have the inequality

$$\frac{1}{k^3 - 5k} \geq \frac{1}{k^3} \geq 0, \quad \text{for all } k \geq 3.$$

Unfortunately, this inequality goes the *wrong way:* we know that $\sum_{k=3}^{\infty} \frac{1}{k^3}$ is a convergent p-series, but since $\sum_{k=3}^{\infty} \frac{1}{k^3 - 5k}$ is "larger" than this convergent series, the Comparison Test says nothing. ■

Think about what happened in example 3.7 this way: while you might observe that

$$k^2 \geq \frac{1}{k^3} \geq 0, \quad \text{for all } k \geq 1$$

and you know that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is convergent, the Comparison Test says nothing about the "larger" series $\sum_{k=1}^{\infty} k^2$. In fact, we know that this last series is divergent (by the kth-term test for divergence, since $\lim_{k\to\infty} k^2 = \infty \neq 0$). To resolve this difficulty for the present problem, we will need to either make a different comparison or use the Limit Comparison Test, which follows.

NOTES

When we say $\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$, we mean that the limit exists and is positive. In particular, we mean that $\lim_{k\to\infty} \frac{a_k}{b_k} \neq \infty$.

THEOREM 3.4 (Limit Comparison Test)

Suppose that $a_k, b_k > 0$ and that for some (finite) value, L, $\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$. Then, either $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ *both* converge or they *both* diverge.

PROOF

If $\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$, this says that we can make $\frac{a_k}{b_k}$ as close to L as desired. So, in particular, we can make $\frac{a_k}{b_k}$ within distance $\frac{L}{2}$ of L. That is, for some number $N > 0$,

$$L - \frac{L}{2} < \frac{a_k}{b_k} < L + \frac{L}{2}, \quad \text{for all } k > N$$

or

$$\frac{L}{2} < \frac{a_k}{b_k} < \frac{3L}{2}. \tag{3.6}$$

Multiplying inequality (3.6) through by b_k (since $b_k > 0$), we get

$$0 < \frac{L}{2} b_k < a_k < \frac{3L}{2} b_k, \quad \text{for } k \geq N.$$

Note that this says that if $\sum_{k=1}^{\infty} a_k$ converges, then the "smaller" series $\sum_{k=1}^{\infty} \left(\frac{L}{2} b_k\right) = \frac{L}{2} \sum_{k=1}^{\infty} b_k$ must also converge, by the Comparison Test. Likewise, if $\sum_{k=1}^{\infty} a_k$ diverges, the "larger" series $\sum_{k=1}^{\infty} \left(\frac{3L}{2} b_k\right) = \frac{3L}{2} \sum_{k=1}^{\infty} b_k$ must also diverge. In the same way, if $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty}\left(\frac{3L}{2}b_k\right)$ converges and so, too, must the "smaller" series $\sum_{k=1}^{\infty} a_k$. Finally, if $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty}\left(\frac{L}{2}b_k\right)$ diverges and hence, the "larger" series $\sum_{k=1}^{\infty} a_k$ must diverge, also. ■

We can now use the Limit Comparison Test to test the series from example 3.7 whose convergence we have so far been unable to confirm.

EXAMPLE 3.8 Using the Limit Comparison Test

Investigate the convergence or divergence of the series $\sum_{k=3}^{\infty} \frac{1}{k^3 - 5k}$.

Solution Recall that we had already observed in example 3.7 that the general term $a_k = \frac{1}{k^3 - 5k}$ "looks like" $b_k = \frac{1}{k^3}$, for k large. We then consider the limit

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty}\left(a_k \frac{1}{b_k}\right) = \lim_{k\to\infty} \frac{1}{(k^3 - 5k)} \frac{1}{\left(\frac{1}{k^3}\right)} = \lim_{k\to\infty} \frac{1}{1 - \frac{5}{k^2}} = 1 > 0.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series ($p = 3 > 1$), the Limit Comparison Test says that $\sum_{k=3}^{\infty} \frac{1}{k^3 - 5k}$ is also convergent, as we had originally suspected. ■

The Limit Comparison Test can be used to resolve convergence questions for a great many series. The first step in using this (like the Comparison Test) is to find another series (whose convergence or divergence is known) that "looks like" the series in question.

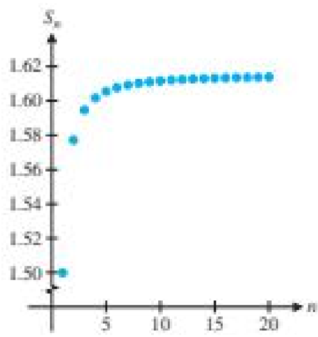


FIGURE 5.29
$S_n = \sum_{k=1}^{n} \frac{k^2 - 2k + 7}{k^5 + 5k^4 - 3k^3 + 2k - 1}$

n	$S_n = \sum_{k=1}^{n} \frac{k^2 - 2k + 7}{k^5 + 5k^4 - 3k^3 + 2k - 1}$
5	1.60522
10	1.61145
20	1.61365
50	1.61444
75	1.61453
100	1.61457

EXAMPLE 3.9 Using the Limit Comparison Test

Investigate the convergence or divergence of the series

$$\sum_{k=1}^{\infty} \frac{k^2 - 2k + 7}{k^5 + 5k^4 - 3k^3 + 2k - 1}.$$

Solution The graph of the first 20 partial sums in Figure 5.29 suggests that the series converges to a limit of about 1.61. The accompanying table of partial sums supports this conjecture.

Notice that for k large, the general term looks like $\frac{k^2}{k^5} = \frac{1}{k^3}$ (since the terms with the largest exponents tend to dominate the expression, for large values of k). From the Limit Comparison Test, for $b_k = \frac{1}{k^3}$, we have

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{k^2 - 2k + 7}{k^5 + 5k^4 - 3k^3 + 2k - 1} \frac{1}{\left(\frac{1}{k^3}\right)} \\
&= \lim_{k\to\infty} \frac{(k^2 - 2k + 7)}{(k^5 + 5k^4 - 3k^3 + 2k - 1)} \frac{k^3}{1} \\
&= \lim_{k\to\infty} \frac{(k^5 - 2k^4 + 7k^3)}{(k^5 + 5k^4 - 3k^3 + 2k - 1)} \frac{\left(\frac{1}{k^5}\right)}{\left(\frac{1}{k^5}\right)} \\
&= \lim_{k\to\infty} \frac{1 - \frac{2}{k} + \frac{7}{k^2}}{1 + \frac{5}{k} - \frac{3}{k^2} + \frac{2}{k^4} - \frac{1}{k^5}} = 1 > 0.
\end{aligned}$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series ($p = 3 > 1$), the Limit Comparison Test says that $\sum_{k=1}^{\infty} \frac{k^2 - 2k + 7}{k^5 + 5k^4 - 3k^3 + 2k - 1}$ converges, also. Finally, now that we have established that the series is, in fact, convergent, we can use our table of computed partial sums to approximate the sum of the series as 1.61457. ■

BEYOND FORMULAS

Keeping track of the many convergence tests arising in the study of infinite series can be somewhat challenging. We need all of these convergence tests because there is not a single test that works for all series (although more than one test may be used for a given series). Keep in mind that each test works only for specific types of series. As a result, you must be able to distinguish one type of infinite series (such as a geometric series) from another (such as a p-series), in order to determine the right test to use.

EXERCISES 5.3

WRITING EXERCISES

1. Notice that the Comparison Test doesn't always give us information about convergence or divergence. If $a_k \le b_k$ for each k and $\sum_{k=1}^{\infty} b_k$ diverges, explain why you can't tell whether or not $\sum_{k=1}^{\infty} a_k$ diverges.
2. Explain why the Limit Comparison Test works. In particular, if $\lim_{k\to\infty} \frac{a_k}{b_k} = 1$, explain how a_k and b_k compare and conclude that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ either both converge or both diverge.
3. In the Limit Comparison Test, if $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum_{k=1}^{\infty} a_k$ converges, explain why you can't tell whether or not $\sum_{k=1}^{\infty} b_k$ converges.
4. A p-series converges if $p > 1$ and diverges if $p < 1$. What happens for $p = 1$? If your friend knows that the harmonic series diverges, explain an easy way to remember the rest of the conclusion of the p-series test.

In exercises 1–20, determine convergence or divergence of the series.

1. (a) $\sum_{k=1}^{\infty} \frac{4}{\sqrt[3]{K}}$ (b) $\sum_{k=1}^{\infty} k^{-9/10}$
2. (a) $\sum_{k=4}^{\infty} k^{-11/10}$ (b) $\sum_{k=6}^{\infty} \frac{4}{\sqrt{k}}$
3. (a) $\sum_{k=3}^{\infty} \frac{k+1}{k^2 + 2k + 3}$ (b) $\sum_{k=0}^{\infty} \frac{\sqrt{k}}{k^2 + 1}$
4. (a) $\sum_{k=8}^{\infty} \frac{4}{2 + 4k}$ (b) $\sum_{k=6}^{\infty} \frac{4}{(2 + 4k)^2}$
5. (a) $\sum_{k=2}^{\infty} \frac{2}{k \ln k}$ (b) $\sum_{k=2}^{\infty} \frac{3}{k(\ln k)^2}$
6. (a) $\sum_{k=1}^{\infty} \frac{2k}{k^2 + 1}$ (b) $\sum_{k=2}^{\infty} \frac{k^2 + 1}{k^3 + 3k + 2}$
7. (a) $\sum_{k=3}^{\infty} \frac{e^{1/k}}{k^2}$ (b) $\sum_{k=4}^{\infty} \frac{\sqrt{1 + 1/k}}{k^2}$
8. (a) $\sum_{k=1}^{\infty} \frac{e^{-\sqrt{k}}}{\sqrt{k}}$ (b) $\sum_{k=1}^{\infty} \frac{k e^{-k^2}}{4 + e^{-k}}$
9. (a) $\sum_{k=1}^{\infty} \frac{2k^2}{k^{5/2} + 2}$ (b) $\sum_{k=0}^{\infty} \frac{2}{\sqrt{k^2 + 4}}$
10. (a) $\sum_{k=0}^{\infty} \frac{4}{\sqrt{k^3 + 1}}$ (b) $\sum_{k=0}^{\infty} \frac{k^2 + 1}{\sqrt{k^5 + 1}}$
11. (a) $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1 + k^2}$ (b) $\sum_{k=1}^{\infty} \frac{\sin^{-1}(1/k)}{k^2}$
12. (a) $\sum_{k=1}^{\infty} \frac{1}{\cos^2 k}$ (b) $\sum_{k=1}^{\infty} \frac{e^{1/k} + 1}{k^3}$
13. (a) $\sum_{k=2}^{\infty} \frac{\ln k}{k}$ (b) $\sum_{k=1}^{\infty} \frac{2 + \cos k}{k}$
14. (a) $\sum_{k=4}^{\infty} \frac{k^4 + 2k - 1}{k^5 + 3k^2 + 1}$ (b) $\sum_{k=6}^{\infty} \frac{k^3 + 2k + 3}{k^4 + 2k^2 + 4}$
15. (a) $\sum_{k=3}^{\infty} \frac{k+1}{k^2 + 2}$ (b) $\sum_{k=2}^{\infty} \frac{\sqrt{k+1}}{k^2 + 2}$
16. (a) $\sum_{k=8}^{\infty} \frac{k+1}{k^3 + 2}$ (b) $\sum_{k=5}^{\infty} \frac{\sqrt{k+1}}{\sqrt{k^3 + 2}}$

17. (a) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k\sqrt{k}+k\sqrt{k+1}}$ (b) $\displaystyle\sum_{k=1}^{\infty}\frac{2k+1}{k\sqrt{k}+k^2\sqrt{k+1}}$

18. (a) $\displaystyle\sum_{k=4}^{\infty} ke^{-k}$ (b) $\displaystyle\sum_{k=3}^{\infty}\frac{k^3}{e^k}$

19. (a) $\displaystyle\sum_{k=2}^{\infty}\frac{1}{\ln k}$ (b) $\displaystyle\sum_{k=3}^{\infty}\frac{1}{\ln(\ln k)}$

(c) $\displaystyle\sum_{k=3}^{\infty}\frac{1}{\ln(k\ln k)}$ (d) $\displaystyle\sum_{k=2}^{\infty}\frac{1}{\ln(k!)}$

20. (a) $\displaystyle\sum_{k=1}^{\infty}\tan^{-1} k$ (b) $\displaystyle\sum_{k=1}^{\infty}\frac{\tan^{-1} k}{k}$

(c) $\displaystyle\sum_{k=1}^{\infty}\frac{\tan^{-1} k}{k^2}$ (d) $\displaystyle\sum_{k=2}^{\infty}\frac{\sec^{-1} k}{k^2\sqrt{1-1/k^2}}$

In exercises 21–24, determine all values of p for which the series converges.

21. $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k(\ln k)^p}$

22. $\displaystyle\sum_{k=0}^{\infty}\frac{1}{(a+bk)^p}$, $a>0$, $b>0$

23. $\displaystyle\sum_{k=2}^{\infty}\frac{\ln k}{k^p}$

24. $\displaystyle\sum_{k=1}^{\infty}k^{p-1}e^{kp}$

In exercises 25–30, estimate the error in using the indicated partial sum S_n to approximate the sum of the series.

25. S_{100}, $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^4}$

26. S_{100}, $\displaystyle\sum_{k=1}^{\infty}\frac{4}{k^2}$

27. S_{50}, $\displaystyle\sum_{k=1}^{\infty}\frac{6}{k^8}$

28. S_{80}, $\displaystyle\sum_{k=1}^{\infty}\frac{2}{k^2+1}$

29. S_{40}, $\displaystyle\sum_{k=1}^{\infty}ke^{-k^2}$

30. S_{200}, $\displaystyle\sum_{k=1}^{\infty}\frac{\tan^{-1} k}{1+k^2}$

In exercises 31–34, determine the number of terms needed to obtain an approximation accurate to within 10^{-4}.

31. $\displaystyle\sum_{k=1}^{\infty}\frac{3}{k^5}$

32. $\displaystyle\sum_{k=1}^{\infty}\frac{2}{k^2}$

33. $\displaystyle\sum_{k=1}^{\infty}ke^{-k^2}$

34. $\displaystyle\sum_{k=1}^{\infty}\frac{4}{k^5}$

In exercises 35 and 36, answer with "converges" or "diverges" or "can't tell." Assume that $a_k>0$ and $b_k>0$.

35. Assume that $\displaystyle\sum_{k=1}^{\infty}a_k$ converges and fill in the blanks.

(a) If $b_k\ge a_k$ for $k\ge 10$, then $\displaystyle\sum_{k=1}^{\infty}b_k$ _____.

(b) If $\displaystyle\lim_{k\to\infty}\frac{b_k}{a_k}=0$, then $\displaystyle\sum_{k=1}^{\infty}b_k$ _____.

(c) If $b_k\le a_k$ for $k\ge 6$, then $\displaystyle\sum_{k=1}^{\infty}b_k$ _____.

(d) If $\displaystyle\lim_{k\to\infty}\frac{b_k}{a_k}=\infty$, then $\displaystyle\sum_{k=1}^{\infty}b_k$ _____.

36. Assume that $\displaystyle\sum_{k=1}^{\infty}a_k$ diverges and fill in the blanks.

(a) If $b_k\ge a_k$ for $k\ge 10$, then $\displaystyle\sum_{k=1}^{\infty}b_k$ _____.

(b) If $\displaystyle\lim_{k\to\infty}\frac{b_k}{a_k}=0$, then $\displaystyle\sum_{k=1}^{\infty}b_k$ _____.

(c) If $b_k\le a_k$ for $k\ge 6$, then $\displaystyle\sum_{k=1}^{\infty}b_k$ _____.

(d) If $\displaystyle\lim_{k\to\infty}\frac{b_k}{a_k}=\infty$, then $\displaystyle\sum_{k=1}^{\infty}b_k$ _____.

37. Prove the following extensions of the Limit Comparison Test:

(a) if $\displaystyle\lim_{k\to\infty}\frac{a_k}{b_k}=0$ and $\displaystyle\sum_{k=1}^{\infty}b_k$ converges, then $\displaystyle\sum_{k=1}^{\infty}a_k$ converges.

(b) if $\displaystyle\lim_{k\to\infty}\frac{a_k}{b_k}=\infty$ and $\displaystyle\sum_{k=1}^{\infty}b_k$ diverges, then $\displaystyle\sum_{k=1}^{\infty}a_k$ diverges.

38. If $a_k>0$ and $\displaystyle\sum_{k=1}^{\infty}a_k$ converge, prove that $\displaystyle\sum_{k=1}^{\infty}a_k^2$ converges.

39. Prove that if $\displaystyle\sum_{k=1}^{\infty}a_k^2$ and $\displaystyle\sum_{k=1}^{\infty}b_k^2$ converge, then $\displaystyle\sum_{k=1}^{\infty}|a_kb_k|$ converges.

40. Prove that for $a_k>0$, $\displaystyle\sum_{k=1}^{\infty}a_k$ converges if and only if $\displaystyle\sum_{k=1}^{\infty}\frac{a_k}{1+a_k}$ converges. $\left(\text{Hint: If } x<1\text{, then } x<\dfrac{2x}{1+x}.\right)$

41. Prove that the every-other-term harmonic series $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots$ diverges. (Hint: Write the series as $\displaystyle\sum_{k=0}^{\infty}\frac{1}{2k+1}$ and use the Limit Comparison Test.)

42. Would the every-third-term harmonic series $1+\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\cdots$ diverge? How about the every-fourth-term harmonic series $1+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\cdots$? Make as general a statement as possible about such series.

43. Show that $\displaystyle\sum_{k=2}^{\infty}\frac{1}{(\ln k)^{\ln k}}$ and $\displaystyle\sum_{k=2}^{\infty}\frac{1}{(\ln k)^k}$ both converge.

44. Show that $\displaystyle\sum_{k=2}^{\infty}\frac{1}{(\ln k)^n}$ diverges for any integer $n>0$. Compare this result to exercise 43.

45. Use your CAS to evaluate $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^p}$ for $p=2, 4, 6, 8$ and 10. Can your CAS evaluate the sum for odd values of p?

46. **The Riemann zeta function** is defined by $\zeta(x)=\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^x}$ for $x>1$. Explain why the restriction $x>1$ is necessary. Leonhard Euler, considered to be one of the greatest mathematicians ever, proved the remarkable result that $\zeta(x)=\displaystyle\prod_{p=\text{prime}}\frac{1}{\left(1-\frac{1}{p^x}\right)}$.

47. (a) Explain why the Trapezoidal Rule approximation of $\int_0^n x^n\,dx$ will be larger than the integral, for any integer $n>1$.

(b) Show that the Trapezoidal Rule approximation, with $h=1$ equals $1^n+2^n+\cdots+(n-1)^n+\dfrac{1}{2}n^n$.

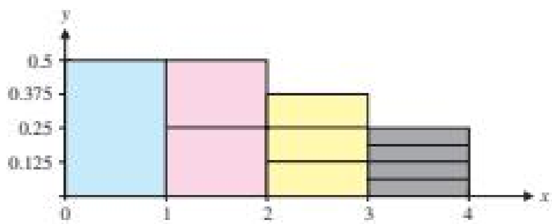
(c) Conclude that $\dfrac{1^n+2^n+\cdots+n^n}{n^n}>\dfrac{3n+1}{2n+2}$.

48. Numerically investigate the p-series $\sum_{k=1}^{\infty} \frac{1}{k^{0.9}}$ and $\sum_{k=1}^{\infty} \frac{1}{k^{1.1}}$ and for other values of p close to 1. Can you distinguish convergent from divergent series numerically?

APPLICATIONS

1. Suppose that you toss a fair coin until you get heads. How many times would you expect to toss the coin? To answer this, notice that the probability of getting heads on the first toss is $\frac{1}{2}$, getting tails then heads is $\left(\frac{1}{2}\right)^2$, getting two tails then heads is $\left(\frac{1}{2}\right)^3$ and so on. The mean number of tosses is $\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k$. Use the Integral Test to prove that this series converges and estimate the sum numerically.

2. The series $\sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k$ can be visualized as the area shown in the figure. In columns of width one, we see one rectangle of height $\frac{1}{2}$, two rectangles of height $\frac{1}{4}$, three rectangles of height $\frac{1}{8}$ and so on. Start the sum by taking one rectangle from each column. The combined area of the first rectangles is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. Show that this is a convergent series with sum 1. Next, take the second rectangle from each column that has at least two rectangles. The combined area of the second rectangles is $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$. Show that this is a convergent series with sum $\frac{1}{2}$. Next, take the third rectangle from each column that has at least three rectangles. The combined area from the third rectangles is $\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$. Show that this is a convergent series with sum $\frac{1}{4}$. Continue this process and show that the total area of all rectangles is $1 + \frac{1}{2} + \frac{1}{4} + \cdots$. Find the sum of this convergent series.



3. The **coupon collectors' problem** is faced by collectors of trading cards. If there are n different cards that make a complete set and you randomly obtain one at a time, how many cards would you expect to obtain before having a complete set? (By random, we mean that each different card has the same probability of $\frac{1}{n}$ of being the next card obtained.) In exercises 3–5, we find the answer for $n = 10$. The first step is simple; to collect one card you need to obtain one card. Now, given that you have one card, how many cards do you need to obtain to get a second (different) card? If you're lucky, the next card is it (this has probability $\frac{9}{10}$). But your next card might be a duplicate, then you get a new card (this has probability $\frac{1}{10} \cdot \frac{9}{10}$). Or you might get two duplicates and then a new card (this has probability $\frac{1}{10} \cdot \frac{1}{10} \cdot \frac{9}{10}$); and so on. The mean is $1 \cdot \frac{9}{10} + 2 \cdot \frac{1}{10} \cdot \frac{9}{10} + 3 \cdot \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{9}{10} + \cdots$ or $\sum_{k=1}^{\infty} k\left(\frac{1}{10}\right)^{k-1}\left(\frac{9}{10}\right) = \sum_{k=1}^{\infty} \frac{9k}{10^k}$. Using the same trick as in exercise 2, show that this is a convergent series with sum $\frac{10}{9}$.

4. In the situation of exercise 3, if you have two different cards out of ten, the average number of cards to get a third distinct card is $\sum_{k=1}^{\infty} \frac{8k2^{k-1}}{10^k}$; show that this is a convergent series with sum $\frac{10}{8}$.

5. (a) Extend the results of exercises 3 and 4 to find the average number of cards you need to obtain to complete the set of ten different cards.

   (b) Compute the ratio of cards obtained to cards in the set. That is, for a set of 10 cards, on the average you need to obtain _____ times 10 cards to complete the set.

6. (a) Generalize exercise 5 in the case of n cards in the set ($n > 2$).

   (b) Use the divergence of the harmonic series to state the unfortunate fact about the ratio of cards obtained to cards in the set as n increases.

EXPLORATORY EXERCISES

1. In this exercise, you explore the convergence of the infinite product $P = 2^{1/4}3^{1/9}4^{1/16} \cdots$. This can be written in the form $P = \prod_{k=2}^{\infty} k^{1/k^2}$. For the partial product $P_n = \prod_{k=2}^{n} k^{1/k^2}$, use the natural logarithm to write

$$P_n = e^{\ln P_n} = e^{\ln[2^{1/4}3^{1/9}4^{1/16} \cdots n^{1/n^2}]} = e^{S_n}, \text{ where}$$
$$S_n = \ln[2^{1/4}3^{1/9}4^{1/16} \cdots n^{1/n^2}]$$
$$= \frac{1}{4}\ln 2 + \frac{1}{9}\ln 3 + \frac{1}{16}\ln 4 + \cdots + \frac{1}{n^2}\ln n.$$

By comparing to an appropriate integral and showing that the integral converges, show that $\{S_n\}$ converges. Show that $\{P_n\}$ converges to a number between 2.33 and 2.39. Use a CAS or calculator to compute P_n for large n and see how accurate the computation is.

2. Define a function $f(x)$ in the following way for $0 \le x \le 1$. Write out the binary expansion of x. That is,

$$x = \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \cdots$$

where each a_i is either 0 or 1. Prove that this infinite series converges. Then $f(x)$ is the corresponding ternary expansion, given by

$$f(x) = \frac{a_1}{3} + \frac{a_2}{9} + \frac{a_3}{27} + \cdots$$

Prove that this series converges. There is a subtle issue here of whether the function is well defined or not. Show that $\frac{1}{2}$ can be written with $a_1 = 1$ and $a_k = 0$ for $k \ge 2$ and also with $a_1 = 0$ and $a_k = 1$ for $k \ge 2$. Show that you get different values of $f(x)$ with different representations. In such cases, we choose the representation with as few 1's as possible. Show that $f(2x) = 3f(x)$ and $f\left(x + \frac{1}{2}\right) = \frac{1}{3} + f(x)$ for $0 \le x \le \frac{1}{2}$. Use these facts to compute $\int_0^1 f(x)dx$. Generalize the result for any base n conversion

$$f(x) = \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \cdots,$$

where n is an integer greater than 1.

5.4 ALTERNATING SERIES

So far, we have focused our attention on positive-term series, that is, series for which all the terms are positive. In this section, we examine *alternating series*, that is, series whose terms alternate back and forth from positive to negative.

An **alternating series** is any series of the form

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots,$$

where $a_k > 0$, for all k.

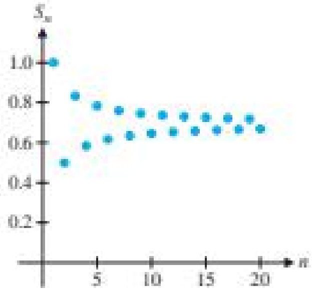


FIGURE 5.30

$$S_n = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}$$

n	$S_n = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}$
1	1
2	0.5
3	0.83333
4	0.58333
5	0.78333
6	0.61667
7	0.75952
8	0.63452
9	0.74563
10	0.64563
11	0.73654
12	0.65321
13	0.73013
14	0.65871
15	0.72537
16	0.66287
17	0.7217
18	0.66614
19	0.71877
20	0.66877

EXAMPLE 4.1 The Alternating Harmonic Series

Investigate the convergence or divergence of the **alternating harmonic series**

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

Solution The graph of the first 20 partial sums seen in Figure 5.30 suggests that the series might converge to about 0.7. We now calculate the first few partial sums by hand. Note that

$$S_1 = 1, \qquad S_2 = 1 - \frac{1}{2} = \frac{1}{2},$$

$$S_3 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}, \qquad S_4 = \frac{5}{6} - \frac{1}{4} = \frac{7}{12},$$

$$S_5 = \frac{7}{12} + \frac{1}{5} = \frac{47}{60}, \qquad S_6 = \frac{47}{60} - \frac{1}{6} = \frac{37}{60},$$

and so on. We have plotted the first 8 partial sums on the number line shown in Figure 5.31.

FIGURE 5.31

Partial sums of $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$

Notice that the partial sums are bouncing back and forth, but seem to be zeroing in on some value. This should not be surprising, since each new term that is added or subtracted is less than the term added or subtracted to get the previous partial sum. You should notice this same zeroing-in process in the accompanying table displaying the first 20 partial sums of the series. Based on the behavior of the partial sums, it is reasonable to conjecture that the series converges to some value between 0.66877 and 0.71877. We can resolve the question of convergence definitively with Theorem 4.1. ■

THEOREM 4.1 (Alternating Series Test)

Suppose that $\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \le a_k$ for all $k \ge 1$. Then, the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges.

Before considering the proof of Theorem 4.1, make sure that you have a clear idea what it is saying. In the case of an alternating series satisfying the hypotheses of the theorem, we start with 0 and add $a_1 > 0$ to get the first partial sum S_1. To get the next partial sum, S_2, we subtract a_2 from S_1, where $a_2 \le a_1$. This says that S_2 will be between 0 and S_1. We illustrate this situation in Figure 5.32.

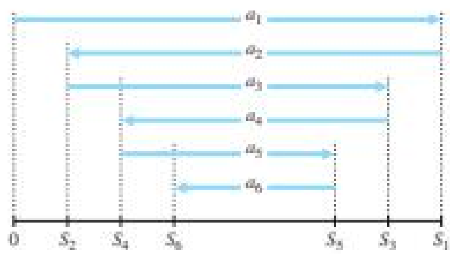


FIGURE 5.32
Convergence of the partial sums of an alternating series

Continuing in this fashion, we add a_3 to S_2 to get S_3. Since $a_3 \le a_2$, we must have that $S_2 \le S_3 \le S_1$. Referring to Figure 5.32, notice that

$$S_2 \le S_4 \le S_6 \le \cdots \le S_5 \le S_3 \le S_1.$$

In particular, this says that *all* of the odd-indexed partial sums (i.e., S_{2n+1}, for $n = 0, 1, 2, \ldots$) are larger than *all* of the even-indexed partial sums (i.e., S_{2n}, for $n = 1, 2, \ldots$). As the partial sums oscillate back and forth, they should be drawing closer and closer to some limit S, somewhere between all of the even-indexed partial sums and the odd-indexed partial sums,

$$S_2 \le S_4 \le S_6 \le \cdots \le S \le \cdots \le S_5 \le S_3 \le S_1. \tag{4.1}$$

PROOF

Notice from Figure 5.32 that the even- and odd-indexed partial sums seem to behave somewhat differently. First, we consider the even-indexed partial sums. We have

$$S_2 = a_1 - a_2 \ge 0$$

and

$$S_4 = S_2 + (a_3 - a_4) \ge S_2,$$

since $(a_3 - a_4) \ge 0$. Likewise, for any n, we can write

$$S_{2n} = S_{2n-2} + (a_{2n-1} - a_{2n}) \ge S_{2n-2},$$

since $(a_{2n-1} - a_{2n}) \ge 0$. This says that the sequence of even-indexed partial sums $\{S_{2n}\}_{n=1}^{\infty}$ is increasing (as we saw in Figure 5.32). Further, observe that

$$0 \le S_{2n} = a_1 + (-a_2 + a_3) + (-a_4 + a_5) + \cdots + (-a_{2n-2} + a_{2n-1}) - a_{2n} \le a_1,$$

for all n, since every term in parentheses is negative. Thus, $\{S_{2n}\}_{n=1}^{\infty}$ is both bounded (by a_1) and monotonic (increasing). By Theorem 1.4, $\{S_{2n}\}_{n=1}^{\infty}$ must be convergent to some number, say L.

Turning to the sequence of odd-indexed partial sums, notice that we have

$$S_{2n+1} = S_{2n} + a_{2n+1}.$$

From this, we have

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + a_{2n+1}) = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = L + 0 = L,$$

since $\lim_{n\to\infty} a_n = 0$. Since both the sequence of odd-indexed partial sums $\{S_{2n+1}\}_{n=0}^{\infty}$ and the sequence of even-indexed partial sums $\{S_{2n}\}_{n=1}^{\infty}$ converge to the same limit, L, we have that

$$\lim_{n\to\infty} S_n = L,$$

also. ■

EXAMPLE 4.2 Using the Alternating Series Test

Reconsider the convergence of the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$.

Solution Notice that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0.$$

Further,
$$0 < a_{k+1} = \frac{1}{k+1} \le \frac{1}{k} = a_k, \quad \text{for all } k \ge 1.$$

By the Alternating Series Test, the series converges. (The calculations from example 4.1 give an approximate sum. An exact sum is found in exercise 45.) ■

The Alternating Series Test is straightforward, but you will sometimes need to work a bit to verify the hypotheses.

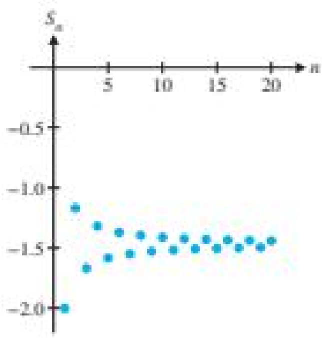


FIGURE 5.33
$S_n = \sum_{k=1}^{n} \frac{(-1)^k(k+3)}{k(k+1)}$

EXAMPLE 4.3 Using the Alternating Series Test

Investigate the convergence or divergence of the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k(k+3)}{k(k+1)}$.

Solution The graph of the first 20 partial sums seen in Figure 5.33 suggests that the series converges to some value around −1.5. The following tables showing some select partial sums suggests the same conclusion.

n	$S_n = \sum_{k=1}^{n} \frac{(-1)^k(k+3)}{k(k+1)}$
50	−1.45545
100	−1.46066
200	−1.46322
300	−1.46406
400	−1.46448

n	$S_n = \sum_{k=1}^{n} \frac{(-1)^k(k+3)}{k(k+1)}$
51	−1.47581
101	−1.47076
201	−1.46824
301	−1.46741
401	−1.46699

We can verify that the series converges by first checking that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{(k+3)}{k(k+1)} \frac{\frac{1}{k^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{\frac{1}{k} + \frac{3}{k^2}}{1 + \frac{1}{k}} = 0.$$

Next, consider the ratio of two consecutive terms:

$$\frac{a_{k+1}}{a_k} = \frac{(k+4)}{(k+1)(k+2)} \frac{k(k+1)}{(k+3)} = \frac{k^2+4k}{k^2+5k+6} < 1,$$

for all $k \ge 1$. From this, it follows that $a_{k+1} < a_k$, for all $k \ge 1$ and so, by the Alternating Series Test, the series converges. Finally, from the preceding tables, we can see that the series converges to a sum between −1.46448 and −1.46699. (How can you be sure that the sum is in this interval?) ■

EXAMPLE 4.4 A Divergent Alternating Series

Determine whether the alternating series $\sum_{k=3}^{\infty} \frac{(-1)^k k}{k+2}$ converges or diverges.

Solution First, notice that

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{k+2} = 1 \ne 0.$$

So, this alternating series is divergent, since by the kth-term test for divergence, the terms must tend to zero in order for the series to be convergent. ■

Estimating the Sum of an Alternating Series

So far, we have found approximate sums of convergent series by calculating a number of partial sums of the series. As was the case for positive-term series to which the Integral Test applies, we can say something very precise for alternating series. First, note that the error in approximating the sum S by the nth partial sum S_n is $S - S_n$.

Look back at Figure 5.32 and observe that all of the even-indexed partial sums S_n of the convergent alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ lie below the sum S, while all of the odd-indexed partial sums lie above S. That is [as in (4.1)],

$$S_2 \le S_4 \le S_6 \le \cdots \le S \le \cdots \le S_5 \le S_3 \le S_1.$$

This says that for n even, $$S_n \le S \le S_{n+1}.$$

Subtracting S_n from all terms, we get

$$0 \le S - S_n \le S_{n+1} - S_n = a_{n+1}.$$

Since $a_{n+1} > 0$, we have $$-a_{n+1} < 0 \le S - S_n \le a_{n+1},$$

or $$|S - S_n| \le a_{n+1}, \quad \text{for } n \text{ even.} \tag{4.2}$$

Similarly, for n odd, we have that $$S_{n+1} \le S \le S_n.$$

Again subtracting S_n, we get

$$-a_{n+1} = S_{n+1} - S_n \le S - S_n \le 0 < a_{n+1}$$

or $$|S - S_n| \le a_{n+1}, \quad \text{for } n \text{ odd.} \tag{4.3}$$

Since (4.2) and (4.3) (called **error bounds**) are the same, we have the same error bound whether n is even or odd. This establishes the following result.

THEOREM 4.2

Suppose that $\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \le a_k$ for all $k \ge 1$. Then, the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1} a_k$ converges to some number S and the error in approximating S by the nth partial sum S_n satisfies

$$|S - S_n| \le a_{n+1}. \tag{4.4}$$

Theorem 4.2 says that the absolute value of the error in approximating S by S_n does not exceed a_{n+1} (the absolute value of the first neglected term).

EXAMPLE 4.5 Estimating the Sum of an Alternating Series

Approximate the sum of the alternating series $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^4}$ by the 40th partial sum and estimate the error in this approximation.

Solution We leave it as an exercise to show that this series is convergent. We then approximate the sum by

$$S \approx S_{40} \approx 0.9470326439.$$

From our error estimate (4.4), we have

$$|S - S_{40}| \le a_{41} = \frac{1}{41^4} \approx 3.54 \times 10^{-7}.$$

This says that our approximation $S \approx 0.9470326439$ is off by no more than $\pm 3.54 \times 10^{-7}$. ■

A much more interesting question than the one asked in example 4.5 is the following. For a given convergent alternating series, how many terms must we take in order to obtain an approximation with a given accuracy? We illustrate this in example 4.6.

EXAMPLE 4.6 Finding the Number of Terms Needed for a Given Accuracy

For the convergent alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$, how many terms are needed to guarantee that S_n is within 1×10^{-10} of the actual sum S?

Solution In this case, we want to find the number of terms n for which

$$|S - S_n| \le 1 \times 10^{-10}.$$

From (4.4), we have that $$|S - S_n| \le a_{n+1} = \frac{1}{(n+1)^4}.$$

So, we look for n such that $$\frac{1}{(n+1)^4} \le 1 \times 10^{-10}.$$

Solving for n, we get $$10^{10} \le (n+1)^4,$$

so that $$\sqrt[4]{10^{10}} \le n + 1$$

or $$n \ge \sqrt[4]{10^{10}} - 1 \approx 315.2.$$

So, if we take $n \ge 316$, we will guarantee an error of no more than 1×10^{-10}. Using the suggested number of terms, we get the approximate sum

$$S \approx S_{316} \approx 0.947032829447,$$

which we now know to be correct to within 1×10^{-10}. ■

BEYOND FORMULAS

When you think about infinite series, you must understand the interplay between sequences and series. Our tests for convergence involve sequences and are completely separate from the question of finding the *sum* of the series. It is important to keep reminding yourself that the sum of a convergent series is the limit of the sequence of partial sums. Often, the best we can do is to approximate the sum of a series by adding together a number of terms. In this case, it becomes important to determine the accuracy of the approximation. For alternating series, this is found by examining the first neglected term. When finding an approximation with a specified accuracy, you first use the error bound in Theorem 4.2 to find how many terms you need to add. You then get an approximation with the desired accuracy by adding together that many terms.

EXERCISES 5.4

WRITING EXERCISES

1. If $a_k \geq 0$ and $\lim_{k\to\infty} a_k = 0$, explain in terms of partial sums why $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ is more likely to converge than $\sum_{k=1}^{\infty} a_k$.

2. Explain why in Theorem 4.2 we need the assumption that $a_{k+1} \leq a_k$. That is, what would go wrong with the proof if $a_{k+1} > a_k$?

3. The Alternating Series Test was stated for the series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$. Explain the difference between $\sum_{k=1}^{\infty} (-1)^k a_k$ and $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ and explain why we could have stated the theorem for $\sum_{k=1}^{\infty} (-1)^k a_k$.

4. A common mistake is to think that if $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ converges. Explain why this is not true for positive-term series. This is also not true for alternating series *unless* you add one more hypothesis. State the extra hypothesis. (Exercise 43 shows why it is needed.)

In exercises 1–24, determine whether the series is convergent or divergent.

1. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3}{k}$

2. $\sum_{k=1}^{\infty} (-1)^k \frac{2}{k^2}$

3. $\sum_{k=1}^{\infty} (-1)^k \frac{4}{\sqrt{k}}$

4. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{k+1}$

5. $\sum_{k=2}^{\infty} (-1)^k \frac{k}{k^2+2}$

6. $\sum_{k=7}^{\infty} (-1)^k \frac{2k-1}{k^3}$

7. $\sum_{k=5}^{\infty} (-1)^{k+1} \frac{k}{2^k}$

8. $\sum_{k=4}^{\infty} (-1)^{k+1} \frac{3^k}{k}$

9. $\sum_{k=1}^{\infty} (-1)^k \frac{4^k}{k^2}$

10. $\sum_{k=1}^{\infty} (-1)^k \frac{k+2}{4^k}$

11. $\sum_{k=1}^{\infty} \frac{3}{2+k}$

12. $\sum_{k=2}^{\infty} \frac{3}{2^k}$

13. $\sum_{k=3}^{\infty} (-1)^k \frac{3}{\sqrt{k+1}}$

14. $\sum_{k=4}^{\infty} (-1)^k \frac{k+1}{k^3}$

15. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k!}$

16. $\sum_{k=3}^{\infty} (-1)^{k+1} \frac{k!}{3^k}$

17. $\sum_{k=1}^{\infty} \frac{4k}{k^2+2k+2}$

18. $\sum_{k=3}^{\infty} \frac{4k^2}{k^4+2k+2}$

19. $\sum_{k=5}^{\infty} (-1)^{k+1} 2e^{-k}$

20. $\sum_{k=6}^{\infty} (-1)^{k+1} 3e^{1/k}$

21. $\sum_{k=2}^{\infty} (-1)^k \ln k$

22. $\sum_{k=2}^{\infty} (-1)^k \frac{1}{\ln k}$

23. $\sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{2^k}$

24. $\sum_{k=0}^{\infty} (-1)^{k+1} 2^k$

In exercises 25–32, estimate the sum of each convergent series to within 0.01.

25. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{4}{k^3}$

26. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k^3}$

27. $\sum_{k=3}^{\infty} (-1)^k \frac{k}{2^k}$

28. $\sum_{k=4}^{\infty} (-1)^k \frac{k^2}{10^k}$

29. $\sum_{k=0}^{\infty} (-1)^k \frac{3}{k!}$

30. $\sum_{k=0}^{\infty} (-1)^{k+1} \frac{2}{k!}$

31. $\sum_{k=2}^{\infty} (-1)^{k+1} \frac{4}{k^4}$

32. $\sum_{k=3}^{\infty} (-1)^{k+1} \frac{3}{k^3}$

In exercises 33–36, determine how many terms are needed to estimate the sum of the series to within 0.0001.

33. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k}$

34. $\sum_{k=0}^{\infty} (-1)^k \frac{2^k}{k!}$

35. $\sum_{k=0}^{\infty} (-1)^k \frac{10^k}{k!}$

36. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k!}{k^k}$

In exercises 37–40, explain why Theorem 4.1 does not directly apply. Conjecture the convergence or divergence of the series.

37. $\sum_{k=1}^{\infty} (-1)^k e^{-k} \sin k$

38. $\sum_{k=2}^{\infty} (-1)^k \frac{|\sin(k\pi/2)|}{k}$

39. $\sum_{k=3}^{\infty} (-1)^k \frac{1+(-1)^k}{\sqrt{k}}$

40. $\sum_{k=4}^{\infty} (-1)^{2k} \frac{\sin k}{k^2}$

41. In the text, we showed you one way to verify that a sequence is decreasing. As an alternative, explain why if $a_k = f(k)$ and $f'(k) < 0$, then the sequence a_k is decreasing. Use this method to prove that $a_k = \frac{k}{k^2+2}$ is decreasing.

42. Verify that the series $\sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ converges. It can be shown that the sum of this series is $\frac{\pi}{4}$. Given this result, we could use this series to obtain an approximation of π. How many terms would be necessary to get eight digits of π correct?

43. In this exercise, you will discover why the Alternating Series Test requires that $a_{k+1} \leq a_k$. If $a_k = \begin{cases} 1/k & \text{if } k \text{ is odd} \\ 1/k^2 & \text{if } k \text{ is even} \end{cases}$, argue that $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ diverges to ∞. Thus, an alternating series can diverge even if $\lim_{k\to\infty} a_k = 0$.

44. (a) Find a counterexample to show that the following statement is false (not always true). If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, then $\sum_{k=1}^{\infty} a_k b_k$ converges. (b) Find assumptions that can be made (for example, $a_k > 0$) that make the statement in part (a) true.

45. For the alternating harmonic series, show that $S_{2n} = \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{n+k} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+k/n}$. Identify this as a Riemann sum and show that the alternating harmonic series converges to ln 2.

46. Find all values of p such that the series $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^p}$ converges. Compare your result to the p-series of section 5.3.

APPLICATIONS

1. A person starts walking from home (at $x = 0$) toward a friend's house (at $x = 1$). Three-fourths of the way there, he changes his mind and starts walking back home. Three-fourths of the way home, he changes his mind again and starts walking back to his friend's house. If he continues this pattern of indecision, always turning around at the three-fourths mark, what will be the eventual outcome? A similar problem appeared in a national magazine and created a minor controversy due to the ambiguous wording of the problem. It is clear that the first turnaround is at $x = \frac{3}{4}$ and the second turnaround is at $\frac{3}{4} - \frac{3}{4}\left(\frac{3}{4}\right) = \frac{3}{16}$. But is the third turnaround three-fourths of the way to $x = 1$ or $x = \frac{3}{4}$? The magazine writer assumed the latter. Show that with this assumption, the person's location forms a geometric series. Find the sum of the series and state where the person ends up.

2. If the problem of Exercise 1 is interpreted differently, a more interesting answer results. As before, let $x_1 = \frac{3}{4}$ and $x_2 = \frac{3}{16}$. If the next turnaround is three-fourths of the way from x_2 to 1, then $x_3 = \frac{3}{16} + \frac{3}{4}\left(1 - \frac{3}{16}\right) = \frac{3}{4} + \frac{1}{4}x_2 = \frac{51}{64}$. Three-fourths of the way back to $x = 0$ would put us at $x_4 = x_3 - \frac{3}{4}x_3 = \frac{1}{4}x_3 = \frac{51}{256}$. Show that if n is even, then $x_{n+1} = \frac{3}{4} + \frac{1}{4}x_n$ and $x_{n+2} = \frac{1}{4}x_{n+1}$. Show that the person ends up walking back and forth between two specific locations.

EXPLORATORY EXERCISES

1. In this exercise, you will determine whether or not the improper integral $\int_0^1 \sin(1/x)\, dx$ converges. Argue that $\int_{1/\pi}^1 \sin(1/x)\, dx$, $\int_{1/(2\pi)}^{1/\pi} \sin(1/x)\, dx$, $\int_{1/(3\pi)}^{1/(2\pi)} \sin(1/x)\, dx, \ldots$ exist and that (if it exists),

$$\int_0^1 \sin(1/x)\, dx = \int_{1/\pi}^1 \sin(1/x)\, dx + \int_{1/(2\pi)}^{1/\pi} \sin(1/x)\, dx + \int_{1/(3\pi)}^{1/(2\pi)} \sin(1/x)\, dx + \cdots.$$

Verify that the series is an alternating series and show that the hypotheses of the Alternating Series Test are met. Thus, the series and the improper integral both converge.

2. Consider the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$, where x is a constant. Show that the series converges for $x = 1/2$; $x = -1/2$; any x such that $-1 < x \le 1$. Show that the series diverges if $x = -1$, $x < -1$ or $x > 1$. We see in Exercise 5 of section 5.7 that when the series converges, it converges to $\ln(1 + x)$. Verify this numerically for $x = 1/2$ and $x = -1/2$.

5.5 ABSOLUTE CONVERGENCE AND THE RATIO TEST

Outside of the Alternating Series Test presented in section 5.4, our other tests for convergence of series (i.e., the Integral Test and the two comparison tests) apply only to series all of whose terms are *positive*. So, what do we do if we're faced with a series that has both positive and negative terms, but that is not an alternating series? For instance, the series

$$\sum_{k=1}^{\infty} \frac{\sin k}{k^3} = \sin 1 + \frac{1}{8}\sin 2 + \frac{1}{27}\sin 3 + \frac{1}{64}\sin 4 + \cdots$$

has both positive and negative terms, but the terms do not alternate signs. (Calculate the first five or six terms of the series to see this for yourself.) For any such series $\sum_{k=1}^{\infty} a_k$, we check whether the series of absolute values $\sum_{k=1}^{\infty} |a_k|$ is convergent. When this happens, we say that the original series $\sum_{k=1}^{\infty} a_k$ is **absolutely convergent** (or **converges absolutely**). Note that to test the convergence of the series of absolute values $\sum_{k=1}^{\infty} |a_k|$ (all of whose terms are positive), we have all of our earlier tests for positive-term series available to us.

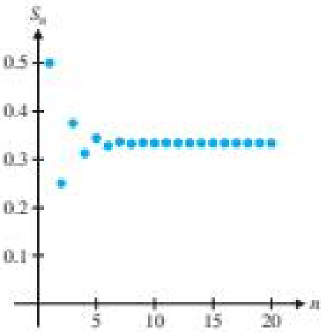


FIGURE 5.34

$$S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2^k}$$

EXAMPLE 5.1 Testing for Absolute Convergence

Determine whether $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k}$ is absolutely convergent.

Solution It is easy to show that this alternating series converges to approximately 0.35. (See Figure 5.34.) To determine absolute convergence, we need to check whether or not the series of absolute values is convergent. We have

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{2^k} \right| = \sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k,$$

which you should recognize as a convergent geometric series ($|r| = \frac{1}{2} < 1$). This says that the original series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k}$ converges absolutely. ■

We'll prove shortly that every absolutely convergent series is also convergent. However, the reverse is not true; there are many series that are convergent, but not absolutely convergent. These are called **conditionally convergent** series. Can you think of an example of such a series? If so, it's probably the example that follows.

EXAMPLE 5.2 A Conditionally Convergent Series

Determine whether the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is absolutely convergent.

Solution In example 4.2, we showed that this series is convergent. To test this for absolute convergence, we consider the series of absolute values,

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$$

(the harmonic series), which diverges. This says that $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges conditionally (i.e., it converges, but does not converge absolutely). ■

THEOREM 5.1

If $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

This result says that if a series converges absolutely, then it must also converge. Because of this, when we test series, we first test for absolute convergence. If a series converges absolutely, then we need not test any further to establish convergence.

PROOF

Notice that for any real number, x, we can say that $-|x| \le x \le |x|$. So, for any k, we have

$$-|a_k| \le a_k \le |a_k|.$$

Adding $|a_k|$ to all the terms, we get

$$0 \le a_k + |a_k| \le 2|a_k|. \tag{5.1}$$

Since $\sum_{k=1}^{\infty} |a_k|$ is convergent, we have that $\sum_{k=1}^{\infty} 2|a_k| = 2\sum_{k=1}^{\infty} |a_k|$ is convergent. Define $b_k = a_k + |a_k|$. From (5.1),

$$0 \le b_k \le 2|a_k|$$

and so, by the Comparison Test, $\sum_{k=1}^{\infty} b_k$ is convergent. Observe that we may write

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}(a_k + |a_k| - |a_k|) = \sum_{k=1}^{\infty} \underbrace{(a_k + |a_k|)}_{b_k} - \sum_{k=1}^{\infty} |a_k|$$
$$= \sum_{k=1}^{\infty} b_k - \sum_{k=1}^{\infty} |a_k|.$$

Since the two series on the right-hand side are convergent, it follows that $\sum_{k=1}^{\infty} a_k$ must also be convergent. ■

EXAMPLE 5.3 Testing for Absolute Convergence

Determine whether $\sum_{k=1}^{\infty} \frac{\sin k}{k^3}$ is convergent or divergent.

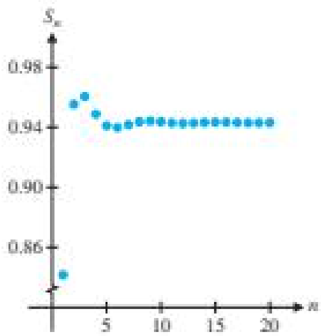


FIGURE 5.35

$S_n = \sum_{k=1}^{n} \frac{\sin k}{k^3}$

Solution Notice that while this is not a positive-term series, neither is it an alternating series. Because of this, our only choice is to test the series for absolute convergence. From the graph of the first 20 partial sums seen in Figure 5.35, it appears that the series is converging to some value around 0.94. To test for absolute convergence, we consider the series of absolute values, $\sum_{k=1}^{\infty} \left|\frac{\sin k}{k^3}\right|$. Notice that

$$\left|\frac{\sin k}{k^3}\right| = \frac{|\sin k|}{k^3} \le \frac{1}{k^3}, \tag{5.2}$$

since $|\sin k| \le 1$, for all k. Of course, $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series ($p = 3 > 1$). By the Comparison Test and (5.2), $\sum_{k=1}^{\infty} \left|\frac{\sin k}{k^3}\right|$ converges, too. Consequently, the original series $\sum_{k=1}^{\infty} \frac{\sin k}{k^3}$ converges absolutely and, hence, converges. ■

○ The Ratio Test

We next introduce a very powerful tool for testing a series for absolute convergence. This test can be applied to a wide range of series, including the extremely important case of power series, which we discuss in section 5.6. As you'll see, this test is remarkably easy to use.

THEOREM 5.2 (Ratio Test)

Given $\sum_{k=1}^{\infty} a_k$, with $a_k \neq 0$ for all k, suppose that

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L.$$

Then,

(i) if $L < 1$, the series converges absolutely,
(ii) if $L > 1$ (or $L = \infty$), the series diverges and
(iii) if $L = 1$, there is no conclusion.

PROOF

(i) For $L < 1$, pick any number r with $L < r < 1$. Then, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = L < r.$$

For this to occur, there must be some number $N > 0$, such that for $k \ge N$,

$$\left|\frac{a_{k+1}}{a_k}\right| < r. \tag{5.3}$$

Multiplying both sides of (5.3) by $|a_k|$ gives us

$$|a_{k+1}| < r|a_k|.$$

In particular, taking $k = N$ gives us

$$|a_{N+1}| < r|a_N|$$

and taking $k = N + 1$ gives us

$$|a_{N+2}| < r|a_{N+1}| < r^2|a_N|.$$

Likewise,

$$|a_{N+3}| < r|a_{N+2}| < r^3|a_N|$$

and so on. We have

$$|a_{N+k}| < r^k|a_N|, \quad \text{for } k = 1, 2, 3, \cdots.$$

Notice that $\sum_{k=1}^{\infty}|a_N|r^k = |a_N|\sum_{k=1}^{\infty}r^k$ is a convergent geometric series, since $0 < r < 1$. By the Comparison Test, it follows that $\sum_{k=1}^{\infty}|a_{N+k}| = \sum_{n=N+1}^{\infty}|a_n|$ converges, too. This says that $\sum_{n=N+1}^{\infty}a_n$ converges absolutely. Finally, since

$$\sum_{n=1}^{\infty}a_n = \sum_{n=1}^{N}a_n + \sum_{n=N+1}^{\infty}a_n,$$

we also get that $\sum_{n=1}^{\infty}a_n$ converges absolutely.

(ii) For $L > 1$, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = L > 1.$$

This says that there must be some number $N > 0$, such that for $k \ge N$,

$$\left|\frac{a_{k+1}}{a_k}\right| > 1. \tag{5.4}$$

Multiplying both sides of (5.4) by $|a_k|$, we get

$$|a_{k+1}| > |a_k| > 0, \quad \text{for all } k \ge N.$$

Notice that if this is the case, then

$$\lim_{k\to\infty} a_k \ne 0.$$

By the kth-term test for divergence, we now have that $\sum_{k=1}^{\infty}a_k$ diverges. ■

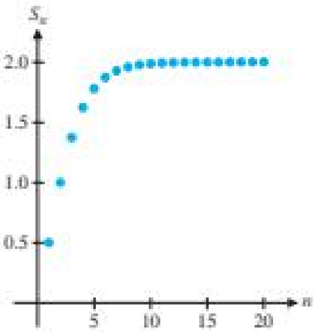


FIGURE 5.36

$S_n = \sum_{k=1}^{n} \frac{k}{2^k}$

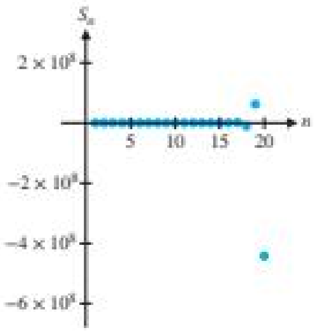


FIGURE 5.37

$S_n = \sum_{k=0}^{n-1} \frac{(-1)^k k!}{e^k}$

EXAMPLE 5.4 Using the Ratio Test

Test $\sum_{k=1}^{\infty} \frac{(-1)^k k}{2^k}$ for convergence.

Solution The graph of the first 20 partial sums of the series of absolute values, $\sum_{k=1}^{\infty} \frac{k}{2^k}$, seen in Figure 5.36, suggests that the series of absolute values converges to about 2. From the Ratio Test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\frac{k+1}{2^{k+1}}}{\frac{k}{2^k}} = \lim_{k\to\infty}\frac{k+1}{2^{k+1}}\frac{2^k}{k} = \frac{1}{2}\lim_{k\to\infty}\frac{k+1}{k} = \frac{1}{2} < 1$$

Since $2^{k+1} = 2^k \cdot 2^1$.

and so, the series converges absolutely, as expected from Figure 5.36. ■

The Ratio Test is particularly useful when the general term of a series contains an exponential term, as in example 5.4, or a factorial, as in example 5.5.

EXAMPLE 5.5 Using the Ratio Test

Test $\sum_{k=0}^{\infty} \frac{(-1)^k k!}{e^k}$ for convergence.

Solution The graph of the first 20 partial sums of the series seen in Figure 5.37 suggests that the series diverges. We can confirm this suspicion with the Ratio Test. We have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\frac{(k+1)!}{e^{k+1}}}{\frac{k!}{e^k}} = \lim_{k\to\infty}\frac{(k+1)!}{e^{k+1}}\frac{e^k}{k!}$$

$$= \lim_{k\to\infty}\frac{(k+1)k!}{ek!} = \frac{1}{e}\lim_{k\to\infty}\frac{k+1}{1} = \infty.$$

Since $(k+1)! = (k+1)\cdot k!$ and $e^{k+1} = e^k \cdot e^1$.

By the Ratio Test, the series diverges, as we suspected. ■

HISTORICAL NOTES

Srinivasa Ramanujan (1887–1920) Indian mathematician whose incredible discoveries about infinite series still amaze mathematicians. Largely self-taught, Ramanujan filled notebooks with conjectures about series, continued fractions and the Riemann-zeta function. Ramanujan rarely gave a proof or even justification of his results. Nevertheless, the famous English mathematician G. H. Hardy said, "They must be true because, if they weren't true, no one would have had the imagination to invent them." (See exercise 61, to come.)

Recall that in the statement of the Ratio Test we said that if

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1,$$

then the Ratio Test yields no conclusion. By this, we mean that in such cases the series may or may not converge and further testing is required.

EXAMPLE 5.6 A Divergent Series for Which the Ratio Test Is Inconclusive

Use the Ratio Test for the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$.

Solution We have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\frac{1}{k+1}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{k}{k+1} = 1.$$

In this case, the Ratio Test yields no conclusion, although we already know that the harmonic series diverges. ■

Courtesy of Alain Connes

TODAY IN MATHEMATICS

Alain Connes (1947–Present) A French mathematician who earned a Fields Medal in 1983 for his spectacular results in the classification of operator algebras. As a student, Connes developed a very personal understanding of mathematics. He has explained, "I first began to work in a very small place in the mathematical geography . . . . I had my own system, which was very strange because when the problems the teacher was asking fell into my system, then of course I would have an instant answer, but when they didn't—and many problems, of course, didn't fall into my system—then I would be like an idiot and I wouldn't care."* As Connes' personal mathematical system expanded, he found more and more "instant answers" to important problems.

*Sabbagh, K. (2002) Dr Riemann's Zeros. London: Atlantic Books.

EXAMPLE 5.7 A Convergent Series for Which the Ratio Test Is Inconclusive

Use the Ratio Test to test the series $\sum_{k=0}^{\infty} \frac{1}{k^2}$.

Solution Here, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{1}{(k+1)^2}\frac{k^2}{1} = \lim_{k\to\infty}\frac{k^2}{k^2+2k+1} = 1.$$

So again, the Ratio Test yields no conclusion, although we already know that this is a convergent p-series ($p = 2 > 1$). ■

Carefully examine examples 5.6 and 5.7 and you should recognize that the Ratio Test will be inconclusive for any p-series.

The Root Test

We now present one final test for convergence of series.

THEOREM 5.3 (Root Test)

Given $\sum_{k=1}^{\infty} a_k$, suppose that $\lim_{k\to\infty} \sqrt[k]{|a_k|} = L$. Then,

(i) if $L < 1$, the series converges absolutely,
(ii) if $L > 1$ (or $L = \infty$), the series diverges and
(iii) if $L = 1$, there is no conclusion.

Notice how similar the conclusion is to the conclusion of the Ratio Test. The proof is also similar to that of the Ratio Test and we leave this as an exercise.

EXAMPLE 5.8 Using the Root Test

Use the Root Test to determine the convergence or divergence of the series $\sum_{k=1}^{\infty}\left(\frac{2k+4}{5k-1}\right)^k$.

Solution In this case, we consider

$$\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\sqrt[k]{\left|\frac{2k+4}{5k-1}\right|^k} = \lim_{k\to\infty}\frac{2k+4}{5k-1} = \frac{2}{5} < 1.$$

By the Root Test, the series is absolutely convergent. ■

Summary of Convergence Tests

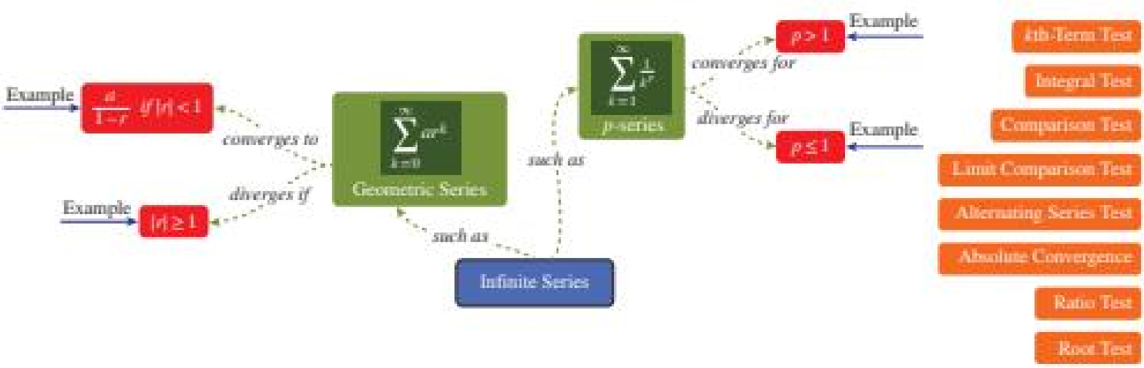
By this point in your study of series, it may seem as if we have thrown at you a dizzying array of different series and tests for convergence or divergence. Just how are you to keep all of these straight? The best suggestion we have is that you work through *many* problems. We provide a good assortment in the exercise set that follows this section. Some of these require the methods of this section, while others are drawn from the preceding sections (just to keep you thinking about the big picture). For the sake of convenience, we summarize our convergence tests in the table that follows.

Test	*When to Use*	*Conclusions*	*Section*
Geometric Series	$\sum_{k=0}^{\infty} ar^k$	Converges to $\frac{a}{1-r}$ if $\lvert r\rvert < 1$; diverges if $\lvert r\rvert \ge 1$.	5.2
***k*th-Term Test**	All series	If $\lim_{k\to\infty} a_k \ne 0$, the series diverges.	5.2

Test	*When to Use*	*Conclusions*	*Section*
Integral Test	$\sum_{k=1}^{\infty} a_k$ where $f(k) = a_k$, f is continuous and decreasing and $f(x) \geq 0$	$\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)dx$ *both* converge or *both* diverge.	5.3
***p*-series**	$\sum_{k=1}^{\infty} \frac{1}{k^p}$	Converges for $p > 1$; diverges for $p \leq 1$.	5.3
Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $0 \leq a_k \leq b_k$	If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.	5.3
Limit Comparison Test	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, where $a_k, b_k > 0$ and $\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$	$\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ *both* converge or *both* diverge.	5.3
Alternating Series Test	$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ where $a_k > 0$ for all k	If $\lim_{k\to\infty} a_k = 0$ and $a_{k+1} \leq a_k$ for all k, then the series converges.	5.4
Absolute Convergence	Series with some positive and some negative terms (including alternating series)	If $\sum_{k=1}^{\infty} \lvert a_k \rvert$ converges, then $\sum_{k=1}^{\infty} a_k$ converges absolutely.	5.5
Ratio Test	Any series (especially those involving exponentials and/or factorials)	For $\lim_{k\to\infty} \left\lvert \frac{a_{k+1}}{a_k} \right\rvert = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.	5.5
Root Test	Any series (especially those involving exponentials)	For $\lim_{k\to\infty} \sqrt[k]{\lvert a_k \rvert} = L$, if $L < 1$, $\sum_{k=1}^{\infty} a_k$ converges absolutely if $L > 1$, $\sum_{k=1}^{\infty} a_k$ diverges, if $L = 1$, no conclusion.	5.5

Concept Mapping

At this point of your study, we offer you another alternative to the summary in the above table. Concept maps are useful to lessen the effect of the roller coaster ride as you try to get your head around the different series and tests for convergence or divergence. It is not about us and our work—it is about you and your understanding of the different ideas presented here. We hope that you will try to expand, modify or change the concept map below as you see fit. To the right-hand side of the map, we left you some concepts that might be useful to integrate so that the bigger picture does not elude you.



EXERCISES 5.5

WRITING EXERCISES

1. Suppose that $a_k \geq 0$, $\lim_{k\to\infty} a_k = 0$, $\lim_{k\to\infty} b_k = 0$ and $\{b_k\}$ contains both positive and negative terms. Explain why $\sum_{k=1}^{\infty} b_k$ is more likely to converge than $\sum_{k=1}^{\infty} a_k$. In light of this, explain why Theorem 5.1 is true.

2. In the Ratio Test, if $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1$, which is (eventually) bigger, $|a_{k+1}|$ or $|a_k|$? Explain why this implies that the series $\sum_{k=1}^{\infty} a_k$ diverges.

3. In the Ratio Test, if $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L < 1$, which is bigger, $|a_{k+1}|$ or $|a_k|$? This inequality could also hold if $L = 1$. Compare the relative sizes of $|a_{k+1}|$ and $|a_k|$ if $L = 0.8$ versus $L = 1$. Explain why $L = 0.8$ would be more likely to correspond to a convergent series than $L = 1$.

4. In many series of interest, the terms of the series involve powers of k (e.g., k^2), exponentials (e.g., 2^k) or factorials (e.g., $k!$). For which type(s) of terms is the Ratio Test likely to produce a result (i.e., a limit different from 1)? Briefly explain.

In exercises 1–40, determine whether the series is absolutely convergent, conditionally convergent or divergent.

1. $\sum_{k=0}^{\infty} (-1)^k \frac{3}{k!}$
2. $\sum_{k=0}^{\infty} (-1)^k \frac{6}{k!}$
3. $\sum_{k=0}^{\infty} (-1)^k 2^k$
4. $\sum_{k=0}^{\infty} (-1)^k \frac{2}{3^k}$
5. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2+1}$
6. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2+1}{k}$
7. $\sum_{k=3}^{\infty} (-1)^k \frac{3^k}{k!}$
8. $\sum_{k=4}^{\infty} (-1)^k \frac{10^k}{k!}$
9. $\sum_{k=2}^{\infty} (-1)^{k+1} \frac{k}{2k+1}$
10. $\sum_{k=3}^{\infty} (-1)^{k+1} \frac{4}{2k+1}$
11. $\sum_{k=6}^{\infty} (-1)^k \frac{k\,2^k}{3^k}$
12. $\sum_{k=1}^{\infty} (-1)^k \frac{k^2 3^k}{2^k}$
13. $\sum_{k=1}^{\infty} \left(\frac{4k}{5k+1}\right)^k$
14. $\sum_{k=5}^{\infty} \left(\frac{1-3k}{4k}\right)^k$
15. $\sum_{k=1}^{\infty} \frac{-2}{k}$
16. $\sum_{k=1}^{\infty} \frac{4}{k}$
17. $\sum_{k=0}^{\infty} (-1)^{k+1} \frac{\sqrt{k}}{k+1}$
18. $\sum_{k=2}^{\infty} (-1)^{k+1} \frac{k}{k^3+1}$
19. $\sum_{k=7}^{\infty} \frac{k^2}{e^k}$
20. $\sum_{k=1}^{\infty} k^3 e^{-k}$
21. $\sum_{k=2}^{\infty} \frac{e^{3k}}{k^{3k}}$
22. $\sum_{k=4}^{\infty} \left(\frac{e^k}{k^2}\right)^k$
23. $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$
24. $\sum_{k=1}^{\infty} \frac{\cos k}{k^3}$
25. $\sum_{k=1}^{\infty} \frac{\cos k\pi}{k}$
26. $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k}$
27. $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$
28. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$
29. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k\sqrt{k}}$
30. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$
31. $\sum_{k=3}^{\infty} \frac{3}{k^k}$
32. $\sum_{k=8}^{\infty} \frac{2k}{3^k}$
33. $\sum_{k=6}^{\infty} (-1)^{k+1} \frac{k!}{4^k}$
34. $\sum_{k=4}^{\infty} (-1)^{k+1} \frac{k^2 4^k}{k!}$
35. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^{10}}{(2k)!}$
36. $\sum_{k=0}^{\infty} (-1)^k \frac{4^k}{(2k+1)!}$
37. $\sum_{k=0}^{\infty} \frac{(-2)^k(k+1)}{5^k}$
38. $\sum_{k=1}^{\infty} \frac{(-3)^k}{k^2 4^k}$
39. $\sum_{k=1}^{\infty} \frac{\cos(k\pi/5)}{k!}$
40. $\sum_{k=1}^{\infty} (1+1/k)^{k^2}$

In exercises 41–60, name the method by identifying a test that will determine whether the series converges or diverges.

41. $\sum_{k=1}^{\infty} \frac{e^k}{k!}$
42. $\sum_{k=2}^{\infty} \frac{e^k}{(2+1/k)^k}$
43. $\sum_{k=2}^{\infty} (-1)^k \frac{1+1/k}{k}$
44. $\sum_{k=0}^{\infty} \frac{k^2+5}{\sqrt{k^5+2}}$
45. $\sum_{k=3}^{\infty} k^{-4/5}$
46. $\sum_{k=1}^{\infty} (-4/5)^k$
47. $\sum_{k=0}^{\infty} \frac{k^2+1}{4^k}$
48. $\sum_{k=1}^{\infty} (-1)^k \frac{e^k}{k!}$
49. $\sum_{k=1}^{\infty} k^2 e^{-k^3}$
50. $\sum_{k=1}^{\infty} k^3 e^{-k^2}$
51. $\sum_{k=0}^{\infty} (-1)^k \left(\frac{4+k}{3+2k}\right)^k$
52. $\sum_{k=0}^{\infty} (-1)^k \frac{4+k}{3+2k}$
53. $\sum_{k=1}^{\infty} \frac{\ln k^2}{k^3}$
54. $\sum_{k=2}^{\infty} \frac{k^2}{\sqrt{k^3+1}}$
55. $\sum_{k=2}^{\infty} \frac{\cos k}{k^2}$
56. $\sum_{k=1}^{\infty} \frac{\cos k\pi}{k}$
57. $\sum_{k=2}^{\infty} \frac{2}{k\sqrt{\ln k + 1}}$
58. $\sum_{k=3}^{\infty} (-1)^k \ln(2+1/k)$
59. $\sum_{k=1}^{\infty} \frac{3^k}{(k!)^2}$
60. $\sum_{k=2}^{\infty} \frac{3^k}{k}$

61. (a) In the 1910s, the Indian mathematician Srinivasa Ramanujan discovered the formula

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26{,}390k)}{(k!)^4 396^{4k}}.$$

Approximate the series with only the $k = 0$ term and show that you get 6 digits of π correct. Approximate the series using the $k = 0$ and $k = 1$ terms and show that you get 14 digits of π correct. In general, each term of this remarkable series increases the accuracy by 8 digits.

(b) Prove that Ramanujan's series in part (a) converges.

62. To show that $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ converges, use the Ratio Test and the fact that

$$\lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e.$$

63. (a) Find all values of p such that $\sum_{k=1}^{\infty} \frac{p^k}{k}$ converges.

(b) Find all values of p such that $\sum_{k=1}^{\infty} \frac{p^k}{k^2}$ converges.

64. Determine whether $\sum_{k=1}^{\infty} \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$ converges or diverges.

EXPLORATORY EXERCISES

1. One reason that it is important to distinguish absolute from conditional convergence of a series is the rearrangement of series, to be explored in this exercise. Show that the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k}$ is absolutely convergent and find its sum S. Find the sum S_+ of the positive terms of the series. Find the sum S_- of the negative terms of the series. Verify that $S = S_+ + S_-$. However, you cannot separate the positive and negative terms for conditionally convergent series. For example, show that $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ converges (conditionally) but that the series of positive terms and the series of negative terms both diverge. Thus, the order of terms matters for conditionally convergent series. Amazingly, for conditionally convergent series, you can reorder the terms so that the partial sums converge to *any* real number. To illustrate this, suppose we want to reorder the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ so that the partial sums converge to $\frac{\pi}{2}$. Start by pulling out positive terms $\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right)$ such that the partial sum is within 0.1 of $\frac{\pi}{2}$. Next, take the first negative term $\left(-\frac{1}{2}\right)$ and positive terms such that the partial sum is within 0.05 of $\frac{\pi}{2}$. Then take the next negative term $\left(-\frac{1}{4}\right)$ and positive terms such that the partial sum is within 0.01 of $\frac{\pi}{2}$. Argue that you could continue in this fashion to reorder the terms so that the partial sums converge to $\frac{\pi}{2}$. (Especially explain why you will never "run out of" positive terms.) Then explain why you cannot do the same with the absolutely convergent series $\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k}$.

2. In this exercise, you show that the Root Test is more general than the Ratio Test. To be precise, show that if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r \neq 1$ then $\lim_{n\to\infty} |a_n|^{1/n} = r$ by considering $\lim_{n\to\infty} \ln\left|\frac{a_{n+1}}{a_n}\right|$ and $\lim_{n\to\infty} \ln|a_n|^{1/n} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \ln\left|\frac{a_{k+1}}{a_k}\right|$. Interpret this result in terms of how likely the Ratio Test or Root Test is to give a definite conclusion. Show that the result is not "if and only if" by finding a sequence for which $\lim_{n\to\infty} |a_n|^{1/n} < 1$ but $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ does not exist. In spite of this, give one reason why the Ratio Test might be preferable to the Root Test.

5.6 POWER SERIES

We now expand our discussion of series to the case where the terms of the series are functions of the variable x. Pay close attention, as the primary reason for studying series is that we can use them to represent functions. This opens up numerous possibilities for us, from approximating the values of transcendental functions to calculating derivatives and integrals of such functions, to studying differential equations. As well, *defining* functions as convergent series produces an explosion of new functions available to us, including many important functions, such as the Bessel functions. We take the first few steps in this section.

As a start, consider the series

$$\sum_{k=0}^{\infty} (x-2)^k = 1 + (x-2) + (x-2)^2 + (x-2)^3 + \cdots.$$

Notice that for each fixed x, this is a geometric series with $r = (x-2)$, which will converge whenever $|r| = |x-2| < 1$ and diverge whenever $|r| = |x-2| \geq 1$. Further, for each x with $|x-2| < 1$ (i.e., $1 < x < 3$), the series converges to

$$\frac{a}{1-r} = \frac{1}{1-(x-2)} = \frac{1}{3-x}.$$

That is, for each x in the interval $(1, 3)$, we have

$$\sum_{k=0}^{\infty} (x-2)^k = \frac{1}{3-x}.$$

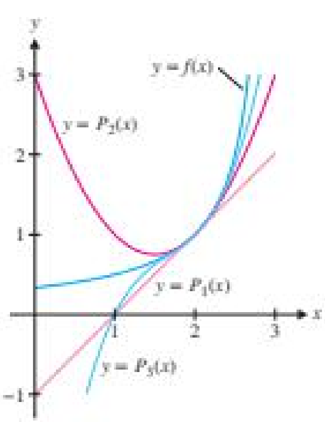


FIGURE 5.38
$y = \frac{1}{3-x}$ and the first three partial sums of $\sum_{k=0}^{\infty} (x-2)^k$

For all other values of x, the series diverges. In Figure 5.38, we show a graph of $f(x) = \frac{1}{3-x}$, along with the first three partial sums P_n of this series, where

$$P_n(x) = \sum_{k=0}^{n} (x-2)^k = 1 + (x-2) + (x-2)^2 + \cdots + (x-2)^n,$$

on the interval $[0, 3]$. Notice that as n gets larger, $P_n(x)$ appears to get closer to $f(x)$, for any given x in the interval $(1, 3)$. Further, as n gets larger, $P_n(x)$ tends to stay close to $f(x)$ for a larger range of x-values.

Here, we noticed that a series is equivalent to (i.e., it converges to) a *known* function on a certain interval. Alternatively, imagine what benefits you would derive if for a given function (one that you don't know much about) you could find an equivalent series representation. This is precisely what we do in section 5.7. For instance, we will show that for all x,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots. \tag{6.1}$$

As one immediate use of (6.1), suppose that you wanted to calculate $e^{1.234567}$. Using (6.1), for any given x, we can compute an approximation to e^x, simply by summing the first few terms of the equivalent power series. This is easy to do, since the partial sums of the series are simply polynomials.

In general, any series of the form

Power Series

$$\sum_{k=0}^{\infty} b_k(x-c)^k = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots$$

is called a **power series** in powers of $(x - c)$. We refer to the constants b_k, $k = 0, 1, 2, \ldots$, as the **coefficients** of the series. The first question is: for what values of x does the series converge? Saying this another way, the power series $\sum_{k=0}^{\infty} b_k(x-c)^k$ defines a function of x and its domain is the set of all x for which the series converges. The primary tool for investigating the convergence or divergence of a power series is the Ratio Test.

EXAMPLE 6.1 Determining Where a Power Series Converges

Determine the values of x for which the power series $\sum_{k=0}^{\infty} \frac{k}{3^{k+1}} x^k$ converges.

Solution Using the Ratio Test, we have

$$\begin{aligned}
\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty} \left|\frac{(k+1)x^{k+1}}{3^{k+2}} \frac{3^{k+1}}{kx^k}\right| \\
&= \lim_{k\to\infty} \frac{(k+1)|x|}{3k} = \frac{|x|}{3} \lim_{k\to\infty} \frac{k+1}{k} \\
&= \frac{|x|}{3} < 1,
\end{aligned}$$

Since $x^{k+1} = x^k \cdot x^1$ and $3^{k+2} = 3^{k+1} \cdot 3^1$.

for $|x| < 3$ or $-3 < x < 3$. So, the series converges absolutely for $-3 < x < 3$ and diverges for $|x| > 3$ (i.e., for $x > 3$ or $x < -3$). Since the Ratio Test gives no conclusion for the endpoints $x = \pm 3$, we must test these separately.

For $x = 3$, we have the series

$$\sum_{k=0}^{\infty} \frac{k}{3^{k+1}} x^k = \sum_{k=0}^{\infty} \frac{k}{3^{k+1}} 3^k = \sum_{k=0}^{\infty} \frac{k}{3}.$$

Since

$$\lim_{k\to\infty} \frac{k}{3} = \infty \neq 0,$$

the series diverges by the kth-term test for divergence. The series diverges when $x = -3$, for the same reason. Thus, the power series converges for all x in the interval $(-3, 3)$ and diverges for all x outside this interval. ■

Observe that in example 6.1, as well as in our introductory example, the series have the form $\sum_{k=0}^{\infty} b_k(x-c)^k$ and there is an interval of the form $(c - r, c + r)$ on which the series converges and outside of which the series diverges. (In the case of example 6.1, notice that $c = 0$.) This interval on which a power series converges is called the **interval of**

convergence. The constant r is called the **radius of convergence** (i.e., r is half the length of the interval of convergence). As we see in the following result, there is such an interval for every power series.

NOTES

In part (iii) of Theorem 6.1, the series may converge at neither, one or both of the endpoints $x = c - r$ and $x = c + r$. Because the interval of convergence is centered at $x = c$, we refer to c as the **center** of the power series.

THEOREM 6.1

Given any power series, $\sum_{k=0}^{\infty} b_k(x - c)^k$, there are exactly three possibilities:

(i) The series converges absolutely for *all* $x \in (-\infty, \infty)$ and the radius of convergence is $r = \infty$;
(ii) The series converges *only* for $x = c$ (and diverges for all other values of x) and the radius of convergence is $r = 0$; or
(iii) The series converges absolutely for $x \in (c - r, c + r)$ and diverges for $x < c - r$ and for $x > c + r$, for some number r with $0 < r < \infty$.

The proof of the theorem can be found in the appendix.

EXAMPLE 6.2 Finding the Interval and Radius of Convergence

Determine the interval and radius of convergence for the power series

$$\sum_{k=0}^{\infty} \frac{10^k}{k!}(x - 1)^k.$$

Solution From the Ratio Test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{10^{k+1}(x-1)^{k+1}}{(k+1)!}\frac{k!}{10^k(x-1)^k}\right|$$

$$= 10|x - 1|\lim_{k\to\infty}\frac{k!}{(k+1)k!}$$

Since $(x-1)^{k+1} = (x-1)^k(x-1)^1$ and $(k+1)! = (k+1)k!$.

$$= 10|x - 1|\lim_{k\to\infty}\frac{1}{k+1} = 0 < 1,$$

for *all* x. This says that the series converges absolutely for all x. Thus, the interval of convergence for this series is $(-\infty, \infty)$ and the radius of convergence is $r = \infty$. ■

The interval of convergence for a power series can be a closed interval, an open interval or a half-open interval, as in example 6.3.

EXAMPLE 6.3 A Half-Open Interval of Convergence

Determine the interval and radius of convergence for the power series $\sum_{k=1}^{\infty}\frac{x^k}{k4^k}$.

Solution From the Ratio Test, we have

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+1)4^{k+1}}\frac{k4^k}{x^k}\right|$$

$$= \frac{|x|}{4}\lim_{k\to\infty}\frac{k}{k+1} = \frac{|x|}{4} < 1.$$

So, we are guaranteed absolute convergence for $|x| < 4$ and divergence for $|x| > 4$. It remains only to test the endpoints of the interval: $x = \pm 4$. For $x = 4$, we have

$$\sum_{k=1}^{\infty}\frac{x^k}{k4^k} = \sum_{k=1}^{\infty}\frac{4^k}{k4^k} = \sum_{k=1}^{\infty}\frac{1}{k},$$

NOTES

In example 6.3, the series for the endpoints of the interval of convergence are closely related. One is a positive term series of the form $\sum_{k=1}^{\infty} a_k$, while the other is the corresponding alternating series $\sum_{k=1}^{\infty} (-1)^k a_k$. This is a common occurrence.

which you will recognize as the harmonic series, which diverges. For $x = -4$, we have

$$\sum_{k=1}^{\infty} \frac{x^k}{k4^k} = \sum_{k=1}^{\infty} \frac{(-4)^k}{k4^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k},$$

which is the alternating harmonic series, which we know converges conditionally. (See example 5.2.) So, in this case, the interval of convergence is the half-open interval $[-4, 4)$ and the radius of convergence is $r = 4$. ■

Notice that (as stated in Theorem 6.1) every power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges at least for $x = c$ since for $x = c$, we have the trivial case

$$\sum_{k=0}^{\infty} a_k(x-c)^k = \sum_{k=0}^{\infty} a_k(c-c)^k = a_0 + \sum_{k=1}^{\infty} a_k 0^k = a_0 + 0 = a_0.$$

EXAMPLE 6.4 A Power Series That Converges at Only One Point

Determine the radius of convergence for the power series $\sum_{k=0}^{\infty} k!(x-5)^k$.

Solution From the Ratio Test, we have

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{(k+1)!(x-5)^{k+1}}{k!(x-5)^k}\right| \\ &= \lim_{k\to\infty}\frac{(k+1)k!|x-5|}{k!} \\ &= \lim_{k\to\infty}[(k+1)|x-5|] \\ &= \begin{cases}0, & \text{if } x = 5 \\ \infty, & \text{if } x \neq 5\end{cases}.\end{aligned}$$

Thus, this power series converges only for $x = 5$ and so its radius of convergence is $r = 0$. ■

Suppose that the power series $\sum_{k=0}^{\infty} b_k(x-c)^k$ has radius of convergence $r > 0$. Then the series converges absolutely for all x in the interval $(c-r, c+r)$ and so defines a function f on the interval $(c-r, c+r)$,

$$f(x) = \sum_{k=0}^{\infty} b_k(x-c)^k = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots.$$

It turns out that such a function is continuous and differentiable, although the proof is beyond the level of this course. In fact, we differentiate exactly the way you might expect,

Differentiating a power series

$$\begin{aligned}f'(x) &= \frac{d}{dx}[b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + \cdots] \\ &= b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + \cdots = \sum_{k=1}^{\infty} b_k k(x-c)^{k-1},\end{aligned}$$

where the radius of convergence of the resulting series is also r. Since we find the derivative by differentiating each term in the series, we call this **term-by-term** differentiation. Likewise, we can integrate a convergent power series term-by-term, Integrating a power series

$$\int f(x)dx = \int \sum_{k=0}^{\infty} b_k(x-c)^k\,dx = \sum_{k=0}^{\infty} b_k \int (x-c)^k\,dx$$
$$= \sum_{k=0}^{\infty} b_k \frac{(x-c)^{k+1}}{k+1} + K,$$

where the radius of convergence of the resulting series is again r and where K is a constant of integration. The proof of these two results can be found in a text on advanced calculus. It's important to recognize that these two results are *not* obvious. They are not simply an application of the rule that a derivative or integral of a sum is the sum of the derivatives or integrals, respectively, since a series is not a sum, but rather a limit of a sum. Further, these results are true for power series, but are *not* true for series in general.

EXAMPLE 6.5 A Convergent Series Whose Series of Derivatives Diverges

Find the interval of convergence of the series $\sum_{k=1}^{\infty} \frac{\sin(k^3 x)}{k^2}$ and show that the series of derivatives does not converge for any x.

Solution Notice that

$$\left|\frac{\sin(k^3 x)}{k^2}\right| \le \frac{1}{k^2}, \text{ for all } x,$$

since $|\sin(k^3x)| \le 1$. Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series ($p = 2 > 1$), it follows from the Comparison Test that $\sum_{k=0}^{\infty} \frac{\sin(k^3 x)}{k^2}$ converges absolutely, for all x. On the other hand, the series of derivatives (found by differentiating the series term-by-term) is

$$\sum_{k=1}^{\infty} \frac{d}{dx}\left[\frac{\sin(k^3 x)}{k^2}\right] = \sum_{k=1}^{\infty} \frac{k^3\cos(k^3 x)}{k^2} = \sum_{k=1}^{\infty} [k\cos(k^3 x)],$$

which *diverges* for all x, by the kth-term test for divergence, since the terms do not tend to zero as $k \to \infty$, for any x. ■

Keep in mind that $\sum_{k=1}^{\infty} \frac{\sin(k^3x)}{k^2}$ is not a power series. (Why not?) The result of example 6.5, a convergent series whose series of derivatives diverges, *cannot* occur with any power series with radius of convergence $r > 0$.

In example 6.6, we find that once we have a convergent power series representation for a given function, we can use this to obtain power series representations for any number of other functions by substitution or by differentiating and integrating the series term-by-term.

EXAMPLE 6.6 Differentiating and Integrating a Power Series

Use the power series $\sum_{k=0}^{\infty} (-1)^k x^k$ to find power series representations of $\frac{1}{(1+x)^2}$, $\frac{1}{1+x^2}$ and $\tan^{-1}x$.

Solution Notice that $\sum_{k=0}^{\infty}(-1)^k x^k = \sum_{k=0}^{\infty}(-x)^k$ is a geometric series with ratio $r = -x$. This series converges, then, whenever $|r| = |-x| = |x| < 1$, to

$$\frac{a}{1-r} = \frac{1}{1-(-x)} = \frac{1}{1+x}.$$

That is, for $-1 < x < 1$,

$$\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k. \tag{6.2}$$

Differentiating both sides of (6.2), we get

$$\frac{-1}{(1+x)^2} = \sum_{k=0}^{\infty}(-1)^k k x^{k-1}, \quad \text{for } -1 < x < 1.$$

Multiplying both sides by -1 gives us a new power series representation:

$$\frac{1}{(1+x)^2} = \sum_{k=0}^{\infty}(-1)^{k+1} k x^{k-1},$$

valid for $-1 < x < 1$. Notice that we can also obtain a new power series from (6.2) by substitution. For instance, if we replace x with x^2, we get

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k (x^2)^k = \sum_{k=0}^{\infty}(-1)^k x^{2k}, \tag{6.3}$$

valid for $-1 < x^2 < 1$ (which is equivalent to having $x^2 < 1$ or $-1 < x < 1$).

Integrating both sides of (6.3) gives us

$$\int \frac{1}{1+x^2}\,dx = \sum_{k=0}^{\infty}(-1)^k \int x^{2k}\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1} + c. \tag{6.4}$$

You should recognize the integral on the left-hand side of (6.4) as $\tan^{-1}x$. That is,

$$\tan^{-1}x = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1} + c, \quad \text{for } -1 < x < 1. \tag{6.5}$$

Taking $x = 0$ gives us

$$\tan^{-1}0 = \sum_{k=0}^{\infty}\frac{(-1)^k 0^{2k+1}}{2k+1} + c = c,$$

so that $c = \tan^{-1}0 = 0$. Equation (6.5) now gives us a power series representation for $\tan^{-1}x$, namely:

$$\tan^{-1}x = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots, \quad \text{for } -1 < x < 1.$$

In this case, the series also converges at the endpoint $x = 1$. ■

Notice that working as in example 6.6, we can produce power series representations of any number of functions. In section 5.7, we present a systematic method for producing power series representations for a wide range of functions.

BEYOND FORMULAS

You should think of a power series as a different form for writing functions. Just as $\frac{x}{e^x}$ can be rewritten as xe^{-x}, many functions can be rewritten as power series. In general, having an alternative for writing a function gives you one more option to consider when trying to solve a problem. Further, power series representations are often easier to work with than other representations and have the advantage of having derivatives and integrals that are easy to compute.

EXERCISES 5.6

WRITING EXERCISES

1. Power series have the form $\sum_{k=0}^{\infty} a_k(x-c)^k$. Explain why the farther x is from c, the larger the terms of the series are and the less likely the series is to converge. Describe how this general trend relates to the radius of convergence.

2. Applying the Ratio Test to $\sum_{k=0}^{\infty} a_k(x-c)^k$ requires you to evaluate $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}(x-c)\right|$. As $x \neq c$ increases or decreases, $|x-c|$ increases. If the series has a finite radius of convergence $r > 0$, what is the value of the limit when $|x-c| = r$? Explain how the limit changes when $|x-c| < r$ and $|x-c| > r$ and how this determines the convergence or divergence of the series.

3. As shown in example 6.2, $\sum_{k=0}^{\infty} \frac{10^k}{k!}(x-1)^k$ converges for all x. If $x = 1001$, the value of $(x-1)^k = 1000^k$ gets very large very fast, as k increases. Explain why, for the series to converge, the value of $k!$ must get large faster than 1000^k. To illustrate how fast the factorial grows, compute 50!, 100! and 200! (if your calculator can).

4. In a power series representation of $\sqrt{x+1}$ about $c = 0$, explain why the radius of convergence cannot be greater than 1. (Think about the domain of $\sqrt{x+1}$.)

In exercises 1–16, determine the radius and interval of convergence.

1. $\sum_{k=0}^{\infty} \frac{2^k}{k!}(x-2)^k$
2. $\sum_{k=0}^{\infty} \frac{3^k}{k!}x^k$
3. $\sum_{k=0}^{\infty} \frac{k}{4^k}x^k$
4. $\sum_{k=0}^{\infty} \frac{k}{2^k}x^k$
5. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k3^k}(x-1)^k$
6. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k4^k}(x+2)^k$
7. $\sum_{k=0}^{\infty} k!(x+1)^k$
8. $\sum_{k=1}^{\infty} \frac{1}{k}(x-1)^{2k+1}$
9. $\sum_{k=2}^{\infty} (k+3)^2(2x-3)^k$
10. $\sum_{k=4}^{\infty} \frac{1}{k^2}(3x+2)^k$
11. $\sum_{k=1}^{\infty} \frac{4^k}{\sqrt{k}}(2x+1)^k$
12. $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}(3x-1)^k$
13. $\sum_{k=1}^{\infty} \frac{k^2}{2^k}(x+2)^k$
14. $\sum_{k=0}^{\infty} \frac{k^2}{k!}(x+1)^k$
15. $\sum_{k=3}^{\infty} \frac{k!}{(2k)!}x^k$
16. $\sum_{k=2}^{\infty} \frac{(k!)^2}{(2k)!}x^{2k+1}$

In exercises 17 and 18, graph partial sums $P_6(x)$, $P_9(x)$ and $P_{12}(x)$. Discuss the behavior of the partial sums both inside and outside the radius of convergence from each of the following exercises.

17. Exercise 3
18. Exercise 5

In exercises 19–24, determine the interval of convergence and the function to which the given power series converges.

19. $\sum_{k=0}^{\infty} (x+2)^k$
20. $\sum_{k=0}^{\infty} (x-3)^k$
21. $\sum_{k=0}^{\infty} (2x-1)^k$
22. $\sum_{k=0}^{\infty} (3x+1)^k$
23. $\sum_{k=0}^{\infty} (-1)^k\left(\frac{x}{2}\right)^k$
24. $\sum_{k=0}^{\infty} 3\left(\frac{x}{4}\right)^k$

In exercises 25–32, find a power series representation of $f(x)$ about $c = 0$ (refer to example 6.6). Also, determine the radius and interval of convergence, and graph $f(x)$ together with the partial sums $\sum_{k=0}^{3} a_k x^k$ and $\sum_{k=0}^{6} a_k x^k$.

25. $f(x) = \frac{2}{1-x}$
26. $f(x) = \frac{3}{x-1}$
27. $f(x) = \frac{3}{1+x^2}$
28. $f(x) = \frac{2}{1-x^2}$
29. $f(x) = \frac{2x}{1-x^3}$
30. $f(x) = \frac{3x}{1+x^2}$
31. $f(x) = \frac{2}{4+x}$
32. $f(x) = \frac{3}{6-x}$

In exercises 33–38, find a power series representation and radius of convergence by integrating or differentiating one of the series from exercises 25–32.

33. $f(x) = 3\tan^{-1} x$
34. $f(x) = 2\ln(1-x)$
35. $f(x) = \frac{2x}{(1-x^2)^2}$
36. $f(x) = \frac{3}{(x-1)^2}$
37. $f(x) = \ln(1+x^2)$
38. $f(x) = \ln(4+x)$

In exercises 39–42, find the interval of convergence of the (non-power) series and the corresponding series of derivatives.

39. $\sum_{k=1}^{\infty} \frac{\cos(k^3 x)}{k^2}$
40. $\sum_{k=1}^{\infty} \frac{\cos(x/k)}{k}$
41. $\sum_{k=0}^{\infty} e^{kx}$
42. $\sum_{k=0}^{\infty} e^{-2kx}$

43. For any constants a and $b > 0$, determine the interval and radius of convergence of $\sum_{k=0}^{\infty} \frac{(x-a)^k}{b^k}$.

44. Prove that if $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence r, with $0 < r < \infty$, then $\sum_{k=0}^{\infty} a_k x^{2k}$ has radius of convergence $\sqrt{r}$.

45. If $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence r, with $0 < r < \infty$, determine the radius of convergence of $\sum_{k=0}^{\infty} a_k (x-c)^k$ for any constant c.

46. If $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence r, with $0 < r < \infty$, determine the radius of convergence of $\sum_{k=0}^{\infty} a_k \left(\frac{x}{b}\right)^k$ for any constant $b \neq 0$.

47. Show that $f(x) = \frac{x+1}{(1-x)^2} = \frac{\frac{3x}{1-x}+1}{1-x}$ has the power series representation $f(x) = 1 + 3x + 5x^2 + 7x^3 + 9x^4 + \cdots$. Find the radius of convergence. Set $x = \frac{1}{1000}$ and discuss the interesting decimal representation of $\frac{1{,}001{,}000}{998{,}001}$.

48. Show that the long division algorithm produces $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$. Explain why this equation is not valid for all x.

49. Define $f(x) = \int_0^x \frac{2t}{1-t^3}\,dt$. Find a power series for f and determine its radius of convergence. Graph f.

50. Define $f(x) = \int_0^x \frac{2}{1+t^4}\,dt$. Find a power series for f and determine its radius of convergence. Graph f.

51. Evaluate $\int_0^1 \frac{1+x^2}{1+x^4}\,dx$ by (a) integrating a power series and (b) rewriting the integrand as $\frac{1}{1+(1-\sqrt{2}x)^2} + \frac{1}{1+(1+\sqrt{2}x)^2}$.

