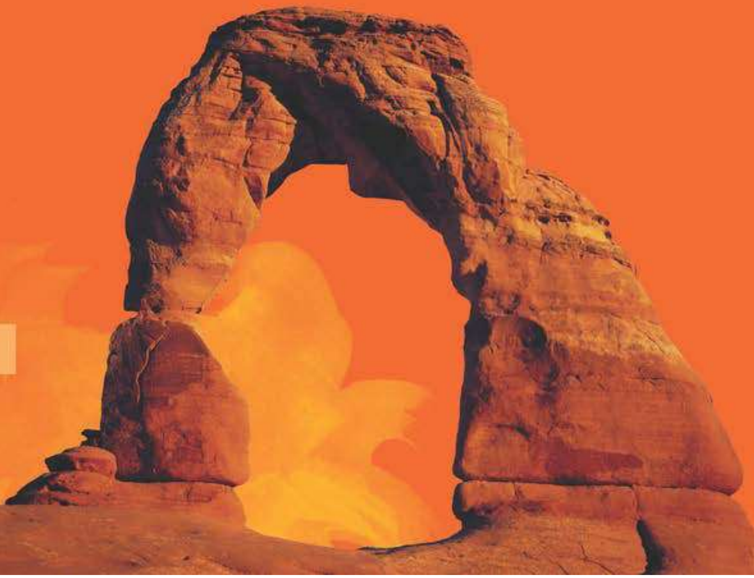
Teacher Edition

Volume 2

Reveal MATH™

Integrated III



McGraw Hill

Teacher Edition

Volume 2

Reveal MATH®

Integrated III

Mc Graw Hill

www.my.mheducation.com

Cover: (t to b, l to r) Ivan Kurmyshov/Shutterstock, Mike Watson Images/Getty Images

Send all inquiries to:
McGraw-Hill Education
8787 Orion Place
Columbus, OH 43240

ISBN: 978-0-07-703207-4
MHID: 0-07-703207-1

Printed in the United States of America.

3 4 5 6 7 8 9 10 11 WEB 26 25 24 23 22 21 20

Contents in Brief

Reveal Math Guiding Principles

Academic research and the science of learning provide the foundation for this powerful K–12 math program designed to help reveal the mathematician in every student.

> *Reveal Math* is built on a solid foundation of **RESEARCH** that shaped the **PEDAGOGY** of the program.

Reveal Math Integrated I, Integrated II, and Integrated III (Reveal Math Integrated) used findings from research on teaching and learning mathematics to develop its instructional model. Based on analyses of research findings, these areas form the foundational structure of the program:

- Rigor
- Productive Struggle
- Formative Assessment
- Rich Tasks
- Mathematical Discourse
- Collaborative Learning

Instructional Model

1 Launch

WARM UP

During the **Warm Up,** students complete exercises to activate prior knowledge and review prerequisite concepts and skills.

LAUNCH THE LESSON

In **Launch the Lesson**, students view a real-world scenario and image to pique their interest in the lesson content. They are introduced to questions that they will be able to answer at the end of the lesson.

EXPLORE

During the **Explore** activity, students work in partners or small groups to explore a rich mathematical problem related to the lesson content.

INDIVIDUAL ACTIVITY
GROUP ACTIVITY
CLASS ACTIVITY

Reveal the full potential
in every student!

2 Explore and Develop

LEARN

In the **Learn** section, students gain the foundational knowledge needed to actively work through upcoming Examples.

EXAMPLES & CHECK

Students work through **Examples** related to the key concepts and engage in mathematical discourse.

Students complete a **Check** after several Examples as a quick formative assessment to help teachers adjust instruction as needed.

3 Reflect and Practice

EXIT TICKET

The **Exit Ticket** gives students an opportunity to convey their understanding of the lesson concepts.

PRACTICE

Students complete **Practice** exercises individually or collaboratively to solidify their understanding of lesson concepts and build proficiency with lesson skills.

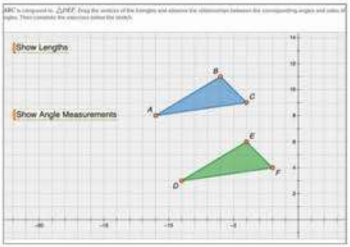
Klaus Vedfelt/DigitalVision/Getty Images

Reveal Math Key Areas of Focus

Reveal Math Integrated I, II, III (Reveal Math Integrated) have a strong focus on rigor—especially the development of conceptual understanding—an emphasis on student mindset, and ongoing formative assessment feedback loops.

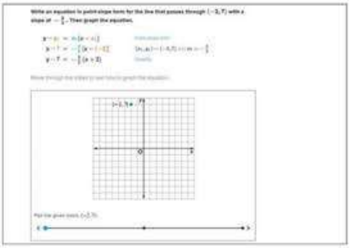
Rigor

Reveal Math Integrated has been thoughtfully designed to incorporate a balance of the three elements of rigor: conceptual understanding, procedural skills and fluency, and application.