52. Even great mathematicians can make mistakes. Leonhard Euler started with the equation $\frac{x}{x-1} + \frac{x}{1-x} = 0$, rewrote it as $\frac{1}{1-1/x} + \frac{x}{1-x} = 0$, found power series representations for each function and concluded that $\cdots + \frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2 + \cdots = 0$. Substitute $x = 1$ to show that the conclusion is false, then find the mistake in Euler's derivation.

53. For $0 < p < 1$, evaluate $\sum_{k=2}^{\infty} k(k-1)p^{k-2}$ and $\sum_{k=3}^{\infty} k(k-1)(k-2)p^{k-3}$. Generalize to $\sum_{k=3}^{\infty} \binom{k}{n} p^{k-n}$ for positive integers n.

54. For each series $f(x)$, compare the intervals of convergence of $f(x)$ and $\int f(x)\,dx$, where the antiderivative is taken term-by-term. (a) $f(x) = \sum_{k=0}^{\infty} (-1)^k x^k$; (b) $f(x) = \sum_{k=0}^{\infty} \sqrt{k} x^k$; (c) $f(x) = \sum_{k=0}^{\infty} \frac{1}{k} x^k$. Based on the examples in this exercise, does integration make it more or less likely that the series will converge at the endpoints?

APPLICATIONS

1. A discrete random variable that assumes value k with probability p_k for $k = 1, 2, \cdots$, has expected value $\sum_{k=1}^{\infty} k p_k$. A **generating function** for the random variable is $F(x) = \sum_{k=1}^{\infty} p_k x^k$. Show that $F'(1)$ equals the expected value.

2. An **electric dipole** consists of a charge q at $x = 1$ and a charge $-q$ at $x = -1$. The electric field at any $x > 1$ is given by $E(x) = \frac{kq}{(x-1)^2} - \frac{kq}{(x+1)^2}$, for some constant k. Find a power series representation for $E(x)$.

EXPLORATORY EXERCISES

1. Note that the radius of convergence in each of exercises 25–29 is 1. Given that the functions in exercises 25, 26, 28 and 29 are undefined at $x = 1$, explain why the radius of convergence can't be larger than 1. The restricted radius in exercise 27 can be understood using complex numbers. Show that $1 + x^2 = 0$ for $x = \pm i$, where $i = \sqrt{-1}$. In general, a complex number $a + b\,i$ is associated with the point (a, b). Find the "distance" between the complex numbers 0 and i by finding the distance between the associated points $(0, 0)$ and $(0, 1)$. Discuss how this compares to the radius of convergence. Then use the ideas in this exercise to quickly conjecture the radius of convergence of power series with center $c = 0$ for the functions $f(x) = \frac{4}{1+4x}$, $f(x) = \frac{2}{4+x}$ and $f(x) = \frac{2}{4+x^2}$.

2. Let $\{f_k(x)\}$ be a sequence of functions defined on a set E. The **Weierstrass M-test** states that if there exist constants M_k such that $|f_k(x)| \le M_k$ for each x and $\sum_{k=1}^{\infty} M_k$ converges, then $\sum_{k=1}^{\infty} f_k(x)$ converges (uniformly) for each x in E. Prove that $\sum_{k=1}^{\infty} \frac{1}{k^2+x^2}$ and $\sum_{k=1}^{\infty} x^2 e^{-kx}$ converge (uniformly) for all x. "Uniformly" in this exercise refers to the rate at which the infinite series converges to its sum. A precise definition can be found in an advanced calculus book. We explore the main idea of the definition in this exercise. Explain why you would expect the convergence of the series $\sum_{k=1}^{\infty} \frac{1}{k^2+x^2}$ to be slowest at $x = 0$. Now, numerically explore the following question. Defining $f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2+x^2}$ and $S_n(x) = \sum_{k=1}^{n} \frac{1}{k^2+x^2}$, is there an integer N such that if $n > N$ then $|f(x) - S_n(x)| < 0.01$ for all x?

5.7 TAYLOR SERIES

Representation of Functions as Power Series

In this section, we develop a compelling reason for considering series. They are not merely a mathematical curiosity but, rather, are an essential means for exploring and computing with transcendental functions (e.g., $\sin x$, $\cos x$, $\ln x$, e^x, etc.).

Suppose that the power series $\sum_{k=0}^{\infty} b_k(x-c)^k$ has radius of convergence $r > 0$. As we've observed, this means that the series converges absolutely to some function f on the interval $(c-r, c+r)$. We have

$$f(x) = \sum_{k=0}^{\infty} b_k(x-c)^k = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + b_4(x-c)^4 + \cdots,$$

for each $x \in (c-r, c+r)$. Differentiating term-by-term, we get that

$$f'(x) = \sum_{k=0}^{\infty} b_k k(x-c)^{k-1} = b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + 4b_4(x-c)^3 + \cdots,$$

again, for each $x \in (c-r, c+r)$. Likewise, we get

$$f''(x) = \sum_{k=0}^{\infty} b_k k(k-1)(x-c)^{k-2} = 2b_2 + 3\cdot 2b_3(x-c) + 4\cdot 3b_4(x-c)^2 + \cdots$$

and $$f'''(x) = \sum_{k=0}^{\infty} b_k k(k-1)(k-2)(x-c)^{k-3} = 3\cdot 2b_3 + 4\cdot 3\cdot 2b_4(x-c) + \cdots$$

and so on (all valid for $c - r < x < c + r$). Notice that if we substitute $x = c$ in each of the above derivatives, all the terms of the series drop out, except one. We get

$$f(c) = b_0,$$
$$f'(c) = b_1,$$
$$f''(c) = 2b_2,$$
$$f'''(c) = 3!b_3$$

and so on. Observe that in general, we have

$$f^{(k)}(c) = k!\, b_k. \tag{7.1}$$

Solving (7.1) for b_k, we have

$$b_k = \frac{f^{(k)}(c)}{k!}, \text{ for } k = 0, 1, 2, \cdots.$$

To summarize, we found that if $\sum_{k=0}^{\infty} b_k(x-c)^k$ is a convergent power series with radius of convergence $r > 0$, then the series converges to some function f that we can write as

$$f(x) = (x-c)^k = (x-c)^k, \text{ for } x \in (c-r, c+r)\,.\ \sum_{k=0}^{\infty} b_k \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}$$

Now, think about this problem from another angle. Instead of starting with a series, suppose that you start with an infinitely differentiable function, f (i.e. f can be differentiated infinitely often). Then, we can construct the series

Taylor series expansion of $f(x)$ about $x = c$

$$\boxed{\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k,}$$

called a **Taylor series** expansion for f. (See the historical note on Brook Taylor in section 6.2.) There are two important questions we need to answer.

- Does a series constructed in this way converge and, if so, what is its radius of convergence?
- If the series converges, it converges to a function. Does it converge to f?

We can answer the first of these questions on a case-by-case basis, usually by applying the Ratio Test. The second question will require further insight.

EXAMPLE 7.1 Constructing a Taylor Series Expansion

Construct the Taylor series expansion for $f(x) = e^x$, about $x = 0$ (i.e., take $c = 0$).

Solution Here, we have the extremely simple case where

$$f'(x) = e^x,\ f''(x) = e^x \text{ and so on, } f^{(k)}(x) = e^x, \text{ for } k = 0, 1, 2, \cdots.$$

This gives us the Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{\infty} \frac{e^0}{k!}x^k = \sum_{k=0}^{\infty} \frac{1}{k!}x^k.$$

From the Ratio Test, we have

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{|x|^{k+1}}{(k+1)!}\frac{k!}{|x|^k} = |x| \lim_{k\to\infty} \frac{k!}{(k+1)k!}$$

$$= |x| \lim_{k\to\infty} \frac{1}{k+1} = |x|(0) = 0 < 1, \qquad \text{for all } x.$$

So, the Taylor series $\sum_{k=0}^{\infty} \frac{1}{k!}x^k$ converges absolutely for all real numbers x. At this point, though, we do not know the function to which the series converges. (Could it be e^x?) ■

REMARK 7.1

The special case of a Taylor series expansion about $x = 0$ is often called a **Maclaurin series**. (See the historical note about Colin Maclaurin in section 5.3.) That is, the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$ is the Maclaurin series expansion for f.

Before we present any further examples of Taylor series, let's see if we can determine the function to which a given Taylor series converges. First, notice that the partial sums of a Taylor series (like those for any power series) are simply polynomials. We define

Taylor polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Observe that $P_n(x)$ is a polynomial of degree n, as $\frac{f^{(k)}(c)}{k!}$ is a constant for each k. We refer to P_n as the **Taylor polynomial of degree *n*** for f expanded about $x = c$.

EXAMPLE 7.2 Constructing and Graphing Taylor Polynomials

For $f(x) = e^x$, find the Taylor polynomial of degree n expanded about $x = 0$.

Solution As in example 7.1, we have that $f^{(k)}(x) = e^x$, for all k. So, we have that the nth-degree Taylor polynomial is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{n} \frac{e^0}{k!}x^k$$

$$= \sum_{k=0}^{n} \frac{1}{k!}x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

Since we established in example 7.1 that the Taylor series for $f(x) = e^x$ about $x = 0$ converges for all x, this says that the sequence of partial sums (i.e., the sequence of Taylor polynomials) converges for all x. In an effort to determine the function to which the Taylor polynomials are converging, we have plotted $P_1(x)$, $P_2(x)$, $P_3(x)$ and $P_4(x)$, together with the graph of $f(x) = e^x$ in Figures 5.39a–d, respectively.

Notice that as n gets larger, the graphs of $P_n(x)$ appear (at least on the interval displayed) to be approaching the graph of $f(x) = e^x$. Since we know that the Taylor series converges and the graphical evidence suggests that the partial sums of the series are approaching $f(x) = e^x$, it is reasonable to conjecture that the series converges to e^x.

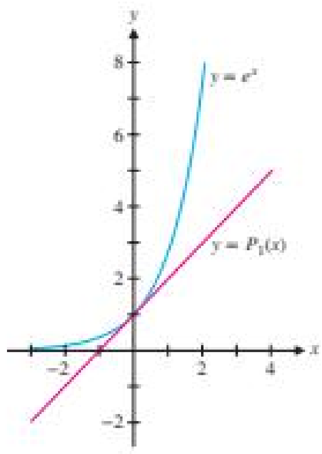


FIGURE 5.39a
$y = e^x$ and $y = P_1(x)$

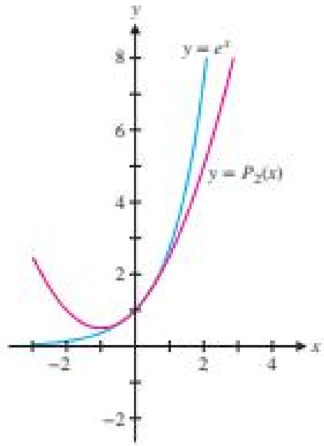


FIGURE 5.39b
$y = e^x$ and $y = P_2(x)$

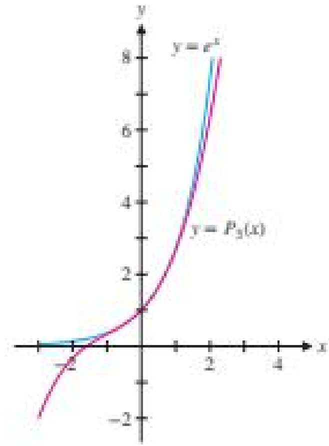


FIGURE 5.39c
$y = e^x$ and $y = P_3(x)$

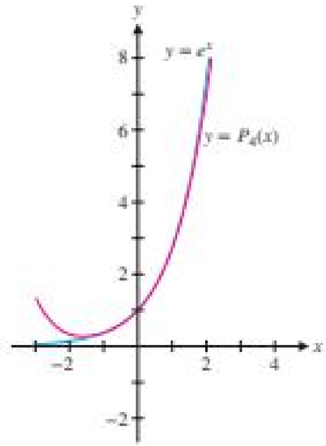


FIGURE 5.39d
$y = e^x$ and $y = P_4(x)$

This is, in fact, exactly what is happening, as we can prove using Theorems 7.1 and 7.2. ■

THEOREM 7.1 (Taylor's Theorem)

Suppose that f has $(n + 1)$ derivatives on the interval $(c - r, c + r)$, for some $r > 0$. Then, for $x \in (c - r, c + r)$, $f(x) \approx P_n(x)$ and the error in using $P_n(x)$ to approximate $f(x)$ is

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}, \tag{7.2}$$

for some number z between x and c.

REMARK 7.2

Observe that for $n = 0$, Taylor's Theorem simplifies to a very familiar result. We have

$$R_0(x) = f(x) - P_0(x) = \frac{f'(z)}{(0 + 1)!}(x - c)^{0+1}.$$

Since $P_0(x) = f(c)$, we have simply

$$f(x) - f(c) = f'(z)(x - c).$$

Dividing by $(x - c)$, gives us

$$\frac{f(x) - f(c)}{x - c} = f'(z),$$

which is the conclusion of the Mean Value Theorem. In this way, observe that Taylor's Theorem is a generalization of the Mean Value Theorem.

The error term $R_n(x)$ in (7.2) is often called the **remainder term**. Note that this term looks very much like the first neglected term of the Taylor series, except that $f^{(n+1)}$ is evaluated at some (unknown) number z between x and c, instead of at c. This remainder term serves two purposes: it enables us to obtain an estimate of the error in using a Taylor polynomial to approximate a given function and, as we'll see in Theorem 7.2, it gives us the means to prove that a Taylor series for a given function f converges to f.

The proof of Taylor's Theorem is somewhat technical and so we leave it for the end of the section.

Note: If we could show that

$$\lim_{n\to\infty} R_n(x) = 0, \quad \text{for } all\ x \text{ in } (c - r, c + r),$$

then we would have that

$$0 = \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} [f(x) - P_n(x)] = f(x) - \lim_{n\to\infty} P_n(x)$$

or

$$\lim_{n\to\infty} P_n(x) = f(x), \quad \text{for } all\ x \in (c - r, c + r).$$

That is, the sequence of partial sums of the Taylor series (i.e., the sequence of Taylor polynomials) converges to $f(x)$ for each $x \in (c - r, c + r)$. We summarize this in Theorem 7.2.

THEOREM 7.2

Suppose that f has derivatives of all orders in the interval $(c - r, c + r)$, for some $r > 0$ and $\lim_{n\to\infty} R_n(x) = 0$, for all x in $(c - r, c + r)$. Then, the Taylor series for f expanded about $x = c$ converges to $f(x)$, that is,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k,$$

for all x in $(c - r, c + r)$.

We now return to the Taylor series expansion of $f(x) = e^x$ about $x = 0$, constructed in example 7.1 and investigated further in example 7.2, and prove that it converges to e^x, as we had suspected.

EXAMPLE 7.3 Proving That a Taylor Series Converges to the Desired Function

Show that the Taylor series for $f(x) = e^x$ expanded about $x = 0$ converges to e^x.

Solution We already found the indicated Taylor series, $\sum_{k=0}^{\infty} \frac{1}{k!}x^k$ in example 7.1. Here, we have $f^{(k)}(x) = e^x$, for all $k = 0, 1, 2, \cdots$. This gives us the remainder term

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - 0)^{n+1} = \frac{e^z}{(n+1)!}x^{n+1}, \tag{7.3}$$

where z is somewhere between x and 0 (and depends also on the value of n). We first find a bound on the size of e^z. Notice that if $x > 0$, then $0 < z < x$ and so,

$$e^z < e^x.$$

If $x \le 0$, then $x \le z \le 0$, so that

$$e^z \le e^0 = 1.$$

We define M to be the larger of these two bounds on e^z. That is, we let

$$M = \max\{e^x, 1\}.$$

Then, for any x and any n, we have

$$e^z \le M.$$

Together with (7.3), this gives us the error estimate

$$|R_n(x)| = \frac{e^z}{(n+1)!}|x|^{n+1} \le M\frac{|x|^{n+1}}{(n+1)!}. \tag{7.4}$$

To prove that the Taylor series converges to e^x, we want to use (7.4) to show that $\lim_{n\to\infty} R_n(x) = 0$, for all x. However, for any given x, we cannot compute $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!}$ directly. Instead, we use the following indirect approach. We test the *series* $\sum_{n=0}^{\infty} \frac{|x|^{n+1}}{(n+1)!}$ using the Ratio Test. We have

$$\lim_{k\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|^{n+2}}{(n+2)!}\frac{(n+1)!}{|x|^{n+1}} = |x| \lim_{n\to\infty} \frac{1}{n+2} = 0 < 1,$$

for all x. This says that the series $\sum_{n=0}^{\infty} \frac{|x|^{n+1}}{(n+1)!}$ converges absolutely for all x. By the kth-term test for divergence, it then follows that the general term must tend to 0 as

$n \to \infty$, for all x. That is,

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

and so, from (7.4), $\lim_{n\to\infty} R_n(x) = 0$, for all x. From Theorem 7.2, we now conclude that the Taylor series converges to e^x for all x. That is,

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots. \tag{7.5}$$

■

When constructing a Taylor series expansion, first *accurately* calculate enough derivatives for you to recognize the general form of the nth derivative. Then, show that $R_n(x) \to 0$, as $n \to \infty$, for all x, to ensure that the series converges to the function you are expanding.

One of the reasons for calculating Taylor series is that we can use their partial sums to compute approximate values of a function.

M	$\sum_{k=0}^{M} \frac{1}{k!}$
5	2.716666667
10	2.718281801
15	2.718281828
20	2.718281828

EXAMPLE 7.4 Using a Taylor Series to Obtain an Approximation of e

Use the Taylor series for e^x in (7.5) to obtain an approximation to the number e.

Solution We have

$$e = e^1 = \sum_{k=0}^{\infty} \frac{1}{k!} 1^k = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

We list some partial sums of this series in the accompanying table. From this we get the very accurate approximation

$$e \approx 2.718281828. \quad ■$$

EXAMPLE 7.5 A Taylor Series Expansion of $\sin x$

Find the Taylor series for $f(x) = \sin x$, expanded about $x = \frac{\pi}{2}$ and prove that the series converges to $\sin x$ for all x.

Solution In this case, the Taylor series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(\frac{\pi}{2})}{k!} \left(x - \frac{\pi}{2}\right)^k.$$

First, we compute some derivatives and their value at $x = \frac{\pi}{2}$. We have

$$\begin{aligned} f(x) &= \sin x & f(\tfrac{\pi}{2}) &= 1, \\ f'(x) &= \cos x & f'(\tfrac{\pi}{2}) &= 0, \\ f''(x) &= -\sin x & f''(\tfrac{\pi}{2}) &= -1, \\ f'''(x) &= -\cos x & f'''(\tfrac{\pi}{2}) &= 0, \\ f^{(4)}(x) &= \sin x & f^{(4)}(\tfrac{\pi}{2}) &= 1 \end{aligned}$$

and so on. Recognizing that every other term is zero and every other term is ± 1, we see that the Taylor series is

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{f^{(k)}(\frac{\pi}{2})}{k!} \left(x - \frac{\pi}{2}\right)^k &= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(x - \frac{\pi}{2}\right)^{2k}. \end{aligned}$$

To test this series for convergence, we consider the remainder term

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} \left(x - \frac{\pi}{2} \right)^{n+1} \right|, \tag{7.6}$$

for some z between x and $\frac{\pi}{2}$. From our derivative calculation, note that

$$f^{(n+1)}(z) = \begin{cases} \pm\cos z, & \text{if } n \text{ is even} \\ \pm\sin z, & \text{if } n \text{ is odd} \end{cases}.$$

From this, it follows that $$|f^{(n+1)}(z)| \le 1,$$

for every n. (Notice that this is true whether n is even or odd.) From (7.6), we now have

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} \right| \left| x - \frac{\pi}{2} \right|^{n+1} \le \frac{1}{(n+1)!} \left| x - \frac{\pi}{2} \right|^{n+1} \to 0,$$

as $n \to \infty$, for every x, as in example 7.3. This says that

$$\boxed{\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(x - \frac{\pi}{2} \right)^{2k} = 1 - \frac{1}{2} \left(x - \frac{\pi}{2} \right)^2 + \frac{1}{4!} \left(x - \frac{\pi}{2} \right)^4 - \cdots,}$$

for all x. In Figures 5.40a–d, we show graphs of $f(x) = \sin x$ together with the Taylor polynomials $P_2(x)$, $P_4(x)$, $P_6(x)$ and $P_8(x)$ (the first few partial sums of the series). Notice that the higher the degree of the Taylor polynomial, the larger the interval over which the polynomial provides a close approximation to $f(x) = \sin x$.

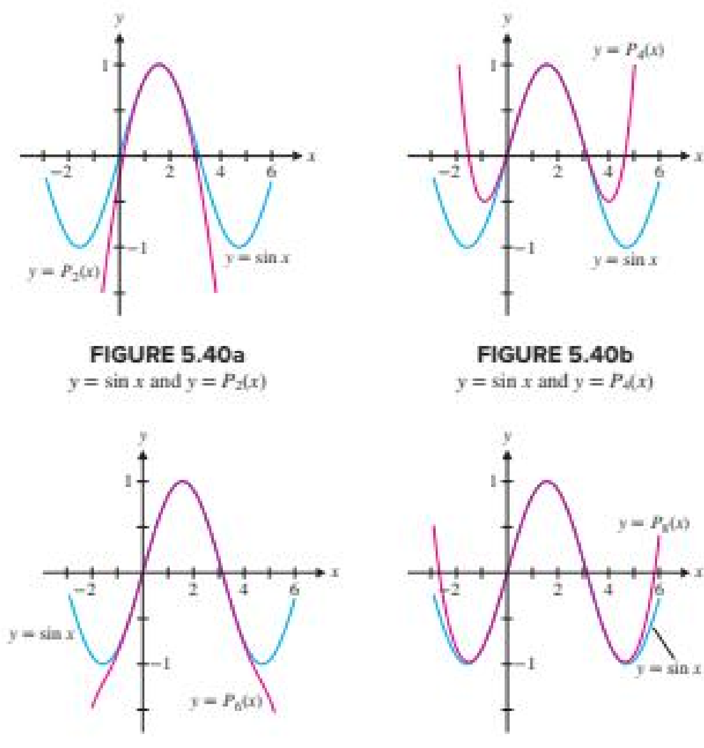


FIGURE 5.40a
$y = \sin x$ and $y = P_2(x)$

FIGURE 5.40b
$y = \sin x$ and $y = P_4(x)$

FIGURE 5.40c
$y = \sin x$ and $y = P_6(x)$

FIGURE 5.40d
$y = \sin x$ and $y = P_8(x)$

■

In example 7.6, we illustrate how to use Taylor's Theorem to estimate the error in using a Taylor polynomial to approximate the value of a function.

EXAMPLE 7.6 Estimating the Error in a Taylor Polynomial Approximation

Expand $f(x) = \ln x$ in a Taylor series about a convenient point and use a Taylor polynomial of degree 4 to approximate the value of ln(1.1). Then, estimate the error in this approximation.

Solution First, note that since ln 1 is known exactly and 1 is close to 1.1 (why would this matter?), we expand $f(x) = \ln x$ in a Taylor series about $x = 1$. We compute an adequate number of derivatives so that the pattern becomes clear. We have

$$\begin{aligned} f(x) &= \ln x & f(1) &= 0 \\ f'(x) &= x^{-1} & f'(x) &= 1 \\ f''(x) &= -x^{-2} & f''(x) &= -1 \\ f'''(x) &= 2x^{-3} & f'''(x) &= 2 \\ f^{(4)}(x) &= -3\cdot 2x^{-4} & f^{(4)}(x) &= -3! \\ f^{(5)}(x) &= 4!\,x^{-5} & f^{(5)}(x) &= 4! \\ &\vdots & &\vdots \\ f^{(k)}(x) &= (-1)^{k+1}(k-1)!\,x^{-k} & f^{(k)}(1) &= (-1)^{k+1}(k-1)! \end{aligned}$$

We get the Taylor series

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \cdots + (-1)^{k+1}\frac{(k-1)!}{k!}(x-1)^k + \cdots \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k. \end{aligned}$$

We leave it as an exercise to show that the series converges to $f(x) = \ln x$, for $0 < x < 2$. The Taylor polynomial $P_4(x)$ is then

$$\begin{aligned} P_4(x) &= \sum_{k=1}^{4} \frac{(-1)^{k+1}}{k}(x-1)^k \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4. \end{aligned}$$

We show a graph of $y = \ln x$ and $y = P_4(x)$ in Figure 5.41. Taking $x = 1.1$ gives us the approximation

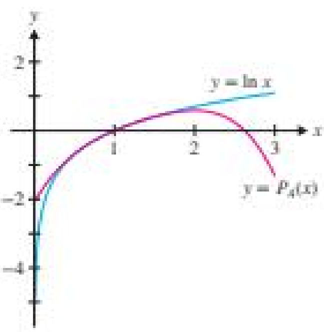


FIGURE 5.41
$y = \ln x$ and $y = P_4(x)$

$$\ln(1.1) \approx P_4(1.1) = 0.1 - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 \approx 0.095308333.$$

We can use the remainder term to estimate the error in this approximation. We have

$$\begin{aligned} |Error| &= |\ln(1.1) - P_4(1.1)| = |R_4(1.1)| \\ &= \left|\frac{f^{(4+1)}(z)}{(4+1)!}(1.1-1)^{4+1}\right| = \frac{4!|z|^{-5}}{5!}(0.1)^5, \end{aligned}$$

where z is between 1 and 1.1. This gives us the following bound on the error:

$$|Error| = \frac{(0.1)^5}{5z^5} < \frac{(0.1)^5}{5(1^5)} = 0.000002,$$

since $1 < z < 1.1$ implies that $\frac{1}{z} < \frac{1}{1} = 1$. This says that the approximation $\ln(1.1) \approx 0.095308333$ is off by no more than ± 0.000002. ■

A better question related to example 7.6 is to determine how many terms of the Taylor series are needed in order to guarantee a given accuracy. We use the remainder term to accomplish this in example 7.7.

EXAMPLE 7.7 Finding the Number of Terms Needed for a Given Accuracy

Find the number of terms in the Taylor series for $f(x) = \ln x$ expanded about $x = 1$ that will guarantee an accuracy of at least 1×10^{-10} in the approximation of (a) ln(1.1) and (b) ln(1.5).

Solution (a) From our calculations in example 7.6 and (7.2), we have that for some number z between 1 and 1.1,

$$|R_n(1.1)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}(1.1-1)^{n+1}\right|$$
$$= \frac{n!|z|^{-n-1}}{(n+1)!}(0.1)^{n+1} = \frac{(0.1)^{n+1}}{(n+1)z^{n+1}} < \frac{(0.1)^{n+1}}{n+1}.$$

Further, since we want the error to be less than 1×10^{-10}, we require that

$$|R_n(1.1)| < \frac{(0.1)^{n+1}}{n+1} < 1 \times 10^{-10}.$$

You can solve this inequality for n by trial and error, to find that $n = 9$ will guarantee the required accuracy. Notice that larger values of n will also guarantee this accuracy, since $\frac{(0.1)^{n+1}}{n+1}$ is a decreasing function of n. We then have the approximation

$$\ln(1.1) \approx P_9(1.1) = \sum_{k=0}^{9}\frac{(-1)^{k+1}}{k}(1.1-1)^k$$
$$= (0.1) - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 + \frac{1}{5}(0.1)^5$$
$$- \frac{1}{6}(0.1)^6 + \frac{1}{7}(0.1)^7 - \frac{1}{8}(0.1)^8 + \frac{1}{9}(0.1)^9$$
$$\approx 0.095310179813,$$

which from our error estimate we know is correct to within $\pm 1 \times 10^{-10}$. We show a graph of $y = \ln x$ and $y = P_9(x)$ in Figure 5.42. In comparing Figure 5.42 with Figure 8.41, observe that while $P_9(x)$ provides an improved approximation to $P_4(x)$ over the interval of convergence (0, 2), it does not provide a better approximation outside of this interval.

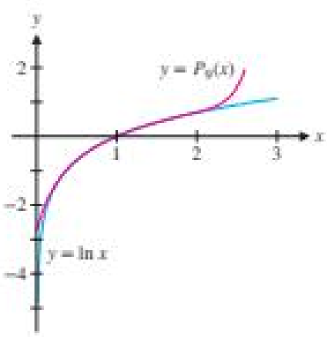


FIGURE 5.42
$y = \ln x$ and $y = P_9(x)$

(b) Similarly, notice that for some number z between 1 and 1.5,

$$|R_n(1.5)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}(1.5-1)^{n+1}\right| = \frac{n!|z|^{-n-1}}{(n+1)!}(0.5)^{n+1}$$
$$= \frac{(0.5)^{n+1}}{(n+1)z^{n+1}} < \frac{(0.5)^{n+1}}{n+1},$$

since $1 < z < 1.5$ implies that $\frac{1}{z} < \frac{1}{1} = 1$. So, here we require that

$$|R_n(1.5)| < \frac{(0.5)^{n+1}}{n+1} < 1 \times 10^{-10}.$$

Solving this by trial and error shows that $n = 28$ will guarantee the required accuracy. Observe that this says that to obtain the same accuracy, many more terms are needed to approximate $f(1.5)$ than for $f(1.1)$. This further illustrates the general principle that the farther away x is from the point about which we expand, the slower the convergence of the Taylor series. ■

For your convenience, we have compiled a list of common Taylor series in the following table.

Taylor Series	*Interval of Convergence*	*Where to Find*
$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$	$(-\infty, \infty)$	examples 7.1 and 7.3
$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$	$(-\infty, \infty)$	exercise 2
$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k} = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$	$(-\infty, \infty)$	exercise 1
$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x - \frac{\pi}{2}\right)^{2k} = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \cdots$	$(-\infty, \infty)$	example 7.5
$\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$	$(0, 2]$	examples 7.6, 7.7
$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}x^{2k+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$	$(-1, 1)$	example 6.6

Notice that once you have found a Taylor series expansion for a given function, you can find any number of other Taylor series simply by making a substitution.

EXAMPLE 7.8 Finding New Taylor Series from Old Ones

Find Taylor series in powers of x for e^{2x}, e^{x^2} and e^{-2x}.

Solution Rather than compute the Taylor series for these functions from scratch, recall that we had established in example 7.3 that

$$e^t = \sum_{k=0}^{\infty} \frac{1}{k!}t^k = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots, \tag{7.7}$$

for all $t \in (-\infty, \infty)$. Taking $t = 2x$ in (7.7), we get the new Taylor series:

$$e^{2x} = \sum_{k=0}^{\infty} \frac{1}{k!}(2x)^k = \sum_{k=0}^{\infty} \frac{2^k}{k!}x^k = 1 + 2x + \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 + \cdots.$$

Similarly, setting $t = x^2$ in (7.7), we get the Taylor series

$$e^{x^2} = \sum_{k=0}^{\infty} \frac{1}{k!}(x^2)^k = \sum_{k=0}^{\infty} \frac{1}{k!}x^{2k} = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \cdots.$$

Finally, taking $t = -2x$ in (7.7), we get

$$e^{-2x} = \sum_{k=0}^{\infty} \frac{1}{k!}(-2x)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}2^k x^k = 1 - 2x + \frac{2^2}{2!}x^2 - \frac{2^3}{3!}x^3 + \cdots.$$

Notice that all of these last three series converge for all $x \in (-\infty, \infty)$. (Why is that?) ■

Proof of Taylor's Theorem

Recall that we had observed that the Mean Value Theorem is a special case of Taylor's Theorem. As it turns out, the proof of Taylor's Theorem parallels that of the Mean Value Theorem, as both make use of Rolle's Theorem. As with the proof of the Mean Value

Theorem, for a *fixed* $x \in (c - r, c + r)$, we define the function

$$g(t) = f(x) - f(t) - f'(t)(x - t) - \frac{1}{2!}f''(t)(x - t)^2 - \frac{1}{3!}f'''(t)(x - t)^3 - \cdots - \frac{1}{n!}f^{(n)}(t)(x - t)^n - R_n(x)\frac{(x - t)^{n+1}}{(x - c)^{n+1}},$$

where $R_n(x)$ is the remainder term, $R_n(x) = f(x) - P_n(x)$. If we take $t = x$, notice that

$$g(x) = f(x) - f(x) - 0 - 0 - \cdots - 0 = 0$$

and if we take $t = c$, we get

$$\begin{aligned} g(c) &= f(x) - f(c) - f'(c)(x - c) - \frac{1}{2!}f''(c)(x - c)^2 - \frac{1}{3!}f'''(c)(x - c)^3 \\ &\quad - \cdots - \frac{1}{n!}f^{(n)}(c)(x - c)^n - R_n(x)\frac{(x - c)^{n+1}}{(x - c)^{n+1}} \\ &= f(x) - P_n(x) - R_n(x) = R_n(x) - R_n(x) = 0. \end{aligned}$$

By Rolle's Theorem, there must be some number z between x and c for which $g'(z) = 0$. Differentiating our expression for $g(t)$ (with respect to t), we get (beware of all the product rules!)

$$\begin{aligned} g'(t) &= 0 - f'(t) - f'(t)(-1) - f''(t)(x - t) - \frac{1}{2}f''(t)(2)(x - t)(-1) \\ &\quad - \frac{1}{2}f'''(t)(x - t)^2 - \cdots - \frac{1}{n!}f^{(n)}(t)(n)(x - t)^{n-1}(-1) \\ &\quad - \frac{1}{n!}f^{(n+1)}(t)(x - t)^n - R_n(x)\frac{(n + 1)(x - t)^n(-1)}{(x - c)^{n+1}} \\ &= -\frac{1}{n!}f^{(n+1)}(t)(x - t)^n + R_n(x)\frac{(n + 1)(x - t)^n}{(x - c)^{n+1}}, \end{aligned}$$

after most of the terms cancel. So, taking $t = z$, we have that

$$0 = g'(z) = -\frac{1}{n!}f^{(n+1)}(z)(x - z)^n + R_n(x)\frac{(n + 1)(x - z)^n}{(x - c)^{n+1}}.$$

Solving this for the term containing $R_n(x)$, we get

$$R_n(x)\frac{(n + 1)(x - z)^n}{(x - c)^{n+1}} = \frac{1}{n!}f^{(n+1)}(z)(x - z)^n$$

and finally,

$$\begin{aligned} R_n(x) &= \frac{1}{n!}f^{(n+1)}(z)(x - z)^n\frac{(x - c)^{n+1}}{(n + 1)(x - z)^n} \\ &= \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}, \end{aligned}$$

as we had claimed.

BEYOND FORMULAS

You should think of this section as giving a general procedure for finding power series representations, where section 5.6 solves that problem only for special cases. Further, if a function is the sum of a convergent power series, then we can approximate the function with a partial sum of that series. Taylor's Theorem provides us with an estimate of the error in a given approximation and tells us that (in general) the approximation is improved by taking more terms.

EXERCISES 5.7

WRITING EXERCISES

1. Describe how the Taylor polynomial of degree $n = 1$ compares to the linear approximation. (See section 3.1.) Give an analogous interpretation of the Taylor polynomial of degree $n = 2$. How do various graphical properties (position, slope, concavity) of the Taylor polynomial compare with those of the function $f(x)$ at $x = c$?

2. Briefly discuss how a computer might use Taylor polynomials to compute sin(1.2). In particular, how would the computer know how many terms to compute? How would the number of terms necessary to compute sin(1.2) compare to the number needed to compute sin(100)? Describe a trick that would make it much easier for the computer to compute sin(100). (Hint: The sine function is periodic.)

3. Taylor polynomials are built up from a knowledge of $f(c)$, $f'(c)$, $f''(c)$ and so on. Explain in graphical terms why information at one point (e.g., position, slope, concavity, etc.) can be used to construct the graph of the function on the entire interval of convergence.

4. If $f(c)$ is the position of an object at time $t = c$, then $f'(c)$ is the object's velocity and $f''(c)$ is the object's acceleration at time c. Explain in physical terms how knowledge of these values at one time [plus $f'''(c)$, etc.] can be used to predict the position of the object on the interval of convergence.

5. Our table of common Taylor series lists two different series for sin x. Explain how the same function could have two different Taylor series representations. For a given problem (e.g., approximate sin 2), explain how you would choose which Taylor series to use.

6. Explain why the Taylor series with center $c = 0$ of $f(x) = x^2 - 1$ is simply $x^2 - 1$.

In exercises 1–8, find the Maclaurin series (i.e., Taylor series about $c = 0$) and its interval of convergence.

1. $f(x) = \cos x$
2. $f(x) = \sin x$
3. $f(x) = e^{2x}$
4. $f(x) = \cos 2x$
5. $f(x) = \ln(1 + x)$
6. $f(x) = e^{-x}$
7. $f(x) = 1/(1 + x)^2$
8. $f(x) = 1/(1 - x)$

In exercises 9–14, find the Taylor series about the indicated center and determine the interval of convergence.

9. $f(x) = e^{x-1}$, $c = 1$
10. $f(x) = \cos x$, $c = -\pi/2$
11. $f(x) = \ln x$, $c = e$
12. $f(x) = e^x$, $c = 2$
13. $f(x) = \dfrac{1}{x}$, $c = 1$
14. $f(x) = \dfrac{1}{x+5}$, $c = 0$

In exercises 15–20, graph $f(x)$ and the Taylor polynomials for the indicated center c and degree n.

15. $f(x) = \sqrt{x}$, $c = 1$, $n = 3$; $n = 6$
16. $f(x) = \dfrac{1}{1+x}$, $c = 0$, $n = 4$; $n = 8$
17. $f(x) = e^x$, $c = 2$, $n = 3$; $n = 6$
18. $f(x) = \cos x$, $c = \pi/2$, $n = 4$; $n = 8$
19. $f(x) = \sin^{-1} x$, $c = 0$, $n = 3$; $n = 5$
20. $f(x) = \dfrac{2}{x-5}$, $c = 6$, $n = 4$; $n = 8$

In exercises 21–24, prove that the Taylor series converges to $f(x)$ by showing that $R_n(x) \to 0$ as $n \to \infty$.

21. $\sin x = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
22. $\cos x = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$
23. $\ln x = \displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}$, $1 \le x \le 2$
24. $e^{-x} = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$

In exercises 25–28, (a) use a Taylor polynomial of degree 4 to approximate the given number, (b) estimate the error in the approximation and (c) estimate the number of terms needed in a Taylor polynomial to guarantee an accuracy of 10^{-10}.

25. ln(1.05)
26. ln(0.9)
27. $\sqrt{1.1}$
28. $\sqrt{1.2}$

In exercises 29–34, use a known Taylor series to find the Taylor series about $c = 0$ for the given function and find its radius of convergence.

29. $f(x) = e^{-3x}$
30. $f(x) = \dfrac{e^x - 1}{x}$
31. $f(x) = xe^{-x^2}$
32. $f(x) = \sin x^2$
33. $f(x) = x \sin 2x$
34. $f(x) = \cos x^3$

In exercises 35–38, use a Taylor series to verify the given formula.

35. $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{k!} = e^2$
36. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{(2k+1)!} = 0$
37. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$
38. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln 2$

In exercises 39–42, name the method by stating the best method (geometric series, substitution, differentiation/antidifferentiation, Taylor series) for finding a power series representation of the function.

39. **(a)** $f(x) = (1-x)^{-2}$ **(b)** $f(x) = \dfrac{2}{x+x^2}$

40. **(a)** $f(x) = \cos(x^3)$ **(b)** $f(x) = \cos^3 x$

41. **(a)** $f(x) = \tanh^{-1} x$ **(b)** $f(x) = \tan^{-1} x$

42. **(a)** $f(x) = \dfrac{1}{e^{2x}}$ **(b)** $f(x) = e^{2(x+1)}$

43. You may have wondered why it is necessary to show that $\lim_{n\to\infty} R_n(x) = 0$ to conclude that a Taylor series converges to $f(x)$. Consider $f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. Show that $f'(0) = f''(0) = 0$. (Hint: Use the fact that $\lim_{h\to 0} \dfrac{e^{-1/h^2}}{h^n} = 0$ for any positive integer n.) It turns out that $f^{(n)}(0) = 0$ for all n. Thus, the Taylor series of $f(x)$ about $c = 0$ equals 0, a convergent "series" that does not converge to $f(x)$.

44. In many applications, the **error function** $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-u^2}\,du$ is important. Compute and graph the fourth-order Taylor polynomial for $\operatorname{erf}(x)$ about $c = 0$.

Exercises 45–48 involve the binomial expansion.

45. Show that the Maclaurin series for $(1+x)^r$ is $1 + \displaystyle\sum_{k=1}^{\infty} \frac{r(r-1)\cdots(r-k+1)}{k!}x^k$, for any constant r.

46. Simplify the series in exercise 45 for $r = 2$; $r = 3$; r is a positive integer.

47. Use the result of exercise 45 to write out the Maclaurin series for $f(x) = \sqrt{1+x}$.

48. Use the result of exercise 45 to write out the Maclaurin series for $f(x) = (1+x)^{3/2}$.

49. Find the Taylor series of $f(x) = |x|$ with center $c = 1$. Argue that the radius of convergence is ∞. However, show that the Taylor series of $f(x)$ does not converge to $f(x)$ for all x.

50. Find the Maclaurin series of $f(x) = \sqrt{a^2+x^2} - \sqrt{a^2-x^2}$ for some non-zero constant a.

51. Prove that if f and g are functions such that $f''(x)$ and $g''(x)$ exist for all x and $\lim_{x\to a} \dfrac{f(x)-g(x)}{(x-a)^2} = 0$, then $f(a) = g(a)$, $f'(a) = g'(a)$ and $f''(a) = g''(a)$. What does this imply about the Taylor series for $f(x)$ and $g(x)$?

52. Generalize exercise 51 by proving that if f and g are functions such that for some positive integer n, $f^{(n)}(x)$ and $g^{(n)}(x)$ exist for all x and $\lim_{x\to a} \dfrac{f(x)-g(x)}{(x-a)^n} = 0$, then $f^{(k)}(a) = g^{(k)}(a)$ for $0 \le k \le n$.

53. Find the first five terms in the Taylor series about $c = 0$ for $f(x) = e^x \sin x$ and compare to the product of the Taylor polynomials about $c = 0$ of e^x and $\sin x$.

54. Find the first five terms in the Taylor series about $c = 0$ for $f(x) = \tan x$ and compare to the quotient of the Taylor polynomials about $c = 0$ of $\sin x$ and $\cos x$.

55. Find the first four non-zero terms in the Maclaurin series of $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ and compare to the Maclaurin series for $\sin x$.

56. Find the Taylor series of $f(x) = x \ln x$ about $c = 1$. Compare to the Taylor series for $\ln x$ about $c = 1$.

57. Find the Maclaurin series of $f(x) = \cosh x$ and $f(x) = \sinh x$. Compare to the Maclaurin series of $\cos x$ and $\sin x$.

58. Use the Maclaurin series for $\tan x$ and the result of exercise 57 to conjecture the Maclaurin series for $\tanh x$.

59. We have seen that $\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} + \cdots$. Determine how many terms are needed to approximate $\sin 1$ to within 10^{-5}. Show that $\sin 1 = \int_0^1 \cos x\,dx$. Determine how many points are needed for Simpson's Rule to approximate this integral to within 10^{-5}. Compare the efficiency of Maclaurin series and Simpson's Rule for this problem.

60. As in exercise 59, compare the efficiency of Maclaurin series and Simpson's Rule in estimating e to within 10^{-5}.

EXPLORATORY EXERCISES

1. Almost all of our series results apply to series of complex numbers. Defining $i = \sqrt{-1}$, show that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and so on. Replacing x with ix in the Maclaurin series for e^x, separate terms containing i from those that don't contain i (after the simplifications indicated above) and derive **Euler's formula:** $e^{ix} = \cos x + i \sin x$. Show that $\cos(ix) = \cosh x$ and $\sin(ix) = i \sinh x$. That is, the trig functions and their hyperbolic counterparts are closely related as functions of complex variables.

2. The method used in examples 7.3, 7.5, 7.6 and 7.7 does not require us to actually find $R_n(x)$, but to approximate it with a worst-case bound. Often this approximation is fairly close to $R_n(x)$, but this is not always true. As an extreme example of this, show that the bound on $R_n(x)$ for $f(x) = \ln x$ about $c = 1$ (see exercise 23) increases without bound for $0 < x < \frac{1}{2}$. Explain why this does not necessarily mean that the actual error increases without bound. In fact, $R_n(x) \to 0$ for $0 < x < \frac{1}{2}$ but we must show this using some other method. Use integration of an appropriate power series to show that $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1}\frac{(x-1)^k}{k}$ converges to $\ln x$ for $0 < x < \frac{1}{2}$.

3. Verify numerically that if a_1 is close to π, the sequence $a_{n+1} = a_n + \sin a_n$ converges to π. (In other words, if a_n is an approximation of π, then $a_n + \sin a_n$ is a better approximation.) To prove this, find the Taylor series for $\sin x$ about $c = \pi$. Use this to show that if $\pi < a_n < 2\pi$, then $\pi < a_{n+1} < a_n$. Similarly, show that if $0 < a_n < \pi$, then $a_n < a_{n+1} < \pi$.

5.8 APPLICATIONS OF TAYLOR SERIES

In this section, we expand on our earlier presentation of Taylor series, by giving a few examples of how these are used to approximate the values of transcendental functions, evaluate limits and integrals, and define important new functions. These represent a small sampling of the important applications of Taylor series.

First, consider how calculators and computers might calculate values of transcendental functions, such as sin(1.234567). We illustrate this in example 8.1.

EXAMPLE 8.1 Using Taylor Polynomials to Approximate a Sine Value

Use a Taylor series to approximate sin(1.234567) accurate to within 10^{-11}.

Solution In section 5.7, we left it as an exercise to show that the Taylor series expansion for $f(x) = \sin x$ about $x = 0$ is

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,$$

where the interval of convergence is $(-\infty, \infty)$. Taking $x = 1.234567$, we have

$$\sin 1.234567 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(1.234567)^{2k+1},$$

which is an alternating series. We can use a partial sum of this series to approximate the desired value, but how many terms will we need for the desired accuracy? Recall that for alternating series, the error in a partial sum is bounded by the absolute value of the first neglected term. (Note that you could also use the remainder term from Taylor's Theorem to bound the error.) To ensure that the error is less than 10^{-11}, we must find an integer k such that $\frac{1.234567^{2k+1}}{(2k+1)!} < 10^{-11}$. By trial and error, we find that

$$\frac{1.234567^{17}}{17!} \approx 1.010836 \times 10^{-13} < 10^{-11},$$

so that $k = 8$ will do. This says that the first neglected term corresponds to $k = 8$ and so, we compute the partial sum

$$\begin{aligned} \sin 1.234567 &\approx \sum_{k=0}^{7} \frac{(-1)^k}{(2k+1)!}(1.234567)^{2k+1} \\ &= 1.234567 - \frac{1.234567^3}{3!} + \frac{1.234567^5}{5!} - \frac{1.234567^7}{7!} + \cdots - \frac{1.234567^{15}}{15!} \\ &\approx 0.94400543137. \end{aligned}$$

Check your calculator or computer to verify that this matches your calculator's estimate. ■

In example 8.1, while we produced an approximation with the desired accuracy, we did not do this in the most efficient fashion, as we simply grabbed the most handy Taylor series expansion of $f(x) = \sin x$. We illustrate a more efficient choice in example 8.2.

EXAMPLE 8.2 Choosing a More Appropriate Taylor Series Expansion

Repeat example 8.1, but this time make a more appropriate choice for the Taylor series.

Solution Recall that Taylor series converge much faster close to the point about which you expand than they do far away. Given this and the fact that we know the exact value of $\sin x$ at only a few points, you should quickly recognize that a series expanded about $x = \frac{\pi}{2} \approx 1.57$ is a better choice for computing sin 1.234567 than one expanded about $x = 0$. (Another reasonable choice is the Taylor series expansion about $x = \frac{\pi}{3}$.) In example 7.5, recall that we had found that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x - \frac{\pi}{2}\right)^{2k} = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \cdots,$$

where the interval of convergence is $(-\infty, \infty)$. Taking $x = 1.234567$ gives us

$$\begin{aligned}\sin 1.234567 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(1.234567 - \frac{\pi}{2}\right)^{2k} \\ &= 1 - \frac{1}{2}\left(1.234567 - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(1.234567 - \frac{\pi}{2}\right)^4 - \cdots,\end{aligned}$$

which is again an alternating series. Using the remainder term from Taylor's Theorem to bound the error, we have that

$$\begin{aligned}|R_n(1.234567)| &= \left|\frac{f^{(2n+2)}(z)}{(2n+2)!}\right|\left|1.234567 - \frac{\pi}{2}\right|^{2n+2} \\ &\le \frac{\left|1.234567 - \frac{\pi}{2}\right|^{2n+2}}{(2n+2)!}.\end{aligned}$$

(Note that we get the same error bound if we use the error bound for an alternating series.) By trial and error, you can find that

$$\frac{\left|1.234567 - \frac{\pi}{2}\right|^{2n+2}}{(2n+2)!} < 10^{-11}$$

for $n = 4$, so that an approximation with the required degree of accuracy is

$$\begin{aligned}\sin 1.234567 &\approx \sum_{k=0}^{4} \frac{(-1)^k}{(2k)!}\left(1.234567 - \frac{\pi}{2}\right)^{2k} \\ &= 1 - \frac{1}{2}\left(1.234567 - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(1.234567 - \frac{\pi}{2}\right)^4 \\ &\quad - \frac{1}{6!}\left(1.234567 - \frac{\pi}{2}\right)^6 + \frac{1}{8!}\left(1.234567 - \frac{\pi}{2}\right)^8 \\ &\approx 0.94400543137.\end{aligned}$$

Compare this result to example 8.1, where we needed many more terms of the Taylor series to obtain the same degree of accuracy. ■

We can also use Taylor series to quickly conjecture the value of difficult limits. Be careful, though: the theory of when these conjectures are guaranteed to be correct is beyond the level of this text. However, we can certainly obtain helpful hints about certain limits.

EXAMPLE 8.3 Using Taylor Polynomials to Conjecture the Value of a Limit

Use Taylor series to conjecture $\lim_{x \to 0} \frac{\sin x^3 - x^3}{x^9}$.

Solution Again recall that the Maclaurin series for $\sin x$ is

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,$$

where the interval of convergence is $(-\infty, \infty)$. Substituting x^3 for x gives us

$$\sin x^3 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(x^3)^{2k+1} = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots$$

This gives us

$$\frac{\sin x^3 - x^3}{x^9} = \frac{\left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots\right) - x^3}{x^9} = -\frac{1}{3!} + \frac{x^6}{5!} + \cdots$$

and so, computing the limit as $x \to 0$ of the above polynomial, we conjecture that

$$\lim_{x \to 0} \frac{\sin x^3 - x^3}{x^9} = -\frac{1}{3!} = -\frac{1}{6}.$$

You can verify that this limit is correct using l'Hôpital's Rule (three times, simplifying each time). ■

Since Taylor polynomials are used to approximate functions on a given interval and since polynomials are easy to integrate, we can use a Taylor polynomial to obtain an approximation of a definite integral. It turns out that such an approximation is often better than that obtained from the numerical methods developed in section 4.7. We illustrate this in example 8.4.

EXAMPLE 8.4 Using Taylor Series to Approximate a Definite Integral

Use a Taylor polynomial with $n = 8$ to approximate $\int_{-1}^{1} \cos(x^2)\,dx$.

Solution Since we do not know an antiderivative of $\cos(x^2)$, we must rely on a numerical approximation of the integral. Since we are integrating on the interval $(-1, 1)$, a Maclaurin series expansion (i.e. a Taylor series expansion about $x = 0$) is a good choice. We have

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots,$$

which converges on all of $(-\infty, \infty)$. Replacing x by x^2 gives us

$$\cos(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{4k} = 1 - \frac{1}{2}x^4 + \frac{1}{4!}x^8 - \frac{1}{6!}x^{12} + \cdots,$$

so that

$$\cos(x^2) \approx 1 - \frac{1}{2}x^4 + \frac{1}{4!}x^8.$$

This leads us to the approximation

$$\begin{aligned}\int_{-1}^{1} \cos(x^2)\,dx &\approx \int_{-1}^{1} \left(1 - \frac{1}{2}x^4 + \frac{1}{4!}x^8\right)dx \\ &= \left(x - \frac{x^5}{10} + \frac{x^9}{216}\right)\Bigg|_{x=-1}^{x=1} \\ &= \frac{977}{540} \approx 1.809259.\end{aligned}$$

Our CAS gives us $\int_{-1}^{1} \cos(x^2)\,dx \approx 1.809048$, so our approximation appears to be very accurate. ■

You might reasonably argue that we don't need Taylor series to obtain approximations like those in example 8.4, as you could always use other, simpler numerical methods like Simpson's Rule to do the job. That's often true, but just try to use Simpson's Rule on the integral in example 8.5.

EXAMPLE 8.5 Using Taylor Series to Approximate the Value of an Integral

Use a Taylor polynomial with $n = 5$ to approximate $\int_{-1}^{1} \frac{\sin x}{x}\, dx$.

Solution Note that you do not know an antiderivative of $\frac{\sin x}{x}$. Further, while the integrand is discontinuous at $x = 0$, this does *not* need to be treated as an improper integral, since $\lim_{x\to 0} \frac{\sin x}{x} = 1$. (That is, the integrand can be extended to a continuous function g, with $g(0) = 1$.) From the first few terms of the Maclaurin series for $f(x) = \sin x$, we have the Taylor polynomial approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

so that

$$\frac{\sin x}{x} \approx 1 - \frac{x^2}{3!} + \frac{x^4}{5!}.$$

Consequently,

$$\begin{aligned}\int_{-1}^{1} \frac{\sin x}{x}\, dx &\approx \int_{-1}^{1} \left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right) dx \\ &= \left(x - \frac{x^3}{18} + \frac{x^5}{600}\right)\Bigg|_{x=-1}^{x=1} \\ &= \left(1 - \frac{1}{18} + \frac{1}{600}\right) - \left(-1 + \frac{1}{18} - \frac{1}{600}\right) \\ &= \frac{1703}{900} \approx 1.89222.\end{aligned}$$

Our CAS gives us $\int_{-1}^{1} \frac{\sin x}{x}\, dx \approx 1.89216$, so our approximation is quite good. On the other hand, if you try to apply Simpson's Rule or Trapezoidal Rule, the algorithm will not work, as they will attempt to evaluate $\frac{\sin x}{x}$ at $x = 0$. ■

While you have now calculated Taylor series expansions of many familiar functions, many other functions are actually *defined* by a power series. These include many functions in the very important class of **special functions** that frequently arise in physics and engineering applications. One important family of special functions are the Bessel functions, which arise in the study of fluid mechanics, acoustics, wave propagation and other areas of applied mathematics. The **Bessel function of order** p is defined by the power series

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+p}}{2^{2k+p} k!(k+p)!}, \tag{8.1}$$

for non-negative integers p. Bessel functions arise in the solution of the differential equation $x^2y'' + xy' + (x^2 - p^2)y = 0$. In examples 8.6 and 8.7, we explore several interesting properties of Bessel functions.

EXAMPLE 8.6 The Radius of Convergence of a Bessel Function

Find the radius of convergence for the series defining the Bessel function $J_0(x)$.

Solution From equation (8.1) with $p = 0$, we have $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$. The Ratio Test gives us

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{2k+2}}{2^{2k+2}[(k+1)!]^2}\frac{2^{2k}(k!)^2}{x^{2k}}\right| = \lim_{k\to\infty}\left|\frac{x^2}{4(k+1)^2}\right| = 0 < 1,$$

for all x. The series then converges absolutely for all x and so the radius of convergence is ∞. ■

In example 8.7, we explore an interesting relationship between the zeros of two Bessel functions.

EXAMPLE 8.7 The Zeros of Bessel Functions

Verify graphically that on the interval [0, 10], the zeros of $J_0(x)$ and $J_1(x)$ alternate.

Solution Unless you have a CAS with these Bessel functions available as built-in functions, you will need to graph partial sums of the defining series:

$$J_0(x) \approx \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2} \quad\text{and}\quad J_1(x) \approx \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}.$$

Before graphing these, you must first determine how large n should be in order to produce a reasonable graph. Notice that for each fixed $x > 0$, both of the defining series are alternating series. Consequently, the error in using a partial sum to approximate the function is bounded by the first neglected term. That is,

$$\left|J_0(x) - \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}\right| \le \frac{x^{2n+2}}{2^{2n+2}[(n+1)!]^2}$$

and

$$\left|J_1(x) - \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}\right| \le \frac{x^{2n+3}}{2^{2n+3}(n+1)!(n+2)!},$$

with the maximum error in each occurring at $x = 10$. Notice that for $n = 12$, we have that

$$\left|J_0(x) - \sum_{k=0}^{12} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}\right| \le \frac{x^{2(12)+2}}{2^{2(12)+2}[(12+1)!]^2} \le \frac{10^{26}}{2^{26}(13!)^2} < 0.04$$

and

$$\left|J_1(x) - \sum_{k=0}^{12} \frac{(-1)^k x^{2k+1}}{2^{2k+1}k!(k+1)!}\right| \le \frac{x^{2(12)+3}}{2^{2(12)+3}(12+1)!(12+2)!} \le \frac{10^{27}}{2^{27}(13!)(14!)} < 0.04.$$

So, in either case, using a partial sum with $n = 12$ results in an approximation that is within 0.04 of the correct value for each x in the interval [0, 10]. This is sufficiently accurate for our present purposes. Figure 5.43 shows graphs of partial sums with $n = 12$ for $J_0(x)$ and $J_1(x)$.

Notice that $J_1(0) = 0$ and in the figure, you can clearly see that $J_0(x) = 0$ at about $x = 2.4$, $J_1(x) = 0$ at about $x = 3.9$, $J_0(x) = 0$ at about $x = 5.6$, $J_1(x) = 0$ at about $x = 7.0$ and $J_0(x) = 0$ at about $x = 8.8$. From this, it is now apparent that the zeros of $J_0(x)$ and $J_1(x)$ do indeed alternate on the interval [0, 10]. ■

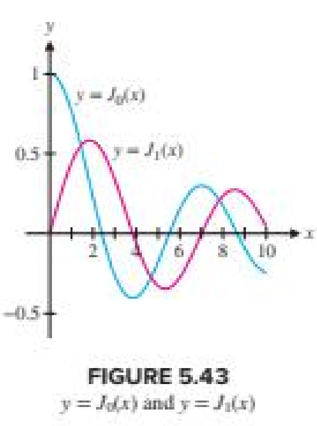


FIGURE 5.43
$y = J_0(x)$ and $y = J_1(x)$

It turns out that the result of example 8.7 generalizes to any interval of positive numbers and any two Bessel functions of consecutive order. That is, between consecutive zeros of $J_p(x)$ is a zero of $J_{p+1}(x)$ and between consecutive zeros of $J_{p+1}(x)$ is a zero of $J_p(x)$. We explore this further in the exercises.

The Binomial Series

You are already familiar with the Binomial Theorem, which states that for any positive integer n,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.$$

We often write this as $$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k,$$

where we use the shorthand notation $\binom{n}{k}$ to denote the binomial coefficient, defined by

$$\binom{n}{0} = 1, \quad \binom{n}{1} = n, \quad \binom{n}{2} = \frac{n(n-1)}{2} \quad \text{and}$$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}, \quad \text{for } k \geq 3.$$

For the case where $a = 1$ and $b = x$, the Binomial Theorem simplifies to

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k.$$

Newton discovered that this result could be extended to include values of n other than positive integers. What resulted is a special type of power series known as the *binomial series,* which has important applications in statistics and physics. We begin by deriving the Maclaurin series for $f(x) = (1+x)^n$, for some constant $n \neq 0$. Computing derivatives and evaluating these at $x = 0$, we have

$$\begin{aligned}
f(x) &= (1+x)^n & f(0) &= 1\\
f'(x) &= n(1+x)^{n-1} & f'(0) &= n\\
f''(x) &= n(n-1)(1+x)^{n-2} & f''(0) &= n(n-1)\\
&\vdots & &\vdots\\
f^{(k)}(x) &= n(n-1)\cdots(n-k+1)(1+x)^{n-k} & f^{(k)}(0) &= n(n-1)\cdots(n-k+1).
\end{aligned}$$

We call the resulting Maclaurin series the **binomial series**, given by

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k &= 1 + nx + n(n-1)\frac{x^2}{2!} + \cdots + n(n-1)\cdots(n-k+1)\frac{x^k}{k!} + \cdots\\
&= \sum_{k=0}^{\infty}\binom{n}{k}x^k.
\end{aligned}$$

From the Ratio Test, we have

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{n(n-1)\cdots(n-k+1)(n-k)x^{k+1}}{(k+1)!}\,\frac{k!}{n(n-1)\cdots(n-k+1)x^k}\right|\\
&= |x|\lim_{k\to\infty}\frac{|n-k|}{k+1} = |x|,
\end{aligned}$$

so that the binomial series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. By showing that the remainder term $R_k(x)$ tends to zero as $k \to \infty$, we can confirm that the binomial series converges to $(1+x)^n$ for $|x| < 1$. We state this formally in Theorem 8.1.

THEOREM 8.1 (Binomial Series)

For any real number r, $(1+x)^r = \sum_{k=0}^{\infty}\binom{r}{k}x^k$, for $-1 < x < 1$.

As seen in the exercises, for some values of the exponent r, the binomial series also converges at one or both of the endpoints $x = \pm 1$.

EXAMPLE 8.8 Using the Binomial Series

Using the binomial series, find a Maclaurin series for $f(x) = \sqrt{1+x}$ and use it to approximate $\sqrt{17}$ accurate to within 0.000001.

Solution From the binomial series with $r = \frac{1}{2}$, we have

$$\sqrt{1+x} = (1+x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots$$
$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots,$$

for $-1 < x < 1$. To use this to approximate $\sqrt{17}$, we first rewrite it in a form involving $\sqrt{1+x}$, for $-1 < x < 1$. Observe that we can do this by writing

$$\sqrt{17} = \sqrt{16 \cdot \frac{17}{16}} = 4\sqrt{\frac{17}{16}} = 4\sqrt{1 + \frac{1}{16}}.$$

Since $x = \frac{1}{16}$ is in the interval of convergence, $-1 < x < 1$, the binomial series gives us

$$\sqrt{17} = 4\sqrt{1 + \frac{1}{16}} = 4\left[1 + \frac{1}{2}\left(\frac{1}{16}\right) - \frac{1}{8}\left(\frac{1}{16}\right)^2 + \frac{1}{16}\left(\frac{1}{16}\right)^3 - \frac{5}{128}\left(\frac{1}{16}\right)^4 + \cdots\right].$$

Since this is an alternating series, the error in using the first n terms to approximate the sum is bounded by the first neglected term. So, if we use only the first three terms of the series, the error is bounded by $\frac{1}{16}\left(\frac{1}{16}\right)^3 \approx 0.000015 > 0.000001$. However, if we use the first four terms of the series to approximate the sum, the error is bounded by $\frac{5}{128}\left(\frac{1}{16}\right)^4 \approx 0.0000006 < 0.000001$, as desired. So, we can achieve the desired accuracy by summing the first four terms of the series:

$$\sqrt{17} \approx 4\left[1 + \frac{1}{2}\left(\frac{1}{16}\right) - \frac{1}{8}\left(\frac{1}{16}\right)^2 + \frac{1}{16}\left(\frac{1}{16}\right)^3\right] \approx 4.1231079,$$

where this approximation is accurate to within the desired accuracy. ■

EXERCISES 5.8

WRITING EXERCISES

1. In example 8.2, we showed that an expansion about $x = \frac{\pi}{2}$ is more accurate for approximating sin 1.234567 than an expansion about $x = 0$ with the same number of terms. Explain why an expansion about $x = 1.2$ would be even more efficient, but is not practical.

2. Assuming that you don't need to rederive the Maclaurin series for $\cos x$, compare the amount of work done in example 8.4 to the work needed to compute a Simpson's Rule approximation with $n = 16$.

3. In equation (8.1), we defined the Bessel functions as series. This may seem like a convoluted way of defining a function, but compare the levels of difficulty doing the following with a Bessel function versus $\sin x$: computing $f(0)$, computing $f(1.2)$, evaluating $f(2x)$, computing $f'(x)$, computing $\int f(x)\,dx$ and computing $\int_0^1 f(x)\,dx$.

4. Discuss how you might estimate the error in the approximation of example 8.4.

In exercises 1–6, use an appropriate Taylor series to approximate the given value, accurate to within 10^{-11}.

1. $\sin 1.61$	2. $\sin 6.32$	3. $\cos 0.34$
4. $\cos 3.04$	5. $e^{-0.2}$	6. $e^{0.4}$

In exercises 7–12, use a known Taylor series to conjecture the value of the limit.

7. $\lim_{x \to 0} \frac{\cos x^2 - 1}{x^4}$

8. $\lim_{x \to 0} \frac{\sin x^2 - x^2}{x^6}$

9. $\lim_{x \to 1} \frac{\ln x - (x-1)}{(x-1)^2}$ **10.** $\lim_{x \to 0} \frac{\tan^{-1} x - x}{x^3}$

11. $\lim_{x \to 0} \frac{e^x - 1}{x}$ **12.** $\lim_{x \to 0} \frac{e^{-2x} - 1}{x}$

In exercises 13–18, use a known Taylor polynomial with n non-zero terms to estimate the value of the integral.

13. $\int_{-1}^{1} \frac{\sin x}{x}\, dx,\ n = 3$ **14.** $\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \cos x^2\, dx,\ n = 4$

15. $\int_{-1}^{1} e^{-x^2}\, dx,\ n = 5$ **16.** $\int_{0}^{1} \tan^{-1} x\, dx,\ n = 5$

17. $\int_{1}^{2} \ln x\, dx,\ n = 5$ **18.** $\int_{0}^{1} e^{\sqrt{x}}\, dx,\ n = 4$

19. Find the radius of convergence of $J_1(x)$.

20. Find the radius of convergence of $J_2(x)$.

21. Find the number of terms needed to approximate $J_2(x)$ within 0.04 for x in the interval $[0, 10]$.

22. Show graphically that the zeros of $J_1(x)$ and $J_2(x)$ alternate on the interval $(0, 10]$.

In exercises 23–26, use the Binomial Theorem to find the first five terms of the Maclaurin series.

23. $f(x) = \frac{1}{\sqrt{1-x}}$ **24.** $f(x) = \sqrt[3]{1+2x}$

25. $f(x) = \frac{6}{\sqrt[3]{1+3x}}$ **26.** $f(x) = (1+x^2)^{4/5}$

In exercises 27 and 28, use the Binomial Theorem to approximate the value to within 10^{-6}.

27. (a) $\sqrt{26}$ (b) $\sqrt{24}$ **28.** (a) $\frac{2}{\sqrt[3]{9}}$ (b) $\sqrt[4]{17}$

29. Apply the Binomial Theorem to $(x+4)^3$ and $(1-2x)^4$. Determine the number of non-zero terms in the binomial expansion for any positive integer n.

30. If n and k are positive integers with $n > k$, show that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

31. Use exercise 23 to find the Maclaurin series for $\frac{1}{\sqrt{1-x^2}}$ and use it to find the Maclaurin series for $\sin^{-1} x$.

32. Use the Binomial Theorem to find the Maclaurin series for $(1+2x)^{4/3}$ and compare this series to that of exercise 24.

APPLICATIONS

1. Einstein's theory of relativity states that the mass of an object traveling at velocity v is $m(v) = m_0/\sqrt{1 - v^2/c^2}$, where m_0 is the rest mass of the object and c is the speed of light. (a) Show that $m \approx m_0 + \left(\frac{m_0}{2c^2}\right)v^2$. (b) Use this approximation to estimate how large v would need to be to increase the mass by 10%. (c) Find the fourth-degree Taylor polynomial expanded about $v = 0$. Use it to repeat part (b).

2. Show that $\frac{mt}{\sqrt{m^2c^2 + t^2}} \approx \frac{1}{c}t$ for small t.

3. The weight (force due to gravity) of an object of mass m and altitude x km above the surface of the earth is $w(x) = \frac{mgR^2}{(R+x)^2}$, where R is the radius of the earth and g is the acceleration due to gravity. (a) Show that $w(x) \approx mg(1 - 2x/R)$. (b) Estimate how large x would need to be to reduce the weight by 10%. (c) Find the second-degree Taylor polynomial expanded about $x = 0$ for $w(x)$. Use it to repeat part (b).

4. (a) Based on your answers to exercise 3, is weight significantly different at a high-altitude location (e.g., 2300 m) compared to sea level? (b) The radius of the earth is up to 480 km larger at the equator than it is at the poles. Which would have a larger effect on weight, altitude or latitude?

In exercises 5–8, use the Maclaurin series expansion $\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \cdots$.

5. The tangential component of the space shuttle's velocity during re-entry is approximately $v(t) = v_c \tanh\left(\frac{g}{v_c}t + \tanh^{-1}\frac{v_0}{v_c}\right)$, where v_0 is the velocity at time 0 and v_c is the terminal velocity (see Long and Weiss, 1999[1]). If $\tanh^{-1}\frac{v_0}{v_c} = \frac{1}{2}$, show that $v(t) \approx gt + \frac{1}{2}v_c$. Is this estimate of $v(t)$ too large or too small?

6. Show that in exercise 5, $v(t) \to v_c$ as $t \to \infty$. Use the approximation in exercise 5 to estimate the time needed to reach 90% of the terminal velocity.

7. The downward velocity of a sky diver of mass m is $v(t) = \sqrt{40mg}\tanh\left(\sqrt{\frac{g}{40m}}\,t\right)$. Show that $v(t) \approx gt - \frac{g^2}{120m}t^3$.

8. The velocity of a water wave of length L in water of depth h satisfies the equation $v^2 = \frac{gL}{2\pi}\tanh\frac{2\pi h}{L}$. Show that $v \approx \sqrt{gh}$.

9. The energy density of electromagnetic radiation at wavelength λ from a black body at temperature T (degrees Kelvin) is given by **Planck's law** of black body radiation: $f(\lambda) = \frac{8\pi hc}{\lambda^5(e^{hc/\lambda kT} - 1)}$, where h is Planck's constant, c is the speed of light and k is Boltzmann's constant. To find the wavelength of peak emission, maximize $f(\lambda)$ by minimizing $g(\lambda) = \lambda^5(e^{hc/\lambda kT} - 1)$. Use a Taylor polynomial for e^x with $n = 7$ to expand the expression in parentheses and find the critical number of the resulting function. (Hint: Use $\frac{hc}{k} \approx 0.014$.) Compare this to **Wien's law**: $\lambda_{max} = \frac{0.002898}{T}$. Wien's law is accurate for small λ. Discuss the flaw in our use of Maclaurin series.

[1] Long, L. N. and Weiss, H. (1999). The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians. *The American Mathematical Monthly*, 106(2), 127–135.

10. Use a Taylor polynomial for e^x to expand the denominator in Planck's law of exercise 9 and show that $f(\lambda) \approx \frac{8\pi kT}{\lambda^4}$. State whether this approximation is better for small or large wavelengths λ. This is known in physics as the **Rayleigh–Jeans law**.

11. The power of a reflecting telescope is proportional to the surface area S of the parabolic reflector, where $S = \frac{8\pi}{3}c^2\left[\left(\frac{d^2}{16c^2}+1\right)^{3/2} - 1\right]$. Here, d is the diameter of the parabolic reflector, which has depth k with $c = \frac{d^2}{4k}$. Expand the term $\left(\frac{d^2}{16c^2}+1\right)^{3/2}$ and show that if $\frac{d^2}{16c^2}$ is small, then $S \approx \frac{\pi d^2}{4}$.

12. A disk of radius a has a charge of constant density σ. Point P lies at a distance r directly above the disk. The **electrical potential** at point P is given by $V = 2\pi\sigma(\sqrt{r^2+a^2} - r)$. Show that for large r, $V \approx \frac{\pi a^2 \sigma}{r}$.

EXPLORATORY EXERCISES

1. The Bessel functions and **Legendre polynomials** are examples of the so-called special functions. For non-negative integers n, the Legendre polynomials are defined by

$$P_n(x) = 2^{-n}\sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!}{(n-k)!k!(n-2k)!}x^{n-2k}.$$

Here, $[n/2]$ is the greatest integer less than or equal to $n/2$ (for example, $[1/2] = 0$ and $[2/2] = 1$). Show that $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$. Show that for these three functions,

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0, \quad \text{for } m \neq n.$$

This fact, which is true for all Legendre polynomials, is called the **orthogonality condition**. Orthogonal functions are commonly used to provide simple representations of complicated functions.

2. Suppose that p is an approximation of π with $|p - \pi| < 0.001$. Explain why p has two digits of accuracy and has a decimal expansion that starts $p = 3.14\ldots$. Use Taylor's Theorem to show that $p + \sin p$ has six digits of accuracy. In general, if p has n digits of accuracy, show that $p + \sin p$ has $3n$ digits of accuracy. Compare this to the accuracy of $p - \tan p$.

5.9 FOURIER SERIES

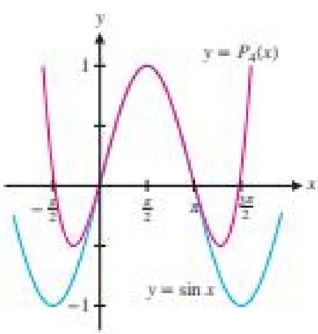


FIGURE 5.44
$y = \sin x$ and $y = P_4(x)$

Many phenomena we encounter in the world around us are periodic in nature. That is, they repeat themselves over and over again. For instance, light, sound, radio waves and x-rays are all periodic. For such phenomena, Taylor polynomial approximations have inherent shortcomings. Recall that as x gets farther away from c (the point about which we expanded), the difference between the function and a given Taylor polynomial grows. Such behavior is illustrated in Figure 5.44 for the case of $f(x) = \sin x$ expanded about $x = \frac{\pi}{2}$.

Because Taylor polynomials provide an accurate approximation only in the vicinity of c, we say that they are accurate *locally*. In many situations, notably in communications, we need to find an approximation to a given periodic function that is valid *globally* (i.e. for all x). For this reason, we construct a different type of series expansion for periodic functions, one where each of the terms in the expansion is periodic.

Recall that we say that a function f is **periodic** of **period** $T > 0$ if $f(x + T) = f(x)$, for all x in the domain of f. The most familiar periodic functions are $\sin x$ and $\cos x$, which are both periodic of period 2π. Further, $\sin(2x)$, $\cos(2x)$, $\sin(3x)$, $\cos(3x)$ and so on are all periodic of period 2π. In fact,

$$\sin(kx) \text{ and } \cos(kx), \qquad \text{for } k = 1, 2, 3, \ldots$$

are all periodic of period 2π, as follows. For any integer k, let $f(x) = \sin(kx)$. We then have

$$f(x + 2\pi) = \sin[k(x + 2\pi)] = \sin(kx + 2k\pi) = \sin(kx) = f(x).$$

Likewise, you can show that $\cos(kx)$ has period 2π.

So, if you wanted to expand a periodic function of period 2π in a series, you might consider a series each of whose terms has period 2π, for instance

FOURIER SERIES

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)].$$

Notice that if the series converges, it will converge to a periodic function of period 2π, since every term in the series has period 2π. The coefficients of the series, $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, are called the **Fourier coefficients**. You may have noticed the unusual way in which we wrote the leading term of the series $\left(\frac{a_0}{2}\right)$. We did this in order to simplify the formulas for computing these coefficients, as we'll see later.

There are a number of important questions we must address:

- What functions can be expanded in a Fourier series?
- How do we compute the Fourier coefficients?
- Does the Fourier series converge? If so, to what function does the series converge?

We begin our investigation much as we did with power series. Suppose that a given Fourier series converges on the interval $[-\pi, \pi]$. It then represents a function f on that interval,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)], \tag{9.1}$$

where f must be periodic outside of $[-\pi, \pi]$. Although some of the details of the proof are beyond the level of this course, we want to give you some idea of how the Fourier coefficients are computed. If we integrate both sides of equation (9.1) with respect to x on the interval $[-\pi, \pi]$, we get

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi}^{\pi} \frac{a_0}{2}dx + \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)]dx \\ &= \int_{-\pi}^{\pi} \frac{a_0}{2}dx + \sum_{k=1}^{\infty} \left[a_k \int_{-\pi}^{\pi} \cos(kx)\,dx + b_k \int_{-\pi}^{\pi} \sin(kx)\,dx\right], \end{aligned} \tag{9.2}$$

assuming we can interchange the order of integration and summation. (In general, the order may *not* be interchanged—this is beyond the level of this course—but for many Fourier series, doing so is permissible.) Observe that for every $k = 1, 2, 3, \ldots$, we have

$$\int_{-\pi}^{\pi} \cos(kx)\,dx = \frac{1}{k}\sin(kx)\Big|_{-\pi}^{\pi} = \frac{1}{k}[\sin(k\pi) - \sin(-k\pi)] = 0$$

and

$$\int_{-\pi}^{\pi} \sin(kx)\,dx = -\frac{1}{k}\cos(kx)\Big|_{-\pi}^{\pi} = -\frac{1}{k}[\cos(k\pi) - \cos(-k\pi)] = 0.$$

This reduces equation (9.2) to simply

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}dx = a_0\,\pi.$$

Solving this for a_0, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx. \tag{9.3}$$

HISTORICAL NOTES

Jean-Baptiste Joseph Fourier (1768–1830) French mathematician who invented Fourier series. Fourier was heavily involved in French politics, becoming a member of the Revolutionary Committee, serving as scientific advisor to Napoleon and establishing educational facilities in Egypt. Fourier held numerous offices, including secretary of the Cairo Institute and Prefect of Grenoble. Fourier introduced his trigonometric series as an essential technique for developing his highly original and revolutionary theory of heat.

Similarly, if we multiply both sides of equation (9.1) by $\cos(nx)$ (where n is an integer, $n \geq 1$) and then integrate with respect to x on the interval $[-\pi, \pi]$, we get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx &= \int_{-\pi}^{\pi} \frac{a_0}{2}\cos(nx)\,dx \\ &\quad + \int_{-\pi}^{\pi} \sum_{k=1}^{\infty}[a_k\cos(kx)\cos(nx) + b_k\sin(kx)\cos(nx)]\,dx \\ &= \frac{a_0}{2}\int_{-\pi}^{\pi}\cos(nx)dx \\ &\quad + \sum_{k=1}^{\infty}\left[a_k\int_{-\pi}^{\pi}\cos(kx)\cos(nx)dx + b_k\int_{-\pi}^{\pi}\sin(kx)\cos(nx)\,dx\right], \end{aligned} \tag{9.4}$$

again assuming we can interchange the order of integration and summation. Next, recall that

$$\int_{-\pi}^{\pi}\cos(nx)dx = 0, \qquad \text{for all } n = 1, 2, \cdots.$$

It's a straightforward, yet lengthy exercise to show that

$$\int_{-\pi}^{\pi}\sin(kx)\cos(nx)dx = 0, \qquad \text{for all } n = 1, 2, \ldots \text{ and for all } k = 1, 2, \ldots,$$

and that

$$\int_{-\pi}^{\pi}\cos(kx)\cos(nx)dx = \begin{cases} 0, & \text{if } n \neq k \\ \pi, & \text{if } n = k \end{cases}.$$

Notice that this says that every term in the series in equation (9.4) except one (the term corresponding to $k = n$) is zero. Equation (9.4) reduces to simply

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = a_n\pi.$$

This gives us (after substituting k for n)

Fourier coefficients

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx, \quad \text{for } k = 1, 2, 3, \cdots. \tag{9.5}$$

Similarly, multiplying both sides of equation (9.1) by $\sin(nx)$ and integrating from $-\pi$ to π leads us to

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx, \quad \text{for } k = 1, 2, 3, \cdots. \tag{9.6}$$

Equations (9.3), (9.5) and (9.6) are called the **Euler–Fourier formulas**. Notice that equation (9.3) is the same as (9.5) with $k = 0$. (This was the reason we chose the leading term of the series to be $\frac{a_0}{2}$, instead of simply a_0.)

To summarize what we've done so far, we have observed that if a Fourier series converges on the interval $[-\pi, \pi]$, then it converges to a function f where the Fourier coefficients satisfy the Euler–Fourier formulas (9.3), (9.5) and (9.6).

Certainly, given many functions f, we can compute the coefficients in (9.3), (9.5) and (9.6) and write down a Fourier series. But, will the series converge and, if it does, to what function will it converge? We'll answer these questions shortly. For the moment, we simply compute a Fourier series to see what we can observe.

EXAMPLE 9.1 Finding a Fourier Series Expansion

Find the Fourier series corresponding to the **square-wave** function

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x \le 0 \\ 1, & \text{if } 0 < x \le \pi \end{cases},$$

where f is assumed to be periodic outside of the interval $[-\pi, \pi]$. (See the graph in Figure 5.45.)

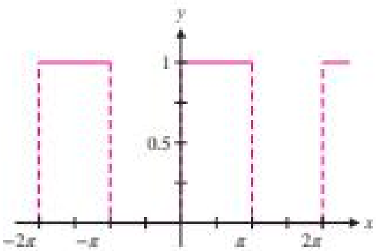


FIGURE 5.45
Square-wave function

Solution Even though a_0 satisfies the same formula as a_k for $k \ge 1$, we must always compute a_0 separately from the rest of the a_k's. From equation (9.3), we get

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = 0 + \frac{\pi}{\pi} = 1.$$

From (9.5), we also have that for $k \ge 1$,

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (0)\cos(kx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} (1)\cos(kx)\,dx \\ &= \frac{1}{\pi k}\sin(kx)\Big|_0^{\pi} = \frac{1}{\pi k}[\sin(k\pi) - \sin(0)] = 0. \end{aligned}$$

Finally, from (9.6), we have

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (0)\sin(kx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} (1)\sin(kx)\,dx \\ &= -\frac{1}{\pi k}\cos(kx)\Big|_0^{\pi} = -\frac{1}{\pi k}[\cos(k\pi) - \cos(0)] = -\frac{1}{\pi k}[(-1)^k - 1] \\ &= \begin{cases} 0, & \text{if } k \text{ is even} \\ \dfrac{2}{\pi k}, & \text{if } k \text{ is odd} \end{cases}. \end{aligned}$$

Notice that we can write the even- and odd-indexed coefficients separately as $b_{2k} = 0$, for $k = 1, 2, \ldots$ and $b_{2k-1} = \frac{2}{(2k-1)\pi}$, for $k = 1, 2, \ldots$. We then have the Fourier series

$$\begin{aligned} \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k\cos(kx) + b_k\sin(kx)] &= \frac{1}{2} + \sum_{k=1}^{\infty} b_k\sin(kx) = \frac{1}{2} + \sum_{k=1}^{\infty} b_{2k-1}\sin[(2k-1)x] \\ &= \frac{1}{2} + \sum_{k=1}^{\infty}\frac{2}{(2k-1)\pi}\sin[(2k-1)x] \\ &= \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin(3x) + \frac{2}{5\pi}\sin(5x) + \cdots. \end{aligned}$$

TODAY IN MATHEMATICS

Ingrid Daubechies (1954–Present) A Belgian physicist and mathematician who pioneered the use of wavelets, which extend the ideas of Fourier series. In a talk on the relationship between algorithms and analysis, she explained that her wavelet research was of a type "stimulated by the requirements of engineering design rather than natural science problems, but equally interesting and possibly far-reaching."* To meet the needs of an efficient image compression algorithm, she created the first continuous wavelet corresponding to a fast algorithm. The Daubechies wavelets are now the most commonly used wavelets in applications and were instrumental in the explosion of wavelet applications in areas as diverse as FBI fingerprinting, magnetic resonance imaging (MRI) and digital storage formats such as JPEG-2000.

*Daubechies I., *The interplay between analysis and algorithms*. Accessible at: http://jointmathematicsmeetings.org/meetings/national/jmm/1003-94-8.pdf

Unfortunately, none of our existing convergence tests apply to this series. Instead, we consider the graphs of the first few partial sums of the series defined by

$$F_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \frac{2}{(2k-1)\pi} \sin[(2k-1)x].$$

In Figures 5.46a–d, we graph a number of these partial sums.

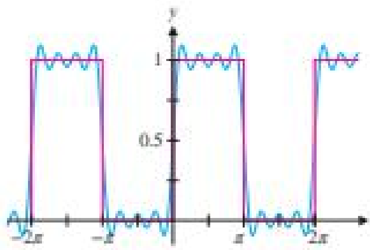


FIGURE 5.46a
$y = F_4(x)$ and $y = f(x)$

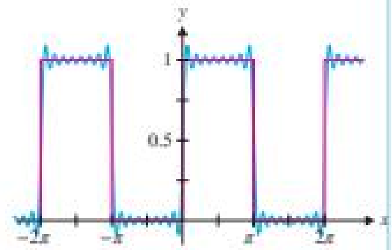


FIGURE 5.46b
$y = F_8(x)$ and $y = f(x)$

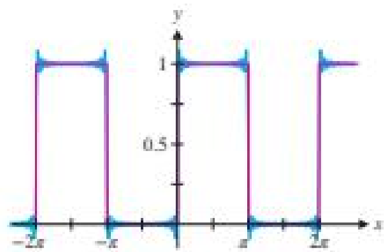


FIGURE 5.46c
$y = F_{20}(x)$ and $y = f(x)$

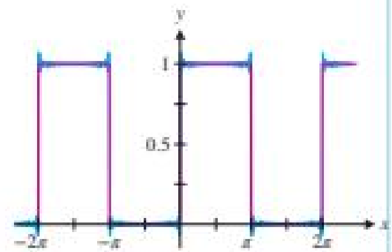


FIGURE 5.46d
$y = F_{50}(x)$ and $y = f(x)$

Notice that as n gets larger and larger, the graph of $F_n(x)$ appears to be approaching the graph of the square-wave function $f(x)$ shown in red and seen in Figure 5.45. Based on this, we might conjecture that the Fourier series converges to the function $f(x)$. As it turns out, this is not quite correct. We'll soon see that the series converges to $f(x)$ everywhere, *except* at points of discontinuity. ■

Next, we give an example of constructing a Fourier series for another common waveform.

EXAMPLE 9.2 A Fourier Series Expansion for the Triangular-Wave Function

Find the Fourier series expansion of $f(x) = |x|$, for $-\pi \le x \le \pi$, where f is assumed to be periodic, of period 2π, outside of the interval $[-\pi, \pi]$.

Solution In this case, f is the **triangular-wave** function graphed in Figure 5.47. From the Euler–Fourier formulas, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -x\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$
$$= -\frac{1}{\pi}\frac{x^2}{2}\bigg|_{-\pi}^{0} + \frac{1}{\pi}\frac{x^2}{2}\bigg|_{0}^{\pi} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Similarly, for each $k \ge 1$, we get

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(kx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (-x)\cos(kx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(kx)\,dx.$$

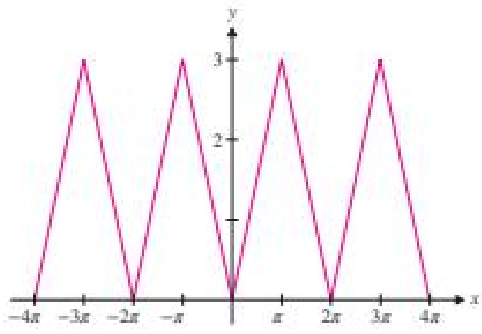


FIGURE 5.47
Triangular-wave function

Both integrals require the same integration by parts. We let

$$u = x \qquad dv = \cos(kx)\,dx$$
$$du = dx \qquad v = \frac{1}{k}\sin(kx)$$

so that

$$\begin{aligned}
a_k &= -\frac{1}{\pi}\int_{-\pi}^{0} x\cos(kx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(kx)\,dx \\
&= -\frac{1}{\pi}\left[\frac{x}{k}\sin(kx)\right]_{-\pi}^{0} + \frac{1}{\pi k}\int_{-\pi}^{0}\sin(kx)\,dx + \frac{1}{\pi}\left[\frac{x}{k}\sin(kx)\right]_{0}^{\pi} - \frac{1}{\pi k}\int_{0}^{\pi}\sin(kx)\,dx \\
&= -\frac{1}{\pi}\left[0 + \frac{\pi}{k}\sin(-\pi k)\right] - \frac{1}{\pi k^2}\cos(kx)\Big|_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{\pi}{k}\sin(\pi k) - 0\right] + \frac{1}{\pi k^2}\cos(kx)\Big|_{0}^{\pi} \\
&= 0 - \frac{1}{\pi k^2}[\cos 0 - \cos(-k\pi)] + 0 + \frac{1}{\pi k^2}[\cos(k\pi) - \cos 0] \\
&= \frac{2}{\pi k^2}[\cos(k\pi) - 1] = \begin{cases} 0, & \text{if } k \text{ is even} \\ \dfrac{-4}{\pi k^2}, & \text{if } k \text{ is odd} \end{cases}.
\end{aligned}$$

Since $\sin \pi k = 0$ and $\sin(-\pi k) = 0$.

Since $\cos(k\pi) = 1$ when k is even, and $\cos(k\pi) = -1$ when k is odd.

Writing the even- and odd-indexed coefficients separately, we have $a_{2k} = 0$, for $k = 1, 2, \ldots$ and $a_{2k-1} = \dfrac{-4}{\pi(2k-1)^2}$, for $k = 1, 2, \ldots$. We leave it as an exercise to show that

$$b_k = 0, \qquad \text{for all } k.$$

This gives us the Fourier series

$$\begin{aligned}
\frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k\cos(kx) + b_k\sin(kx)] &= \frac{\pi}{2} + \sum_{k=1}^{\infty} a_k\cos(kx) = \frac{\pi}{2} + \sum_{k=1}^{\infty} a_{2k-1}\cos[(2k-1)x] \\
&= \frac{\pi}{2} - \sum_{k=1}^{\infty}\frac{4}{\pi(2k-1)^2}\cos[(2k-1)x] \\
&= \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos(3x) - \frac{4}{25\pi}\cos(5x) - \cdots.
\end{aligned}$$

You can show that this series converges absolutely for all x, by using the Comparison Test, since

$$|a_k| = \left|\frac{4}{\pi(2k-1)^2}\cos(2k-1)x\right| \le \frac{4}{\pi(2k-1)^2}$$

and the series $\sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2}$ converges. (Hint: Compare this last series to the convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, using the Limit Comparison Test.) To get an idea of the function to which the series converges, we plot several of the partial sums of the series,

$$F_n(x) = \frac{\pi}{2} - \sum_{k=1}^{n} \frac{4}{\pi(2k-1)^2} \cos\left[(2k-1)x\right].$$

See if you can conjecture the sum of the series by looking at Figures 5.48a–d. Notice how quickly the partial sums of the series appear to converge to the triangular-wave function f (shown in red; also see Figure 5.47). We'll see later that the Fourier series converges to $f(x)$ for all x. There's something further to note here: the accuracy of the approximation is fairly uniform. That is, the difference between a given partial sum and $f(x)$ is roughly the same for each x. Take care to distinguish this behavior from that of Taylor polynomial approximations, where the farther you get away from the point about which you've expanded, the worse the approximation tends to get.

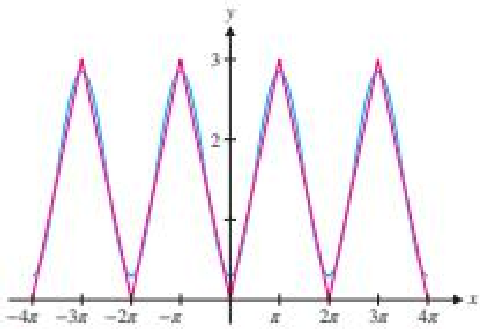


FIGURE 5.48a
$y = F_1(x)$ and $y = f(x)$

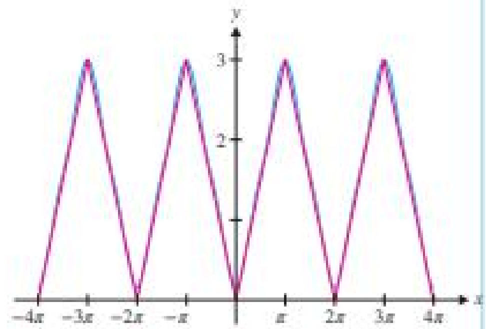


FIGURE 5.48b
$y = F_2(x)$ and $y = f(x)$

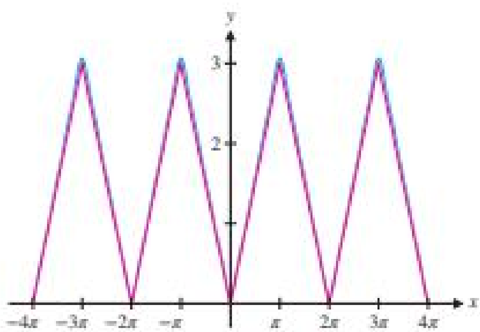


FIGURE 5.48c
$y = F_4(x)$ and $y = f(x)$

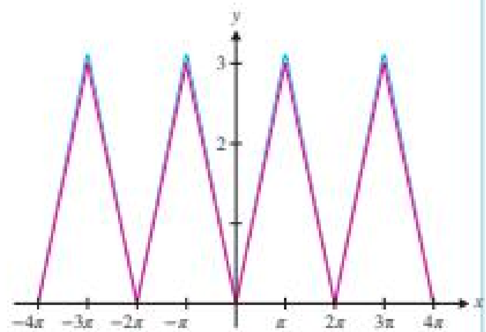


FIGURE 5.48d
$y = F_8(x)$ and $y = f(x)$

Functions of Period Other Than 2π

For a function f that is periodic of period $T \neq 2\pi$, we want to expand f in a series of simple functions of period T. First, define $l = \frac{T}{2}$ and notice that

$$\cos\left(\frac{k\pi x}{l}\right) \quad \text{and} \quad \sin\left(\frac{k\pi x}{l}\right)$$

are periodic of period $T = 2l$, for each $k = 1, 2, \ldots$. The Fourier series expansion of f of period $2l$ is then

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k \cos\left(\frac{k\pi x}{l}\right) + b_k \sin\left(\frac{k\pi x}{l}\right)\right].$$

We leave it as an exercise to show that the Fourier coefficients in this case are given by the Euler–Fourier formulas:

$$a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{k\pi x}{l}\right)dx, \text{ for } k = 0, 1, 2, \ldots \quad (9.7)$$

and

$$b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{k\pi x}{l}\right)dx, \text{ for } k = 1, 2, 3, \ldots. \quad (9.8)$$

Notice that (9.3), (9.5) and (9.6) are equivalent to (9.7) and (9.8) with $l = \pi$.

EXAMPLE 9.3 A Fourier Series Expansion for a Square-Wave Function

Find a Fourier series expansion for the function

$$f(x) = \begin{cases} -2, & \text{if } -1 < x \le 0 \\ 2, & \text{if } 0 < x \le 1 \end{cases},$$

where f is defined so that it is periodic of period 2 outside of the interval $[-1, 1]$.

Solution The graph of f is the square wave seen in Figure 5.49.

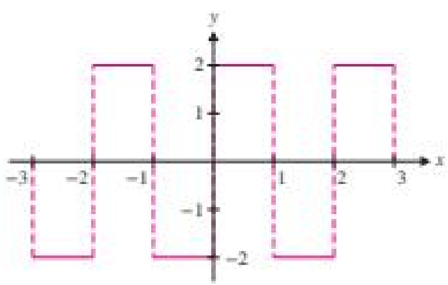


FIGURE 5.49
Square wave

From the Euler–Fourier formulas (9.7) and (9.8) with $l = 1$, we have

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0}(-2)\,dx + \int_{0}^{1} 2\,dx = 0.$$

Likewise, we get

$$a_k = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{k\pi x}{1}\right)dx = 0, \quad \text{for } k = 1, 2, 3, \ldots.$$

Finally, we have

$$\begin{aligned} b_k &= \frac{1}{1}\int_{-1}^{1} f(x)\sin\left(\frac{k\pi x}{1}\right)dx = \int_{-1}^{0}(-2)\sin(k\pi x)\,dx + \int_{0}^{1} 2\sin(k\pi x)\,dx \\ &= \frac{2}{k\pi}\cos(k\pi x)\Big|_{-1}^{0} - \frac{2}{k\pi}\cos(k\pi x)\Big|_{0}^{1} = \frac{4}{k\pi}[\cos 0 - \cos(k\pi)] \\ &= \frac{4}{k\pi}[1 - \cos(k\pi)] = \begin{cases} 0, & \text{if } k \text{ is even} \\ \dfrac{8}{k\pi}, & \text{if } k \text{ is odd} \end{cases}. \end{aligned}$$

Since $\cos(k\pi) = 1$ when k is even, and $\cos(k\pi) = -1$ when k is odd.

This gives us the Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k\cos(k\pi x) + b_k\sin(k\pi x)] = \sum_{k=1}^{\infty} b_k\sin(k\pi x) = \sum_{k=1}^{\infty} b_{2k-1}\sin[(2k-1)\pi x]$$

$$= \sum_{k=1}^{\infty}\frac{8}{(2k-1)\pi}\sin[(2k-1)\pi x].$$

Since $b_{2k} = 0$ and $b_{2k-1} = \frac{8}{(2k-1)\pi}$

Although we as yet have no tools for determining the convergence or divergence of this series, we graph a few of the partial sums of the series,

$$F_n(x) = \sum_{k=1}^{n}\frac{8}{(2k-1)\pi}\sin[(2k-1)\pi x]$$

in Figures 5.50a–d. From the graphs, it appears that the series is converging to the square-wave function f, except at the points of discontinuity, $x = 0, \pm 1, \pm 2, \pm 3, \ldots$. At those points, the series appears to converge to 0. You can easily verify this by observing that the terms of the series are

$$\frac{8}{(2k-1)\pi}\sin[(2k-1)\pi x] = 0, \quad \text{for integer values of } x.$$

Since each term in the series is zero, the series converges to 0 at all integer values of x. You might think of this as follows: at the points where f is discontinuous, the series converges to the average of the two function values on either side of the discontinuity. As we will see, this is typical of the convergence of Fourier series.

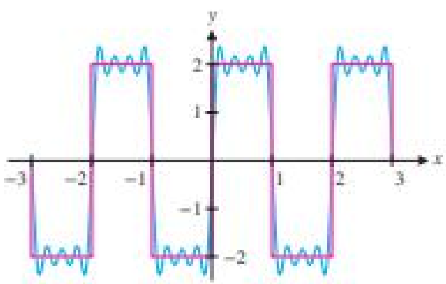


FIGURE 5.50a
$y = F_4(x)$ and $y = f(x)$

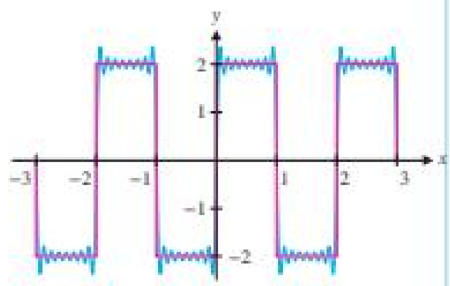


FIGURE 5.50b
$y = F_8(x)$ and $y = f(x)$

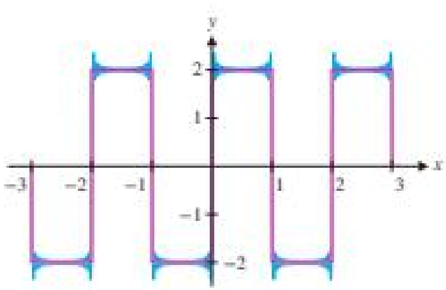


FIGURE 5.50c
$y = F_{20}(x)$ and $y = f(x)$

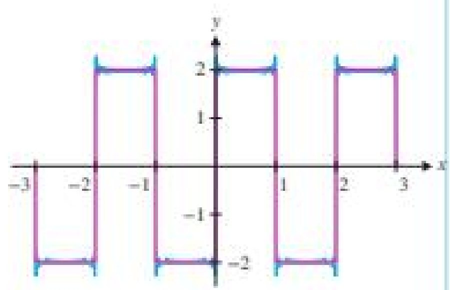


FIGURE 5.50d
$y = F_{50}(x)$ and $y = f(x)$

We now state the following major result on the convergence of Fourier series.

REMARK 9.1

The Fourier Convergence Theorem says that a Fourier series may converge to a discontinuous function, even though every term in the series is continuous (and differentiable) for all x.

THEOREM 9.1 (Fourier Convergence Theorem)

Suppose that f is periodic of period $2l$ and that f and f' are continuous on the interval $[-l, l]$, except for at most a finite number of jump discontinuities. Then, f has a convergent Fourier series expansion. Further, the series converges to $f(x)$, when f is continuous at x and to

$$\frac{1}{2}\left[\lim_{t\to x^+} f(t) + \lim_{t\to x^-} f(t)\right]$$

at any points x where f is discontinuous.

The proof of the theorem is beyond the level of this text and can be found in texts on advanced calculus or Fourier analysis.

EXAMPLE 9.4 Proving Convergence of a Fourier Series

Use the Fourier Convergence Theorem to prove that the Fourier series expansion of period 2π,

$$\frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi} \cos[(2k-1)x],$$

derived in example 9.2, for $f(x) = |x|$, for $-\pi \le x \le \pi$ and periodic outside of $[-\pi, \pi]$, converges to $f(x)$ everywhere.

Solution First, note that f is continuous everywhere. (See Figure 5.47.) We also have that since

$$f(x) = |x| = \begin{cases} -x, & \text{if } -\pi \le x < 0 \\ x, & \text{if } 0 \le x < \pi \end{cases}$$

and it is periodic outside $[-\pi, \pi]$, then

$$f'(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}.$$

So, f' is also continuous on $[-\pi, \pi]$, except for jump discontinuities at $x = 0$ and $x = \pm\pi$. From the Fourier Convergence Theorem, we now have that the Fourier series converges to $f(x)$ everywhere (since f is continuous everywhere). Given this, we write

$$f(x) = \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi} \cos(2k-1)x,$$

for all x. ■

As you can see from the Fourier Convergence Theorem, Fourier series do not always converge to the function you are expanding.

EXAMPLE 9.5 Investigating Convergence of a Fourier Series

Use the Fourier Convergence Theorem to investigate the convergence of the Fourier series

$$\sum_{k=1}^{\infty} \frac{8}{(2k-1)\pi} \sin[(2k-1)\pi x],$$

derived as an expansion of the square-wave function

$$f(x) = \begin{cases} -2, & \text{if } -1 < x \le 0 \\ 2, & \text{if } 0 < x \le 1 \end{cases},$$

where f is taken to be periodic outside of $[-1, 1]$. (See example 9.3.)

Solution First, note that f is continuous, except for jump discontinuities at $x = 0, \pm 1, \pm 2, \cdots$. Further,

$$f'(x) = \begin{cases} 0, & \text{if } -1 < x < 0 \\ 0, & \text{if } 0 < x < 1 \end{cases}$$

and it is periodic outside of $[-1, 1]$. Thus, f' is also continuous everywhere, except at integer values of x where f' is undefined. From the Fourier Convergence Theorem, the Fourier series will converge to $f(x)$ everywhere, except at the discontinuities, $x = 0, \pm 1, \pm 2, \ldots$, where the series converges to the average of the one-sided limits, that is, 0. A graph of the function to which the series converges is shown in Figure 5.51. Since the series does not converge to f everywhere, we cannot say that the function and the series are *equal*. In this case, we usually write

$$f(x) \sim \sum_{k=1}^{\infty} \frac{8}{(2k-1)\pi} \sin\left[(2k-1)\pi x\right]$$

to indicate that the series *corresponds* to f (but is not necessarily equal to f). In the context of Fourier series, this says that the series converges to $f(x)$ at every x where f is continuous and to the average of the one-sided limits at any jump discontinuities. Notice that this is the behavior seen in the graphs of the partial sums of the series seen in Figures 5.50a–d.

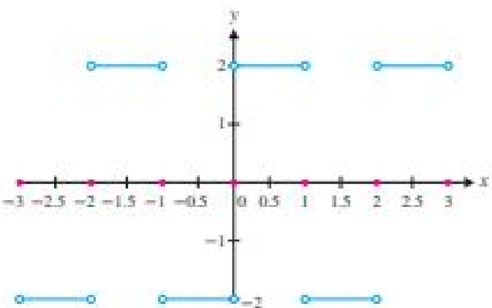


FIGURE 5.51

$$\sum_{k=1}^{\infty} \frac{8}{(2k-1)\pi} \sin\left[(2k-1)\pi x\right]$$

○ Fourier Series and Music Synthesizers

Fourier series are widely used in such field as engineering, physics and chemistry. We give you a sense of how they are used with the following brief discussion of music synthesizers and a variety of exercises.

A pure tone can be modeled by $A \sin \omega t$, where the amplitude A determines the volume and the frequency ω determines the pitch. For example, for a music synthesizer to mimic a saxophone, it must match the saxophone's characteristic waveform. (See Figure 5.52.) The shape of the waveform affects the **timbre** of the tone, a quality most humans readily discern. (A saxophone *sounds* different than a trumpet, doesn't it?)

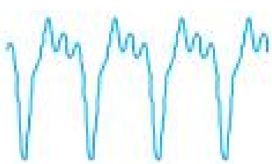


FIGURE 5.52
Saxophone waveform

Given a waveform such as the one shown in Figure 5.52, can you add together several pure tones of the form $A \sin \omega t$ to approximate the waveform? Note that if the pure tones are of the form $b_1 \sin t$, $b_2 \sin 2t$, $b_3 \sin 3t$, and so on, this is essentially a Fourier series problem. That is, we want to approximate a given wave function $f(t)$ by a sum of these pure tones, as follows:

$$f(t) \approx b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \cdots + b_n \sin nt.$$

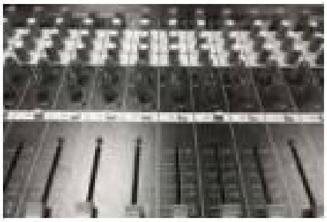


FIGURE 5.53
A graphic equalizer
KRIT GONNGON/Shutterstock

Although the cosine terms are all missing, notice that this is the partial sum of a Fourier series. (Such series are called **Fourier sine series** and are explored in the exercises.) For music synthesizers, the Fourier coefficients are simply the amplitudes of the various harmonics in a given waveform. In this context, you can think of the bass and treble controls on a stereo as manipulating the amplitudes of different terms in a Fourier series. Cranking up the bass emphasizes low-frequency terms (i.e. increases the coefficients of the first few terms of the Fourier series), while turning up the treble emphasizes the high-frequency terms. An equalizer (see Figure 5.53) gives you more direct control of individual frequencies.

In general, the idea of analyzing a wave phenomenon by breaking the wave down into its component frequencies is essential to much of modern science and engineering. This type of *spectral analysis* is used in numerous scientific disciplines.

BEYOND FORMULAS

Fourier series provide an alternative to power series for representing functions; which representation is more useful depends on the specifics of the problem you are solving. Fourier series and their extensions (including wavelets) are used to represent wave phenomena such as sight and sound. In our digital age, such applications are everywhere.

EXERCISES 5.9

WRITING EXERCISES

1. Explain why the Fourier series of $f(x) = 1 + 3\cos x - \sin 2x$ on the interval $[-\pi, \pi]$ is simply $1 + 3\cos x - \sin 2x$. (Hint: Explain what the goal of a Fourier series representation is and note that in this case no work needs to be done.) Would this change if the interval were $[-1, 1]$ instead?

2. Polynomials are built up from the basic operations of arithmetic. We often use Taylor series to rewrite an awkward function (e.g., $\sin x$) into arithmetic form. Many natural phenomena are waves, which are well modeled by sines and cosines. Discuss the extent to which the following statement is true: Fourier series allow us to rewrite algebraic functions (e.g. x^2) into a natural (wave) form.

3. Theorem 9.1 states that a Fourier series may converge to a function with jump discontinuities. In examples 9.1 and 9.3, identify the locations of the jump discontinuities and the values to which the Fourier series converges at these points. In what way are these values reasonable compromises?

4. Carefully examine Figures 5.46 and 5.50. For which x's does the Fourier series seem to converge rapidly? Slowly? Note that for every n, the partial sum $F_n(x)$ passes *exactly* through the limiting point for jump discontinuities. Describe the behavior of the partial sums *near* the jump discontinuities. This overshoot/undershoot behavior is referred to as the **Gibbs phenomenon**.

In exercises 1–8, find the Fourier series of the function on the interval $[-\pi, \pi]$. Graph the function and the partial sums $F_4(x)$ and $F_8(x)$ on the interval $[-2\pi, 2\pi]$.

1. $f(x) = x$
2. $f(x) = x^2$
3. $f(x) = 2|x|$
4. $f(x) = 3x$
5. $f(x) = \begin{cases} 1, & \text{if } -\pi < x < 0 \\ -1, & \text{if } 0 < x < \pi \end{cases}$
6. $f(x) = \begin{cases} 1, & \text{if } -\pi < x < 0 \\ 0, & \text{if } 0 < x < \pi \end{cases}$
7. $f(x) = 3\sin 2x$
8. $f(x) = 2\sin 3x$

In exercises 9–14, find the Fourier series of the function on the given interval.

9. $f(x) = -x$, $[-1, 1]$
10. $f(x) = |x|$, $[-1, 1]$
11. $f(x) = x^2$, $[-1, 1]$
12. $f(x) = 3x$, $[-2, 2]$
13. $f(x) = \begin{cases} 0, & \text{if } -1 < x < 0 \\ x, & \text{if } 0 < x < 1 \end{cases}$
14. $f(x) = \begin{cases} 0, & \text{if } -1 < x < 0 \\ 1 - x, & \text{if } 0 < x < 1 \end{cases}$

In exercises 15–20, do not compute the Fourier series, but graph the function to which the Fourier series converges, showing at least three full periods.

15. $f(x) = x$, $[-2, 2]$
16. $f(x) = x^2$, $[-3, 3]$
17. $f(x) = \begin{cases} -x, & \text{if } -1 < x < 0 \\ 0, & \text{if } 0 < x < 1 \end{cases}$
18. $f(x) = \begin{cases} 1, & \text{if } -2 < x < 0 \\ 3, & \text{if } 0 < x < 2 \end{cases}$
19. $f(x) = \begin{cases} -1, & \text{if } -2 < x < -1 \\ 0, & \text{if } -1 < x < 1 \\ 1, & \text{if } 1 < x < 2 \end{cases}$

20. $f(x) = \begin{cases} 2, & \text{if } -2 < x < -1 \\ -2, & \text{if } -1 < x < 1 \\ 0, & \text{if } 1 < x < 2 \end{cases}$

21. Substitute $x = 1$ into the Fourier series formula of exercise 11 to prove that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

22. Use the Fourier series of example 9.1 to prove that $\sum_{k=1}^{\infty} \frac{\sin(2k-1)}{2k-1} = \frac{\pi}{4}$.

23. Use the Fourier series of example 9.2 to prove that $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$.

24. Combine the results of exercises 21 and 23 to find $\sum_{k=1}^{\infty} \frac{1}{(2k)^2}$.

In exercises 25–28, use the Fourier Convergence Theorem to investigate the convergence of the Fourier series in the given exercise.

25. exercise 5

26. exercise 7

27. exercise 9

28. exercise 17

29. You have undoubtedly noticed that many Fourier series consist of only cosine or only sine terms. This can be easily understood in terms of even and odd functions. A function f is **even** if $f(-x) = f(x)$ for all x. A function is **odd** if $f(-x) = -f(x)$ for all x. Show that $\cos x$ is even, $\sin x$ is odd and $\cos x + \sin x$ is neither.

30. If f is even, show that $g(x) = f(x) \cos x$ is even and $h(x) = f(x) \sin x$ is odd.

31. If f is odd, show that $g(x) = f(x) \cos x$ is odd and $h(x) = f(x) \sin x$ is even. If f and g are even, what can you say about $f\,g$?

32. If f is even and g is odd, what can you say about $f\,g$? If f and g are odd, what can you say about $f\,g$?

33. Prove the general Euler–Fourier formulas (9.7) and (9.8).

34. If g is an odd function (see exercise 29), show that $\int_{-l}^{l} g(x)\,dx = 0$ for any (positive) constant l. [Hint: Compare $\int_{-l}^{0} g(x)\,dx$ and $\int_{0}^{l} g(x)\,dx$. You will need to make the change of variable $t = -x$ in one of the integrals.] Using the results of exercise 30, show that if f is even, then $b_k = 0$ for all k and the Fourier series of $f(x)$ consists only of a constant and cosine terms. If f is odd, show that $a_k = 0$ for all k and the Fourier series of $f(x)$ consists only of sine terms.

In exercises 35–38, use the even/odd properties of $f(x)$ to predict (don't compute) whether the Fourier series will contain only cosine terms, only sine terms or both.

35. $f(x) = x^3$

36. $f(x) = x^4$

37. $f(x) = e^x$

38. $f(x) = |x|$

39. The function $f(x) = \begin{cases} -1, & \text{if } -2 < x < 0 \\ 3, & \text{if } 0 < x < 2 \end{cases}$ is neither even nor odd, but can be written as $f(x) = g(x) + 1$ where $g(x) = \begin{cases} -2, & \text{if } -2 < x < 0 \\ 2, & \text{if } 0 < x < 2 \end{cases}$. Explain why the Fourier series of $f(x)$ will contain sine terms and the constant 1, but no cosine terms.

40. Suppose that you want to find the Fourier series of $f(x) = x + x^2$. Explain why to find b_k you would need only to integrate $x \sin\left(\frac{k\pi x}{l}\right)$ and to find a_k you would need only to integrate $x^2 \cos\left(\frac{k\pi x}{l}\right)$.

APPLICATIONS

Exercises 1 and 2 are adapted from the owner's manual of a high-end music synthesizer.

1. A fundamental choice to be made when generating a new tone on a music synthesizer is the waveform. The options are sawtooth, square and pulse. You worked with the sawtooth wave in exercise 9. Graph the limiting function for the function in exercise 9 on the interval $[-4, 4]$. Explain why "sawtooth" is a good name. A square wave is shown in Figure 5.49. A pulse wave of period 2 with width $1/n$ is generated by $f(x) = \begin{cases} -2, & \text{if } 1/n < |x| < 1 \\ 2, & \text{if } |x| \le 1/n \end{cases}$. Graph pulse waves of width 1/3 and 1/4 on the interval $[-4, 4]$.

2. The **harmonic content** of a wave equals the ratio of integral harmonic waves to the fundamental wave. To understand what this means, write the Fourier series of exercise 9 as $\frac{2}{\pi}(-\sin \pi x + \frac{1}{2}\sin 2\pi x - \frac{1}{3}\sin 3\pi x + \frac{1}{4}\sin 4\pi x - \cdots)$. The harmonic content of the sawtooth wave is $\frac{1}{n}$. Explain how this relates to the relative sizes of the Fourier coefficients. The harmonic content of the square wave is $\frac{1}{n}$ with even-numbered harmonics missing. Compare this description to the Fourier series of example 9.3. The harmonic content of the pulse wave of width $\frac{1}{3}$ is $\frac{1}{n}$ with every third harmonic missing. Without computing the Fourier coefficients, write out the general form of the Fourier series of $f(x) = \begin{cases} -2, & \text{if } 1/3 < |x| < 1 \\ 2, & \text{if } |x| \le 1/3 \end{cases}$.

3. Piano tuning is relatively simple, due to the phenomenon studied in this exercise. Compare the graphs of $\sin 8t + \sin 8.2t$ and $2 \sin 8t$. Note especially that the amplitude of $\sin 8t + \sin 8.2t$ appears to slowly rise and fall. In the trigonometric identity $\sin 8t + \sin 8.2t = [2\cos(0.2t)]\sin(8.1t)$, think of $2\cos(0.2t)$ as the amplitude of $\sin(8.1t)$ and explain why the amplitude varies slowly. Piano tuners often start by striking a tuning fork of a certain pitch (e.g., $\sin 8t$) and then striking the corresponding piano note. If the piano is slightly out-of-tune (e.g., $\sin 8.2t$), the tuning fork plus piano produces a combined tone that noticeably increases and decreases in volume. Use your graph to explain why this occurs.

4. The function $\sin 8\pi t$ represents a 4-Hz signal (1 Hz equals 1 cycle per second) if t is measured in seconds. If you received this signal, your task might be to take your measurements of the signal and try to reconstruct the function. For example, if you measured three samples per second, you would have the data $f(0) = 0, f(1/3) = \sqrt{3}/2, f(2/3) = -\sqrt{3}/2$ and $f(1) = 0$. Knowing the signal is of the form $A \sin Bt$, you would use the data to try to solve for A and B. In this case, you don't have enough information to guarantee getting the right values for A and B. Prove this by finding several values of A and B with $B \neq 8\pi$ that match the data. A famous result of H. Nyquist from 1928 states that to reconstruct a signal of frequency f you need at least $2f$ samples.

5. The energy of a signal $f(x)$ on the interval $[-\pi, \pi]$ is defined by $E = \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx$. If $f(x)$ has a Fourier series $f(x) = \frac{a_0}{2} + \sum_{k=0}^{\infty} [a_k \cos kx + b_k \sin kx]$, show that $E = A_0^2 + A_1^2 + A_2^2 + \cdots$, where $A_k = \sqrt{a_k^2 + b_k^2}$. The sequence $\{A_k\}$ is called the **energy spectrum** of $f(x)$.

6. Carefully examine the graphs in Figure 5.46. There is a Gibbs phenomenon at $x = 0$. Does it appear that the size of the Gibbs overshoot changes as the number of terms increases? We examine that here. For the partial sum $F_n(x)$ as defined in example 9.1, it can be shown that the absolute maximum occurs at $\frac{\pi}{2n}$. Evaluate $F_n\left(\frac{\pi}{2n}\right)$ for $n = 4$, $n = 6$ and $n = 8$. Show that for large n, the size of the bump is $\left|F_n\left(\frac{\pi}{2n}\right) - f\left(\frac{\pi}{2n}\right)\right| \approx 0.09$. Gibbs showed that, in general, the size of the bump at a jump discontinuity is about 0.09 times the size of the jump.

7. Some fixes have been devised to reduce the Gibbs phenomenon. Define the **σ-factors** by $\sigma_k = \dfrac{\sin\left(\frac{k\pi}{n}\right)}{\frac{k\pi}{n}}$ for $k = 1, 2, \ldots, n$ and consider the modified Fourier sum $\frac{a_0}{2} + \sum_{k=0}^{n} [a_k \sigma_k \cos kx + b_k \sigma_k \sin kx]$. For example 9.1, plot the modified sums for $n = 4$ and $n = 8$ and compare to Figure 5.46: $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 < x < \pi \end{cases}$. F_{2n-1} has critical point at $\pi/2n$ and $\lim_{n\to\infty} F_{2n-1}(\frac{\pi}{2n}) = \frac{2}{\pi}\int_0^{\pi} \frac{\sin x}{x} dx \approx 1.18$.

EXPLORATORY EXERCISES

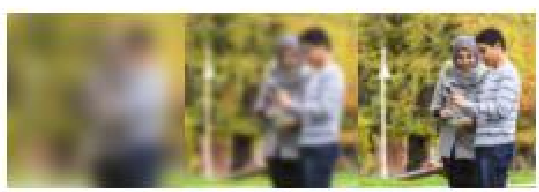
1. Suppose that you wanted to approximate a waveform with sine functions (no cosines), as in the music synthesizer problem. Such a **Fourier sine series** will be derived in this exercise. You essentially use Fourier series with a trick to guarantee sine terms only. Start with your waveform as a function defined on the interval $[0, l]$, for some length l. Then define a function $g(x)$ that equals $f(x)$ on $[0, l]$ *and* that is an odd function. Show that $g(x) = \begin{cases} f(x), & \text{if } 0 \le x \le l \\ -f(-x), & \text{if } -l < x < 0 \end{cases}$ works. Explain why the Fourier series expansion of $g(x)$ on $[-l, l]$ would contain sine terms only. This series is the sine series expansion of $f(x)$. Show the following helpful shortcut: the sine series coefficients are

$$b_k = \frac{1}{l}\int_{-l}^{l} g(x)\sin\left(\frac{k\pi}{l}\right)dx = \frac{2}{l}\int_0^{l} f(x)\sin\left(\frac{k\pi}{l}\right)dx.$$

Then compute the sine series expansion of $f(x) = x^2$ on $[0, 1]$ and graph the limit function on $[-3, 3]$. Analogous to the above, develop a **Fourier cosine series** and find the cosine series expansion of $f(x) = x$ on $[0, 1]$.

2. Fourier series are a part of the field of **Fourier analysis**, which is central to many engineering applications. Fourier analysis includes the Fourier transforms (and the FFT or Fast Fourier Transform) and inverse Fourier transforms, to which you will get a brief introduction in this exercise. Given measurements of a signal (waveform), the goal is to construct the Fourier series of a function. To start with a simple version of the problem, suppose the signal has the form $f(x) = \frac{a_0}{2} + a_1\cos \pi x + a_2\cos 2\pi x + b_1\sin \pi x + b_2\sin 2\pi x$ and you have the measurements $f(-1) = 0, f(-\frac{1}{2}) = 1, f(0) = 2, f(\frac{1}{2}) = 1$ and $f(1) = 0$. Substituting into the general equation for $f(x)$, show that $f(-1) = \frac{a_0}{2} - a_1 + a_2 = 0$. Similarly, $\frac{a_0}{2} - a_2 - b_1 = 1$, $\frac{a_0}{2} + a_1 + a_2 = 2$, $\frac{a_0}{2} - a_2 + b_1 = 1$, and $\frac{a_0}{2} - a_1 + a_2 = 0$. Note that the first and last equations are identical and that b_2 never appears in an equation. Thus, you have four equations and four unknowns. Solve the equations. You should conclude that $f(x) = 1 + \cos\pi x + b_2\sin\pi x$, with no information about b_2. Fortunately, there is an easier way of determining the Fourier coefficients. Recall that $a_k = \int_{-1}^{1} f(x)\cos k\pi x\, dx$ and $b_k = \int_{-1}^{1} f(x)\sin k\pi x\, dx$. You can estimate the integral using function values at $x = -1/2$, $x = 0$, $x = 1/2$ and $x = 1$. Find a version of a Riemann sum approximation that gives $a_0 = 2$, $a_1 = 1$, $a_2 = 0$ and $b_1 = 0$. What value is given for b_2? Use this Riemann sum rule to find the appropriate coefficients for the data $f(-\frac{3}{4}) = \frac{3}{4}, f(-\frac{1}{2}) = \frac{1}{2}, f(-\frac{1}{4}) = \frac{1}{4}, f(0) = 0, f(\frac{1}{4}) = -\frac{1}{4}$, $f(\frac{1}{2}) = -\frac{1}{2}, f(\frac{3}{4}) = -\frac{3}{4}$ and $f(1) = -1$. Compare to the Fourier series of exercise 9.

3. Fourier series have been used extensively in processing digital information, including digital photographs and music synthesis. A digital photograph stored in "bitmap" format can be thought of as three functions $f_R(x, y)$, $f_G(x, y)$ and $f_B(x, y)$. For example, $f_R(x, y)$ could be the amount of red content in the pixel that contains the point (x, y). Briefly explain what $f_G(x, y)$ and $f_B(x, y)$ would represent and how the three functions could be combined to create a color picture. A sine series for a function $f(x)$ on the interval $[0, L]$ is $\sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{L}\right)$ where $b_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx$. Describe what a sine series for a function $f(x, y)$ with $0 \le x \le L$ and $0 \le y \le M$ would look like. If possible, take your favorite photograph in bitmap format and write a program to find Fourier approximations. The accompanying images were created in this way. The first three images show Fourier approximations with 2, 10 and 50 terms, respectively. Notice that while the 50-term approximation is fairly sharp, there are some ripples (or "ghosts") outlining the two people; the ripples are more obvious in the 10-term image. Briefly explain how these ripples relate to the Gibbs phenomenon.



2 terms 10 terms 50 terms

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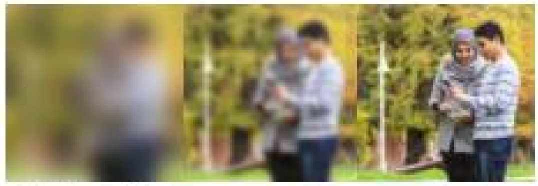
In Applications exercise 7 a σ-correction is introduced that reduces the Gibbs phenomenon. The next three images show the same picture using σ-corrected Fourier approximations with 2, 10 and 50 terms, respectively. Describe how the correction of the Gibbs phenomenon shows up in the images. Based on these images, how does the rate of convergence for Fourier series compare to σ-corrected Fourier series?



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Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Sequence	Limit of sequence	Squeeze Theorem
Infinite series	Partial sum	Series converges
Series diverges	Geometric series	kth-term test for divergence
Harmonic series	Integral Test	
Comparison Test	Limit Comparison Test	p-Series
Conditional convergence	Absolute convergence	Alternating Series Test
Ratio Test	Root Test	Alternating harmonic series
Radius of convergence	Taylor series	Power series
Taylor's Theorem	Fourier series	Taylor polynomial

TRUE OR FALSE

State whether each statement is true or false, and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. An increasing sequence diverges to infinity.
2. As n increases, $n!$ increases faster than 10^n.
3. If the sequence a_n diverges, then the series $\sum_{k=1}^{\infty} a_k$ diverges.
4. If a_k decreases to 0 as $k \to \infty$, then $\sum_{k=1}^{\infty} a_k$ diverges.
5. If $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{k=1}^{\infty} a_k$ converges for $a_k = f(k)$.
6. If the Comparison Test can be used to determine the convergence or divergence of a series, then the Limit Comparison Test can also determine the convergence or divergence of the series.
7. Using the Alternating Series Test, if $\lim_{k\to\infty} a_k \neq 0$, then you can conclude that $\sum_{k=1}^{\infty} a_k$ diverges.
8. The difference between a partial sum of a convergent series and its sum is less than the first neglected term in the series.
9. If a series is conditionally convergent, then the Ratio Test will be inconclusive.
10. A series with all negative terms cannot be conditionally convergent.
11. If $\sum_{k=1}^{\infty} |a_k|$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.
12. A series may be integrated term-by-term and the interval of convergence will remain the same.
13. A Taylor series of a function f is simply a power series representation of f.
14. The more terms in a Taylor polynomial, the better the approximation.
15. The Fourier series of x^2 converges to x^2 for all x.

In exercises 1–8, determine whether the sequence converges or diverges. If it converges, give the limit.

1. $a_n = \dfrac{4}{3+n}$
2. $a_n = \dfrac{3n}{1+n}$
3. $a_n = (-1)^n \dfrac{n}{n^2+4}$
4. $a_n = (-1)^n \dfrac{n}{n+4}$
5. $a_n = \dfrac{4^n}{n!}$
6. $a_n = \dfrac{n!}{n^n}$
7. $a_n = \cos \pi n$
8. $a_n = \dfrac{\cos n\pi}{n}$

In exercises 9–18, answer with "converges," "diverges" or "can't tell."

9. If $\lim_{k\to\infty} a_k = 1$, then $\sum_{k=1}^{\infty} a_k$ _____.
10. If $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ _____.

Review Exercises

11. If $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1$, then $\sum_{k=1}^{\infty} a_k$ ______.

12. If $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 0$, then $\sum_{k=1}^{\infty} a_k$ ______.

13. If $\lim_{k\to\infty} a_k = \frac{1}{2}$, then $\sum_{k=1}^{\infty} a_k$ ______.

14. If $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \frac{1}{2}$, then $\sum_{k=1}^{\infty} a_k$ ______.

15. If $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \frac{1}{2}$, then $\sum_{k=1}^{\infty} a_k$ ______.

16. If $\lim_{k\to\infty} k^2 a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ ______.

17. If $p > 1$, then $\sum_{k=1}^{\infty} \frac{8}{k^p}$ ______.

18. If $r > 1$, then $\sum_{k=1}^{\infty} ar^k$ ______.

In exercises 19–22, find the sum of the convergent series.

19. $\sum_{k=0}^{\infty} 4\left(\frac{1}{2}\right)^k$

20. $\sum_{k=1}^{\infty} \frac{4}{k(k+2)}$

21. $\sum_{k=0}^{\infty} 4^{-k}$

22. $\sum_{k=0}^{\infty} (-1)^k \frac{3}{4^k}$

In exercises 23 and 24, estimate the sum of the series to within 0.01.

23. $\sum_{k=0}^{\infty} (-1)^k \frac{k}{k^4+1}$

24. $\sum_{k=0}^{\infty} (-1)^{k+1} \frac{3}{k!}$

In exercises 25–44, determine whether the series converges or diverges.

25. $\sum_{k=0}^{\infty} \frac{2k}{k+3}$

26. $\sum_{k=0}^{\infty} (-1)^k \frac{2k}{k+3}$

27. $\sum_{k=0}^{\infty} (-1)^k \frac{4}{\sqrt{k+1}}$

28. $\sum_{k=0}^{\infty} \frac{4}{\sqrt{k+1}}$

29. $\sum_{k=1}^{\infty} 3k^{-7/8}$

30. $\sum_{k=1}^{\infty} 2k^{-8/7}$

31. $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^3+1}$

32. $\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^3+1}}$

33. $\sum_{k=1}^{\infty} (-1)^k \frac{4^k}{k!}$

34. $\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k}$

35. $\sum_{k=1}^{\infty} (-1)^k \ln\left(1+\frac{1}{k}\right)$

36. $\sum_{k=1}^{\infty} \frac{\cos k\pi}{\sqrt{k}}$

37. $\sum_{k=1}^{\infty} \frac{2}{(k+3)^2}$

38. $\sum_{k=2}^{\infty} \frac{4}{k \ln k}$

39. $\sum_{k=1}^{\infty} \frac{k!}{3^k}$

40. $\sum_{k=1}^{\infty} \frac{k}{3^k}$

41. $\sum_{k=1}^{\infty} \frac{e^{1/k}}{k^2}$

42. $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{\ln k + 1}}$

43. $\sum_{k=1}^{\infty} \frac{4^k}{(k!)^2}$

44. $\sum_{k=1}^{\infty} \frac{k^2+4}{k^3+3k+1}$

In exercises 45–48, determine whether the series converges absolutely, converges conditionally or diverges.

45. $\sum_{k=1}^{\infty} (-1)^k \frac{k}{k^2+1}$

46. $\sum_{k=1}^{\infty} (-1)^k \frac{3}{k+1}$

47. $\sum_{k=1}^{\infty} \frac{\sin k}{k^{3/2}}$

48. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3}{\ln k + 1}$

In exercises 49 and 50, find all values of p for which the series converges.

49. $\sum_{k=1}^{\infty} \frac{2}{(3+k)^p}$

50. $\sum_{k=1}^{\infty} e^{kp}$

In exercises 51 and 52, determine the number of terms necessary to estimate the sum of the series to within 10^{-6}.

51. $\sum_{k=1}^{\infty} (-1)^k \frac{3}{k^2}$

52. $\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k!}$

In exercises 53–56, find a power series representation for the function. Find the radius of convergence.

53. $\frac{1}{4+x}$

54. $\frac{2}{6-x}$

55. $\frac{3}{3+x^2}$

56. $\frac{2}{1+4x^2}$

In exercises 57 and 58, use the series from exercises 53 and 54 to find a power series and its radius of convergence.

57. $\ln(4+x)$

58. $\ln(6-x)$

In exercises 59–66, find the interval of convergence.

59. $\sum_{k=0}^{\infty} (-1)^k 2x^k$

60. $\sum_{k=0}^{\infty} (-1)^k (2x)^k$

61. $\sum_{k=1}^{\infty} (-1)^k \frac{2}{k} x^k$

62. $\sum_{k=1}^{\infty} \frac{-3}{\sqrt{k}}\left(\frac{x}{2}\right)^k$

Review Exercises

63. $\sum_{k=0}^{\infty} \frac{4}{k!}(x-2)^k$

64. $\sum_{k=0}^{\infty} k^2(x+3)^k$

65. $\sum_{k=0}^{\infty} 3^k(x-2)^k$

66. $\sum_{k=0}^{\infty} \frac{k}{4^k}(x+1)^k$

In exercises 67 and 68, derive the Taylor series of $f(x)$ about the center $x = c$.

67. $f(x) = \sin x,\ c = 0$

68. $f(x) = \frac{1}{x},\ c = 1$

In exercises 69 and 70, find the Taylor polynomial $P_4(x)$. Graph $f(x)$ and $P_4(x)$.

69. $f(x) = \ln x,\ c = 1$

70. $f(x) = \frac{1}{\sqrt{x}},\ c = 1$

In exercises 71 and 72, use the Taylor polynomials from exercises 69 and 70 to estimate the given values. Determine the order of the Taylor polynomial needed to estimate the value to within 10^{-8}.

71. $\ln 1.2$

72. $\frac{1}{\sqrt{1.1}}$

In exercises 73 and 74, use a known Taylor series to find a Taylor series of the function and find its radius of convergence.

73. e^{-3x^2}

74. $\sin 4x$

In exercises 75 and 76, use the first five non-zero terms of a known Taylor series to estimate the value of the integral.

75. $\int_0^1 \tan^{-1} x\,dx$

76. $\int_0^2 e^{-3x^2}\,dx$

In exercises 77 and 78, derive the Fourier series of the function.

77. $f(x) = x,\ -2 \le x \le 2$

78. $f(x) = \begin{cases} 0, & \text{if } -\pi < x \le 0 \\ 1, & \text{if } 0 < x \le \pi \end{cases}$

In exercises 79–82, graph at least three periods of the function to which the Fourier series converges.

79. $f(x) = x^2,\ -1 \le x \le 1$

80. $f(x) = 2x,\ -2 \le x \le 2$

81. $f(x) = \begin{cases} -1, & \text{if } -1 < x \le 0 \\ 1, & \text{if } 0 < x \le 1 \end{cases}$

82. $f(x) = \begin{cases} 0, & \text{if } -2 < x \le 0 \\ x, & \text{if } 0 < x \le 2 \end{cases}$

83. Suppose you and your friend take turns tossing a coin. The first one to get a head wins. Obviously, the person who goes first has an advantage, but how much of an advantage is it? If you go first, the probability that you win on your first toss is $\frac{1}{2}$, the probability that you win on your second toss is $\frac{1}{8}$, the probability that you win on your third toss is $\frac{1}{32}$ and so on. Sum a geometric series to find the probability that you win.

84. In a game similar to that of exercise 83, the first one to roll a 4 on a six-sided die wins. Is this game more fair than the previous game? The probabilities of winning on the first, second and third roll are $\frac{1}{6}$, $\frac{25}{216}$ and $\frac{625}{7776}$, respectively. Sum a geometric series to find the probability that you win.

85. Recall the Fibonacci sequence defined by $a_0 = 1$, $a_1 = 1$, $a_2 = 2$ and $a_{n+1} = a_n + a_{n-1}$. Prove the following fact: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1+\sqrt{5}}{2}$. (This number, known to the ancient Greeks, is called the **golden ratio**.) (Hint: Start with $a_{n+1} = a_n + a_{n-1}$ and divide by a_n. If $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, argue that $\lim_{n\to\infty} \frac{a_{n-1}}{a_n} = \frac{1}{r}$ and then solve the equation $r = 1 + \frac{1}{r}$.)

86. The Fibonacci sequence can be visualized with the following construction. Start with two side-by-side squares of side 1 (Figure A). Above them, draw a square (Figure B), which will have side 2. To the left of that, draw a square (Figure C), which will have side 3. Continue to spiral around, drawing squares that have sides given by the Fibonacci sequence. For each bounding rectangle in Figures A–C, compute the ratio of the sides of the rectangle. (Hint: Start with $\frac{2}{1}$ and then $\frac{3}{2}$.) Find the limit of the ratios as the construction process continues. The Greeks proclaimed this to be the most "pleasing" of all rectangles, building the Parthenon and other important buildings with these proportions.[2]

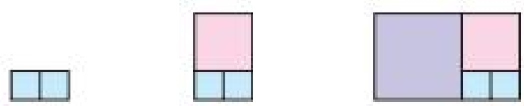

FIGURE A FIGURE B FIGURE C

87. Another type of sequence studied by mathematicians is the **continued fraction**. Numerically explore the sequence $1 + \frac{1}{1}$, $1 + \frac{1}{1+\frac{1}{1}}$, $1 + \frac{1}{1+\frac{1}{1+\frac{1}{1}}}$ and so on. Show that the limit L satisfies the equation $L = 1 + \frac{1}{L}$. Show that the limit equals the golden ratio! Viscount Brouncker, a seventeenth-century English mathematician, showed that the sequence $1 + \frac{1^2}{2}$, $1 + \frac{1^2}{2+\frac{3^2}{2}}$, $1 + \frac{1^2}{2+\frac{3^2}{2+\frac{5^2}{2}}}$ and so on, converges to $\frac{4}{\pi}$.[3] Explore this sequence numerically.

[2]See Huntley, H. E. (2012). *The Divine Proportion: A Study in Mathematical Beauty* (New York: Dover Publications).

[3]See Beckmann, P. (2015). *A History of pi* (New York: St. Martin's Griffin).

Review Exercises

88. For the power series $\frac{1}{1-x-x^2} = c_1 + c_2x + c_3x^2 + \cdots$, show that the constants c_i are the Fibonacci numbers. Substitute $x = \frac{1}{1000}$ to find the interesting decimal representation for $\frac{1{,}000{,}000}{998{,}999}$.

89. If $0 < r < \frac{1}{2}$, show that $1 + 2r + 4r^2 + \cdots + (2r)^n + \cdots = \frac{1}{1-2r}$. Replace r with $\frac{1}{1000}$ and discuss what's interesting about the decimal representation of $\frac{500}{499}$.

EXPLORATORY EXERCISES

1. The challenge here is to determine $\sum_{k=1}^{\infty} \frac{x^k}{k(k+1)}$ as completely as possible. Start by finding the interval of convergence. Find the sum for the special cases (a) $x = 0$ and (b) $x = 1$. For $0 < x < 1$, do the following: (c) Rewrite the series using the partial fractions expansion of $\frac{1}{k(k+1)}$. (d) Because the series converges absolutely, it is legal to rearrange terms. Do so and rewrite the series as $x + \frac{x-1}{x}\left[\frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots\right]$. (e) Identify the series in brackets as $\int\left(\sum_{k=1}^{\infty} x^k\right) dx$, evaluate the series and then integrate term-by-term. (f) Replace the term in brackets in part (d) with its value obtained in part (e). (g) The next case is for $-1 < x < 0$. Use the technique in parts (c)–(f) to find the sum. (h) Evaluate the sum at $x = -1$ using the fact that the alternating harmonic series sums to ln 2. (Used by permission of the Virginia Tech Mathematics Contest. Solution suggested by Gregory Minton.)

2. You have used Fourier series to show that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. Here, you will use a version of **Vièta's formula** to give an alternative derivation. Start by using a Maclaurin series for $\sin x$ to derive a series for $f(x) = \frac{\sin\sqrt{x}}{\sqrt{x}}$. Then find the zeros of $f(x)$. Vièta's formula states that the sum of the reciprocals of the zeros of $f(x)$ equals the negative of the coefficient of the linear term in the Maclaurin series of $f(x)$ divided by the constant term. Take this equation and multiply by π^2 to get the desired formula. Use the same method with a different function to show that $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$.

Parametric Equations and Polar Coordinates

CHAPTER

6

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Chapter Topics

6.1 Plane Curves and Parametric Equations
6.2 Calculus and Parametric Equations
6.3 Arc Length and Surface Area in Parametric Equations
6.4 Polar Coordinates
6.5 Calculus and Polar Coordinates
6.6 Conic Sections
6.7 Conic Sections in Polar Coordinates

You are all familiar with sonic booms, those loud crashes of noise caused by aircraft flying faster than the speed of sound. You may have even heard a sonic boom, but you have probably never *seen* one. The remarkable photograph here shows water vapor outlining the surface of a shock wave created by an F-18 jet flying supersonically. (Note that there is also a small cone of water vapor trailing the back of the cockpit.)

You may be surprised at the apparently conical shape, but a mathematical analysis verifies that the shape of the shock waves is indeed conical. (You will have an opportunity to explore this in the exercises in section 6.1.) To visualize how sound waves propagate, imagine an exploding firecracker. If you think of this in two dimensions, you'll recognize that the sound waves propagate in a series of ever-expanding concentric circles that reach everyone standing a given distance away from the firecracker at the same time.

In this chapter, we extend the concepts of calculus to curves described by parametric equations and polar coordinates. For instance, to describe the motion of an object such as an airplane in two dimensions, we would need to describe the object's position (x, y) as a function of the parameter t (time). That is, we write the position in the form $(x, y) = (x(t), y(t))$, where $x(t)$ and $y(t)$ are functions to which our existing techniques of calculus can be applied. The equations $x = x(t)$ and $y = y(t)$ are called *parametric equations.* Additionally, we'll explore how to use polar coordinates to represent curves, not as a set of points (x, y), but rather, by specifying the points by the distance from the origin to the point, together with an angle corresponding to the direction from the origin to the point. Polar coordinates are especially convenient for describing circles such as those that occur in propagating sound waves.

These alternative descriptions of curves bring us needed flexibility for attacking many problems. Often, even very complicated looking curves have a simple description in terms of parametric equations or polar coordinates.

6.1 PLANE CURVES AND PARAMETRIC EQUATIONS

We often find it convenient to describe the location of a point (x, y) in the plane in terms of a parameter. For instance, for a moving object, we would naturally give its location in terms of time. In this way, we not only specify the path the object follows, but we also know *when* it passes through each point.

Given any pair of functions $x(t)$ and $y(t)$ defined on the same domain D, the equations

$$x = x(t), \qquad y = y(t)$$

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are called **parametric equations**. Notice that for each choice of t, the parametric equations specify a point $(x, y) = (x(t), y(t))$ in the xy-plane. The collection of all such points is called the **graph** of the parametric equations. In the case where $x(t)$ and $y(t)$ are continuous functions and D is an interval of the real line, the graph is a curve in the xy-plane, referred to as a **plane curve**.

The choice of the letter t to denote the independent variable (called the **parameter**) should make you think of *time*, which is often what the parameter represents. In fact, you might recall that in section 5.5, we used a pair of equations of this type to describe two-dimensional projectile motion. In general, a parameter can be *any* quantity that is convenient for describing the relationship between x and y. In example 1.1, we simplify our discussion by eliminating the parameter.

EXAMPLE 1.1 Graphing a Plane Curve

Sketch the plane curve defined by the parametric equations $x = 6 - t^2$, $y = t/2$, for $-2 \le t \le 4$.

t	x	y
−2	2	−1
−1	5	$-\frac{1}{2}$
0	6	0
1	5	$\frac{1}{2}$
2	2	1
3	−3	$\frac{3}{2}$
4	−10	2

Solution In the accompanying table, we list a number of values of the parameter t and the corresponding values of x and y. We have plotted these points and connected them with a smooth curve in Figure 6.1. You might also notice that we can easily eliminate the parameter here, by solving for t in terms of y. We have $t = 2y$, so that $x = 6 - 4y^2$. The graph of this last equation is a parabola opening to the left. However, the plane curve we're looking for is the portion of this parabola corresponding to $-2 \le t \le 4$. From the table, notice that this corresponds to $-1 \le y \le 2$, so that the plane curve is the portion of the parabola indicated in Figure 6.1.

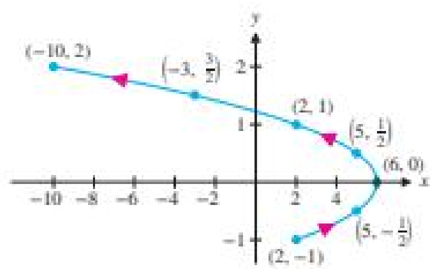

FIGURE 6.1
$x = 6 - t^2, y = \frac{t}{2}, -2 \le t \le 4$

You probably noticed the small arrows drawn on top of the plane curve in Figure 6.1. These indicate the **orientation** of the curve (i.e. the direction of increasing t). If t represents time and the curve represents the path of an object, the orientation indicates the direction followed by the object as it traverses the path, as in example 1.2.

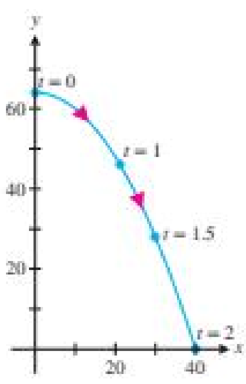

FIGURE 6.2
Path of projectile

EXAMPLE 1.2 The Path of a Projectile

Find the path of a projectile thrown horizontally with initial speed of 4.9 m/s from a height of 122.5 m.

Solution Following our discussion in section 5.5, the path is defined by the parametric equations

$$x = 4.9t, \qquad y = 122.5 - 4.9t^2, \qquad \text{for } 0 \le t \le 5,$$

where t represents time (in seconds). This describes the plane curve shown in Figure 6.2. Note that in this case, the orientation indicated in the graph gives the direction of motion. If we eliminate the parameter, as in example 1.1, the corresponding x-y equation

$y = 122.5 - 4.9\left(\frac{x}{4.9}\right)^2$ describes the path followed by the projectile. However, the parametric equations provide us with additional information, as they also tell us *when* the object is located at a given point and indicate the *direction* of motion. ■

Graphing calculators and computer algebra systems sketch a plane curve by plotting points corresponding to a large number of values of the parameter t and then connecting the plotted points with a curve. The appearance of the resulting graph depends greatly on the graphing window used *and also* on the particular choice of t-values. This can be seen in example 1.3.

EXAMPLE 1.3 Parametric Equations Involving Sines and Cosines

Sketch the plane curve defined by the parametric equations

$$x = 2\cos t, \quad y = 2\sin t, \quad \text{for (a) } 0 \le t \le 2\pi \text{ and (b) } 0 \le t \le \pi. \tag{1.1}$$

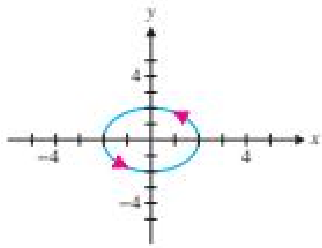

FIGURE 6.3a
$x = 2\cos t, y = 2\sin t$

Solution (a) The default graph produced by most graphing calculators looks something like the curve shown in Figure 6.3a (where we have added arrows indicating the orientation). We can improve this sketch by noticing that since $x = 2\cos t$, x ranges between -2 and 2. Similarly, y ranges between -2 and 2. Changing the graphing window to $-2.1 \le x \le 2.1$ and $-2.1 \le y \le 2.1$ produces the curve shown in Figure 6.3b. The curve looks like an ellipse, but with some thought we can identify it as a circle. Rather than eliminate the parameter by solving for t in terms of either x or y, instead notice from (1.1) that

$$x^2 + y^2 = 4\cos^2 t + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4.$$

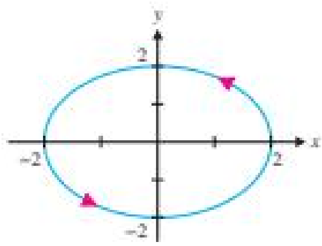

FIGURE 6.3b
$x = 2\cos t, y = 2\sin t$

So, the plane curve lies on the circle of radius 2 centered at the origin. It's not hard to see that this is the entire circle, traversed counterclockwise. (Simply plot some points, or better yet, recognize that t corresponds to the angle measured from the positive x-axis to the line segment joining (x, y) to the origin, noting that t ranges from 0 to 2π.) A "square" graphing window—one with the same scale on the x- and y-axes, though not necessarily the same x and y ranges—gives us the circle seen in Figure 6.3c.

(b) Since we've identified t as the angle as measured from the positive x-axis, limiting the domain to $0 \le t \le \pi$ will give the top half of the circle, as shown in Figure 6.3d.

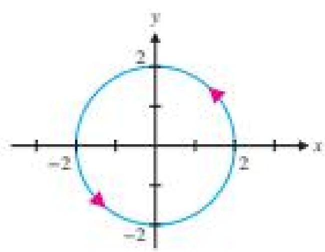

FIGURE 6.3c
A circle

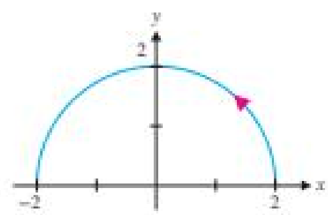

FIGURE 6.3d
Top semicircle

■

REMARK 1.1

To sketch a parametric graph on a CAS, you may need to write the equations in *vector format*. For instance, in the case of example 1.3, instead of entering $x = 2\cos t$ and $y = 2\sin t$, you would enter the ordered pair of functions $(2\cos t, 2\sin t)$.

Simple modifications to the parametric equations in example 1.3 will produce a variety of circles and ellipses. We explore this in example 1.4 and the exercises.

EXAMPLE 1.4 More Circles and Ellipses Defined by Parametric Equations

Identify the plane curves (a) $x = 2\sin t$, $y = 3\cos t$, (b) $x = 2 + 4\cos t$, $y = 3 + 4\sin t$ and (c) $x = 3\cos 2t$, $y = 3\sin 2t$, all for $0 \le t \le 2\pi$.

REMARK 1.2

Look carefully at the plane curves in examples 1.3 and 1.4 until you can identify the roles of each of the constants in the equations $x = a + b\cos ct$, $y(t) = d + e\sin ct$. These interpretations are important in applications.

Solution A computer-generated sketch of (a) is shown in Figure 6.4a. Observe that this is not a circle, since the parametric equations produce x-values between -2 and 2 and y-values between -3 and 3. To verify that this is an ellipse, observe that

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{4\sin^2 t}{4} + \frac{9\cos^2 t}{9} = \sin^2 t + \cos^2 t = 1.$$

A computer-generated sketch of (b) is shown in Figure 6.4b. You should verify that this is the circle $(x-2)^2 + (y-3)^2 = 16$, by eliminating the parameter. Finally, a computer sketch of (c) is shown in Figure 6.4c. You should verify that this is the circle $x^2 + y^2 = 9$, but what is the role of the 2 in the argument of cosine and sine? If you sketched this on a calculator, you may have noticed that the circle was completed long before the calculator finished graphing. Given the 2, a complete circle corresponds to $0 \le 2t \le 2\pi$ or $0 \le t \le \pi$. With the domain $0 \le t \le 2\pi$, the circle is traced out twice. You might say that the factor of 2 in the argument doubles the speed with which the curve is traced.

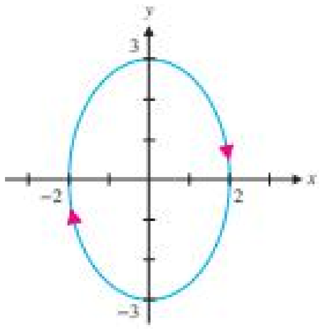

FIGURE 6.4a
$x = 2\sin t,\ y = 3\cos t$

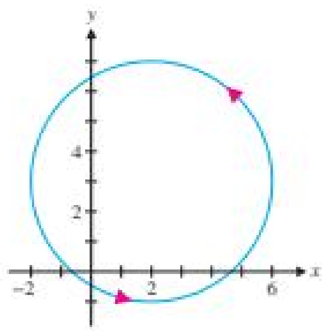

FIGURE 6.4b
$x = 2 + 4\cos t,\ y = 3 + 4\sin t$

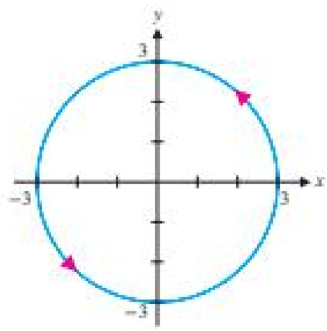

FIGURE 6.4c
$x = 3\cos 2t,\ y = 3\sin 2t$

In example 1.5, we see how to find parametric equations for a line segment.

REMARK 1.3

There are infinitely many choices of parameters that produce a given curve. For instance, you can verify that

$$x = -2 + 3t, \quad y = -3 + 5t, \quad \text{for } 1 \le t \le 2$$

and

$$x = t, \quad y = \frac{1 + 5t}{3}, \quad \text{for } 1 \le t \le 4$$

both produce the line segment from example 1.5. We say that each of these pairs of parametric equations is a different **parameterization** of the curve.

EXAMPLE 1.5 Parametric Equations for a Line Segment

Find parametric equations for the line segment joining the points (1, 2) and (4, 7).

Solution For a line segment, notice that the parametric equations can be chosen to be linear functions. That is,

$$x = a + bt, \qquad y = c + dt,$$

for some constants a, b, c and d. (Eliminate the parameter t to see why this generates a line.) The simplest way to choose these constants is to have $t = 0$ correspond to the starting point (1, 2). Note that if $t = 0$, the equations reduce to $x = a$ and $y = c$. To start our segment at $x = 1$ and $y = 2$, we set $a = 1$ and $c = 2$. Taking $t = 1$ to correspond to the endpoint (4, 7), we have $a + b = 4$ and $c + d = 7$. Since $a = 1$ and $c = 2$, we get $b = 3$ and $d = 5$, so that

$$x = 1 + 3t, \qquad y = 2 + 5t, \qquad \text{for } 0 \le t \le 1$$

is a pair of parametric equations describing the line segment. ■

In general, for parametric equations of the form $x = a + bt$, $y = c + dt$, notice that you can always choose a and c to be the x- and y-coordinates, respectively, of the starting point (since $x = a$, $y = c$ corresponds to $t = 0$). In this case, b is the difference in x-coordinates (endpoint minus starting point) and d is the difference in y-coordinates. With these choices, the line segment is always sketched out for $0 \le t \le 1$.

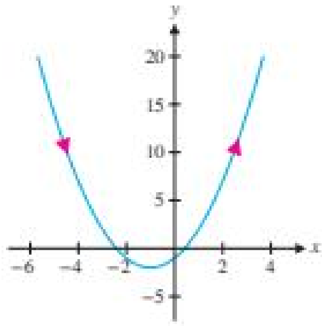

FIGURE 6.5a
$y = (x + 1)^2 - 2$

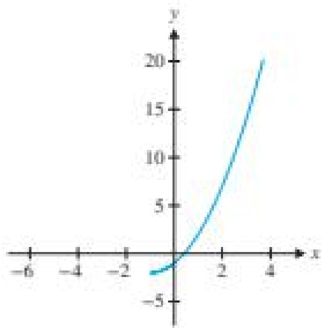

FIGURE 6.5b
$x = t^2 - 1, y = t^4 - 2$

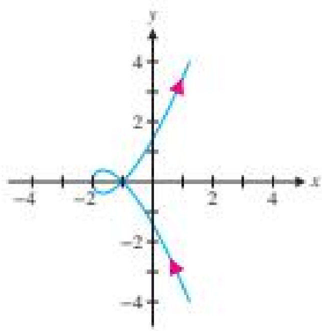

FIGURE 6.6a
$x = t^2 - 2, y = t^3 - t$

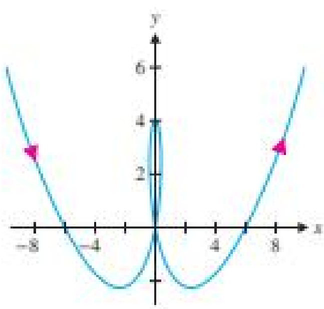

FIGURE 6.6b
$x = t^3 - t, y = t^4 - 5t^2 + 4$

As we illustrate in example 1.6, every equation of the form $y = f(x)$ can be simply expressed using parametric equations.

EXAMPLE 1.6 Parametric Equations from an *x-y* Equation

Find parametric equations for the portion of the parabola $y = x^2$ from $(-1, 1)$ to $(3, 9)$.

Solution Any equation of the form $y = f(x)$ can be converted to parametric form simply by defining $t = x$. Here, this gives us $y = x^2 = t^2$, so that

$$x = t, \quad y = t^2, \quad \text{for } -1 \le t \le 3,$$

is a parametric representation of the curve. ■

Besides indicating an orientation, parametric representations of curves often also carry with them a built-in restriction on the portion of the curve included, as we see in example 1.7.

EXAMPLE 1.7 Parametric Representations of a Curve with a Subtle Difference

Sketch the plane curves (a) $x = t - 1$, $y = t^2 - 2$ and (b) $x = t^2 - 1$, $y = t^4 - 2$.

Solution Since there is no restriction placed on t, we can assume that t can be any real number. Eliminating the parameter in (a), we get $t = x + 1$, so that the parametric equations in (a) correspond to the parabola $y = (x + 1)^2 - 2$, shown in Figure 6.5a. Notice that the graph includes the entire parabola, since t and hence, $x = t - 1$ can be any real number. (If your calculator sketch doesn't show both sides of the parabola, adjust the range of t-values in the plot.) The importance of this check is shown by (b). When we eliminate the parameter, we get $t^2 = x + 1$ and so, $y = (x + 1)^2 - 2$. This gives the same parabola as in (a). However, the initial computer sketch of the parametric equations shown in Figure 6.5b shows only the right half of the parabola. To verify that this is correct, note that since $x = t^2 - 1$, we have that $x \ge -1$ for every real number t. Therefore, the curve is only the right half of the parabola $y = (x + 1)^2 - 2$, as shown. Note that we do not indicate an orientation here, since the curve is traversed in one direction for $t > 0$ and the opposite direction for $t < 0$. ■

Many plane curves described parametrically are unlike anything you've seen so far in your study of calculus. Many of these are difficult to draw by hand, but can be easily plotted with a graphing calculator or CAS.

EXAMPLE 1.8 Some Unusual Plane Curves

Sketch the plane curves (a) $x = t^2 - 2$, $y = t^3 - t$ and (b) $x = t^3 - t$, $y = t^4 - 5t^2 + 4$.

Solution A sketch of (a) is shown in Figure 6.6a. From the vertical line test, this is not the graph of any function. Further, converting to an x-y equation here is messy and not particularly helpful. (Try this to see why.) However, note that $x = t^2 - 2 \ge -2$ for all t and $y = t^3 - t$ has no maximum or minimum. (Think about why.)

A computer sketch of (b) is shown in Figure 6.6b. Again, this is not a familiar graph. To get an idea of the scope of the graph, note that $x = t^3 - t$ has no maximum or minimum. To find the minimum of $y = t^4 - 5t^2 + 4$, note that critical numbers are at $t = 0$ and $t = \pm\sqrt{\frac{5}{2}}$ with corresponding function values 4 and $-\frac{9}{4}$, respectively. You should conclude that $y \ge -\frac{9}{4}$, as indicated in Figure 6.6b. ■

You should now have some idea of the flexibility of parametric equations. Significantly, a large number of applications translate simply into parametric equations.

BEYOND FORMULAS

When thinking of parametric equations, it is often helpful to think of t as representing time and the graph as representing the path of a moving particle. However, it is important to realize that the parameter can be anything. For example, in equations of circles and ellipses, the parameter may represent the angle as you rotate around the oval. Allowing the parameter to change from problem to problem gives us incredible flexibility to describe the relationship between x and y in the most convenient way possible.

EXERCISES 6.1

WRITING EXERCISES

1. Interpret in words the roles of each of the constants in the parametric equations $\begin{cases} x = a_1 + b_1 \cos(ct) \\ y = a_2 + b_2 \sin(ct) \end{cases}$.

2. An algorithm was given in example 1.5 for finding parametric equations of a line segment. Discuss the advantages that this method has over the other methods presented in remark 1.3.

3. As indicated in remark 1.3, a given curve can be described by numerous sets of parametric equations. Explain why several different equations can all be correct. (Hint: Emphasize the fact that t is a dummy variable.)

In exercises 1–12, sketch the plane curve defined by the given parametric equations and find an x-y equation for the curve.

1. $\begin{cases} x = 2 \cos t \\ y = 3 \sin t \end{cases}$

2. $\begin{cases} x = 1 + 2 \cos t \\ y = -2 + 2 \sin t \end{cases}$

3. $\begin{cases} x = -1 + 2t \\ y = 3t \end{cases}$

4. $\begin{cases} x = 4 + 3t \\ y = 2 - 4t \end{cases}$

5. $\begin{cases} x = 1 + t \\ y = t^2 + 2 \end{cases}$

6. $\begin{cases} x = 2 - t \\ y = t^2 + 1 \end{cases}$

7. $\begin{cases} x = t^2 - 1 \\ y = 2t \end{cases}$

8. $\begin{cases} x = t^2 - 1 \\ y = t^2 + 1 \end{cases}$

9. $\begin{cases} x = \sin^{-1} t \\ y = \sin t \end{cases}$

10. $\begin{cases} x = \tan^{-1} t \\ y = 4/\sqrt{t+1} \end{cases}$

11. $\begin{cases} x = \sqrt{\ln t} \\ y = 1/t \end{cases}$

12. $\begin{cases} x = e^t \\ y = e^{-2t} \end{cases}$

In exercises 13–20, use a graphing calculator to sketch the plane curves defined by the parametric equations.

13. $\begin{cases} x = t^3 - 2t \\ y = t^2 - 3 \end{cases}$

14. $\begin{cases} x = t^3 - 2t \\ y = t^2 - 3t \end{cases}$

15. $\begin{cases} x = \cos 2t \\ y = \sin 7t \end{cases}$

16. $\begin{cases} x = \cos 2t \\ y = \sin \pi t \end{cases}$

17. $\begin{cases} x = 3 \cos 2t + \sin 5t \\ y = 3 \sin 2t + \cos 5t \end{cases}$

18. $\begin{cases} x = 3 \cos 2t + \sin 6t \\ y = 3 \sin 2t + \cos 6t \end{cases}$

19. $\begin{cases} x = (2 - t/\sqrt{t^2 + 1}) \cos 32t \\ y = (1 + t/\sqrt{t^2 + 1}) \sin 32t \end{cases}$

20. $\begin{cases} x = 8t \cos 4t/\sqrt{t^2 + 1} \\ y = 8t \sin 4t/\sqrt{t^2 + 1} \end{cases}$

In exercises 21–28, find parametric equations describing the given curve.

21. The line segment from (0, 1) to (3, 4)

22. The line segment from (3, 1) to (1, 3)

23. The line segment from (−2, 4) to (6, 1)

24. The line segment from (4, −2) to (2, −1)

25. The portion of the parabola $y = x^2 + 1$ from (1, 2) to (2, 5)

26. The portion of the parabola $y = 2 - x^2$ from (2, −2) to (0, 2)

27. The circle of radius 3 centered at (2, 1), counterclockwise

28. The circle of radius 5 centered at (−1, 3), counterclockwise

In exercises 29–32, find parametric equations for the path of a projectile launched from height h with initial speed v at angle θ from the horizontal.

29. $h = 16$ m, $v = 12$ m/s, (a) $\theta = 0°$ (b) $\theta = 6°$ up

30. $h = 100$ m, $v = 24$ m/s, (a) $\theta = 0°$ (b) $\theta = 4°$ down

31. $h = 10$ m, $v = 2$m/s, (a) $\theta = 0°$ (b) $\theta = 8°$ down

32. $h = 40$ m, $v = 8$ m/s, (a) $\theta = 0°$ (b) $\theta = 6°$ up

In exercises 33–42, find all points of intersection of the two curves.

33. $\begin{cases} x = t \\ y = t^2 - 1 \end{cases}$ and $\begin{cases} x = 1 + s \\ y = 4 - s \end{cases}$

34. $\begin{cases} x = t^2 \\ y = t + 1 \end{cases}$ and $\begin{cases} x = 2 + s \\ y = 1 - s \end{cases}$

35. $\begin{cases} x = t + 3 \\ y = t^2 \end{cases}$ and $\begin{cases} x = 1 + s \\ y = 2 - s \end{cases}$

36. $\begin{cases} x = t^2 + 3 \\ y = t^3 + t \end{cases}$ and $\begin{cases} x = 2 + s \\ y = 1 - s \end{cases}$

37. Conjecture the difference between the graphs of $\begin{cases} x = \cos 2t \\ y = \sin kt \end{cases}$, where k is an integer compared to when k is an irrational number. (Hint: Try $k = 3$, $k = \sqrt{3}$, $k = 5$, $k = \sqrt{5}$ and other values.)

38. Compare the graphs of $\begin{cases} x = \cos 3t \\ y = \sin kt \end{cases}$ for $k = 1$, $k = 2$, $k = 3$, $k = 4$ and $k = 5$, and describe the role that k plays in the graph.

39. Compare the graphs of $\begin{cases} x = \cos t - \frac{1}{2}\cos kt \\ y = \sin t - \frac{1}{2}\sin kt \end{cases}$ for $k = 2$, $k = 3$, $k = 4$ and $k = 5$, and describe the role that k plays in the graph.

40. Describe the role that r plays in the graph of $\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$ and then describe how to sketch the graph of $\begin{cases} x = t\cos t \\ y = t\sin t \end{cases}$.

41. Compare the graphs of $\begin{cases} x = \cos 2t \\ y = \sin t \end{cases}$ and $\begin{cases} x = \cos t \\ y = \sin 2t \end{cases}$. Use the identities $\cos 2t = \cos^2 t - \sin^2 t$ and $\sin 2t = 2\cos t \sin t$ to find x-y equations for each graph.

42. Determine parametric equations for the curves defined by $x^{2n} + y^{2n} = r^{2n}$ for integers n. [Hint: Start with $n = 1$, $x^2 + y^2 = r^2$, then think of the general equation as $(x^n)^2 + (y^n)^2 = r^{2n}$.] Sketch the graphs for $n = 1$, $n = 2$ and $n = 3$, and predict what the curve will look like for large values of n.

In exercises 43–48, match the parametric equations with the corresponding plane curve displayed in Figures A–F. Give reasons for your choices.

43. $\begin{cases} x = t^2 - 1 \\ y = t^4 \end{cases}$

44. $\begin{cases} x = t - 1 \\ y = t^3 \end{cases}$

45. $\begin{cases} x = t^2 - 1 \\ y = \sin t \end{cases}$

46. $\begin{cases} x = t^2 - 1 \\ y = \sin 2t \end{cases}$

47. $\begin{cases} x = \cos 3t \\ y = \sin 2t \end{cases}$

48. $\begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases}$

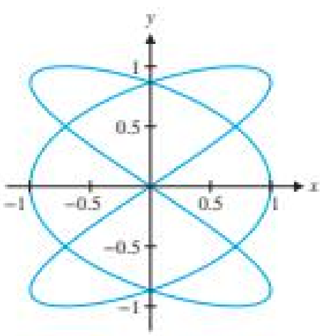

FIGURE A

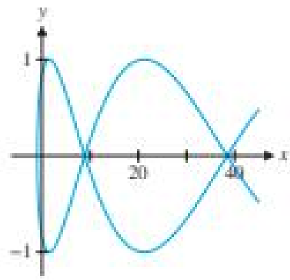

FIGURE B

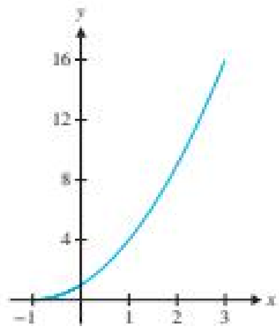

FIGURE C

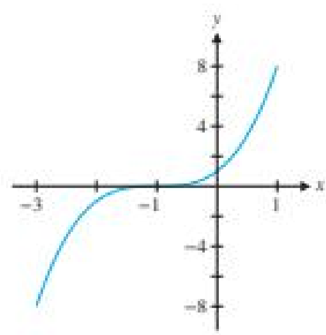

FIGURE D

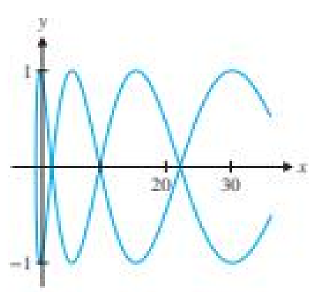

FIGURE E

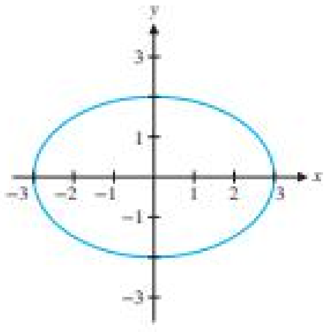

FIGURE F

APPLICATIONS

Exercises 1 and 2 explore the sound barrier problem discussed in the chapter introduction. Define 1 unit to be the distance traveled by sound in 1 second.

1. (a) Suppose a sound wave is emitted from the origin at time 0. After t seconds ($t > 0$), explain why the position in units of the sound wave is modeled by $x = t\cos\theta$ and $y = t\sin\theta$, where the dummy parameter θ has range $0 \le \theta \le 2\pi$.

(b) Find parametric equations for the position at time t seconds ($t > 0$) of a sound wave emitted at time c seconds from the point (a, b).

(c) Suppose that a jet has speed 0.8 unit per second (i.e., Mach 0.8) with position function $x(t) = 0.8t$ and $y(t) = 0$. To model the position at time $t = 5$ seconds of various sound waves emitted by the jet, do the following on one set of axes. (1) Graph the position after 5 seconds of the sound wave emitted from (0, 0); (2) graph the position after 4 seconds of the sound wave emitted from (0.8, 0); (3) graph the position after 3 seconds of the sound wave emitted from (1.6, 0); (4) graph the position after 2 seconds of the sound wave emitted from (2.4, 0); (5) graph the position after 1 second of the sound wave emitted from (3.2, 0); (6) mark the position of the jet at time $t = 5$.

(d) Repeat part (c) for a jet with speed 1.0 unit per second (Mach 1). The sound waves that intersect at the jet's location are the "sound barrier" that must be broken.

(e) Repeat part (c) for a jet with speed 1.4 units per second (Mach 1.4).

(f) In part (e), the sound waves intersect each other. The intersections form the "shock wave" that we hear as a sonic boom. Theoretically, the angle θ between the shock wave and the x-axis satisfies the equation $\sin\theta = \frac{1}{m}$, where m is the Mach speed of the jet. Show that for $m = 1.4$, the theoretical shock wave is formed by the lines $x(t) = 7 - \sqrt{0.96}t$, $y(t) = t$ and $x(t) = 7 - \sqrt{0.96}t$, $y(t) = -t$. Superimpose these lines onto the graph of part (e).

(g) In part (f), the shock wave of a jet at Mach 1.4 is modeled by two lines. Argue that in three dimensions, the shock wave has circular cross-sections. Describe the three-dimensional figure formed by revolving the lines in part (f) about the x-axis.

2. If a pebble is dropped into water, a wave spreads out in an expanding circle. Let v be the speed of the propagation of the wave. If a boat moves through this water with speed $1.4v$, argue that the boat's wake will be described by the graphs of part (f) of exercise 1. Graph the wake of a boat with speed $1.6v$.

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Exercise 3 shows that a celestial object can incorrectly appear to be moving faster than the speed of light.

3. (a) A bright object is at position $(0, D)$ at time 0, where D is a very large positive number. The object moves toward the positive x-axis with constant speed $v < c$ at an angle θ from the vertical. Find parametric equations for the position of the object at time t.

(b) Let $s(t)$ be the distance from the object to the origin at time t. Then $L(t) = \frac{s(t)}{c}$ gives the amount of time it takes for light emitted by the object at time t to reach the origin. Show that $L'(t) = \frac{1}{c}\frac{v^2t - Dv\cos\theta}{s(t)}$.

(c) An observer stands at the origin and tracks the horizontal movement of the object. Light received at time T was emitted by the object at time t, where $T = t + L(t)$. Similarly, light received at time $T + \Delta T$ was emitted at time $t + dt$, where typically $dt \ne \Delta T$. The apparent x-coordinate of the object at time T is $x_a(T) = x(t)$. The apparent horizontal speed of the object at time T as measured by the observer is $h(T) = \lim_{\Delta T \to 0} \frac{x_a(T + \Delta T) - x_a(T)}{\Delta T}$. Tracing back to time t, show that $h(t) = \lim_{dt \to 0} \frac{x(t + dt) - x(t)}{\Delta T} = \frac{v\sin\theta}{T'(t)} = \frac{v\sin\theta}{1 + L'(t)}$.

(d) Show that $h(0) = \frac{cv\sin\theta}{c - v\cos\theta}$.

(e) Show that for a constant speed v, the maximum apparent horizontal speed $h(0)$ occurs when the object moves at an angle with $\cos\theta = \frac{v}{c}$. Find this speed in terms of v and $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$.

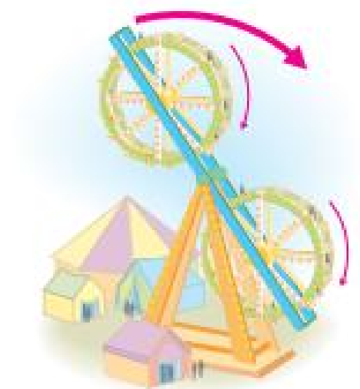
(f) Show that as v approaches c, the apparent horizontal speed can exceed c, causing the observer to measure an object moving faster than the speed of light! As v approaches c, show that the angle producing the maximum apparent horizontal speed decreases to 0. Discuss why this is paradoxical.

(g) If $\frac{v}{c} > 1$, show that $h(0)$ has no maximum.

4. Let r_E and r_M model the paths of Earth and Mars, respectively, around the Sun, where $r_E = \begin{cases} x = \cos 2\pi t \\ y = \sin 2\pi t \end{cases}$ and $r_M = \begin{cases} x = 1.5 \cos \pi t \\ y = 1.5 \sin \pi t \end{cases}$. According to this model, how do the radii and periods of the orbits compare? How accurate is this? The orbit of Mars relative to Earth is modeled by $r_M - r_E$. Graph this and identify the retrograde motion of Mars as seen from Earth.

EXPLORATORY EXERCISE

1. Many carnivals have a version of the double Ferris wheel. A large central arm rotates clockwise. At each end of the central arm is a Ferris wheel that rotates clockwise around the arm. Assume that the central arm has length 60 m and rotates about its center. Also assume that the wheels have radius 12 m and rotate at the same speed as the central arm. Find parametric equations for the position of a rider and graph the rider's path. Adjust the speed of rotation of the wheels to improve the ride.



6.2 CALCULUS AND PARAMETRIC EQUATIONS

Our initial aim in this section is to find a way to determine the slopes of tangent lines to curves that are defined parametrically. First, recall that for a differentiable function $y = f(x)$, the slope of the tangent line at the point $x = a$ is given by $f'(a)$. Written in Leibniz notation, the slope is $\frac{dy}{dx}(a)$. In the case of a curve defined parametrically, both x and y are functions of the parameter t. Notice that if $x = x(t)$ and $y = y(t)$ both have derivatives that are continuous at $t = c$, the chain rule gives us

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

As long as $\frac{dx}{dt}(c) \neq 0$, we then have

$$\boxed{\frac{dy}{dx}(a) = \frac{\frac{dy}{dt}(c)}{\frac{dx}{dt}(c)} = \frac{y'(c)}{x'(c)},} \qquad (2.1)$$

where $a = x(c)$. In the case where $x'(c) = y'(c) = 0$, we define

$$\boxed{\frac{dy}{dx}(a) = \lim_{t \to c} \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \lim_{t \to c} \frac{y'(t)}{x'(t)},} \qquad (2.2)$$

provided the limit exists.

REMARK 2.1

Be careful how you interpret equation (2.1). The primes on the right side of the equation refer to derivatives with respect to the parameter t. We recommend that you (at least initially) use the Leibniz notation, which also gives you a simple way to accurately remember the chain rule.

CAUTION

Look carefully at (2.3) and convince yourself that

$$\frac{d^2y}{dx^2} \neq \frac{\dfrac{d^2y}{dt^2}}{\dfrac{d^2x}{dt^2}}.$$

Equating these two expressions is a common error. You should be careful to avoid this trap.

We can use (2.1) to calculate second (as well as higher) order derivatives. Notice that if we replace y by $\frac{dy}{dx}$, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}. \tag{2.3}$$

The Scrambler is a popular carnival ride consisting of two sets of rotating arms. (See Figure 6.7a). If the inner arms have length 2 and rotate counterclockwise, we can describe the location (x_i, y_i) of the end of one of the inner arms by the parametric equations $x_i = 2\cos t$, $y_i = 2\sin t$. At the end of each inner arm, a set of outer arms rotate clockwise at roughly twice the speed. If the outer arms have length 1, parametric equations describing the outer arm rotation are $x_o = \sin 2t$, $y_o = \cos 2t$. Here, the reversal of sine and cosine terms indicates that the rotation is clockwise and the factor of 2 inside the sine and cosine terms indicates that the speed of the rotation is double that of the inner arms. The position of a person riding the Scrambler is the sum of the two component motions; that is,

$$x = 2\cos t + \sin 2t, \qquad y = 2\sin t + \cos 2t.$$

The graph of these parametric equations is shown in Figure 6.7b.

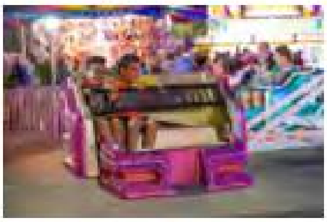


FIGURE 6.7a
The Scrambler
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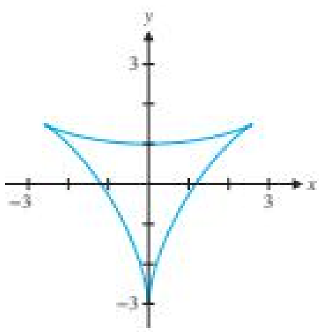


FIGURE 6.7b
Path of a Scrambler rider

EXAMPLE 2.1 Slopes of Tangent Lines to a Parametric Curve

Find the slope of the tangent line to the Scrambler path described by $x = 2\cos t + \sin 2t$, $y = 2\sin t + \cos 2t$ at (a) $t = 0$ and (b) the point $(0, -3)$.

Solution (a) First, note that

$$\frac{dx}{dt} = -2\sin t + 2\cos 2t \quad \text{and} \quad \frac{dy}{dt} = 2\cos t - 2\sin 2t.$$

From (2.1), the slope of the tangent line at $t = 0$ is then

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{\dfrac{dy}{dt}(0)}{\dfrac{dx}{dt}(0)} = \frac{2\cos 0 - 2\sin 0}{-2\sin 0 + 2\cos 0} = 1.$$

(b) To determine the slope at the point $(0, -3)$, we must first determine a value of t that corresponds to the point. In this case, notice that $t = 3\pi/2$ gives $x = 0$ and $y = -3$. Here, we have

$$\frac{dx}{dt}\left(\frac{3\pi}{2}\right) = \frac{dy}{dt}\left(\frac{3\pi}{2}\right) = 0$$

and consequently, we must use (2.2) to compute $\frac{dy}{dx}$. Since the limit has the indeterminate form $\frac{0}{0}$, we use l'Hôpital's Rule, to get

$$\frac{dy}{dx}\left(\frac{3\pi}{2}\right) = \lim_{t \to 3\pi/2} \frac{2\cos t - 2\sin 2t}{-2\sin t + 2\cos 2t} = \lim_{t \to 3\pi/2} \frac{-2\sin t - 4\cos 2t}{-2\cos t - 4\sin 2t},$$

which does not exist, since the limit in the numerator is 6 and the limit in the denominator is 0. This says that the slope of the tangent line at $t = 3\pi/2$ is undefined. In Figure 6.8, we have drawn in the tangent lines at $t = 0$ and $3\pi/2$. Notice that the tangent line at the point $(0, -3)$ is vertical. ■

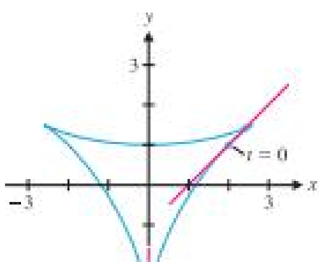


FIGURE 6.8
Tangent lines to the Scrambler path

Finding slopes of tangent lines can help us identify many points of interest.

EXAMPLE 2.2 Finding Vertical and Horizontal Tangent Lines

Identify all points at which the plane curve $x = \cos 2t$, $y = \sin 3t$ has a horizontal or vertical tangent line.

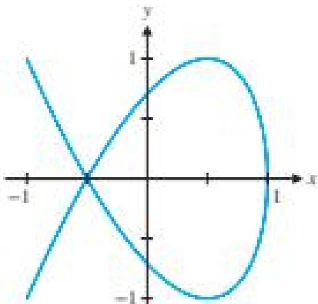


FIGURE 6.9
$x = \cos 2t$, $y = \sin 3t$

Solution The sketch of the curve shown in Figure 6.9 suggests that there are two locations—the top and bottom of the bow—with horizontal tangent lines and one point—the far right edge of the bow—with a vertical tangent line. Recall that horizontal tangent lines occur where $\frac{dy}{dx} = 0$. From (2.1), we then have $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = 0$, which can occur only when

$$0 = y'(t) = 3 \cos 3t,$$

provided that $x'(t) = -2 \sin 2t \neq 0$ for the same value of t. Since $\cos \theta = 0$ only when θ is an odd multiple of $\frac{\pi}{2}$, we have that $y'(t) = 3 \cos 3t = 0$, only when $3t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$ and so, $t = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \dots$. The corresponding points on the curve are then

$$\left(x\left(\frac{\pi}{6}\right), y\left(\frac{\pi}{6}\right)\right) = \left(\cos\frac{\pi}{3}, \sin\frac{\pi}{2}\right) = \left(\frac{1}{2}, 1\right),$$

$$\left(x\left(\frac{3\pi}{6}\right), y\left(\frac{3\pi}{6}\right)\right) = \left(\cos \pi, \sin\frac{3\pi}{2}\right) = (-1, -1),$$

$$\left(x\left(\frac{7\pi}{6}\right), y\left(\frac{7\pi}{6}\right)\right) = \left(\cos\frac{7\pi}{3}, \sin\frac{7\pi}{2}\right) = \left(\frac{1}{2}, -1\right)$$

and

$$\left(x\left(\frac{9\pi}{6}\right), y\left(\frac{9\pi}{6}\right)\right) = \left(\cos 3\pi, \sin\frac{9\pi}{2}\right) = (-1, 1).$$

Note that $t = \frac{5\pi}{6}$ and $t = \frac{11\pi}{6}$ reproduce the first and third points, respectively, and so on. The points $(\frac{1}{2}, 1)$ and $(\frac{1}{2}, -1)$ are on the top and bottom of the bow, respectively, where there clearly are horizontal tangents. The points $(-1, -1)$ and $(-1, 1)$ should not seem quite right, though. These points are on the extreme ends of the bow and certainly don't look like they have vertical or horizontal tangents. In fact, they don't. Notice that at both $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$, we have $x'(t) = y'(t) = 0$ and so, the slope must be computed as a limit using (2.2). We leave it as an exercise to show that the slopes at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ are $\frac{9}{4}$ and $-\frac{9}{4}$, respectively.

To find points where there is a vertical tangent, we need to see where $x'(t) = 0$ but $y'(t) \neq 0$. Setting $0 = x'(t) = -2 \sin 2t$, we get $\sin 2t = 0$, which occurs if $2t = 0, \pi, 2\pi, \dots$ or $t = 0, \frac{\pi}{2}, \pi, \dots$. The corresponding points are

$$(x(0), y(0)) = (\cos 0, \sin 0) = (1, 0),$$

$$(x(\pi), y(\pi)) = (\cos 2\pi, \sin 3\pi) = (1, 0)$$

and the points corresponding to $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$, which we have already discussed (where $y'(t) = 0$, also). Since $y'(t) = 3 \cos 3t \neq 0$, for $t = 0$ or $t = \pi$, there is a vertical tangent line only at the point $(1, 0)$. ■

Theorem 2.1 generalizes what we observed in example 2.2.

THEOREM 2.1

Suppose that $x'(t)$ and $y'(t)$ are continuous. Then for the curve defined by the parametric equations $x = x(t)$ and $y = y(t)$,

(i) if $y'(c) = 0$ and $x'(c) \neq 0$, there is a horizontal tangent line at the point $(x(c), y(c))$;

(ii) if $x'(c) = 0$ and $y'(c) \neq 0$, there is a vertical tangent line at the point $(x(c), y(c))$.

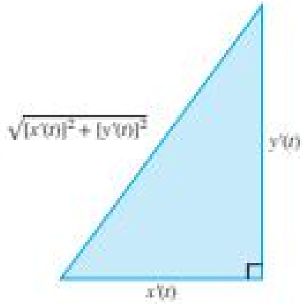


FIGURE 6.10
Horizontal and vertical components of velocity and speed

PROOF

The proof depends on the calculation of derivatives for parametric curves and is left as an exercise. ■

An interesting question about the Scrambler is whether the rider ever comes to a complete stop. To answer this question, we need to be able to compute velocities. Recall that if the position of an object moving along a straight line is given by the differentiable function $f(t)$, the object's velocity is given by $f'(t)$. The situation with parametric equations is completely analogous. If the position is given by $(x(t), y(t))$, for differentiable functions $x(t)$ and $y(t)$, then the **horizontal component of velocity** is given by $x'(t)$ and the **vertical component of velocity** is given by $y'(t)$. (See Figure 6.10.) We define the **speed** to be $\sqrt{[x'(t)]^2 + [y'(t)]^2}$. From this, note that the speed is 0 if and only if $x'(t) = y'(t) = 0$. In this event, there is no horizontal or vertical motion.

EXAMPLE 2.3 Velocity of the Scrambler

For the path of the Scrambler $x = 2\cos t + \sin 2t$, $y = 2\sin t + \cos 2t$, find the horizontal and vertical components of velocity and speed at times $t = 0$ and $t = \frac{\pi}{2}$, and indicate the direction of motion. Also determine all times at which the speed is zero.

Solution Here, the horizontal component of velocity is $\dfrac{dx}{dt} = -2\sin t + 2\cos 2t$ and the vertical component is $\dfrac{dy}{dt} = 2\cos t - 2\sin 2t$. At $t = 0$, the horizontal and vertical components of velocity both equal 2 and the speed is $\sqrt{4+4} = \sqrt{8}$. The rider is located at the point $(x(0), y(0)) = (2, 1)$ and is moving to the right [since $x'(0) > 0$] and up [since $y'(0) > 0$]. At $t = \frac{\pi}{2}$, the velocity has components -4 (horizontal) and 0 (vertical) and the speed is $\sqrt{16+0} = 4$. At this time, the rider is located at the point $(0, 1)$ and is moving to the left [since $x'\left(\frac{\pi}{2}\right) < 0$].

In general, the speed $s(t)$ of the rider at time t is given by

$$\begin{aligned} s(t) &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-2\sin t + 2\cos 2t)^2 + (2\cos t - 2\sin 2t)^2} \\ &= \sqrt{4\sin^2 t - 8\sin t\cos 2t + 4\cos^2 2t + 4\cos^2 t - 8\cos t\sin 2t + 4\sin^2 2t} \\ &= \sqrt{8 - 8\sin t\cos 2t - 8\cos t\sin 2t} \\ &= \sqrt{8 - 8\sin 3t}, \end{aligned}$$

using the identities $\sin^2 t + \cos^2 t = 1$, $\cos^2 2t + \sin^2 2t = 1$ and $\sin t\cos 2t + \sin 2t\cos t = \sin 3t$. So, the speed is 0 whenever $\sin 3t = 1$. This occurs when $3t = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$, or $t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6}, \ldots$. The corresponding points on the curve are $\left(x\left(\frac{\pi}{6}\right), y\left(\frac{\pi}{6}\right)\right) = \left(\frac{3}{2}\sqrt{3}, \frac{3}{2}\right)$, $\left(x\left(\frac{5\pi}{6}\right), y\left(\frac{5\pi}{6}\right)\right) = \left(\frac{3}{2}\sqrt{3}, \frac{3}{2}\right)$ and $\left(x\left(\frac{9\pi}{6}\right), y\left(\frac{9\pi}{6}\right)\right) = (0, -3)$. You can easily verify that these points are the three tips of the path seen in Figure 6.7b. ■

We just showed that riders in the Scrambler of Figure 6.7b actually come to a brief stop at the outside of each loop. As you will explore in the exercises, for similar Scrambler paths, the riders slow down but have a positive speed at the outside of each loop. This is true of the Scrambler at most carnivals, for which a more complicated path makes up for the lack of stopping.

Note that any curve that begins and ends at the same point will enclose an area. Finding the area enclosed by such a curve is a straightforward extension of our original development of integration. Recall that for a continuous function f defined on $[a, b]$, where $f(x) \geq 0$ on $[a, b]$, the area under the curve $y = f(x)$ for $a \leq x \leq b$ is given by

$$A = \int_a^b f(x)\,dx = \int_a^b y\,dx.$$

Now, suppose that this same curve is described parametrically by $x = x(t)$ and $y = y(t)$, where the curve is traversed exactly once for $c \leq t \leq d$. We can then compute the area by making the substitution $x = x(t)$. It then follows that $dx = x'(t)\,dt$ and so the area is given by

$$A = \int_a^b \underbrace{y}_{y(t)} \underbrace{dx}_{x'(t)dt} = \int_c^d y(t)x'(t)\,dt,$$

where you should notice that we have also changed the limits of integration to match the new variable of integration. We generalize this result in Theorem 2.2.

THEOREM 2.2 (Area Enclosed by a Curve Defined Parametrically)

Suppose that the parametric equations $x = x(t)$ and $y = y(t)$, with $c \leq t \leq d$, describe a curve that is traced out *clockwise* exactly once and where the curve does not intersect itself, except that the initial and terminal points are the same [i.e. $x(c) = x(d)$ and $y(c) = y(d)$]. Then the enclosed area is given by

$$A = \int_c^d y(t)x'(t)\,dt = -\int_c^d x(t)y'(t)\,dt. \tag{2.4}$$

If the curve is traced out *counterclockwise*, then the enclosed area is given by

$$A = -\int_c^d y(t)x'(t)\,dt = \int_c^d x(t)y'(t)\,dt. \tag{2.5}$$

PROOF

This result is a special case of Green's Theorem, which we will develop in section 14.4. ■

The new area formulas given in Theorem 2.2 turn out to be quite useful, as we illustrate in example 2.4.

EXAMPLE 2.4 Finding the Area Enclosed by a Curve

Find the area enclosed by the path of the Scrambler $x = 2\cos t + \sin 2t$, $y = 2\sin t + \cos 2t$.

Solution Notice that the curve is traced out counterclockwise once for $0 \leq t \leq 2\pi$. From (2.5), the area is then

$$\begin{aligned} A &= \int_0^{2\pi} x(t)y'(t)\,dt = \int_0^{2\pi} (2\cos t + \sin 2t)(2\cos t - 2\sin 2t)\,dt \\ &= \int_0^{2\pi} (4\cos^2 t - 2\cos t \sin 2t - 2\sin^2 2t)\,dt = 2\pi, \end{aligned}$$

where we evaluated the integral using a CAS. ■

In example 2.5, we use Theorem 2.2 to derive a formula for the area enclosed by an ellipse. Pay particular attention to how much easier this is to do with parametric equations than it is to do with the original x-y equation.

EXAMPLE 2.5 Finding the Area Enclosed by an Ellipse

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (for constants $a, b > 0$).

Solution One way to compute the area is to solve the equation for y to obtain $y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$ and then integrate:

$$A = \int_{-a}^{a} \left[b\sqrt{1 - \frac{x^2}{a^2}} - \left(-b\sqrt{1 - \frac{x^2}{a^2}} \right) \right] dx.$$

You can evaluate this integral by trigonometric substitution or by using a CAS, but a simpler, more elegant way to compute the area is to use parametric equations. Notice that the ellipse is described parametrically by $x = a\cos t$, $y = b\sin t$, for $0 \le t \le 2\pi$. The ellipse is then traced out counterclockwise exactly once for $0 \le t \le 2\pi$, so that the area is given by (2.5) to be

$$A = -\int_0^{2\pi} y(t)x'(t)\,dt = -\int_0^{2\pi} (b\sin t)(-a\sin t)\,dt = ab\int_0^{2\pi} \sin^2 t\,dt = ab\pi,$$

where this last integral can be evaluated by using the half-angle formula:

$$\sin^2 t = \frac{1}{2}(1 - \cos 2t).$$

We leave the details of this calculation as an exercise. ■

BEYOND FORMULAS

Many of the formulas in this section are not new, but are simply modifications of the well-established rules for differentiation and integration. If you think of them this way, they are not complicated memorization exercises, but instead are old standards expressed in a slightly different way.

EXERCISES 6.2

WRITING EXERCISES

1. In the derivation of parametric equations for the Scrambler, we used the fact that reversing the sine and cosine functions to $\begin{cases} x = \sin t \\ y = \cos t \end{cases}$ causes the circle to be traced out clockwise. Explain why this is so by starting at $t = 0$ and following the graph as t increases to 2π.

2. Explain why Theorem 2.1 makes sense. (Hint: If $y'(c) = 0$, what does that say about the change in y-coordinates on the graph? Why do you also need $x'(c) \ne 0$ to guarantee a horizontal tangent?)