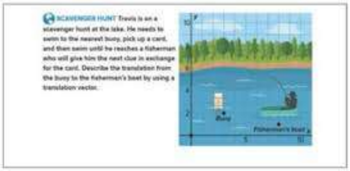
Conceptual Understanding

Explore activities give all students an opportunity to work collaboratively and discuss their thinking as they build conceptual understanding of new concepts. In the **Explore** activity to the left, students use **Web Sketchpad**® to build understanding of the relationships between corresponding sides and angles in congruent triangles.



Procedural Skills and Fluency

Students use different strategies and tools to build procedural fluency. In the **Example** shown, students build proficiency with writing equations in point-slope form.



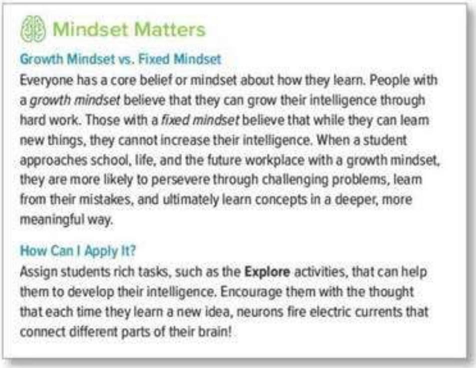
SCAVENGER HUNT Travis is on a scavenger hunt at the lake. He needs to swim to the nearest buoy, pick up a card, and then swim until he reaches a fisherman who will give him the next clue in exchange for the card. Describe the translation from the buoy to the fisherman's boat by using a translation vector.

Application

Real-world examples and practice problems are opportunities for students to apply their learning to new situations. In the real-world example shown, students apply their understanding by solving a multi-step problem with translations.

Student Mindset

Mindset Matters tips located in each module provide specific examples of how Reveal Math Integrated content can be used to promote a growth mindset in all students. Another feature focused on promoting a growth mindset is **Ignite! Activities** developed by Dr. Raj Shah to spark student curiosity about why the math works. An **Ignite!** delivers problem sets that are flexible enough so that students with varying background knowledge can engage with the content and motivates them to ask questions, solve complex problems, and develop a can-do attitude toward math.



Mindset Matters

Growth Mindset vs. Fixed Mindset

Everyone has a core belief or mindset about how they learn. People with a *growth mindset* believe that they can grow their intelligence through hard work. Those with a *fixed mindset* believe that while they can learn new things, they cannot increase their intelligence. When a student approaches school, life, and the future workplace with a growth mindset, they are more likely to persevere through challenging problems, learn from their mistakes, and ultimately learn concepts in a deeper, more meaningful way.

How Can I Apply It?

Assign students rich tasks, such as the **Explore** activities, that can help them to develop their intelligence. Encourage them with the thought that each time they learn a new idea, neurons fire electric currents that connect different parts of their brain!

Teacher Edition Mindset Tip

IGNITE!

Student Ignite! Activity

Formative Assessment

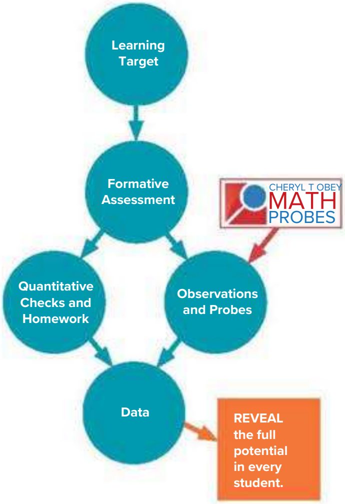
The key to reaching all learners is to adjust instruction based on each student's understanding. Reveal Math Integrated offers powerful formative assessment tools that help teachers to efficiently and effectively differentiate instruction for all students.

Math Probes

Each module includes a **Cheryl Tobey Formative Assessment Math Probe** that is focused on addressing student misconceptions about key math topics. Students can complete these probes at the beginning, middle, or end of a module. The teacher support includes a list of recommended differentiated resources that teachers assign based on students' responses.

Example Checks

After multiple examples, a formative assessment **Check** that students complete on their own allows teachers to gauge students' understanding of the concept or skill presented. When students complete the Check online, the teacher receives resource recommendations which can be assigned to students.



A Powerful Blended Learning Experience

The *Reveal Math Integrated I, II, III* (Reveal Math Integrated) blended learning experience was designed to include purposeful print and digital components focused on sparking student curiosity and providing teachers with flexible implementation options.

Reveal Math Integrated has been thoughtfully developed to provide a rich learning experience no matter where a district, school, or classroom falls on the digital spectrum. All of the instructional content can be projected or can be accessed via desktop, laptop, or tablet.