3. Imagine an object with position given by $x(t)$ and $y(t)$. If a right triangle has a horizontal leg of length $x'(t)$ and a vertical leg of length $y'(t)$, what would the length of the hypotenuse represent? Explain why this makes sense.

4. Explain why the sign ($\pm$) of $\int_c^d y(t)x'(t)\,dt$ in Theorem 2.2 is different for curves traced out clockwise and counterclockwise.

In exercises 1–6, find the slopes of the tangent lines to the given curves at the indicated points.

1. $\begin{cases} x = t^2 - 2 \\ y = t^3 - t \end{cases}$ (a) $t = -1$ (b) $t = 1$ (c) $(-2, 0)$

2. $\begin{cases} x = t^3 - t \\ y = t^4 - 5t^2 + 4 \end{cases}$ (a) $t = -1$ (b) $t = 1$ (c) $(0, 4)$

3. $\begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases}$ (a) $t = \frac{\pi}{4}$ (b) $t = \frac{\pi}{2}$ (c) $(0, 3)$

4. $\begin{cases} x = \cos 2t \\ y = \sin 4t \end{cases}$ (a) $t = \frac{\pi}{4}$ (b) $t = \frac{\pi}{2}$ (c) $\left(\frac{\sqrt{2}}{2}, 1\right)$

5. $\begin{cases} x = t\cos t \\ y = t\sin t \end{cases}$ (a) $t = 0$ (b) $t = \frac{\pi}{2}$ (c) $(\pi, 0)$

6. $\begin{cases} x = \sqrt{t^2 + 1} \\ y = \sin t \end{cases}$ (a) $t = -\pi$ (b) $t = \pi$ (c) $(0, 0)$

In exercises 7 and 8, sketch the graph and find the slope of the curve at the given point.

7. $\begin{cases} x = t^2 - 2 \\ y = t^3 - t \end{cases}$ at $(-1, 0)$

8. $\begin{cases} x = t^3 - t \\ y = t^4 - 5t^2 + 4 \end{cases}$ at $(0, 0)$

In exercises 9–14, identify all points at which the curve has (a) a horizontal tangent and (b) a vertical tangent.

9. $\begin{cases} x = \cos 2t \\ y = \sin 4t \end{cases}$

10. $\begin{cases} x = \cos 2t \\ y = \sin 7t \end{cases}$

11. $\begin{cases} x = t^2 - 1 \\ y = t^4 - 4t \end{cases}$

12. $\begin{cases} x = t^2 - 1 \\ y = t^4 - 4t^2 \end{cases}$

13. $\begin{cases} x = 2\cos t + \sin 2t \\ y = 2\sin t + \cos 2t \end{cases}$

14. $\begin{cases} x = 2\cos 2t + \sin t \\ y = 2\sin 2t + \cos t \end{cases}$

In exercises 15–20, parametric equations for the position of an object are given. Find the object's velocity and speed at the given times and describe its motion.

15. $\begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases}$ (a) $t = 0$ (b) $t = \frac{\pi}{2}$

16. $\begin{cases} x = 2\sin 2t \\ y = 3\cos 2t \end{cases}$ (a) $t = 0$ (b) $t = \frac{\pi}{2}$

17. $\begin{cases} x = 20t \\ y = 30 - 2t - 16t^2 \end{cases}$ (a) $t = 0$ (b) $t = 2$

18. $\begin{cases} x = 40t + 5 \\ y = 20 + 3t - 16t^2 \end{cases}$ (a) $t = 0$ (b) $t = 2$

19. $\begin{cases} x = 2\cos 2t + \sin 5t \\ y = 2\sin 2t + \cos 5t \end{cases}$ (a) $t = 0$ (b) $t = \frac{\pi}{2}$

20. $\begin{cases} x = 3\cos t + \sin 3t \\ y = 3\sin t + \cos 3t \end{cases}$ (a) $t = 0$ (b) $t = \frac{\pi}{2}$.

In exercises 21–28, find the area enclosed by the given curve.

21. $\begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases}$

22. $\begin{cases} x = 6\cos t \\ y = 2\sin t \end{cases}$

23. $\begin{cases} x = \frac{1}{2}\cos t - \frac{1}{4}\cos 2t \\ y = \frac{1}{2}\sin t - \frac{1}{4}\sin 2t \end{cases}$

24. $\begin{cases} x = 2\cos 2t + \cos 4t \\ y = 2\sin 2t + \sin 4t \end{cases}$

25. $\begin{cases} x = \cos t \\ y = \sin 2t \end{cases}, \frac{\pi}{2} \le t \le \frac{3\pi}{2}$

26. $\begin{cases} x = t\sin t \\ y = t\cos t \end{cases}, -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

27. $\begin{cases} x = t^3 - 4t \\ y = t^2 - 3 \end{cases}, -2 \le t \le 2$

28. $\begin{cases} x = t^3 - 4t \\ y = t^4 - 1 \end{cases}, -2 \le t \le 2$

In exercises 29 and 30, find the speed and acceleration of the object each time it crosses the x-axis.

29. $\begin{cases} x = 2\cos^2 t + 2\cos t - 1 \\ y = 2(1 - \cos t)\sin t \end{cases}$

30. $\begin{cases} x = 6\cos t + 6\cos^3 t \\ y = 6\sin t - 6\sin^3 t \end{cases}$

31. Suppose that $x = 2\cos t$ and $y = 2\sin t$. Compare $\frac{d^2y}{dx^2}(\sqrt{3})$ and $\dfrac{\frac{d^2y}{dt^2}(\pi/6)}{\frac{d^2x}{dt^2}(\pi/6)}$ and show that they are not equal.

32. For $x = at^2$ and $y = bx^2$ for non-zero constants a and b, determine whether there are any values of t such that $\frac{d^2y}{dx^2}(x(t)) = \dfrac{\frac{d^2y}{dt^2}(t)}{\frac{d^2x}{dt^2}(t)}$.

33. Suppose you are standing at the origin watching an object that has position $(x(t), y(t))$ at time t. Show that, from your perspective, the object is moving clockwise if $\left(\frac{y(t)}{x(t)}\right)' < 0$ and is moving counterclockwise if $\left(\frac{y(t)}{x(t)}\right)' > 0$.

34. In the Ptolemaic model of planetary motion, the earth was at the center of the solar system and the sun and planets orbited the earth. Circular orbits, which were preferred for aesthetic reasons, could not account for the actual motion of the planets as viewed from the earth. Ptolemy modified the circles into epicycloids, which are circles on circles similar to the Scrambler of example 2.1. Suppose that a planet's motion is given by $\begin{cases} x = 10\cos 16\pi t + 20\cos 4\pi t \\ y = 10\sin 16\pi t + 20\sin 4\pi t \end{cases}$. Using the result of exercise 33, find the intervals in which the planet rotates clockwise and the intervals in which the planet rotates counterclockwise.

35. Find parametric equations for the path traced out by a specific point on a circle of radius r rolling from left to right at a constant speed $v > r$. Assume that the point starts at (r, r) at time $t = 0$. (Hint: First, find parametric equations for the center of the circle. Then, add on parametric equations for the point going around the center of the circle.) Find the minimum and maximum speeds of the point and the locations where each occurs. Graph the curve for $v = 3$ and $r = 2$. This curve is called a **cycloid**.

36. Find parametric equations for the path traced out by a specific point inside the circle as the circle rolls from left to right. (Hint: If r is the radius of the circle, let $d < r$ be the distance from the point to the center.) Find the minimum and maximum speeds of the point and the locations where each occurs. Graph the curve for $v = 3$, $r = 2$ and $d = 1$. This curve is called a **trochoid**.

37. A **hypocycloid** is the path traced out by a point on a smaller circle of radius b that is rolling inside a larger circle of radius $a > b$ (see accompanying figure). Find parametric equations for the hypocycloid and graph it for $a = 5$ and $b = 3$. Find an equation in terms of the parameter t for the slope of the tangent line to the hypocycloid and determine one point at which the tangent line is vertical. What interesting simplification occurs if $a = 2b$?

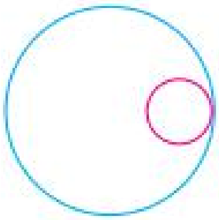


Figure for exercise 37

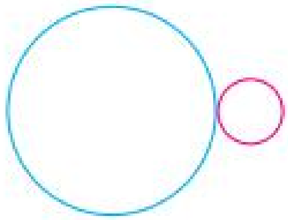


Figure for exercise 38

38. An **epicycloid** is the path traced out by a point on a smaller circle of radius b that is rolling outside a larger circle of radius $a > b$ (see accompanying figure). Find parametric equations for the epicycloid and graph it for $a = 8$ and $b = 5$. Find an equation in terms of the parameter t for the slope of the tangent line to the epicycloid and determine one point at which the slope is vertical. What interesting simplification occurs if $a = 2b$?

APPLICATIONS

1. Suppose an object follows the path $\begin{cases} x = \sin 4t \\ y = -\cos 4t \end{cases}$. Show that its speed is constant. Show that, at any time t, the tangent line is perpendicular to a line connecting the origin and the object.

2. A Ferris wheel has height 100 m and completes one revolution in 3 minutes at a constant speed. Compute the speed of a rider on the Ferris wheel.

3. A modification of the Scrambler in example 2.1 is $\begin{cases} x = 2\cos 3t + \sin 5t \\ y = 2\sin 3t + \cos 5t \end{cases}$. In example 2.1, the ratio of the speed of the outer arms to the speed of the inner arms is 2-to-1. What is the ratio in this version of the Scrambler? Sketch a graph showing the motion of this new Scrambler.

4. Compute the speed of the Scrambler in exercise 3. Using trigonometric identities as in example 2.3, show that the speed is at a minimum when $\sin 8t = 1$ but that the speed is never zero. Show that the minimum speed is reached at the outer points of the path.

5. Find parametric equations for a Scrambler that is the same as in example 2.1 except that the outer arms rotate three times as fast as the inner arms. Sketch a graph of its motion and determine its minimum and maximum speeds.

6. Find parametric equations for a Scrambler that is the same as in example 2.1 except that the inner arms have length 3. Sketch a graph of its motion and determine its minimum and maximum speeds.

EXPLORATORY EXERCISES

1. By varying the speed of the outer arms, the Scrambler of example 2.1 can be generalized to $\begin{cases} x = 2\cos t + \sin kt \\ y = 2\sin t + \cos kt \end{cases}$ for some positive constant k. Show that the minimum speed for any such Scrambler is reached at the outside of a loop. Show that the only value of k that actually produces a speed of 0 is $k = 2$. By varying the lengths of the arms, you can further generalize the Scrambler to $\begin{cases} x = r\cos t + \sin kt \\ y = r\sin t + \cos kt \end{cases}$ for positive constants $r > 1$ and k. Sketch the paths for several such Scramblers and determine the relationship between r and k needed to produce a speed of 0. Find the "best" Scrambler as judged by complexity of path and variation in passenger speed.

2. **Bézier curves** are essential in almost all areas of modern engineering design. (For instance, Bézier curves were used for sketching many of the figures for this book.) One version of a Bézier curve starts with control points at (a, y_a), (b, y_b), (c, y_c) and (d, y_d). The Bézier curve passes through the points (a, y_a) and (d, y_d). The tangent line at $x = a$ passes through (b, y_b) and the tangent line at $x = d$ passes through (c, y_c). Show that these criteria are met, for $0 \le t \le 1$, with

$$\begin{cases} x = (a + b - c - d)t^3 + (2d - 2b + c - a)t^2 \\ \quad + (b - a)t + a \\ y = (y_a + y_b - y_c - y_d)t^3 + (2y_d - 2y_b + y_c - y_a)t^2 \\ \quad + (y_b - y_a)t + y_a \end{cases}$$

Use this formula to find and graph the Bézier curve with control points (0, 0), (1, 2), (2, 3) and (3, 0). Explore the effect of moving the middle control points, for example moving them up to (1, 3) and (2, 4), respectively.

6.3 ARC LENGTH AND SURFACE AREA IN PARAMETRIC EQUATIONS

In this section, we investigate arc length and surface area for curves defined parametrically. Along the way, we explore one of the most famous and interesting curves in mathematics.

Let C be the curve defined by the parametric equations $x = x(t)$ and $y = y(t)$, for $a \le t \le b$ (see Figure 6.11a), where x, x', y and y' are continuous on the interval $[a, b]$. We further assume that the curve does not intersect itself, except possibly at a finite number

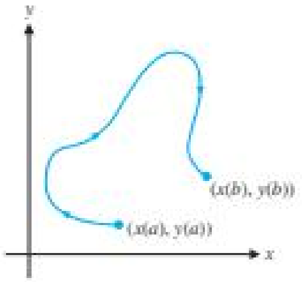


FIGURE 6.11a
The plane curve C

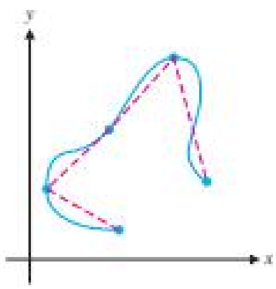


FIGURE 6.11b
Approximate arc length

of points. Our goal is to compute the length of the curve (the **arc length**). Once again, we begin by constructing an approximation and then improve the approximation.

First, we divide the t-interval $[a, b]$ into n subintervals of equal length, Δt:

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b,$$

where $t_i - t_{i-1} = \Delta t = \dfrac{b-a}{n}$, for each $i = 1, 2, 3, \ldots, n$. For each subinterval $[t_{i-1}, t_i]$, we approximate the arc length s_i of the portion of the curve joining the point $(x(t_{i-1}), y(t_{i-1}))$ to the point $(x(t_i), y(t_i))$ with the length of the line segment joining these points, as shown in Figure 6.11b for the case where $n = 4$. We have

$$\begin{aligned} s_i &\approx d\{(x(t_{i-1}), y(t_{i-1})), (x(t_i), y(t_i))\} \\ &= \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2}. \end{aligned}$$

Recall that from the Mean Value Theorem (see section 2.10 and make sure you know why we can apply it here), we have that

$$x(t_i) - x(t_{i-1}) = x'(c_i)(t_i - t_{i-1}) = x'(c_i)\,\Delta t$$

and

$$y(t_i) - y(t_{i-1}) = y'(d_i)(t_i - t_{i-1}) = y'(d_i)\,\Delta t,$$

where c_i and d_i are some points in the interval (t_{i-1}, t_i). This gives us

$$\begin{aligned} s_i &\approx \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2} \\ &= \sqrt{[x'(c_i)\,\Delta t]^2 + [y'(d_i)\,\Delta t]^2} \\ &= \sqrt{[x'(c_i)]^2 + [y'(d_i)]^2}\,\Delta t. \end{aligned}$$

Notice that if Δt is small, then c_i and d_i are close together. So, we can make the further approximation

$$s_i \approx \sqrt{[x'(c_i)]^2 + [y'(c_i)]^2}\,\Delta t,$$

for each $i = 1, 2, \ldots, n$. The total arc length is then approximately

$$s \approx \sum_{i=1}^{n} \sqrt{[x'(c_i)]^2 + [y'(c_i)]^2}\,\Delta t.$$

Taking the limit as $n \to \infty$ then gives us the exact arc length, which you should recognize as an integral:

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[x'(c_i)]^2 + [y'(c_i)]^2}\,\Delta t = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt.$$

We summarize this discussion in Theorem 3.1.

THEOREM 3.1 (Arc Length for a Curve Defined Parametrically)

For the curve defined parametrically by $x = x(t)$, $y = y(t)$, $a \le t \le b$, if x' and y' are continuous on $[a, b]$ and the curve does not intersect itself (except possibly at a finite number of points), then the arc length s of the curve is given by

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt. \qquad (3.1)$$

In example 3.1, we illustrate the use of (3.1) to find the arc length of the Scrambler curve from example 2.1.

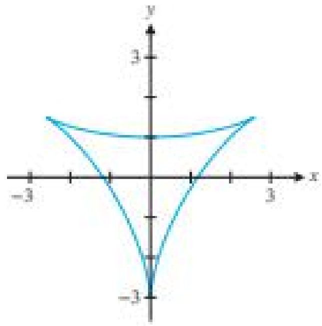


FIGURE 6.12
$x = 2\cos t + \sin 2t$,
$y = 2\sin t + \cos 2t$,
$0 \le t \le 2\pi$

EXAMPLE 3.1 Finding the Arc Length of a Plane Curve

Find the arc length of the Scrambler curve $x = 2\cos t + \sin 2t$, $y = 2\sin t + \cos 2t$, for $0 \le t \le 2\pi$. Also, find the average speed of the Scrambler over this interval.

Solution The curve is shown in Figure 6.12. First, note that x, x', y and y' are all continuous on the interval $[0, 2\pi]$. From (3.1), we then have

$$\begin{aligned} s &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{2\pi} \sqrt{(-2\sin t + 2\cos 2t)^2 + (2\cos t - 2\sin 2t)^2}\,dt \\ &= \int_0^{2\pi} \sqrt{4\sin^2 t - 8\sin t\cos 2t + 4\cos^2 2t + 4\cos^2 t - 8\cos t\sin 2t + 4\sin^2 2t}\,dt \\ &= \int_0^{2\pi} \sqrt{8 - 8\sin t\cos 2t - 8\cos t\sin 2t}\,dt = \int_0^{2\pi} \sqrt{8 - 8\sin 3t}\,dt \approx 16, \end{aligned}$$

since $\sin^2 t + \cos^2 t = 1$, $\cos^2 2t + \sin^2 2t = 1$ and $\sin t\cos 2t + \sin 2t\cos t = \sin 3t$ and where we have approximated the last integral numerically. To find the average speed over the given interval, we simply divide the arc length (i.e. the distance traveled), by the total time, 2π, to obtain

$$s_{ave} \approx \frac{16}{2\pi} \approx 2.5. \blacksquare$$

We want to emphasize that Theorem 3.1 allows the curve to intersect itself at a *finite* number of points, but not to intersect itself over an entire interval of values of the parameter t. To see why this requirement is needed, notice that the parametric equations $x = \cos t$, $y = \sin t$, for $0 \le t \le 4\pi$, describe the circle of radius 1 centered at the origin. However, the circle is traversed *twice* as t ranges from 0 to 4π. If you were to apply (3.1) to this curve, you would obtain

$$\int_0^{4\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{4\pi} \sqrt{(-\sin t)^2 + \cos^2 t}\,dt = 4\pi,$$

which corresponds to twice the arc length (circumference) of the circle. As you can see, if a curve intersects itself over an entire interval of values of t, the arc length of such a portion of the curve is counted twice by (3.1).

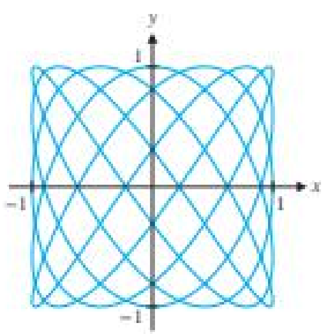


FIGURE 6.13
A Lissajous curve

EXAMPLE 3.2 Finding the Arc Length of a Complicated Plane Curve

Find the arc length of the plane curve $x = \cos 5t$, $y = \sin 7t$, for $0 \le t \le 2\pi$.

Solution This unusual curve (an example of a **Lissajous curve**) is sketched in Figure 6.13. We leave it as an exercise to verify that the hypotheses of Theorem 3.1 are met. From (3.1), we then have that

$$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{2\pi} \sqrt{(-5\sin 5t)^2 + (7\cos 7t)^2}\,dt \approx 36.5,$$

where we have approximated the integral numerically. This is a long curve to be confined within the rectangle $-1 \le x \le 1$, $-1 \le y \le 1$! $\blacksquare$

The arc length formula (3.1) should seem familiar to you. Parametric equations for a curve $y = f(x)$ are $x = t$, $y = f(t)$ and from (3.1), the arc length of this curve for $a \le x \le b$ is then

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b \sqrt{1 + [f'(t)]^2}\,dt,$$

which is the arc length formula derived in section 5.4. Thus, the formula developed in section 5.4 is a special case of (3.1).

Observe that the speed of the Scrambler calculated in example 2.3 and the length of the Scrambler curve found in example 3.1 both depend on the same quantity: $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. Observe that if the parameter t represents time, then $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ represents speed and from Theorem 3.1, the arc length (i.e. the distance traveled) is the integral of the speed with respect to time.

Arc length is a key ingredient in a famous problem called the **brachistochrone problem**. We state this problem in the context of a ski slope consisting of a tilted plane, where a skier wishes to get from a point A at the top of the slope to a point B down the slope (but *not* directly beneath A) in the least time possible. (See Figure 6.14.) Suppose the path taken by the skier is given by the parametric equations $x = x(u)$ and $y = y(u)$, $0 \le u \le 1$, where x and y determine the position of the skier in the plane of the ski slope. (For simplicity, we orient the positive y-axis so that it points down. Also, we name the parameter u since u will, in general, *not* represent time.)

FIGURE 6.14
Downhill skier

To derive a formula for the time required to get from point A to point B, start with the familiar formula distance = rate · time. As seen in the derivation of the arc length formula (3.1), for a small section of the curve, the distance is approximately $\sqrt{[x'(u)]^2 + [y'(u)]^2}\,du$. The rate is harder to identify since we aren't given position as a function of time. For simplicity, we assume that the only effect of friction is to keep the skier on the path and that $y(t) \ge 0$. In this case, using the principle of conservation of energy, it can be shown that the skier's speed is given by $\frac{\sqrt{y(u)}}{k}$ for some constant $k \ge 0$. Putting the pieces together, the total time from point A to point B is given by

$$\text{Time} = \int_0^1 k\sqrt{\frac{[x'(u)]^2 + [y'(u)]^2}{y(u)}}\,du. \tag{3.2}$$

Your first thought might be that the shortest path from point A to point B is along a straight line. If you're thinking of *short* in terms of *distance*, you're right, of course. However, if you think of *short* in terms of *time* (how most skiers would think of it), this is not true. In example 3.3, we show that the fastest path from point A to point B is, in fact, *not* along a straight line, by exhibiting a faster path.

EXAMPLE 3.3 Skiing a Curved Path That is Faster Than Skiing a Straight Line

If point A in our skiing example is $(0, 0)$ and point B is $(\pi, 2)$, show that skiing along the **cycloid** defined by

$$x = \pi u - \sin \pi u, \qquad y = 1 - \cos \pi u$$

is faster than skiing along the line segment connecting the points. Explain the result in physical terms.

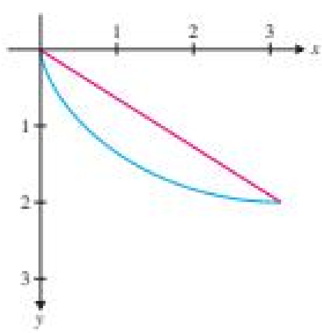


FIGURE 6.15
Two skiing paths

Solution First, note that the line segment connecting the points is given by $x = \pi u$, $y = 2u$, for $0 \le u \le 1$. Further, both curves meet the endpoint requirements that $(x(0), y(0)) = (0, 0)$ and $(x(1), y(1)) = (\pi, 2)$. For the cycloid, we have from (3.2) that

$$\begin{aligned}\text{Time} &= \int_0^1 k\sqrt{\frac{[x'(u)]^2 + [y'(u)]^2}{y(u)}}\,du \\ &= k\int_0^1 \sqrt{\frac{(\pi - \pi\cos \pi u)^2 + (\pi \sin \pi u)^2}{1 - \cos \pi u}}\,du \\ &= k\sqrt{2}\,\pi \int_0^1 \sqrt{\frac{1 - \cos \pi u}{1 - \cos \pi u}}\,du \\ &= k\sqrt{2}\pi.\end{aligned}$$

HISTORICAL NOTES

Jacob Bernoulli (1655–1705) and Johann Bernoulli (1667–1748)

Swiss mathematicians who were instrumental in the development of the calculus. Jacob was the first of several generations of Bernoullis to make important contributions to mathematics. He was active in probability, series and the calculus of variations and introduced the term "integral." Johann followed his brother into mathematics while also earning a doctorate in medicine. Johann first stated l'Hôpital's Rule, one of many results over which he fought bitterly (usually with his brother, but, after Jacob's death, also with his own son Daniel) to receive credit. Both brothers were sensitive, irritable, egotistical (Johann had his tombstone inscribed, "The Archimedes of his age") and quick to criticize others. Their competitive spirit accelerated the development of calculus.

Similarly, for the line segment, we have that

$$\begin{aligned}\text{Time} &= \int_0^1 k\sqrt{\frac{[x'(u)]^2 + [y'(u)]^2}{y(u)}}\,du \\ &= k\int_0^1 \sqrt{\frac{\pi^2 + 2^2}{2u}}\,du \\ &= k\sqrt{2}\sqrt{\pi^2 + 4}.\end{aligned}$$

Notice that the cycloid route is faster since $\pi < \sqrt{\pi^2 + 4}$. The two paths are shown in Figure 6.15. Observe that the cycloid is very steep at the beginning, which would allow a skier to go faster following the cycloid than following the straight line. As it turns out, the greater speed of the cycloid more than compensates for the longer distance of its path. ■

We will ask you to construct some skiing paths of your own in the exercises. However, it has been proved that the cycloid is the plane curve with the shortest time (which is what the Greek root words for **brachistochrone** mean). In addition, we will give you an opportunity to discover another remarkable property of the cycloid, relating to another famous problem, the **tautochrone problem**. Both problems have an interesting history focused on brothers Jacob and Johann Bernoulli, who solved the problem in 1697 (along with Newton, Leibniz and l'Hôpital) and argued incessantly about who deserved credit.

Much as we did in section 5.4, we can use our arc length formula to find a formula for the surface area of a surface of revolution. Recall that if the curve $y = f(x)$ for $c \le x \le d$ is revolved about the x-axis (see Figure 6.16), the surface area is given by

$$\text{Surface Area} = \int_c^d 2\pi \underbrace{|f(x)|}_{\text{radius}} \underbrace{\sqrt{1 + [f'(x)]^2}}_{\text{arc length}}\,dx.$$

Let C be the curve defined by the parametric equations $x = x(t)$ and $y = y(t)$ with $a \le t \le b$, where x, x', y and y' are continuous and where the curve does not intersect itself for $a \le t \le b$. We leave it as an exercise to derive the corresponding formula for parametric equations:

$$\text{Surface Area} = \int_a^b 2\pi \underbrace{|y(t)|}_{\text{radius}} \underbrace{\sqrt{[x'(t)]^2 + [y'(t)]^2}}_{\text{arc length}}\,dt.$$

More generally, we have that if the curve is revolved about the line $y = c$, the surface area is given by

$$\text{Surface Area} = \int_a^b 2\pi \underbrace{|y(t) - c|}_{\text{radius}} \underbrace{\sqrt{[x'(t)]^2 + [y'(t)]^2}}_{\text{arc length}}\,dt. \tag{3.3}$$

Likewise, if we revolve the curve about the line $x = d$, the surface area is given by

$$\text{Surface Area} = \int_a^b 2\pi \underbrace{|x(t) - d|}_{\text{radius}} \underbrace{\sqrt{[x'(t)]^2 + [y'(t)]^2}}_{\text{arc length}}\,dt. \tag{3.4}$$

Look carefully at what all of the surface area formulas have in common. That is, in each case, the surface area is given by

SURFACE AREA

$$\text{Surface Area} = \int_a^b 2\pi(\text{radius})(\text{arc length})\,dt. \tag{3.5}$$

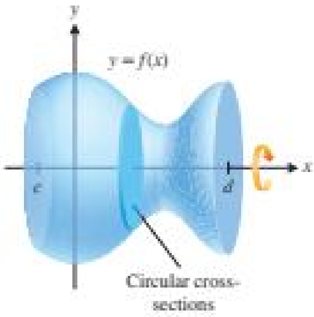


FIGURE 6.16
Surface of revolution

Look carefully at the graph of the curve and the axis about which you are revolving, to see how to fill in the blanks in (3.5). As we observed in section 5.4, it is very important that you draw a picture here.

EXAMPLE 3.4 Finding Surface Area with Parametric Equations

Find the surface area of the surface formed by revolving the half-ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, $y \geq 0$, about the x-axis. (See Figure 6.17.)

Solution It would truly be a mess to set up the integral for $y = f(x) = 2\sqrt{1 - x^2/9}$. (Think about this!) Instead, notice that you can represent the curve by the parametric equations $x = 3\cos t$, $y = 2\sin t$, for $0 \leq t \leq \pi$. From (3.3), the surface area is then given by

$$\begin{aligned}\text{Surface Area} &= \int_0^{\pi} 2\pi \underbrace{(2\sin t)}_{\text{radius}} \underbrace{\sqrt{(-3\sin t)^2 + (2\cos t)^2}\,dt}_{\text{arc length}} \\ &= 4\pi \int_0^{\pi} \sin t \sqrt{9\sin^2 t + 4\cos^2 t}\,dt \\ &= 4\pi \frac{9\sqrt{5}\sin^{-1}(\sqrt{5}/3) + 10}{5} \approx 67.7,\end{aligned}$$

where we used a CAS to evaluate the integral. ■

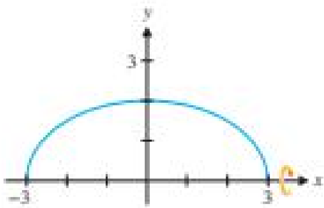


FIGURE 6.17
$y = 2\sqrt{\frac{1 - x^2}{9}}$

EXAMPLE 3.5 Revolving about a Line Other Than a Coordinate Axis

Find the surface area of the surface formed by revolving the curve $x = \sin 2t$, $y = \cos 3t$, for $0 \leq t \leq \pi/3$, about the line $x = 2$.

Solution A sketch of the curve is shown in Figure 6.18. Since the x-values on the curve are all less than 2, the radius of the solid of revolution is $2 - x = 2 - \sin 2t$ and so, from (3.4), the surface area is given by

$$\text{Surface Area} = \int_0^{\pi/3} 2\pi \underbrace{(2 - \sin 2t)}_{\text{radius}} \underbrace{\sqrt{[2\cos 2t]^2 + [-3\sin 3t]^2}\,dt}_{\text{arc length}} \approx 20.1,$$

where we have approximated the value of the integral numerically. ■

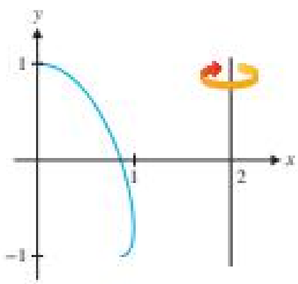


FIGURE 6.18
$x = \sin 2t$, $y = \cos 3t$

In example 3.6, we model a physical process with parametric equations. Since the modeling process is itself of great importance, be sure that you understand all of the steps. Also, see if you can find an alternative approach to this problem.

EXAMPLE 3.6 Arc Length for a Falling Ladder

An 8 m tall ladder stands vertically against a wall. The bottom of the ladder is pulled along the floor, with the top remaining in contact with the wall, until the ladder rests flat on the floor. Find the distance traveled by the midpoint of the ladder.

Solution We first find parametric equations for the position of the midpoint of the ladder. We orient the x- and y-axes as shown in Figure 6.19.

Let x denote the distance from the wall to the bottom of the ladder and let y denote the distance from the floor to the top of the ladder. Since the ladder is 8 m long, observe that $x^2 + y^2 = 64$. Defining the parameter $t = x$, we have $y = \sqrt{64 - t^2}$.

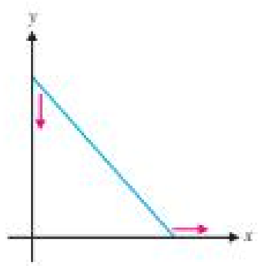


FIGURE 6.19
Ladder sliding down a wall

The midpoint of the ladder has coordinates $\left(\frac{x}{2}, \frac{y}{2}\right)$ and so, parametric equations for the midpoint are

$$\begin{cases} x(t) = \frac{1}{2}t \\ y(t) = \frac{1}{2}\sqrt{64 - t^2} \end{cases}$$

When the ladder stands vertically against the wall, we have $x = 0$ and when it lies flat on the floor, $x = 4$. So, $0 \le t \le 8$. From (3.1), the arc length is then given by

$$s = \int_0^8 \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\frac{-t}{\sqrt{64 - t^2}}\right)^2}\, dt = \int_0^8 \sqrt{\frac{1}{4}\left(1 + \frac{t^2}{64 - t^2}\right)}\, dt$$

$$= \int_0^8 \frac{1}{2}\sqrt{\frac{64}{64 - t^2}}\, dt = \int_0^8 \frac{1}{2}\sqrt{\frac{1}{1 - (t/8)^2}}\, dt.$$

Substituting $u = \frac{t}{8}$ gives us $du = \frac{1}{8}dt$ or $dt = 8\, du$. For the limits of integration, note that when $t = 0$, $u = 0$ and when $t = 8$, $u = 1$. The arc length is then

$$s = \int_0^8 \frac{1}{2}\sqrt{\frac{1}{1 - (t/8)^2}}\, dt = \int_0^1 \frac{1}{2}\sqrt{\frac{1}{1 - u^2}}\, 8\, du = 4 \sin^{-1} u \Big|_{u=0}^{u=1}$$

$$= 4\left(\frac{\pi}{2} - 0\right) = 2\pi.$$

Since this is a rare arc length integral that can be evaluated exactly, you might be suspicious that there is an easier way to find the arc length. We explore this in the exercises. ■

EXERCISES 6.3

WRITING EXERCISES

1. In the derivation preceding Theorem 3.1, we justified the equation

$$g(t_i) - g(t_{i-1}) = g'(c_i)\,\Delta t.$$

Thinking of $g(t)$ as position and $g'(t)$ as velocity, explain why this makes sense.

2. The curve in example 3.2 was a long curve contained within a small rectangle. What would you guess would be the maximum length for a curve contained in such a rectangle? Briefly explain.

3. In example 3.3, we noted that the steeper initial slope of the cycloid would allow the skier to build up more speed than the straight-line path. The cycloid takes this idea to the limit by having a vertical tangent line at the origin. Explain why, despite the vertical tangent line, it is still physically possible for the skier to stay on this slope. (Hint: How do the two dimensions of the path relate to the three dimensions of the ski slope?)

4. The tautochrone problem discussed in exploratory exercise 2 involves starting on the same curve at two different places and comparing the times required to reach the end. For the cycloid, compare the speed of a skier starting at the origin versus one starting halfway to the bottom. Explain why it is not clear whether starting halfway down would get you to the bottom faster.

In exercises 1–8, find the arc length of each curve; compute one exactly and approximate the other numerically.

1. (a) $\begin{cases} x = 2\cos t \\ y = 4\sin t \end{cases}$ (b) $\begin{cases} x = 1 - 2\cos t \\ y = 2 + 2\sin t \end{cases}$

2. (a) $\begin{cases} x = t^3 - 4 \\ y = t^2 - 3 \end{cases}, \ -2 \le t \le 2$

(b) $\begin{cases} x = t^3 - 4t \\ y = t^2 - 3t \end{cases}, \ -2 \le t \le 2$

3. (a) $\begin{cases} x = \cos 4t \\ y = \sin 4t \end{cases}$ (b) $\begin{cases} x = \cos 7t \\ y = \sin 11t \end{cases}$

4. (a) $\begin{cases} x = t\cos t \\ y = t\sin t \end{cases}, \ -1 \le t \le 1$

(b) $\begin{cases} x = t^2\cos t \\ y = t^2\sin t \end{cases}, \ -1 \le t \le 1$

5. (a) $\begin{cases} x = \sin t\cos t \\ y = \sin^2 t \end{cases}, \ 0 \le t \le \pi/2$

(b) $\begin{cases} x = \sin 4t\cos t \\ y = \sin 4t\sin t \end{cases}, \ 0 \le t \le \pi/2$

6. (a) $\begin{cases} x = \sin t \\ y = \sin \pi t \end{cases}, \ 0 \le t \le \pi$

(b) $\begin{cases} x = \sin t \\ y = \pi\sin t \end{cases}, \ 0 \le t \le \pi$

7. (a) $\begin{cases} x = e^{2t} + e^{-2t} \\ y = 4t \end{cases}$, $0 \le t \le 4$

(b) $\begin{cases} x = e^{2t} \\ y = t^2 \end{cases}$, $0 \le t \le 4$

8. (a) $\begin{cases} x = 4t \\ y = \sqrt{t^2 + 4} \end{cases}$, $1 \le t \le 2$

(b) $\begin{cases} x = 36t + 2 \\ y = t^3 + 3/t \end{cases}$, $1 \le t \le 2$

In exercises 9–12, (a) show that the curve starts at the origin at $t = 0$ and reaches the point $(\pi, 2)$ at $t = 1$. (b) Use the time formula (3.2) to determine how long it would take a skier to take the given path. (c) Find the slope at the origin and the arc length for the curve in the indicated exercise.

9. $\begin{cases} x = \pi t \\ y = 2\sqrt{t} \end{cases}$

10. $\begin{cases} x = \pi t \\ y = 2\sqrt[4]{t} \end{cases}$

11. $\begin{cases} x = -\frac{1}{2}\pi(\cos \pi t - 1) \\ y = 2t + \frac{7}{10}\sin \pi t \end{cases}$

12. $\begin{cases} x = \pi t - 0.6 \sin \pi t \\ y = 2t + 0.4 \sin \pi t \end{cases}$

In exercises 13–18, compute the surface area of the surface obtained by revolving the given curve about the indicated axis.

13. $\begin{cases} x = t^2 - 1 \\ y = t^3 - 4t \end{cases}$, $-2 \le t \le 0$ (a) about the x-axis (b) about $x = -1$

14. $\begin{cases} x = t^2 - 1 \\ y = t^3 - 4t \end{cases}$, $0 \le t \le 2$ (a) about the x-axis (b) about $x = 3$

15. $\begin{cases} x = t^3 - 4t \\ y = t^2 - 3 \end{cases}$, $0 \le t \le 2$ (a) about the y-axis (b) about $y = 2$

16. $\begin{cases} x = 4t \\ y = \sqrt{t^2 + 4} \end{cases}$, $1 \le t \le 2$ (a) about the x-axis (b) about $x = 4$

17. $\begin{cases} x = 2t \\ y = 2\cos t \end{cases}$, $0 \le t \le \frac{\pi}{2}$ (a) about the y-axis (b) about $y = 3$

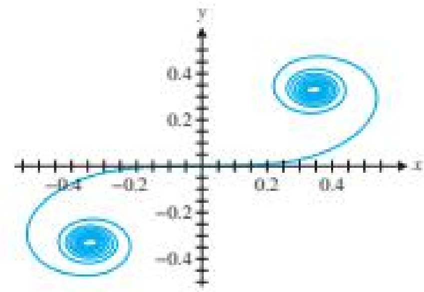
18. $\begin{cases} x = \ln t \\ y = e^{-t} \end{cases}$, $1 \le t \le 2$ (a) about the x-axis (b) about the y-axis

APPLICATIONS

1. An 8 m tall ladder stands vertically against a wall. The top of the ladder is pulled directly away from the wall, with the bottom remaining in contact with the wall, until the ladder rests on the floor. Find parametric equations for the position of the midpoint of the ladder. Find the distance traveled by the midpoint of the ladder.

2. The answer in exercise 1 equals the circumference of a quarter-circle of radius 4. Discuss whether this is a coincidence or not. Compare this value to the arc length in example 3.6. Discuss whether or not this is a coincidence.

3. The figure shown here is called **Cornu's spiral**. It is defined by the parametric equations $x = \int_0^t \cos \pi s^2\, ds$ and $y = \int_0^t \sin \pi s^2\, ds$. Each of these integrals is important in the study of Fresnel diffraction. Find the arc length of the spiral for (a) $-2\pi \le t \le 2\pi$ and (b) general $a \le t \le b$. Use this result to discuss the rate at which the spiraling occurs.



4. A cycloid is the curve traced out by a point on a circle as the circle rolls along the x-axis. Suppose the circle has radius 4, the point we are following starts at (0, 8) and the circle rolls from left to right. Find parametric equations for the cycloid and find the arc length as the circle completes one rotation.

EXPLORATORY EXERCISES

1. For the brachistochrone problem, two criteria for the fastest curve are: (1) steep slope at the origin and (2) concave down (note in Figure 6.15 that the positive y-axis points downward). Explain why these criteria make sense. Identify other reasonable criteria. Then find parametric equations for a curve (different from the cycloid or those of exercises 9–12) that meet all the criteria. Use the formula of example 3.3 to find out how fast your curve is. You can't beat the cycloid, but get as close as you can!

2. The **tautochrone** problem is another surprising problem that was studied and solved by the same seventeenth-century mathematicians as the brachistochrone problem. (See *Journey through Genius* by William Dunham[1] for a description of this interesting piece of history, featuring the brilliant yet combative Bernoulli brothers.) Recall that the cycloid of example 3.3 runs from (0, 0) to $(\pi, 2)$. It takes the skier $k\sqrt{2}\pi = \pi/g$ seconds to ski the path. How long would it take the skier starting partway down the path, for instance at $(\pi/2 - 1, 1)$? Find the slope of the cycloid at this point and compare it to the slope at (0, 0). Explain why the skier would build up less speed starting at this new point. Graph the speed function for the cycloid with $0 \le u \le 1$ and explain why the farther down the slope you start, the less speed you'll have. To see how speed and distance balance, use the time formula

$$T = \frac{\pi}{g}\int_a^1 \frac{\sqrt{1 - \cos \pi u}}{\sqrt{\cos \pi a - \cos \pi u}}\, du$$

for the time it takes to ski the cycloid starting at the point $(\pi a - \sin \pi a,\ 1 - \cos \pi a)$, $0 < a < 1$. What is the remarkable property that the cycloid has?

[1]Dunham, W. (1990). *Journey through Genius* (New York: Wiley).

6.4 POLAR COORDINATES

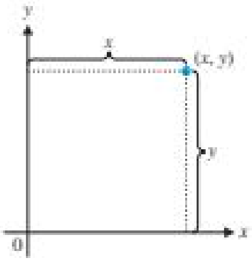


FIGURE 6.20
Rectangular coordinates

The familiar x-y coordinate system is often referred to as a system of **rectangular coordinates**, because a point is described in terms of its horizontal and vertical distances from the origin. (See Figure 6.20.)

An alternative description of a point in the xy-plane consists of specifying the distance r from the point to the origin and an angle θ (in radians) measured from the positive x-axis counterclockwise to the ray connecting the point and the origin. (See Figure 6.21.) We describe the point by the ordered pair (r, θ) and refer to r and θ as **polar coordinates** for the point. For convenience, we allow r to be negative, with the understanding that in this case, the point is in the opposite direction from that indicated by the angle θ.

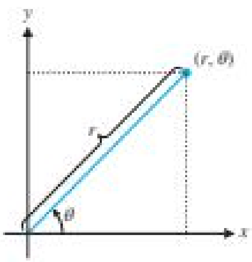


FIGURE 6.21
Polar coordinates

EXAMPLE 4.1 Plotting Points in Polar Coordinates

Plot the points with the indicated polar coordinates (r, θ) and determine the corresponding rectangular coordinates (x, y) for: (a) $(2, 0)$, (b) $(3, \frac{\pi}{2})$, (c) $(-3, \frac{\pi}{2})$ and (d) $(2, \pi)$.

Solution (a) Notice that the angle $\theta = 0$ locates the point on the positive x-axis. At a distance of $r = 2$ units from the origin, this corresponds to the point $(2, 0)$ in rectangular coordinates. (See Figure 6.22a.)

(b) The angle $\theta = \frac{\pi}{2}$ locates points on the positive y-axis. At a distance of $r = 3$ units from the origin, this corresponds to the point $(0, 3)$ in rectangular coordinates. (See Figure 6.22b.)

(c) The angle is the same as in (b), but a negative value of r indicates that the point is located 3 units in the opposite direction, at the point $(0, -3)$ in rectangular coordinates. (See Figure 6.22b.)

(d) The angle $\theta = \pi$ corresponds to the negative x-axis. The distance of $r = 2$ units from the origin gives us the point $(-2, 0)$ in rectangular coordinates. (See Figure 6.22c.)

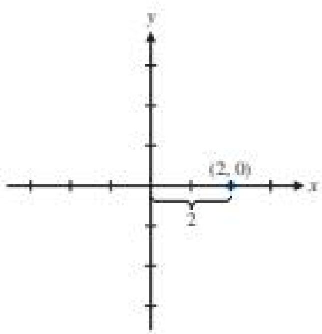


FIGURE 6.22a
The point $(2, 0)$ in polar coordinates

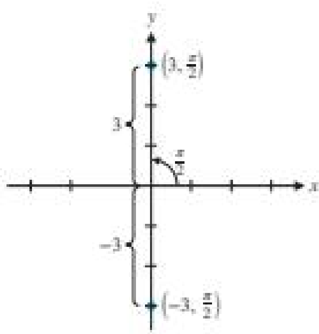


FIGURE 6.22b
The points $\left(3, \frac{\pi}{2}\right)$ and $\left(-3, \frac{\pi}{2}\right)$ in polar coordinates

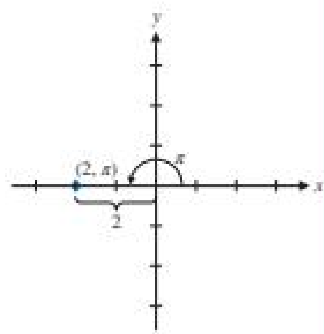


FIGURE 6.22c
The point $(2, \pi)$ in polar coordinates

■

EXAMPLE 4.2 Converting from Rectangular to Polar Coordinates

Find a polar coordinate representation of the rectangular point $(1, 1)$.

Solution From Figure 6.23a, notice that the point lies on the line $y = x$, which makes an angle of $\frac{\pi}{4}$ with the positive x-axis. From the distance formula, we get that $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. This says that we can write the point as $(\sqrt{2}, \frac{\pi}{4})$ in polar coordinates. Referring to Figure 6.23b, notice that we can specify the same point by using a negative value of r, $r = -\sqrt{2}$, with the angle $\frac{5\pi}{4}$. (Think about this further.) Notice also, that the angle $\frac{9\pi}{4} = \frac{\pi}{4} + 2\pi$ corresponds to the same ray shown in Figure 6.23a. (See Figure 6.23c.) In fact, all of the polar points $(\sqrt{2}, \frac{\pi}{4} + 2n\pi)$ and $(-\sqrt{2}, \frac{5\pi}{4} + 2n\pi)$ for any integer n correspond to the same point in the xy-plane.

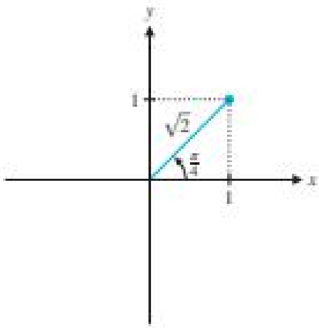


FIGURE 6.23a
Polar coordinates for the point (1, 1)

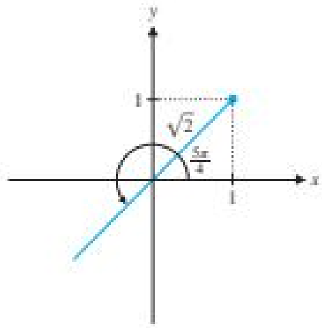


FIGURE 6.23b
An alternative polar representation of (1, 1)

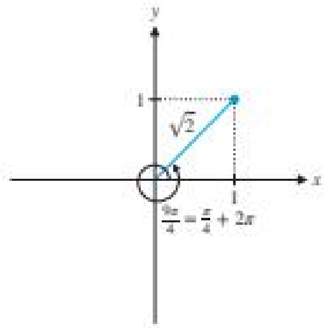


FIGURE 6.23c
Another polar representation of the point (1, 1)

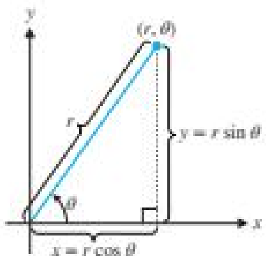


FIGURE 6.24
Converting from polar to rectangular coordinates

Referring to Figure 6.24, notice that it is a simple matter to find the rectangular coordinates (x, y) of a point specified in polar coordinates as (r, θ). From the usual definitions for $\sin\theta$ and $\cos\theta$, we get

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{4.1}$$

From the equations in (4.1), notice that for a point (x, y) in the plane,

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

and for $x \neq 0$,

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

That is, every polar coordinate representation (r, θ) of the point (x, y), where $x \neq 0$ must satisfy

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}. \tag{4.2}$$

REMARK 4.1

As we saw in example 4.2, each point (x, y) in the plane has infinitely many polar coordinate representations. For a given angle θ, the angles $\theta \pm 2\pi$, $\theta \pm 4\pi$ and so on, all correspond to the same ray. For convenience, we use the notation $\theta + 2n\pi$ (for any integer n) to represent all of these possible angles.

Notice that since there's more than one choice of r and θ, we cannot actually solve the equations in (4.2) to produce formulas for r and θ. In particular, while you might be tempted to write $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, this is not the only possible choice. Remember that for (r, θ) to be a polar representation of the point (x, y), θ can be *any* angle for which $\tan\theta = \frac{y}{x}$, while $\tan^{-1}\left(\frac{y}{x}\right)$ gives you an angle θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Finding polar coordinates for a given point is typically a process involving some graphing and some thought.

EXAMPLE 4.3 Converting from Rectangular to Polar Coordinates

Find all polar coordinate representations for the rectangular points (a) (2, 3) and (b) (−3, 1).

Solution (a) With $x = 2$ and $y = 3$, we have from (4.2) that

$$r^2 = x^2 + y^2 = 2^2 + 3^2 = 13,$$

REMARK 4.2

Notice that for any point (x, y) specified in rectangular coordinates $(x \neq 0)$, we can always write the point in polar coordinates using either of the polar angles $\tan^{-1}\left(\frac{y}{x}\right)$ or $\tan^{-1}\left(\frac{y}{x}\right) + \pi$. You can determine which angle corresponds to $r = \sqrt{x^2 + y^2}$ and which corresponds to $r = -\sqrt{x^2 + y^2}$ by looking at the quadrant in which the point lies.

so that $r = \pm\sqrt{13}$. Also,

$$\tan\theta = \frac{y}{x} = \frac{3}{2}.$$

One angle is then $\theta = \tan^{-1}\left(\frac{3}{2}\right) \approx 0.98$ radian. To determine which choice of r corresponds to this angle, note that (2, 3) is located in the first quadrant. (See Figure 6.25a.) Since 0.98 radian also puts you in the first quadrant, this angle corresponds to the positive value of r, so that $\left(\sqrt{13}, \tan^{-1}\left(\frac{3}{2}\right)\right)$ is one polar representation of the point. The negative choice of r corresponds to an angle one half-circle (i.e., π radians) away (see Figure 6.25b), so that another representation is $\left(-\sqrt{13}, \tan^{-1}\left(\frac{3}{2}\right) + \pi\right)$. Every other polar representation is found by adding multiples of 2π to the two angles used above. That is, every polar representation of the point (2, 3) must have the form $\left(\sqrt{13}, \tan^{-1}\left(\frac{3}{2}\right) + 2n\pi\right)$ or $\left(-\sqrt{13}, \tan^{-1}\left(\frac{3}{2}\right) + \pi + 2n\pi\right)$, for some integer choice of n.

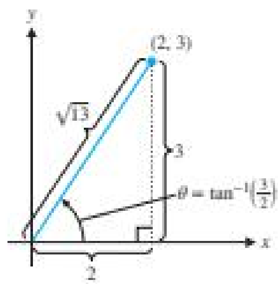


FIGURE 6.25a
The point (2, 3)

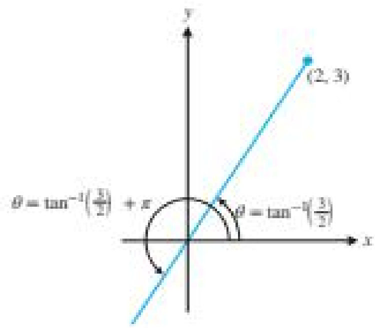


FIGURE 6.25b
Negative value of r

(b) For the point (−3, 1), we have $x = -3$ and $y = 1$. From (4.2), we have

$$r^2 = x^2 + y^2 = (-3)^2 + 1^2 = 10,$$

so that $r = \pm\sqrt{10}$. Further,

$$\tan\theta = \frac{y}{x} = \frac{1}{-3},$$

so that the most obvious choice for the polar angle is $\theta = \tan^{-1}\left(-\frac{1}{3}\right) \approx -0.32$, which lies in the fourth quadrant. Since the point (−3, 1) is in the second quadrant, this choice of the angle corresponds to the negative value of r. (See Figure 6.26.) The positive value of r then corresponds to the angle $\theta = \tan^{-1}\left(-\frac{1}{3}\right) + \pi$. Observe that all polar coordinate representations must then be of the form $\left(-\sqrt{10}, \tan^{-1}\left(-\frac{1}{3}\right) + 2n\pi\right)$ or $\left(\sqrt{10}, \tan^{-1}\left(-\frac{1}{3}\right) + \pi + 2n\pi\right)$, for some integer choice of n. ■

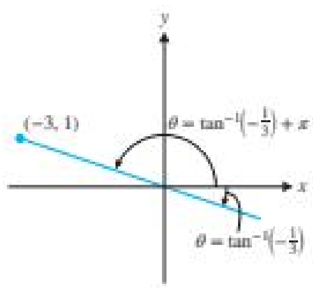


FIGURE 6.26
The point (−3, 1)

The conversion from polar coordinates to rectangular coordinates is completely straightforward, as seen in example 4.4.

EXAMPLE 4.4 Converting from Polar to Rectangular Coordinates

Find the rectangular coordinates for the polar points (a) $\left(3, \frac{\pi}{6}\right)$ and (b) (−2, 3).

Solution For (a), we have from (4.1) that

$$x = r\cos\theta = 3\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2}$$

and

$$y = r\sin\theta = 3\sin\frac{\pi}{6} = \frac{3}{2}.$$

The rectangular point is then $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. For (b), we have

$$x = r\cos\theta = -2\cos 3 \approx 1.98$$

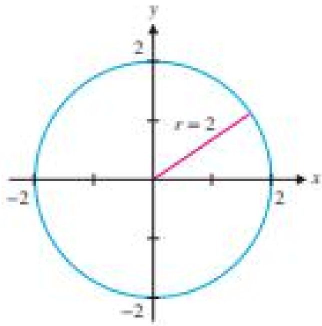


FIGURE 6.27a
The circle $r = 2$

and $$y = r\sin\theta = -2\sin 3 \approx -0.28.$$

The rectangular point is then $(-2\cos 3, -2\sin 3)$, which is located at approximately $(1.98, -0.28)$. ■

The **graph** of a polar equation $r = f(\theta)$ is the set of all points (x, y) for which $x = r\cos\theta$, $y = r\sin\theta$ and $r = f(\theta)$. In other words, the graph of a polar equation is a graph in the xy-plane of all those points whose polar coordinates satisfy the given equation. We begin by sketching two very simple (and familiar) graphs. The key to drawing the graph of a polar equation is to always keep in mind what the polar coordinates represent.

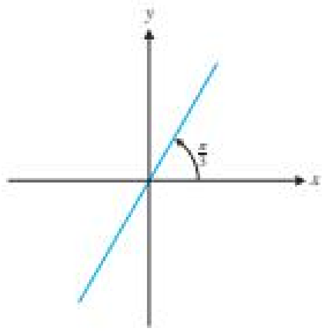


FIGURE 6.27b
The line $\theta = \frac{\pi}{3}$

EXAMPLE 4.5 Some Simple Graphs in Polar Coordinates

Sketch the graphs of (a) $r = 2$ and (b) $\theta = \pi/3$.

Solution For (a), notice that $2 = r = \sqrt{x^2 + y^2}$ and so, we want all points whose distance from the origin is 2 (with *any* polar angle θ). Of course, this is the definition of a circle of radius 2 with center at the origin. (See Figure 6.27a.) For (b), notice that $\theta = \pi/3$ specifies all points with a polar angle of $\pi/3$ from the positive x-axis (at *any* distance r from the origin). Including negative values for r, this defines a line with slope $\tan \pi/3 = \sqrt{3}$. (See Figure 6.27b.) ■

It turns out that many familiar curves have simple polar equations.

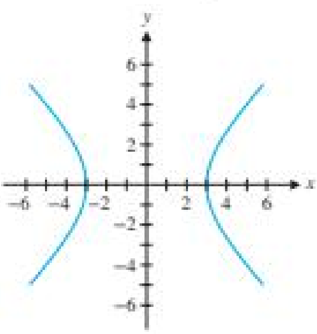


FIGURE 6.28
$x^2 - y^2 = 9$

EXAMPLE 4.6 Converting an Equation from Rectangular to Polar Coordinates

Find the polar equation(s) corresponding to the hyperbola $x^2 - y^2 = 9$. (See Figure 6.28.)

Solution From (4.1), we have

$$9 = x^2 - y^2 = r^2\cos^2\theta - r^2\sin^2\theta$$
$$= r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta.$$

Solving for r, we get

$$r^2 = \frac{9}{\cos 2\theta} = 9\sec 2\theta,$$

so that $$r = \pm 3\sqrt{\sec 2\theta}.$$

Notice that in order to keep $\sec 2\theta > 0$, we can restrict 2θ to lie in the interval $-\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$, so that $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Observe that with this range of values of θ, the hyperbola is drawn exactly once, where $r = 3\sqrt{\sec 2\theta}$ corresponds to the right branch of the hyperbola and $r = -3\sqrt{\sec 2\theta}$ corresponds to the left branch. ■

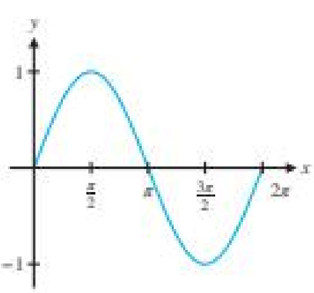


FIGURE 6.29a
$y = \sin x$ plotted in *rectangular* coordinates

EXAMPLE 4.7 A Surprisingly Simple Polar Graph

Sketch the graph of the polar equation $r = \sin\theta$.

Solution For reference, we first sketch a graph of the sine function in *rectangular* coordinates on the interval $[0, 2\pi]$. (See Figure 6.29a.) Notice that on the interval $0 \le \theta \le \frac{\pi}{2}$, $\sin\theta$ increases from 0 to its maximum value of 1. This corresponds to a polar arc in the first quadrant from the origin ($r = 0$) to 1 unit up on the y-axis. Then, on the interval $\frac{\pi}{2} \le \theta \le \pi$, $\sin\theta$ decreases from 1 to 0, corresponding to an arc in the second quadrant, from 1 unit up on the y-axis back to the origin. Next, on the interval $\pi \le \theta \le \frac{3\pi}{2}$, $\sin\theta$ decreases from 0 to its minimum value of -1. Since the values of r are negative, remember that this means that the points are plotted in the *opposite* quadrant (i.e. the first quadrant), tracing out the same curve in the first quadrant as we've already drawn for $0 \le \theta \le \frac{\pi}{2}$. Likewise, taking θ in the interval $\frac{3\pi}{2} \le \theta \le 2\pi$ retraces the portion

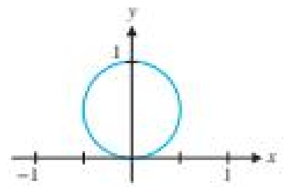


FIGURE 6.29b
The circle $r = \sin \theta$

of the curve in the second quadrant. Since $\sin \theta$ is periodic of period 2π, taking further values of θ simply retraces portions of the curve that we have already drawn. A sketch of the polar graph is shown in Figure 6.29b. We now verify that this curve is actually a circle. Multiplying the polar equation through by r, we get

$$r^2 = r \sin \theta.$$

From (4.1) and (4.2) we have that $y = r \sin \theta$ and $r^2 = x^2 + y^2$, which gives us the rectangular equation

$$x^2 + y^2 = y$$

or

$$0 = x^2 + y^2 - y.$$

Completing the square, we get

$$0 = x^2 + \left(y^2 - y + \frac{1}{4}\right) - \frac{1}{4}$$

and adding $\frac{1}{4}$ to both sides, we get

$$\left(\frac{1}{2}\right)^2 = x^2 + \left(y - \frac{1}{2}\right)^2.$$

This is the rectangular equation for the circle of radius $\frac{1}{2}$ centered at the point $\left(0, \frac{1}{2}\right)$, which is what we see in Figure 6.29b. ■

The graphs of many polar equations are not the graphs of *any* functions of the form $y = f(x)$, as in example 4.8.

EXAMPLE 4.8 An Archimedian Spiral

Sketch the graph of the polar equation $r = \theta$, for $\theta \geq 0$.

Solution Notice that here, as θ increases, so too does r. That is, as the polar angle increases, the distance from the origin also increases accordingly. This produces the spiral (an example of an **Archimedian spiral**) seen in Figure 6.30. ■

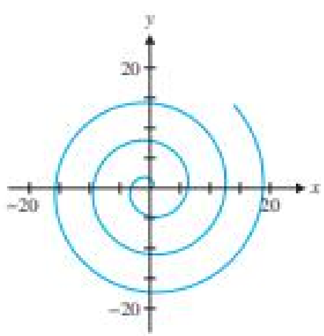


FIGURE 6.30
The spiral $r = \theta, \theta \geq 0$

The graphs shown in examples 4.9, 4.10 and 4.11 are all in the general class known as **limaçons**. This class of graphs is defined by $r = a \pm b \sin \theta$ or $r = a \pm b \cos \theta$, for positive constants a and b. If $a = b$, the graphs are called **cardioids**.

EXAMPLE 4.9 A Limaçon

Sketch the graph of the polar equation $r = 3 + 2 \cos \theta$.

Solution We begin by sketching the graph of $y = 3 + 2 \cos x$ in rectangular coordinates on the interval $[0, 2\pi]$, to use as a reference. (See Figure 6.31.) Notice that in this case, we have $r = 3 + 2 \cos \theta > 0$ for all values of θ. Further, the maximum value of r is 5 (corresponding to when $\cos \theta = 1$ at $\theta = 0, 2\pi$, etc.) and the minimum value of r is 1 (corresponding to when $\cos \theta = -1$ at $\theta = \pi, 3\pi$, etc.). In this case, the polar graph is traced out with $0 \leq \theta \leq 2\pi$. We summarize the intervals of increase and decrease for r in the following table.

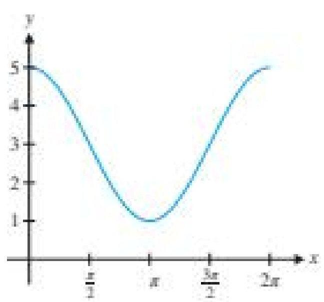


FIGURE 6.31
$y = 3 + 2 \cos x$ in rectangular coordinates

Interval	$\cos \theta$	$r = 3 + 2 \cos \theta$
$\left[0, \frac{\pi}{2}\right]$	Decreases from 1 to 0	Decreases from 5 to 3
$\left[\frac{\pi}{2}, \pi\right]$	Decreases from 0 to −1	Decreases from 3 to 1
$\left[\pi, \frac{3\pi}{2}\right]$	Increases from −1 to 0	Increases from 1 to 3
$\left[\frac{3\pi}{2}, 2\pi\right]$	Increases from 0 to 1	Increases from 3 to 5

In Figures 6.32a–6.32d, we show how the sketch progresses through each interval indicated in the table, with the completed figure (called a **limaçon**) shown in Figure 6.32d.

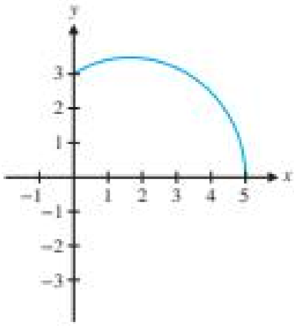


FIGURE 6.32a
$0 \le \theta \le \frac{\pi}{2}$

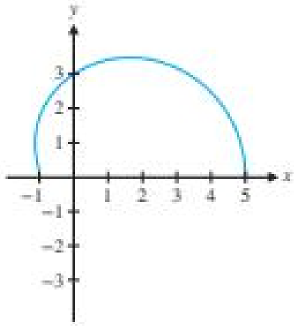


FIGURE 6.32b
$0 \le \theta \le \pi$

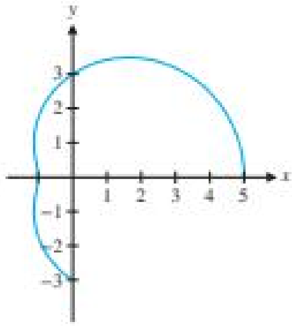


FIGURE 6.32c
$0 \le \theta \le \frac{3\pi}{2}$

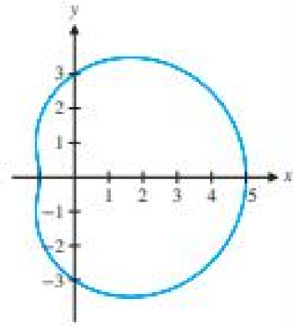


FIGURE 6.32d
$0 \le \theta \le 2\pi$

EXAMPLE 4.10 The Graph of a Cardioid

Sketch the graph of the polar equation $r = 2 - 2\sin\theta$.

Solution As we have done several times now, we first sketch a graph of $y = 2 - 2\sin x$ in rectangular coordinates, on the interval $[0, 2\pi]$, as in Figure 6.33. We summarize the intervals of increase and decrease in the following table.

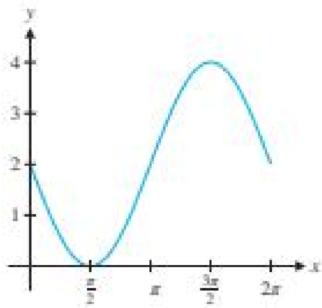


FIGURE 6.33
$y = 2 - 2\sin x$ in rectangular coordinates

Interval	$\sin\theta$	$r = 2 - 2\sin\theta$
$\left[0, \frac{\pi}{2}\right]$	Increases from 0 to 1	Decreases from 2 to 0
$\left[\frac{\pi}{2}, \pi\right]$	Decreases from 1 to 0	Increases from 0 to 2
$\left[\pi, \frac{3\pi}{2}\right]$	Decreases from 0 to −1	Increases from 2 to 4
$\left[\frac{3\pi}{2}, 2\pi\right]$	Increases from −1 to 0	Decreases from 4 to 2

Again, we sketch the graph in stages, corresponding to each of the intervals indicated in the table, as seen in Figures 6.34a–6.34d.

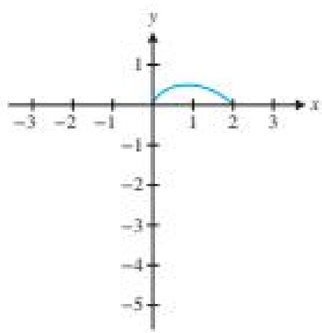


FIGURE 6.34a
$0 \le \theta \le \frac{\pi}{2}$

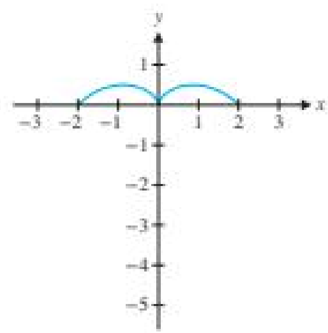


FIGURE 6.34b
$0 \le \theta \le \pi$

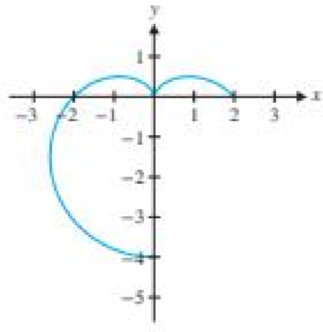


FIGURE 6.34c
$0 \le \theta \le \frac{3\pi}{2}$

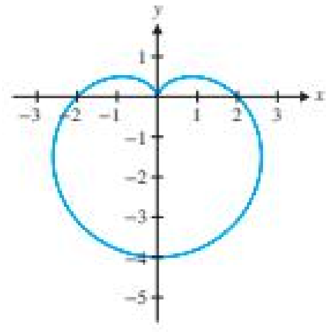


FIGURE 6.34d
$0 \le \theta \le 2\pi$

The completed graph appears in Figure 6.34d and is sketched out for $0 \le \theta \le 2\pi$. You can see why this figure is called a **cardioid** ("heartlike"). ■

EXAMPLE 4.11 The Graph of a Limaçon with a Loop

Sketch the graph of the polar equation $r = 1 - 2\sin\theta$.

Solution We again begin by sketching a graph of $y = 1 - 2\sin x$ in rectangular coordinates, as in Figure 6.35. We summarize the intervals of increase and decrease in the following table:

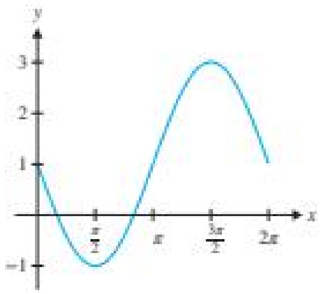


FIGURE 6.35
$y = 1 - 2\sin x$ in rectangular coordinates

Interval	$\sin\theta$	$r = 1 - 2\sin\theta$
$\left[0, \frac{\pi}{2}\right]$	Increases from 0 to 1	Decreases from 1 to −1
$\left[\frac{\pi}{2}, \pi\right]$	Decreases from 1 to 0	Increases from −1 to 1
$\left[\pi, \frac{3\pi}{2}\right]$	Decreases from 0 to −1	Increases from 1 to 3
$\left[\frac{3\pi}{2}, 2\pi\right]$	Increases from −1 to 0	Decreases from 3 to 1

Notice that since r assumes both positive and negative values in this case, we need to exercise a bit more caution, as negative values for r cause us to draw that portion of the graph in the *opposite* quadrant. Observe that $r = 0$ when $1 - 2\sin\theta = 0$, that is, when $\sin\theta = \frac{1}{2}$. This will occur when $\theta = \frac{\pi}{6}$ and when $\theta = \frac{5\pi}{6}$. For this reason, we expand the above table, to include more intervals and where we also indicate the quadrant where the graph is to be drawn, as follows:

Interval	$\sin\theta$	$r = 1 - 2\sin\theta$	*Quadrant*
$\left[0, \frac{\pi}{6}\right]$	Increases from 0 to $\frac{1}{2}$	Decreases from 1 to 0	First
$\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$	Increases from $\frac{1}{2}$ to 1	Decreases from 0 to −1	Third
$\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$	Decreases from 1 to $\frac{1}{2}$	Increases from −1 to 0	Fourth
$\left[\frac{5\pi}{6}, \pi\right]$	Decreases from $\frac{1}{2}$ to 0	Increases from 0 to 1	Second
$\left[\pi, \frac{3\pi}{2}\right]$	Decreases from 0 to −1	Increases from 1 to 3	Third
$\left[\frac{3\pi}{2}, 2\pi\right]$	Increases from −1 to 0	Decreases from 3 to 1	Fourth

We sketch the graph in stages in Figures 6.36a–6.36f, corresponding to each of the intervals indicated in the table.

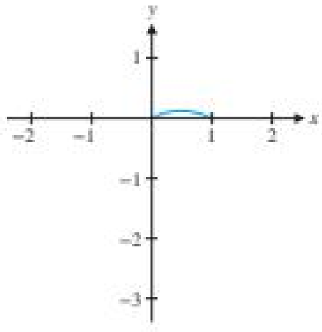


FIGURE 6.36a
$0 \le \theta \le \frac{\pi}{6}$

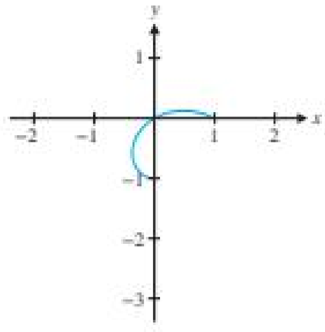


FIGURE 6.36b
$0 \le \theta \le \frac{\pi}{2}$

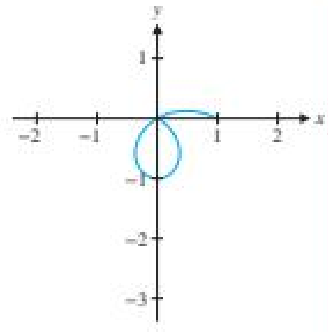


FIGURE 6.36c
$0 \le \theta \le \frac{5\pi}{6}$

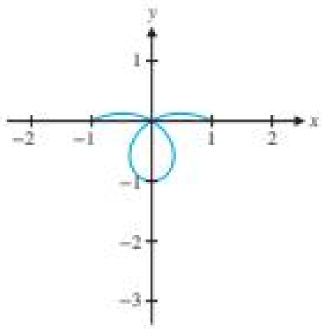


FIGURE 6.36d
$0 \le \theta \le \pi$

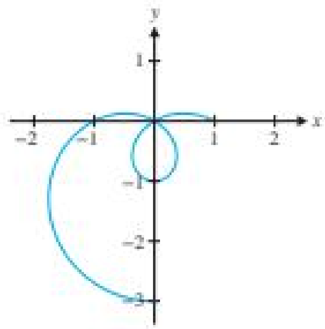


FIGURE 6.36e
$0 \le \theta \le \frac{3\pi}{2}$

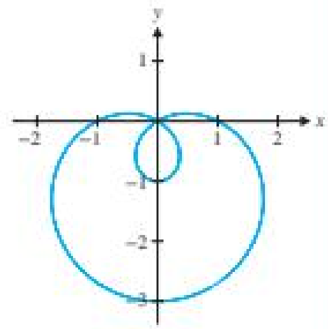


FIGURE 6.36f
$0 \le \theta \le 2\pi$

The completed graph appears in Figure 6.36f and is sketched out for $0 \le \theta \le 2\pi$. You should observe from this the importance of determining where $r = 0$, as well as where r is increasing and decreasing. ■

EXAMPLE 4.12 A Four-Leaf Rose

Sketch the graph of the polar equation $r = \sin 2\theta$.

Solution As usual, we will first draw a graph of $y = \sin 2x$ in rectangular coordinates on the interval $[0, 2\pi]$, as seen in Figure 6.37. Notice that the period of $\sin 2\theta$ is only π. We summarize the intervals on which the function is increasing and decreasing in the following table.

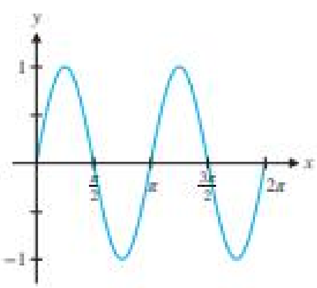


FIGURE 6.37
$y = \sin 2x$ in rectangular coordinates

Interval	$r = \sin 2\theta$	*Quadrant*
$\left[0, \frac{\pi}{4}\right]$	Increases from 0 to 1	First
$\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$	Decreases from 1 to 0	First
$\left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$	Decreases from 0 to −1	Fourth
$\left[\frac{3\pi}{4}, \pi\right]$	Increases from −1 to 0	Fourth
$\left[\pi, \frac{5\pi}{4}\right]$	Increases from 0 to 1	Third
$\left[\frac{5\pi}{4}, \frac{3\pi}{2}\right]$	Decreases from 1 to 0	Third
$\left[\frac{3\pi}{2}, \frac{7\pi}{4}\right]$	Decreases from 0 to −1	Second
$\left[\frac{7\pi}{4}, 2\pi\right]$	Increases from −1 to 0	Second

We sketch the graph in stages in Figures 6.38a–6.38h, corresponding to the intervals indicated in the table, where we have also indicated the lines $y = \pm x$, as a guide.

This is an interesting curve known as a **four-leaf rose**. Notice again the significance of the points corresponding to $r = 0$, or $\sin 2\theta = 0$. Also, notice that r reaches a maximum of 1 when $2\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$ or $\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$ and r reaches a minimum of -1 when $2\theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$ or $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$. Again, you must keep in mind that when r is negative, we draw the graph in the opposite quadrant.

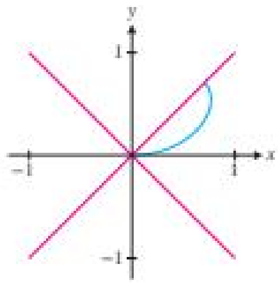


FIGURE 6.38a
$0 \leq \theta \leq \frac{\pi}{4}$

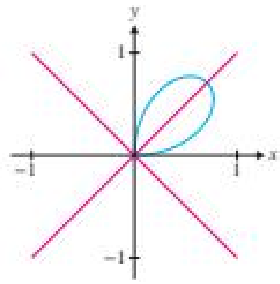


FIGURE 6.38b
$0 \leq \theta \leq \frac{\pi}{2}$

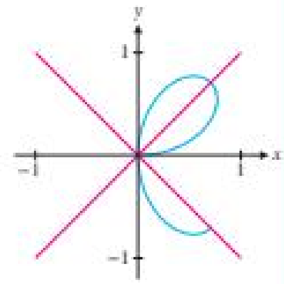


FIGURE 6.38c
$0 \leq \theta \leq \frac{3\pi}{4}$

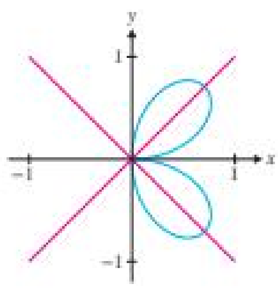


FIGURE 6.38d
$0 \leq \theta \leq \pi$

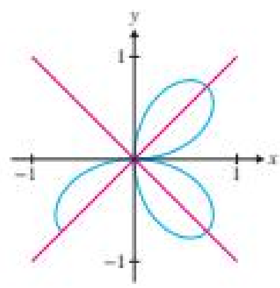


FIGURE 6.38e
$0 \leq \theta \leq \frac{5\pi}{4}$

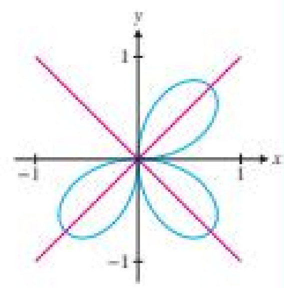


FIGURE 6.38f
$0 \leq \theta \leq \frac{3\pi}{2}$

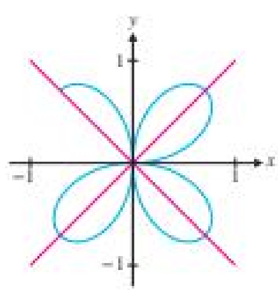


FIGURE 6.38g
$0 \leq \theta \leq \frac{7\pi}{4}$

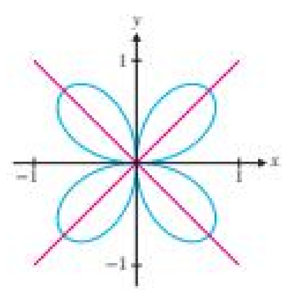


FIGURE 6.38h
$0 \leq \theta \leq 2\pi$

Note that in example 4.12, even though the period of the function $\sin 2\theta$ is π, it took θ-values ranging from 0 to 2π to sketch the entire curve $r = \sin 2\theta$. By contrast, the period of the function $\sin \theta$ is 2π, but the circle $r = \sin \theta$ in example 4.7 was completed with $0 \leq \theta \leq \pi$. To determine the range of values of θ that produces a graph, you need to carefully identify important points as we did in example 4.12. The Trace feature found on graphing calculators can be very helpful for getting an idea of the θ-range, but remember that such Trace values are only approximates.

You will explore a variety of other interesting graphs in the exercises.

BEYOND FORMULAS

The graphics in Figures 6.34, 6.36 and 6.38 provide a good visual model of how to think of polar graphs. Most polar graphs $r = f(\theta)$ can be sketched as a sequence of connected arcs, where the arcs start and stop at places where $r = 0$ or where a new quadrant is entered. By breaking the larger graph into small arcs, you can use the properties of f to determine where each arc starts and stops.

EXERCISES 6.4

WRITING EXERCISES

1. Suppose a point has polar representation (r, θ). Explain why another polar representation of the same point is $(-r, \theta + \pi)$.
2. After working with rectangular coordinates for so long, the idea of polar representations may seem slightly awkward. However, polar representations are entirely natural in many settings. For instance, if you were on a ship at sea and another ship was approaching you, explain whether you would use a polar representation (distance and bearing) or a rectangular representation (distance east–west and distance north–south).
3. In example 4.7, the graph (a circle) of $r = \sin\theta$ is completely traced out with $0 \le \theta \le \pi$. Explain why graphing $r = \sin\theta$ with $\pi \le \theta \le 2\pi$ would produce the same full circle.
4. Two possible advantages of introducing a new coordinate system are making previous problems easier to solve and allowing new problems to be solved. Give two examples of graphs for which the polar equation is simpler than the rectangular equation. Give two examples of polar graphs for which you have not seen a rectangular equation.

In exercises 1–6, plot the given polar points (r, θ) and find their rectangular representation.

1. $(2, 0)$ 2. $(2, \pi)$ 3. $(-2, \pi)$
4. $\left(-3, \frac{3\pi}{2}\right)$ 5. $(3, -\pi)$ 6. $\left(5, -\frac{\pi}{2}\right)$

In exercises 7–12, find all polar coordinate representations of the given rectangular point.

7. $(2, -2)$ 8. $(-1, 1)$ 9. $(0, 3)$
10. $(2, -1)$ 11. $(3, 4)$ 12. $(-2, -\sqrt{5})$

In exercises 13–18, find rectangular coordinates for the given polar point.

13. $\left(2, -\frac{\pi}{3}\right)$ 14. $\left(-1, \frac{\pi}{3}\right)$ 15. $(0, 3)$
16. $\left(3, \frac{\pi}{8}\right)$ 17. $\left(4, \frac{\pi}{10}\right)$ 18. $(-3, 1)$

In exercises 19–26, sketch the graph of the polar equation and find a corresponding x-y equation.

19. $r = 4$ 20. $r = \sqrt{3}$ 21. $\theta = \pi/6$
22. $\theta = 3\pi/4$ 23. $r = \cos\theta$ 24. $r = 2\cos\theta$
25. $r = 3\sin\theta$ 26. $r = 2\sin\theta$

In exercises 27–40, sketch the graph and identify all values of θ where $r = 0$ and a range of values of θ that produces one copy of the graph.

27. $r = \cos 2\theta$ 28. $r = \cos 3\theta$
29. $r = \sin 3\theta$ 30. $r = \sin 2\theta$
31. $r = 3 + 2\sin\theta$ 32. $r = 2 - 2\cos\theta$
33. $r = 2 - 4\sin\theta$ 34. $r = 2 + 4\cos\theta$
35. $r = 2 + 2\sin\theta$ 36. $r = 3 - 6\cos\theta$
37. $r = \frac{1}{4}\theta$ 38. $r = e^{\theta/4}$
39. $r = 2\cos(\theta - \pi/4)$ 40. $r = 2\sin(3\theta - \pi)$

In exercises 41–50, sketch the graph and identify all values of θ where $r = 0$.

41. $r = \cos\theta + \sin\theta$ 42. $r = \cos\theta + \sin 2\theta$
43. $r = \tan^{-1} 2\theta$ 44. $r = \theta/\sqrt{\theta^2 + 1}$
45. $r = 2 + 4\cos 3\theta$ 46. $r = 2 - 4\sin 4\theta$
47. $r = \dfrac{2}{1 + \sin\theta}$ 48. $r = \dfrac{3}{1 - \sin\theta}$
49. $r = \dfrac{2}{1 + \cos\theta}$ 50. $r = \dfrac{3}{1 - \cos\theta}$

In exercises 51–56, find a polar equation corresponding to the given rectangular equation.

51. $y^2 - x^2 = 4$ 52. $x^2 + y^2 = 9$
53. $x^2 + y^2 = 16$ 54. $x^2 + y^2 = x$
55. $y = 3$ 56. $x = 2$

In exercises 57–62, sketch the graph for several different values of a and describe the effects of changing a.

57. $r = a\cos\theta$

58. $r = a\sin\theta$

59. $r = \cos(a\theta)$

60. $r = \sin(a\theta)$

61. $r = 1 + a\cos\theta$

62. $r = 1 + a\sin\theta$

63. Graph $r = 4\cos\theta\sin^2\theta$ and explain why there is no curve to the left of the y-axis.

64. Graph $r = \theta\cos\theta$ for $-2\pi \le \theta \le 2\pi$. Explain why this is called the Garfield curve.

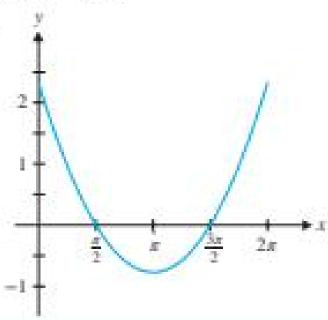
jazzzz/Shutterstock

65. Sketch the graph of $r = \cos\frac{11}{12}\theta$ first for $0 \le \theta \le \pi$, then for $0 \le \theta \le 2\pi$, then for $0 \le \theta \le 3\pi$, . . . , and finally for $0 \le \theta \le 24\pi$. Discuss any patterns that you find and predict what will happen for larger domains.

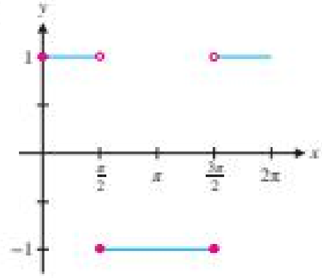
66. Sketch the graph of $r = \cos\pi\theta$ first for $0 \le \theta \le 1$, then for $0 \le \theta \le 2$, then for $0 \le \theta \le 3$, . . . and finally for $0 \le \theta \le 20$. Discuss any patterns that you find and predict what will happen for larger domains.

In exercises 67 and 68, use the graph of $y = f(x)$ to sketch a graph of $r = f(\theta)$.

67.



68.



EXPLORATORY EXERCISES

1. In this exercise, you will explore the roles of the constants a, b and c in the graph of $r = af(b\theta + c)$. To start, sketch $r = \sin\theta$ followed by $r = 2\sin\theta$ and $r = 3\sin\theta$. What does the constant a affect? Then sketch $r = \sin(\theta + \pi/2)$ and $r = \sin(\theta - \pi/4)$. What does the constant c affect? Now for the tough one. Sketch $r = \sin 2\theta$ and $r = \sin 3\theta$. What does the constant b seem to affect? Test all of your hypotheses on the base function $r = 1 + 2\cos\theta$ and several functions of your choice.

2. The polar curve $r = ae^{b\theta}$ is an **equiangular curve**. Sketch the curve and then show that $\frac{dr}{d\theta} = br$. A somewhat complicated geometric argument shows that $\frac{dr}{d\theta} = r\cot\alpha$, where α is the angle between the tangent line and the line connecting the point on the curve to the origin. Comparing equations, conclude that the angle α is constant (hence "equiangular"). To illustrate this property, compute α for the points at $\theta = 0$ and $\theta = \pi$ for $r = e^{\theta}$. This type of spiral shows up often in nature, including shellfish (shown here is an ammonite fossil from about 350 million years ago) and the florets of the common daisy. Other examples, including the connection to sunflowers, the Fibonacci sequence and the musical scale, can be found in H. E. Huntley's *The Divine Proportion*.

bezergheanu mircea/Alamy

6.5 CALCULUS AND POLAR COORDINATES

Having introduced polar coordinates and looked at a variety of polar graphs, our next step is to extend the techniques of calculus to the case of polar coordinates. In this section, we focus on tangent lines, area and arc length. Surface area and other applications will be examined in the exercises.

Notice that you can think of the graph of the polar equation $r = f(\theta)$ as the graph of the parametric equations $x = f(t)\cos t$, $y(t) = f(t)\sin t$ (where we have used the parameter $t = \theta$), since from (4.2)

$$x = r\cos\theta = f(\theta)\cos\theta \tag{5.1}$$

and

$$y = r\sin\theta = f(\theta)\sin\theta. \tag{5.2}$$

In view of this, we can now take any results already derived for parametric equations and extend these to the special case of polar coordinates.

In section 6.2, we showed that the slope of the tangent line at the point corresponding to $\theta = a$ is given [from (2.1)] to be

$$\boxed{\left.\frac{dy}{dx}\right|_{\theta=a} = \frac{\frac{dy}{d\theta}(a)}{\frac{dx}{d\theta}(a)},} \tag{5.3}$$

as long as $\frac{dx}{d\theta}(a) \neq 0$. From the product rule, (5.1) and (5.2), we have

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

and

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta.$$

Putting these together with (5.3), we get

$$\left.\frac{dy}{dx}\right|_{\theta=a} = \frac{f'(a)\sin a + f(a)\cos a}{f'(a)\cos a - f(a)\sin a}, \tag{5.4}$$

as long as the denominator is non-zero.

EXAMPLE 5.1 Finding the Slope of the Tangent Line to a Three-Leaf Rose

Find the slope of the tangent line to the three-leaf rose $r = \sin 3\theta$ at $\theta = \frac{\pi}{4}$.

Solution A sketch of the curve is shown in Figure 6.39a. From (4.1), we have

$$y = r\sin\theta = \sin 3\theta \sin\theta$$

and

$$x = r\cos\theta = \sin 3\theta\cos\theta.$$

Using (5.3), we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{(3\cos 3\theta)\sin\theta + \sin 3\theta(\cos\theta)}{(3\cos 3\theta)\cos\theta - \sin 3\theta(\sin\theta)}.$$

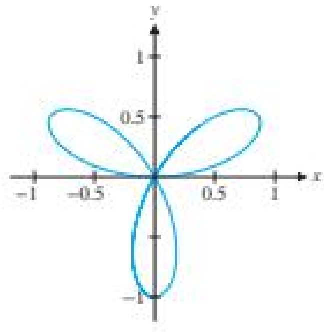


FIGURE 6.39a
Three-leaf rose

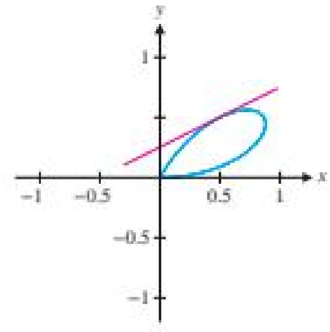


FIGURE 6.39b
The tangent line at $\theta = \frac{\pi}{4}$

Taking $\theta = \frac{\pi}{4}$ gives us

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{\left(3\cos\frac{3\pi}{4}\right)\sin\frac{\pi}{4} + \sin\frac{3\pi}{4}\left(\cos\frac{\pi}{4}\right)}{\left(3\cos\frac{3\pi}{4}\right)\cos\frac{\pi}{4} - \sin\frac{3\pi}{4}\left(\sin\frac{\pi}{4}\right)} = \frac{-\frac{3}{2}+\frac{1}{2}}{-\frac{3}{2}-\frac{1}{2}} = \frac{1}{2}.$$

In Figure 6.39b, we show the section of $r = \sin 3\theta$ for $0 \le \theta \le \frac{\pi}{3}$, along with the tangent line at $\theta = \frac{\pi}{4}$. ■

For polar graphs, it's important to find places where r has reached a maximum or minimum, but these may or may not correspond to a horizontal tangent line. We explore this idea further in example 5.2.

EXAMPLE 5.2 Polar Graphs and Horizontal Tangent Lines

For the three-leaf rose $r = \sin 3\theta$, find the locations of all horizontal tangent lines. Also, at the three points where $|r|$ is a maximum, show that the tangent line is perpendicular to the line segment connecting the point to the origin.

Solution From (5.3) and (5.4), we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Here, $f(\theta) = \sin 3\theta$ and so, to have $\frac{dy}{dx} = 0$, we must have

$$0 = \frac{dy}{d\theta} = 3\cos 3\theta \sin\theta + \sin 3\theta \cos\theta.$$

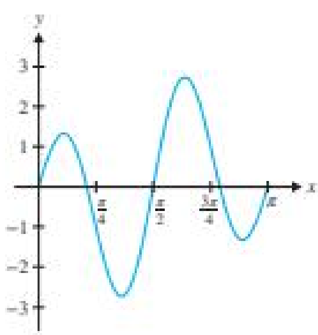


FIGURE 6.40a
$y = 3\cos 3x \sin x + \sin 3x \cos x$

From the graph of $f(x) = 3\cos 3x \sin x + \sin 3x \cos x$ with $0 \le x \le \pi$ (in Figure 6.40a), observe that there appear to be five solutions. Three of the solutions can be found exactly: $\theta = 0$, $\theta = \frac{\pi}{2}$ and $\theta = \pi$. You can find the remaining two numerically: $\theta \approx 0.659$ and $\theta \approx 2.48$. (You can also use trig identities to arrive at $\sin^2\theta = \frac{3}{8}$.) The corresponding points on the curve $r = \sin 3\theta$ (specified in rectangular coordinates) are (0, 0), (0.73, 0.56), (0, −1), (−0.73, 0.56) and (0, 0). The point (0, −1) lies at the bottom of a leaf. This is the familiar situation of a horizontal tangent line at a local (and in fact, absolute) minimum. The tangent lines at these points are shown in Figure 6.40b. Note that these points correspond to points where the y-coordinate is a maximum. However, the tips of the leaves represent the extreme points of most interest. Notice that the tips are where $|r|$ is a maximum. For $r = \sin 3\theta$, this occurs when $\sin 3\theta = \pm 1$, that is, where $3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$, or $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \ldots$. From (5.4), the slope of the tangent line to the curve at $\theta = \frac{\pi}{6}$ is given by

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{\left(3\cos\frac{3\pi}{6}\right)\sin\frac{\pi}{6} + \sin\frac{3\pi}{6}\left(\cos\frac{\pi}{6}\right)}{\left(3\cos\frac{3\pi}{6}\right)\cos\frac{\pi}{6} - \sin\frac{3\pi}{6}\left(\sin\frac{\pi}{6}\right)} = \frac{0 + \frac{\sqrt{3}}{2}}{0 - \frac{1}{2}} = -\sqrt{3}.$$

The rectangular point corresponding to $\theta = \frac{\pi}{6}$ is given by

$$\left(1\cos\frac{\pi}{6}, 1\sin\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

The slope of the line segment joining this point to the origin is then $\frac{1}{\sqrt{3}}$, making this line segment perpendicular to the tangent line, since the product of the slopes is -1. This is illustrated in Figure 6.40c. Similarly, the slope of the tangent line at $\theta = \frac{5\pi}{6}$ is $\sqrt{3}$, which again makes the tangent line at that point perpendicular to the line segment from the origin to the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Finally, we have already shown that the slope of the tangent line at $\theta = \frac{\pi}{2}$ is 0 and a horizontal tangent line is perpendicular to the vertical line from the origin to the point $(0, -1)$.

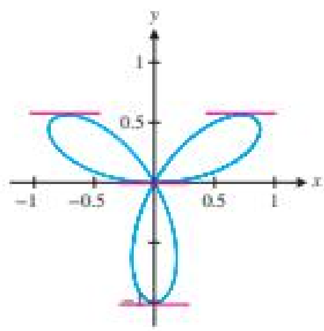


FIGURE 6.40b
Horizontal tangent lines

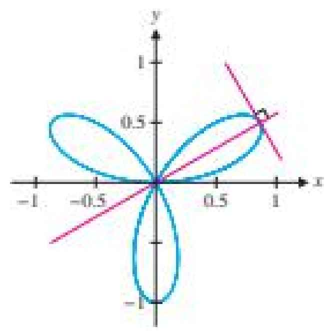


FIGURE 6.40c
The tangent line at the tip of a leaf

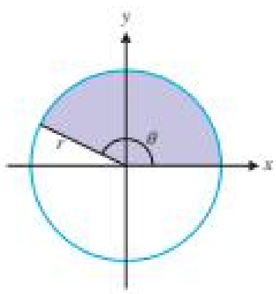


FIGURE 6.41
Circular sector

Next, for polar curves like the three-leaf rose seen in Figure 6.39a, we would like to compute the area enclosed by the curve. Since such a graph is *not* the graph of a function of the form $y = f(x)$, we cannot use the usual area formulas developed in Chapter 5. While we can convert our area formulas for parametric equations (from Theorem 2.2) into polar coordinates, a simpler approach uses the following geometric argument.

Observe that a sector of a circle of radius r and central angle θ, measured in radians (see Figure 6.41) contains a fraction $\left(\frac{\theta}{2\pi}\right)$ of the area of the entire circle. So, the area of the sector is given by

$$A = \pi r^2 \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta.$$

Now, consider the area enclosed by the polar curve defined by the equation $r = f(\theta)$ and the rays $\theta = a$ and $\theta = b$ (see Figure 6.42a), where f is continuous and positive on the interval $a \leq \theta \leq b$. As we did when we defined the definite integral, we begin by partitioning the θ-interval into n equal pieces:

$$a = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = b.$$

The width of each of these subintervals is then $\Delta\theta = \theta_i - \theta_{i-1} = \frac{b-a}{n}$. (Does this look familiar?) On each subinterval $[\theta_{i-1}, \theta_i]$ $(i = 1, 2, \ldots, n)$, we approximate the curve with the circular arc $r = f(\theta_i)$. (See Figure 6.42b.)

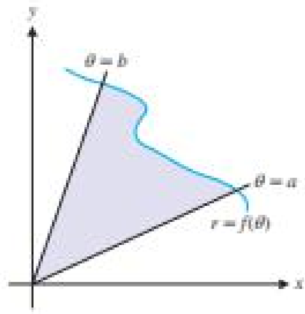


FIGURE 6.42a
Area of a polar region

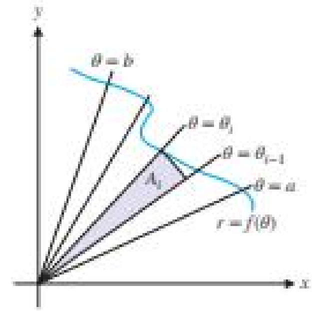


FIGURE 6.42b
Approximating the area of a polar region

The area A_i enclosed by the curve on this subinterval is then approximately the same as the area of the circular sector of radius $f(\theta_i)$ and central angle $\Delta\theta$:

$$A_i \approx \frac{1}{2}r^2\,\Delta\theta = \frac{1}{2}[f(\theta_i)]^2\,\Delta\theta.$$

The total area A enclosed by the curve is then approximately the same as the sum of the areas of all such circular sectors:

$$A \approx \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2\,\Delta\theta.$$

As we have done numerous times now, we can improve the approximation by making n larger. Taking the limit as $n \to \infty$ gives us a definite integral:

Area in polar coordinates

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2\,\Delta\theta = \int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta.} \tag{5.5}$$

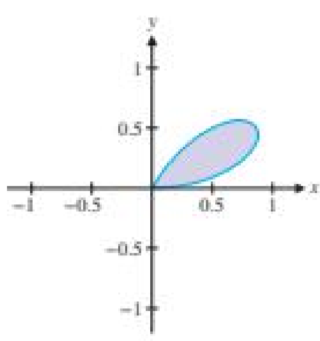


FIGURE 6.43
One leaf of $r = \sin 3\theta$

EXAMPLE 5.3 The Area of One Leaf of a Three-Leaf Rose

Find the area of one leaf of the rose $r = \sin 3\theta$.

Solution Notice that one leaf of the rose is traced out with $0 \le \theta \le \frac{\pi}{3}$. (See Figure 6.43.) From (5.5), the area is given by

$$\begin{aligned} A &= \int_0^{\pi/3} \frac{1}{2}(\sin 3\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/3} \sin^2 3\theta\,d\theta \\ &= \frac{1}{4}\int_0^{\pi/3} (1 - \cos 6\theta)\,d\theta = \frac{1}{4}\left(\theta - \frac{1}{6}\sin 6\theta\right)\Bigg|_0^{\pi/3} = \frac{\pi}{12}, \end{aligned}$$

where we have used the half-angle formula $\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$ to simplify the integrand. ■

Often, the most challenging part of finding the area of a polar region is determining the limits of integration.

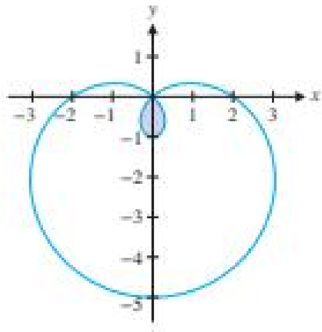


FIGURE 6.44
$r = 2 - 3\sin\theta$

EXAMPLE 5.4 The Area of the Inner Loop of a Limaçon

Find the area of the inner loop of the limaçon $r = 2 - 3\sin\theta$.

Solution A sketch of the limaçon is shown in Figure 6.44. Starting at $\theta = 0$, the curve starts at the point (2, 0), passes through the origin, traces out the inner loop, passes back through the origin and finally traces out the outer loop. Thus, the inner loop is formed by θ-values between the first and second occurrences of $r = 0$ with $\theta > 0$. Solving $r = 0$, we get $\sin\theta = \frac{2}{3}$. The two smallest positive solutions are $\theta = \sin^{-1}\left(\frac{2}{3}\right)$ and $\theta = \pi - \sin^{-1}\left(\frac{2}{3}\right)$. Numerically, these are approximately equal to $\theta = 0.73$ and $\theta = 2.41$. From (5.5), the area is approximately

$$A \approx \int_{0.73}^{2.41} \frac{1}{2}(2 - 3\sin\theta)^2\,d\theta = \frac{1}{2}\int_{0.73}^{2.41}(4 - 12\sin\theta + 9\sin^2\theta)d\theta$$
$$= \frac{1}{2}\int_{0.73}^{2.41}\left[4 - 12\sin\theta + \frac{9}{2}(1 - \cos 2\theta)\right]d\theta \approx 0.44,$$

where we have used the half-angle formula $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ to simplify the integrand. ■

When finding the area lying between two polar graphs, we use the familiar device of subtracting one area from another. Although the calculations in example 5.5 aren't too messy, finding the points of intersection of two polar curves often provides the greatest challenge.

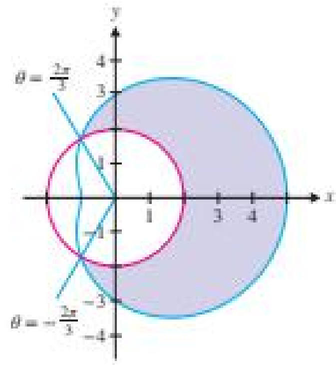


FIGURE 6.45a
$r = 3 + 2\cos\theta$ and $r = 2$

EXAMPLE 5.5 Finding the Area between Two Polar Graphs

Find the area inside the limaçon $r = 3 + 2\cos\theta$ and outside the circle $r = 2$.

Solution We show a sketch of the two curves in Figure 6.45a. Notice that the limits of integration correspond to the values of θ where the two curves intersect. So, we must first solve the equation $3 + 2\cos\theta = 2$. Notice that since $\cos\theta$ is periodic, there are infinitely many solutions of this equation. Consequently, it is essential to consult the graph to determine which solutions you need. In this case, we want the least negative and the smallest positive solutions. (Look carefully at Figure 6.45b, where we have shaded the area between the graphs corresponding to θ between $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, the first two positive solutions. This portion of the graphs corresponds to the area *outside* the limaçon and *inside* the circle!) With $3 + 2\cos\theta = 2$, we have $\cos\theta = -\frac{1}{2}$, which occurs at $\theta = -\frac{2\pi}{3}$ and $\theta = \frac{2\pi}{3}$. From (5.5), the area enclosed by the portion of the limaçon on this interval is given by

$$\int_{-2\pi/3}^{2\pi/3} \frac{1}{2}(3 + 2\cos\theta)^2\,d\theta = \frac{33\sqrt{3} + 44\pi}{6}.$$

Similarly, the area enclosed by the circle on this interval is given by

$$\int_{-2\pi/3}^{2\pi/3} \frac{1}{2}(2)^2\,d\theta = \frac{8\pi}{3}.$$

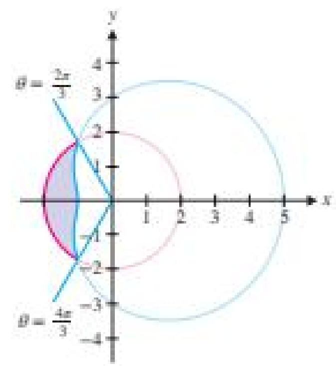


FIGURE 6.45b
$\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$

The area *inside* the limaçon and *outside* the circle is then given by

$$A = \int_{-2\pi/3}^{2\pi/3} \frac{1}{2}(3 + 2\cos\theta)^2\,d\theta - \int_{-2\pi/3}^{2\pi/3} \frac{1}{2}(2)^2\,d\theta$$
$$= \frac{33\sqrt{3} + 44\pi}{6} - \frac{8\pi}{3} = \frac{33\sqrt{3} + 28\pi}{6} \approx 24.2.$$

Here, we have left the (routine) details of the integrations to you. ■

In cases where r takes on both positive and negative values, finding the intersection points of two curves is more complicated.

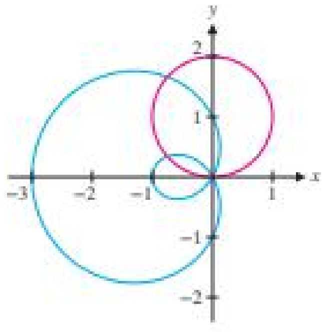


FIGURE 6.46a
$r = 1 - 2\cos\theta$ and $r = 2\sin\theta$

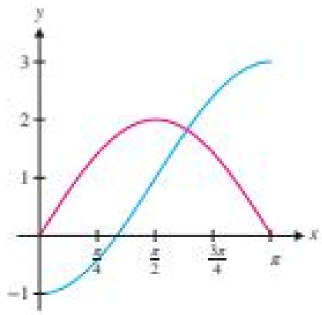


FIGURE 6.46b
Rectangular plot: $y = 1 - 2\cos x$, $y = 2\sin x$, $0 \le x \le \pi$

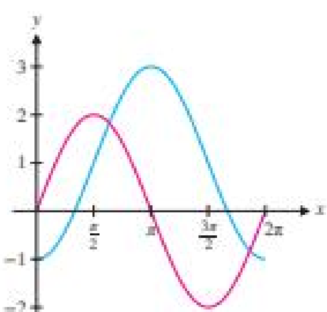


FIGURE 6.46c
Rectangular plot: $y = 1 - 2\cos x$, $y = 2\sin x$, $0 \le x \le 2\pi$

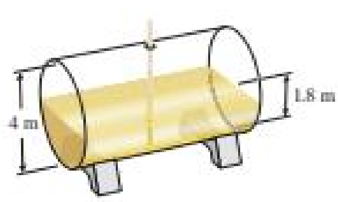


FIGURE 6.47a
A cylindrical oil tank

EXAMPLE 5.6 Finding Intersections of Polar Curves Where r Can Be Negative

Find all intersections of the limaçon $r = 1 - 2\cos\theta$ and the circle $r = 2\sin\theta$.

Solution We show a sketch of the two curves in Figure 6.46a. Notice from the sketch that there are three intersections of the two curves. Since $r = 2\sin\theta$ is completely traced with $0 \le \theta \le \pi$, you might reasonably expect to find three solutions of the equation $1 - 2\cos\theta = 2\sin\theta$ on the interval $0 \le \theta \le \pi$. However, if we draw a rectangular plot of the two curves $y = 1 - 2\cos x$ and $y = 2\sin x$, on the interval $0 \le x \le \pi$ (see Figure 6.46b), we can clearly see that there is only one solution in this range, at approximately $\theta \approx 1.99$. (Use Newton's method or your calculator's solver to obtain an accurate approximation.) The corresponding rectangular point is $(r\cos\theta, r\sin\theta) \approx (-0.74, 1.67)$. Looking at Figure 6.46a, observe that there is another intersection located below this point. From a rectangular plot of the two curves corresponding to an expanded range of values of θ (see Figure 6.46c), notice that there is a second solution of the equation $1 - 2\cos\theta = 2\sin\theta$, near $\theta = 5.86$, which corresponds to the point $(-0.74, 0.34)$. Note that this point is on the inner loop of $r = 1 - 2\cos\theta$ and corresponds to a negative value of r. Finally, there appears to be a third intersection at or near the origin. Notice that this does not arise from any solution of the equation $1 - 2\cos\theta = 2\sin\theta$. This is because, while both curves pass through the origin (you should verify this!), they each do so for *different* values of θ. (Keep in mind that the origin corresponds to the point $(0, \theta)$, in polar coordinates, for *any* angle θ.) Notice that $1 - 2\cos\theta = 0$ for $\theta = \frac{\pi}{3}$ and $2\sin\theta = 0$ for $\theta = 0$. So, although the curves intersect at the origin, they each pass through the origin for different values of θ. ■

REMARK 5.1

To find points of intersection of two polar curves $r = f(\theta)$ and $r = g(\theta)$, you must keep in mind that points have more than one representation in polar coordinates. In particular, this says that points of intersection need not correspond to solutions of $f(\theta) = g(\theta)$.

In example 5.7, we see an application that is far simpler to set up in polar coordinates than in rectangular coordinates.

EXAMPLE 5.7 Finding the Volume of a Partially Filled Cylinder

A cylindrical oil tank with a radius of 2 m is lying on its side. A measuring stick shows that the oil is 1.8 m deep. (See Figure 6.47a.) What percentage of a full tank is left?

Solution Notice that since we wish to find the *percentage* of oil remaining in the tank, the length of the tank has no bearing on this problem. (Think about this some.) We need only consider a cross-section of the tank, which we represent as a circle of radius 2 centered at the origin. The proportion of oil remaining is given by the area of that portion of the circle lying beneath the line $y = -0.2$, divided by the total area of the circle. The area of the circle is 4π, so we need only find the area of the shaded region in Figure 6.47b. Computing this area in rectangular coordinates is a mess (try it!), but it is straightforward in polar coordinates. First, notice that the line $y = -0.2$ corresponds to $r\sin\theta = -0.2$ or $r = -0.2\csc\theta$. The area beneath the line and inside the circle is then given by (5.5) as

$$\text{Area} = \int_{\theta_1}^{\theta_2} \frac{1}{2}(2)^2\,d\theta - \int_{\theta_1}^{\theta_2} \frac{1}{2}(-0.2\csc\theta)^2\,d\theta,$$

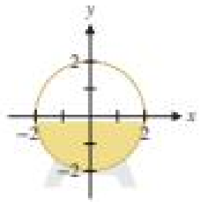


FIGURE 6.47b
Cross-section of a tank

where θ_1 and θ_2 are the appropriate intersections of $r = 2$ and $r = -0.2 \csc \theta$. Using Newton's method, the first two positive solutions of $2 = -0.2 \csc \theta$ are $\theta_1 \approx 3.242$ and $\theta_2 \approx 6.183$. The area is then

$$\begin{aligned}\text{Area} &= \int_{\theta_1}^{\theta_2} \frac{1}{2}(2)^2\, d\theta - \int_{\theta_1}^{\theta_2} \frac{1}{2}(-0.2 \csc \theta)^2\, d\theta \\ &= (2\theta + 0.02 \cot \theta)\Big|_{\theta_1}^{\theta_2} \approx 5.485.\end{aligned}$$

The fraction of oil remaining in the tank is then approximately $5.485/4\pi \approx 0.43648$ or about 43.6% of the total capacity of the tank. ■

We close this section with a brief discussion of arc length for polar curves. Recall that from (3.1), the arc length of a curve defined parametrically by $x = x(t)$, $y = y(t)$, for $a \le t \le b$, is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{5.6}$$

Once again thinking of a polar curve as a parametric representation (where the parameter is θ), we have that for the polar curve $r = f(\theta)$,

$$x = r \cos \theta = f(\theta) \cos \theta \qquad \text{and} \qquad y = r \sin \theta = f(\theta) \sin \theta.$$

This gives us

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= [f'(\theta)\cos \theta - f(\theta)\sin \theta]^2 + [f'(\theta)\sin \theta + f(\theta)\cos \theta]^2 \\ &= [f'(\theta)]^2(\cos^2 \theta + \sin^2 \theta) + f'(\theta)f(\theta)(-2 \cos \theta \sin \theta + 2 \sin \theta \cos \theta) \\ &\quad + [f(\theta)]^2(\cos^2 \theta + \sin^2 \theta) \\ &= [f'(\theta)]^2 + [f(\theta)]^2.\end{aligned}$$

From (5.6), the arc length is then

Arc length in polar coordinates

$$s = \int_a^b \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}\, d\theta. \tag{5.7}$$

EXAMPLE 5.8 Arc Length of a Polar Curve

Find the arc length of the cardioid $r = 2 - 2 \cos \theta$.

Solution A sketch of the cardioid is shown in Figure 6.49. First, notice that the curve is traced out with $0 \le \theta \le 2\pi$. From (5.7), the arc length is given by

$$\begin{aligned}s &= \int_a^b \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}\, d\theta = \int_0^{2\pi} \sqrt{(2 \sin \theta)^2 + (2 - 2 \cos \theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4 \sin^2 \theta + 4 - 8 \cos \theta + 4 \cos^2 \theta}\, d\theta = \int_0^{2\pi} \sqrt{8 - 8 \cos \theta}\, d\theta = 16,\end{aligned}$$

where we leave the details of the integration as an exercise. (Hint: Use the half-angle formula $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ to simplify the integrand. Be careful: remember that $\sqrt{x^2} = |x|$!) ■

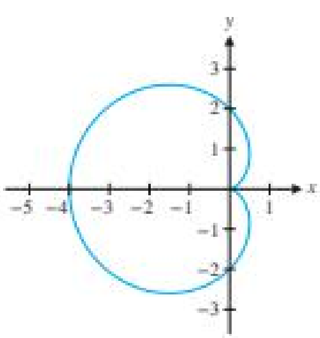


FIGURE 6.48
$r = 2 - 2 \cos \theta$

EXERCISES 6.5

WRITING EXERCISES

1. Explain why the tangent line is perpendicular to the radius line at any point at which r is a local maximum. (See example 5.2.) In particular, if the tangent and radius are not perpendicular at (r, θ), explain why r is not a local maximum.

2. In example 5.5, explain why integrating from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$ would give the area shown in Figure 6.45b and not the desired area.

3. Referring to example 5.6, explain why intersections can occur in each of the cases $f(\theta) = g(\theta)$, $f(\theta) = -g(\theta + \pi)$ and $f(\theta_1) = g(\theta_2) = 0$.

4. In example 5.7, explain why the length of the tank doesn't matter. If the problem were to compute the *amount* of oil left, would the length matter?

In exercises 1–6, find the slope of the tangent line to the polar curve at the given point.

1. $r = \sin 3\theta$ at (a) $\theta = \frac{\pi}{3}$ (b) $\theta = \frac{\pi}{2}$
2. $r = \cos 2\theta$ at (a) $\theta = 0$ (b) $\theta = \frac{\pi}{4}$
3. $r = 3\sin\theta$ at (a) $\theta = 0$ (b) $\theta = \frac{\pi}{2}$
4. $r = \sin 4\theta$ at (a) $\theta = \frac{\pi}{4}$ (b) $\theta = \frac{\pi}{16}$
5. $r = e^{2\theta}$ at (a) $\theta = 0$ (b) $\theta = 1$
6. $r = \ln\theta$ at (a) $\theta = e$ (b) $\theta = 4$

In exercises 7–10, (a) find all points at which $|r|$ is a maximum and show that the tangent line is perpendicular to the radius connecting the point to the origin. (b) Find all points at which there is a horizontal tangent and determine the concavity of the curve at each point.

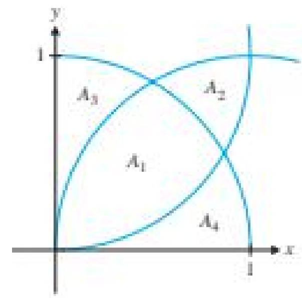
7. $r = \sin 3\theta$
8. $r = \cos 4\theta$
9. $r = 2 - 4\sin 2\theta$
10. $r = 2 + 4\sin 2\theta$

In exercises 11–26, find the area of the indicated region.

11. One leaf of $r = \cos 3\theta$
12. One leaf of $r = \sin 4\theta$
13. Bounded by $r = 2\cos\theta$
14. Bounded by $r = 2 - 2\cos\theta$
15. Small loop of $r = 1 + 2\sin 2\theta$
16. Large loop of $r = 1 + 2\sin 2\theta$
17. Inner loop of $r = 3 - 4\sin\theta$
18. Inner loop of $r = 1 - 3\cos\theta$
19. Inner loop of $r = 2 + 3\sin 3\theta$
20. Outer loop of $r = 2 + 3\sin 3\theta$
21. Inside of $r = 3 + 2\sin\theta$ and outside of $r = 2$
22. Inside of $r = 2$ and outside of $r = 2 - 2\sin\theta$
23. Inside of $r = 2$ and outside of both loops of $r = 1 + 2\sin\theta$
24. Inside of $r = 2\sin 2\theta$ and outside $r = 1$
25. Inside of both $r = 1 + \cos\theta$ and $r = 1$
26. Inside of both $r = 1 + \sin\theta$ and $r = 1 + \cos\theta$

In exercises 27–30, find the area of the indicated region formed by $r = 1$, $r = 2\cos\theta$ and $r = 2\sin\theta$. (Suggested by Tim Pennings.)

27. A_1
28. A_2
29. A_3
30. A_4



In exercises 31–34, find all points at which the two curves intersect.

31. $r = 1 - 2\sin\theta$ and $r = 2\cos\theta$
32. $r = 1 + 3\cos\theta$ and $r = -2 + 5\sin\theta$
33. $r = 1 + \sin\theta$ and $r = 1 + \cos\theta$
34. $r = 1 + \sqrt{3}\sin\theta$ and $r = 1 + \cos\theta$

In exercises 35–40, find the arc length of the given curve.

35. $r = 2 - 2\sin\theta$
36. $r = 3 - 3\cos\theta$
37. $r = \sin 3\theta$
38. $r = 2\cos 3\theta$
39. $r = 1 + 2\sin 2\theta$
40. $r = 2 + 3\sin 3\theta$

41. Repeat example 5.7 for the case where the oil stick shows a depth of (a) 1.4 (b) 2.4.

42. Repeat example 5.7 for the case where the oil stick shows a depth of (a) 1.0 (b) 2.6.

43. The problem of finding the slope of $r = \sin 3\theta$ at the point (0, 0) is not a well-defined problem.

(a) To see what we mean, show that the curve passes through the origin at $\theta = 0$, $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$, and find the slopes at these angles.

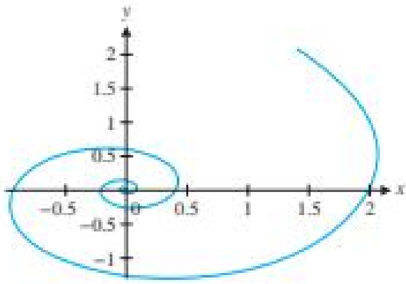
(b) For each of the three slopes, illustrate with a sketch of $r = \sin 3\theta$ for θ-values near the given values (e.g., $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$ to see the slope at $\theta = 0$).

44. Find and illustrate all slopes of $r = 2 - 3\sin\theta$ at the origin.

45. If the polar curve $r = f(\theta)$, $a \le \theta \le b$, has length L, show that $r = cf(\theta)$, $a \le \theta \le b$, has length $|c|L$ for any constant c.

46. If the polar curve $r = f(\theta)$, $a \le \theta \le b$, encloses area A, show that for any constant c, $r = cf(\theta)$, $a \le \theta \le b$, encloses area c^2A.

47. A logarithmic spiral is the graph of $r = ae^{b\theta}$ for positive constants a and b. The accompanying figure shows the case where $a = 2$ and $b = \frac{1}{4}$ with $\theta \le 1$. Although the graph never reaches the origin, the limit of the arc length from $\theta = d$ to a given point with $\theta = c$, as d decreases to $-\infty$, exists. Show that this total arc length equals $\dfrac{\sqrt{b^2+1}}{b}R$, where R is the distance from the starting point to the origin.



48. For the logarithmic spiral of exercise 47, if the starting point P is on the x-axis, show that the total arc length to the origin equals the distance from P to the y-axis along the tangent line to the curve at P.

EXPLORATORY EXERCISES

1. In this exercise, you will discover a remarkable property about the area underneath the graph of $y = \frac{1}{x}$. First, show that a polar representation of this curve is $r^2 = \frac{1}{\sin\theta\cos\theta}$. We will find the area bounded by $y = \frac{1}{x}$, $y = mx$ and $y = 2mx$ for $x > 0$, where m is a positive constant. Sketch graphs for $m = 1$ (the area bounded by $y = \frac{1}{x}$, $y = x$ and $y = 2x$) and $m = 2$ (the area bounded by $y = \frac{1}{x}$, $y = 2x$ and $y = 4x$). Which area looks larger? To find out, you should integrate. Explain why this would be a very difficult integration in rectangular coordinates. Then convert all curves to polar coordinates and compute the polar area. You should discover that the area equals $\frac{1}{2}\ln 2$ for any value of m. (Are you surprised?)

2. In the study of biological oscillations (e.g., the beating of heart cells), an important mathematical term is **limit cycle.** A simple example of a limit cycle is produced by the polar coordinates initial value problem $\dfrac{dr}{dt} = ar(1-r)$, $r(0) = r_0$ and $\dfrac{d\theta}{dt} = 2\pi$, $\theta(0) = \theta_0$. Here, a is a positive constant. In section 7.2, we showed that the solution of the initial value problem $\dfrac{dr}{dt} = ar(1-r)$, $r(0) = r_0$ is

$$r(t) = \frac{r_0}{r_0 - (r_0 - 1)e^{-at}}$$

and it is not hard to show that the solution of the initial value problem $\dfrac{d\theta}{dt} = 2\pi$, $\theta(0) = \theta_0$ is $\theta(t) = 2\pi t + \theta_0$. In rectangular coordinates, the solution of the combined initial value problem has parametric equations $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$. Graph the solution in the cases (a) $a = 1$, $r_0 = \frac{1}{2}$, $\theta_0 = 0$; (b) $a = 1$, $r_0 = \frac{3}{2}$, $\theta_0 = 0$; (c) your choice of $a > 0$, your choice of r_0 with $0 < r_0 < 1$, your choice of θ_0; (d) your choice of $a > 0$, your choice of r_0 with $r_0 > 1$, your choice of θ_0. As t increases, what is the limiting behavior of the solution? Explain what is meant by saying that this system has a limit cycle of $r = 1$.

6.6 CONIC SECTIONS

Among the most important curves you will encounter are the **conic sections**, which we explore here. The conic sections include parabolas, ellipses and hyperbolas, which are undoubtedly already familiar to you. In this section, we focus on geometric properties that are most easily determined in rectangular coordinates.

We visualize each conic section as the intersection of a plane with a right circular cone. (See Figures 6.49a–6.49c.) Depending on the orientation of the plane, the resulting curve can be a parabola, an ellipse or a hyperbola.

Parabolas

We define a **parabola** (see Figure 6.50) to be the set of all points that are equidistant from a fixed point (called the **focus**) and a line (called the **directrix**).

FIGURE 6.49a
Parabola

FIGURE 6.49b
Ellipse

FIGURE 6.49c
Hyperbola

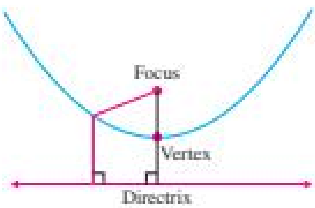


FIGURE 6.50
Parabola

A special point on the parabola is the **vertex**, the midpoint of the perpendicular line segment from the focus to the directrix.

A parabola whose directrix is a horizontal line has a simple rectangular equation.

EXAMPLE 6.1 Finding the Equation of a Parabola

Find an equation of the parabola with focus at the point (0, 2) whose directrix is the line $y = -2$.

Solution By definition, any point (x, y) on the parabola must be equidistant from the focus and the directrix. (See Figure 6.51.) From the distance formula, the distance from (x, y) to the focus is given by $\sqrt{x^2 + (y-2)^2}$ and the distance to the directrix is $|y - (-2)|$.

Setting these equal and squaring both sides, we get

$$x^2 + (y-2)^2 = (y+2)^2.$$

Expanding this out and simplifying, we get

$$x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$$

or

$$y = \frac{1}{8}x^2.$$

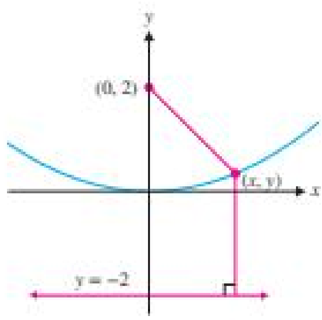


FIGURE 6.51
The parabola with focus at (0, 2) and directrix $y = -2$

In general, the following relationship holds.

THEOREM 6.1

The parabola with vertex at the point (b, c), focus at $(b, c + \frac{1}{4a})$ and directrix given by the line $y = c - \frac{1}{4a}$ is described by the equation $y = a(x-b)^2 + c$.

PROOF

Given the focus $(b, c + \frac{1}{4a})$ and directrix $y = c - \frac{1}{4a}$, the vertex is the midpoint (b, c). (See Figure 6.52.) For any point (x, y) on the parabola, its distance to the focus is given by $\sqrt{(x-b)^2 + \left(y - c - \frac{1}{4a}\right)^2}$, while its distance to the directrix is given by $|y - c + \frac{1}{4a}|$. Setting these equal and squaring as in example 6.1, we have

$$(x-b)^2 + \left(y - c - \frac{1}{4a}\right)^2 = \left(y - c + \frac{1}{4a}\right)^2.$$

Expanding this out and simplifying, we get the more familiar form of the equation: $y = a(x-b)^2 + c$, as desired. ■

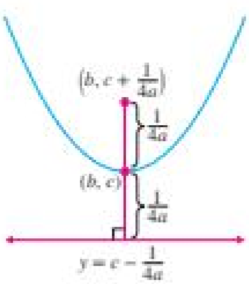


FIGURE 6.52
Parabola

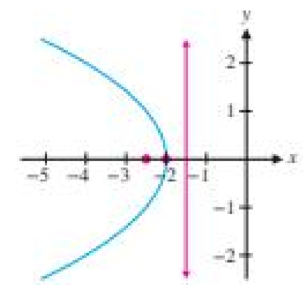


FIGURE 6.53
Parabola with focus at $(-\frac{5}{2}, 0)$ and directrix $x = -\frac{3}{2}$

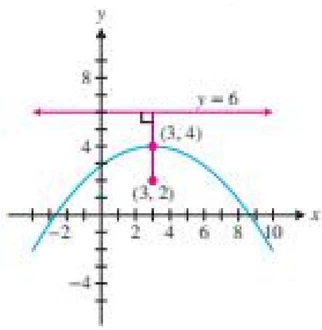


FIGURE 6.54
Parabola with focus at (3, 2) and directrix $y = 6$

FIGURE 6.55
Reflection of rays.

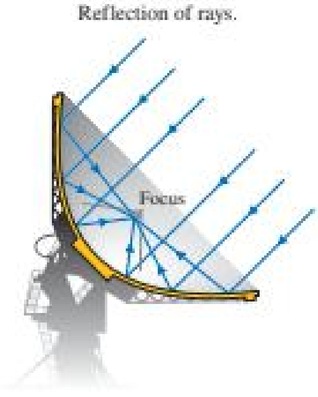


FIGURE 6.56
The reflective property.

We simply reverse the roles of x and y to obtain the following result, whose proof is left as an exercise.

THEOREM 6.2

The parabola with vertex at the point (c, b), focus at $(c + \frac{1}{4a}, b)$ and directrix given by the line $x = c - \frac{1}{4a}$ is described by the equation $x = a(y - b)^2 + c$.

We illustrate Theorem 6.2 in example 6.2.

EXAMPLE 6.2 A Parabola Opening to the Left

For the parabola $4x + 2y^2 + 8 = 0$, find the vertex, focus and directrix.

Solution Solving for x, we have $x = -\frac{1}{2}y^2 - 2$. The vertex is then at $(-2, 0)$. The focus and directrix are shifted left and right, respectively, from the vertex by $\frac{1}{4a} = -\frac{1}{2}$. This puts the focus at $(-2 - \frac{1}{2}, 0) = (-\frac{5}{2}, 0)$ and the directrix at $x = -2 - (-\frac{1}{2}) = -\frac{3}{2}$. We show a sketch of the parabola in Figure 6.53. ■

EXAMPLE 6.3 Finding the Equation of a Parabola

Find an equation relating all points that are equidistant from the point (3, 2) and the line $y = 6$.

Solution Referring to Figure 6.54, notice that the vertex must be at the point (3, 4) (i.e., the midpoint of the perpendicular line segment connecting the focus to the directrix) and the parabola opens down. From the vertex, the focus is shifted vertically by $\frac{1}{4a} = -2$ units, so that $a = \frac{1}{(-2)4} = -\frac{1}{8}$. An equation is then

$$y = -\frac{1}{8}(x - 3)^2 + 4.$$ ■

You see parabolas nearly every day. A very useful property of parabolas is their **reflective property**. This can be seen as follows. For the parabola $x = ay^2$ indicated in Figure 6.55, draw a horizontal line that intersects the parabola at the point A. Then, one can show that the acute angle α between the horizontal line and the tangent line at A is the same as the acute angle β between the tangent line and the line segment joining A to the focus. You may already have recognized that light rays are reflected from a surface in exactly the same fashion (since the angle of incidence must equal the angle of reflection). In Figure 6.55, we indicate a number of rays (you can think of them as light rays, although they could represent other forms of energy) traveling horizontally until they strike the parabola. As indicated, *all* rays striking the parabola are reflected *through the focus* of the parabola.

Due to this reflective property, satellite dishes are usually built with a parabolic shape and have a microphone located at the focus to receive all signals. (See Figure 6.56.) This reflective property works in both directions. That is, energy emitted from the focus will reflect off the parabola and travel in parallel rays. For this reason, flashlights utilize parabolic reflectors to direct their light in a beam of parallel rays.

EXAMPLE 6.4 Design of a Flashlight

A parabolic reflector for a flashlight has the shape $x = 2y^2$. Where should the lightbulb be located?

Solution Based on the reflective property of parabolas, the lightbulb should be located at the focus of the parabola. The vertex is at (0, 0) and the focus is shifted to the right from the vertex $\frac{1}{4a} = \frac{1}{8}$ units, so the lightbulb should be located at the point $\left(\frac{1}{8}, 0\right)$. ■

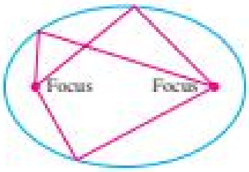


FIGURE 6.57a
Definition of ellipse

○ Ellipses

We define an **ellipse** to be the set of all points for which the sum of the distances to two fixed points (called **foci**, the plural of focus) is constant. This definition is illustrated in Figure 6.57a. We define the **center** of an ellipse to be the midpoint of the line segment connecting the foci.

The familiar equation of an ellipse can be derived from this definition. For convenience, we assume that the foci lie at the points $(c, 0)$ and $(-c, 0)$, for some positive constant c (i.e., they lie on the x-axis, at the same distance from the origin). For any point (x, y) on the ellipse, the distance from (x, y) to the focus $(c, 0)$ is $\sqrt{(x-c)^2+y^2}$ and the distance to the focus $(-c, 0)$ is $\sqrt{(x+c)^2+y^2}$. The sum of these distances must equal a constant that we'll call k. We then have

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=k.$$

Subtracting the first square root from both sides and then squaring, we get

$$\left(\sqrt{(x+c)^2+y^2}\right)^2=\left(k-\sqrt{(x-c)^2+y^2}\right)^2$$

or $$x^2+2cx+c^2+y^2=k^2-2k\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2.$$

Now, solving for the remaining term with the radical and squaring gives us

$$\left[2k\sqrt{(x-c)^2+y^2}\right]^2=(k^2-4cx)^2,$$

so that $$4k^2x^2-8k^2cx+4k^2c^2+4k^2y^2=k^4-8k^2cx+16c^2x^2$$

or $$(4k^2-16c^2)x^2+4k^2y^2=k^4-4k^2c^2.$$

Setting $k = 2a$, we obtain

$$(16a^2-16c^2)x^2+16a^2y^2=16a^4-16a^2c^2.$$

Notice that since $2a$ is the sum of the distances from (x, y) to $(c, 0)$ and from (x, y) to $(-c, 0)$ and the distance from $(c, 0)$ to $(-c, 0)$ is $2c$, we must have $2a > 2c$, so that $a > c > 0$. Dividing both sides of the equation by 16 and defining $b^2 = a^2 - c^2$, we get

$$b^2x^2+a^2y^2=a^2b^2.$$

Finally, dividing by a^2b^2 leaves us with the familiar equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

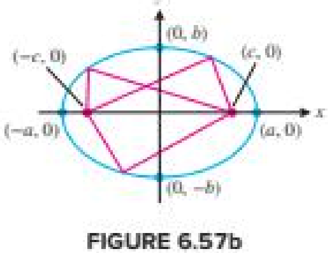


FIGURE 6.57b
Ellipse with foci at $(c, 0)$ and $(-c, 0)$

In this equation, notice that x can assume values from $-a$ to a and y can assume values from $-b$ to b. The points $(a, 0)$ and $(-a, 0)$ are called the **vertices** of the ellipse. (See Figure 6.57b.) Since $a > b$, we call the line segment joining the vertices the **major axis** and we call the line segment joining the points $(0, b)$ and $(0, -b)$ the **minor axis**. Notice that the length of the major axis is $2a$ and the length of the minor axis is $2b$.

We state the general case in Theorem 6.3.

THEOREM 6.3

The equation

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1 \qquad (6.1)$$

with $a > b > 0$ describes an ellipse with foci at $(x_0 - c, y_0)$ and $(x_0 + c, y_0)$, where $c = \sqrt{a^2 - b^2}$. The **center** of the ellipse is at the point (x_0, y_0) and the **vertices** are located at $(x_0 \pm a, y_0)$ on the major axis. The endpoints of the minor axis are located at $(x_0, y_0 \pm b)$.

The equation

$$\frac{(x-x_0)^2}{b^2}+\frac{(y-y_0)^2}{a^2}=1 \qquad (6.2)$$

with $a > b > 0$ describes an ellipse with foci at $(x_0, y_0 - c)$ and $(x_0, y_0 + c)$ where $c = \sqrt{a^2 - b^2}$. The **center** of the ellipse is at the point (x_0, y_0) and the **vertices** are located at $(x_0, y_0 \pm a)$ on the major axis. The endpoints of the minor axis are located at $(x_0 \pm b, y_0)$.

In example 6.5, we use Theorem 6.3 to identify the features of an ellipse.

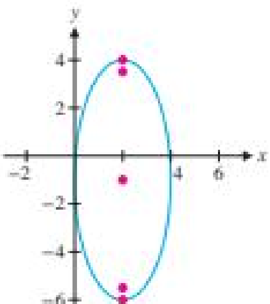


FIGURE 6.58
$\frac{(x-2)^2}{4}+\frac{(y+1)^2}{25}=1$

EXAMPLE 6.5 Identifying the Features of an Ellipse

Identify the center, foci and vertices of the ellipse $\frac{(x-2)^2}{4}+\frac{(y+1)^2}{25}=1$.

Solution From (6.2), the center is at $(2, -1)$. The values of a^2 and b^2 are 25 and 4, respectively, so that $c = \sqrt{21}$. Since the major axis is parallel to the y-axis, the foci are shifted c units above and below the center, at $(2, -1 - \sqrt{21})$ and $(2, -1 + \sqrt{21})$. Notice that in this case, the vertices are the intersections of the ellipse with the line $x = 2$. With $x = 2$, we have $(y + 1)^2 = 25$, so that $y = -1 \pm 5$ and the vertices are $(2, -6)$ and $(2, 4)$. Finally, the endpoints of the minor axis are found by setting $y = -1$. We have $(x - 2)^2 = 4$, so that $x = 2 \pm 2$ and these endpoints are $(0, -1)$ and $(4, -1)$. The ellipse is sketched in Figure 6.58. ■

EXAMPLE 6.6 Finding an Equation of an Ellipse

Find an equation of the ellipse with foci at (2, 3) and (2, 5) and vertices (2, 2) and (2, 6).

Solution Here, the center is the midpoint of the foci, (2, 4). You can now see that the foci are shifted $c = 1$ unit from the center. The vertices are shifted $a = 2$ units from the center. From $c^2 = a^2 - b^2$, we get $b^2 = 4 - 1 = 3$. Notice that the major axis is parallel to the y-axis, so that $a^2 = 4$ is the divisor of the y-term. From (6.2), the ellipse has the equation

$$\frac{(x-2)^2}{3}+\frac{(y-4)^2}{4}=1. \ ■$$

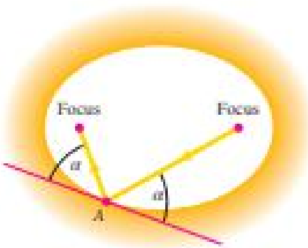


FIGURE 6.59
The reflective property of ellipses

Much like parabolas, ellipses have some useful reflective properties. As illustrated in Figure 6.59, a line segment joining one focus to a point A on the ellipse makes the same acute angle with the tangent line at A as does the line segment joining the other focus to A. Again, this is the same way in which light and sound reflect off a surface, so that a ray originating at one focus will always reflect off the ellipse toward the other focus. A surprising application of this principle is found in the so-called "whispering gallery" of the U.S. Senate. The ceiling of this room is elliptical, so that by standing at one focus you can hear everything said on the other side of the room at the other focus. (You probably never imagined how much of a role mathematics could play in political intrigue!)

EXAMPLE 6.7 A Medical Application of the Reflective Property of Ellipses

A medical procedure called **shockwave lithotripsy** is used to break up kidney stones that are too large or irregular to be passed. In this procedure, shockwaves emanating from a transducer located at one focus are bounced off of an elliptical reflector to the kidney stone located at the other focus. Suppose that the reflector is described by the equation $\frac{x^2}{112} + \frac{y^2}{48} = 1$ (in units of cm). Where should the transducer be placed?

Solution In this case,

$$c = \sqrt{a^2 - b^2} = \sqrt{112 - 48} = 8,$$

so that the foci are 16 cm apart. Since the transducer must be located at one focus, it should be placed 16 cm away from the kidney stone and aligned so that the line segment from the kidney stone to the transducer lies along the major axis of the elliptical reflector. ■

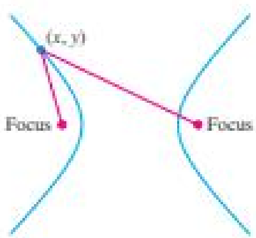


FIGURE 6.60
Definition of hyperbola

Hyperbolas

We define a **hyperbola** to be the set of all points such that the *difference* of the distances between two fixed points (called the **foci**) is a constant, as illustrated in Figure 6.60. Notice that it is nearly identical to the definition of the ellipse, except that we subtract the distances instead of add them.

The familiar equation of the hyperbola can be derived from the definition. The derivation is almost identical to that of the ellipse, except that the quantity $a^2 - c^2$ is now negative. We leave the details of the derivation of this as an exercise. An equation of the hyperbola with foci at $(\pm c, 0)$ and parameter $2a$ (equal to the difference of the distances) is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

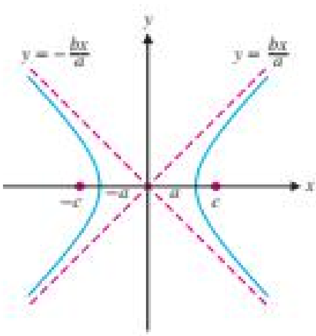


FIGURE 6.61
Hyperbola, shown with its asymptotes

where $b^2 = c^2 - a^2$. For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, notice that $y^2 = \frac{b^2}{a^2}x^2 - b^2$, so that

$$\lim_{x \to \pm\infty} \frac{y^2}{x^2} = \lim_{x \to \pm\infty} \left(\frac{b^2}{a^2} - \frac{b^2}{x^2} \right) = \frac{b^2}{a^2}.$$

That is, as $x \to \pm\infty$, $\frac{y^2}{x^2} \to \frac{b^2}{a^2}$, so that $\frac{y}{x} \to \pm\frac{b}{a}$ and so, $y = \pm\frac{b}{a}x$ are the (slant) asymptotes, as shown in Figure 6.61.

We state the general case in Theorem 6.4.

THEOREM 6.4

The equation

$$\frac{(x-x_0)^2}{a^2}-\frac{(y-y_0)^2}{b^2}=1 \tag{6.3}$$

describes a hyperbola with foci at the points (x_0-c, y_0) and (x_0+c, y_0), where $c=\sqrt{a^2+b^2}$. The **center** of the hyperbola is at the point (x_0, y_0) and the **vertices** are located at $(x_0 \pm a, y_0)$. The asymptotes are $y=\pm\frac{b}{a}(x-x_0)+y_0$.

The equation

$$\frac{(y-y_0)^2}{a^2}-\frac{(x-x_0)^2}{b^2}=1 \tag{6.4}$$

describes a hyperbola with foci at the points (x_0, y_0-c) and (x_0, y_0+c), where $c=\sqrt{a^2+b^2}$. The **center** of the hyperbola is at the point (x_0, y_0) and the **vertices** are located at $(x_0, y_0 \pm a)$. The asymptotes are $y=\pm\frac{a}{b}(x-x_0)+y_0$.

In example 6.8, we use Theorem 6.4 to identify the features of a hyperbola.

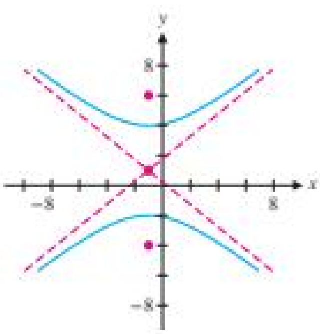


FIGURE 6.62
$\frac{(y-1)^2}{9}-\frac{(x+1)^2}{16}=1$

EXAMPLE 6.8 Identifying the Features of a Hyperbola

For the hyperbola $\frac{(y-1)^2}{9}-\frac{(x+1)^2}{16}=1$, find the center, vertices, foci and asymptotes.

Solution Notice that from (6.4), the center is at $(-1, 1)$. Setting $x=-1$, we find that the vertices are shifted vertically by $a=3$ units from the center, to $(-1, -2)$ and $(-1, 4)$. The foci are shifted vertically by $c=\sqrt{a^2+b^2}=\sqrt{25}=5$ units from the center, to $(-1, -4)$ and $(-1, 6)$. The asymptotes are $y=\pm\frac{3}{4}(x+1)+1$. A sketch of the hyperbola is shown in Figure 6.62. ■

EXAMPLE 6.9 Finding the Equation of a Hyperbola

Find an equation of the hyperbola with center at $(-2, 0)$, vertices at $(-4, 0)$ and $(0, 0)$ and foci at $(-5, 0)$ and $(1, 0)$.

Solution Notice that since the center, vertices and foci all lie on the x-axis, the hyperbola must have an equation of the form of (6.3). Here, the vertices are shifted $a=2$ units from the center and the foci are shifted $c=3$ units from the center. Then, we have $b^2=c^2-a^2=5$. Following (6.3), we have the equation

$$\frac{(x+2)^2}{4}-\frac{y^2}{5}=1. \quad ■$$

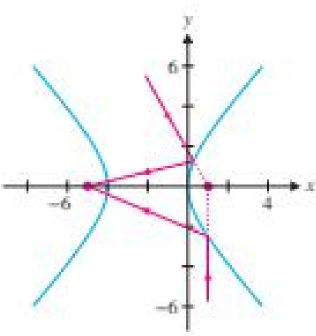


FIGURE 6.63
The reflective property of hyperbolas

Much like parabolas and ellipses, hyperbolas have a reflective property that is useful in applications. It can be shown that a ray directed toward one focus will reflect off the hyperbola toward the other focus, as illustrated in Figure 6.63.

EXAMPLE 6.10 An Application to Hyperbolic Mirrors

A hyperbolic mirror is constructed in the shape of the top half of the hyperbola $(y+2)^2-\frac{x^2}{3}=1$. Toward what point will light rays following the paths $y=kx$ reflect (where k is a constant)?

Solution For the given hyperbola, we have $c = \sqrt{a^2 + b^2} = \sqrt{1 + 3} = 2$. Notice that the center is at $(0, -2)$ and the foci are at $(0, 0)$ and $(0, -4)$. Since rays of the form $y = kx$ will pass through the focus at $(0, 0)$, they will be reflected toward the focus at $(0, -4)$. ■

A clever use of parabolic and hyperbolic mirrors in telescope design is illustrated in Figure 6.64, where a parabolic mirror to the left and a hyperbolic mirror to the right are arranged so that they have a common focus at the point F. The vertex of the parabola is located at the other focus of the hyperbola, at the point E, where there is an opening for the eye or a camera. Notice that light entering the telescope from the right (and passing around the hyperbolic mirror) will reflect off the parabola directly toward its focus at F. Since F is also a focus of the hyperbola, the light will reflect off the hyperbola toward its other focus at E. In combination, the mirrors focus all incoming light at the point E.

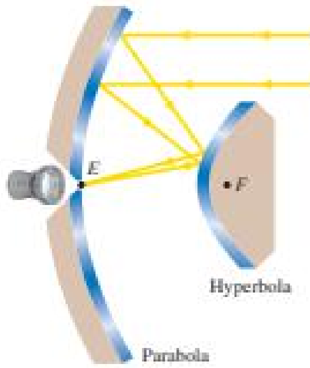


FIGURE 6.64
A combination of parabolic and hyperbolic mirrors

EXERCISES 6.6

WRITING EXERCISES

1. Each fixed point referred to in the definitions of the conic sections is called a **focus.** Use the reflective properties of the conic sections to explain why this is an appropriate name.

2. A hyperbola looks somewhat like a pair of parabolas facing opposite directions. Discuss the differences between a parabola and one half of a hyperbola (recall that hyperbolas have asymptotes).

3. Carefully explain why in example 6.6 (or for any other ellipse) the sum of the distances from a point on the ellipse to the two foci equals $2a$.

4. Imagine playing a game of pool on an elliptical pool table with a single hole located at one focus. If a ball rests near the other focus, which is clearly marked, describe an easy way to hit the ball into the hole.

In exercises 1–12, find an equation for the indicated conic section.

1. Parabola with focus $(0, -1)$ and directrix $y = 1$
2. Parabola with focus $(1, 2)$ and directrix $y = -2$
3. Parabola with focus $(3, 0)$ and directrix $x = 1$
4. Parabola with focus $(2, 0)$ and directrix $x = -2$
5. Ellipse with foci $(0, 1)$ and $(0, 5)$ and vertices $(0, -1)$ and $(0, 7)$
6. Ellipse with foci $(1, 2)$ and $(1, 4)$ and vertices $(1, 1)$ and $(1, 5)$
7. Ellipse with foci $(2, 1)$ and $(6, 1)$ and vertices $(0, 1)$ and $(8, 1)$
8. Ellipse with foci $(3, 2)$ and $(5, 2)$ and vertices $(2, 2)$ and $(6, 2)$
9. Hyperbola with foci $(0, 0)$ and $(4, 0)$ and vertices $(1, 0)$ and $(3, 0)$
10. Hyperbola with foci $(-2, 2)$ and $(6, 2)$ and vertices $(0, 2)$ and $(4, 2)$
11. Hyperbola with foci $(2, 2)$ and $(2, 6)$ and vertices $(2, 3)$ and $(2, 5)$
12. Hyperbola with foci $(0, -2)$ and $(0, 4)$ and vertices $(0, 0)$ and $(0, 2)$

In exercises 13–24, identify the conic section and find each vertex, focus and directrix.

13. $y = 2(x + 1)^2 - 1$

14. $y = -2(x + 2)^2 - 1$

15. $\frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} = 1$

16. $\frac{(x+2)^2}{16} + \frac{y^2}{4} = 1$

17. $\frac{(x-1)^2}{9} - \frac{y^2}{4} = 1$

18. $\frac{(x+1)^2}{4} - \frac{(y-3)^2}{4} = 1$

19. $\frac{(y+1)^2}{16} - \frac{(x+2)^2}{4} = 1$

20. $\frac{y^2}{4} - \frac{(x+2)^2}{9} = 1$

21. $(x - 2)^2 + 9y^2 = 9$

22. $4x^2 + (y + 1)^2 = 16$

23. $(x + 1)^2 - 4(y - 2) = 16$

24. $4(x + 2) - (y - 1)^2 = -4$

In exercises 25–30, graph the conic section and find an equation.

25. All points equidistant from the point (2, 1) and the line $y = -3$

26. All points equidistant from the point (−1, 0) and the line $y = 4$

27. All points such that the sum of the distances to the points (0, 2) and (4, 2) equals 8

28. All points such that the sum of the distances to the points (3, 1) and (−1, 1) equals 6

29. All points such that the difference of the distances to the points (0, 4) and (0, −2) equals 4

30. All points such that the difference of the distances to the points (2, 2) and (6, 2) equals 2

31. A parabolic flashlight reflector has the shape $x = 4y^2$. Where should the lightbulb be placed?

32. A parabolic flashlight reflector has the shape $x = \frac{1}{2}y^2$. Where should the lightbulb be placed?

33. A parabolic satellite dish has the shape $y = 2x^2$. Where should the microphone be placed?

34. A parabolic satellite dish has the shape $y = 4x^2$. Where should the microphone be placed?

35. In example 6.7, if the shape of the reflector is $\frac{x^2}{124} + \frac{y^2}{24} = 1$, how far from the kidney stone should the transducer be placed?

36. In example 6.7, if the shape of the reflector is $\frac{x^2}{44} + \frac{y^2}{125} = 1$, how far from the kidney stone should the transducer be placed?

37. If a hyperbolic mirror is in the shape of the top half of $(y + 4)^2 - \frac{x^2}{15} = 1$, to which point will light rays following the path $y = cx$ $(y < 0)$ reflect?

38. If a hyperbolic mirror is in the shape of the bottom half of $(y - 3)^2 - \frac{x^2}{8} = 1$, to which point will light rays following the path $y = cx$ $(y > 0)$ reflect?

39. If a hyperbolic mirror is in the shape of the right half of $\frac{x^2}{3} - y^2 = 1$, to which point will light rays following the path $y = c(x - 2)$ reflect?

40. If a hyperbolic mirror is in the shape of the left half of $\frac{x^2}{8} - y^2 = 1$, to which point will light rays following the path $y = c(x + 3)$ reflect?

APPLICATIONS

1. If the ceiling of a room has the shape $\frac{x^2}{400} + \frac{y^2}{100} = 1$, where should you place the desks so that you can sit at one desk and hear everything said at the other desk?

2. If the ceiling of a room has the shape $\frac{x^2}{900} + \frac{y^2}{100} = 1$, where should you place two desks so that you can sit at one desk and hear everything said at the other desk?

3. A spectator at the 2000 Summer Olympic Games throws an object. After 2 seconds, the object is 28 meters from the spectator. After 4 seconds, the object is 48 meters from the spectator. If the object's distance from the spectator is a quadratic function of time, find an equation for the position of the object. Sketch a graph of the path. What is the object?

4. Halley's comet follows an elliptical path with $a = 17.79$ Au (astronomical units) and $b = 4.53$ (Au). Compute the distance the comet travels in one orbit. Given that Halley's comet completes an orbit in approximately 76 years, what is the average speed of the comet?

EXPLORATORY EXERCISES

1. All of the equations of conic sections that we have seen so far have been of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$. In this exercise, you will classify the conic sections for different values of the constants. First, assume that $A > 0$ and $C > 0$. Which conic section will you get? Next, try $A > 0$ and $C < 0$. Which conic section is it this time? How about $A < 0$ and $C > 0$? $A < 0$ and $C < 0$? Finally, suppose that either A or C (not both) equals 0; which conic section is it? In all cases, the

values of the constants D, E and F do not affect the classification. Explain what effect these constants have.

2. In this exercise, you will generalize the results of exercise 1 by exploring the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. (In exercise 1, the coefficient of xy was 0.) You will need to have software that will graph such equations. Make up several examples with $B^2 - 4AC = 0$ (e.g., $B = 2$, $A = 1$ and $C = 1$). Which conic section results? Now, make up several examples with $B^2 - 4AC < 0$ (e.g., $B = 1$, $A = 1$ and $C = 1$). Which conic section do you get? Finally, make up several examples with $B^2 - 4AC > 0$ (e.g., $B = 4$, $A = 1$ and $C = 1$). Which conic section is this?

6.7 CONIC SECTIONS IN POLAR COORDINATES

An alternative definition of the conic sections utilizes an important quantity called *eccentricity* and is especially convenient for studying conic sections in polar coordinates. We introduce this concept in this section and review some options for parametric representations of conic sections.

For a fixed point P (the **focus**) and a fixed line l not containing P (the **directrix**), consider the set of all points whose distance to the focus is a constant multiple of their distance to the directrix. The constant multiple $e > 0$ is called the **eccentricity**. Note that if $e = 1$, this is the usual definition of a parabola. For other values of e, we get the other conic sections, as we see in Theorem 7.1.

THEOREM 7.1

The set of all points whose distance to the focus is the product of the eccentricity e and the distance to the directrix is

(i) an ellipse if $0 < e < 1$,
(ii) a parabola if $e = 1$ or
(iii) a hyperbola if $e > 1$.

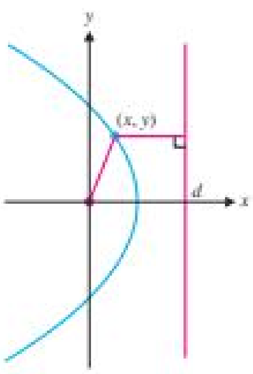


FIGURE 6.65
Focus and directrix

PROOF

We can simplify the algebra greatly by assuming that the focus is located at the origin and the directrix is the line $x = d > 0$. (We illustrate this in Figure 6.65 for the case of a parabola.) For any point (x, y) on the curve, observe that the distance to the focus is given by $\sqrt{x^2 + y^2}$ and the distance to the directrix is $d - x$. We then have

$$\sqrt{x^2 + y^2} = e(d - x). \tag{7.1}$$

Squaring both sides gives us

$$x^2 + y^2 = e^2(d^2 - 2dx + x^2).$$

Gathering together the like terms, we get

$$x^2(1 - e^2) + 2de^2x + y^2 = e^2d^2. \tag{7.2}$$

Note that (7.2) has the form of the equation of a conic section. In particular, if $e = 1$, (7.2) becomes

$$2dx + y^2 = d^2,$$

which is the equation of a parabola. If $0 < e < 1$, notice that $(1 - e^2) > 0$ and so, (7.2) is the equation of an ellipse (with center shifted to the left by the x-term). Finally, if $e > 1$, then $(1 - e^2) < 0$ and so (7.2) is the equation of a hyperbola. ■

Notice that the original form of the defining equation (7.1) of these conic sections includes the term $\sqrt{x^2 + y^2}$, which should make you think of polar coordinates. Recall that in polar coordinates, $r = \sqrt{x^2 + y^2}$ and $x = r \cos \theta$. Equation (7.1) now becomes

$$r = e(d - r \cos \theta).$$

Solving for r, we have

$$\boxed{r = \frac{ed}{e \cos \theta + 1},}$$

which is the polar form of an equation for the conic sections with focus and directrix oriented as in Figure 6.65. As you will show in the exercises, different orientations of the focus and directrix can produce different forms of the polar equation. We summarize the possibilities in Theorem 7.2.

THEOREM 7.2

The conic section with eccentricity $e > 0$, focus at (0, 0) and the indicated directrix has the polar equation

(i) $r = \dfrac{ed}{e \cos \theta + 1}$, if the directrix is the line $x = d > 0$,

(ii) $r = \dfrac{ed}{e \cos \theta - 1}$, if the directrix is the line $x = d < 0$,

(iii) $r = \dfrac{ed}{e \sin \theta + 1}$, if the directrix is the line $y = d > 0$ or

(iv) $r = \dfrac{ed}{e \sin \theta - 1}$, if the directrix is the line $y = d < 0$.

Notice that we proved part (i) above. The remaining parts are derived in similar fashion and are left as exercises. In example 7.1, we illustrate how the eccentricity affects the graph of a conic section.

EXAMPLE 7.1 The Effect of Various Eccentricities

Find polar equations of the conic sections with focus (0, 0), directrix $x = 4$ and eccentricities (a) $e = 0.4$, (b) $e = 0.8$, (c) $e = 1$, (d) $e = 1.2$ and (e) $e = 2$.

Solution By Theorem 7.1, observe that (a) and (b) are ellipses, (c) is a parabola and (d) and (e) are hyperbolas. By Theorem 7.2, all have polar equations of the form $r = \dfrac{4e}{e \cos \theta + 1}$. The graphs of the ellipses $r = \dfrac{1.6}{0.4 \cos \theta + 1}$ and $r = \dfrac{3.2}{0.8 \cos \theta + 1}$ are shown in Figure 6.66a. Note that the ellipse with the smaller eccentricity is much more nearly circular than the ellipse with the larger eccentricity. Further, the ellipse with $e = 0.8$ opens up much farther to the left. In fact, as the value of e approaches 1, the ellipse will open up farther to the left, approaching the parabola with $e = 1$, $r = \dfrac{4}{\cos \theta + 1}$, also shown in Figure 6.66a. For values of $e > 1$, the graph is a hyperbola, opening up to the right and left. For instance, with $e = 1.2$ and $e = 2$, we have the hyperbolas $r = \dfrac{4.8}{1.2 \cos \theta + 1}$ and $r = \dfrac{8}{2 \cos \theta + 1}$ (shown in Figure 6.66b), where we also indicate the parabola with $e = 1$. Notice how the second hyperbola approaches its asymptotes much more rapidly than the first.

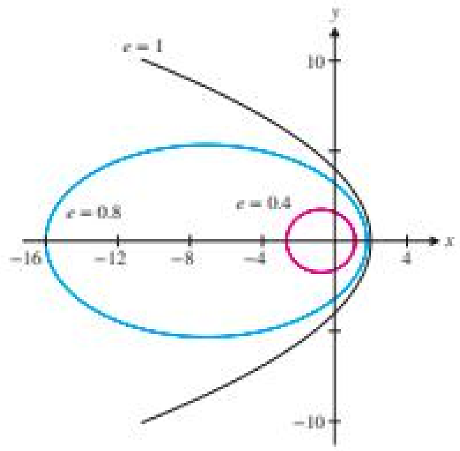


FIGURE 6.66a
$e = 0.4$, $e = 0.8$ and $e = 1.0$

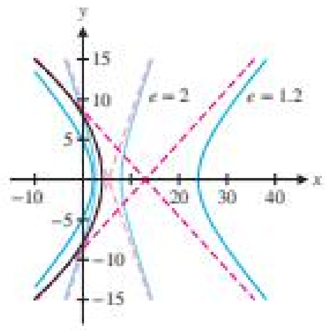


FIGURE 6.66b
$e = 1.0$, $e = 1.2$ and $e = 2.0$

EXAMPLE 7.2 The Effect of Various Directrixes

Find polar equations of the conic sections with focus (0, 0), eccentricity $e = 0.5$ and directrix given by (a) $y = 2$, (b) $y = -3$ and (c) $x = -2$.

Solution First, note that with an eccentricity of $e = 0.5$, each of these conic sections is an ellipse. From Theorem 7.2, we know that (a) has the form $r = \frac{1}{0.5 \sin\theta + 1}$. A sketch is shown in Figure 6.67a.

For (b), we have $r = \frac{-1.5}{0.5 \sin\theta - 1}$ and show a sketch in Figure 6.67b. For (c), the directrix is parallel to the x-axis and so, from Theorem 7.2, we have $r = \frac{-1}{0.5 \cos\theta - 1}$. A sketch is shown in Figure 6.67c.

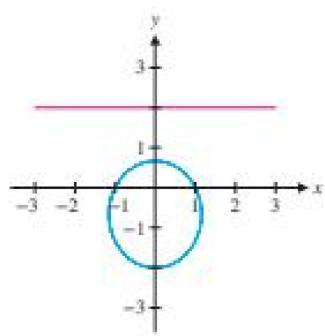


FIGURE 6.67a
Directrix: $y = 2$

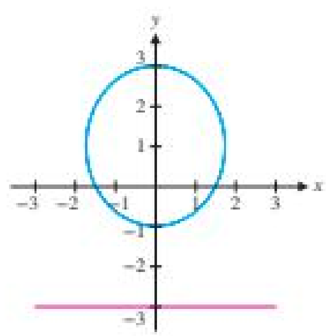


FIGURE 6.67b
Directrix: $y = -3$

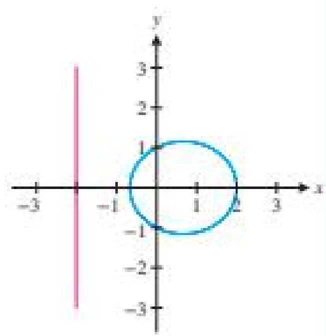


FIGURE 6.67c
Directrix: $x = -2$

The results of Theorem 7.2 apply only to conic sections with a focus at the origin. Recall that in rectangular coordinates, it's easy to translate the center of a conic section. Unfortunately, this is not true in polar coordinates.

In example 7.3, we see how to write some conic sections parametrically.

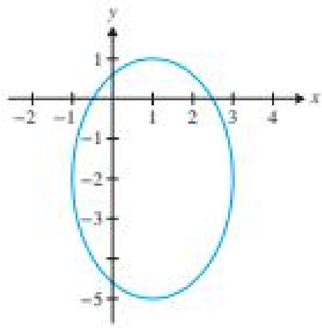


FIGURE 6.68a
$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{9}=1$

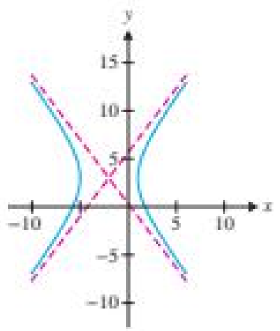


FIGURE 6.68b
$\frac{(x+2)^2}{9}-\frac{(y-3)^2}{16}=1$

EXAMPLE 7.3 Parametric Equations for Some Conic Sections

Find parametric equations of the conic sections (a) $\frac{(x-1)^2}{4}+\frac{(y+2)^2}{9}=1$ and (b) $\frac{(x+2)^2}{9}-\frac{(y-3)^2}{16}=1$.

Solution Notice that the curve in (a) is an ellipse with center at $(1, -2)$ and major axis parallel to the y-axis. Parametric equations for the ellipse are

$$\begin{cases} x = 2\cos t + 1 \\ y = 3\sin t - 2 \end{cases} \quad \text{with } 0 \le t \le 2\pi.$$

We show a sketch in Figure 6.68a.

You should recognize that the curve in (b) is a hyperbola. It is convenient to use hyperbolic functions in its parametric representation. The parameters are $a^2 = 9$ ($a = 3$) and $b^2 = 16$ ($b = 4$) and the center is $(-2, 3)$. Parametric equations are

$$\begin{cases} x = 3\cosh t - 2 \\ y = 4\sinh t + 3 \end{cases},$$

for the right half of the hyperbola and

$$\begin{cases} x = -3\cosh t - 2 \\ y = 4\sinh t + 3 \end{cases},$$

for the left half. We leave it as an exercise to verify that this is a correct parameterization. We sketch the hyperbola in Figure 6.68b. ■

In 1543, the astronomer Copernicus shocked the world with the publication of his theory that the Earth and the other planets revolve in circular orbits about the Sun. This stood in sharp contrast to the age-old belief that the Sun and other planets revolved around the Earth. By the early part of the seventeenth century, Johannes Kepler had analyzed 20 years' worth of painstaking observations of the known planets made by Tycho Brahe (before the invention of the telescope). He concluded that, in fact, each planet moves in an elliptical orbit, with the Sun located at one focus. About 100 years later, Isaac Newton used his newly created calculus to show that Kepler's conclusions follow directly from Newton's **universal law of gravitation**. Although we must delay a more complete presentation of Kepler's laws until Chapter 11, we are now in a position to illustrate one of these. Kepler's second law states that, measuring from the Sun to a planet, equal areas are swept out in equal times. As we see in example 7.4, this implies that planets speed up and slow down as they orbit the Sun.

EXAMPLE 7.4 Kepler's Second Law of Planetary Motion

Suppose that a planet's orbit follows the elliptical path $r = \frac{2}{\sin\theta + 2}$ with the Sun located at the origin (one of the foci), as illustrated in Figure 6.69a. Show that roughly equal areas are swept out from $\theta = 0$ to $\theta = \pi$ and from $\theta = \frac{3\pi}{2}$ to $\theta = 5.224895$. Then, find the corresponding arc lengths and compare the average speeds of the planet on these arcs.

Solution First, note that the area swept out by the planet from $\theta = 0$ to $\theta = \pi$ is the area bounded by the polar graphs $r = f(\theta) = \frac{2}{\sin\theta + 2}$, $\theta = 0$ and $\theta = \pi$. (See Figure 6.69b.) From (5.5), this is given by

$$A = \frac{1}{2}\int_0^\pi [f(\theta)]^2\,d\theta = \frac{1}{2}\int_0^\pi \left(\frac{2}{\sin\theta + 2}\right)^2 d\theta \approx 0.9455994.$$

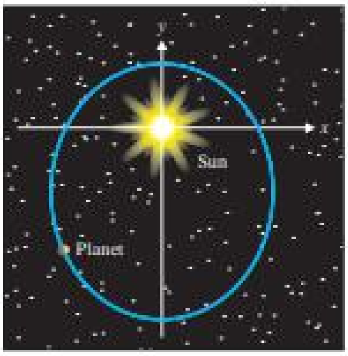


FIGURE 6.69a
Elliptical orbit

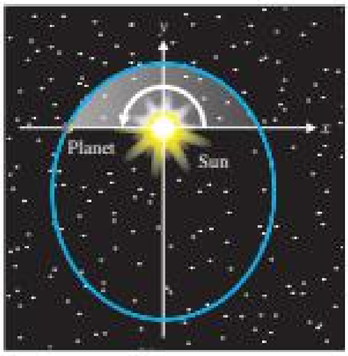


FIGURE 6.69b
Area swept out by the orbit from $\theta = 0$ to $\theta = \pi$

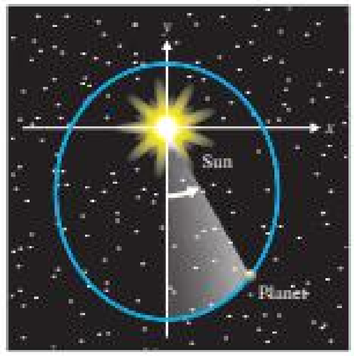


FIGURE 6.69c
Area swept out by the orbit from $\theta = \frac{3\pi}{2}$ to $\theta = 5.224895$

Similarly, the area swept out from $\theta = \frac{3\pi}{2}$ to $\theta = 5.224895$ (see Figure 6.69c) is given by

$$A = \frac{1}{2}\int_{3\pi/2}^{5.224895} \left(\frac{2}{\sin\theta + 2}\right)^2 d\theta \approx 0.9455995.$$

From (5.7), the arc length of the portion of the curve on the interval from $\theta = 0$ to $\theta = \pi$ is given by

$$\begin{aligned} s_1 &= \int_0^{\pi} \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}\, d\theta \\ &= \int_0^{\pi} \sqrt{\frac{4\cos^2\theta}{(\sin\theta + 2)^4} + \frac{4}{(\sin\theta + 2)^2}}\, d\theta \approx 2.53, \end{aligned}$$

while the arc length of the portion of the curve on the interval from $\theta = \frac{3\pi}{2}$ to $\theta = 5.224895$ is given by

$$s_2 = \int_{3\pi/2}^{5.224895} \sqrt{\frac{4\cos^2\theta}{(\sin\theta + 2)^4} + \frac{4}{(\sin\theta + 2)^2}}\, d\theta \approx 1.02.$$

Since these arcs are traversed in the same time, this says that the average speed on the portion of the orbit from $\theta = 0$ to $\theta = \pi$ is roughly two-and-a-half times the average speed on the portion of the orbit from $\theta = \frac{3\pi}{2}$ to $\theta = 5.224895$. ■

EXERCISES 6.7

WRITING EXERCISES

1. Based on Theorem 7.1, we might say that parabolas are the rarest of the conic sections, since they occur only for $e = 1$ exactly. Referring to Figure 6.49, explain why it takes a fairly precise cut of the cone to produce a parabola.

2. Describe how the ellipses in Figure 6.66a "open up" into a parabola as e increases to $e = 1$. What happens as e decreases to $e = 0$?

In exercises 1–16, find polar equations for and graph the conic section with focus (0, 0) and the given directrix and eccentricity.

1. Directrix $x = 2$, $e = 0.6$
2. Directrix $x = 2$, $e = 1.2$
3. Directrix $x = 2$, $e = 1$
4. Directrix $x = 2$, $e = 2$
5. Directrix $y = 2$, $e = 0.6$
6. Directrix $y = 2$, $e = 1.2$
7. Directrix $y = 2$, $e = 1$
8. Directrix $y = 2$, $e = 2$
9. Directrix $x = -2$, $e = 0.4$
10. Directrix $x = -2$, $e = 1$
11. Directrix $x = -2$, $e = 2$
12. Directrix $x = -2$, $e = 4$
13. Directrix $y = -2$, $e = 0.4$

14. Directrix $y = -2$, $e = 0.9$

15. Directrix $y = -2$, $e = 1$

16. Directrix $y = -2$, $e = 1.1$

In exercises 17–22, graph and interpret the conic section.

17. $r = \dfrac{4}{2\cos(\theta - \pi/6) + 1}$

18. $r = \dfrac{4}{4\sin(\theta - \pi/6) + 1}$

19. $r = \dfrac{-6}{\sin(\theta - \pi/4) - 2}$

20. $r = \dfrac{-4}{\cos(\theta - \pi/4) - 4}$

21. $r = \dfrac{-3}{2\cos(\theta + \pi/4) - 2}$

22. $r = \dfrac{3}{2\cos(\theta + \pi/4) + 2}$

In exercises 23–28, find parametric equations of the conic sections.

23. $\dfrac{(x+1)^2}{9} + \dfrac{(y-1)^2}{4} = 1$

24. $\dfrac{(x-2)^2}{9} - \dfrac{(y+1)^2}{16} = 1$

25. $\dfrac{(x+1)^2}{16} - \dfrac{y^2}{9} = 1$

26. $\dfrac{x^2}{4} + y^2 = 1$

27. $\dfrac{x^2}{4} + y = 1$

28. $x - \dfrac{y^2}{4} = 1$

29. Repeat example 7.4 with $0 \le \theta \le \frac{\pi}{2}$ and $\frac{3\pi}{2} \le \theta \le 4.953$.

30. Repeat example 7.4 with $\frac{\pi}{2} \le \theta \le \pi$ and $4.471 \le \theta \le \frac{3\pi}{2}$.

31. Prove Theorem 7.2 (ii).

32. Prove Theorem 7.2 (iii).

33. Prove Theorem 7.2 (iv).

EXPLORATORY EXERCISES

1. If Neptune's orbit is given by

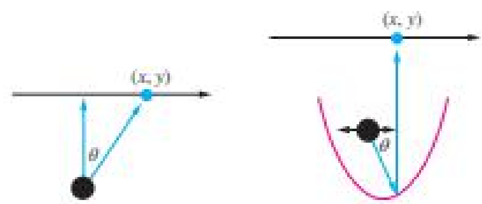
$$r = \frac{1.82 \times 10^{14}}{343\cos(\theta - 0.77) + 40{,}000}$$

and Pluto's orbit is given by

$$r = \frac{5.52 \times 10^{13}}{2481\cos(\theta - 3.91) + 10{,}000},$$

show that Pluto is sometimes closer and sometimes farther from the Sun than Neptune. Based on these equations, will the planets ever collide?

2. Vision has proved to be one of the biggest challenges for building functional robots. Robot vision either can be designed to mimic human vision or can follow a different design. Two possibilities are analyzed here. In the diagram to the left, a camera follows an object directly from left to right. If the camera is at the origin, the object moves with speed 1 m/s and the line of motion is at $y = c$, find an expression for θ' as a function of the position of the object. In the diagram to the right, the camera looks down into a curved mirror and indirectly views the object. Assume that the mirror has equation $r = \dfrac{1 - \sin\theta}{2\cos^2\theta}$. Show that the mirror is parabolic and find its focus and directrix. With $x = r\cos\theta$, find an expression for θ' as a function of the position of the object. Compare values of θ' at $x = 0$ and other x-values. If a large value of θ' causes the image to blur, which camera system is better? Does the distance $y = c$ affect your preference?



Review Exercises

WRITING EXERCISES

The following list includes terms that are defined in this chapter. For each term, (1) give a precise definition, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Parametric equations	Velocity	Focus
Polar coordinates	Arc length	Vertex
Slope in parametric equations	Surface area	Directrix
Slope in polar coordinates	Parabola	Eccentricity
Area in parametric equations	Ellipse	Kepler's laws
Area in polar coordinates	Hyperbola	

Review Exercises

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to make a new statement that is true.

1. For a given curve, there is exactly one set of parametric equations that describes it.
2. In parametric equations, circles are always sketched counterclockwise.
3. In parametric equations, the derivative equals slope and velocity.
4. If a point has polar coordinates (r, θ), then it also has polar coordinates $(-r, \theta + \pi)$.
5. If f is periodic with fundamental period 2π, then one copy of $r = f(\theta)$ is traced out with $0 \le \theta \le 2\pi$.
6. In polar coordinates, you can describe circles but not lines.
7. To find all intersections of polar curves $r = f(\theta)$ and $r = g(\theta)$, solve $f(\theta) = g(\theta)$.
8. The focus, vertex and directrix of a parabola all lie on the same line.
9. The equation of any conic section can be written in the form $r = \dfrac{ed}{e\cos\theta + 1}$.

In exercises 1–4, sketch the plane curve defined by the parametric equations and find a corresponding x-y equation for the curve.

1. $\begin{cases} x = -1 + 3\cos t \\ y = 2 + 3\sin t \end{cases}$
2. $\begin{cases} x = 2 - t \\ y = 1 + 3t \end{cases}$
3. $\begin{cases} x = t^2 + 1 \\ y = t^4 \end{cases}$
4. $\begin{cases} x = \cos t \\ y = \cos^2 t - 1 \end{cases}$

In exercises 5–8, sketch the plane curves defined by the parametric equations.

5. $\begin{cases} x = \cos 2t \\ y = \sin 6t \end{cases}$
6. $\begin{cases} x = \cos 6t \\ y = \sin 2t \end{cases}$
7. $\begin{cases} x = \cos 2t \cos t \\ y = \cos 2t \sin t \end{cases}$
8. $\begin{cases} x = \cos 2t \cos 3t \\ y = \cos 2t \sin 3t \end{cases}$

In exercises 9–12, match the parametric equations with the corresponding plane curve.

9. $\begin{cases} x = t^2 - 1 \\ y = t^3 \end{cases}$
10. $\begin{cases} x = t^3 \\ y = t^2 - 1 \end{cases}$
11. $\begin{cases} x = \cos 2t \cos t \\ y = \cos 2t \sin t \end{cases}$
12. $\begin{cases} x = \cos(t + \cos t) \\ y = \cos(t + \sin t) \end{cases}$

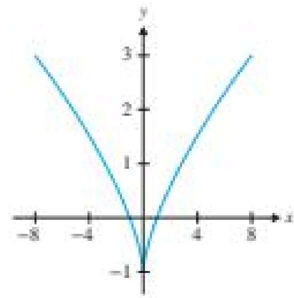


FIGURE A

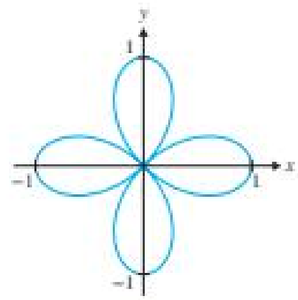


FIGURE B

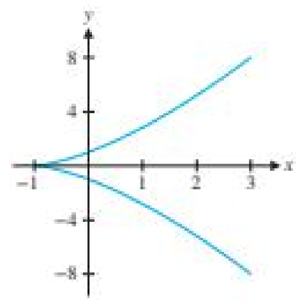


FIGURE C

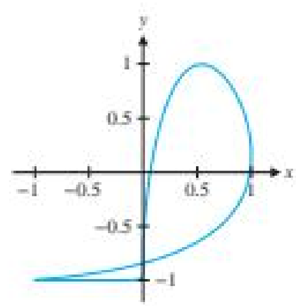


FIGURE D

Review Exercises

In exercises 13 and 14, find parametric equations for the given curve.

13. The line segment from (2, 1) to (4, 7)

14. The portion of the parabola $y = x^2 + 1$ from (1, 2) to (3, 10)

In exercises 15 and 16, find the slopes of the curves at the points (a) $t = 0$, (b) $t = 1$ and (c) (2, 3).

15. $\begin{cases} x = t^3 - 3t \\ y = t^2 - t + 1 \end{cases}$

16. $\begin{cases} x = t^2 - 2 \\ y = t + 2 \end{cases}$

In exercises 17 and 18, parametric equations for the position of an object are given. Find the object's velocity and speed at time $t = 0$ and describe its motion.

17. $\begin{cases} x = t^3 - 3t \\ y = t^2 + 2t \end{cases}$

18. $\begin{cases} x = t^3 - 3t \\ y = t^2 + 2 \end{cases}$

In exercises 19–22, find the area enclosed by the curve.

19. $\begin{cases} x = 3\sin t \\ y = 2\cos t \end{cases}$

20. $\begin{cases} x = 4\sin 3t \\ y = 3\cos 3t \end{cases}$

21. $\begin{cases} x = \cos 2t \\ y = \sin \pi t \end{cases}, -1 \le t \le 1$

22. $\begin{cases} x = t^2 - 1 \\ y = t^3 - t \end{cases}, -1 \le t \le 1$

In exercises 23–26, find the arc length of the curve (approximate numerically, if needed).

23. $\begin{cases} x = \cos 2t \\ y = \sin \pi t \end{cases}, -1 \le t \le 1$

24. $\begin{cases} x = t^2 - 1 \\ y = t^3 - 4t \end{cases}, -1 \le t \le 1$

25. $\begin{cases} x = \cos 4t \\ y = \sin 5t \end{cases}$

26. $\begin{cases} x = \sin 10t \\ y = t^2 - 1 \end{cases}, -\pi \le t \le \pi$

In exercises 27 and 28, compute the surface area of the surface obtained by revolving the curve about the indicated axis.

27. $\begin{cases} x = t^3 - 4t \\ y = t^4 - 4t \end{cases}, -1 \le t \le 1$, about the x-axis

28. $\begin{cases} x = t^3 - 4t \\ y = t^4 - 4t \end{cases}, -1 \le t \le 1$, about $y = 2$

In exercises 29 and 30, sketch the graph of the polar equation and find a corresponding x-y equation.

29. $r = 3\cos\theta$

30. $r = 2\sec\theta$

In exercises 31–38, sketch the graph and identify all values of θ where $r = 0$ and a range of values of θ that produces one copy of the graph.

31. $r = 2\sin\theta$

32. $r = 2 - 2\sin\theta$

33. $r = 2 - 3\sin\theta$

34. $r = \cos 3\theta + \sin 2\theta$

35. $r^2 = 4\sin 2\theta$

36. $r = e^{\cos\theta} - 2\cos 4\theta$

37. $r = \dfrac{2}{1 + 2\sin\theta}$

38. $r = \dfrac{2}{1 + 2\cos\theta}$

In exercises 39 and 40, find a polar equation corresponding to the rectangular equation.

39. $x^2 + y^2 = 9$

40. $(x - 3)^2 + y^2 = 9$

In exercises 41 and 42, find the slope of the tangent line to the polar curve at the given point.

41. $r = \cos 3\theta$ at $\theta = \frac{\pi}{6}$

42. $r = 1 - \sin\theta$ at $\theta = 0$

In exercises 43–48, find the area of the indicated region.

43. One leaf of $r = \sin 5\theta$

44. One leaf of $r = \cos 2\theta$

45. Inner loop of $r = 1 - 2\sin\theta$

46. Bounded by $r = 3\sin\theta$

47. Inside of $r = 1 + \sin\theta$ and outside of $r = 1 + \cos\theta$

48. Inside of $r = 1 + \cos\theta$ and outside of $r = 1 + \sin\theta$

In exercises 49 and 50, find the arc length of the curve.

49. $r = 3 - 4\sin\theta$

50. $r = \sin 4\theta$

In exercises 51–53, find an equation for the conic section.

51. Parabola with focus (1, 2) and directrix $y = 0$

52. Ellipse with foci (2, 1) and (2, 3) and vertices (2, 0) and (2, 4)

53. Hyperbola with foci (2, 0) and (2, 4) and vertices (2, 1) and (2, 3)

In exercises 54–58, identify the conic section and find each vertex, focus and directrix.

54. $y = 3(x - 2)^2 + 1$

55. $\dfrac{(x + 1)^2}{9} + \dfrac{(y - 3)^2}{25} = 1$

Review Exercises

56. $\frac{x^2}{9} - \frac{(y+2)^2}{4} = 1$

57. $(x-1)^2 + y = 4$

58. $(x-1)^2 + 4y^2 = 4$

59. A parabolic satellite dish has the shape $y = \frac{1}{2}x^2$. Where should the microphone be placed?

60. If a hyperbolic mirror is in the shape of the top half of $(y+2)^2 - \frac{x^2}{3} = 1$, to which point will light rays following the path $y = cx$ $(y < 0)$ reflect?

In exercises 61–64, find a polar equation and graph the conic section with focus (0, 0) and the given directrix and eccentricity.

61. Directrix $x = 3$, $e = 0.8$

62. Directrix $y = 3$, $e = 1$

63. Directrix $y = 2$, $e = 1.4$

64. Directrix $x = 1$, $e = 2$

In exercises 65 and 66, find parametric equations for the conic sections.

65. $\frac{(x+1)^2}{9} + \frac{(y-3)^2}{25} = 1$

66. $\frac{x^2}{9} - \frac{(y+2)^2}{4} = 1$

EXPLORATORY EXERCISE

1. Sketch several polar graphs of the form $r = 1 + a\cos\theta$ and $r = 1 + a\sin\theta$ using some constants a that are positive and some that are negative, greater than 1, equal to 1 and less than 1 (for example, $a = -2$, $a = -1$, $a = -1/2$, $a = 1/2$, $a = 1$ and $a = 2$). Discuss all patterns you find.

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CHAPTER 7

Vectors and Vector-Valued Functions

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Auto racing at all levels has become a highly technological competition. Formula One race cars are engineered with large wings on the back and with undersides shaped like upside-down airplane wings, to use the air flow to help hold the cars to the track even at speeds up to 322 km/h. The engineering of stock cars is less obvious to the untrained eye, but is no less important. Stock car racers are challenged by, among other things, intricate rules that severely limit the extent to which the cars can be modified. Nevertheless, at the Bristol Motor Speedway, the track is an oval only 0.858 km in length and racers regularly exceed 193 km per hour, completing a lap in just over 15 seconds. These speeds would be unsafe if the track were not specially designed for high-speed racing. In particular, the Bristol track is steeply banked, with a 16-degree bank on straightaways and a spectacular 36-degree bank in the corners.

Chapter Topics

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As you will see in the exercises in section 7.1, the banking of a road changes the role of gravity. In effect, part of the weight of the car is diverted into a force that helps a car make its turn safely. In this chapter, we introduce vectors and develop the calculations needed to resolve vectors into components. This is a fundamental tool for engineers designing race cars and racetracks.

This chapter represents a crossroads from the primarily two-dimensional world of first-year calculus to the three-dimensional world of many important scientific and engineering problems. The rest of the calculus we develop in this book builds directly on the basic ideas developed here.

7.1 THE DOT PRODUCT

You may recall previously defining vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ and examining many of the properties of vectors, including how to add and subtract two vectors. It turns out that two different kinds of products involving vectors have proved to be useful: the dot (or scalar) product and the cross (or vector) product. We introduce the first of these two products in this section.

DEFINITION 1.1

The **dot product** of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ in V_3 is defined by

$$\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3. \quad (1.1)$$

Likewise, the dot product of two vectors in V_2 is defined by

$$\mathbf{a} \cdot \mathbf{b} = \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2.$$

Be sure to notice that the dot product of two vectors is a *scalar* (i.e. a number, not a vector). For this reason, the dot product is also called the **scalar product**.

EXAMPLE 1.1 Computing a Dot Product in $\mathbb{R}^3$

Compute the dot product $\mathbf{a} \cdot \mathbf{b}$ for $\mathbf{a} = \langle 1, 2, 3 \rangle$ and $\mathbf{b} = \langle 5, -3, 4 \rangle$.

Solution We have

$$\mathbf{a} \cdot \mathbf{b} = \langle 1, 2, 3 \rangle \cdot \langle 5, -3, 4 \rangle = (1)(5) + (2)(-3) + (3)(4) = 11. \blacksquare$$

Certainly, dot products are very simple to compute, whether a vector is written in component form or written in terms of the standard basis vectors, as in example 1.2.

EXAMPLE 1.2 Computing a Dot Product in $\mathbb{R}^2$

Find the dot product of the two vectors $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} + 6\mathbf{j}$.

Solution We have

$$\mathbf{a} \cdot \mathbf{b} = (2)(3) + (-5)(6) = 6 - 30 = -24. \blacksquare$$

The dot product in V_2 or V_3 satisfies the following simple properties.

REMARK 1.1

Since vectors in V_2 can be thought of as special cases of vectors in V_3 (where the third component is zero), all of the results we prove for vectors in V_3 hold equally for vectors in V_2.

THEOREM 1.1

For vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and any scalar d, the following hold:

(i) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (commutativity)
(ii) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (distributive law)
(iii) $(d\mathbf{a}) \cdot \mathbf{b} = d(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (d\mathbf{b})$
(iv) $\mathbf{0} \cdot \mathbf{a} = 0$ and
(v) $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$.

PROOF

We prove (i) and (v) for $\mathbf{a}, \mathbf{b} \in V_3$. The remaining parts are left as exercises.

(i) For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, we have from (1.1) that

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= b_1 a_1 + b_2 a_2 + b_3 a_3 = \mathbf{b} \cdot \mathbf{a}, \end{aligned}$$

since multiplication of real numbers is commutative.

(v) For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, we have

$$\mathbf{a} \cdot \mathbf{a} = \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle = a_1^2 + a_2^2 + a_3^2 = \|\mathbf{a}\|^2. \blacksquare$$

Notice that properties (i)–(iv) of Theorem 1.1 are also properties of multiplication of real numbers. This is why we use the word *product* in dot product. However, there

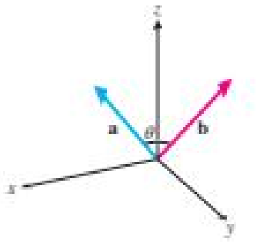


FIGURE 7.1a
The angle between two vectors

are some properties of multiplication of real numbers not shared by the dot product. For instance, we will see that $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$ does not imply that either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.

For two *non-zero* vectors **a** and **b** in V_3, we define the **angle** θ $(0 \le \theta \le \pi)$ **between the vectors** to be the smaller angle between **a** and **b**, formed by placing their initial points at the same point, as illustrated in Figure 7.1a.

Notice that if **a** and **b** have the *same* direction, then $\theta = 0$. If **a** and **b** have *opposite* directions, then $\theta = \pi$. We say that **a** and **b** are **orthogonal** (or **perpendicular**) if $\theta = \frac{\pi}{2}$. We consider the zero vector $\mathbf{0}$ to be orthogonal to every vector. The general case is stated in Theorem 1.2.

THEOREM 1.2

Let θ be the angle between non-zero vectors **a** and **b**. Then,

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta. \tag{1.2}$$

PROOF

We must prove the theorem for three separate cases.

(i) If **a** and **b** have the *same direction*, then $\mathbf{b} = c\mathbf{a}$, for some scalar $c > 0$ and the angle between **a** and **b** is $\theta = 0$. This says that

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (c\mathbf{a}) = c\mathbf{a} \cdot \mathbf{a} = c\|\mathbf{a}\|^2.$$

Further,

$$\|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta = \|\mathbf{a}\||c|\|\mathbf{a}\|\,\cos 0 = c\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{b},$$

since for $c > 0$, we have $|c| = c$.

(ii) If **a** and **b** have the opposite direction, the proof is nearly identical to case (i) above and we leave the details as an exercise.

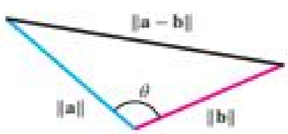


FIGURE 7.1b
The angle between two vectors

(iii) If **a** and **b** are not parallel, then we have that $0 < \theta < \pi$, as shown in Figure 7.1b. Recall that the Law of Cosines allows us to relate the lengths of the sides of triangles like the one in Figure 7.1b. We have

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta. \tag{1.3}$$

Now, observe that

$$\begin{aligned}\|\mathbf{a} - \mathbf{b}\|^2 &= \|\langle a_1 - b_1, a_2 - b_2, a_3 - b_3\rangle\|^2 \\ &= (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 \\ &= (a_1^2 - 2a_1b_1 + b_1^2) + (a_2^2 - 2a_2b_2 + b_2^2) + (a_3^2 - 2a_3b_3 + b_3^2) \\ &= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - 2(a_1b_1 + a_2b_2 + a_3b_3) \\ &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a} \cdot \mathbf{b} \end{aligned} \tag{1.4}$$

Equating the right-hand sides of (1.3) and (1.4), we get (1.2), as desired. ■

We can use (1.2) to find the angle between two vectors, as in example 1.3.

EXAMPLE 1.3 Finding the Angle between Two Vectors

Find the angle between the vectors $\mathbf{a} = \langle 2, 1, -3\rangle$ and $\mathbf{b} = \langle 1, 5, 6\rangle$.

Solution From (1.2), we have

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \frac{-11}{\sqrt{14}\sqrt{62}}.$$

It follows that $\theta = \cos^{-1}\left(\frac{-11}{\sqrt{14}\sqrt{62}}\right) \approx 1.953$(radians)

(or about 112°), since $0 \le \theta \le \pi$ and the inverse cosine function returns an angle in this range. ■

The following result is an immediate and important consequence of Theorem 1.2.

COROLLARY 1.1

Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

PROOF

First, observe that if either **a** or **b** is the zero vector, then $\mathbf{a} \cdot \mathbf{b} = 0$ and **a** and **b** are orthogonal, as the zero vector is considered orthogonal to every vector. If **a** and **b** are non-zero vectors and if θ is the angle between **a** and **b**, we have from Theorem 1.2 that

$$\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta = \mathbf{a} \cdot \mathbf{b} = 0$$

if and only if $\cos\theta = 0$ (since neither **a** nor **b** is the zero vector). This occurs if and only if $\theta = \frac{\pi}{2}$, which is equivalent to having **a** and **b** orthogonal and so, the result follows. ■

EXAMPLE 1.4 Determining Whether Two Vectors Are Orthogonal

Determine whether the following pairs of vectors are orthogonal: (a) $\mathbf{a} = \langle 1, 3, -5 \rangle$ and $\mathbf{b} = \langle 2, 3, 10 \rangle$ and (b) $\mathbf{a} = \langle 4, 2, -1 \rangle$ and $\mathbf{b} = \langle 2, 3, 14 \rangle$.

Solution For (a), we have:

$$\mathbf{a} \cdot \mathbf{b} = 2 + 9 - 50 = -39 \ne 0,$$

so that **a** and **b** are *not* orthogonal.

For (b), we have

$$\mathbf{a} \cdot \mathbf{b} = 8 + 6 - 14 = 0,$$

so that **a** and **b** are orthogonal, in this case. ■

The following two results provide us with some powerful tools for comparing the magnitudes of vectors.

THEOREM 1.3 (Cauchy–Schwartz Inequality)

For any vectors **a** and **b**,

$$|\mathbf{a} \cdot \mathbf{b}| \le \|\mathbf{a}\|\|\mathbf{b}\|. \tag{1.5}$$

PROOF

If either **a** or **b** is the zero vector, notice that (1.5) simply says that $0 \le 0$, which is certainly true. On the other hand, if neither **a** nor **b** is the zero vector, we have from (1.2) that

$$|\mathbf{a} \cdot \mathbf{b}| = \|\mathbf{a}\|\|\mathbf{b}\||\cos\theta| \le \|\mathbf{a}\|\|\mathbf{b}\|,$$

since $|\cos\theta| \le 1$ for all values of θ. ■

One benefit of the Cauchy–Schwartz Inequality is that it allows us to prove the following very useful result. If you were going to learn only one inequality in your lifetime, this is probably the one you would want to learn.

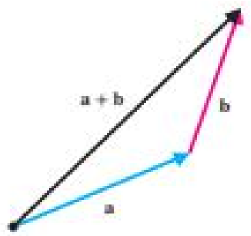


FIGURE 7.2
The Triangle Inequality

> **THEOREM 1.4** (The Triangle Inequality)
> For any vectors **a** and **b**,
> $$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|. \quad (1.6)$$

Before we prove the theorem, consider the triangle formed by the vectors **a**, **b** and **a** + **b**, shown in Figure 7.2. Notice that the Triangle Inequality says that the length of the vector **a** + **b** never exceeds the sum of the individual lengths of **a** and **b**.

PROOF

From Theorem 1.1 (i), (ii) and (v), we have

$$\begin{aligned}\|\mathbf{a} + \mathbf{b}\|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= \|\mathbf{a}\|^2 + 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2.\end{aligned}$$

From the Cauchy–Schwartz Inequality (1.5), we have $\mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|$ and so, we have

$$\begin{aligned}\|\mathbf{a} + \mathbf{b}\|^2 &= \|\mathbf{a}\|^2 + 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2 \\ &\leq \|\mathbf{a}\|^2 + 2\|\mathbf{a}\|\|\mathbf{b}\| + \|\mathbf{b}\|^2 = (\|\mathbf{a}\| + \|\mathbf{b}\|)^2.\end{aligned}$$

Taking square roots gives us (1.6). ■

○ Components and Projections

Think about the case where a vector represents a force. Often, it's impractical to exert a force in the direction you'd like. For instance, in pulling a child's wagon, we exert a force in the direction determined by the position of the handle, instead of in the direction of motion. (See Figure 7.3.) An important question is whether there is a force of smaller magnitude that can be exerted in a different direction and still produce the same effect on the wagon. Notice that it is the horizontal portion of the force that most directly contributes to the motion of the wagon. (The vertical portion of the force only acts to reduce friction.) We now consider how to compute such a component of a force.

FIGURE 7.3
Pulling a wagon

For any two non-zero position vectors **a** and **b**, let θ be the angle between the vectors. If we drop a perpendicular line segment from the terminal point of **a** to the line containing the vector **b**, then from elementary trigonometry, the base of the triangle (in the case where $0 < \theta < \frac{\pi}{2}$) has length given by $\|\mathbf{a}\| \cos\theta$. (See Figure 7.4a.)

> **TODAY IN MATHEMATICS**
>
> **Lene Vestergaard Hau (1959–Present)** A Danish mathematician and physicist known for her experiments to slow down and stop light. Although neither of her parents had a background in science or mathematics, she says that as a student, "I loved mathematics. I would rather do mathematics than go to the movies in those days. But after a while I discovered quantum mechanics and I've been hooked ever since." Hau credits a culture of scientific achievement with her success. "I was lucky to be a Dane. Denmark has a long scientific tradition that included the great Niels Bohr. . . . In Denmark, physics is widely respected by laymen as well as scientists, and laymen contribute to physics."*
>
> * Browne, M. W. (1999), Lene Vestergaard Hau: She Puts the Brakes on Light (The New York Times)

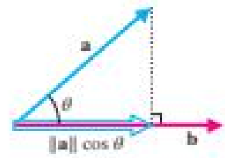


FIGURE 7.4a
$\text{comp}_{\mathbf{b}}\,\mathbf{a}$, for $0 < \theta < \frac{\pi}{2}$

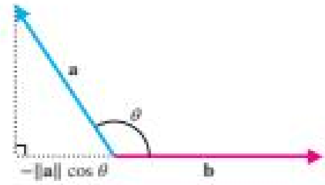


FIGURE 7.4b
$\text{comp}_{\mathbf{b}}\,\mathbf{a}$, for for $\frac{\pi}{2} < \theta < \pi$

On the other hand, notice that if $\frac{\pi}{2} < \theta < \pi$, the length of the base is given by $-\|\mathbf{a}\|\cos\theta$. (See Figure 7.4b.) In either case, we refer to $\|\mathbf{a}\|\cos\theta$ as the **component** of **a** along **b**, denoted $\text{comp}_{\mathbf{b}}\,\mathbf{a}$. Using (1.2), observe that we can rewrite this as

$$\begin{aligned}\text{comp}_{\mathbf{b}}\,\mathbf{a} &= \|\mathbf{a}\|\cos\theta = \frac{\|\mathbf{a}\|\,\|\mathbf{b}\|}{\|\mathbf{b}\|}\cos\theta \\ &= \frac{1}{\|\mathbf{b}\|}\|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta = \frac{1}{\|\mathbf{b}\|}\mathbf{a}\cdot\mathbf{b}\end{aligned}$$

Component of a along b

or

$$\boxed{\text{comp}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}.} \tag{1.7}$$

Notice that $\text{comp}_{\mathbf{b}}\,\mathbf{a}$ is a scalar and that we divide the dot product in (1.7) by $\|\mathbf{b}\|$ and not by $\|\mathbf{a}\|$. One way to keep this straight is to recognize that the components in Figures 7.4a and 7.4b depend on how long **a** is but not on how long **b** is. We can view (1.7) as the dot product of the vector **a** and a unit vector in the direction of **b**, given by $\frac{\mathbf{b}}{\|\mathbf{b}\|}$.

Once again, consider the case where the vector **a** represents a force. Rather than the component of **a** along **b**, we are often interested in finding a force vector parallel to **b** having the same component along **b** as **a**. We call this vector the **projection** of **a** onto **b**, denoted $\mathbf{proj}_{\mathbf{b}}\,\mathbf{a}$, as indicated in Figures 7.5a and 7.5b. Since the projection has magnitude $|\text{comp}_{\mathbf{b}}\,\mathbf{a}|$ and points in the direction of **b**, for $0 < \theta < \frac{\pi}{2}$ and opposite **b**, for $\frac{\pi}{2} < \theta < \pi$, we have from (1.7) that

$$\mathbf{proj}_{\mathbf{b}}\,\mathbf{a} = (\text{comp}_{\mathbf{b}}\,\mathbf{a})\frac{\mathbf{b}}{\|\mathbf{b}\|} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}\right)\frac{\mathbf{b}}{\|\mathbf{b}\|},$$

Projection of a onto b

or

$$\boxed{\mathbf{proj}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b},} \tag{1.8}$$

where $\frac{\mathbf{b}}{\|\mathbf{b}\|}$ represents a unit vector in the direction of **b**.

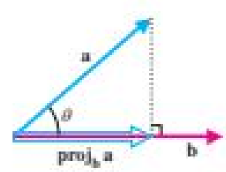


FIGURE 7.5a
$\mathbf{proj}_{\mathbf{b}}\,\mathbf{a}$, for $0 < \theta < \frac{\pi}{2}$

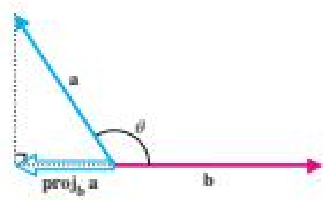


FIGURE 7.5b
$\mathbf{proj}_{\mathbf{b}}\,\mathbf{a}$, for $\frac{\pi}{2} < \theta < \pi$

In example 1.5, we illustrate the process of finding components and projections.

CAUTION

Be careful to distinguish between the *projection* of **a** onto **b** (a vector) and the *component* of **a** along **b** (a scalar). It is very common to confuse the two.

EXAMPLE 1.5 Finding Components and Projections

For $\mathbf{a} = \langle 2, 3\rangle$ and $\mathbf{b} = \langle -1, 5\rangle$, find the component of **a** along **b** and the projection of **a** onto **b**.

Solution From (1.7), we have

$$\text{comp}_{\mathbf{b}}\,\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|} = \frac{\langle 2,3\rangle\cdot\langle -1,5\rangle}{\|\langle -1,5\rangle\|} = \frac{-2+15}{\sqrt{1+5^2}} = \frac{13}{\sqrt{26}}.$$

Similarly, from (1.8), we have

$$\begin{aligned}\mathbf{proj_b\,a} &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}\right)\frac{\mathbf{b}}{\|\mathbf{b}\|} = \left(\frac{13}{\sqrt{26}}\right)\frac{\langle -1,5\rangle}{\sqrt{26}}\\ &= \frac{13}{26}\langle -1,5\rangle = \frac{1}{2}\langle -1,5\rangle = \left\langle -\frac{1}{2}, \frac{5}{2}\right\rangle.\end{aligned}$$

We leave it as an exercise to show that, in general, $\text{comp}_{\mathbf{b}}\,\mathbf{a} \neq \text{comp}_{\mathbf{a}}\,\mathbf{b}$ and $\mathbf{proj_b\,a} \neq \mathbf{proj_a\,b}$. One reason for needing to consider components of a vector in a given direction is to compute work, as we see in example 1.6.

FIGURE 7.6
Pulling a wagon

EXAMPLE 1.6 Calculating Work

You exert a constant force of 180 newtons in the direction of the handle of the wagon pictured in Figure 7.6. If the handle makes an angle of $\frac{\pi}{4}$ with the horizontal and you pull the wagon along a flat surface for 1 km (1000 m), find the work done.

Solution First, recall from our discussion in Chapter 5 that if we apply a constant force F for a distance d, the work done is given by $W = Fd$. In this case, the force exerted in the direction of motion is not given. However, since the magnitude of the force is 180, the force vector must be

$$\mathbf{F} = 180\left\langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right\rangle = 180\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle = \langle 90\sqrt{2}, 90\sqrt{2}\rangle.$$

The force exerted in the direction of motion is simply the component of the force along the vector **i** (that is, the horizontal component of **F**) or $90\sqrt{2}$. The work done is then

$$W = Fd = 90\sqrt{2}(1000) \approx 127279 \text{ J}$$

More generally, if a constant force **F** moves an object from point P to point Q, we refer to the vector $\mathbf{d} = \overrightarrow{PQ}$ as the **displacement vector.** The work done is the product of the component of **F** along **d** and the distance:

$$\begin{aligned} W &= \text{comp}_{\mathbf{d}}\mathbf{F}\|\mathbf{d}\| \\ &= \frac{\mathbf{F}\cdot\mathbf{d}}{\|\mathbf{d}\|}\cdot\|\mathbf{d}\| = \mathbf{F}\cdot\mathbf{d}.\end{aligned}$$

Here, this gives us

$$W = \langle 90\sqrt{2}, 90\sqrt{2}\rangle\cdot\langle 1000, 0\rangle = 90\sqrt{2}(1000), \text{ as before.}$$

BEYOND FORMULAS

The dot product gives us a shortcut for computing components and projections. The dot product test for perpendicular vectors follows directly from this interpretation. In general, components and projections are used to isolate a particular portion of a larger problem for detailed analysis. This sort of reductionism is central to much of modern science.

EXERCISES 7.1

WRITING EXERCISES

1. Explain in words why the Triangle Inequality is true.
2. The dot product is called a "product" because the properties listed in Theorem 1.1 are true for multiplication of real numbers. Two other properties of multiplication of real numbers involve factoring: (1) if $ab = ac$ ($a \neq 0$) then $b = c$ and (2) if $ab = 0$ then $a = 0$ or $b = 0$. Discuss the extent to which these properties are true for the dot product.
3. To understand the importance of unit vectors, identify the simplification in formulas for finding the angle between vectors and for finding the component of a vector, if the vectors are unit vectors. There is also a theoretical benefit to using unit vectors. Compare the number of vectors in a particular direction to the number of unit vectors in that direction. (For this reason, unit vectors are sometimes called **direction vectors**.)
4. It is important to understand why work is computed using only the component of force in the direction of motion. Suppose you push on a door to close it. If you are pushing on the edge of the door straight at the door hinges, are you accomplishing anything useful? In this case, the work done would be zero. If you change the angle at which you push very slightly, what happens? As the angle increases, discuss how the component of force in the direction of motion changes and how the work done changes.

In exercises 1–6, compute $\mathbf{a} \cdot \mathbf{b}$.

1. $\mathbf{a} = \langle 3, 1 \rangle$, $\mathbf{b} = \langle 2, 4 \rangle$
2. $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j}$
3. $\mathbf{a} = \langle 2, -1, 3 \rangle$, $\mathbf{b} = \langle 0, 2, -4 \rangle$
4. $\mathbf{a} = \langle 3, 2, 0 \rangle$, $\mathbf{b} = \langle -2, 4, 3 \rangle$
5. $\mathbf{a} = 2\mathbf{i} - \mathbf{k}$, $\mathbf{b} = 4\mathbf{j} - \mathbf{k}$
6. $\mathbf{a} = 3\mathbf{i} + 3\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + \mathbf{j}$

In exercises 7–10, compute the angle between the vectors.

7. $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$
8. $\mathbf{a} = \langle 2, 0, -2 \rangle$, $\mathbf{b} = \langle 0, -2, 4 \rangle$
9. $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
10. $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{k}$

In exercises 11–14, determine whether the vectors are orthogonal.

11. $\mathbf{a} = \langle 2, -1 \rangle$, $\mathbf{b} = \langle 2, 4 \rangle$
12. $\mathbf{a} = 6\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = -\mathbf{i} + 3\mathbf{j}$
13. $\mathbf{a} = 3\mathbf{i}$, $\mathbf{b} = 6\mathbf{j} - 2\mathbf{k}$
14. $\mathbf{a} = \langle 4, -1, 1 \rangle$, $\mathbf{b} = \langle 2, 4, 4 \rangle$

In exercises 15–18, (a) find a 3-dimensional vector perpendicular to the given vector and (b) find a vector of the form $\langle a, 2, -3 \rangle$ that is perpendicular to the given vector.

15. $\langle 2, -1, 0 \rangle$
16. $\langle 4, -1, 1 \rangle$
17. $6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
18. $2\mathbf{i} - 3\mathbf{k}$

In exercises 19–24, find $\text{comp}_{\mathbf{b}}\,\mathbf{a}$ and $\text{proj}_{\mathbf{b}}\,\mathbf{a}$.

19. $\mathbf{a} = \langle 2, 1 \rangle$, $\mathbf{b} = \langle 3, 4 \rangle$
20. $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{b} = 4\mathbf{i} - 3\mathbf{j}$
21. $\mathbf{a} = \langle 2, -1, 3 \rangle$, $\mathbf{b} = \langle 1, 2, 2 \rangle$
22. $\mathbf{a} = \langle 1, 4, 5 \rangle$, $\mathbf{b} = \langle -2, 1, 2 \rangle$
23. $\mathbf{a} = \langle 2, 0, -2 \rangle$, $\mathbf{b} = \langle 0, -3, 4 \rangle$
24. $\mathbf{a} = \langle 3, 2, 0 \rangle$, $\mathbf{b} = \langle -2, 2, 1 \rangle$

25. Repeat example 1.6 with an angle of $\frac{\pi}{3}$ with the horizontal.
26. Repeat example 1.6 with an angle of $\frac{\pi}{6}$ with the horizontal.
27. Explain why the answers to exercises 25 and 26 aren't the same, even though the force exerted is the same. In this setting, explain why a larger amount of work corresponds to a more efficient use of the force.
28. Find the force needed in exercise 25 to produce the same amount of work as in example 1.6.

29. A constant force of $\langle 30, 20\rangle$ N moves an object in a straight line from the point $(0, 0)$ to the point $(24, 10)$. Compute the work done.

30. A constant force of $\langle 60, -30\rangle$ N moves an object in a straight line from the point $(0, 0)$ to the point $(10, -10)$. Compute the work done.

31. Label each statement as true or false. If it is true, briefly explain why; if it is false, give a counterexample.

(a) If $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.

(b) If $\mathbf{b} = \mathbf{c}$, then $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c}$.

(c) $\mathbf{a}\cdot\mathbf{a} = \|\mathbf{a}\|^2$.

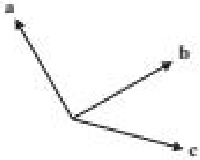
(d) If $\|\mathbf{a}\| > \|\mathbf{b}\|$ then $\mathbf{a}\cdot\mathbf{c} > \mathbf{b}\cdot\mathbf{c}$.

(e) If $\|\mathbf{a}\| = \|\mathbf{b}\|$ then $\mathbf{a} = \mathbf{b}$.

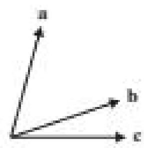
32. To compute $\mathbf{a}\cdot\mathbf{b}$, where $\mathbf{a} = \langle 2, 5\rangle$ and $\mathbf{b} = \frac{\langle 4, 1\rangle}{\sqrt{17}}$, you can first compute $\langle 2, 5\rangle \cdot \langle 4, 1\rangle$ and then divide the result (13) by $\sqrt{17}$. Which property stated in Theorem 1.1 is being used?

In exercises 33 and 34, use the figures to sequence a · b, a · c and b · c in increasing order.

33.



34.



35. If $\mathbf{a} = \langle 2, 1\rangle$, find a vector $\mathbf{b}$ such that (a) $\text{comp}_{\mathbf{b}}\,\mathbf{a} = 1$; (b) $\text{comp}_{\mathbf{a}}\,\mathbf{b} = -1$.

36. If $\mathbf{a} = \langle 4, -2\rangle$, find a vector $\mathbf{b}$ such that (a) $\text{proj}_{\mathbf{b}}\,\mathbf{a} = \langle 4, 0\rangle$; (b) $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \langle 4, -2\rangle$.

37. Find the angles in the triangle with vertices $(1, 2, 0)$, $(3, 0, -1)$ and $(1, 1, 1)$.

38. Find the angles in the quadrilateral $ABCD$ with vertices $A = (2, 0, 1)$, $B = (2, 1, 4)$, $C = (4, -2, 5)$ and $D = (4, 0, 2)$.

39. The distance from a point P to a line L is the length of the line segment connecting P to L at a right angle. Show that the distance from (x_1, y_1) to the line $ax + by + c = 0$ equals $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.

40. Prove that the distance between lines $ax + by + c = 0$ and $ax + by + d = 0$ equals $\frac{|d - c|}{\sqrt{a^2 + b^2}}$.

41. (a) Find the angle between the diagonal of a square and an adjacent side. (b) Find the angle between the diagonal of a cube and an adjacent side. (c) Extend the results of parts (a) and (b) to a hypercube of dimension $n \geq 4$.

42. Prove that $\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2$. State this result in terms of properties of the parallelogram formed by vectors $\mathbf{a}$ and $\mathbf{b}$.

Exercises 43–54 involve the Cauchy–Schwartz and Triangle Inequalities.

43. By the Cauchy–Schwartz Inequality, $|\mathbf{a}\cdot\mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|$. What relationship must exist between $\mathbf{a}$ and $\mathbf{b}$ to have $|\mathbf{a}\cdot\mathbf{b}| = \|\mathbf{a}\|\|\mathbf{b}\|$?

44. By the Triangle Inequality, $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$. What relationship must exist between $\mathbf{a}$ and $\mathbf{b}$ to have $\|\mathbf{a} + \mathbf{b}\| = \|\mathbf{a}\| + \|\mathbf{b}\|$?

45. Use the Triangle Inequality to prove that $\|\mathbf{a} - \mathbf{b}\| \geq \|\mathbf{a}\| - \|\mathbf{b}\|$.

46. Prove parts (ii) and (iii) of Theorem 1.1.

47. For vectors $\mathbf{a}$ and $\mathbf{b}$, use the Cauchy–Schwartz Inequality to find the maximum value of $\mathbf{a}\cdot\mathbf{b}$ if $\|\mathbf{a}\| = 3$ and $\|\mathbf{b}\| = 5$.

48. Find a formula for $\mathbf{a}$ in terms of $\mathbf{b}$ if $\|\mathbf{a}\| = 3$, $\|\mathbf{b}\| = 5$ and $\mathbf{a}\cdot\mathbf{b}$ is maximum.

49. Use the Cauchy–Schwartz Inequality in n dimensions to show that $\left(\sum_{k=1}^{n}|a_k b_k|\right)^2 \leq \left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}b_k^2\right)$. If both $\sum_{k=1}^{\infty}a_k^2$ and $\sum_{k=1}^{\infty}b_k^2$ converge, what can be concluded? Apply the result to $a_k = \frac{1}{k}$ and $b_k = \frac{1}{k^2}$.

50. Show that $\sum_{k=1}^{n}|a_k b_k| \leq \frac{1}{2}\sum_{k=1}^{n}a_k^2 + \frac{1}{2}\sum_{k=1}^{n}b_k^2$. If both $\sum_{k=1}^{\infty}a_k^2$ and $\sum_{k=1}^{\infty}b_k^2$ converge, what can be concluded? Apply the result to $a_k = \frac{1}{k}$ and $b_k = \frac{1}{k^2}$. Is this bound better or worse than the bound found in exercise 49?

51. (a) Use the Cauchy–Schwartz Inequality in n dimensions to show that $\sum_{k=1}^{n}|a_k| \leq \sqrt{n}\left(\sum_{k=1}^{n}a_k^2\right)^{1/2}$. (b) If $p_1, p_2, \ldots, p_n$ are non-negative numbers that sum to 1, show that $\sum_{k=1}^{n}p_k^2 \geq \frac{1}{n}$. (c) Among all sets of non-negative numbers $p_1, p_2, \ldots, p_n$ that sum to 1, find the choice of $p_1, p_2, \ldots, p_n$ that minimizes $\sum_{k=1}^{n}p_k^2$.

52. Use the Cauchy–Schwartz Inequality in n dimensions to show that $\sum_{k=1}^{n}|a_k| \leq \left(\sum_{k=1}^{n}|a_k|^{2/3}\right)^{1/2}\left(\sum_{k=1}^{n}|a_k|^{4/3}\right)^{1/2}$.

53. Show that $\sum_{k=1}^{n}a_k^2 b_k^2 \leq \left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}b_k^2\right)$ and then $\left(\sum_{k=1}^{n}a_k b_k c_k\right)^2 \leq \left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}b_k^2\right)\left(\sum_{k=1}^{n}c_k^2\right)$.

54. Show that $\sqrt{\frac{x+y}{x+y+z}} + \sqrt{\frac{y+z}{x+y+z}} + \sqrt{\frac{x+z}{x+y+z}} \leq \sqrt{6}$.

55. Prove that $\text{comp}_{\mathbf{c}}(\mathbf{a} + \mathbf{b}) = \text{comp}_{\mathbf{c}}\,\mathbf{a} + \text{comp}_{\mathbf{c}}\,\mathbf{b}$ for any non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

56. The **orthogonal projection** of vector $\mathbf{a}$ along vector $\mathbf{b}$ is defined as $\mathbf{orth}_{\mathbf{b}}\,\mathbf{a} = \mathbf{a} - \mathbf{proj}_{\mathbf{b}}\,\mathbf{a}$. Sketch a picture showing vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{proj}_{\mathbf{b}}\,\mathbf{a}$ and $\mathbf{orth}_{\mathbf{b}}\,\mathbf{a}$, and explain what is orthogonal about $\mathbf{orth}_{\mathbf{b}}\,\mathbf{a}$.

57. Write the given vector as $\mathbf{a} + \mathbf{b}$, where $\mathbf{a}$ is parallel to $\langle 1, 2, 3\rangle$ and $\mathbf{b}$ is perpendicular to $\langle 1, 2, 3\rangle$, for (a) $\langle 3, -1, 2\rangle$ and (b) $\langle 0, 4, 2\rangle$.

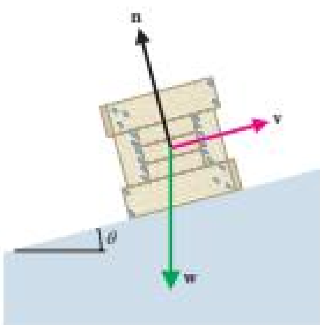
58. (a) For the Mandelbrot set and associated Julia sets, functions of the form $f(x) = x^2 - c$ are analyzed for various constants c. The iterates of the function increase if $|x^2 - c| > |x|$. Show that this is true if $|x| > \frac{1}{2} + \sqrt{\frac{1}{4} + c}$. (b) Show that the vector analog of part (a) is also true. For vectors $\mathbf{x}$, $\mathbf{x}_2$ and $\mathbf{c}$, if $\|\mathbf{x}\| > \frac{1}{2} + \sqrt{\frac{1}{4} + \|c\|}$ and $\|\mathbf{x}_2\| = \|\mathbf{x}\|^2$, then $\|\mathbf{x}_2 - \mathbf{c}\| > \|\mathbf{x}\|$.

APPLICATIONS

1. In a methane molecule (CH_4), a carbon atom is surrounded by four hydrogen atoms. Assume that the hydrogen atoms are at (0, 0, 0), (1, 1, 0), (1, 0, 1) and (0, 1, 1) and the carbon atom is at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Compute the **bond angle**, the angle from hydrogen atom to carbon atom to hydrogen atom.

2. Suppose that a beam of an oil rig is installed in a direction parallel to $\langle 10, 1, 5\rangle$. (a) If a wave exerts a force of $\langle 0, -200, 0\rangle$ newtons, find the component of this force along the beam. (b) Repeat with a force of $\langle 13, -190, -61\rangle$ newtons. The forces in parts (a) and (b) have nearly identical magnitudes. Explain why the force components are different.

3. In the diagram, a crate of weight w N is placed on a ramp inclined at angle θ above the horizontal. The vector **v** along the ramp is given by $\mathbf{v} = \langle \cos\theta, \sin\theta\rangle$ and the **normal** vector by $\mathbf{n} = \langle -\sin\theta, \cos\theta\rangle$. (a) Show that **v** and **n** are perpendicular. Find the component of $\mathbf{w} = \langle 0, -w\rangle$ along **v** and the component of **w** along **n**.



(b) If the coefficient of static friction between the crate and ramp equals μ_s, the crate will slide down the ramp if the component of **w** along **v** is greater than the product of μ_s and the component of **w** along **n**. Show that this occurs if the angle θ is steep enough that $\theta > \tan^{-1}\mu_s$.

4. A weight of 500 N is supported by two ropes that exert forces of $\mathbf{a} = \langle -100, 200\rangle$ N and $\mathbf{b} = \langle 100, 300\rangle$ N. Find the angles α, β and θ between the ropes.

5. A car makes a turn on a banked road. If the road is banked at 10°, show that a vector parallel to the road is $\langle \cos 10°, \sin 10°\rangle$.

   (a) If the car has weight 8000 N, find the component of the weight vector along the road vector. This component of weight provides a force that helps the car turn. Compute the ratio of the component of weight along the road to the component of weight into the road. Discuss why it might be dangerous if this ratio is very small or very large.

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   (b) Repeat part (a) for a 10,000 N car on a 15° bank.

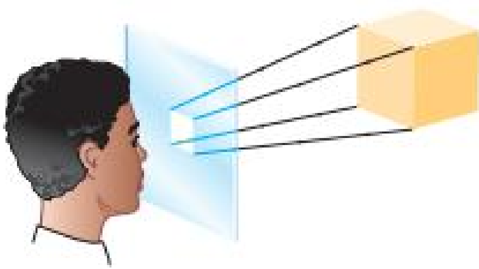
6. The racetrack at Bristol, Tennessee, is famous for its short length and its steeply banked curves. The track is an oval of length 0.858 km and the corners are banked at 36°. Circular motion at a constant speed v requires a centripetal force of $F = \frac{mv^2}{r}$, where r is the radius of the circle and m is the mass of the car. For a track banked at angle A, the weight of the car provides a centripetal force of $mg \sin A$, where g is the gravitational constant. Setting the two equal gives $\frac{v^2}{r} = g \sin A$. Assuming that the Bristol track is circular (it's not really) and using $g = 9.8$ m/s², find the speed supported by the Bristol bank. Cars actually complete laps at over 190 km/h. Discuss where the additional force for this higher speed might come from.

7. Suppose a small business sells three products. In a given month, if 3000 units of product A are sold, 2000 units of product B are sold and 4000 units of product C are sold, then the **sales vector** for that month is defined by $\mathbf{s} = \langle 3000, 2000, 4000\rangle$. If the prices of products A, B and C are \$20, \$15 and \$25, respectively, then the **price vector** is defined by $\mathbf{p} = \langle 20, 15, 25\rangle$. Compute $\mathbf{s} \cdot \mathbf{p}$ and discuss how it relates to monthly revenue.

8. Suppose that in a particular county, ice cream sales (in thousands of liters) for a year are given by the vector $\mathbf{s} = \langle 3, 5, 12, 40, 60, 100, 120, 160, 110, 50, 10, 2\rangle$. That is, 3000 liters were sold in January, 5000 liters were sold in February, and so on. In the same county, suppose that festivals for the year are given by the vector $\mathbf{m} = \langle 2, 0, 1, 6, 4, 8, 10, 13, 8, 2, 0, 6\rangle$. Show that the average monthly ice cream sales are $\bar{s} = 56{,}000$ liters and that the average monthly number of festivals is $\bar{m} = 5$. Compute the vectors **a** and **b**, where the components of **a** equal the components of **s** with the mean 56 subtracted (so that $\mathbf{a} = \langle -53, -51, -44, \ldots\rangle$) and the components of **b** equal the components of **m** with the mean 5 subtracted. The correlation between ice cream sales and festivals is defined as $\rho = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$. Often, a positive correlation is incorrectly interpreted as meaning that **a** "causes" **b**. (In fact, correlation should *never* be used to infer a cause-and-effect relationship.) Explain why such a conclusion would be invalid in this case.

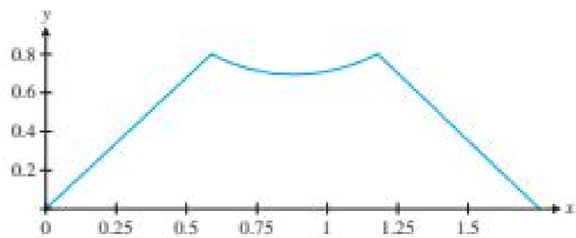
EXPLORATORY EXERCISES

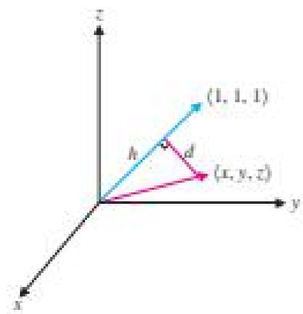
1. This exercise develops a basic principle used in computer graphics. In the drawing, an artist traces the image of an object onto a pane of glass. Explain why the trace will be distorted unless the artist keeps the pane of glass perpendicular to the line of sight. The trace is thus a projection of the object onto the pane of glass. To make this precise, suppose that the artist is at the point (100, 0, 0) and the point $P_1 = (2, 1, 3)$ is part of the object being traced. Find the projection $\mathbf{p}_1$ of the position vector $\langle 2, 1, 3\rangle$ along the artist's position vector $\langle 100, 0, 0\rangle$. Then find the vector $\mathbf{q}_1$ such that $\langle 2, 1, 3\rangle = \mathbf{p}_1 + \mathbf{q}_1$. Which of the vectors $\mathbf{p}_1$ and $\mathbf{q}_1$ does the artist actually see and which one is hidden? Repeat this with the point $P_2 = (-2, 1, 3)$ and find vectors $\mathbf{p}_2$ and $\mathbf{q}_2$ such that $\langle -2, 1, 3\rangle = \mathbf{p}_2 + \mathbf{q}_2$. The artist would plot both points P_1 and P_2 at the same point on the pane of glass. Identify which of the vectors $\mathbf{p}_1$, $\mathbf{q}_1$, $\mathbf{p}_2$ and $\mathbf{q}_2$ correspond to this point. From the artist's perspective, one of the points P_1 or P_2 is hidden behind the other. Identify which point is hidden and explain how the information in the vectors $\mathbf{p}_1$, $\mathbf{q}_1$, $\mathbf{p}_2$ and $\mathbf{q}_2$ can be used to determine which point is hidden.



2. Take a cube and spin it around a diagonal.

If you spin it rapidly, you will see a curved outline appear in the middle. (See the figure below.) How does a cube become curved? This exercise answers that question. Suppose that the cube is a unit cube with $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le 1$, and we rotate about the diagonal from (0, 0, 0) to (1, 1, 1). Spinning the cube, we see the combination of points on the cube at their maximum distance from the diagonal. The points on the edge of the cube have the maximum distance. If (x, y, z) is a point on an edge of the cube, define h to be the component of the vector $\langle x, y, z\rangle$ along the diagonal $\langle 1, 1, 1\rangle$. The distance d from (x, y, z) to the diagonal is then $d = \sqrt{\|\langle x, y, z\rangle\|^2 - h^2}$, as in the diagram below. The curve is produced by the edge from (0, 0, 1) to (0, 1, 1). Parametric equations for this segment are $x = 0$, $y = t$ and $z = 1$, for $0 \le t \le 1$. For the vector $\langle 0, t, 1\rangle$, compute h and then d. Graph $d(t)$. You should see a curve similar to the middle of the outline shown below. Show that this curve is actually part of a hyperbola. Then find the outline created by other sides of the cube. Which ones produce curves and which produce straight lines?





7.2 THE CROSS PRODUCT

In this section, we define a second type of product of vectors, the *cross product* or *vector product*, which has many important applications, from physics and engineering mechanics to space travel. We first need a few definitions.

DEFINITION 2.1

The **determinant** of a 2 × 2 matrix of real numbers is defined by

$$\underbrace{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}_{2\times 2 \text{ matrix}} = a_1b_2 - a_2b_1. \tag{2.1}$$

EXAMPLE 2.1 Computing a 2 × 2 Determinant

Evaluate the determinant $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$.

Solution From (2.1), we have

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2. \quad \blacksquare$$

DEFINITION 2.2

The **determinant** of a 3 × 3 matrix of real numbers is defined as a combination of three 2 × 2 determinants, as follows:

$$\underbrace{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}_{3\times 3 \text{ matrix}} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{2.2}$$

Equation (2.2) is referred to as an **expansion** of the determinant **along the first row**. Notice that the multipliers of each of the 2 × 2 determinants are the entries of the first row of the 3 × 3 matrix. Each 2 × 2 determinant is the determinant you get if you eliminate the row and column in which the corresponding multiplier lies. That is, for the *first* term, the multiplier is a_1 and the 2 × 2 determinant is found by eliminating the *first* row and *first* column from the 3 × 3 matrix:

$$\begin{vmatrix} \not{a_1} & \not{a_2} & \not{a_3} \\ \not{b_1} & b_2 & b_3 \\ \not{c_1} & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}.$$

Likewise, the *second* 2 × 2 determinant is found by eliminating the first row and the *second* column from the 3 × 3 determinant:

$$\begin{vmatrix} \not{a_1} & \not{a_2} & \not{a_3} \\ b_1 & \not{b_2} & b_3 \\ c_1 & \not{c_2} & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}.$$

Be certain to notice the minus sign in front of this term. Finally, the *third* determinant is found by eliminating the first row and the *third* column from the 3 × 3 determinant:

$$\begin{vmatrix} \not{a_1} & \not{a_2} & \not{a_3} \\ b_1 & b_2 & \not{b_3} \\ c_1 & c_2 & \not{c_3} \end{vmatrix} = \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

EXAMPLE 2.2 Evaluating a 3 × 3 Determinant

Evaluate the determinant $\begin{vmatrix} 1 & 2 & 4 \\ -3 & 3 & 1 \\ 3 & -2 & 5 \end{vmatrix}$.

Solution Expanding along the first row, we have:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 4 \\ -3 & 3 & 1 \\ 3 & -2 & 5 \end{vmatrix} &= (1)\begin{vmatrix} 3 & 1 \\ -2 & 5 \end{vmatrix} - (2)\begin{vmatrix} -3 & 1 \\ 3 & 5 \end{vmatrix} + (4)\begin{vmatrix} -3 & 3 \\ 3 & -2 \end{vmatrix} \\ &= (1)[(3)(5)-(1)(-2)] - (2)[(-3)(5)-(1)(3)] \\ &\quad + (4)[(-3)(-2)-(3)(3)] \\ &= 41. \end{aligned}$$

We use determinant notation as a convenient device for defining the cross product, as follows.

DEFINITION 2.3

For two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ in V_3, we define the **cross product** (or **vector product**) of **a** and **b** to be

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}. \tag{2.3}$$

Notice that $\mathbf{a} \times \mathbf{b}$ is also a vector in V_3. To compute $\mathbf{a} \times \mathbf{b}$, you must write the components of **a** in the second row and the components of **b** in the third row (the order is important!) Also note that while we've used the determinant notation, the 3 × 3 determinant indicated in (2.3) is not really a determinant, in the sense in which we defined them, since the entries in the first row are vectors instead of scalars. Nonetheless, we find this slight abuse of notation convenient for computing cross products and we use it routinely.

EXAMPLE 2.3 Computing a Cross Product

Compute $\langle 1, 2, 3 \rangle \times \langle 4, 5, 6 \rangle$.

Solution From (2.3), we have

$$\begin{aligned} \langle 1, 2, 3 \rangle \times \langle 4, 5, 6 \rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}\mathbf{k} \\ &= -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k} = \langle -3, 6, -3 \rangle. \end{aligned}$$

REMARK 2.1

The cross product is defined only for vectors in V_3. There is no corresponding operation for vectors in V_2.

THEOREM 2.1

For any vector $\mathbf{a} \in V_3$, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ and $\mathbf{a} \times \mathbf{0} = \mathbf{0}$.

PROOF

We prove the first of these two results. The second, we leave as an exercise. For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, we have from (2.3) that

$$\begin{aligned} \mathbf{a} \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ a_2 & a_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ a_1 & a_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} \mathbf{k} \\ &= (a_2a_3 - a_3a_2)\mathbf{i} - (a_1a_3 - a_3a_1)\mathbf{j} + (a_1a_2 - a_2a_1)\mathbf{k} = 0. \blacksquare \end{aligned}$$

Let's take a brief look back at the result of example 2.3. There, we saw that

$$\langle 1, 2, 3 \rangle \times \langle 4, 5, 6 \rangle = \langle -3, 6, -3 \rangle.$$

There is something rather interesting to observe here. Note that

$$\langle 1, 2, 3 \rangle \cdot \langle -3, 6, -3 \rangle = 0$$

and

$$\langle 4, 5, 6 \rangle \cdot \langle -3, 6, -3 \rangle = 0.$$

That is, both $\langle 1, 2, 3 \rangle$ and $\langle 4, 5, 6 \rangle$ are orthogonal to their cross product. As it turns out, this is true in general, as we see in Theorem 2.2.

THEOREM 2.2

For any vectors **a** and **b** in V_3, $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**.

PROOF

Recall that two vectors are orthogonal if and only if their dot product is zero. Now, using (2.3), we have

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) &= \langle a_1, a_2, a_3 \rangle \cdot \left[\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \right] \\ &= a_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - a_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + a_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= a_1[a_2b_3 - a_3b_2] - a_2[a_1b_3 - a_3b_1] + a_3[a_1b_2 - a_2b_1] \\ &= a_1a_2b_3 - a_1a_3b_2 - a_1a_2b_3 + a_2a_3b_1 + a_1a_3b_2 - a_2a_3b_1 \\ &= 0, \end{aligned}$$

so that **a** and $(\mathbf{a} \times \mathbf{b})$ are orthogonal. We leave it as an exercise to show that $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$, also. ■

Notice that for non-zero and nonparallel vectors **a** and **b**, since $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**, it is also orthogonal to every vector lying in the plane containing **a** and **b**. (We also say that $\mathbf{a} \times \mathbf{b}$ is orthogonal to the plane, in this case.) But, given a plane, out of which side of the plane does $\mathbf{a} \times \mathbf{b}$ point? We can get an idea by computing some simple cross products.

Notice that

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} = \mathbf{k}.$$

Likewise,

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}.$$

These are illustrations of the **right-hand rule**: If you align the fingers of your *right* hand along the vector **a** and bend your fingers around in the direction of rotation from **a** toward **b** (through an angle of less than 180°), your thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$, as in

HISTORICAL NOTES

Josiah Willard Gibbs (1839–1903)
American physicist and mathematician who introduced and named the dot product and the cross product. A graduate of Yale, Gibbs published important papers in thermodynamics, statistical mechanics and the electromagnetic theory of light. Gibbs used vectors to determine the orbit of a comet from only three observations. Originally produced as printed notes for his students, Gibbs' vector system greatly simplified the original system developed by Hamilton. Gibbs was well liked but not famous in his lifetime. One biographer wrote of Gibbs that, "The greatness of his intellectual achievements will never overshadow the beauty and dignity of his life."*

*Bumstead, H. A. (1903), Josiah Willard Gibbs, American Journal of Science (4) (XVI)

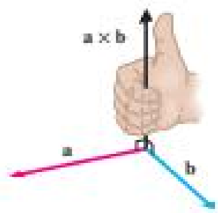


FIGURE 7.7a
$\mathbf{a} \times \mathbf{b}$

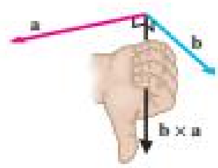


FIGURE 7.7b
$\mathbf{b} \times \mathbf{a}$

Figure 7.7a. Now, following the right-hand rule, $\mathbf{b} \times \mathbf{a}$ will point in the direction opposite $\mathbf{a} \times \mathbf{b}$. (See Figure 7.7b.) In particular, notice that

$$\mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k}.$$

We leave it as an exercise to show that

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i},$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \quad \text{and} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}.$$

Take the time to think through the right-hand rule for each of these cross products.

There are several other unusual things to observe here. Notice that

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \neq -\mathbf{k} = \mathbf{j} \times \mathbf{i},$$

which says that the cross product is *not* commutative. Further, notice that

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i},$$

while

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0},$$

so that the cross product is also *not* associative. That is, in general,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c}).$$

Since the cross product does not follow several of the rules you might expect a product to satisfy, you might ask what rules the cross product *does* satisfy. We summarize these in Theorem 2.3.

THEOREM 2.3

For any vectors **a, b** and **c** in V_3 and any scalar d, the following hold:

(i) $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$	(anticommutativity)
(ii) $(d\mathbf{a}) \times \mathbf{b} = d(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (d\mathbf{b})$	
(iii) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$	(distributive law)
(iv) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$	(distributive law)
(v) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$	(scalar triple product) and
(vi) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$	(vector triple product).

PROOF

We prove parts (i) and (iii) only. The remaining parts are left as exercises.

(i) For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, we have from (2.3) that

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} \\ &= -\begin{vmatrix} b_2 & b_3 \\ a_2 & a_3 \end{vmatrix}\mathbf{i} + \begin{vmatrix} b_1 & b_3 \\ a_1 & a_3 \end{vmatrix}\mathbf{j} - \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}\mathbf{k} = -(\mathbf{b} \times \mathbf{a}), \end{aligned}$$

since swapping two rows in a 2×2 matrix (or in a 3×3 matrix, for that matter) changes the sign of its determinant.

(iii) For $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, we have

$$\mathbf{b} + \mathbf{c} = \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$$

and so,
$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}.$$

Looking only at the **i** component of this, we have

$$\begin{aligned}\begin{vmatrix} a_2 & a_3 \\ b_2+c_2 & b_3+c_3 \end{vmatrix} &= a_2(b_3+c_3) - a_3(b_2+c_2) \\ &= (a_2b_3 - a_3b_2) + (a_2c_3 - a_3c_2) \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix},\end{aligned}$$

which you should note is also the **i** component of $\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$. Similarly, you can show that the **j** and **k** components also match, which establishes the result. ■

Always keep in mind that vectors are specified by two things: magnitude and direction. We have already shown that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**. In Theorem 2.4, we make a general (and quite useful) statement about $\|\mathbf{a} \times \mathbf{b}\|$.

THEOREM 2.4

For non-zero vectors **a** and **b** in V_3, if θ is the angle between **a** and **b** ($0 \le \theta \le \pi$), then

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta. \tag{2.4}$$

PROOF

From (2.3), we get

$$\begin{aligned}\|\mathbf{a} \times \mathbf{b}\|^2 &= [a_2b_3 - a_3b_2]^2 + [a_1b_3 - a_3b_1]^2 + [a_1b_2 - a_2b_1]^2 \\ &= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_1^2b_3^2 - 2a_1a_3b_1b_3 + a_3^2b_1^2 \\ &\quad + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - \|\mathbf{a}\|^2\|\mathbf{b}\|^2\cos^2\theta \qquad \text{From Theorem 1.2} \\ &= \|\mathbf{a}\|^2\|\mathbf{b}\|^2(1 - \cos^2\theta) \\ &= \|\mathbf{a}\|^2\|\mathbf{b}\|^2\sin^2\theta.\end{aligned}$$

Taking square roots, we get

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta,$$

since $\sin\theta \ge 0$, for $0 \le \theta \le \pi$. ■

The following characterization of parallel vectors is an immediate consequence of Theorem 2.4.

COROLLARY 2.1

Two non-zero vectors $\mathbf{a}, \mathbf{b} \in V_3$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

PROOF

Recall that **a** and **b** are parallel if and only if the angle θ between them is either 0 or π. In either case, $\sin\theta = 0$ and so, by Theorem 2.4,

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta = \|\mathbf{a}\|\|\mathbf{b}\|(0) = 0.$$

The result then follows from the fact that the only vector with zero magnitude is the zero vector. ■

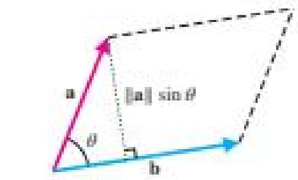


FIGURE 7.8
Parallelogram

Theorem 2.4 also provides us with the following interesting geometric interpretation of the cross product. For any two non-zero vectors **a** and **b**, as long as **a** and **b** are not parallel, they form two adjacent sides of a parallelogram, as seen in Figure 7.8. Notice that the area of the parallelogram is given by the product of the base and the altitude. We have

$$\begin{aligned}\text{Area} &= (\text{base})(\text{altitude}) \\ &= \|\mathbf{b}\|\,\|\mathbf{a}\|\sin\theta = \|\mathbf{a}\times\mathbf{b}\|,\end{aligned} \tag{2.5}$$

from Theorem 2.4. That is, the magnitude of the cross product of two vectors gives the area of the parallelogram with two adjacent sides formed by the vectors.

EXAMPLE 2.4 Finding the Area of a Parallelogram Using the Cross Product

Find the area of the parallelogram with two adjacent sides formed by the vectors $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle 4, 5, 6\rangle$.

Solution First notice that

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 3\\ 4 & 5 & 6\end{vmatrix} = \mathbf{i}\begin{vmatrix}2 & 3\\ 5 & 6\end{vmatrix} - \mathbf{j}\begin{vmatrix}1 & 3\\ 4 & 6\end{vmatrix} + \mathbf{k}\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix} = \langle -3, 6, -3\rangle.$$

From (2.5), the area of the parallelogram is given by

$$\|\mathbf{a}\times\mathbf{b}\| = \|\langle -3, 6, -3\rangle\| = \sqrt{54} \approx 7.348. \blacksquare$$

We can also use Theorem 2.4 to find the distance from a point to a line in $\mathbb{R}^3$, as follows. Let d represent the distance from the point Q to the line through the points P and R. From elementary trigonometry, we have that

$$d = \|\overrightarrow{PQ}\|\sin\theta,$$

where θ is the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. (See Figure 7.9.) From (2.4), we have

$$\|\overrightarrow{PQ}\times\overrightarrow{PR}\| = \|\overrightarrow{PQ}\|\,\|\overrightarrow{PR}\|\sin\theta = \|\overrightarrow{PR}\|(d).$$

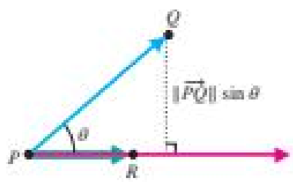


FIGURE 7.9
Distance from a point to a line

Solving this for d, we get

$$d = \frac{\|\overrightarrow{PQ}\times\overrightarrow{PR}\|}{\|\overrightarrow{PR}\|}. \tag{2.6}$$

EXAMPLE 2.5 Finding the Distance from a Point to a Line

Find the distance from the point $Q(1, 2, 1)$ to the line through the points $P(2, 1, -3)$ and $R(2, -1, 3)$.

Solution First, the position vectors corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\overrightarrow{PQ} = \langle -1, 1, 4\rangle \quad \text{and} \quad \overrightarrow{PR} = \langle 0, -2, 6\rangle,$$

and

$$\langle -1, 1, 4\rangle \times \langle 0, -2, 6\rangle = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 1 & 4\\ 0 & -2 & 6\end{vmatrix} = \langle 14, 6, 2\rangle.$$

We then have from (2.6) that

$$d = \frac{\|\overrightarrow{PQ}\times\overrightarrow{PR}\|}{\|\overrightarrow{PR}\|} = \frac{\|\langle 14, 6, 2\rangle\|}{\|\langle 0, -2, 6\rangle\|} = \frac{\sqrt{236}}{\sqrt{40}} \approx 2.429. \blacksquare$$

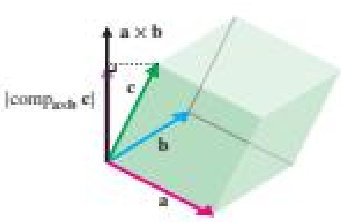


FIGURE 7.10
Parallelepiped formed by the vectors **a**, **b** and **c**

For any three non-zero and noncoplanar vectors **a**, **b** and **c** (i.e., three vectors that do not lie in a single plane), consider the parallelepiped formed using the vectors as three adjacent edges. (See Figure 7.10.) Recall that the volume of such a solid is given by

$$\text{Volume} = (\text{Area of base})(\text{altitude}).$$

Further, since two adjacent sides of the base are formed by the vectors **a** and **b**, we know that the area of the base is given by $\|\mathbf{a} \times \mathbf{b}\|$. Referring to Figure 7.10, notice that the altitude is given by

$$|\text{comp}_{\mathbf{a}\times\mathbf{b}}\mathbf{c}| = \frac{|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|}{\|\mathbf{a} \times \mathbf{b}\|},$$

from (1.7). The volume of the parallelepiped is then

$$\text{Volume} = \|\mathbf{a} \times \mathbf{b}\| \frac{|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|}{\|\mathbf{a} \times \mathbf{b}\|} = |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|.$$

The scalar $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ is called the **scalar triple product** of the vectors **a**, **b** and **c**. It turns out that we can evaluate the scalar triple product by computing a single determinant, as follows. Note that for $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, we have

$$\begin{aligned}
\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) &= \mathbf{c} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \langle c_1, c_2, c_3 \rangle \cdot \left(\mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \\
&= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\
&= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \qquad (2.7)
\end{aligned}$$

EXAMPLE 2.6 Finding the Volume of a Parallelepiped Using the Cross Product

Find the volume of the parallelepiped with three adjacent edges formed by the vectors $\mathbf{a} = \langle 1, 2, 3 \rangle$, $\mathbf{b} = \langle 4, 5, 6 \rangle$ and $\mathbf{c} = \langle 7, 8, 0 \rangle$.

Solution First, note that Volume $= |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|$. From (2.7), we have that

$$\begin{aligned}
\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) &= \begin{vmatrix} 7 & 8 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 7\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - 8\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \\
&= 7(-3) - 8(-6) = 27.
\end{aligned}$$

So, the volume of the parallelepiped is Volume $= |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| = |27| = 27$. ■

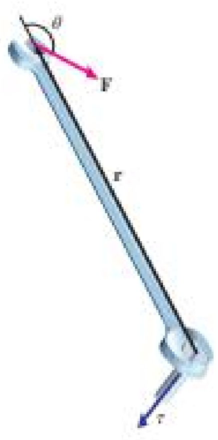


FIGURE 7.11
Torque, τ

Consider the action of a wrench on a bolt, as shown in Figure 7.11. In order to tighten the bolt, we apply a force **F** at the end of the handle, in the direction indicated in the figure. This force creates a **torque** τ acting along the axis of the bolt, drawing it in tight. Notice that the torque acts in the direction perpendicular to both **F** and the position vector **r** for the handle as indicated in Figure 7.11. In fact, using the right-hand rule, the torque acts in the same direction as $\mathbf{r} \times \mathbf{F}$ and physicists define the torque vector to be

$$\tau = \mathbf{r} \times \mathbf{F}.$$

In particular, this says that

$$\|\tau\| = \|\mathbf{r} \times \mathbf{F}\| = \|\mathbf{r}\|\|\mathbf{F}\| \sin\theta, \qquad (2.8)$$

from (2.4). There are several observations we can make from this. First, this says that the farther away from the axis of the bolt we apply the force (i.e., the larger $\|\mathbf{r}\|$ is), the greater the magnitude of the torque. So, a longer wrench produces a greater torque, for a given amount of force applied. Second, notice that $\sin\theta$ is maximized when $\theta = \frac{\pi}{2}$, so that from (2.8) the magnitude of the torque is maximized when $\theta = \frac{\pi}{2}$ (when the force vector **F** is orthogonal to the position vector **r**). If you've ever spent any time using a wrench, this should fit well with your experience.

EXAMPLE 2.7 Finding the Torque Applied by a Wrench

If you apply a force of magnitude 110 N at the end of a 40 cm-long wrench, at an angle of $\frac{\pi}{3}$ to the wrench, find the magnitude of the torque applied to the bolt. What is the maximum torque that a force of 110 N applied at that point can produce?

Solution From (2.8), we have

$$\|\tau\| = \|\mathbf{r}\|\|\mathbf{F}\|\sin\theta = \left(\frac{40}{100}\right)110\sin\frac{\pi}{3}$$
$$= \left(\frac{40}{100}\right)110\frac{\sqrt{3}}{2} \approx 38\text{ N}\cdot\text{m}$$

Further, the maximum torque is obtained when the angle between the wrench and the force vector is $\frac{\pi}{2}$. This would give us a maximum torque of

$$\|\tau\| = \|\mathbf{r}\|\|\mathbf{F}\|\sin\theta = \left(\frac{40}{100}\right)110(1) = 44\text{ N}\cdot\text{m}$$

FIGURE 7.12
Spinning ball
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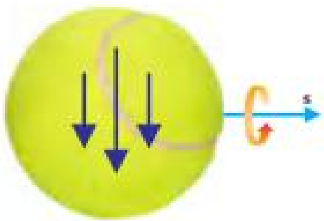

FIGURE 7.13a
Backspin
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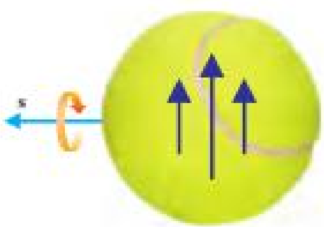

FIGURE 7.13b
Topspin
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In many sports, the action is at least partially influenced by the motion of a spinning ball. For instance, in cricket, batters must contend with pitchers' curveballs, and in golf, players try to control their slice. In tennis, players hit shots with topspin, while in basketball, players improve their shooting by using backspin. The list goes on and on. These are all examples of the **Magnus force**, which we describe below.

Suppose that a ball is spinning with angular velocity ω, measured in radians per second (i.e., ω is the rate of change of the rotational angle). The ball spins about an axis, as shown in Figure 7.12. We define the spin vector $\mathbf{s}$ to have magnitude ω and direction parallel to the spin axis. We use a right-hand rule to distinguish between the two directions parallel to the spin axis: curl the fingers of your right hand around the ball in the direction of the spin, and your thumb will point in the correct direction. Two examples are shown in Figures 7.13a and 7.13b. The motion of the ball disturbs the air through which it travels, creating a Magnus force $\mathbf{F}_m$ acting on the ball. For a ball moving with velocity $\mathbf{v}$ and spin vector $\mathbf{s}$, $\mathbf{F}_m$ is given by

$$\mathbf{F}_m = c(\mathbf{s} \times \mathbf{v}),$$

for some positive constant c. Suppose the balls in Figure 7.13a and Figure 7.13b are moving into the page and away from you. Using the usual sports terminology, the first ball has backspin and the second ball has topspin. Using the right-hand rule, we see that the Magnus force acting on the first ball acts in the upward direction, as indicated in Figure 7.14a. This says that backspin (for example, on a basketball or golf shot) produces an upward force that helps the ball land more softly than a ball with no spin. Similarly, the Magnus force acting on the second ball acts in the downward direction (see Figure 7.14b), so that topspin (for example, on a tennis shot or cricket hit) produces a downward force that causes the ball to drop to the ground more quickly than a ball with no spin.

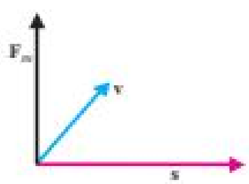

FIGURE 7.14a
Magnus force for a ball with backspin

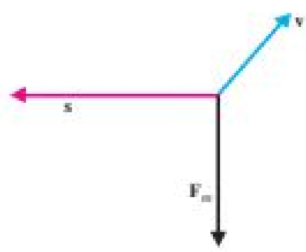

FIGURE 7.14b
Magnus force for a ball with topspin

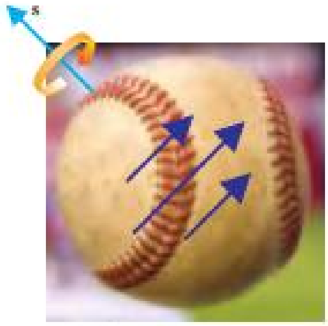

Figure 7.15a
Right-hand curveball
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Figure 7.15b
Right-hand golf shot
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EXAMPLE 2.8 Finding the Direction of a Magnus Force

The balls shown in Figures 7.15a and 7.15b are moving into the page and away from you with spin as indicated. The first ball represents a right-handed baseball pitcher's curveball, while the second ball represents a right-handed golfer's shot. Determine the direction of the Magnus force and discuss the effects on the ball.

Solution For the first ball, notice that the spin vector points up and to the left, so that $\mathbf{s} \times \mathbf{v}$ points down and to the left as shown in Figure 7.16a. Such a ball will curve to the left and drop faster than a ball that is not spinning, making it more difficult to hit. For the second ball, the spin vector points down and to the right, so $\mathbf{s} \times \mathbf{v}$ points up and to the right. Such a ball will move to the right (a "slice") and stay in the air longer than a ball that is not spinning. (See Figure 7.16b.)

Figure 7.16a
Magnus force for a right-handed curveball

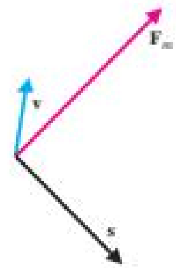


FIGURE 7.16b
Magnus force for a right-handed golf shot

EXERCISES 7.2

WRITING EXERCISES

1. In this chapter, we have developed several tests for geometric relationships. Briefly describe how to test whether two vectors are (a) parallel; (b) perpendicular. Briefly describe how to test whether (c) three points are colinear; (d) four points are coplanar.

2. The flip side of the problems in exercise 1 is to construct vectors with desired properties. Briefly describe how to construct a vector (a) parallel to a given vector; (b) perpendicular to a given vector. (c) Given a vector, describe how to construct two other vectors such that the three vectors are mutually perpendicular.

3. In example 2.7, how would the torque change if the force $\mathbf{F}$ were replaced with the force $-\mathbf{F}$? Answer both in mathematical terms and in physical terms.

4. Sketch a picture and explain in geometric terms why $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$.

In exercises 1–4, compute the given determinant.

1. $\begin{vmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -2 & -1 & 1 \end{vmatrix}$

2. $\begin{vmatrix} 0 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & 1 & 2 \end{vmatrix}$

3. $\begin{vmatrix} 2 & 3 & -1 \\ 0 & 1 & 0 \\ -2 & -1 & 3 \end{vmatrix}$

4. $\begin{vmatrix} -2 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 2 \end{vmatrix}$

In exercises 5–10, compute the cross product $\mathbf{a} \times \mathbf{b}$.

5. $\mathbf{a} = \langle 1, 2, -1 \rangle$, $\mathbf{b} = \langle 1, 0, 2 \rangle$

6. $\mathbf{a} = \langle 3, 0, -1 \rangle$, $\mathbf{b} = \langle 1, 2, 2 \rangle$

7. $\mathbf{a} = \langle 0, 1, 4 \rangle$, $\mathbf{b} = \langle -1, 2, -1 \rangle$

8. $\mathbf{a} = \langle 2, -2, 0 \rangle$, $\mathbf{b} = \langle 3, 0, 1 \rangle$

9. $\mathbf{a} = 2\mathbf{i} - \mathbf{k}$, $\mathbf{b} = 4\mathbf{j} + \mathbf{k}$

10. $\mathbf{a} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{j} - \mathbf{k}$

In exercises 11–16, find two unit vectors orthogonal to the two given vectors.

11. $\mathbf{a} = \langle 1, 0, 4\rangle$, $\mathbf{b} = \langle 1, -4, 2\rangle$

12. $\mathbf{a} = \langle 2, -2, 1\rangle$, $\mathbf{b} = \langle 0, 0, -2\rangle$

13. $\mathbf{a} = \langle 2, -1, 0\rangle$, $\mathbf{b} = \langle 1, 0, 3\rangle$

14. $\mathbf{a} = \langle 0, 2, 1\rangle$, $\mathbf{b} = \langle 1, 0, -1\rangle$

15. $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$, $\mathbf{b} = 4\mathbf{j} + \mathbf{k}$

16. $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{k}$

In exercises 17–20, find the distance from the point Q to the given line.

17. $Q = (1, 2, 0)$, line through $(0, 1, 2)$ and $(3, 1, 1)$

18. $Q = (2, 0, 1)$, line through $(1, -2, 2)$ and $(3, 0, 2)$

19. $Q = (3, -2, 1)$, line through $(2, 1, -1)$ and $(1, 1, 1)$

20. $Q = (1, 3, 1)$, line through $(1, 3, -2)$ and $(1, 0, -2)$

In exercises 21–26, find the indicated area or volume.

21. Area of the parallelogram with two adjacent sides formed by $\langle 2, 3\rangle$ and $\langle 1, 4\rangle$

22. Area of the parallelogram with two adjacent sides formed by $\langle -2, 1\rangle$ and $\langle 1, 3\rangle$

23. Area of the triangle with vertices $(0, 0, 0)$, $(2, 3, -1)$ and $(3, -1, 4)$

24. Area of the triangle with vertices $(1, 1, 0)$, $(0, -2, 1)$ and $(1, -3, 0)$

25. Volume of the parallelepiped with three adjacent edges formed by $\langle 2, 1, 0\rangle$, $\langle -1, 2, 0\rangle$ and $\langle 1, 1, 2\rangle$

26. Volume of the parallelepiped with three adjacent edges formed by $\langle 0, -1, 0\rangle$, $\langle 0, 2, -1\rangle$ and $\langle 1, 0, 2\rangle$

27. If you apply a force of magnitude 80 N at the end of a 20 cm-long wrench at an angle of $\frac{\pi}{4}$ to the wrench, find the magnitude of the torque applied to the bolt.

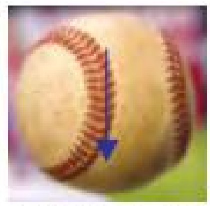
28. If you apply a force of magnitude 160 N at the end of a 0.5 m-long wrench at an angle of $\frac{\pi}{3}$ to the wrench, find the magnitude of the torque applied to the bolt.

29. Use the torque formula $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ to explain the positioning of doorknobs. In particular, explain why the knob is placed as far as possible from the hinges and at a height that makes it possible for most people to push or pull on the door at a right angle to the door.

30. In the diagram, a foot applies a force $\mathbf{F}$ vertically to a bicycle pedal. Compute the torque on the sprocket in terms of θ and $\mathbf{F}$. Determine the angle θ at which the torque is maximized. When helping a young person to learn to ride a bicycle, most people rotate the sprocket so that the pedal sticks straight out to the front. Explain why this is helpful.

In exercises 31–34, assume that the balls are moving into the page (and away from you) with the indicated spin. Determine the direction of the spin vector and of the Magnus force.

31. (a)



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(b)

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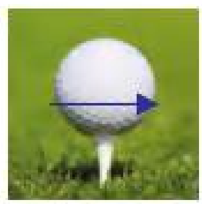
32. (a)



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(b)

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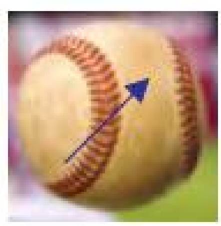
33. (a)



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(b)

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34. (a)

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(b)

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In exercises 35–40, label each statement as true or false. If it is true, briefly explain why. If it is false, give a counterexample.

35. If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.
36. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
37. $\mathbf{a} \times \mathbf{a} = \|\mathbf{a}\|^2$
38. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$
39. If the force is doubled, the torque doubles.
40. If the spin rate is doubled, the Magnus force is doubled.

In exercises 41–44, use the cross product to determine the angle θ between the vectors, assuming that $0 \le \theta \le \frac{\pi}{2}$.

41. $\mathbf{a} = \langle 1, 0, 4\rangle$, $\mathbf{b} = \langle 2, 0, 1\rangle$
42. $\mathbf{a} = \langle 2, 2, 1\rangle$, $\mathbf{b} = \langle 0, 0, 2\rangle$
43. $\mathbf{a} = 3\mathbf{i} + \mathbf{k}$, $\mathbf{b} = 4\mathbf{j} + \mathbf{k}$
44. $\mathbf{a} = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j}$

In exercises 45–50, draw pictures to identify the cross product (do not compute!).

45. $\mathbf{i} \times (3\mathbf{k})$
46. $\mathbf{k} \times (2\mathbf{i})$
47. $\mathbf{i} \times (\mathbf{j} \times \mathbf{k})$
48. $\mathbf{j} \times (\mathbf{j} \times \mathbf{k})$
49. $\mathbf{j} \times (\mathbf{j} \times \mathbf{i})$
50. $(\mathbf{j} \times \mathbf{i}) \times \mathbf{k}$

In exercises 51–54, use the parallelepiped volume formula to determine whether the vectors are coplanar.

51. $\langle 2, 3, 1\rangle$, $\langle 1, 0, 2\rangle$ and $\langle 0, 3, -3\rangle$
52. $\langle 1, -3, 1\rangle$, $\langle 2, -1, 0\rangle$ and $\langle 0, -5, 2\rangle$
53. $\langle 1, 0, -2\rangle$, $\langle 3, 0, 1\rangle$ and $\langle 2, 1, 0\rangle$
54. $\langle 1, 1, 2\rangle$, $\langle 0, -1, 0\rangle$ and $\langle 3, 2, 4\rangle$

55. Show that $\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2$.
56. Show that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.
57. Show that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}$.
58. Prove parts (ii), (iv), (v) and (vi) of Theorem 2.3.
59. In each of the situations shown here, $\|\mathbf{a}\| = 3$ and $\|\mathbf{b}\| = 4$. In which case is $\|\mathbf{a} \times \mathbf{b}\|$ larger? What is the maximum possible value for $\|\mathbf{a} \times \mathbf{b}\|$?

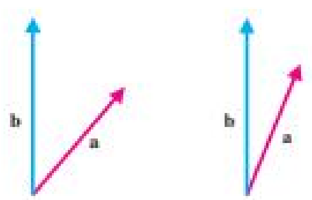


FIGURE A FIGURE B

60. In Figures A and B, if the angles between **a** and **b** are 50° and 20°, respectively, find the exact values for $\|\mathbf{a} \times \mathbf{b}\|$. Also, state whether $\mathbf{a} \times \mathbf{b}$ points into or out of the page.
61. Identify the expressions that are undefined.
    (a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ (b) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$
    (c) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$ (d) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
62. Explain why each equation is true.
    (a) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ (b) $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{a}) = 0$

APPLICATIONS

In exercises 1–8, a sports situation is described, with the typical ball spin shown in the indicated exercise. Discuss the effects on the ball and how the game is affected.

1. Baseball overhand fastball, spin in exercise 31(a)
2. Baseball right-handed curveball, spin in exercise 33(a)
3. Tennis topspin groundstroke, spin in exercise 34(a)
4. Tennis left-handed slice serve, spin in exercise 32(b)
5. Soccer spiral pass, spin in exercise 34(b)
6. Soccer left-footed "curl" kick, spin in exercise 31(b)
7. Golf "pure" hit, spin in exercise 32(a)
8. Golf right-handed "hook" shot, spin in exercise 33(b)

EXPLORATORY EXERCISES

1. Devise a test that quickly determines whether $\|\mathbf{a} \times \mathbf{b}\| < |\mathbf{a} \cdot \mathbf{b}|$, $\|\mathbf{a} \times \mathbf{b}\| > |\mathbf{a} \cdot \mathbf{b}|$ or $\|\mathbf{a} \times \mathbf{b}\| = |\mathbf{a} \cdot \mathbf{b}|$. Apply your test to the following vectors: (a) $\langle 2, 1, 1\rangle$ and $\langle 3, 1, 2\rangle$, (b) $\langle 2, 1, -1\rangle$ and $\langle -1, -2, 1\rangle$ and (c) $\langle 2, 1, 1\rangle$ and $\langle -1, 2, 2\rangle$. For randomly chosen vectors, which of the three cases is the most likely?

2. In this exercise, we explore the equation of motion for a general projectile in three dimensions. Newton's second law is $\mathbf{F} = m\mathbf{a}$. Three forces that could affect the motion of the projectile are gravity, air drag and the Magnus force. Orient the axes such that positive z is up, positive x is right and positive y is straight ahead. The force due to gravity is weight, given by $\mathbf{F}_g = \langle 0, 0, -mg\rangle$. Air drag has magnitude proportional to the square of speed and direction opposite that of velocity. Show that if $\mathbf{v}$ is the velocity vector, then $\mathbf{F}_d = -\|\mathbf{v}\|\mathbf{v}$ satisfies both properties. The Magnus force is proportional to $\mathbf{s} \times \mathbf{v}$, where $\mathbf{s}$ is the spin vector. The full model is then
$$\frac{dv}{dt} = \langle 0, 0, -g\rangle - c_d\|\mathbf{v}\|\mathbf{v} + c_m(\mathbf{s} \times \mathbf{v}),$$
for positive constants c_d and c_m. With $\mathbf{v} = \langle v_x, v_y, v_z\rangle$ and $\mathbf{s} = \langle s_x, s_y, s_z\rangle$, expand this equation into separate differential equations for v_x, v_y and v_z. We can't solve these equations, but we can get some information by considering signs. For a golf drive, the spin produced could be pure backspin, in which case the spin vector is $\mathbf{s} = \langle \omega, 0, 0\rangle$ for some large $\omega > 0$. (A golf shot can have spins of 4000 rpm.) The initial velocity of a good shot would be straight ahead with some loft, $\mathbf{v}(0) = \langle 0, b, c\rangle$ for positive constants b and c. At the beginning of the flight, show that $v'_y < 0$ and thus, v_y decreases. If the ball spends approximately the same amount of time going up as coming down, conclude that the ball will travel farther downrange while going up than coming down. Next, consider the case of a ball with some sidespin, so that $s_x > 0$ and $s_y > 0$. By examining the sign of v'_x, determine whether this ball will curve to the right or left. Examine the other equations and determine what other effects this sidespin may have.

7.3 VECTOR-VALUED FUNCTIONS

For the circuitous path of the airplane indicated in Figure 7.17a, it turns out to be convenient to describe the airplane's location at any given time by the endpoint of a vector whose initial point is located at the origin (a **position vector**). (See Figure 7.17b for vectors indicating the location of the plane at a number of times.) Notice that a *function* that gives us a vector in V_3 for each time t would do the job nicely. This is the concept of a vector-valued function, which we define more precisely in Definition 3.1.

DEFINITION 3.1

A **vector-valued function** $\mathbf{r}(t)$ is a mapping from its domain $D \subset \mathbb{R}$ to its range $R \subset V_3$, so that for each t in D, $\mathbf{r}(t) = \mathbf{v}$ for exactly one vector $\mathbf{v} \in R$. We can always write a vector-valued function as

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}, \tag{3.1}$$

for some scalar functions f, g and h (called the **component functions** of $\mathbf{r}$).

For each t, we regard $\mathbf{r}(t)$ as a position vector. The endpoint of $\mathbf{r}(t)$ then can be viewed as tracing out a curve, as illustrated in Figure 7.17b. Observe that for $\mathbf{r}(t)$ as defined in (3.1), this curve is the same as that described by the parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$. In three dimensions, such a curve is referred to as a **space curve**.

FIGURE 7.17a
Airplane's flight path

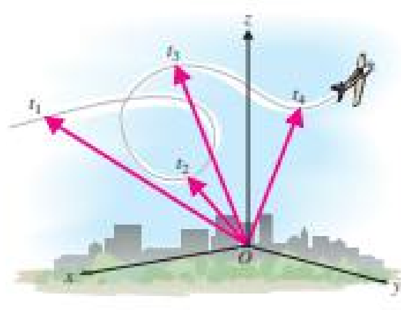


FIGURE 7.17b
Vectors indicating plane's position at several times

We can likewise define a vector-valued function $\mathbf{r}(t)$ in V_2 by

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j},$$

for some scalar functions f and g.

REMARK 3.1

Although any variable would do, we routinely use the variable t to represent the independent variable for vector-valued functions, since in many applications t represents *time*.

EXAMPLE 3.1 Sketching the Curve Defined by a Vector-Valued Function

Sketch a graph of the curve traced out by the endpoint of the two-dimensional vector-valued function

$$\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2 - 2)\mathbf{j}.$$

Solution Substituting some values for t, we have $\mathbf{r}(0) = \mathbf{i} - 2\mathbf{j} = \langle 1, -2 \rangle$, $\mathbf{r}(2) = 3\mathbf{i} + 2\mathbf{j} = \langle 3, 2 \rangle$ and $\mathbf{r}(-2) = \langle -1, 2 \rangle$. We plot these in Figure 7.18a. The endpoints of all position vectors $\mathbf{r}(t)$ lie on the curve C, described parametrically by

$$C: x = t+1, \quad y = t^2 - 2, \quad t \in \mathbb{R}.$$

We can eliminate the parameter by solving for t in terms of x:

$$t = x - 1.$$

The curve is then given by

$$y = t^2 - 2 = (x-1)^2 - 2.$$

Notice that the graph of this is a parabola opening up, with vertex at the point $(1, -2)$, as seen in Figure 7.18b. The small arrows marked on the graph indicate the **orientation**, that is, the direction of increasing values of t. If the curve describes the path of an object, then the orientation indicates the direction in which the object traverses the path. In this case, we can easily determine the orientation from the parametric representation of the curve. Since $x = t + 1$, observe that x increases as t increases.

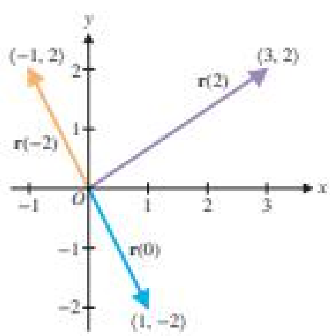


FIGURE 7.18a
Some values of
$\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2 - 2)\mathbf{j}$

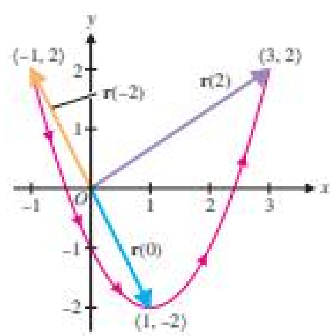


FIGURE 7.18b
Curve defined by
$\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2 - 2)\mathbf{j}$

You may recall from your experience with parametric equations in Chapter 6 that eliminating the parameter from the parametric representation of a curve is not always so easy as it was in example 3.1. We illustrate this in example 3.2.

EXAMPLE 3.2 A Vector-Valued Function Defining an Ellipse

Sketch a graph of the curve traced out by the endpoint of the vector-valued function $\mathbf{r}(t) = 4\cos t\mathbf{i} - 3\sin t\mathbf{j}$, $t \in \mathbb{R}$.

Solution In this case, the curve can be written parametrically as

$$x = 4\cos t, \qquad y = -3\sin t, \qquad t \in \mathbb{R}.$$

Instead of solving for the parameter t, it often helps to look for some relationship between the variables. Here,

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$$

or

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1,$$

which is the equation of an ellipse. (See Figure 7.19.) To determine the orientation of the curve here, you'll need to look carefully at both parametric equations. First, fix a starting place on the curve, for convenience, say, (4, 0). This corresponds to $t = 0$, $\pm 2\pi, \pm 4\pi, \ldots$. As t increases, notice that $\cos t$ (and hence, x) decreases initially, while $\sin t$ increases, so that $y = -3\sin t$ decreases (initially). With both x and y decreasing initially, we get the clockwise orientation indicated in Figure 7.19. ■

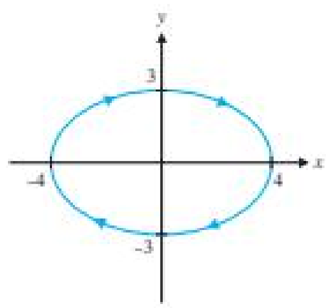


FIGURE 7.19
Curve defined by
$\mathbf{r}(t) = 4\cos t\mathbf{i} - 3\sin t\mathbf{j}$

Just as the endpoint of a vector-valued function in two dimensions traces out a curve, if we were to plot the value of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ for every value of t, the endpoints of the vectors would trace out a curve in three dimensions.

EXAMPLE 3.3 A Vector-Valued Function Defining an Elliptical Helix

Plot the curve traced out by the vector-valued function $\mathbf{r}(t) = \sin t\mathbf{i} - 3\cos t\mathbf{j} + 2t\mathbf{k}$, $t \geq 0$.

Solution The curve is given parametrically by

$$x = \sin t, \quad y = -3\cos t, \quad z = 2t, \quad t \geq 0.$$

While most curves in three dimensions are difficult to recognize, you should notice that there is a relationship between x and y here, namely,

$$x^2 + \left(\frac{y}{3}\right)^2 = \sin^2 t + \cos^2 t = 1. \tag{3.2}$$

In two dimensions, this is the equation of an ellipse. In three dimensions, since the equation does not involve z, (3.2) is the equation of an elliptic cylinder whose axis is the z-axis. This says that every point on the curve defined by $\mathbf{r}(t)$ lies on this cylinder. From the parametric equations for x and y (in two dimensions), the ellipse is traversed in the counterclockwise direction. This says that the curve will wrap itself around the cylinder (counterclockwise, as you look down the positive z-axis toward the origin), as t increases. Finally, since $z = 2t$, z will increase as t increases and so, the curve will wind its way up the cylinder, as t increases. We show the curve and the elliptical cylinder in Figure 7.20a. We call this curve an **elliptical helix**. In Figure 7.20b, we display a computer-generated graph of the same helix. There, rather than the usual x-, y- and z-axes, we show a framed graph, where the values of x, y and z are indicated on three adjacent edges of a box containing the graph. ■

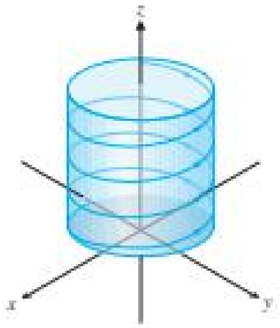


FIGURE 7.20a
Elliptical helix:
$\mathbf{r}(t) = \sin t\mathbf{i} - 3\cos t\mathbf{j} + 2t\mathbf{k}$

We can use vector-valued functions as a convenient representation of some very familiar curves, as we see in example 3.4.

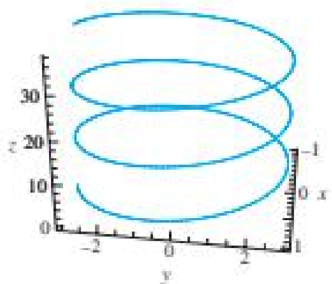


FIGURE 7.20b
Computer sketch:
$\mathbf{r}(t) = \sin t\mathbf{i} - 3\cos t\mathbf{j} + 2t\mathbf{k}$

EXAMPLE 3.4 A Vector-Valued Function Defining a Line

Plot the curve traced out by the vector-valued function

$$\mathbf{r}(t) = \langle 3 + 2t, 5 - 3t, 2 - 4t \rangle, \qquad t \in \mathbb{R}$$

Solution Notice that the curve is given parametrically by

$$x = 3 + 2t, \qquad y = 5 - 3t, \qquad z = 2 - 4t, \qquad t \in \mathbb{R}.$$

You should recognize these equations as parametric equations for the straight line parallel to the vector $\langle 2, -3, -4 \rangle$ and passing through the point $(3, 5, 2)$, as seen in Figure 7.21, where we also note the orientation. ■

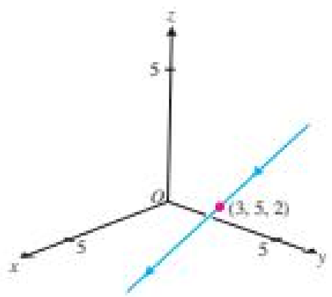


FIGURE 7.21
Straight line:
$\mathbf{r}(t) = \langle 3 + 2t, 5 - 3t, 2 - 4t \rangle$

Most three-dimensional graphs are very challenging to sketch by hand. Although you may want to use computer-generated graphics for most sketches, you will need to be knowledgeable enough to know how to adjust such a graph to uncover any hidden features. You should be able to draw several basic curves by hand, like those in examples 3.3 and 3.4. More importantly, you should be able to recognize the effects various components have on the graph of a three-dimensional curve. In example 3.5, we walk you through matching four vector-valued functions with their computer-generated graphs.

EXAMPLE 3.5 Matching a Vector-Valued Function to Its Graph

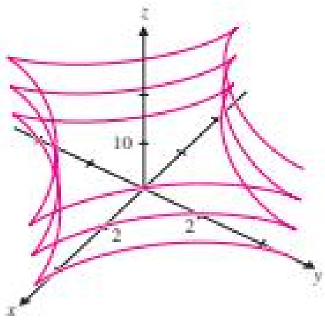
Match each of the vector-valued functions $\mathbf{f}_1(t) = \langle \cos t, \ln t, \sin t \rangle$, $\mathbf{f}_2(t) = \langle t\cos t, t\sin t, t \rangle$, $\mathbf{f}_3(t) = \langle 3\sin 2t, t, t \rangle$ and $\mathbf{f}_4(t) = \langle 5\sin^3 t, 5\cos^3 t, t \rangle$ with the corresponding computer-generated graph.

Solution First, realize that there is no single, correct procedure for solving this problem. Look for familiar functions and match them with familiar graphical properties. From example 3.3, recall that certain combinations of sines and cosines will produce curves that lie on a cylinder. Notice that for the function $\mathbf{f}_1(t)$, $x = \cos t$ and $z = \sin t$, so that

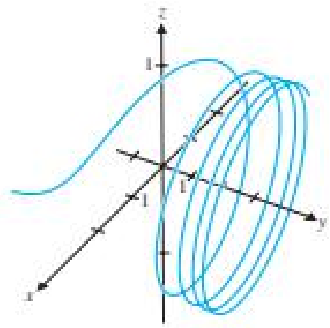
$$x^2 + z^2 = \cos^2 t + \sin^2 t = 1.$$

This says that every point on the curve lies on the cylinder $x^2 + z^2 = 1$ (the right circular cylinder of radius 1 whose axis is the y-axis). Further, the function $y = \ln t$ tends rapidly to $-\infty$ as $t \to 0^+$ and increases slowly as t increases beyond $t = 1$. Notice that the curve in Graph B appears to lie on a right circular cylinder and that the spirals get closer together as you move to the right (as $y \to \infty$) and move very far apart as you move to the left (as $y \to -\infty$). At first glance, you might expect the curve traced out by $\mathbf{f}_2(t)$ also to lie on a right circular cylinder, but look more closely. Here, we have $x = t\cos t$, $y = t\sin t$ and $z = t$, so that

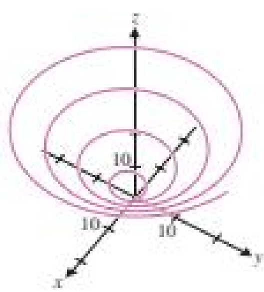
$$x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = z^2.$$



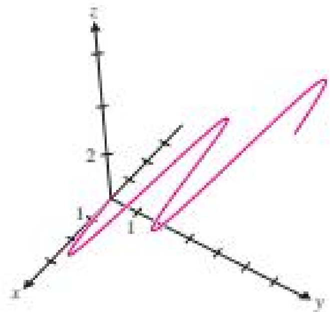
GRAPH A



GRAPH B



GRAPH C



GRAPH D

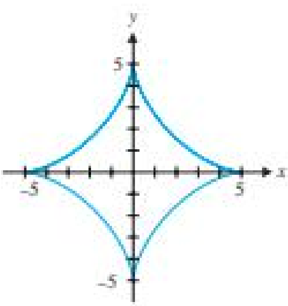


FIGURE 7.22
A cross-section of the cylinder $x = 5\sin^3 t$, $y = 5\cos^3 t$

This says that the curve lies on the surface defined by $x^2 + y^2 = z^2$ (a right circular cone with axis along the z-axis). Notice that only the curve shown in Graph C fits this description. Next, notice that for $\mathbf{f}_3(t)$, the y and z components are identical and so, the curve must lie in the plane $y = z$. Replacing t by y, we have $x = 3\sin 2t = 3\sin 2y$, a sine curve lying in the plane $y = z$. Clearly, the curve in Graph D matches this description. Although Graph A is the only curve remaining to match with $\mathbf{f}_4(t)$, notice that if the cosine and sine terms were not cubed, we would simply have a helix, as in example 3.3. Since $z = t$, each point on the curve lies on the cylinder defined parametrically by $x = 5\sin^3 t$ and $y = 5\cos^3 t$. You need only look at the graph of the cross-section of the cylinder shown in Figure 7.22 (found by graphing the parametric equations $x = 5\sin^3 t$ and $y = 5\cos^3 t$ in two dimensions) to decide that Graph A is the obvious choice. ■

○ Arc Length in $\mathbb{R}^3$

A natural question to ask about a curve is, "How long is it?" Note that the plane curve traced out exactly once by the endpoint of the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t)\rangle$, for $t \in [a, b]$ is the same as the curve defined parametrically by $x = f(t)$, $y = g(t)$. Recall that if f, f', g and g' are all continuous for $t \in [a, b]$, the arc length is given by

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt. \qquad (3.3)$$

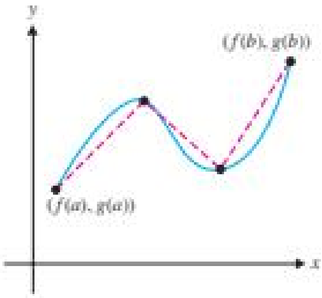


FIGURE 7.23a
Approximate arc length in $\mathbb{R}^2$

We derived this by first breaking the curve into small pieces (i.e., we *partitioned* the interval $[a, b]$) and then approximating the length with the sum of the lengths of small line segments connecting successive points. (See Figure 7.23a.) Finally, we made the approximation exact by taking a limit as the number of points in the partition tended to infinity.

Consider a space curve traced out by the endpoint of the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f, f', g, g', h and h' are all continuous for $t \in [a, b]$ and where the curve is traversed exactly once as t increases from a to b. As we did in the two-dimensional case, we begin by partitioning the interval $[a, b]$ into n subintervals of equal size: $a = t_0 < t_1 < \cdots < t_n = b$, where $t_i - t_{i-1} = \Delta t = \frac{b-a}{n}$, for all $i = 1, 2, \ldots, n$. Next, for each $i = 1, 2, \ldots, n$, we approximate the arc length s_i of that portion of the curve joining the points $(f(t_{i-1}), g(t_{i-1}), h(t_{i-1}))$ and $(f(t_i), g(t_i), h(t_i))$ by the straight-line distance between the points. (See Figure 7.23b for an illustration of the case where $n = 4$.) From the distance formula, we have

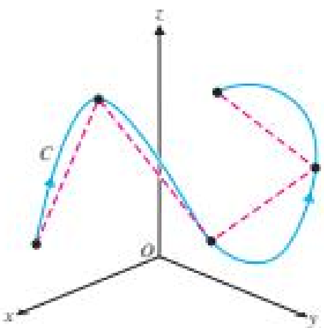


FIGURE 7.23b
Approximate arc length in $\mathbb{R}^3$

$$\begin{aligned} s_i &\approx d\{(f(t_{i-1}), g(t_{i-1}), h(t_{i-1}))(f(t_i), g(t_i), h(t_i))\} \\ &= \sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2 + [h(t_i) - h(t_{i-1})]^2}. \end{aligned}$$

Applying the Mean Value Theorem three times (why can we do this?), we get

$$f(t_i) - f(t_{i-1}) = f'(c_i)(t_i - t_{i-1}) = f'(c_i)\Delta t,$$
$$g(t_i) - g(t_{i-1}) = g'(d_i)(t_i - t_{i-1}) = g'(d_i)\Delta t$$

and
$$h(t_i) - h(t_{i-1}) = h'(e_i)(t_i - t_{i-1}) = h'(e_i)\Delta t,$$

for some points c_i, d_i and e_i in the interval (t_{i-1}, t_i). This gives us

$$\begin{aligned} s_i &\approx \sqrt{[f(t_i) - f(t_{i-1})]^2 + [g(t_i) - g(t_{i-1})]^2 + [h(t_i) - h(t_{i-1})]^2} \\ &= \sqrt{[f'(c_i)\Delta t]^2 + [g'(d_i)\Delta t]^2 + [h'(e_i)\Delta t]^2} \\ &= \sqrt{[f'(c_i)]^2 + [g'(d_i)]^2 + [h'(e_i)]^2}\,\Delta t. \end{aligned}$$

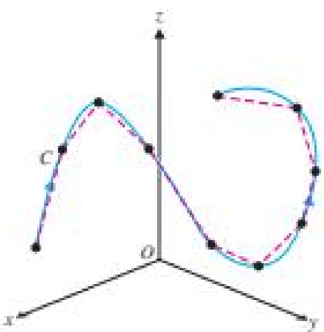


FIGURE 7.23c
Improved arc length approximation

Notice that if Δt is small, then all of c_i, d_i and e_i are very close and we can make the further approximation

$$s_i \approx \sqrt{[f'(c_i)]^2 + [g'(c_i)]^2 + [h'(c_i)]^2}\,\Delta t,$$

for each $i = 1, 2, \ldots, n$. The total arc length is then approximately

$$s \approx \sum_{i=1}^{n} \sqrt{[f'(c_i)]^2 + [g'(c_i)]^2 + [h'(c_i)]^2}\,\Delta t.$$

In Figure 7.23c, we illustrate this approximation for the case where $n = 9$. This suggests that taking the limit as $n \to \infty$ gives the exact arc length:

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(c_i)]^2 + [g'(c_i)]^2 + [h'(c_i)]^2}\,\Delta t,$$

provided the limit exists. You should recognize this as the definite integral

Arc length

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt. \tag{3.4}$$

Observe that the arc length formula for a plane curve (3.3) is a special case of (3.4). As with other formulas for arc length, the integral in (3.4) can only rarely be computed exactly and we must typically be satisfied with a numerical approximation. Example 3.6 illustrates one of the very few arc lengths in $\mathbb{R}^3$ that can be computed exactly.

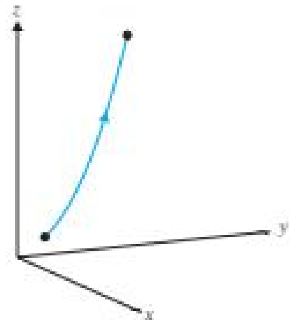


FIGURE 7.24
The curve defined by $\mathbf{r}(t) = \langle 2t, \ln t, t^2 \rangle$

EXAMPLE 3.6 Computing Arc Length in $\mathbb{R}^3$

Find the arc length of the curve traced out by the endpoint of the vector-valued function $\mathbf{r}(t) = \langle 2t, \ln t, t^2 \rangle$, for $1 \le t \le e$.

Solution First, notice that for $x(t) = 2t$, $y(t) = \ln t$ and $z(t) = t^2$, we have $x'(t) = 2$, $y'(t) = \frac{1}{t}$ and $z'(t) = 2t$, and the curve is traversed exactly once for $1 \le t \le e$. (To see why, observe that $x = 2t$ is an increasing function.) From (3.4), we now have

$$\begin{aligned} s &= \int_1^e \sqrt{2^2 + \left(\frac{1}{t}\right)^2 + (2t)^2}\,dt = \int_1^e \sqrt{4 + \frac{1}{t^2} + 4t^2}\,dt \\ &= \int_1^e \sqrt{\frac{1 + 4t^2 + 4t^4}{t^2}}\,dt = \int_1^e \sqrt{\frac{(1 + 2t^2)^2}{t^2}}\,dt \\ &= \int_1^e \frac{1 + 2t^2}{t}\,dt = \int_1^e \left(\frac{1}{t} + 2t\right)dt \\ &= \left(\ln|t| + 2\frac{t^2}{2}\right)\Bigg|_1^e = (\ln e + e^2) - (\ln 1 + 1) = e^2. \end{aligned}$$

We show a graph of the curve for $1 \le t \le e$ in Figure 7.24. ■

The arc length integral in example 3.7 is typical, in that we need a numerical approximation.

EXAMPLE 3.7 Approximating Arc Length in $\mathbb{R}^3$

Find the arc length of the curve traced out by the endpoint of the vector-valued function $\mathbf{r}(t) = \langle e^{2t}, \sin t, t \rangle$, for $0 \le t \le 2$.

Solution First, note that for $x(t) = e^{2t}$, $y(t) = \sin t$ and $z(t) = t$, we have $x'(t) = 2e^{2t}$, $y'(t) = \cos t$ and $z'(t) = 1$, and that the curve is traversed exactly once for $0 \le t \le 2$ (since x is an increasing function of t). From (3.4), we now have

$$s = \int_0^2 \sqrt{(2e^{2t}) + (\cos t)^2 + 1^2}\,dt = \int_0^2 \sqrt{4e^{4t} + \cos^2 t + 1}\,dt.$$

Since you don't know how to evaluate this integral exactly (which is typically the case), you can approximate the integral using Simpson's Rule or the numerical integration routine built into your calculator or computer algebra system, to find that the arc length is approximately $s \approx 53.8$. ■

Often, the curve of interest is determined by the intersection of two surfaces. Parametric equations can give us simple representations of many such curves.

EXAMPLE 3.8 Finding Parametric Equations for an Intersection of Surfaces

Find the arc length of the portion of the curve determined by the intersection of the cone $z = \sqrt{x^2 + y^2}$ and the plane $y + z = 2$ in the first octant.

FIGURE 7.25a
Intersection of cone and plane

Solution The cone and plane are shown in Figure 7.25a. From your knowledge of conic sections, note that this curve could be a parabola or an ellipse. Parametric equations for the curve must satisfy both $z = \sqrt{x^2 + y^2}$ and $y + z = 2$. Eliminating z by solving for it in each equation, we get

$$z = \sqrt{x^2 + y^2} = 2 - y.$$

Squaring both sides and gathering terms, we get

$$x^2 + y^2 = (2 - y)^2 = 4 - 4y + y^2$$

or

$$x^2 = 4 - 4y.$$

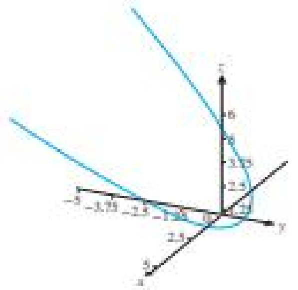


FIGURE 7.25b
Curve of intersection

Solving for y now gives us
$$y = 1 - \frac{x^2}{4},$$
which is clearly the equation of a parabola in two dimensions. To obtain the equation for the three-dimensional parabola, let x be the parameter, which gives us the parametric equations
$$x = t, \quad y = 1 - \frac{t^2}{4}, \quad \text{and} \quad z = \sqrt{t^2 + (1 - t^2/4)} = 1 + \frac{t^2}{4}.$$
A graph is shown in Figure 7.25b. The portion of the parabola in the first octant must have $x \geq 0$ (so $t \geq 0$), $y \geq 0$ (so $t^2 \leq 4$) and $z \geq 0$ (always true). This occurs if $0 \leq t \leq 2$. The arc length is then
$$s = \int_0^2 \sqrt{1 + (-t/2)^2 + (t/2)^2}\, dt = \frac{\sqrt{2}}{2}\ln(\sqrt{2} + \sqrt{3}) + \sqrt{3} \approx 2.54,$$
where we leave the details of the integration to you. ■

BEYOND FORMULAS

If you think that examples 3.1 and 3.2 look very much like parametric equations examples, you're exactly right. The ideas presented there are not new; only the notation and terminology are new. However, the vector notation lets us easily extend these ideas into three dimensions, where the graphs can be more complicated.

EXERCISES 7.3

WRITING EXERCISES

1. Discuss the differences, if any, between the curve traced out by the terminal point of the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ and the curve defined parametrically by $x = f(t)$, $y = g(t)$.

2. In example 3.3, describe the "shadow" of the helix in the xy-plane (the shadow created by shining a light down from the "top" of the z-axis). Equivalently, if the helix is collapsed down into the xy-plane, describe the resulting curve. Compare this curve to the ellipse defined parametrically by $x = \sin t$, $y = -3\cos t$.

3. Discuss how you would compute the arc length of a curve in four or more dimensions. Specifically, for the curve traced out by the terminal point of the n-dimensional vector-valued function $\mathbf{r}(t) = \langle f_1(t), f_2(t), \ldots, f_n(t)\rangle$ for $n \geq 4$, state the arc length formula and discuss how it relates to the n-dimensional distance formula.

4. The helix in Figure 7.20a is shown from a standard viewpoint (above the xy-plane, in between the x- and y-axes). Describe what an observer at the point (0, 0, −1000) would see. Also, describe what observers at the points (1000, 0, 0) and (0, 1000, 0) would see.

In exercises 1–18, sketch the curve traced out by the given vector-valued function by hand.

1. $\mathbf{r}(t) = \langle t - 1, t^2\rangle$
2. $\mathbf{r}(t) = \langle t^2 - 1, 4t\rangle$
3. $\mathbf{r}(t) = 2\cos t\mathbf{i} + (\sin t - 1)\mathbf{j}$
4. $\mathbf{r}(t) = (\sin t - 2)\mathbf{i} + 4\cos t\mathbf{j}$
5. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 3\rangle$
6. $\mathbf{r}(t) = \langle \cos 2t, \sin 2t, 1\rangle$
7. $\mathbf{r}(t) = \langle t, t^2 + 1, -1\rangle$
8. $\mathbf{r}(t) = \langle 3, t, t^2 - 1\rangle$
9. $\mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + 3t^2\mathbf{k}$
10. $\mathbf{r}(t) = (t + 2)\mathbf{i} + (2t - 1)\mathbf{j} + (t + 2)\mathbf{k}$
11. $\mathbf{r}(t) = \langle 4t - 1, 2t + 1, -6t\rangle$
12. $\mathbf{r}(t) = \langle -2t, 2t, 3 - t\rangle$
13. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + t\mathbf{k}$
14. $\mathbf{r}(t) = 2\cos t\mathbf{i} + \sin t\mathbf{j} + 3t\mathbf{k}$
15. $\mathbf{r}(t) = \langle 2\cos t, 2t, 3\sin t\rangle$
16. $\mathbf{r}(t) = \langle -1, 2\cos t, 2\sin t\rangle$
17. $\mathbf{r}(t) = \langle t\cos 2t, t\sin 2t, 2t\rangle$
18. $\mathbf{r}(t) = \langle t\cos t, 2t, t\sin t\rangle$

In exercises 19–26, use graphing technology to sketch the curve traced out by the given vector-valued function. Determine whether the function is periodic; if so, name the period.

19. $\mathbf{r}(t) = \langle \cos 4t, \sin t, \sin 7t\rangle$
20. $\mathbf{r}(t) = \langle 3\cos 2t, \sin 5t, \cos t\rangle$
21. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + (2t^2 - 1)\mathbf{k}$
22. $\mathbf{r}(t) = (t^3 - t)\mathbf{i} + t^2\mathbf{j} + (2t - 4)\mathbf{k}$
23. $\mathbf{r}(t) = \langle \tan t, \sin t^2, \cos t\rangle$
24. $\mathbf{r}(t) = \langle \sin t, -\csc t, \cot t\rangle$

25. (a) $\mathbf{r}(t) = \langle 2\cos t + \sin 2t, 2\sin t + \cos 2t\rangle$

(b) $\mathbf{r}(t) = \langle 2\cos 3t + \sin 5t, 2\sin 3t + \cos 5t\rangle$

(c) How does the graph of $\langle 2\cos at + \sin bt, 2\sin at + \cos bt\rangle$ depend on a and b?

26. (a) $\mathbf{r}(t) = \langle 4\cos 4t - 6\cos t, 4\sin 4t - 6\sin t\rangle$

(b) $\mathbf{r}(t) = \langle 4\cos 7t + 2\cos t, 4\sin 7t + 2\sin t\rangle$

(c) How does the graph of $\langle 4\cos at + b\cos t, 4\sin at + b\sin t\rangle$ depend on a and b?

27. In parts a–f, match the vector-valued function with its graph. Give reasons for your choices.

(a) $\mathbf{r}(t) = \langle \cos t^2, t, t\rangle$

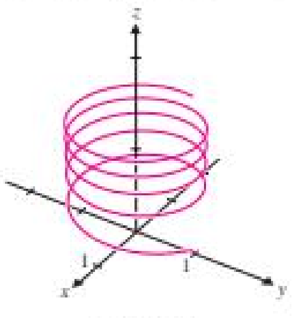
(b) $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t^2\rangle$

(c) $\mathbf{r}(t) = \langle \sin 16\sqrt{t}, \cos 16\sqrt{t}, t\rangle$

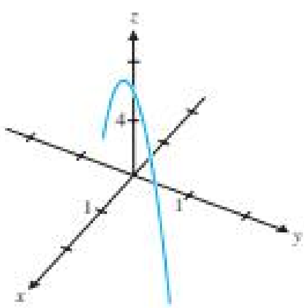
(d) $\mathbf{r}(t) = \langle \sin t^2, \cos t^2, t\rangle$

(e) $\mathbf{r}(t) = \langle t, t, 6 - 4t^2\rangle$

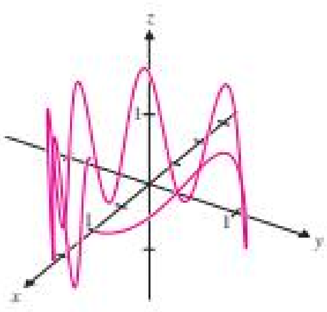
(f) $\mathbf{r}(t) = \langle t^3 - t, 0.5t^2, 2t - 4\rangle$



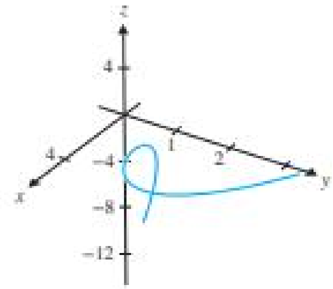
GRAPH 1



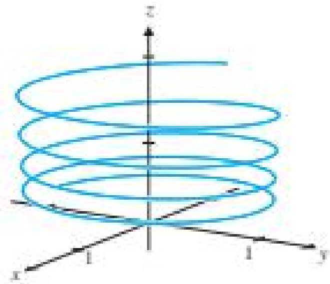
GRAPH 2



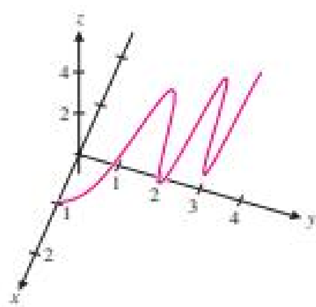
GRAPH 3



GRAPH 4



GRAPH 5



GRAPH 6

28. Of the functions in exercise 27, which are periodic? Which are bounded? Identify all components (x, y, z) that are bounded.

In exercises 29–32, compute the arc length.

29. $\mathbf{r}(t) = \langle t\cos t, t\sin t, \frac{1}{3}(2t)^{3/2}\rangle$, $0 \le t \le 2\pi$

30. $\mathbf{r}(t) = \langle 4t, 3\cos t, 3\sin t\rangle$, $0 \le t \le 3\pi$

31. $\mathbf{r}(t) = \langle 4\ln t, t^2, 4t\rangle$, $1 \le t \le 3$

32. $\mathbf{r}(t) = \langle \tan^{-1} t, \ln(1 + t^2), 2t - 2\tan^{-1} t\rangle$, $0 \le t \le \frac{\pi}{4}$

In exercises 33–38, use a CAS to sketch the curve and estimate its arc length.

33. $\mathbf{r}(t) = \langle \cos t, \sin t, \cos 2t\rangle$, $0 \le t \le 2\pi$

34. $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t + \cos t\rangle$, $0 \le t \le 2\pi$

35. $\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, \cos 16t\rangle$, $0 \le t \le 2$

36. $\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, \cos 16t \rangle, 0 \le t \le 4$

37. $\mathbf{r}(t) = \langle t, t^2 - 1, t^3 \rangle, 0 \le t \le 2$

38. $\mathbf{r}(t) = \langle t^2 + 1, 2t, t^2 - 1 \rangle, 0 \le t \le 2$

39. A spiral staircase makes two complete turns as it rises 3 m between floors. A handrail at the outside of the staircase is located 0.5 m from the center pole of the staircase.

(a) Use parametric equations for a helix to compute the length of the handrail.

(b) Imagine unrolling the staircase so that the handrail is a line segment. Use the formula for the hypotenuse of a right triangle to compute its length.

40. Find the arc length of the section of the helix traced out by $\mathbf{r}(t) = \langle \cos t, \sin t, kt \rangle$ for $0 \le t \le 2\pi$. As in exercise 39, illustrate this as the hypotenuse of a right triangle.

In exercises 41–44, find parametric equations for the indicated curve. If you have access to a graphing utility, graph the surfaces and the resulting curve. Estimate its arc length.

41. The intersection of $z = \sqrt{x^2 + y^2}$ and $z = 2$

42. The intersection of $z = \sqrt{x^2 + y^2}$ and $y + 2z = 2$

43. The intersection of $x^2 + y^2 = 9$ and $y + z = 2$

44. The intersection of $y^2 + z^2 = 9$ and $x = 2$

45. Show that the curve $\mathbf{r}(t) = \langle 2t, 4t^2 - 1, 8t^3 \rangle, 0 \le t \le 1$, has the same arc length as the curve in exercise 37.

46. Show that the curve $\mathbf{r}(t) = \langle t + 1, 2\sqrt{t}, t - 1 \rangle, 0 \le t \le 4$, has the same arc length as the curve in exercise 38.

47. Compare the graphs of $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $\mathbf{g}(t) = \langle \cos t, \cos^2 t, \cos^3 t \rangle$ and $\mathbf{h}(t) = \langle \sqrt{t}, t, t \rangle$. Discuss the similarities and the differences.

48. Compare the graphs of $\mathbf{r}(t) = \langle 2t - 1, t^2, t \rangle$, $\mathbf{g}(t) = \langle 2\sin t - 1, \sin^2 t, \sin t \rangle$ and $\mathbf{h}(t) = \langle 2e^t - 1, e^{2t}, e^t \rangle$. Discuss the similarities and the differences.

49. Show that the curve in exercise 33 lies on the hyperbolic paraboloid $z = x^2 - y^2$. Use a CAS to sketch both the surface and the curve.

50. Show that the curve in exercise 34 lies on the plane $z = x + y$. Use a CAS to sketch both the plane and the curve.

51. (a) Use a graphing utility to sketch the graph of $\mathbf{r}(t) = \langle \cos t, \cos t, \sin t \rangle$ with $0 \le t \le 2\pi$. Explain why the graph should be the same with $0 \le t \le T$, for any $T \ge 2\pi$. Try several larger domains ($0 \le t \le 2\pi, 0 \le t \le 10\pi, 0 \le t \le 50\pi$, etc.) with your graphing utility. Eventually, the ellipse should start looking thicker and for large enough domains you will see a mess of jagged lines. Explain what has gone wrong with the graphing utility.

(b) It may surprise you that this curve is not a circle. Show that the shadows in the xz-plane and yz-plane are circles. Show that the curve lies in the plane $x = y$. Sketch a graph showing the plane $x = y$ and a circular shadow in the yz-plane. To draw a curve in the plane $x = y$ with the circular shadow, explain why the curve must be wider in the xy-direction than in the z-direction. In other words, the curve is not circular.

52. The graph of $\mathbf{r}(t) = \langle \cos t, \cos t, \sqrt{2} \sin t \rangle$ *is* a circle. To verify this, start by showing that $\|\mathbf{r}(t)\| = \sqrt{2}$, for all t. Then observe that the curve lies in the plane $x = y$. Explain why this proves that the graph is a (portion of a) circle.

EXPLORATORY EXERCISES

1. More insight into exercise 52 can be gained by looking at basis vectors. The circle traced out by $\mathbf{r}(t) = \langle \cos t, \cos t \sqrt{2} \sin t \rangle$ lies in the plane $x = y$, which contains the vector $\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1, 1, 0 \rangle$. The plane $x = y$ also contains the vector $\mathbf{v} = \langle 0, 0, 1 \rangle$. Show that *any* vector $\mathbf{w}$ in the plane $x = y$ can be written as $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$ for some constants c_1 and c_2. Also, show that $\mathbf{r}(t) = (\sqrt{2}\cos t)\mathbf{u} + (\sqrt{2}\sin t)v$. Recall that in two dimensions, a circle of radius r centered at the origin can be written parametrically as $(r \cos t)\mathbf{i} + (r \sin t)\mathbf{j}$. In general, suppose that $\mathbf{u}$ and $\mathbf{v}$ are any orthogonal unit vectors. If $\mathbf{r}(t) = (r\cos t)\mathbf{u} + (r \sin t)\mathbf{v}$, show that $\mathbf{r}(t) \cdot \mathbf{r}(t) = r^2$.

2. Examine the graphs of several vector-valued functions of the form $\mathbf{r}(t) = \langle a \cos ct + b \cos dt, a \sin ct + b \sin dt \rangle$, for constants a, b, c and d. Determine the values of these constants that produce graphs of different types. For example, starting with the graph of $\langle 4\cos 4t - 6\cos t, 4\sin 4t - 6\sin t \rangle$, change $c = 4$ to $c = 3$, $c = 5$, $c = 2$, etc. Conjecture a relationship between the number of loops and the difference between c and d. Test this conjecture on other vector-valued functions. Returning to $\langle 4\cos 4t - 6\cos t, 4\sin 4t - 6\sin t \rangle$, change $a = 4$ to other values. Conjecture a relationship between the size of the loops and the value of a.

7.4 THE CALCULUS OF VECTOR-VALUED FUNCTIONS

In this section, we begin to explore the calculus of vector-valued functions, beginning with the notion of limit and progressing to continuity, derivatives and, finally, integrals, just as we did with scalar functions. We define everything in this section in terms of vector-valued functions in three dimensions. The definitions can be interpreted for vector-valued functions in two dimensions in the obvious way, by simply dropping the third component everywhere.

For a vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, when we write

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{u},$$

we mean that as t gets closer and closer to a, the vector $\mathbf{r}(t)$ is getting closer and closer to the vector $\mathbf{u}$. For $\mathbf{u} = \langle u_1, u_2, u_3\rangle$, this means that

$$\lim_{t\to a} \mathbf{r}(t) = \lim_{t\to a} = \langle f(t), g(t), h(t)\rangle = \mathbf{u} = \langle u_1, u_2, u_3\rangle.$$

Notice that for this to occur, we must have that $f(t)$ is approaching u_1, $g(t)$ is approaching u_2 and $h(t)$ is approaching u_3. In view of this, we make the following definition.

DEFINITION 4.1

For a vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, the **limit** of $\mathbf{r}(t)$ as t approaches a is given by

$$\lim_{t\to a} \mathbf{r}(t) = \lim_{t\to a} = \langle f(t), g(t), h(t)\rangle = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle, \tag{4.1}$$

provided *all* of the indicated limits exist. If *any* of the limits indicated on the right-hand side of (4.1) fail to exist, then $\lim_{t\to a} \mathbf{r}(t)$ **does not exist**.

In example 4.1, we see that calculating a limit of a vector-valued function simply consists of calculating three separate limits of scalar functions.

EXAMPLE 4.1 Finding the Limit of a Vector-Valued Function

Find $\lim_{t\to 0} \left\langle t^2 + 1, 5\cos t, \sin t\right\rangle$.

Solution Here, each of the component functions is continuous (for all t) and so, we can calculate their limits simply by substituting the value for t. We have

$$\lim_{t\to 0} \left\langle t^2 + 1, 5\cos t, \sin t\right\rangle = \left\langle \lim_{t\to 0}(t^2+1), \lim_{t\to 0}(5\cos t), \lim_{t\to 0} \sin t\right\rangle$$
$$= \langle 1, 5, 0\rangle. \blacksquare$$

EXAMPLE 4.2 A Limit That Does Not Exist

Find $\lim_{t\to 0} \left\langle e^{2t} + 5, t^2 + 2t - 3, 1/t\right\rangle$.

Solution Notice that the limit of the third component is $\lim_{t\to 0} \frac{1}{t}$, which does not exist. So, even though the limits of the first two components exist, the limit of the vector-valued function does not exist. ■

Recall that for a scalar function f, we say that f is *continuous* at a if and only if

$$\lim_{t\to a} f(t) = f(a).$$

That is, a scalar function is continuous at a point whenever the limit and the value of the function are the same. We define the continuity of vector-valued functions in the same way.

DEFINITION 4.2

The vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is **continuous** at $t = a$ whenever

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$$

(i.e. whenever the limit exists and equals the value of the vector-valued function).

Notice that in terms of the components of $\mathbf{r}$, this says that $\mathbf{r}(t)$ is continuous at $t = a$ whenever

$$\lim_{t\to a}\langle f(t), g(t), h(t)\rangle = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle = \langle f(a), g(a), h(a)\rangle.$$

It then follows that $\mathbf{r}$ is continuous at $t = a$ if and only if

$$\lim_{t\to a} f(t) = f(a), \quad \lim_{t\to a} g(t) = g(a) \quad \text{and} \quad \lim_{t\to a} h(t) = h(a).$$

Look carefully at what we have just said, and observe that we just proved the following theorem.

THEOREM 4.1

A vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is continuous at $t = a$ if and only if *all* of f, g and h are continuous at $t = a$.

Theorem 4.1 says that to determine where a vector-valued function is continuous, you need only check the continuity of each component function (something you already know how to do). We demonstrate this in examples 4.3 and 4.4.

EXAMPLE 4.3 Determining Where a Vector-Valued Function is Continuous

Determine the values of t for which the vector-valued function $\mathbf{r}(t) = \langle e^{5t}, \ln(t+1), \cos t\rangle$ is continuous.

Solution From Theorem 4.1, $\mathbf{r}(t)$ will be continuous wherever *all* its components are continuous. We have: e^{5t} is continuous for all t, $\ln(t+1)$ is continuous for $t > -1$ and $\cos t$ is continuous for all t. So, $\mathbf{r}(t)$ is continuous for $t > -1$. ■

EXAMPLE 4.4 A Vector-Valued Function with Infinitely Many Gaps in its Domain

Determine the values of t for which the vector-valued function $\mathbf{r}(t) = \left\langle \tan t, |t+3|, \frac{1}{t-2}\right\rangle$ is continuous.

Solution First, note that $\tan t$ is continuous, except at $t = \dfrac{(2n+1)\pi}{2}$, for $n = 0, \pm 1, \pm 2, \ldots$ (i.e. except at odd multiples of $\frac{\pi}{2}$). The second component $|t+3|$ is continuous for all t (although it's not differentiable at $t = -3$). Finally, the third component $\dfrac{1}{t-2}$ is continuous except at $t = 2$. Since all three components must be continuous in order for $\mathbf{r}(t)$ to be continuous, we have that $\mathbf{r}(t)$ is continuous, except at $t = 2$ and $t = \dfrac{(2n+1)\pi}{2}$, for $n = 0, \pm 1, \pm 2, \ldots$. ■

Recall that in Grade 11, we defined the derivative of a scalar function f to be

$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}.$$

We replace h by Δt, to emphasize that Δt is an **increment** of the variable t. We then have

$$f'(t) = \lim_{\Delta t\to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

We now define the derivative of a vector-valued function in the expected way.

DEFINITION 4.3

The **derivative** $\mathbf{r}'(t)$ of the vector-valued function $\mathbf{r}(t)$ is defined by

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}, \tag{4.2}$$

for any values of t for which the limit exists. When the limit exists for $t = a$, we say that $\mathbf{r}$ is **differentiable** at $t = a$.

Fortunately, you will not need to learn any new differentiation rules, as the derivative of a vector-valued function is found directly from the derivatives of the individual components, as we see in Theorem 4.2.

THEOREM 4.2

Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and suppose that the components f, g and h are all differentiable for some value of t. Then $\mathbf{r}$ is also differentiable at that value of t and its derivative is given by

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle. \tag{4.3}$$

PROOF

From the definition of derivative of a vector-valued function (4.2), we have

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} [\langle f(t + \Delta t), g(t + \Delta t), h(t + \Delta t) \rangle - \langle f(t), g(t), h(t) \rangle] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle f(t + \Delta t) - f(t), g(t + \Delta t) - g(t), h(t + \Delta t) - h(t) \rangle, \\
&= \lim_{\Delta t \to 0} \left\langle \frac{f(t + \Delta t) - f(t)}{\Delta t}, \frac{g(t + \Delta t) - g(t)}{\Delta t}, \frac{h(t + \Delta t) - h(t)}{\Delta t} \right\rangle \\
&= \left\langle \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \to 0} \frac{h(t + \Delta t) - h(t)}{\Delta t} \right\rangle \\
&= \langle f'(t), g'(t), h'(t) \rangle,
\end{aligned}$$

where we have used the definition of limit of a vector-valued function (4.1) and where in the last step we recognized the definition of the derivatives of each of the component functions f, g and h. ■

We illustrate this in example 4.5.

EXAMPLE 4.5 Finding the Derivative of a Vector-Valued Function

Find the derivative of $\mathbf{r}(t) = \langle \sin(t^2), e^{\cos t}, t \ln t \rangle$.

Solution Applying the chain rule to the first two components and the product rule to the third, we have (for $t > 0$):

$$\begin{aligned}
\mathbf{r}'(t) &= \left\langle \frac{d}{dt}[\sin(t^2)], \frac{d}{dt}(e^{\cos t}), \frac{d}{dt}(t \ln t) \right\rangle \\
&= \left\langle \cos(t^2)\frac{d}{dt}(t^2), e^{\cos t}\frac{d}{dt}(\cos t), \frac{d}{dt}(t) \ln t + t\frac{d}{dt}(\ln t) \right\rangle \\
&= \left\langle \cos(t^2)(2t), e^{\cos t}(-\sin t), (1) \ln t + t\frac{1}{t} \right\rangle \\
&= \langle 2t\cos(t^2), -\sin t\, e^{\cos t}, \ln t + 1 \rangle.
\end{aligned}$$

For the most part, to compute derivatives of vector-valued functions, we need only use the already familiar rules for differentiation of scalar functions. There are several special derivative rules, however, which we state in Theorem 4.3.

THEOREM 4.3

Suppose that $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are differentiable vector-valued functions, $f(t)$ is a differentiable scalar function and c is any scalar constant. Then

1. $\frac{d}{dt}[\mathbf{r}(t) + s(t)] = \mathbf{r}'(t) + s'(t)$
2. $\frac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{r}(t)] = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$
4. $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{s}(t)] = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$ and
5. $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{s}(t)] = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t)$.

Notice that parts (iii), (iv) and (v) are the product rules for the various kinds of products we can define. In each of these three cases, it's important to recognize that these follow the same pattern as the usual product rule for the derivative of the product of two scalar functions.

PROOF

(i) For $\mathbf{r}(t) = \langle f_1(t), g_1(t), h_1(t) \rangle$ and $\mathbf{s}(t) = \langle f_2(t), g_2(t), h_2(t) \rangle$, we have from (4.3) and the rules for vector addition that

$$\begin{aligned}
\frac{d}{dt}[\mathbf{r}(t) + s(t)] &= \frac{d}{dt}[\langle f_1(t), g_1(t), h_1(t) \rangle + \langle f_2(t), g_2(t), h_2(t) \rangle] \\
&= \frac{d}{dt}\langle f_1(t) + f_2(t), g_1(t) + g_2(t), h_1(t) + h_2(t) \rangle \\
&= \langle f'_1(t) + f'_2(t), g'_1(t) + g'_2(t), h'_1(t) + h'_2(t) \rangle \\
&= \langle f'_1(t), g'_1(t), h'_1(t) \rangle + \langle f'_2(t), g'_2(t), +h'_2(t) \rangle \\
&= \mathbf{r}'(t) + s'(t).
\end{aligned}$$

(iv) From the definition of dot product and the usual product rule for the product of two scalar functions, we have

$$\begin{aligned}\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{s}(t)] &= \frac{d}{dt}[\langle f_1(t), g_1(t), h_1(t)\rangle \cdot \langle f_2(t), g_2(t), h_2(t)\rangle] \\ &= \frac{d}{dt}[f_1(t)f_2(t) + g_1(t)g_2(t) + h_1(t)h_2(t)] \\ &= f_1'(t)f_2(t) + f_1(t)f_2'(t) + g_1'(t)g_2(t) + g_1(t)g_2'(t) \\ &\quad + h_1'(t)h_2(t) + h_1(t)h_2'(t) \\ &= [f_1'(t)f_2(t) + g_1'(t)g_2(t) + h_1'(t)h_2(t)] \\ &\quad + [f_1(t)f_2'(t) + g_1(t)g_2'(t) + h_1(t)h_2'(t)] \\ &= \mathbf{r}'(t)\cdot\mathbf{s}(t) + \mathbf{r}(t)\cdot\mathbf{s}'(t).\end{aligned}$$

We leave the proofs of (ii), (iii) and (v) as exercises. ■

We say that the curve traced out by the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ on an interval I is **smooth** if $\mathbf{r}'$ is continuous on I and $\mathbf{r}'(t) \neq \mathbf{0}$, except possibly at any endpoints of I. Notice that this says that the curve is smooth provided f', g' and h' are all continuous on I and $f'(t)$, $g'(t)$ and $h'(t)$ are not *all* zero at the same point in I.

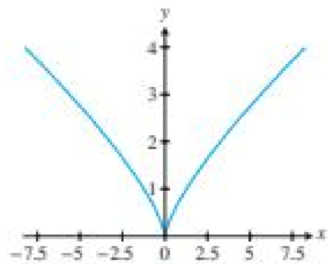


FIGURE 7.26
The curve traced out by $\mathbf{r}(t) = \langle t^3, t^2\rangle$

EXAMPLE 4.6 Determining Where a Curve is Smooth

Determine where the plane curve traced out by the vector-valued function $\mathbf{r}(t) = \langle t^3, t^2\rangle$ is smooth.

Solution We show a graph of the curve in Figure 7.26.

Here, $\mathbf{r}'(t) = \langle 3t^2, 2t\rangle$ is continuous everywhere and $\mathbf{r}'(t) = \mathbf{0}$ if and only if $t = 0$. This says that the curve is smooth in any interval not including $t = 0$. Referring to Figure 7.26, observe that the curve is smooth except at the cusp located at the origin. ■

We next explore an important graphical interpretation of the derivative of a vector-valued function. First, recall that the derivative of a scalar function at a point gives the slope of the tangent line to the curve at that point. For the case of the vector-valued function $\mathbf{r}(t)$, notice that from (4.2), the derivative of $\mathbf{r}(t)$ at $t = a$ is given by

$$\mathbf{r}'(a) = \lim_{\Delta t\to 0}\frac{\mathbf{r}(a+\Delta t) - \mathbf{r}(a)}{\Delta t}.$$

Again, recall that the endpoint of the vector-valued function $\mathbf{r}(t)$ traces out a curve C in $\mathbb{R}^3$. In Figure 7.27a, we show the position vectors $\mathbf{r}(a)$, $\mathbf{r}(a+\Delta t)$ and $\mathbf{r}(a+\Delta t) - \mathbf{r}(a)$, for some fixed $\Delta t > 0$, using our graphical interpretation of vector subtraction.

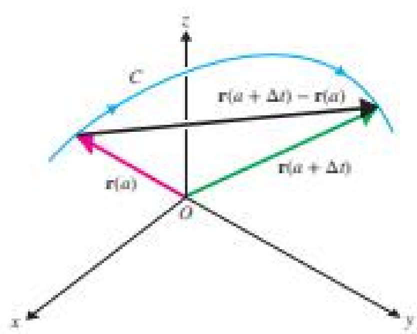


FIGURE 7.27a

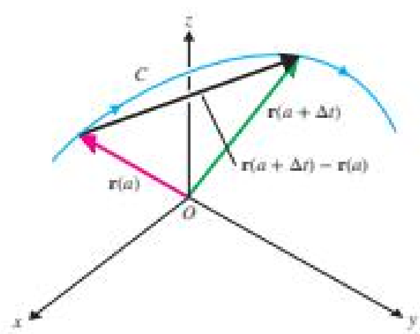


FIGURE 7.27b

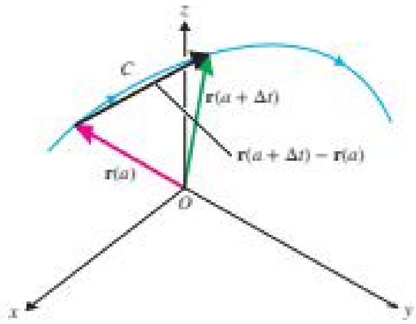


FIGURE 7.27c

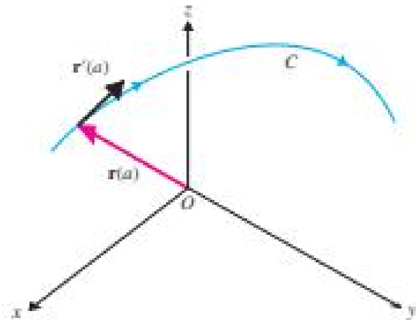


FIGURE 7.27d
The tangent vector $\mathbf{r}'(a)$

(How does the picture differ if $\Delta t < 0$?) Notice that for $\Delta t > 0$, the vector $\frac{\mathbf{r}(a+\Delta t)-\mathbf{r}(a)}{\Delta t}$ points in the same direction as $\mathbf{r}(a+\Delta t)-\mathbf{r}(a)$.

If we take smaller and smaller values of Δt, $\frac{\mathbf{r}(a+\Delta t)-\mathbf{r}(a)}{\Delta t}$ will approach $\mathbf{r}'(a)$. We illustrate this graphically in Figures 7.27b and 7.27c.

As $\Delta t \to 0$, notice that the vector $\frac{\mathbf{r}(a+\Delta t)-\mathbf{r}(a)}{\Delta t}$ approaches a vector that is tangent to the curve C at the terminal point of $\mathbf{r}(a)$, as seen in Figure 7.27d. We refer to $\mathbf{r}'(a)$ as a **tangent vector** to the curve C at the point corresponding to $t = a$. Be sure to observe that $\mathbf{r}'(a)$ lies along the tangent line to the curve at $t = a$ and points in the direction of the orientation of C. (Recognize that Figures 7.27a, 7.27b and 7.27c are all drawn so that $\Delta t > 0$. What changes in each of the figures if $\Delta t < 0$?)

We illustrate this notion for a simple curve in $\mathbb{R}^2$ in example 4.7.

EXAMPLE 4.7 Drawing Position and Tangent Vectors

For $\mathbf{r}(t) = \langle -\cos 2t, \sin 2t\rangle$, plot the curve traced out by the endpoint of $\mathbf{r}(t)$ and draw the position vector and tangent vector at $t = \frac{\pi}{4}$.

Solution First, notice that

$$\mathbf{r}'(t) = \langle 2\sin 2t, 2\sin 2t\rangle.$$

Also, the curve traced out by $\mathbf{r}(t)$ is given parametrically by

$$C: x = -\cos 2t, \quad y = \sin 2t, \quad t \in \mathbb{R}.$$

Observe that here,

$$x^2 + y^2 = \cos^2 2t + \sin^2 t = 1,$$

so that the curve is the circle of radius 1, centered at the origin. Further, from the parameterization, you can see that the orientation is clockwise. The position and tangent vectors at $t = \frac{\pi}{4}$ are given by

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \left\langle -\cos\frac{\pi}{2}, \sin\frac{\pi}{2}\right\rangle = \langle 0, 1\rangle$$

and

$$\mathbf{r}'\left(\frac{\pi}{4}\right) = \left\langle 2\sin\frac{\pi}{2}, 2\cos\frac{\pi}{2}\right\rangle = \langle 2, 0\rangle,$$

respectively. We show the curve, along with the vectors $\mathbf{r}(\frac{\pi}{4})$ and $\mathbf{r}'(\frac{\pi}{4})$ in Figure 7.28 [where we have drawn the initial point of $\mathbf{r}'(\frac{\pi}{4})$ at the terminal point of $\mathbf{r}(\frac{\pi}{4})$]. In particular, you might note that

$$\mathbf{r}\left(\frac{\pi}{4}\right)\cdot\mathbf{r}'\left(\frac{\pi}{4}\right) = 0,$$

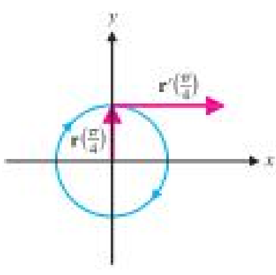


FIGURE 7.28
Position and tangent vectors

so that $\mathbf{r}(\frac{\pi}{4})$ and $\mathbf{r}'(\frac{\pi}{4})$ are orthogonal. In fact, $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal for every t, as follows:

$$\begin{aligned}\mathbf{r}(t)\cdot\mathbf{r}'(t) &= \langle -\cos 2t, \sin 2t\rangle\cdot\langle 2\sin 2t, 2\cos 2t\rangle \\ &= -2\cos 2t\sin 2t + 2\sin 2t\cos 2t = 0.\end{aligned}$$

Were you surprised to find in example 4.7 that the position vector and the tangent vector were orthogonal at every point? As it turns out, this is a special case of a more general result, which we state in Theorem 4.4.

BEYOND FORMULAS

Theorem 4.4 illustrates the importance of good notation. While we could have derived the same result using parametric equations, the vector notation greatly simplifies both the statement and proof of the theorem. The simplicity of the notation allows us to make connections and use our geometric intuition, instead of floundering in a mess of equations. We can visualize the graph of a vector-valued function $\mathbf{r}(t)$ more easily than we can try to keep track of separate expressions for $x(t)$, $y(t)$ and $z(t)$.

THEOREM 4.4

$\|\mathbf{r}(t)\|$ = constant if and only if $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal, for all t.

PROOF

(i) Suppose that $\|\mathbf{r}(t)\| = c$, for some constant c. Recall that

$$\mathbf{r}(t)\cdot\mathbf{r}(t) = \|\mathbf{r}(t)\|^2 = c^2. \tag{4.4}$$

Differentiating both sides of (4.4), we get

$$\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)] = \frac{d}{dt}c^2 = 0.$$

From Theorem 4.3 (4), we now have

$$0 = \frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)] = \mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) = 2\mathbf{r}(t)\cdot\mathbf{r}'(t),$$

so that $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$, as desired.

(ii) We leave the proof of the converse as an exercise. ■

Note that in two dimensions, if $\|\mathbf{r}(t)\| = c$ for all t (where c is a constant), then the curve traced out by the position vector $\mathbf{r}(t)$ must lie on the circle of radius c, centered at the origin. Theorem 4.4 then says that the path traced out by $\mathbf{r}(t)$ lies on a circle centered at the origin if and only if the tangent vector is orthogonal to the position vector at every point on the curve. Likewise, in three dimensions, if $\|\mathbf{r}(t)\| = c$ for all t (where c is a constant), the curve traced out by $\mathbf{r}(t)$ lies on the sphere of radius c centered at the origin. In this case, Theorem 4.4 says that the curve traced out by $\mathbf{r}(t)$ lies on a sphere centered at the origin if and only if the tangent vector is orthogonal to the position vector at every point on the curve.

We conclude this section by making a few straightforward definitions. Recall that when we say that the scalar function $F(t)$ is an antiderivative of the scalar function $f(t)$, we mean that F is any function such that $F'(t) = f(t)$. We now extend this notion to vector-valued functions.

DEFINITION 4.4

The vector-valued function $\mathbf{R}(t)$ is an **antiderivative** of the vector-valued function $\mathbf{r}(t)$ whenever $\mathbf{R}'(t) = \mathbf{r}(t)$.

Notice that if $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ and f, g and h have antiderivatives F, G and H, respectively, then

$$\frac{d}{dt}\langle F(t), G(t), H(t)\rangle = \langle F'(t), G'(t), H'(t)\rangle = \langle f(t), g(t), h(t)\rangle.$$

That is, $\langle F(t), G(t), H(t)\rangle$ is an antiderivative of $\mathbf{r}(t)$. In fact, $\langle F(t)+c_1, G(t)+c_2, H(t)+c_3\rangle$ is also an antiderivative of $\mathbf{r}(t)$, for any choice of constants c_1, c_2 and c_3. This leads us to Definition 4.5.

DEFINITION 4.5

If $\mathbf{R}(t)$ is any antiderivative of $\mathbf{r}(t)$, the **indefinite integral** of $\mathbf{r}(t)$ is defined to be

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector.

As in the scalar case, $\mathbf{R}(t) + \mathbf{c}$ is the most general antiderivative of $\mathbf{r}(t)$. (Why is that?) Notice that this says that

Indefinite integral of a vector-valued function

$$\int \mathbf{r}(t)\,dt = \int \langle f(t), g(t), h(t)\rangle dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle. \tag{4.5}$$

That is, you integrate a vector-valued function by integrating each of the individual components.

EXAMPLE 4.8 Evaluating the Indefinite Integral of a Vector-Valued Function

Evaluate the indefinite integral $\int \left\langle t^2 + 2, \sin 2t, 4te^{t^2} \right\rangle dt$.

Solution From (4.5), we have

$$\begin{aligned}\int \left\langle t^2 + 2, \sin 2t, 4te^{t^2} \right\rangle dt &= \left\langle \int (t^2+2)\,dt, \int \sin 2t\,dt, \int 4te^{t^2}\,dt \right\rangle \\ &= \left\langle \frac{1}{3}t^3 + 2t + c_1, -\frac{1}{2}\cos 2t + c_2, 2e^{t^2} + c_3 \right\rangle \\ &= \left\langle \frac{1}{3}t^3 + 2t, -\frac{1}{2}\cos 2t, 2e^{t^2} \right\rangle + \mathbf{c},\end{aligned}$$

where $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is an arbitrary constant vector. ■

We define the definite integral of a vector-valued function in the obvious way.

DEFINITION 4.6

For the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, we define the **definite integral** of $\mathbf{r}(t)$ on the interval $[a, b]$ by

$$\int_a^b \mathbf{r}(t)\,dt = \int_a^b \langle f(t), g(t), h(t)\rangle dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle. \tag{4.6}$$

This says simply that the definite integral of a vector-valued function $\mathbf{r}(t)$ is the vector whose components are the definite integrals of the corresponding components of $\mathbf{r}(t)$. With this in mind, we now extend the Fundamental Theorem of Calculus to vector-valued functions.

THEOREM 4.5

Suppose that $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$ on the interval $[a, b]$. Then,

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(b) - \mathbf{R}(a).$$

PROOF

The proof is straightforward and we leave this as an exercise. ■

EXAMPLE 4.9 Evaluating the Definite Integral of a Vector-Valued Function

Evaluate $\int_0^1 \langle \sin \pi t, 6t^2 + 4t \rangle dt$.

Solution Notice that an antiderivative for the integrand is

$$\left\langle -\frac{1}{\pi}\cos \pi t, \frac{6t^3}{3} + 4\frac{t^2}{2} \right\rangle = \left\langle -\frac{1}{\pi}\cos \pi t, 2t^3 + 2t^2 \right\rangle.$$

From Theorem 4.5, we have that

$$\int_0^1 \langle \sin \pi t, 6t^2 + 4t \rangle dt = \left\langle -\frac{1}{\pi}\cos \pi t, 2t^3 + 2t^2 \right\rangle \Big|_0^1$$

$$= \left\langle -\frac{1}{\pi}\cos \pi, 2 + 2 \right\rangle - \left\langle -\frac{1}{\pi}\cos 0, 0 \right\rangle$$

$$= \left\langle \frac{1}{\pi} + \frac{1}{\pi}, 4 - 0 \right\rangle = \left\langle \frac{2}{\pi}, 4 \right\rangle.$$

EXERCISES 7.4

WRITING EXERCISES

1. If $\lim_{t\to 0} f(t) = \lim_{t\to 0} g(t) = 0$ and $\lim_{t\to 0} h(t) = \infty$, describe what happens graphically as $t \to 0$ to the curve traced out by $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$. Explain why the limit of $\mathbf{r}(t)$ as $t \to 0$ does not exist.

2. In example 4.3, describe what is happening graphically for $t \le -1$. Explain why we don't say that $\mathbf{r}(t)$ is continuous for $t \le -1$.

3. Suppose that $\mathbf{r}(t)$ is a vector-valued function such that $\mathbf{r}(0) = \langle a, b, c \rangle$ and $\mathbf{r}'(0)$ exists. Imagine zooming in on the curve traced out by $\mathbf{r}(t)$ near the point (a, b, c). Describe what the curve will look like and how it relates to the tangent vector $\mathbf{r}'(0)$.

4. There is a quotient rule corresponding to the product rule in Theorem 4.3, part (3). State this rule and describe in words how you would prove it. Explain why there isn't a quotient rule corresponding to the product rules in parts (4) and (5) of Theorem 4.3.

In exercises 1–6, find the limit if it exists.

1. $\lim_{t\to 0} \langle t^2 - 1, e^{2t}, \sin t \rangle$

2. $\lim_{t\to 1} \left\langle t^2, e^{2t}, \sqrt{t^2 + 2t} \right\rangle$

3. $\lim_{t\to 0} \left\langle \frac{\sin t}{t}, \cos t, \frac{t+1}{t-1} \right\rangle$

4. $\lim_{t\to 1} \left\langle \sqrt{t-1}, t^2 + 3, \frac{t+1}{t-1} \right\rangle$

5. $\lim_{t\to 0} \left\langle \ln t \sqrt{t^2+1}, t - 3 \right\rangle$

6. $\lim_{t\to \pi/2} \langle \cos t, t^2 + 3, \tan t \rangle$

In exercises 7–14, determine all values of t at which the given vector-valued function is continuous.

7. $\mathbf{r}(t) = \left\langle \frac{t+1}{t-2}, \sqrt{t^2 - 1}, 2t \right\rangle$

8. $\mathbf{r}(t) = \left\langle \sin t, \cos t, \frac{3}{t} \right\rangle$

9. $\mathbf{r}(t) = \langle \tan t, \sin t^2, \cos t \rangle$

10. $\mathbf{r}(t) = \langle \cos 5t, \tan t, \ln t \rangle$

11. $\mathbf{r}(t) = \left\langle e^{2t}, \sqrt{t^2 + t}, \frac{2}{t+3} \right\rangle$

12. $\mathbf{r}(t) = \langle \sin t, -\csc t, \cot t \rangle$

13. $\mathbf{r}(t) = \langle \sqrt{t}, \sqrt{4 - t}, \tan t \rangle$

14. $\mathbf{r}(t) = \langle \ln t, \sec t, \sqrt{-t} \rangle$

In exercises 15–20, find the derivative of the given vector-valued function.

15. $\mathbf{r}(t) = \left\langle t^4, \sqrt{t+1}, \frac{3}{t^2} \right\rangle$

16. $\mathbf{r}(t) = \left\langle \frac{t-3}{t+1}, te^{2t}, t^3 \right\rangle$

17. $\mathbf{r}(t) = \langle \sin t, \sin t^2, \cos t \rangle$

18. $\mathbf{r}(t) = \langle \cos 5t, \tan t, 6 \sin t \rangle$

19. $\mathbf{r}(t) = \langle e^{t^2}, t^2 e^{2t}, \sec 2t \rangle$

20. $\mathbf{r}(t) = \left\langle \sqrt{t^2 + 1}, \cos t, e^{-3t} \right\rangle$

In exercises 21–26, determine where the curve traced out by $\mathbf{r}(t)$ is smooth.

21. $\mathbf{r}(t) = \langle t^4 - 2t^2, t^2 - 2t \rangle$

22. $\mathbf{r}(t) = \langle t^2 + t, t^3 \rangle$

23. $\mathbf{r}(t) = \langle \sin t, \cos 2t \rangle$

24. $\mathbf{r}(t) = \langle \cos^2 t, \sin^2 t \rangle$

25. $\mathbf{r}(t) = \langle e^{\sqrt{t}}, t^3 - t \rangle$

26. $\mathbf{r}(t) = \left\langle \frac{t}{t^2 + 4}, \ln(t^2 - 4t) \right\rangle$

In exercises 27–30, sketch the curve traced out by the endpoint of the given vector-valued function and plot position and tangent vectors at the indicated points.

27. $\mathbf{r}(t) = \langle \cos t, \sin t \rangle, t = 0, t = \frac{\pi}{2}, t = \pi$

28. $\mathbf{r}(t) = \langle t, t^2 - 1 \rangle, t = 0, t = 1, t = 2$

29. $\mathbf{r}(t) = \langle \cos t, \sin t \rangle, t = 0, t = \frac{\pi}{2}, t = \pi$

30. $\mathbf{r}(t) = \langle t, t, t^2 - 1 \rangle, t = 0, t = 1, t = 2$

In exercises 31–40, evaluate the given indefinite or definite integral.

31. $\int \langle 3t - 1, \sqrt{t} \rangle dt$

32. $\int \left\langle \frac{3}{t^2}, \frac{4}{t} \right\rangle dt$

33. $\int \langle t \cos 3t, t \sin t^2, e^{2t} \rangle dt$

34. $\int \langle t^2 e^{-t}, \sin^2 t \cos t, \sec^2 t \rangle dt$

35. $\int \left\langle \frac{4}{t^2 - t}, \frac{2t}{t^2 + 1}, \frac{4}{t^2 + 1} \right\rangle dt$

36. $\int \left\langle \frac{2}{\sqrt{1 - t^2}}, \sqrt{t^2 - 1}, t\sqrt{t^2 - 1} \right\rangle dt$

37. $\int_0^1 \langle t^2 - 1, 3t \rangle dt$

38. $\int_1^4 \langle \sqrt{t + 3}, 5(t + 1)^{-1} \rangle dt$

39. $\int_0^2 \left\langle \frac{4}{t + 1}, e^{t-2}, te^t \right\rangle dt$

40. $\int_0^4 \left\langle 2te^{4t}, \frac{4}{t^2 + 5t + 6}, \frac{4t}{t^2 + 1} \right\rangle dt$

In exercises 41–44, find all values of t such that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are perpendicular.

41. $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$

42. $\mathbf{r}(t) = \langle 2 \cos t, \sin t \rangle$

43. $\mathbf{r}(t) = \langle t, t, t^2 - 1 \rangle$

44. $\mathbf{r}(t) = \langle t^2, t, t^2 - 5 \rangle$

45. In each of exercises 41 and 42, show that there are no values of t such that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are parallel.

46. In each of exercises 43 and 44, show that there are no values of t such that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are parallel.

In exercises 47–50, find all values of t such that $\mathbf{r}'(t)$ is parallel to the (a) xy-plane; (b) yz-plane; (c) plane $x = y$.

47. $\mathbf{r}(t) = \langle t, t, t^3 - 3 \rangle$

48. $\mathbf{r}(t) = \langle t^2, t, \sin t^2 \rangle$

49. $\mathbf{r}(t) = \langle \cos t, \sin t, \sin 2t \rangle$

50. $\mathbf{r}(t) = \langle \sqrt{t + 1}, \cos t, t^4 - 8t^2 \rangle$

In exercises 51–54, graph the curve traced out by $\mathbf{r}(t)$.

51. $\mathbf{r}(t) = \langle (2 + \cos 8t)\cos t, (2 + \cos 8t)\sin t, \sin 8t \rangle$

52. $\mathbf{r}(t) = \langle (2 + t \cos 8t)\cos t, (2 + t \cos 8t)\sin t, t \sin 8t \rangle$

53. $\mathbf{r}(t) = \langle 2\cos t \cos 8t, 2\cos t \sin 8t, 2\sin t \rangle$

54. $\mathbf{r}(t) = \langle (-2 + 8 \cos t)\cos(8\sqrt{2}t), (-2 + 8 \cos t)\sin(8\sqrt{2}t), 8 \sin t \rangle$

55. Find all values of a and b for which $\mathbf{r}(t) = \langle \sin t, \sin(at), \sin(bt) \rangle$ is periodic.

56. Find all values of a and b for which $\mathbf{r}(t) = \langle \sin(\pi t), \sin(at), \sin(bt) \rangle$ is periodic.

In exercises 57–60, label as true or false and explain why.

57. If $\mathbf{u}(t) = \frac{1}{\|\mathbf{r}(t)\|}\mathbf{r}(t)$ and $\mathbf{u}(t) \cdot \mathbf{u}'(t) = 0$ then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

58. If $\mathbf{r}(t_0) \cdot \mathbf{r}'(t_0) = 0$ for some t_0, then $\|\mathbf{r}(t)\|$ is constant.

59. If $\int_a^b \mathbf{f}(t) \cdot \mathbf{g}(t)dt = \int_a^b \mathbf{f}(t)dt \cdot \int_a^b \mathbf{g}(t)dt$.

60. If $\mathbf{F}'(t) = \mathbf{f}(t)$, then $\int \mathbf{f}(t)\, dt = \mathbf{F}(t)$.

61. Define the ellipse C with parametric equations $x = a \cos t$ and $y = b \sin t$, for positive constants a and b. For a fixed value of t, define the points $P = (a \cos t, b \sin t)$, $Q = (a \cos(t + \pi/2), b \sin(t + \pi/2))$ and $Q' = (a \cos(t - \pi/2), b \sin(t - \pi/2))$. Show that the vector QQ' (called the **conjugate diameter**) is parallel to the tangent vector to C at the point P. Sketch a graph and show the relationship between P, Q and Q'.

62. Repeat exercise 61 for the general angle θ, so that the points are $P = (a \cos t, b \sin t)$, $Q = (a \cos(t + \theta), b \sin(t + \theta))$ and $Q' = (a \cos(t - \theta), b \sin(t - \theta))$.

63. Find $\frac{d}{dt}[\mathbf{f}(t) \cdot (\mathbf{g}(t) \times \mathbf{h}(t))]$.

64. Find $\frac{d}{dt}[\mathbf{f}(t) \times (\mathbf{g}(t) \times \mathbf{h}(t))]$.

65. Prove Theorem 4.3, part (2).

66. In Theorem 4.3, part (2), replace the scalar product $c\mathbf{r}(t)$ with the dot product $\mathbf{c} \cdot \mathbf{r}(t)$, for a constant vector $\mathbf{c}$ and prove the results.

67. Prove Theorem 4.3, parts (2) and (3).

68. Prove Theorem 4.3, part (5).

69. Prove that if $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal for all t, then $\|\mathbf{r}(t)\| = \text{constant}$ [Theorem 4.4 Proof, part (ii)].

70. Prove Theorem 4.5.

APPLICATION

1. If the curves traced out by $f(t) = \langle t^2 - 4t, \sqrt{t + 5}, 4t \rangle$ and $g(t) = \left\langle \sin(\pi t), \frac{t^2}{t + 1}, 4 + 3t \right\rangle$ represent the paths of two airplanes, determine if they collide.

EXPLORATORY EXERCISES

1. Find all values of t such that $\mathbf{r}'(t) = \mathbf{0}$ for each function: (a) $\mathbf{r}(t) = \langle t, t^2 - 1 \rangle$, (b) $\mathbf{r}(t) = \langle 2\cos t + \sin 2t, 2\sin t + \cos 2t \rangle$, (c) $\mathbf{r}(t) = \langle 2\cos 3t + \sin 5t, 2\sin 3t + \cos 5t \rangle$, (d) $\mathbf{r}(t) = \langle t^2, t^4 - 1 \rangle$ and (e) $\mathbf{r}(t) = \langle t^3, t^6 - 1 \rangle$. Based on your results, conjecture the graphical significance of having the derivative of a vector-valued function equal the zero vector. If $\mathbf{r}(t)$ is the position function of some object in motion, explain the physical significance of having a zero derivative. Explain your geometric interpretation in light of your physical interpretation.

2. A curve C is *smooth* if it is traced out by a vector-valued function $\mathbf{r}(t)$, where $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ for all values of t. Sketch the graph of $r(t) = \left\langle t, \sqrt[3]{t^2} \right\rangle$ and explain why $\mathbf{r}'(t)$ must be continuous. Sketch the graph of $\mathbf{r}(t) = \langle 2\cos t + \sin 2t, 2\sin t + \cos 2t \rangle$ and show that $\mathbf{r}'(0) = \mathbf{0}$. Explain why $\mathbf{r}'(t)$ must be non-zero. Sketch the graph of $\mathbf{r}(t) = \langle 2\cos 3t + \sin 5t, 2\sin 3t + \cos 5t \rangle$ and show that $\mathbf{r}'(t)$ never equals the zero vector. By zooming in on the edges of the graph, show that this curve is accurately described as smooth. Sketch the graphs of $\mathbf{r}(t) = \langle t, t^2 - 1 \rangle$ and $\mathbf{g}(t) = \langle t^2, t^4 - 1 \rangle$ for $t \geq 0$ and observe that they trace out the same curve. Show that $\mathbf{g}'(0) = \mathbf{0}$, but that the curve is smooth at $t = 0$. Explain why this says that the requirement that $\mathbf{r}'(t) \neq \mathbf{0}$ need not hold for *every* $\mathbf{r}(t)$ tracing out the curve. (This requirement needs to hold for only one such $\mathbf{r}(t)$.) Determine which of the following curves are smooth. If the curve is not smooth, identify the graphical characteristic that is "unsmooth": $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, $\mathbf{r}(t) = \left\langle \cos t, \sin t, \sqrt[3]{t^2} \right\rangle$, $\mathbf{r}(t) = \left\langle \tan t, \sin t^2, \cos t \right\rangle$, $\mathbf{r}(t) = \left\langle 5\sin^3 t, \cos^3 t, t \right\rangle$ and $\mathbf{r}(t) = \left\langle \cos t, t^2 e^{-t}, \cos^2 t \right\rangle$.

7.5 MOTION IN SPACE

We are finally at a point where we have sufficient mathematical machinery to describe the motion of an object in a three-dimensional setting. Problems such as this were one of the primary focuses of Newton and many of his contemporaries. Newton used his newly invented calculus to explain all kinds of motion, from the motion of a projectile (such as a ball) hurled through the air, to the motion of the planets. His stunning achievements in this field unlocked mysteries that had eluded the greatest minds for centuries and form the basis of our understanding of mechanics today.

Suppose that an object moves along a curve traced out by the endpoint of the vector-valued function

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle,$$

where t represents time and where $t \in [a, b]$. We observed in section 7.4 that for any given value of t, $\mathbf{r}'(t)$ is a tangent vector pointing in the direction of the orientation of the curve. We can now give another interpretation of this. From (4.3), we have

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle,$$

and the magnitude of this vector-valued function is

$$\|\mathbf{r}'(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}.$$

(Where have you seen this expression before?) Notice that from (3.4), given any number $t_0 \in [a, b]$, the arc length of the portion of the curve from $u = t_0$ up to $u = t$ is given by

$$s(t) = \int_{t_0}^{t} \sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2}\, du. \tag{5.1}$$

Part II of the Fundamental Theorem of Calculus says that if we differentiate both sides of (5.1), we get

$$s'(t) = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} = \|\mathbf{r}'(t)\|.$$

Since $s(t)$ represents arc length, $s'(t)$ gives the instantaneous rate of change of arc length with respect to time, that is, the **speed** of the object as it moves along the curve. So, for any given value of t, $\mathbf{r}'(t)$ is a tangent vector pointing in the direction of the orientation of C (i.e., the direction followed by the object) and whose magnitude gives the speed of the object. So, we call $\mathbf{r}'(t)$ the **velocity** vector, denoted $\mathbf{v}(t)$. Finally, we refer to the derivative of the velocity vector $\mathbf{v}'(t) = \mathbf{r}''(t)$ as the **acceleration** vector, denoted $\mathbf{a}(t)$. When drawing the velocity and acceleration vectors, we locate both of their initial points at the terminal point of $\mathbf{r}(t)$ (i.e., at the point on the curve), as shown in Figure 7.29.

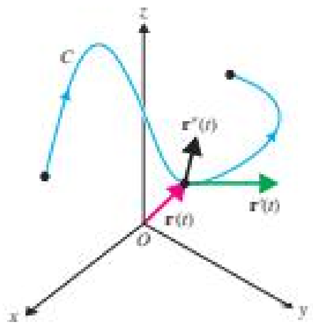


FIGURE 7.29
Position, velocity and acceleration vectors

EXAMPLE 5.1 Finding Velocity and Acceleration Vectors

Find the velocity and acceleration vectors if the position of an object moving in the xy-plane is given by $\mathbf{r}(t) = \langle t^3, 2t^2 \rangle$.

Solution We have

$$v(t) = \mathbf{r}'(t) = \langle 3t^2, 4t \rangle \quad \text{and} \quad a(t) = \mathbf{r}''(t) = \langle 6t, 4 \rangle.$$

In particular, this says that at $t = 1$, we have $\mathbf{r}(1) = \langle 1, 2 \rangle$, $\mathbf{v}(1) = \mathbf{r}'(1) = \langle 3, 4 \rangle$ and $\mathbf{a}(1) = \mathbf{r}''(1) = \langle 6, 4 \rangle$. We plot the curve and these vectors in Figure 7.30. ■

Just as in the case of one-dimensional motion, given the acceleration vector, we can determine the velocity and position vectors, provided we have some additional information.

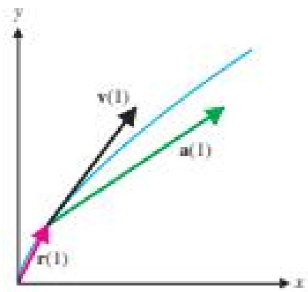


FIGURE 7.30
Position, velocity and acceleration vectors

EXAMPLE 5.2 Finding Velocity and Position from Acceleration

Find the velocity and position of an object at any time t, given that its acceleration is $\mathbf{a}(t) = \langle 6t, 12t + 2, e^t \rangle$, its initial velocity is $\mathbf{v}(0) = \langle 2, 0, 1 \rangle$ and its initial position is $\mathbf{r}(0) = \langle 0, 3, 5 \rangle$.

Solution Since $\mathbf{a}(t) = \mathbf{v}'(t)$, we integrate once to obtain

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}(t)\,dt = \int [6t\mathbf{i} + (12t + 2)\mathbf{j} + e^t\mathbf{k}]\,dt \\ &= 3t^2\mathbf{i} + (6t^2 + 2t)\mathbf{j} + e^t\mathbf{k} + \mathbf{c}_1,\end{aligned}$$

where $\mathbf{c}_1$ is an arbitrary constant vector. To determine the value of $\mathbf{c}_1$, we use the given initial velocity:

$$\langle 2, 0, 1 \rangle = \mathbf{v}(0) = (0)\mathbf{i} + (0)\mathbf{j} + (1)\mathbf{k} + \mathbf{c}_1,$$

so that $\mathbf{c}_1 = \langle 2, 0, 0 \rangle$. This gives us the velocity

$$\mathbf{v}(t) = (3t^2 + 2)\mathbf{i} + (6t^2 + 2t)\mathbf{j} + e^t\mathbf{k}.$$

Since $\mathbf{v}(t) = \mathbf{r}'(t)$, we integrate again, to obtain

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\,dt = \int [(3t^2 + 2)\mathbf{i} + (6t^2 + 2t)\mathbf{j} + e^t\mathbf{k}]dt \\ &= (t^3 + 2t)\mathbf{i} + (2t^3 + t^2)\mathbf{j} + e^t\mathbf{k} + \mathbf{c}_2,\end{aligned}$$

where $\mathbf{c}_2$ is an arbitrary constant vector. We can use the given initial position to determine the value of $\mathbf{c}_2$, as follows:

$$\langle 0, 3, 5 \rangle = \mathbf{r}(0) = (0)\mathbf{i} + (0)\mathbf{j} + (1)\mathbf{k} + \mathbf{c}_2,$$

so that $\mathbf{c}_2 = \langle 0, 3, 4 \rangle$. This gives us the position vector

$$\mathbf{r}(t) = (t^3 + 2t)\mathbf{i} + (2t^3 + t^2 + 3)\mathbf{j} + (e^t + 4)\mathbf{k}.$$

We show the curve and indicate sample vectors for $\mathbf{r}(t)$, $\mathbf{v}(t)$ and $\mathbf{a}(t)$ in Figure 7.31. ■

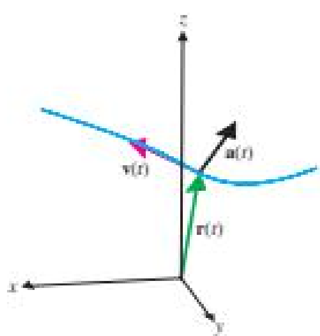


FIGURE 7.31
Position, velocity and acceleration vectors

We have already seen **Newton's second law of motion** several times now. In the case of motion in two or more dimensions, we have the vector form of Newton's second law:

$$\mathbf{F} = m\mathbf{a}.$$

Here, m is the mass, $\mathbf{a}$ is the acceleration vector and $\mathbf{F}$ is the vector representing the net force acting on the object.

EXAMPLE 5.3 Finding the Force Acting on an Object

Find the force acting on an object moving along a circular path of radius b, with constant angular speed.

Solution For simplicity, we will take the circular path to lie in the xy-plane with its center at the origin. Here, by constant **angular speed**, we mean that if θ is the angle made by the position vector and the positive x-axis and t is time (see Figure 7.32a, where the indicated orientation is for the case where $\omega > 0$), then we have that

$$\frac{d\theta}{dt} = \omega \text{ (constant).}$$

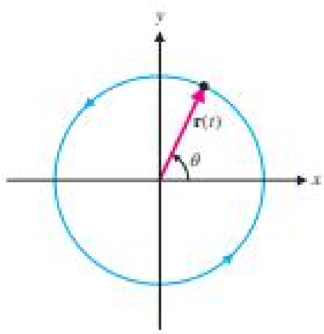


FIGURE 7.32a
Motion along a circle

Notice that this says that $\theta = \omega t + c$, for some constant c. Further, we can think of the circular path as the curve traced out by the endpoint of the vector-valued function

$$\mathbf{r}(t) = \langle b\cos\theta, \text{b} \sin\theta\rangle = \langle \text{b} \cos(\omega \text{t} + \text{c}), \text{b} \sin(\omega \text{t} + \text{c})\rangle.$$

Notice that the path is the same for every value of c. (Think about what the value of c affects.) For simplicity, we take $\theta = 0$ when $t = 0$, so that $\theta = \omega t$ and

$$\mathbf{r}(t) = \langle b\cos\omega t, b\sin\omega t\rangle.$$

Now that we know the position at any time t, we can differentiate to find the velocity and acceleration. We have

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -b\omega\sin\omega t, b\omega\cos\omega t\rangle,$$

so that the speed is $\|\mathbf{v}(t)\| = \omega b$ and

$$\begin{aligned}\mathbf{a}(t) &= \mathbf{v}'(t) = \mathbf{r}''(t) = \langle -b\omega^2\cos\omega t, -b\omega^2\sin\omega t\rangle \\ &= -\omega^2\langle b\cos\omega t, b\sin\omega t\rangle = -\omega^2\mathbf{r}(t).\end{aligned}$$

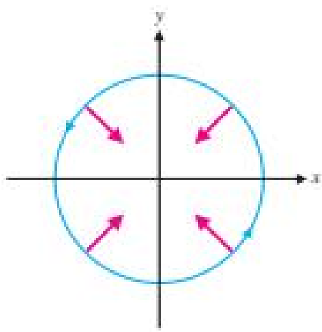


FIGURE 7.32b
Centripetal force

From Newton's second law of motion, we now have

$$\mathbf{F}(t) = m\mathbf{a}(t) = -m\omega^2\mathbf{r}(t).$$

Notice that since $m\omega^2 > 0$, this says that the force acting on the object points in the direction opposite the position vector. That is, at any point on the path, it points in toward the origin. (See Figure 7.32b.) We call such a force a **centripetal** (center-seeking) **force**. ■

Observe that on the circular path of example 5.3, $\|\mathbf{r}(t)\| = b$, so that at every point on the path, the force vector has constant magnitude:

$$\|\mathbf{F}(t)\| = \|-m\omega^2\mathbf{r}(t)\| = m\omega^2\|\mathbf{r}(t)\| = m\omega^2 b.$$

Notice that one consequence of the result $\mathbf{F}(t) = -m\omega^2\mathbf{r}(t)$ from example 5.3 is that the magnitude of the force increases as the rotation rate ω increases. You have experienced this if you have been on a roller coaster with tight turns or loops. The faster you are going, the stronger the force that your seat exerts on you. Alternatively, since the speed is $\|\mathbf{v}(t)\| = \omega b$, the tighter the turn (i.e. the smaller b is), the larger ω must be to obtain a given speed. So, on a roller coaster, a tighter turn requires a larger value of ω, which in turn increases the centripetal force.

Just as we did in the one-dimensional case, we can use Newton's second law of motion to determine the position of an object given only a knowledge of the forces acting on it. We present a simple case in example 5.4.

EXAMPLE 5.4 Analyzing the Motion of a Projectile

A projectile is launched from ground level with an initial speed of 50 meters per second at an angle of $\frac{\pi}{4}$ to the horizontal. Assuming that the only force acting on the object is gravity (i.e. there is no air resistance, etc.), find the maximum altitude, the horizontal range and the speed at impact of the projectile.

TODAY IN MATHEMATICS

Evelyn Boyd Granville (1924–Present)
An American mathematician who has made important contributions to the U.S. space program and the teaching of mathematics. Growing up poor, she and her sister "accepted education as the means to rise above the limitations that a prejudiced society endeavored to place upon us."* She was the second African-American woman to be awarded a PhD in mathematics. Upon graduation, she was employed as a computer programmer and in the early 1960s helped NASA write programs to track the paths of vehicles in space for Project Mercury. She then turned to education, her first love. Granville has coauthored an influential textbook on the teaching of mathematics.

* Granville, E. B. (1989) My Life as a Mathematician, *A Scholarly Journal on Black Women*, Vol. 6, No. 2 (SAGE)

Solution Notice that here, the motion is in a single plane (so that we need only consider two dimensions) and the only force acting on the object is the force of gravity, which acts straight downward. Although this is not constant, it is nearly so at altitudes reasonably close to sea level. We will assume that

$$\mathbf{F}(t) = -mg\mathbf{j},$$

where g is the constant acceleration due to gravity, $g \approx 9.8$ m/s^2 (using the metric value since the initial speed is given in m/s). From Newton's second law of motion, we have

$$-mg\mathbf{j} = \mathbf{F}(t) = m\mathbf{a}(t).$$

We now have

$$\mathbf{v}'(t) = \mathbf{a}(t) = -9.8\mathbf{j}.$$

Integrating this once gives us

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = -9.8t\mathbf{j} + \mathbf{c}_1, \tag{5.2}$$

where $\mathbf{c}_1$ is an arbitrary constant vector. If we knew the initial velocity vector $\mathbf{v}(0)$, we could use this to solve for $\mathbf{c}_1$, but we know only the initial speed (i.e. the magnitude of the velocity vector). Referring to Figure 7.33a, notice that you can read off the components of $\mathbf{v}(0)$, using the definitions of the sine and cosine functions:

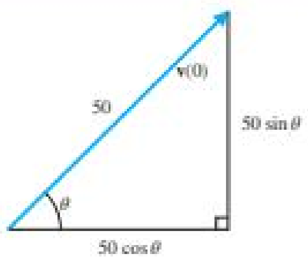


FIGURE 7.33a
Initial velocity vector

$$\mathbf{v}(0) = \left\langle 50\cos\frac{\pi}{4}, 50\sin\frac{\pi}{4}\right\rangle = \left\langle 25\sqrt{2}, 25\sqrt{2}\right\rangle.$$

From (5.2), we now have

$$\left\langle 25\sqrt{2}, 25\sqrt{2}\right\rangle = \mathbf{v}(0) = (-9.8)(0)\mathbf{j} + \mathbf{c}_1 = \mathbf{c}_1.$$

Substituting this back into (5.2), we have

$$\mathbf{v}(t) = -9.8t\mathbf{j} + \left\langle 25\sqrt{2}, 25\sqrt{2}\right\rangle = \left\langle 25\sqrt{2}, 25\sqrt{2} - 9.8t\right\rangle. \tag{5.3}$$

Integrating (5.3) will give us the position vector

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \left\langle 25\sqrt{2}t, 25\sqrt{2}t - 4.9t^2\right\rangle + \mathbf{c}_2,$$

where $\mathbf{c}_2$ is an arbitrary constant vector. Since the initial location was not specified, we choose it to be the origin (for simplicity). This gives us

$$\mathbf{0} = \mathbf{r}(0) = \mathbf{c}_2,$$

so that

$$\mathbf{r}(t) = \left\langle 25\sqrt{2}t, 25\sqrt{2}t - 4.9t^2\right\rangle. \tag{5.4}$$

We show a graph of the path of the projectile in Figure 7.33b. Now that we have found expressions for the position and velocity vectors for any time, we can answer the physical questions. Notice that the maximum altitude occurs at the instant when the object stops moving up (just before it starts to fall). This says that the vertical ($\mathbf{j}$) component of velocity must be zero. From (5.3), we get

$$0 = 25\sqrt{2} - 9.8t,$$

so that the time at the maximum altitude is

$$t = \frac{25\sqrt{2}}{9.8}.$$

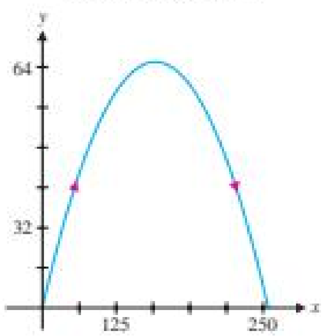


FIGURE 7.33b
Path of a projectile

The maximum altitude is then found from the vertical component of the position vector at this time:

$$\text{Maximum altitude} = 25\sqrt{2}t - 4.9t^2\Big|_{t=\frac{25\sqrt{2}}{9.8}} = 25\sqrt{2}\left(\frac{25\sqrt{2}}{9.8}\right) - 4.9\left(\frac{25\sqrt{2}}{9.8}\right)^2 \text{m}$$

$$= \frac{1250}{19.6}\text{m} = 63.8 \text{ m}.$$

To determine the horizontal range, we first need to determine the instant at which the object strikes the ground. Notice that this occurs when the vertical component of the position vector is zero (i.e., when the height above the ground is zero). From (5.4), we see that this occurs when

$$0 = 25\sqrt{2}t - 4.9t^2 = t(25\sqrt{2} - 4.9t).$$

There are two solutions of this equation: $t = 0$ (the time at which the projectile is launched) and $t = \frac{25\sqrt{2}}{4.9}$ (the time of impact). The horizontal range is then the horizontal component of position at this time:

$$\text{Range} = 25\sqrt{2}t\Big|_{t=\frac{25\sqrt{2}}{4.9}} = (25\sqrt{2})\left(\frac{25\sqrt{2}}{4.9}\right) = \frac{1250}{4.9} = 255.1\text{m}.$$

Finally, the speed at impact is the magnitude of the velocity vector at the time of impact:

$$\left\|\mathbf{v}\left(\frac{25\sqrt{2}}{4.9}\right)\right\| = \left\|\left\langle 25\sqrt{2}, 25\sqrt{2} - 9.8\left(\frac{25\sqrt{2}}{4.9}\right)\right\rangle\right\|$$
$$= \|\langle 25\sqrt{2}, -25\sqrt{2}\rangle\| = 50 \text{ m/s}.$$

You might have noticed in example 5.4 that the speed at impact was the same as the initial speed. Don't expect this to always be the case. Generally, this will be true only for a projectile of constant mass that is fired from ground level and returns to ground level and that is not subject to air resistance or other forces.

Equations of Motion

We now derive the equations of motion for a projectile in a slightly more general setting than that described in example 5.4. Consider a projectile launched from an altitude h above the ground at an angle θ to the horizontal and with initial speed v_0. We can use Newton's second law of motion to determine the position of the projectile at any time t and once we have this, we can answer any questions about the motion.

We again start with Newton's second law and assume that the only force acting on the object is gravity. We have

$$-mg\mathbf{j} = \mathbf{F}(t) = m\mathbf{a}(t).$$

This gives us (as in example 5.4)

$$\mathbf{v}'(t) = \mathbf{a}(t) = -g\mathbf{j}. \tag{5.5}$$

Integrating (5.5) gives us

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = -gt\mathbf{j} + \mathbf{c}_1, \tag{5.6}$$

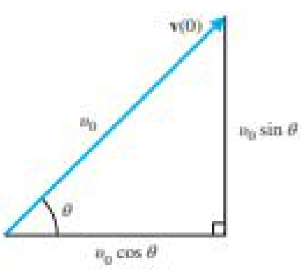


FIGURE 7.34a
Initial velocity

where $\mathbf{c}_1$ is an arbitrary constant vector. In order to solve for $\mathbf{c}_1$, we need the value of $\mathbf{v}(t)$ for some t, but we are given only the initial speed v_0 and the angle at which the projectile is fired. Notice that from the definitions of sine and cosine, we can read off the components of $\mathbf{v}(0)$ from Figure 7.34a. From this and (5.6), we have

$$\langle v_0\cos\theta, v_0\sin\theta\rangle = \mathbf{v}(0) = \mathbf{c}_1.$$

This gives us the velocity vector

$$\mathbf{v}(t) = \langle v_0\cos\theta, v_0\sin\theta - gt\rangle. \tag{5.7}$$

Since $\mathbf{r}'(t) = \mathbf{v}(t)$, we integrate (5.7) to get the position

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \left\langle (v_0\cos\theta)t, (v_0\sin\theta)t - \frac{gt^2}{2}\right\rangle + \mathbf{c}_2.$$

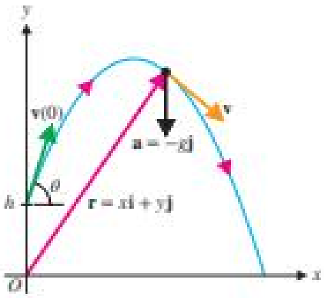


FIGURE 7.34b
Path of the projectile

To solve for $\mathbf{c}_2$, we want to use the initial position $\mathbf{r}(0)$, but we're not given it. We're told only that the projectile starts from an altitude h above the ground. So, we choose the origin to be the point on the ground directly below the launching point, giving us

$$\langle 0, h\rangle = \mathbf{r}(0) = \mathbf{c}_2,$$

so that

$$\mathbf{r}(t) = \left\langle (v_0\cos\theta)t, (v_0\sin\theta)t - \frac{gt^2}{2}\right\rangle + \langle 0, h\rangle$$

$$= \left\langle (v_0\cos\theta)t, h + (v_0\sin\theta)t - \frac{gt^2}{2}\right\rangle. \quad (5.8)$$

Notice that the path traced out by $\mathbf{r}(t)$ (from $t = 0$ until impact) is a portion of a parabola. (See Figure 7.34b.)

Now that we have derived (5.7) and (5.8), we have all we need to answer any further questions about the motion. For instance, the maximum altitude occurs at the time at which the vertical component of velocity is zero (i.e. at the time when the projectile stops rising). From (5.7), we solve

$$0 = v_0\sin\theta - gt,$$

so that the time at which the maximum altitude is reached is given by

Time to reach maximum altitude

$$t_{max} = \frac{v_0\sin\theta}{g}.$$

The maximum altitude is then the vertical component of the position vector at this time. From (5.8), we have

Maximum altitude

$$\text{Maximum altitude} = h + (v_0\sin\theta)t - \frac{gt^2}{2}\bigg|_{t=t_{max}}$$

$$= h + (v_0\sin\theta)\left(\frac{v_0\sin\theta}{g}\right) - \frac{g}{2}\left(\frac{v_0\sin\theta}{g}\right)^2$$

$$= h + \frac{1}{2}\frac{v_0^2\sin^2\theta}{g}.$$

In all of the foregoing analysis, we left the constant acceleration due to gravity as g. You will usually use one of the two approximations:

$$g \approx 32 \text{ ft/s}^2 \quad \text{or} \quad g \approx 9.8 \text{ m/s}^2.$$

When using any other units, simply adjust the units to feet or meters and the time scale to seconds or make the corresponding adjustments to the value of g.

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We next use the calculus to analyze the motion of a body rotating about an axis. (For instance, think about the motion of a gymnast performing a complicated routine.) We use a rotational version of Newton's second law to analyze such motion. Torque (denoted by $\boldsymbol{\tau}$) is defined in section 7.2. In the case of an object rotating in two dimensions, the torque has magnitude (denoted by $\tau = \|\boldsymbol{\tau}\|$) given by the product of the force acting in the direction of the motion and the distance from the rotational center. The moment of inertia I of a body is a measure of how an applied force will cause the object to change its rate of rotation. This is determined by the mass and the distance of the mass from the center of rotation. In rotational motion, the primary variable that we track is an angle of displacement, denoted by θ. For a rotating body, the angle measured from some fixed ray changes with time t, so that the angle is a function $\theta(t)$. We define the **angular velocity** to be $\omega(t) = \theta'(t)$ and the **angular acceleration** to be $\alpha(t) = \omega'(t) = \theta''(t)$. The equation of rotational motion is then

$$\tau = I\alpha. \quad (5.9)$$

Notice how closely this resembles Newton's second law, $F = ma$. The calculus used in example 5.5 should look familiar.

EXAMPLE 5.5 The Rotational Motion of a Merry-Go-Round

A stationary merry-go-round of radius 2 m is started in motion by a push consisting of a force of 50 N on the outside edge, tangent to the circular edge of the merry-go-round, for 1 second. The moment of inertia of the merry-go-round is $I = 25$. Find the resulting angular velocity of the merry-go-round.

Solution We first compute the torque of the push. The force is applied 2 m from the center of rotation, so that the torque has magnitude

$$\tau = \text{(Force)(Distance from axis of rotation)} = (50)(2) = 100 \text{ N.m}$$

From (5.9), we have

$$100 = 25\alpha,$$

so that $\alpha = 4$. Since the force is applied for 1 second, this equation holds for $0 \le t \le 1$. Integrating both sides of the equation $\omega' = \alpha$ from $t = 0$ to $t = 1$, we have by the Fundamental Theorem of Calculus that

$$\omega(1) - \omega(0) = \int_0^1 \alpha \, dt = \int_0^1 4 \, dt = 4. \tag{5.10}$$

If the merry-go-round is initially stationary, then $\omega(0) = 0$ and (5.10) becomes simply $\omega(1) = 4$ rad/s. ■

Notice that we could draw a more general conclusion from (5.10). Even if the merry-go-round is already in motion, applying a force of 50 N tangentially to the edge for 1 second will increase the rotation rate by 4 rad/s.

For rotational motion in three dimensions, the calculations are somewhat more complicated. Recall that we had defined the torque τ due to a force $\mathbf{F}$ applied at position $\mathbf{r}$ to be

$$\tau = \mathbf{r} \times \mathbf{F}.$$

Example 5.6 relates torque to angular momentum. The **linear momentum p** of an object of mass m with velocity $\mathbf{v}$ is given by $\mathbf{p} = m\mathbf{v}$. The **angular momentum L** is defined by $\mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t)$.

EXAMPLE 5.6 Relating Torque and Angular Momentum

Show that torque equals the derivative of angular momentum.

Solution From the definition of angular momentum and the product rule for the derivative of a cross product [Theorem 4.3 (5)], we have

$$\begin{aligned}
\mathbf{L}'(t) &= \frac{d}{dt}[\mathbf{r}(t) \times m\mathbf{v}(t)] \\
&= \mathbf{r}'(t) \times m\mathbf{v}(t) + \mathbf{r}(t) \times m\mathbf{v}'(t) \\
&= \mathbf{v}(t) \times m\mathbf{v}(t) + \mathbf{r}(t) \times m\mathbf{a}(t).
\end{aligned}$$

Notice that the first term on the right-hand side is the zero vector, since it is the cross product of parallel vectors. From Newton's second law, we have $\mathbf{F}(t) = m\mathbf{a}(t)$, so we have

$$\mathbf{L}'(t) = \mathbf{r}(t) \times m\mathbf{a}(t) = \mathbf{r} \times \mathbf{F} = \tau. \quad ■$$

From this result, it is a short step to the principle of **conservation of angular momentum**, which states that, in the absence of torque, angular momentum remains constant. This is left as an exercise.

In example 5.7, we examine a fully three-dimensional projectile motion problem for the first time.

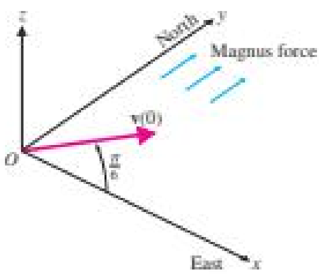


FIGURE 7.35a
The initial velocity and Magnus force vectors

EXAMPLE 5.7 Analyzing the Motion of a Projectile in Three Dimensions

A projectile of mass 1 kg is launched from ground level toward the east at 200 meters/second, at an angle of $\frac{\pi}{6}$ to the horizontal. If the spinning of the projectile applies a steady northerly Magnus force of 2 newtons to the projectile, find the landing location of the projectile and its speed at impact.

Solution Notice that because of the Magnus force, the motion is fully three-dimensional. We orient the x-, y- and z-axes so that the positive y-axis points north, the positive x-axis points east and the positive z-axis points up, as in Figure 7.35a, where we also show the initial velocity vector and vectors indicating the Magnus force. The two forces acting on the projectile are gravity (in the negative z-direction with magnitude $9.8\ m = 9.8$ newtons) and the Magnus force (in the y-direction with magnitude 2 newtons). Newton's second law is $\mathbf{F} = m\mathbf{a} = \mathbf{a}$. We have

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, 2, -9.8 \rangle$$

Integrating gives us the velocity function

$$\mathbf{v}(t) = \langle 0, 2t, -9.8t \rangle + \mathbf{c}_1, \tag{5.11}$$

where $\mathbf{c}_1$ is an arbitrary constant vector. Note that the initial velocity is

$$\mathbf{v}(0) = \left\langle 200 \cos\frac{\pi}{6}, 0, 200 \sin\frac{\pi}{6} \right\rangle = \left\langle 100\sqrt{3}, 0, 100 \right\rangle.$$

From (5.11), we now have

$$\left\langle 100\sqrt{3}, 0, 100 \right\rangle = \mathbf{v}(0) = \mathbf{c}_1,$$

which gives us

$$\mathbf{v}(t) = \left\langle 100\sqrt{3}, 2t, 100 - 9.8t \right\rangle$$

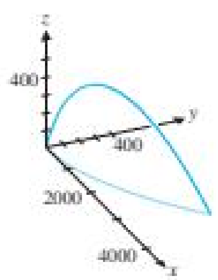


FIGURE 7.35b
Path of the projectile

We integrate this to get the position vector:

$$\mathbf{r}(t) = \left\langle 100\sqrt{3}\,t, t^2, 100t - 4.9t^2 \right\rangle + \mathbf{c}_2,$$

for a constant vector $\mathbf{c}_2$. Taking the initial position to be the origin, we get

$$\mathbf{0} = \mathbf{r}(0) = \mathbf{c}_2,$$

so that

$$\mathbf{r}(t) = \left\langle 100\sqrt{3}t, t^2, 100t - 4.9t^2 \right\rangle. \tag{5.12}$$

Note that the projectile strikes the ground when the $\mathbf{k}$ component of position is zero. From (5.12), we have that this occurs when

$$0 = 100t - 4.9t^2 = t(100 - 4.9t).$$

So, the projectile is on the ground when $t = 0$ (time of launch) and when $t = \frac{100}{4.9} \approx 20.4$ seconds (the time of impact). The location of impact is then the endpoint of the vector $r\left(\frac{100}{4.9}\right) \approx \langle 3534.8, 416.5, 0 \rangle$ and the speed at impact is

$$\left\| \mathbf{v}\left(\frac{100}{4.9}\right) \right\| \approx 204 \text{ m/s}.$$

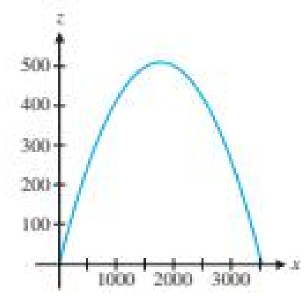


FIGURE 7.35c
Projection of path onto the xz-plane

We show a computer-generated graph of the path of the projectile in Figure 7.35b, where we also indicate the shadow made by the path of the projectile on the ground. In Figure 7.35c, we show the projection of the projectile's path onto the xz-plane. Observe that this parabola is analogous to the parabola shown in Figure 7.33b. ■

EXERCISES 7.5

WRITING EXERCISES

1. Explain why it makes sense in example 5.4 that the speed at impact equals the initial speed. (Hint: What force would slow the object down?) If the projectile were launched from above ground, discuss how the speed at impact would compare to the initial speed.

2. If we had taken air drag into account in example 5.4, discuss how the calculated speed at impact would have changed.

3. In this section, we assumed that the acceleration due to gravity is constant. By contrast, air resistance is a function of velocity. (The faster the object goes, the more air resistance there is.) Explain why including air resistance in our Newton's law model of projectile motion would make the mathematics *much* more complicated.

4. In example 5.7, use the x- and y-components of the position function to explain why the projection of the projectile's path onto the xy-plane would be a parabola. The projection onto the xz-plane is also a parabola. Discuss whether or not the path in Figure 7.35b is a parabola. If you were watching the projectile, would the path appear to be parabolic?

In exercises 1–6, find the velocity and acceleration functions for the given position function.

1. $\mathbf{r}(t) = \langle 5\cos 2t, 5\sin 2t\rangle$

2. $\mathbf{r}(t) = \langle 2\cos t + \sin 2t, 2\sin t + \cos 2t\rangle$

3. $\mathbf{r}(t) = \langle 25t, -16t^2 + 15t + 5\rangle$

4. $\mathbf{r}(t) = \langle 25te^{-2t}, -16t^2 + 10t + 20\rangle$

5. $\mathbf{r}(t) = \left\langle 4te^{-2t}, \sqrt{t^2+1}, t/(t^2+1)\right\rangle$

6. $\mathbf{r}(t) = \langle 3e^{-3t^2}, \sin 2t, e^{-t} + \sin t\rangle$

In exercises 7–14, find the position function from the given velocity or acceleration function.

7. $\mathbf{v}(t) = \langle 10, -32t + 4\rangle$, $\mathbf{r}(0) = \langle 3, 8\rangle$

8. $\mathbf{v}(t) = \langle 4t, t^2 - 1\rangle$, $\mathbf{r}(0) = \langle 10, -2\rangle$

9. $\mathbf{a}(t) = \langle 0, -32\rangle$, $\mathbf{v}(0) = \langle 5, 0\rangle$, $\mathbf{r}(0) = \langle 0, 16\rangle$

10. $\mathbf{a}(t) = \langle t, \sin t\rangle$, $\mathbf{v}(0) = \langle 2, -6\rangle$, $\mathbf{r}(0) = \langle 10, 4\rangle$

11. $\mathbf{v}(t) = \left\langle 12\sqrt{t}, t/(t^2+1), te^{-t}\right\rangle$, $\mathbf{r}(0) = \langle 8, -2, 1\rangle$

12. $\mathbf{v}(t) = \langle te^{-t^2}, 1/(t^2 + I), 2/(t+1)\rangle$, $\mathbf{r}(0) = \langle 4, 0, -3\rangle$

13. $\mathbf{a}(t) = \langle t, 0, -16\rangle$, $\mathbf{v}(0) = \langle 12, -4, 0\rangle$, $\mathbf{r}(0) = \langle 5, 0, 2\rangle$

14. $\mathbf{a}(t) = \langle e^{-3t}, t, \sin t\rangle$, $\mathbf{v}(0) = \langle 4, -2, 4\rangle$, $\mathbf{r}(0) = \langle 0, 4, -2\rangle$

In exercises 15–18, find the magnitude of the net force on an object of mass 10 kg with the given position function (in units of meters and seconds).

15. $\mathbf{r}(t) = \langle 4\cos 2t, 4\sin 2t\rangle$

16. $\mathbf{r}(t) = \langle 3\cos 5t, 3\sin 5t\rangle$

17. $\mathbf{r}(t) = \langle 6\cos 4t, 6\sin 4t\rangle$

18. $\mathbf{r}(t) = \langle 2\cos 3t, 2\sin 3t\rangle$

In exercises 19–22, find the net force acting on an object of mass m with the given position function (in units of meters and seconds).

19. $\mathbf{r}(t) = \langle 3\cos 2t, 5\sin 2t\rangle$, $m = 10$ kg

20. $\mathbf{r}(t) = \langle 3\cos 4t, 2\sin 5t\rangle$, $m = 10$ kg

21. $\mathbf{r}(t) = \langle 3t^2 + t, 3t - 1\rangle$, $m = 20$ kg

22. $\mathbf{r}(t) = \langle 20t - 3, -16t^2 + 2t + 30\rangle$, $m = 20$ kg

In exercises 23–28, a projectile is fired with initial speed v_0 m/s from a height of h meters at an angle of θ above the horizontal. Assuming that the only force acting on the object is gravity, find the maximum altitude, horizontal range and speed at impact.

23. $v_0 = 98$, $h = 0$, $\theta = \frac{\pi}{3}$

24. $v_0 = 98$, $h = 0$, $\theta = \frac{\pi}{6}$

25. $v_0 = 49$, $h = 0$, $\theta = \frac{\pi}{4}$

26. $v_0 = 98$, $h = 0$, $\theta = \frac{\pi}{4}$

27. $v_0 = 60$, $h = 10$, $\theta = \frac{\pi}{3}$

28. $v_0 = 60$, $h = 20$, $\theta = \frac{\pi}{3}$

29. Based on your answers to exercises 25 and 26, what effect does doubling the initial speed have on the horizontal range?

30. The angles $\frac{\pi}{3}$ and $\frac{\pi}{6}$ are symmetric about $\frac{\pi}{4}$; that is, $\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{3} - \frac{\pi}{4}$. Based on your answers to exercises 23 and 24, how do horizontal ranges for symmetric angles compare?

31. Beginning with Newton's second law of motion, derive the equations of motion for a projectile fired from altitude h above the ground at an angle θ to the horizontal and with initial speed v_0.

32. For the general projectile of exercise 31, with $h = 0$, (a) show that the horizontal range is $\frac{v_0^2 \sin 2\theta}{g}$ and (b) find the angle that produces the maximum horizontal range.

33. A force of 100 N is applied to the outside of a stationary merry-go-round of radius 2 N for 0.5 second. The moment of inertia is $I = 10$. Find the resultant change in angular velocity of the merry-go-round.

34. A merry-go-round of radius 2 m and moment of inertia $I = 10$ rotates at 4 rad/s. Find the constant force needed to stop the merry-go-round in 2 seconds.

35. A golfer rotates a club with constant angular acceleration α through an angle of π radians. If the angular velocity increases from 0 to 15 rad/s, find α.

36. For the golf club in exercise 35, find the increase in angular velocity if the club is rotated through an angle of $\frac{3\pi}{2}$ radians

with the same angular acceleration. Describe one advantage of a long swing.

37. Softball pitchers such as Jennie Finch often use a double windmill to generate arm speed. At a constant angular acceleration, compare the speeds obtained rotating through an angle of π versus rotating through an angle of 3π.

38. As the softball in exercise 37 rotates, its linear speed v is related to the angular velocity ω by $v = r\omega$, where r is the distance of the ball from the center of rotation. The pitcher's arm should be fully extended. Explain why this is a good technique for throwing a fast pitch.

39. Use the result of example 5.6 to prove the **Law of Conservation of Angular Momentum**: if there is zero (net) torque on an object, its angular momentum remains constant.

40. Prove the **Law of Conservation of Linear Momentum**: if there is zero (net) force on an object, its linear momentum remains constant.

41. If acceleration is parallel to position ($\mathbf{a} \| \mathbf{r}$), show that there is no torque. Explain this result in terms of the change in angular momentum. (Hint: If $\mathbf{a} \| \mathbf{r}$, would angular velocity or linear velocity be affected?)

42. If the acceleration $\mathbf{a}$ is constant, show that $\mathbf{L}'' = \mathbf{0}$.

43. Suppose an airplane is acted on by three forces: gravity, wind and engine thrust. Assume that the force vector for gravity is $m\mathbf{g} = m\langle 0, 0, -32\rangle$, the force vector for wind is $\mathbf{w} = \langle 0, 1, 0\rangle$ for $0 \le t \le 1$ and $\mathbf{w} = \langle 0, 2, 0\rangle$ for $t > 1$, and the force vector for engine thrust is $\mathbf{e} = \langle 2t, 0, 24\rangle$. Newton's second law of motion gives us $m\mathbf{a} = m\mathbf{g} + \mathbf{w} + \mathbf{e}$. Assume that $m = 1$ and the initial velocity vector is $\mathbf{v}(0) = \langle 100, 0, 10\rangle$. Show that the velocity vector for $0 \le t \le 1$ is $\mathbf{v}(t) = \langle t^2 + 100, t, 10 - 8t\rangle$. For $t > 1$, integrate the equation $\mathbf{a} = \mathbf{g} + \mathbf{w} + \mathbf{e}$, to get $\mathbf{v}(t) = \langle t^2 + a, 2t + b, -8t + c\rangle$, for constants a, b and c. Explain (on physical grounds) why the function $\mathbf{v}(t)$ should be continuous and find the values of the constants that make it so. Show that $\mathbf{v}(t)$ is not differentiable. Given the nature of the force function, why does this make sense?

44. Find the position function for the airplane in exercise 43.

45. Find a vector equation for position and the point of impact if the projectile in exercise 23 is launched in the plane $y = x$.

46. Find a vector equation for position and the point of impact if the projectile in exercise 27 is launched in the plane $y = 2x$.

47. Given that the horizontal range of a projectile launched from the ground is 100 m and the launch angle is $\frac{\pi}{3}$, find the initial speed.

48. Given that the horizontal range of a projectile launched from the ground is 240 m and the launch angle is 30°, find the initial speed.

APPLICATIONS

1. A roller coaster is designed to travel a circular loop of radius 30 m. If the riders feel weightless at the top of the loop, what is the speed of the roller coaster?

2. A roller coaster travels at variable angular speed $\omega(t)$ and radius $r(t)$ but constant speed $c = \omega(t)r(t)$. For the centripetal force $F(t) = m\omega^2(t)r(t)$, show that $F'(t) = m\omega(t)r(t)\omega'(t)$. Conclude that entering a tight curve with $r'(t) < 0$ but maintaining constant speed, the centripetal force increases.

3. A jet pilot executing a circular turn experiences an acceleration of "5 g's" (that is, $\|\mathbf{a}\| = 5\,g$). If the jet's speed is 900 km/h, what is the radius of the turn?

4. For the jet pilot of exercise 3, how many g's would be experienced if the speed were 1800 km/hr?

5. Example 5.3 is a model of a satellite orbiting the Earth. In this case, the force $\mathbf{F}$ is the gravitational attraction of the Earth on the satellite. The magnitude of the force is $\frac{mMG}{b^2}$, where m is the mass of the satellite, M is the mass of the Earth and G is the universal gravitational constant. Using example 5.3, this should be equal to $m\omega^2 b$. For a **geosynchronous orbit**, the frequency ω is such that the satellite completes one orbit in one day. (By orbiting at the same rate as the Earth spins, the satellite can remain directly above the same point on the Earth.) For a sidereal day of 23 hours, 56 minutes and 4 seconds, find ω. Using $MG \approx 39.87187 \times 10^{13}$ N-m²/kg, find b for a geosynchronous orbit (the units of b will be m).

6. Example 5.3 can also model a jet executing a turn. For a jet traveling at 1000 km/h, find the radius b such that the pilot feels 7 g's of force; that is, the magnitude of the force is 7 mg.

EXPLORATORY EXERCISES

1. A ball rolls off a table of height 1.5 m. Its initial velocity is horizontal with speed v_0. Determine where the ball hits the ground and the velocity vector of the ball at the moment of impact. Find the angle between the horizontal and the impact velocity vector. Next, assume that the next bounce of the ball starts with the ball being launched from the ground with initial conditions determined by the impact velocity. The launch speed equals 0.6 times the impact speed (so the ball won't bounce forever) and the launch angle equals the (positive) angle between the horizontal and the impact velocity vector. Using these conditions, determine where the ball next hits the ground. Continue on to find the third point at which the ball bounces.

2. In many sports such as golf and ski jumping, it is important to determine the range of a projectile on a slope. Suppose that the ground passes through the origin and slopes at an angle of α to the horizontal. Show that an equation of the ground is $y = -(\tan\alpha)x$. An object is launched at height $h = 0$ with initial speed v_0 at an angle of θ from the horizontal. Referring to exercise 31, show that the landing condition is now $y = -(\tan\alpha)x$. Find the x-coordinate of the landing point and show that the range (the distance along the ground) is given by $R = \frac{2}{g}v_0^2\sec\alpha\cos\theta(\sin\theta + \tan\alpha\cos\theta)$. Use trigonometric identities to rewrite this as $R = \frac{1}{g}v_0^2\sec^2\alpha[\sin\alpha + \sin(\alpha + 2\theta)]$. Use this formula to find the value of θ that maximizes the range. For flat ground ($\alpha = 0$), the optimal angle is 45°. State an easy way of taking the value of α (say, $\alpha = 10°$ or $\alpha = -8°$) and adjusting from 45° to the optimal angle.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Vector	function	acceleration
Dot product	Angular velocity	Continuous $\mathbf{F}(x)$
Cross product	Arc length	Speed
Orthogonal planes	Velocity vector	Angular
Vector-valued	Angular	momentum

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. The dot product $\mathbf{a} \cdot \mathbf{b} = 0$ implies that either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.
2. If $\mathbf{a} \cdot \mathbf{b} > 0$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is less than $\frac{\pi}{2}$.
3. The cross product can be used to determine the angle between vectors.
4. The graph of the vector-valued function $\langle \cos t, \sin t, f(t) \rangle$, for some function f lies on the circle $x^2 + y^2 = 1$.
5. For vector-valued functions, derivatives are found component by component and all of the usual rules (product, quotient, chain) apply.
6. The derivative of a vector-valued function gives the slope of the tangent line.
7. Newton's laws apply only to straight-line motion and not to rotational motion.

In exercises 1–4, compute the cross product a × b.

1. $\mathbf{a} = \langle 1, -2, 1 \rangle$, $\mathbf{b} = \langle 2, 0, 1 \rangle$
2. $\mathbf{a} = \langle 1, -2, 0 \rangle$, $\mathbf{b} = \langle 1, 0, -2 \rangle$
3. $\mathbf{a} = 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
4. $\mathbf{a} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$

In exercises 5 and 6, find two unit vectors orthogonal to both given vectors.

5. $\mathbf{a} = 2\mathbf{i} + \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ 6. $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$

In exercises 7 and 8, sketch the curve and plot the values of the vector-valued function.

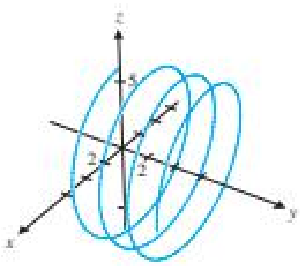
7. $\mathbf{r}(t) = \langle t^2, 2 - t^2, 1 \rangle$, $t = 0$, $t = 1$, $t = 2$
8. $\mathbf{r}(t) = \langle \sin t, 2\cos t, 3 \rangle$, $t = -\pi$, $t = 0$, $t = \pi$

In exercises 9–18, sketch the curve traced out by the given vector-valued function.

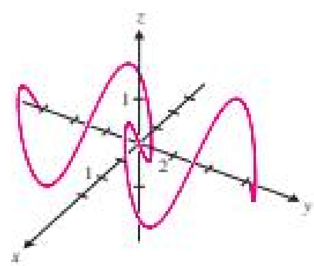
9. $\mathbf{r}(t) = \langle 3\cos t + 1, \sin t \rangle$
10. $\mathbf{r}(t) = \langle 2\sin t, \cos t + 2 \rangle$
11. $\mathbf{r}(t) = \langle 3\cos t + 2\sin 3t, 3\sin t + 2\cos 3t \rangle$
12. $\mathbf{r}(t) = \langle 3\cos t + \sin 3t, 3\sin t + \cos 3t \rangle$
13. $\mathbf{r}(t) = \langle 2\cos t, 3, 3\sin t \rangle$
14. $\mathbf{r}(t) = \langle 3\cos t, -2, 2\sin t \rangle$
15. $\mathbf{r}(t) = \langle 4\cos 3t + 6\cos t, 6\sin t, 4\sin 3t \rangle$
16. $\mathbf{r}(t) = \langle \sin \pi t, \sqrt{t^2 + t^3}, \cos \pi t \rangle$
17. $\mathbf{r}(t) = \langle \tan t, 4\cos t, 4\sin t \rangle$
18. $\mathbf{r}(t) = \langle \cos 5t, \tan t, 6\sin t \rangle$

19. In parts (a)–(f), match the vector-valued function with its graph.

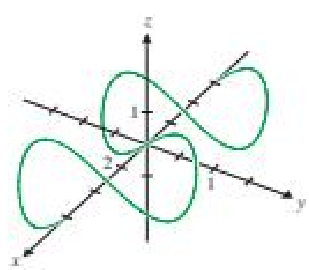
(a) $\mathbf{r}(t) = \langle \sin t, t, \sin 2t \rangle$ (b) $\mathbf{r}(t) = \langle t, \sin t, \sin 2t \rangle$
(c) $\mathbf{r}(t) = \langle 6\sin \pi t, t, 6\cos \pi t \rangle$ (d) $\mathbf{r}(t) = \langle \sin^3 t, \sin^2 t, \cos t \rangle$
(e) $\mathbf{r}(t) = \langle \cos t, 1 - \cos^2 t, \cos t \rangle$
(f) $\mathbf{r}(t) = \langle t^2 + 1, t^2 + 2, t - 1 \rangle$



GRAPH A

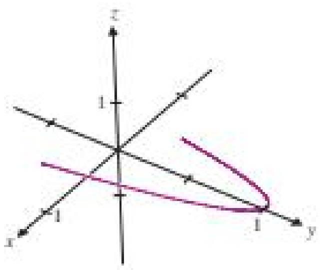


GRAPH B

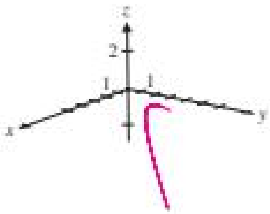


GRAPH C

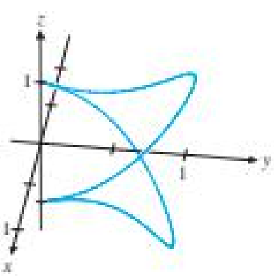
Review Exercises



GRAPH D



GRAPH E



GRAPH F

In exercises 20–22, sketch the curve and find its arc length.

20. $\mathbf{r}(t) = \langle \cos \pi t, \sin \pi t, \cos 4\pi t \rangle, 0 \le t \le 2$

21. $\mathbf{r}(t) = \langle \cos t, \sin t, 6t \rangle, 0 \le t \le 2\pi$

22. $\mathbf{r}(t) = \langle t, 4t - 1, 2 - 6t \rangle, 0 \le t \le 2$

In exercises 23 and 24, find the limit if it exists.

23. $\lim_{t \to 1} \langle t^2 - 1, e^{2t}, \cos \pi t \rangle$

24. $\lim_{t \to 1} \langle e^{-2t}, \csc \pi t, t^3 - 5t \rangle$

In exercises 25 and 26, determine all values of *t* at which the given vector-valued function is continuous.

25. $\mathbf{r}(t) = \langle e^{4t}, \ln t^2, 2t \rangle$

26. $\mathbf{r}(t) = \left\langle \sin t, \tan 2t \frac{3}{t^2 - 1} \right\rangle$

In exercises 27 and 28, find the derivative of the given vector-valued function.

27. $\mathbf{r}(t) = \langle \sqrt{t^2 + 1}, \sin 4t, \ln 4t \rangle$

28. $\mathbf{r}(t) = \langle te^{-2t}, t^3, 5 \rangle$

In exercises 29–32, evaluate the given indefinite or definite integral.

29. $\int \left\langle e^{-4t}, \frac{2}{t^3}, 4t - 1 \right\rangle dt$

30. $\int \left\langle \frac{2t^2}{t^3 + 2}, \sqrt{t + 1} \right\rangle dt$

31. $\int_0^1 \langle \cos \pi t, 4t, 2 \rangle dt$

32. $\int_0^2 \langle e^{-3t}, 6t^2 \rangle dt$

In exercises 33 and 34, find the velocity and acceleration vectors for the given position vector.

33. $\mathbf{r}(t) = \langle 4 \cos 2t, 4 \sin 2t, 4t \rangle$

34. $\mathbf{r}(t) = \langle t^2 + 2, 4, t^3 \rangle$

In exercises 35–38, find the position vector from the given velocity or acceleration vector.

35. $\mathbf{v}(t) = \langle 2t + 4, -32t \rangle, \mathbf{r}(0) = \langle 2, 1 \rangle$

36. $\mathbf{v}(t) = \langle 4, t^2 - 1 \rangle, \mathbf{r}(0) = \langle -4, 2 \rangle$

37. $\mathbf{a}(t) = \langle 0, -32 \rangle, \mathbf{v}(0) = \langle 4, 3 \rangle, \mathbf{r}(0) = \langle 2, 6 \rangle$

38. $\mathbf{a}(t) = \langle t, e^{2t} \rangle, \mathbf{v}(0) = \langle 2, 0 \rangle, \mathbf{r}(0) = \langle 4, 0 \rangle$

In exercises 39 and 40, find the force acting on an object of mass 4 with the given position vector.

39. $\mathbf{r}(t) = \langle 12t, 12 - 16t^2 \rangle$

40. $\mathbf{r}(t) = \langle 3 \cos 2t, 2 \sin 2t \rangle$

In exercises 41 and 42, a projectile is fired with initial speed v_0 meters per second from a height of *h* meters at an angle of θ above the horizontal. Assuming that the only force acting on the object is gravity, find the maximum altitude, horizontal range and speed at impact.

41. $v_0 = 80, h = 0, \theta = \frac{\pi}{12}$

42. $v_0 = 80, h = 6, \theta = \frac{\pi}{4}$

In exercises 43 and 44, find the unit tangent vector to the curve at the indicated points.

43. $\mathbf{r}(t) = \langle e^{-2t}, 2t, 4 \rangle, t = 0, t = 1$

44. $\mathbf{r}(t) = \langle 2, \sin \pi t^2, \ln t \rangle, t = 1, t = 2$

Review Exercises

In exercises 45 and 46, find the unit tangent and principal unit normal vectors at the given points.

45. $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t \rangle$ at $t = 0$

46. $\mathbf{r}(t) = \langle \cos t, \sin t, \sin t \rangle$ at $t = \frac{\pi}{2}$

In exercises 47 and 48, the friction force required to keep a car from skidding on a curve is given by $\mathbf{F}_s(t) = ma_N\mathbf{N}(t)$. Find the friction force needed to keep a car of mass $m = 120$ kg from skidding for the given position vectors.

47. $\mathbf{r}(t) = \langle 80 \cos 6t, 80 \sin 6t \rangle$ **48.** $\mathbf{r}(t) = \langle 80 \cos 4t, 80 \sin 4t \rangle$

EXPLORATORY EXERCISE

1. In this three-dimensional projectile problem, think of the x-axis as pointing to the right, the y-axis as pointing straight ahead and the z-axis as pointing up. Suppose that a projectile of mass $m = 1$ kg is launched from the ground with an initial velocity of 100 m/s in the yz-plane at an angle of $\frac{\pi}{6}$ above the horizontal. The spinning of the projectile produces a constant Magnus force of $\langle 0.1, 0, 0 \rangle$ newtons. Find a vector for the position of the projectile at time $t \geq 0$. Assuming level ground, find the time of flight T for the projectile and find its landing place. Find the curvature for the path of the projectile at time $t \geq 0$. Find the times of minimum and maximum curvature of the path for $0 \leq t \leq T$.

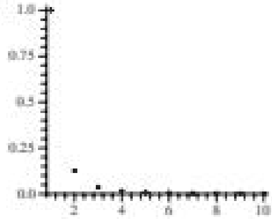
Chapter 5

Exercises 5.1

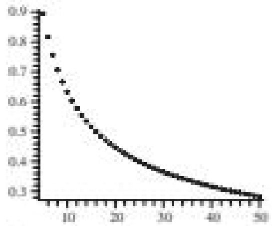
1. $1, \frac{3}{4}, \frac{5}{9}, \frac{7}{16}, \frac{9}{25}, \frac{11}{36}$ **3.** $4, 2, \frac{2}{3}, \frac{1}{6}, \frac{1}{30}, \frac{1}{180}$

5. converges to 0 **7.** converges to 1

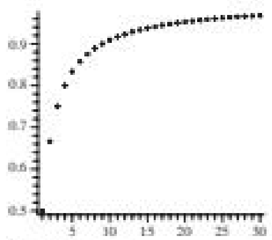
9.



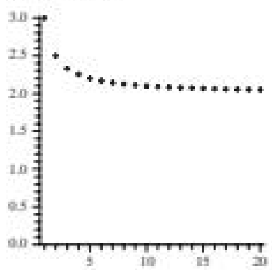
Converges to 0



Converges to 0



Converges to 1



Converges to 2

11. converges to $\frac{3}{2}$ **13.** diverges **15.** diverges

17. converges to 0 **19.** converges to 0 **21.** converges to 0

23. converges to 0 **25.** 1 **27.** ln 2 **29.** 0

35. decreasing **37.** increasing **39.** $|a_n| < 3$

41. $|a_n| < \frac{1}{2}$ **43.** $-\frac{(-2)^n}{16}$ **45.** $\frac{2n-1}{n^2}$

49. $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e^2$; $\lim_{n\to\infty}\left(1-\frac{2}{n}\right) = e^{-2}$

51. 1.8312 **53.** converges to $\frac{1}{2}$; $\int_0^1 x\,dx$

55. (b) $(c_2-c_1)^2+(r_2-r_1)^2=(r_1+r_2)^2$; $|c_2-c_1| = 2\sqrt{r_1 r_2}$;
$(c_3-c_2)^2+(r_3-r_2)^2=(r_2+r_3)^2$; $|c_3-c_2| = 2\sqrt{r_2 r_3}$

(d) $r_n = 1/F_n^2$

57. $S_{48} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}} \approx 0.1308$, $\frac{\pi}{24} \approx 0.1309$

Applications

1. $a_n = 144 - 36\pi \approx 30.9$

Exercises 5.2

1. converges to $\frac{15}{4}$ **3.** converges to $\frac{3}{8}$ **5.** diverges

7. converges to 3 **9.** diverges **11.** diverges

13. converges to 1 **15.** converges to $\frac{2\pi}{\pi-1}$ **17.** diverges

19. converges to $\frac{5}{6}$ **21.** diverges **23.** diverges

25. $-1 < c < 0$ **27.** $c = 0$ **29.** converges

31. converges **33.** (a) $L - \sum_{k=1}^{n-1} a_k$ **37.** 64; 256; 4^{n-1}

39. $\frac{0.9}{1-0.1} = 1$ **41.** $a_k = \frac{1}{k}$ and $b_k = -\frac{1}{k}$

43. $\frac{1}{1-r} > \frac{1}{2}$ if $-1 < r < 1$ **45.** yes **47.** no

Applications

1. $1.3589L$; $n = 4$ **3.** $\frac{p^2}{1-2p(1-p)} > p$ if $p > \frac{1}{2}$; 0.692

5. $2(1-e^{-0.1}) \approx 0.19$ **7.** \$400,000; save \$150,000

9. 1; you eventually win a game **11.** 1.002004008 . . .

Exercises 5.3

1. (a) diverges (b) diverges **3.** (a) diverges (b) converges

5. (a) diverges (b) converges **7.** (a) converges (b) converges

9. (a) diverges (b) diverges **11.** (a) converges (b) converges

13. (a) diverges (b) diverges **15.** (a) diverges (b) converges

17. (a) converges (b) converges

19. (a) diverges (b) diverges (c) diverges (d) diverges

21. $p > 1$ **23.** $p > 1$ **25.** $\frac{1}{3 \cdot 100^3}$ **27.** $\frac{6}{7 \cdot 50^7}$

29. $e^{-1600} \approx 6.73 \times 10^{-696}$ **31.** 101 **33.** 4

35. (a) can't tell (b) converges (c) converges (d) can't tell

45. $\frac{\pi^2}{6}, \frac{\pi^4}{90}, \frac{\pi^6}{945}, \frac{\pi^8}{9450}, \frac{\pi^{10}}{93,555}$

47. (a) $y = x^n$ is concave up for x 70

Applications

1. 2 **5.** (a) $10\sum_{k=1}^{10}\frac{1}{k} \approx 29.3$ (b) 3

Exercises 5.4

1. convergent **3.** convergent **5.** convergent

7. convergent **9.** divergent **11.** diverges

13. convergent **15.** convergent **17.** diverges

19. convergent **21.** divergent **23.** convergent

25. 3.61 **27.** −0.22 **29.** 1.10 **31.** −0.21

33. 20,000 **35.** 34 terms ($k = 0$ to $k = 33$)

37. converges **39.** diverges **41.** $f'(k) < 0$ for $k \geq 2$

43. positives diverge, negatives converge

45. $\int_1^2 \frac{1}{x}\,dx$

Applications

1. $\frac{\frac{3}{4}}{1+\frac{3}{4}} = \frac{3}{7}$

Exercises 5.5

1. absolutely convergent **3.** divergent
5. conditionally convergent **7.** absolutely convergent
9. divergent **11.** absolutely convergent
13. absolutely convergent **15.** divergent
17. conditionally convergent **19.** absolutely convergent
21. absolutely convergent **23.** absolutely convergent
25. conditionally convergent **27.** conditionally convergent
29. absolutely convergent **31.** absolutely convergent
33. divergent **35.** absolutely convergent
37. absolutely convergent **39.** absolutely convergent
41. ratio **43.** alternating **45.** p-series **47.** ratio
49. integral **51.** root **53.** comparison **55.** comparison
57. integral **59.** ratio **63.** (a) $-1 \le p \le 1$

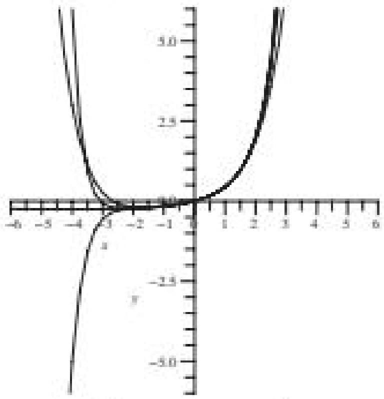
Exercises 5.6

1. $\infty, (-\infty, \infty)$ **3.** $4, (-4, 4)$ **5.** $3, (-2, 4]$ **7.** $0, \{x = -1\}$

9. $r = \frac{1}{2}, (1, 2)$ **11.** $r = \frac{1}{2}, \left(-\frac{5}{8}, -\frac{3}{8}\right)$

13. $r = 2, (-4, 0)$ **15.** $r = \infty$, all x

17.



19. $(-3, -1), \frac{-1}{1+x}$ **21.** $(0, 1), \frac{1}{2-2x}$

23. $(-2, 2), \frac{2}{2+x}$ **25.** $\sum_{k=0}^{\infty} 2x^k, r = 1, (-1, 1)$

27. $\sum_{k=0}^{\infty} (-1)^k 3x^{2k}, r = 1, (-1, 1)$ **29.** $\sum_{k=0}^{\infty} 2x^{2k+1}, r = 1, (-1, 1)$

31. $\sum_{k=0}^{\infty} (-1)^k \frac{1}{2^{2k+1}} x^k, r = 4, (-4, 4)$ **33.** $\sum_{k=0}^{\infty} (-1)^k \frac{3}{2k+1} x^{2k+1}, r = 1$

35. $\sum_{k=0}^{\infty} 2kx^{2k-1}, r = 1$ **37.** $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} x^{2k+2}, r = 1$

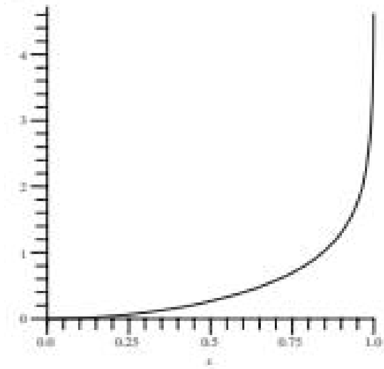
39. $(-\infty, \infty); \sum_{k=0}^{\infty} -k \sin(k^3 x), \{x = n\pi\}$

41. $(-\infty, 0); \sum_{k=0}^{\infty} ke^{kx}, (-\infty, 0)$

43. $(a - b, a + b), r = b$ **45.** r

47. $r = 1$, 1.003005007 . . .

49. $\sum_{k=0}^{\infty} \frac{2x^{3k+2}}{3k+2}; r = 1$



51. (a) $\sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{4k+1} + \frac{1}{4k+3}\right) = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \ldots$

(b) $\frac{1}{\sqrt{2}}\left[\tan^{-1}(1+\sqrt{2}) - \tan^{-1}(1-\sqrt{2})\right]$

53. $\frac{2}{(1-p)^3}, \frac{6}{(1-p)^4}, \frac{1}{(1-p)^{n+1}}$

Exercises 5.7

1. $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, (-\infty, \infty)$ **3.** $\sum_{k=0}^{\infty} \frac{2^k}{k!} x^k, (-\infty, \infty)$

5. $\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}, (-1, 1]$ **7.** $\sum_{k=0}^{\infty} (-1)^k (k+1) x^k, (-1, 1)$

9. $\sum_{k=0}^{\infty} \frac{(x-1)^k}{k!}, (-\infty, \infty)$ **11.** $1 + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{e^{-k}}{k} (x - e)^k, (0, 2e]$

13. $\sum_{k=0}^{\infty} (-1)^k (x-1)^k, (0, 2)$

15. $1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 + \frac{7}{256}(x-1)^5 - \frac{21}{1024}(x-1)^6$

17. $e^2\left[1 + (x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3 + \frac{1}{24}(x-2)^4 + \frac{1}{120}(x-2)^5 + \frac{1}{720}(x-2)^6\right]$

19. $x + \frac{1}{6}x^3 + \frac{3}{40}x^5$

21. $|R_n(x)| \le \frac{3x^{n+1}}{(n+1)!} \to 0$ **23.** $|R_n(x)| = \frac{1}{n+1}\left|\frac{x-1}{z}\right|^{n+1} \to 0$

25. (a) 0.04879; (b) $\frac{7(0.1)^5}{256}$; (c) 7

27. (a) 1.0488 (b) $\frac{7(0.1)^5}{256}$ (b) 9

29. $\sum_{k=0}^{\infty} \frac{(-3)^k}{k!} x^k; r = \infty$ **31.** $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{k!}; r = \infty$

33. $\sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} x^{2k+2}; r = \infty$

35. e^x with $x = 2$ **37.** $\tan^{-1} x$ with $x = 1$

39. (a) differentiation (b) Taylor series

41. (a) antidifferentiation (b) antidifferentiation

43. x; converges to f for all $x \ge 0$

47. $1 + \sum_{k=1}^{\infty} \frac{(-1)^k (2k-3)(2k-1) \cdots (-1)}{2^k k!} x^k$

49. $1 + (x - 1) = x \ne |x|$, for $x < 0$

51. the first three terms of the Taylor series for f and g are identical

53. $x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{90}x^6$; equals the product

55. $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}$; equals to the Maclaurin series for $\sin x$, divided by x

57. $\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$; $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$ **59.** 4; 6; Taylor is more efficient

Exercises 5.8

1. 0.99923163442 **3.** 0.94275466553

5. 0.81873075307 **7.** $-\frac{1}{2}$ **9.** $-\frac{1}{2}$ **11.** 1

13. $\frac{1703}{900}$ **15.** $\frac{5651}{1780}$ **17.** $\frac{2}{3}$ **19.** ∞ **21.** 12

23. $1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 + \cdots$

25. $6 - 6x + 12x^2 - 28x^3 + 70x^4 - \cdots$

27. (a) 5.0990200 (b) 4.8989800

29. $64 + 48x + 12x^2 + x^3$; $1 - 8x + 24x^2 - 32x^3 + 16x^4$; $n + 1$

31. $x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \ldots$

Applications

1. (b) $m = 1.1m_0$ if $V = \sqrt{0.2C} \approx 134,071,263$ m/s

(c) $m_0\left(1 + \frac{1}{2c^2}V^2 + \frac{9}{24c^2}V^4\right)$; $m = 1.1m_0$ if

$V = \sqrt{\frac{4\sqrt{10} - 10}{15}}\ C \approx 125,986,822.3$ m/s

3. (b) $x = \frac{R}{32} \approx 200$ km

(c) $mg\left(1 - \frac{2}{R}x + \frac{3}{R^2}x^2\right)$; $x = R\frac{1-\sqrt{0.7}}{3} \approx 217$ km

5. too large **9.** $\lambda \approx \frac{0.0016}{T}$

Exercises 5.9

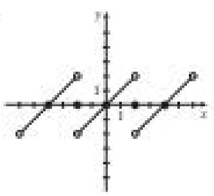
1. $\sum_{k=1}^{\infty} (-1)^{k+1}\frac{2}{k}\sin kx$ **3.** $\pi - \sum_{k=0}^{\infty} \frac{8}{(2k-1)^2\pi}\cos[(2k-1)x]$

5. $\sum_{k=1}^{\infty} \frac{-4}{\pi(2k-1)}\sin[(2k-1)x]$ **7.** $3\sin 2x$

9. $\sum_{k=1}^{\infty} (-1)^k \frac{2}{k\pi}\sin k\pi x$ **11.** $\frac{1}{3} + \sum_{k=1}^{\infty} (-1)^k \frac{4}{k^2\pi^2}\cos k\pi x$

13. $\frac{1}{4} + \sum_{k=1}^{\infty} \frac{-2}{(2k-1)^2\pi^2}\cos(2k-1)\pi x + \sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k\pi}\sin k\pi x$

15.



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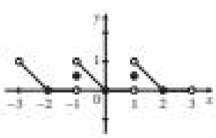

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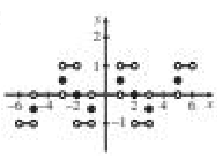
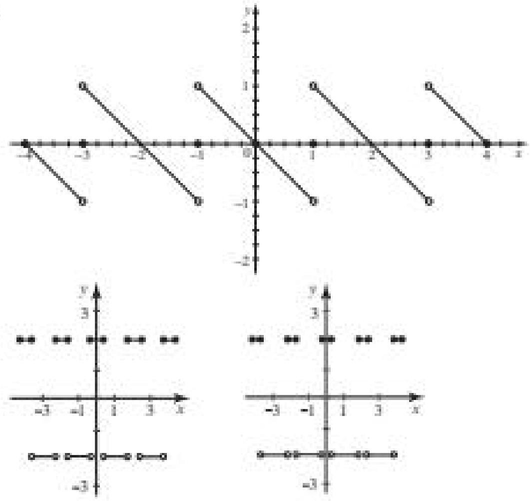


31. If f and g are even then fg is even. **35.** sine **37.** both

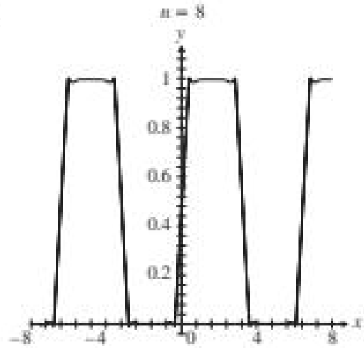
Applications

1.



3. The amplitude varies slowly because the frequency of $2\cos(0.2t)$ is small compared to the frequency of $\sin(8.1t)$. The variation of the amplitude explains why the volume varies, since the volume is proportional to the amplitude.

7.



The modified Fourier series is

$$\frac{1}{2} + \sum_{k=1}^{n} \frac{2n}{[(2k-1)\pi]^2}\sin\left[\frac{(2k-1)\pi}{n}\right]\sin[(2k-1)x].$$

Chapter 5 Review Exercises

1. converges to 0 **3.** converges to 0 **5.** converges to 0

7. diverges **9.** diverges **11.** can't tell **13.** diverges

15. converges **17.** converges **19.** 8 **21.** $\frac{4}{3}$

23. -0.41 **25.** diverges **27.** converges **29.** diverges

31. converges **33.** converges **35.** converges

37. converges **39.** diverges **41.** converges **43.** converges

45. converges conditionally **47.** converges absolutely

49. $p > 1$ **51.** 1732 **53.** $\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{4^{k+1}}$, $r = 4$

55. $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{3^k}$, $r = \sqrt{3}$ **57.** $\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{(k+1)4^{k+1}} + \ln 4$, $r = 4$

59. $(-1, 1)$ **61.** $(-1, 1)$ **63.** $(-\infty, \infty)$

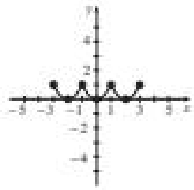
65. $\left(\frac{5}{3}, \frac{7}{3}\right)$ **67.** $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

69. $(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4$

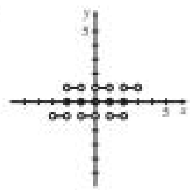
71. 0.1822666, 10 **73.** $\sum_{k=0}^{\infty}(-1)^k\frac{3^k x^{2k}}{k!}$, $r=\infty$ **75.** $\frac{1117}{2520}$

77. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4}{k\pi}\sin\left(\frac{k\pi}{2}x\right)$

79.



81.

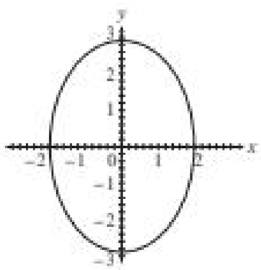


83. $\frac{2}{3}$ **89.** contains powers of 2

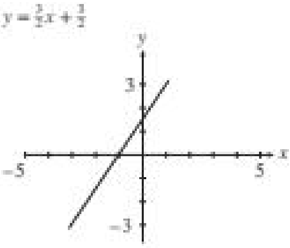
Chapter 6

Exercises 6.1

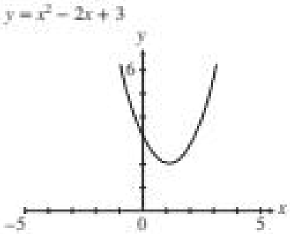
1. $\frac{x^2}{4}+\frac{y^2}{9}=1$



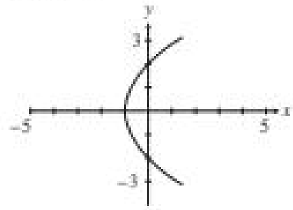
3. $y=\frac{3}{2}x+\frac{3}{2}$



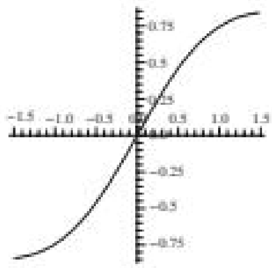
5. $y=x^2-2x+3$



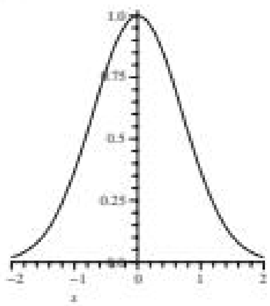
7. $x=\frac{1}{4}y^2-1$



9. $y=\sin(\sin x)$, $\frac{-\pi}{2}\le x\le\frac{\pi}{2}$



11. $y=e^{-t^2}$, $x>0$



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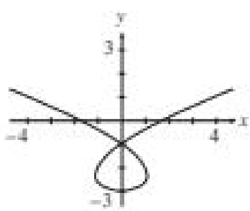

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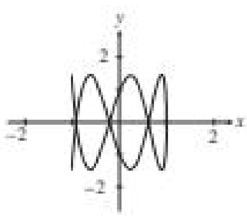

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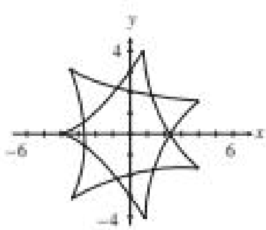

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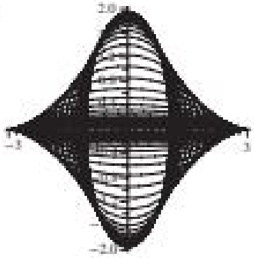


21. $x = 3t,\ y = 1 + 3t,\ 0 \le t \le 1$

23. $x = -2 + 8t,\ y = 4 - 3t,\ 0 \le t \le 1$

25. $x = t,\ y = t^2 + 1,\ 1 \le t \le 2$

27. $x = 2 + 3\cos t,\ y = 1 + 3\sin t,\ 0 \le t \le 2\pi$

29. (a) $x = 12t,\ y = 16 - 16t^2$

(b) $x = (12\cos 6°)t,\ y = 16 + (12\cos 6°)t - 16t^2$

31. (a) $x = 2t,\ y = 10 - 4.9t^2$

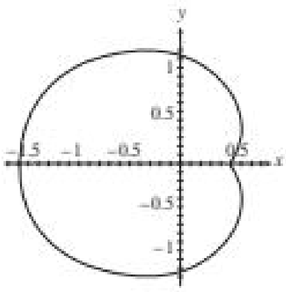
(b) $x = (2\cos 8°)t,\ y = 10 - (2\cos 8°)t - 4.9t^2$

33. (2, 3) and (−3, 8)

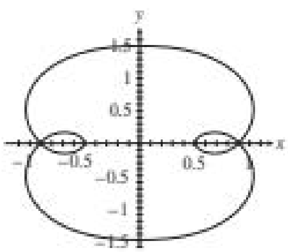
35. (2, 1) and (3, 0)

37. Integer values for k lead to closed curves, but irrational values for k do not.

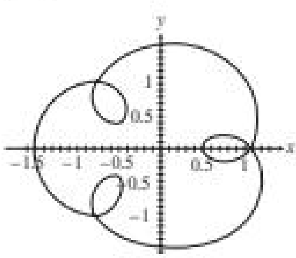
39. $k = 2$



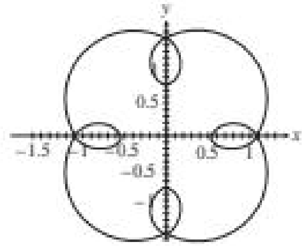
$k = 3$



$k = 4$



$k = 5$



41. $x = 1 - 2y^2,\ -1 \le y \le 1;\ y = \pm 2x\sqrt{1 - x^2}$

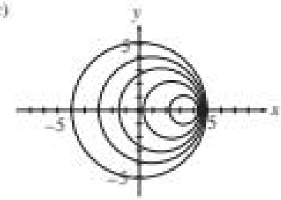
43. C **45.** B **47.** A

Applications

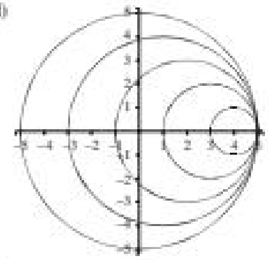
1. (a) a circle with radius t at time t

(b) $x = a + (t - c)\cos\theta,\ y = b + (t - c)\sin\theta$

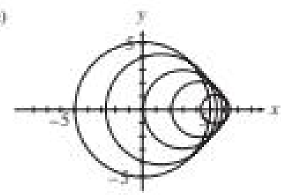
(c)



(d)



(e)



(f) if $\sin\theta = \frac{1}{1.4}$, then $\tan\theta = \frac{1}{\sqrt{0.96}}$ (g) a cone

3. (a) $x = (v\sin\theta)t,\ y = D - (v\cos\theta)t$ (e) $v\gamma$

Exercises 6.2

1. (a) −1 (b) 1 (c) undefined **3.** (a) $-\frac{3}{2}$ (b) 0 (c) 0

5. (a) 0 (b) $-\frac{2}{\pi}$ (c) $-\pi$ **7.** 1 at $t = 1$; −1 at $t = -1$

9. (a) $\left(\frac{\sqrt{2}}{2}, 1\right), \left(\frac{\sqrt{2}}{2}, -1\right), \left(-\frac{\sqrt{2}}{2}, 1\right)\left(-\frac{\sqrt{2}}{2}, -1\right)$

(b) (1, 0), (−1, 0)

11. (a) (0, −3) (b) (−1, 0) **13.** (a) (0, 1) (b) (0, −3)

15. (a) $x' = 0,\ y' = 3$; speed is 3; up (b) $x' = -2;\ y' = 0$; speed is 2; left

17. (a) $x'(0) = 20,\ y'(0) = -2$, speed $= 2\sqrt{101}$, right/down

(b) $x'(2) = 20,\ y'(2) = -66$, speed $= \sqrt{4756}$, right/down

19. (a) $x'(0) = 5,\ y'(0) = 4$, speed $= \sqrt{41}$, right/up

(b) $x'(\frac{\pi}{2}) = 0,\ y'(\frac{\pi}{2}) = -9$, speed = 9, down

21. 6π **23.** $\frac{3\pi}{8}$ **25.** $\frac{4}{3}$ **27.** $\frac{256}{15}$

29. At (3,0), speed is 0 and acceleration is $x'' = -6$, $y'' = 0$; at $(-1, 0)$, speed is 4 and acceleration is $x'' = -2$, $y'' = 0$

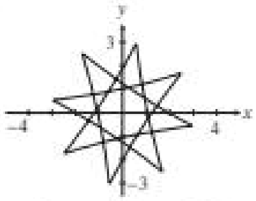
35. $x(t) = vt + r\cos(\frac{v}{r}t)$, $y(t) = r - r\sin(\frac{v}{r}t)$;
min speed = 0 at bottom ($y = 0$), max speed = $2v$ at top ($y = 2r$)

37. $x(t) = (a-b)\cos t + b\cos(\frac{a}{b}t)$, $y(t) = (a-b)\sin t - b\sin(\frac{a}{b}t)$

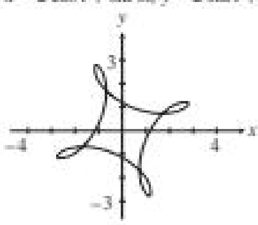
Applications

1. speed = 4, $(\tan 4t)(-\cot 4t) = -1$

3. 5: 3



5. $x = 2\cos t + \sin 3t$, $y = 2\sin t + \cos 3t$;



Min/max speeds: 1, 5

Exercises 6.3

1. (a) 19.38 (b) 4π **3.** (a) 2π (b) 55.09

5. (a) $\frac{\pi}{2}$ (b) 4.29 **7.** (a) $e^8 - e^{-8}$ (b) 2980.2

9. (a) $4.4859k$ (b) $4.4859k$ (c) ∞; 3.89

11. (a) $4.4569k$ (b) $4.4569k$ (c) ∞; 4.07

13. (a) 85.8 (b) 83.92 **15.** (a) 85.8 (b) 162.60

17. (a) 40.30 (b) 43.16

Applications

1. $x = 4u$, $y = 4\sqrt{1-u^2}$; 2π **3.** (a) 4π (b) $b - a$

Exercises 6.4

1. (2, 0) **3.** (2, 0) **5.** (−3, 0)

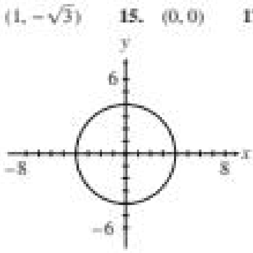
7. $\left(2\sqrt{2}, -\frac{\pi}{4} + 2n\pi\right) \cdot \left(-2\sqrt{2}, \frac{3\pi}{4} + 2n\pi\right)$

9. $\left(3, \frac{\pi}{2} + 2n\pi\right) \cdot \left(-3, \frac{3\pi}{2} + 2n\pi\right)$

11. $\left(5, \tan^{-1}\left(\frac{4}{3}\right) + 2n\pi\right)$, $\left(-5, \tan^{-1}\left(\frac{4}{3}\right) + 2(n+1)\pi\right)$

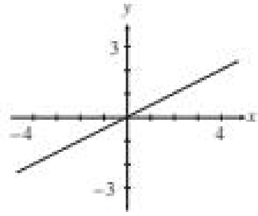
13. $(1, -\sqrt{3})$ **15.** (0, 0) **17.** (3.80, 1.24)

19.



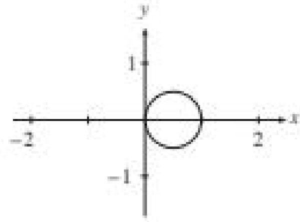
$x^2 + y^2 = 16$

21.



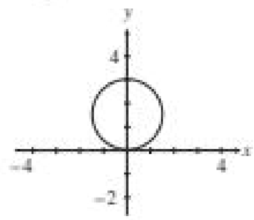
$y = \frac{1}{\sqrt{3}}x$

23.



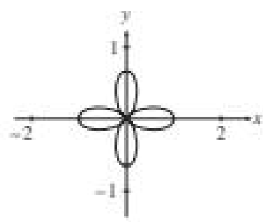
$x^2 + y^2 = x$

25.



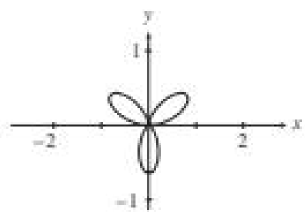
$x^2 + y^2 = 3y$

27.



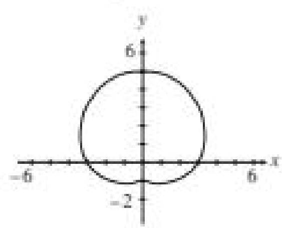
$r = 0$ at $\theta = \frac{k\pi}{4}$(k odd) ,$0 \le \theta \le 2\pi$

29.



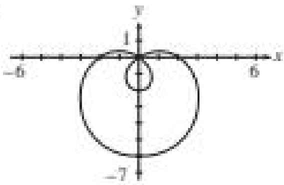
$r = 0$ at $\theta = \frac{n\pi}{3}$, $0 \le \theta \le 2$

31.



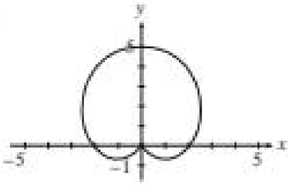
$r > 0$, $0 \le \theta \le 2\pi$

33.



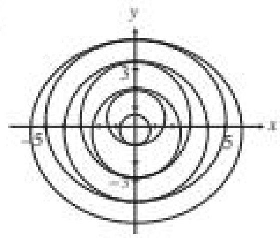
$r = 0$ at $\theta = \frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + \pi n; 0 \le \theta \le 2\pi$

35.



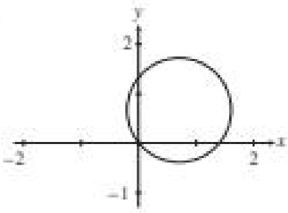
$r = 0$ at $\theta = \frac{3\pi}{2} + 2\pi n, 0 \le \theta \le 2\pi$

37.



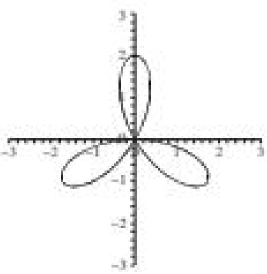
$r = 0$ at $\theta = 0, -\infty < \theta < \infty$

39.



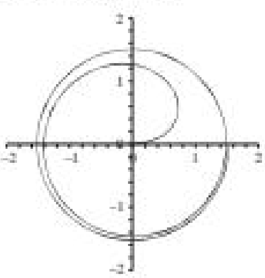
$r = 0$ at $\theta = \frac{3\pi}{2} + \pi n, 0 \le \theta \le \pi$

41.



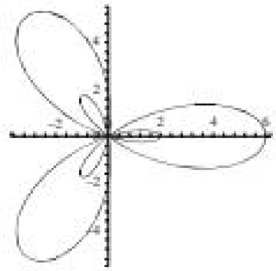
$\theta = \frac{3\pi}{4} + 2\pi n, \frac{7\pi}{4} + 2\pi n$

43.



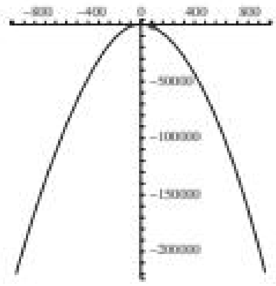
$\theta = 0$

45.



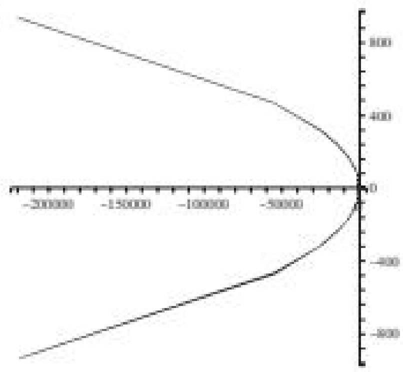
$\theta = \frac{2\pi}{9} + \frac{2\pi}{3}n, \frac{4\pi}{9} + \frac{2\pi}{3}n$

47.



$\theta = -\frac{\pi}{2} + 2\pi n$

49.



$\theta = (2n + 1)\pi$

51. $r = \pm 2\sqrt{-\sec 2\theta}$

53. $r = 4$

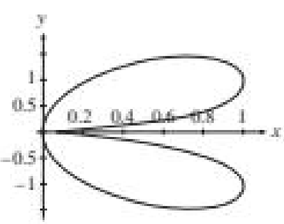
55. $r = 3 \csc\theta$

57. circles of radius $\frac{1}{2}|a|$ and center $(\frac{1}{2}a, 0)$

59. For integer a, rose with $2n$ leaves (n even) or n leaves (n odd)

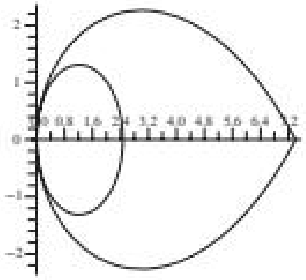
61. For $|a| > 1$, larger $|a|$ gives larger inner loop.

63.



65. n wide overlapping petals on a flower when $0 \le \theta \le n\pi$, up to $n = 24$; graph repeats for larger domains

67.



Exercises 6.5

1. (a) $\sqrt{3}$ **2.** (a) undefined **3.** (a) 0

5. (a) $\frac{1}{2}$ (b) $\frac{2\sin 1+\cos 1}{2\cos 1-\sin 1}$

7. (a) $\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right), \left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right), (0,-1)$
(b) concave up at $(0, 0)$ and $(0, -1)$; concave down at $(\pm 0.73, 0.56)$

9. (a) $(3\sqrt{2}, -3\sqrt{2}), (-3\sqrt{2}, 3\sqrt{2})$
(b) concave up at $(-0.98, -0.13)$, $(-1.22, -1.49)$ and $(2.97, -4.74)$; concave down at $(1.22, 1.49)$, $(0.98, 0.13)$ and $(-2.97, 4.74)$

11. $\frac{\pi}{12}$ **13.** π

15. $\frac{\pi}{2}-\frac{3\sqrt{3}}{4}\approx 0.2718$

17. 0.3806

19. 0.1470 **21.** $\frac{11\sqrt{3}}{2}+\frac{14\pi}{3}\approx 24.187$

23. $\frac{5\pi}{3}+\sqrt{3}\approx 6.9680$

25. $\frac{5\pi}{4}-2\approx 1.9270$

27. $\frac{5\pi}{12}-\frac{\sqrt{3}}{2}$ **29.** $\frac{\sqrt{3}}{4}-\frac{\pi}{12}$

31. $(0, 0)$, $(0.3386, -0.75)$, $(1.6614, -0.75)$

33. $(0, 0)$, $(1.2071, 1.2071)$, $(-0.2071, -0.2071)$

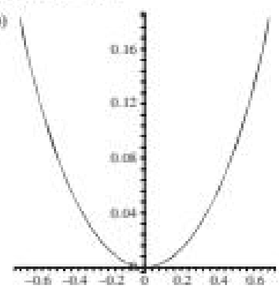
35. 16

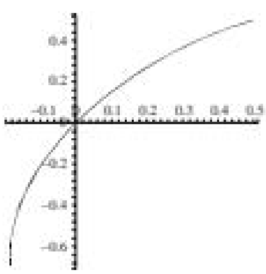
37. 6.6824

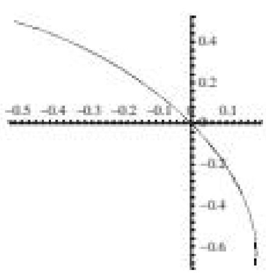
39. 20.0158

41. (a) 31.2% (b) 37.35%

43. (a) 0; $\sqrt{3}$; $-\sqrt{3}$
(b)







Exercises 6.6

1. $y=-\frac{1}{4}x^2$

3. $x=\frac{1}{4}y^2+2$

5. $\frac{x^2}{12}+\frac{(y-3)^2}{16}=1$

7. $\frac{(x-4)^2}{16}+\frac{(y-1)^2}{12}=1$

9. $\frac{(x-2)^2}{1}+\frac{y^2}{3}=1$

11. $\frac{(y-4)^2}{1}+\frac{(x-2)^2}{3}=1$

13. parabola, $(-1, -1)$, $(-1, -\frac{7}{8})$, $y=-\frac{9}{8}$

15. ellipse, $(1, -1)$ and $(1, 5)$, $(1, 2-\sqrt{5})$ and $(1, 2+\sqrt{5})$

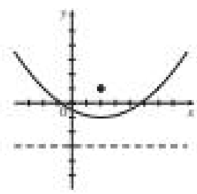
17. hyperbola, $(-2, 0)$ and $(4, 0)$, $(1-\sqrt{13}, 0)$ and $(1+\sqrt{13}, 0)$

19. hyperbola, $(-2, -5)$ and $(-2, 3)$, $(-2, -1-\sqrt{20})$ and $(-2, -1+\sqrt{20})$

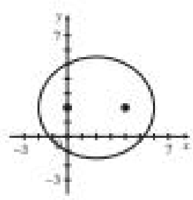
21. ellipse, $(-1, 0)$ and $(5, 0)$, $(2-\sqrt{8}, 0)$ and $(2+\sqrt{8}, 0)$

23. parabola, $(-1, -2)$, $(-1, -1)$, $y=-3$

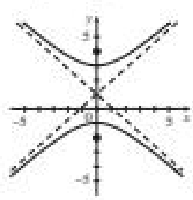
25. $y=\frac{1}{8}(x-2)^2-1$



27. $\frac{(x-2)^2}{16}+\frac{(y-2)^2}{12}=1$



29. $\frac{(y-1)^2}{4}-\frac{x^2}{5}=1$



31. $\left(\frac{1}{16}, 0\right)$

33. $\left(0, \frac{1}{8}\right)$

35. 20 cm

37. $(0, -8)$

39. $(-2, 0)$

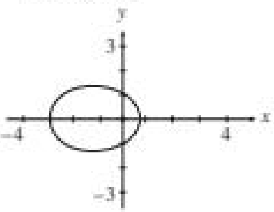
Applications

1. $(\sqrt{300}, 0)$ and $(-\sqrt{300}, 0)$

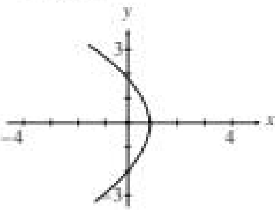
3. $-t^2+16t$, boomerang

Exercises 6.7

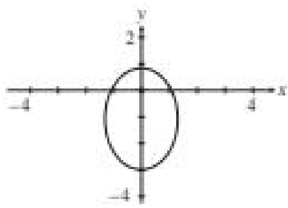
1. $r = \dfrac{1.2}{0.6\cos\theta + 1}$



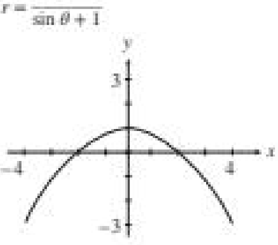
3. $r = \dfrac{2}{\cos\theta + 1}$



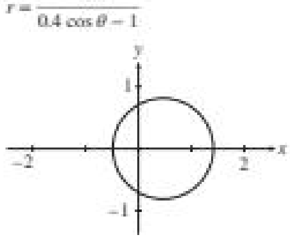
5. $r = \dfrac{1.2}{0.6\sin\theta + 1}$



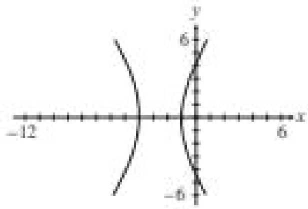
7. $r = \dfrac{2}{\sin\theta + 1}$



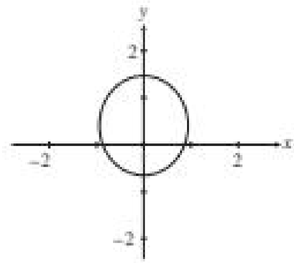
9. $r = \dfrac{-0.8}{0.4\cos\theta - 1}$



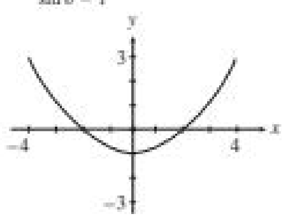
11. $r = \dfrac{-4}{2\cos\theta - 1}$



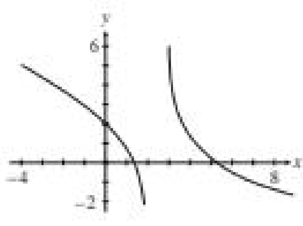
13. $r = \dfrac{-0.8}{0.4\sin\theta - 1}$



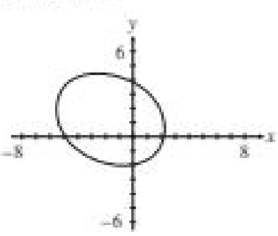
15. $r = \dfrac{-2}{\sin\theta - 1}$



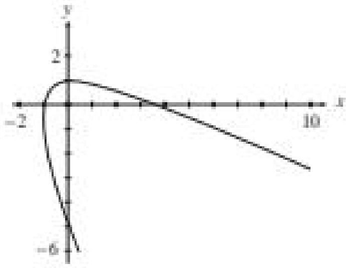
17. rotated hyperbola



19. rotated ellipse



21. rotated parabola



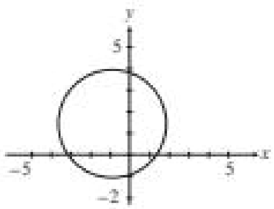
23. $x = -1 + 3\cos t,\ y = 1 + 2\sin t,\ 0 \le t \le 2\pi$

25. $x = -1 + 4\cosh t,\ y = 3\sinh t$ for the right half; $x = -1 - 4\cosh t,\ y = 3\sinh t$ for the left half

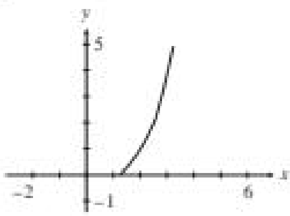
27. $x = t,\ y = -\frac{1}{4}t^2 + 1$ **29.** 2.6 times as fast

Chapter 6 Review Exercises

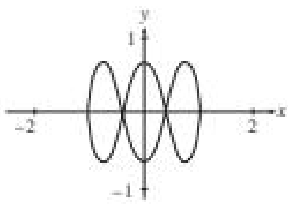
1. $(x+1)^2 + (y-2)^2 = 9$



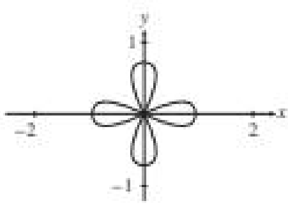
3. $y = x^2 - 2x + 1$



5.



7.



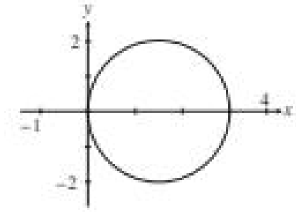
9. C **11.** B **13.** $x = 2 + 2t,\ y = 1 + 6t,\ 0 \le t \le 1$

15. $\frac{1}{3}$ (b) undefined (c) undefined at $t = -1$; $\frac{1}{3}$ at $t = 2$

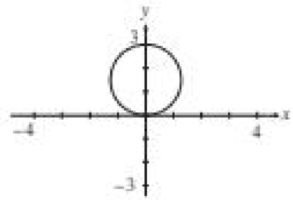
17. $x'(0) = -3,\ y'(0) = 2$, speed $= \sqrt{13}$, left/up **19.** 6π

21. 1.9467 **23.** 5.2495 **25.** 27.18 **27.** 128.075

29. $x^2 + y^2 = 3x$

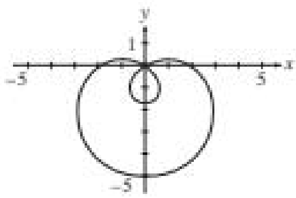


31.



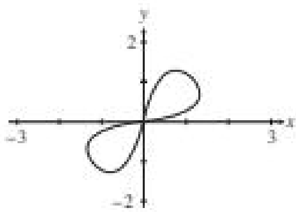
$r = 0$ at $\theta = n\pi$; $0 \le \theta \le \pi$

33.



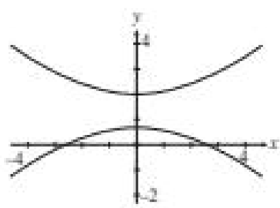
$r = 0$ at $\theta = \sin^{-1}\frac{2}{3} + 2n\pi,\ \pi - \sin^{-1}\frac{2}{3} + 2n\pi$; $0 \le \theta \le 2\pi$

35.



$r = 0$ at $\theta = \frac{n}{2}\pi$; $0 \le \theta \le \frac{\pi}{2}$

37.



$r \ne 0$; $0 \le \theta \le 2\pi$
$(\theta \ne \frac{7\pi}{6}, \frac{11\pi}{6})$

39. $r = 3$ **41.** $\frac{1}{\sqrt{3}}$ **43.** 0.157 **45.** 0.543

47. 2.828 **49.** 28.814 **51.** $y = \frac{1}{4}(x-1)^2 + 1$

53. $\frac{(y-2)^2}{1} - \frac{(x-2)^2}{3} = 1$

55. ellipse, $(-1, -2)$ and $(-1, 8)$, $(-1, -1)$ and $(-1, 7)$

57. parabola, $(1, 4)$, $(1, \frac{15}{4})$, $y = \frac{17}{4}$ **59.** $(0, \frac{1}{2})$

61. $r = \frac{2.4}{0.8\cos\theta + 1}$ **63.** $r = \frac{2.8}{1.4\sin\theta + 1}$

65. $x = -1 + 3\cos t,\ y = 3 + 5\sin t,\ 0 \le t \le 2\pi$

Chapter 7

Exercises 7.1

1. 10 **3.** −14 **5.** 1 **7.** $\cos^{-1}\frac{1}{\sqrt{26}} \approx 1.37$

9. $\cos^{-1}\frac{-8}{\sqrt{234}} \approx 2.12$ **11.** yes **13.** yes

15. (a) one possible answer: $\langle 1, 2, 3\rangle$; (b) $\langle 1, 2, -3\rangle$

17. (a) one possible answer: $\mathbf{i} - 3\mathbf{j}$; (b) $-\frac{7}{5}i + 2j - 3k$

19. $2, \langle \frac{6}{5}, \frac{8}{5}\rangle$ **21.** $2, \frac{2}{3}\langle 1, 2, 2\rangle$ **23.** $-\frac{8}{5}, -\frac{8}{25}\langle 0, -3, 4\rangle$

25. 90,000 J **29.** 920 J

31. (a) false (b) true (c) true (d) false (e) false

33. $\mathbf{a} \cdot \mathbf{c}, \mathbf{a} \cdot \mathbf{b}, \mathbf{b} \cdot \mathbf{c}$

35. (a) $\langle 0, x\rangle$ or $\langle x, -\frac{3}{4}x\rangle$, for any $x > 0$;
(b) $\langle 0, x\rangle$ or $\langle x, -\frac{3}{4}x\rangle$, for any $x < 0$

37. $\cos^{-1}\left(\frac{1}{3\sqrt{2}}\right) \approx 76.4°$, $\cos^{-1}\left(\frac{1}{3\sqrt{2}}\right) \approx 76.4°$,
$\cos^{-1}\left(\frac{8}{9}\right) \approx 27.30°$

41. (a) $\frac{\pi}{4}$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.7°$ (c) $\cos^{-1}\left(\frac{1}{\sqrt{n}}\right)$

43. $\mathbf{a} = c\mathbf{b}$ **47.** 15 **49.** $\sum_{k=1}^{n} \frac{1}{k^3} \le \frac{\pi^3}{6\sqrt{15}}$

51. (c) $P_k = \frac{1}{n}$ for each k

57. (a) $\mathbf{a} = \langle \frac{1}{2}, 1, \frac{3}{2}\rangle$, $\mathbf{b} = \langle \frac{5}{2}, -2, \frac{1}{2}\rangle$;
(b) $\mathbf{a} = \langle 1, 2, 3\rangle$, $\mathbf{b} = \langle -1, 2, -1\rangle$

Applications

1. $\cos^{-1}\left(\frac{1}{3}\right) \approx 109.5°$

3. (a) $\mathbf{v} \cdot \mathbf{n} = 0$, $\text{comp}_{\mathbf{v}}\, w = -w \sin\theta$, $\text{comp}_{\mathbf{n}}\, w = -w\cos\theta$

5. (a) $-8000 \sin 10° \approx -1389$ N, ratio $= \tan 10° \approx 0.18$
(b) $2500 \sin 15° \approx 647$lbs, ratio $= \tan 15° \approx 0.27$

7. \$190,000; monthly revenue

Exercises 7.2

1. 1 **3.** 4 **5.** $\langle 4, -3, -2\rangle$ **7.** $\langle -9, -4, 1\rangle$

9. $\langle 4, -2, 8\rangle$ **11.** $\pm\frac{1}{\sqrt{69}}\langle 8, 1, -2\rangle$ **13.** $\pm\frac{1}{\sqrt{46}}\langle -3, -6, 1\rangle$

15. $\pm\frac{1}{\sqrt{154}}\langle -1, -3, 12\rangle$ **17.** $\sqrt{\frac{7}{2}} \approx 1.87$ **19.** $\sqrt{\frac{61}{5}} \approx 3.49$

21. 5 **23.** $\frac{11\sqrt{3}}{2}$ **25.** 10 **27.** 11.3 N.m

31. (a) spin right, force up
(b) spin down right, force up right

33. (a) spin up left, force down left
(b) spin up right, force up left

35. false **37.** false **39.** true **41.** $\sin^{-1}\frac{7}{\sqrt{85}} \approx 0.86$

43. $\sin^{-1}\frac{13}{\sqrt{170}} \approx 1.49$ **47.** 0 **51.** coplanar

53. not coplanar **55.** $-\mathbf{i}$ **57.** $-3\mathbf{j}$ **59.** Figure A; 12

61. (b) and (c)

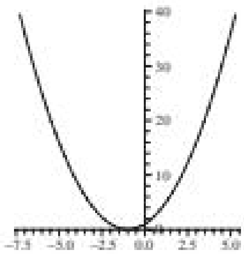
Applications

1. ball rises **3.** ball drops; Figure A:12

5. no effect **7.** ball rises

Exercises 7.3

1.



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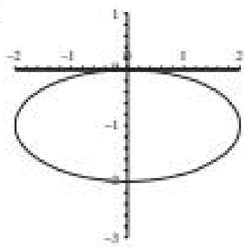

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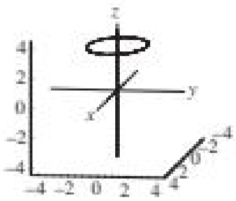

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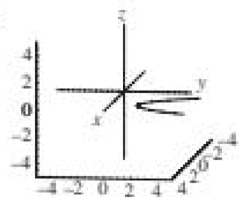

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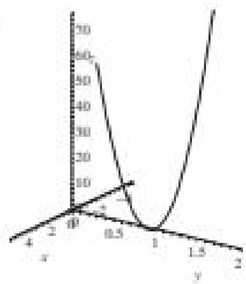

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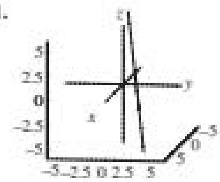

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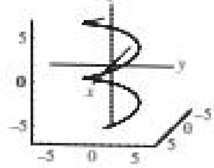

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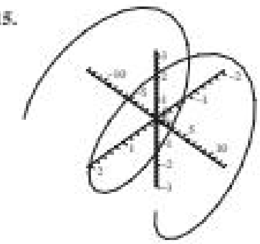

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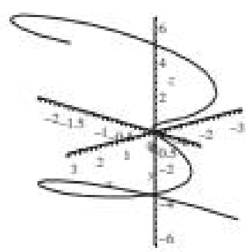

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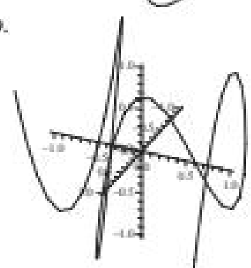
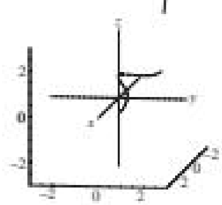


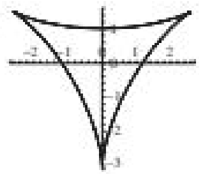
21.

Not Periodic

23.

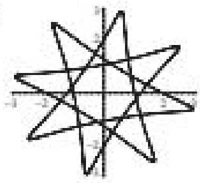


25. (a)



Periodic, 2π

(b)



Periodic, 2π

25. (c) Determines the number and smoothness of endpoints

(i) Depicts a circle when either of a or $b = 0$. If $a = 0$, then the center of the circle lies on the x-axis and if $b = 0$, the center of the circle lies on the y-axis.

(ii) Depicts an ellipse, when $a = ba = b$.

(iii) Depicts either an n-point star of an n-leaves structure where $n = \dfrac{a+b}{\gcd(a, b)}$ for $a, b > 0$.

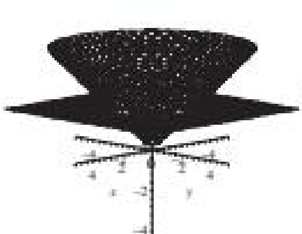
27. (a) 6 (b) 3 (c) 5 (d) 1 (e) 2 (f) 4

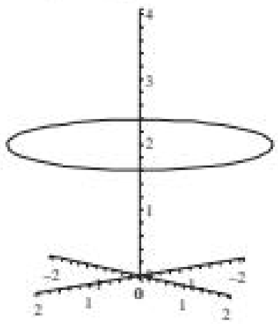
29. $2\pi(\pi + 1)$ **31.** $8 + 4 \ln 3$

33. 10.54 **35.** 21.56 **37.** 9.57

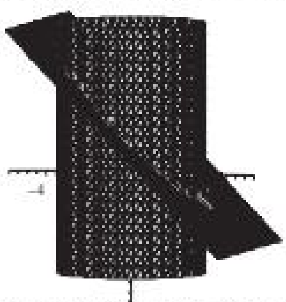
39. (a) 7 m (b) 7 m

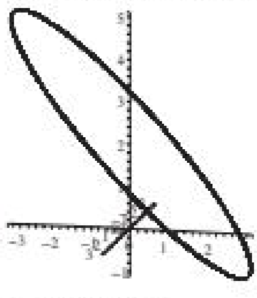
41. $x = 2\cos t,\ y = 2\sin t,\ z = 2,\ 0 < t < 2\pi$, arc length $= 4\pi$





43. $x = 3\cos t,\ y = 3\sin t,\ z = 2 - 3\sin t,\ 0 < t < ln$ arc length $<< 22.9$





47. same except for domains: $-\infty < x < \infty, -1 < x < 1, 0 < x$

49. $\cos 2t = \cos^2 t - \sin^2 t$

Exercises 7.4

1. $\langle -1, 1, 0\rangle$ **3.** $\langle 1, 1, -1\rangle$ **5.** does not exist

7. $t \le -1, 1 \le t < 2, t > 2$ **9.** $t \ne \dfrac{n\pi}{2}$, n odd

11. $t < -3, -3 < t \le -1, t > 0$

13. $0 \le t < \frac{\pi}{2}, \frac{\pi}{2} < t \le 4$ **15.** $\left\langle 4t^3, \dfrac{1}{2\sqrt{t+1}}, -\dfrac{6}{t^3}\right\rangle$

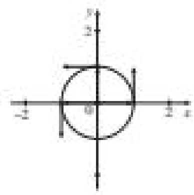
17. $\langle \cos t, 2t\cos t^2, -\sin t\rangle$

19. $\langle 2te^{t^2}, 2te^{2t}(t+1), 2\sec 2t \tan 2t\rangle$

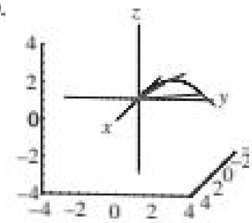
21. Smooth except $t = 1$ **23.** Smooth except $t = \frac{n\pi}{2}$, n odd

25. Smooth for all $t > 0$

27.



29.



31. $\left\langle \dfrac{3}{2}t^2 - t, \dfrac{2}{3}t^{3/2}\right\rangle + c$

33. $\left\langle \dfrac{1}{3}t\sin 3t + \dfrac{1}{9}\cos 9t, -\dfrac{1}{2}\cos t^2, \dfrac{1}{2}e^{2t}\right\rangle + c$

35. $\left\langle 4\ln\left|\dfrac{t-1}{t}\right|, \ln(t^2+1), 4\tan^{-1} t\right\rangle + c$

37. $\left\langle -\dfrac{2}{3}, \dfrac{3}{2}\right\rangle$

39. $\langle 4\ln 3, 1 - e^{-2}, e^2 + 1\rangle$ **41.** all t **43.** $t = 0$

45. In exercise 42 if there existed a t_0 and k different from 0 such that <2 cos t_0; sin t_0/> = k h<−2 sin t_0; cos t_0.>, then we would have $\sin t_0 = k\cos t_0$ and $\cos t_0 = -k\sin t_0$. This would mean that $\sin t_0 = -k^2 \sin t_0$, and this cannot happen for real k unless $\sin t_0 = 0$. But, if $\sin t_0 = 0$, then $\cos t_0$ 6 is different from 0, so this cannot occur.

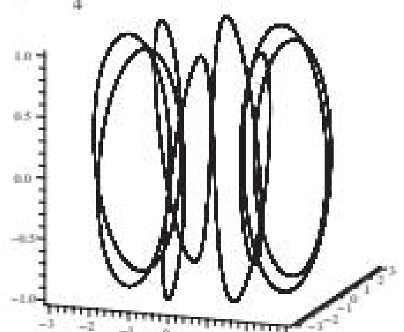
47. (a) $t = 0$ (b) No such t exists (c) For all real t

49. (a) $t = \dfrac{n\pi}{4}$, n odd

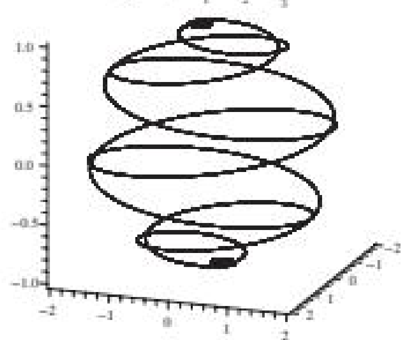
(b) $t = n\pi$, n is an integer

(c) $t = \dfrac{n\pi}{4}$, n odd

51.



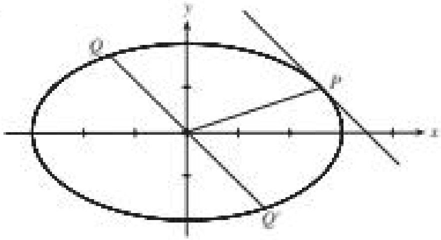
53.



55. Period is 2π for all rational numbers a and b except 0.

57. false **59.** false

61.

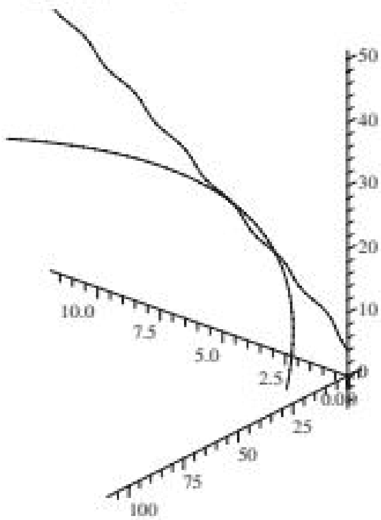


$a = 3, b = 2, t = \frac{\pi}{6}$

63. $\mathbf{f}'(t) \cdot [\mathbf{g}(t) \times \mathbf{h}(t)] + \mathbf{f}(t) \cdot [\mathbf{g}'(t) \times \mathbf{h}(t) + \mathbf{g}(t) \times \mathbf{h}'(t)]$

Applications

1. No $[\mathbf{f}(t) = \mathbf{g}(t)$, for all $t]$



Exercises 7.5

1. $\langle -10 \sin 2t, 10 \cos 2t \rangle, \langle -20 \cos 2t, -20 \sin 2t \rangle$

3. $\langle 25, -32t + 15 \rangle, \langle 0, -32 \rangle$

5. $\left\langle 4(1 - 2t)e^{-2t}, \frac{1}{\sqrt{t^2 + 1}}, \frac{1 - t^2}{(1 + t^2)^2} \right\rangle;$
$\left\langle 16(t - 1)e^{-2t}, \frac{1}{(1 + t^2)^{3/2}}, \frac{2t(t^2 - 3)}{(1 + t^2)^{3/2}} \right\rangle$

7. $\langle 10t + 3, -16t^2 + 4t + 8 \rangle$ **9.** $r(t) = \langle 5t, -16t^2 + 16 \rangle$

11. $\left\langle 8(1 + t^{3/2}), \frac{1}{2} \ln(1 + t^2) - 2, -e^t(t + 1) + 2 \right\rangle$

13. $\left\langle \frac{1}{6}t^3 + 12t + 5, -4t, -8t^2 + 2 \right\rangle$ **15.** $-160\langle \cos 2t, \sin 2t \rangle$

17. $-960\langle \cos 4t, \sin 4t \rangle$ **19.** $\langle -120 \cos 2t, -200 \sin 2t \rangle$

21. $\langle 120, 0 \rangle$

23. Max altitude: 367.5 m; Horizontal range: $490\sqrt{3}$ m; speed at impact: 98 m/s

25. Max altitude: 61.25 m; Horizontal range: 245 m; speed at impact: 49 m/s

27. Max altitude: 147.751 m; Horizontal range: 323.8 m; speed at impact: 61.6119 m/s

29. Quadruples the horizontal range

31. $\mathbf{r}(t) = v_0 \cos \theta t\mathbf{i} + \left(v_0 \sin \theta t - \frac{1}{2}gt^2 + h\right)\mathbf{j}$

33. 10 rad/s **35.** $\frac{225}{2\pi}$ rad/s^2

37. the additional rotation increases speed by a factor of $\sqrt{3}$

43. $a = 100, b = -1, c = 10$

45. $\left\langle r(t) = \frac{49}{\sqrt{2}}t, \frac{49}{\sqrt{2}}t, 49\sqrt{3}t - 4.9t^2 \right\rangle;$

47. 33.64 m/s

Applications

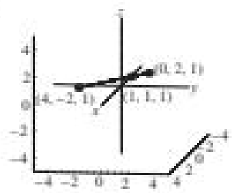
1. 56.57ft/s **3.** 1275.5 m

5. $w = \frac{\pi}{43082}, b \approx 42,168$ km

Chapter 7 Review Exercises

1. $\langle -2, 1, 4 \rangle$ **3.** $-4\mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$ **5.** $\pm\frac{1}{\sqrt{21}}\langle -2, 1, 4 \rangle$

7.



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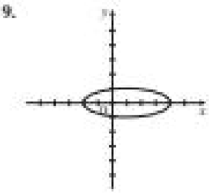

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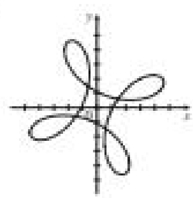

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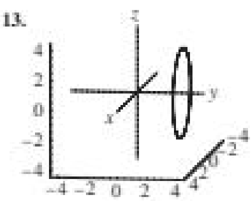

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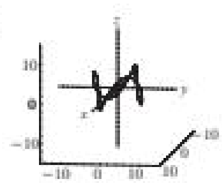

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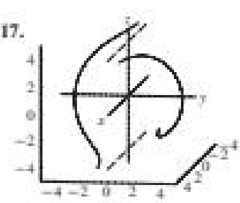
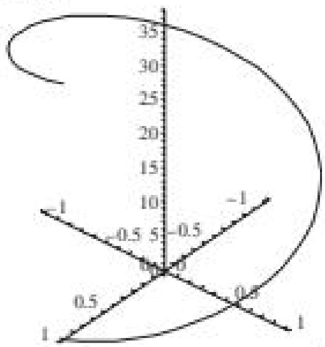


19. (a) B (b) C (c) A (d) F (e) D (f) E

21. $2\pi\sqrt{37}$



23. $\langle 0, e^2, -1\rangle$ **25.** $t \neq 0$ **27.** $\langle \frac{1}{\sqrt{t^2+1}}, 4\cos 4t, \frac{1}{t}\rangle$

29. $\left\langle -\frac{1}{4}e^{-4t}, -t^{-2}, 2t^2 - t\right\rangle + \mathbf{c}$ **31.** $\langle 0, 2, 2\rangle$

33. $\langle -8\sin 2t, 8\cos 2t, 4\rangle, \langle -16\cos 2t, -16\sin 2t, 0\rangle$

35. $\langle t^2 + 4t + 2, -16t^2 + 1\rangle$ **37.** $\langle 4t + 2, -16t^2 + 3t + 6\rangle$

39. $\langle 0, -128\rangle$ **41.** $25(2 - \sqrt{3}) \approx 21.9$ m, 326.5 m, 80 m/s

43. $\frac{1}{\sqrt{2}}\langle -1, 1, 0\rangle, \frac{1}{\sqrt{e^{-4}+1}}\langle -e^{-2}, 1, 0\rangle$

Student Handbook

Symbols, Formulas, and Key Concepts

Chapter Sourced From: EM. End Matter (Glossary), from Calculus: Early Transcendental Functions 5e C © 2018
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Symbols

Algebra	
$\neq$	is not equal to
$\approx$	is approximately equal to
$\sim$	is similar to
$>, \geq$	is greater than, is greater than or equal to
$<, \leq$	is less than, is less than or equal to
$-a$	opposite or additive inverse of *a*
$\lvert a \rvert$	absolute value of *a*
$\sqrt{a}$	principal square root of *a*
$a : b$	ratio of *a* to *b*
(x, y)	ordered pair
(x, y, z)	ordered triple
i	the imaginary unit
$b^{\frac{1}{n}} = \sqrt[n]{b}$	nth root of b
$\mathbb{Q}$	rational numbers
$\mathbb{I}$	irrational numbers
$\mathbb{Z}$	integers
$\mathbb{W}$	whole numbers
$\mathbb{N}$	natural numbers
∞	infinity
$-\infty$	negative infinity
[ ]	endpoint included
( )	endpoints not included
$\log_b x$	logarithm base b of x
$\log x$	common logarithm of x
$\ln x$	natural logarithm of x
ω	omega, angular speed
α	alpha, angle measure
β	beta, angle measure
γ	gamma, angle measure
θ	theta, angle measure
λ	lambda, wavelength
ϕ	phi, angle measure
a	vector **a**
$\lvert \mathbf{a} \rvert$	magnitude of vector **a**

Sets and Logic	
$\in$	is an element of
$\subset$	is a subset of
$\cap$	intersection
$\cup$	union
$\varnothing$	empty set
$\sim p$	negation of *p*, not *p*
$p \wedge q$	conjunction of *p* and *q*
$p \vee q$	disjunction of *p* and *q*
$p \longrightarrow q$	conditional statement, if *p* then *q*
$p \longleftrightarrow q$	biconditional statement, *p* if and only if *q*

Geometry	
$\angle$	angle
$\triangle$	triangle
$^\circ$	degree
π	pi
$\measuredangle$	angles
$m\angle A$	degree measure of $\angle A$
$\overleftrightarrow{AB}$	line containing points *A* and *B*
$\overline{AB}$	segment with endpoints *A* and *B*
$\overrightarrow{AB}$	ray with endpoint *A* containing *B*
AB	measure of $\overline{AB}$, distance between points *A* and *B*
$\parallel$	is parallel to
$\nparallel$	is not parallel to
$\perp$	is perpendicular to
$\triangle$	triangle
▱	parallelogram
n-gon	polygon with *n* sides
$\vec{a}$	vector *a*
$\vec{AB}$	vector from *A* to *B*
$\lvert \vec{AB} \rvert$	magnitude of the vector from *A* to *B*
A'	the image of preimage *A*
$\rightarrow$	is mapped onto
$\odot A$	circle with center *A*
$\overset{\frown}{AB}$	minor arc with endpoints *A* and *B*
$\overset{\frown}{ABC}$	major arc with endpoints *A* and *C*
$m\overset{\frown}{AB}$	degree measure of arc *AB*

Trigonometry	
$\sin x$	sine of *x*
$\cos x$	cosine of *x*
$\tan x$	tangent of *x*
$\sin^{-1} x$	Arcsin *x*
$\cos^{-1} x$	Arccos *x*
$\tan^{-1} x$	Arctan *x*

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Symbols

Functions	
$f(x)$	f of x, the value of f at x
$f(x) = \{$	piecewise-defined function
$f(x) = \lvert x \rvert$	absolute value function
$f(x) = [\![x]\!]$	function of greatest integer not greater than x
$f(x, y)$	f of x and y, a function with two variables, x and y
$[f \circ g](x)$	f of g of x, the composition of functions f and g
$f^{-1}(x)$	inverse of $f(x)$

Calculus	
$\lim\limits_{x \to c}$	limit as x approaches c
m_{sec}	slope of a secant line
$f'(x)$	derivative of $f(x)$
Δ	delta, change
$\int$	indefinite integral
$\int_a^b$	definite integral
$F(x)$	antiderivative of $f(x)$

Probability and Statistics	
$P(a)$	probability of a
$P(n, r)$ or ${}_nP_r$	permutation of n objects taken r at a time
$C(n, r)$ or ${}_nC_r$	combination of n objects taken r at a time
$P(A)$	probability of A
$P(A \mid B)$	the probability of A given that B has already occurred
$n!$	factorial of n (n being a natural number)
Σ	sigma (uppercase), summation
μ	mu, population mean
σ	sigma (lowercase), population standard deviation
σ^2	population variance
s	sample standard deviation
s^2	sample variance
$\sum\limits_{n=1}^{k}$	summation from $n = 1$ to k
$\bar{x}$	x-bar, sample mean
H_0	null hypothesis
H_a	alternative hypothesis

Measures

Metric	Customary
Length	
1 kilometer (km) = 1000 meters (m) 1 meter = 100 centimeters (cm) 1 centimeter = 10 millimeters (mm)	1 mile (mi) = 1760 yards (yd) 1 mile = 5280 feet (ft) 1 yard = 3 feet 1 foot = 12 inches (in) 1 yard = 36 inches
Volume and Capacity	
1 liter (L) = 1000 milliliters (mL) 1 kiloliter (kL) = 1000 liters	1 gallon (gal) = 4 quarts (qt) 1 gallon = 128 fluid ounces (fl oz) 1 quart = 2 pints (pt) 1 pint = 2 cups (c) 1 cup = 8 fluid ounces
Weight and Mass	
1 kilogram (kg) = 1000 grams (g) 1 gram = 1000 milligrams (mg) 1 metric ton (t) = 1000 kilograms	1 ton (T) = 2000 pounds (lb) 1 pound = 16 ounces (oz)

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Arithmetic Operations and Relations

Identity	For any number a, $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
Substitution (=)	If $a = b$, then a may be replaced by b.
Reflexive (=)	$a = a$
Symmetric (=)	If $a = b$, then $b = a$.
Transitive (=)	If $a = b$ and $b = c$, then $a = c$.
Commutative	For any numbers a and b, $a + b = b + a$ and $a \cdot b = b \cdot a$.
Associative	For any numbers a, b, and c, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
Distributive	For any numbers a, b, and c, $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$.
Additive Inverse	For any number a, there is exactly one number $-a$ such that $a + (-a) = 0$.
Multiplicative Inverse	For any number $\frac{a}{b}$, where $a, b \neq 0$, there is exactly one number $\frac{b}{a}$ such that $\frac{a}{b} \cdot \frac{b}{a} = 1$.
Multiplicative (0)	For any number a, $a \cdot 0 = 0 \cdot a = 0$.
Addition (=)	For any numbers a, b, and c, if $a = b$, then $a + c = b + c$.
Subtraction (=)	For any numbers a, b, and c, if $a = b$, then $a - c = b - c$.
Multiplication and Division (=)	For any numbers a, b, and c, with $c \neq 0$, if $a = b$, then $ac = bc$ and $\frac{a}{c} = \frac{b}{c}$.
Addition (>)*	For any numbers a, b, and c, if $a > b$, then $a + c > b + c$.
Subtraction (>)*	For any numbers a, b, and c, if $a > b$, then $a - c > b - c$.
Multiplication and Division (>)*	For any numbers a, b, and c, 1. if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$. 2. if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
Zero Product	For any real numbers a and b, if $ab = 0$, then $a = 0$, $b = 0$, or both a and b equal 0.

* *These properties are also true for* $<$, $\geq$, *and* $\leq$.

Algebraic Formulas and Key Concepts

Matrices

Adding	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$	**Multiplying by a Scalar**	$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$
Subtracting	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$	**Multiplying**	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$

Polynomials

Quadratic Formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a \neq 0$	**Square of a Difference**	$(a - b)^2 = (a - b)(a - b)$ $= a^2 - 2ab + b^2$
Square of a Sum	$(a + b)^2 = (a + b)(a + b)$ $= a^2 + 2ab + b^2$	**Product of Sum and Difference**	$(a + b)(a - b) = (a - b)(a + b)$ $= a^2 - b^2$

Logarithms

Product Property	$\log_x ab = \log_x a + \log_x b$	**Power Property**	$\log_b m^p = p \log_b m$
Quotient Property	$\log_x \frac{a}{b} = \log_x a - \log_x b$, $b \neq 0$	**Change of Base**	$\log_a n = \frac{\log_b n}{\log_b a}$

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Algebraic Formulas and Key Concepts

Exponential and Logarithmic Functions

Compound Interest	$A = P\left(1 + \frac{r}{n}\right)^{nt}$	**Exponential Growth or Decay**	$N = N_0(1 + r)^t$
Continuous Compound Interest	$A = Pe^{rt}$	**Continuous Exponential Growth or Decay**	$N = N_0e^{kt}$
Product Property	$\log_b xy = \log_b x + \log_b y$	**Power Property**	$\log_b x^p = p \log_b x$
Quotient Property	$\log_b \frac{x}{y} = \log_b x - \log_b y$	**Change of Base**	$\log_b x = \frac{\log_a x}{\log_a b}$
Logistic Growth	$f(t) = \frac{c}{1 + ae^{-bt}}$		

Sequences and Series

***n*th term, Arithmetic**	$a_n = a_1 + (n - 1)d$	***n*th term, Geometric**	$a_n = a_1r^{n-1}$
Sum of Arithmetic Series	$S_n = n\left(\frac{a_1 + a_n}{2}\right)$ or $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$	**Sum of Geometric Series**	$S_n = \frac{a_1 - a_1r^n}{1 - r}$ or $S_n = \frac{a_1 - a_nr}{1 - r}, r \neq 1$
Sum of Infinite Geometric Series	$S = \frac{a_1}{1 - r}, \lvert r \rvert < 1$	**Euler's Formula**	$e^{i\theta} = \cos\theta + i\sin\theta$
Power Series	$\sum_{n=0}^{\infty} a_nx^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$	**Exponential Series**	$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
Binomial Theorem	$(a + b)^n = {}_nC_0\, a^nb^0 + {}_nC_1\, a^{n-1}b^1 + {}_nC_2\, a^{n-2}b^2 + \ldots + {}_nC_r\, a^{n-r}b^r + \ldots + {}_nC_n\, a^0b^n$		
Cosine and Sine Power Series	$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^nx^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$		$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^nx^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Vectors

Addition in Plane	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2\rangle$	**Addition in Space**	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$
Subtraction in Plane	$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2\rangle$	**Subtraction in Space**	$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3\rangle$
Scalar Multiplication in Plane	$k\mathbf{a} = \langle ka_1, ka_2\rangle$	**Scalar Multiplication in Space**	$k\mathbf{a} = \langle ka_1, ka_2, ka_3\rangle$
Dot Product in Plane	$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$	**Dot Product in Space**	$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
Angle Between Two Vectors	$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$	**Projection of u onto v**	$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\lvert\mathbf{v}\rvert^2}\right)\mathbf{v}$
Magnitude of a Vector	$\lvert\mathbf{v}\rvert = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	**Triple Scalar Product**	$\mathbf{t} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} t_1 & t_2 & t_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

Equations of a Line on a Coordinate Plane

Slope-intercept form of a line	$y = mx + b$
Point-slope form of a line	$y - y_1 = m(x - x_1)$

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Algebraic Formulas and Key Concepts

Conic Sections			
Parabola	$(x-h)^2 = 4p(y-k)$ or $(y-k)^2 = 4p(x-h)$	**Circle**	$x^2 + y^2 = r^2$ or $(x-h)^2 + (y-k)^2 = r^2$
Ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$	**Hyperbola**	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Rotation of Conics	$x' = x \cos \theta + y \sin \theta$ and $y' = y \cos \theta - x \sin \theta$		

Parametric Equations			
Vertical Position	$y = tv_0 \sin \theta - \frac{1}{2}gt^2 + h_0$	**Horizontal Distance**	$x = tv_0 \cos \theta$

Complex Numbers			
Product Formula	$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$	**Quotient Formula**	$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$
Distinct Roots Formula	$r^{\frac{1}{p}}\left(\cos \frac{\theta + 2n\pi}{p} + i \sin \frac{\theta + 2n\pi}{p}\right)$	**De Moivre's Theorem**	$z^n = [r(\cos \theta + i \sin \theta)]^n$ or $r^n(\cos n\theta + i \sin n\theta)$

Geometric Formulas and Key Concepts

Coordinate Geometry			
Slope	$m = \frac{y_2 - y_1}{x_2 - x_1}, x_2 \neq x_1$	**Distance on a number line**	$d = \|a - b\|$
Distance on a coordinate plane	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	**Arc length**	$\ell = \frac{x}{360} \cdot 2\pi r$
Midpoint on a coordinate plane	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$	**Midpoint on a number line**	$M = \frac{a + b}{2}$
Midpoint in space	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$	**Pythagorean Theorem**	$a^2 + b^2 = c^2$
Distance in space	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$		

Perimeter and Circumference					
Square	$P = 4s$	**Rectangle**	$P = 2\ell + 2w$	**Circle**	$C = 2\pi r$ or $C = \pi d$

Lateral Surface Area			
Prism	$L = Ph$	**Pyramid**	$L = \frac{1}{2}P\ell$
Cylinder	$L = 2\pi rh$	**Cone**	$L = \pi r\ell$

Total Surface Area					
Prism	$S = Ph + 2B$	**Cone**	$S = \pi r\ell + \pi r^2$	**Cylinder**	$S = 2\pi rh + 2\pi r^2$
Pyramid	$S = \frac{1}{2}P\ell + B$	**Sphere**	$S = 4\pi r^2$	**Cube**	$S = 6s^2$

Volume					
Prism	$V = Bh$	**Cone**	$V = \frac{1}{3}\pi r^2 h$	**Cylinder**	$V = \pi r^2 h$
Pyramid	$V = \frac{1}{3}Bh$	**Sphere**	$V = \frac{4}{3}\pi r^3$	**Cube**	$V = s^3$
Rectangular prism	$V = \ell wh$				

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Trigonometric Functions and Identities

Trigonometric Functions			
Trigonometric Functions	$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin\theta}$	$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\cos\theta}$	$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\cos\theta}{\sin\theta}$
Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos A$	$b^2 = a^2 + c^2 - 2ac\cos B$	$c^2 = a^2 + b^2 - 2ab\cos C$
Law of Sines	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	**Heron's Formula**	$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$
Linear Speed	$v = \frac{s}{t}$	**Angular Speed**	$\omega = \frac{\theta}{t}$

Trigonometric Identities			
Reciprocal	$\sin\theta = \frac{1}{\csc\theta}$ $\csc\theta = \frac{1}{\sin\theta}$	$\cos\theta = \frac{1}{\sec\theta}$ $\sec\theta = \frac{1}{\cos\theta}$	$\tan\theta = \frac{1}{\cot\theta}$ $\cot\theta = \frac{1}{\tan\theta}$
Pythagorean	$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$\cot^2\theta + 1 = \csc^2\theta$
Cofunction	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$	$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$	$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$
Odd-Even	$\sin(-\theta) = -\sin\theta$ $\csc(-\theta) = -\csc\theta$	$\cos(-\theta) = \cos\theta$ $\sec(-\theta) = \sec\theta$	$\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$
Sum & Difference	$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$	$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$	
Double-Angle	$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = 2\cos^2\theta - 1$ $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$	$\cos 2\theta = 1 - 2\sin^2\theta$
Power-Reducing	$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$	$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$	$\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$
Half-Angle	$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}$ $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}$	$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$ $\tan\frac{\theta}{2} = \frac{1 - \cos\theta}{\sin\theta}$	$\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta}$
Product-to-Sum	$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$ $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$	$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$ $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$	
Sum-to-Product	$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$ $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$	$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$ $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$	

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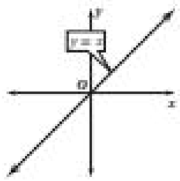
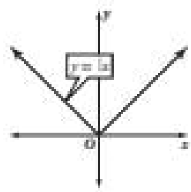
Parent Functions and Function Operations

Parent Functions

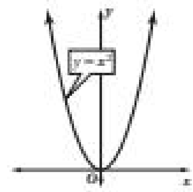
Linear Functions



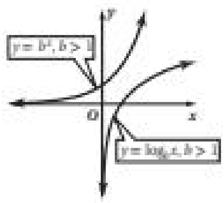
Absolute Value Functions



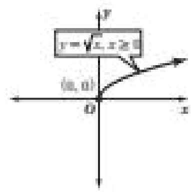
Quadratic Functions



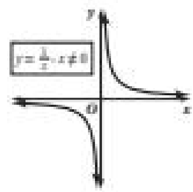
Exponential and Logarithmic Functions



Square Root Functions



Reciprocal and Rational Functions



Function Operations

Addition	$(f + g)(x) = f(x) + g(x)$	**Multiplication**	$(f \cdot g)(x) = f(x) \cdot g(x)$
Subtraction	$(f - g)(x) = f(x) - g(x)$	**Division**	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0$

Calculus

Limits

Sum	$\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$	**Difference**	$\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
Scalar Multiple	$\lim_{x \to c} [k\, f(x)] = k \lim_{x \to c} f(x)$	**Product**	$\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
Quotient	$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, if $\lim_{x \to c} g(x) \neq 0$	**Power**	$\lim_{x \to c} [f(x)^n] = \left[\lim_{x \to c} f(x)\right]^n$
***n*th Root**	$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$, if $\lim_{x \to c} f(x) > 0$ when n is even	**Velocity**	**Average** $v_{avg} = \frac{f(b) - f(a)}{b - a}$ **Instantaneous** $v(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}$

Derivatives

Power Rule	If $f(x) = x^n$, $f'(x) = nx^{n-1}$	**Sum or Difference**	If $f(x) = g(x) \pm h(x)$, then $f'(x) = g'(x) \pm h'(x)$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$	**Quotient Rule**	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Integrals

Indefinite Integral	$\int f(x)\,dx = F(x) + C$	**Fundamental Theorem of Calculus**	$\int_a^b f(x)\,dx = F(b) - F(a)$

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Statistics Formulas and Key Concepts

z-Values	$z = \frac{X - \mu}{\sigma}$	*z*-Value of a Sample Mean	$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$
Binomial Probability	$P(X) = {}_nC_x\, p^x q^{n-x} = \frac{n!}{(n-x)!\,x!}\, p^x q^{n-x}$	Maximum Error of Estimate	$E = z \cdot \sigma_{\bar{x}}$ or $z \cdot \frac{\sigma}{\sqrt{n}}$
Confidence Interval, Normal Distribution	$CI = \bar{x} \pm E$ or $\bar{x} \pm z \cdot \frac{\sigma}{\sqrt{n}}$	Confidence Interval, *t*-Distribution	$CI = \bar{x} \pm t \cdot \frac{s}{\sqrt{n}}$
Correlation Coefficient	$r = \frac{1}{n-1}\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$	*t*-Test for the Correlation Coefficient	$t = r\sqrt{\frac{n-2}{1-r^2}}$, degrees of freedom: $n - 2$

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Cover label guide

Cycle 03 Color

Icon indicates the book type (**Example, Course book**)

Number 3 has been shaded to represent the **volume of Course book**

Number 9 Shows grade level

Shows stream type

- Course book
- Activity book (workbook)
- Reading book (Arabic, social)
- Interactive Student Guide (Math)
- Activity Lab Manual (Science)
- Teacher Guide

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