Lesson

1 Launch

WARM UP	LAUNCH THE LESSON	EXPLORE
The **Warm Up** exercise can be projected on an interactive whiteboard.	**Launch the Lesson** can be projected or assigned to students to access on their own devices.	The **Explore** activity can be projected while students record their observations in the Interactive Student Edition or can be assigned for students to complete on individual devices.

Launch the Lesson

Explore

INDIVIDUAL ACTIVITY

GROUP ACTIVITY

CLASS ACTIVITY

INTERACTIVE PRESENTATION

PRINT STUDENT EDITION

(l)Andrey_Kuzmin/iStock/Getty Images, (r)McGraw-Hill Education

2 Explore and Develop

LEARN

As students are introduced to the key lesson concepts, they can progress through the **Learn** by recording notes in a notebook or on their own devices.

EXAMPLES & CHECK

Either in a notebook or on an individual device, students work through one or more **Examples** related to key lesson concepts.

A **Check** follows several Examples in either the Student Edition or on each student device.

3 Reflect and Practice

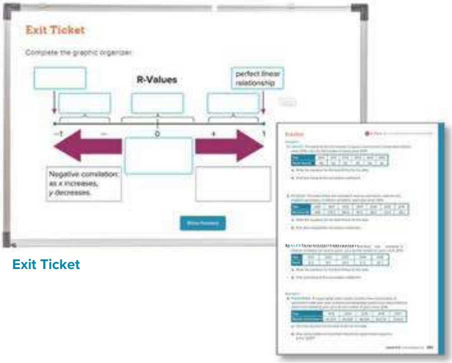
EXIT TICKET

The **Exit Ticket** is projected or accessed via student devices to provide students with lesson closure and an opportunity to revisit the lesson concepts.

PRACTICE

Assign students **Practice** problems from their Student Edition or create a digital assignment for them to work on their device in class or at home to solidify lesson concepts.

Digital Lesson Presentation is aligned to Interactive Student Edition



Exit Ticket

Practice

(l)McGraw-Hill Education, (r)Andrey_Kuzmin/iStock/Getty Images

Supporting All Learners

The *Reveal Math Integrated I, II, III* (Reveal Math Integrated) programs were designed so that all students have access to:

- rich tasks that promote productive struggle,
- opportunities to develop proficiency with the habits of mind and thinking strategies of mathematicians, and
- prompts to promote mathematical discourse and build academic language.

Resources for Differentiating Instruction

When needed, resources are available to differentiate math instruction for students who may need to see a concept in a different way, practice prerequisite skills, or are ready to extend their learning.

AL

Approaching Level Resources

- Remediation Activities
- Extra Examples

BL

Beyond Level Resources

- Beyond Level Differentiated Activities
- Extension Activities

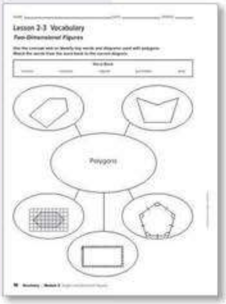
Resources for English Language Learners

Reveal Math Integrated also includes student and teacher resources to support students who are simultaneously learning grade-level math and building their English proficiency. Appropriate, research-based language scaffolds are also provided to support students as they engage in rigorous mathematical tasks and discussions.

ELL

English Language Learners

- Spanish Personal Tutors
- Math Language-Building Activities
- Language Scaffolds
- *Think About It!* and *Talk About It!* Prompts
- Multilingual eGlossary
- Audio
- Graphic Organizers
- Web Sketchpad, Desmos, and eTools



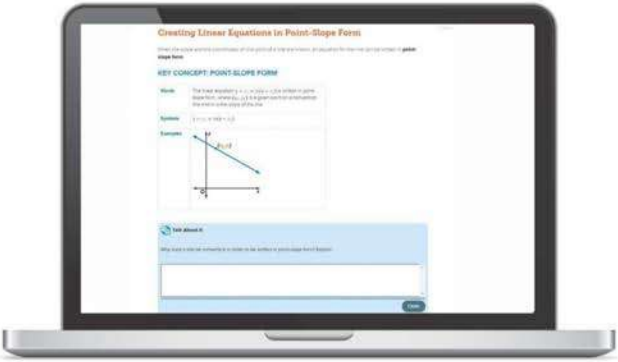
Lesson 2-3 Vocabulary

Two-Dimensional Figures

Polygons

Developing Mathematical Thinking and Strategic Questioning

Reveal Math Integrated I, II, III (Reveal Math Integrated) are comprised of high-quality math content designed to be accessible and relevant to each student. Throughout the program, students are presented with a variety of thoughtfully designed questioning strategies related to the content. Using these questions provides you with an additional, built-in type of formative assessment that can be used to modify instruction. They also strengthen students' ownership of mathematical content knowledge and daily use of the Standards for Mathematical Practice.



Key Concept Introduction followed by a Talk About It question to discuss with a classmate.

You will find these types of questioning strategies throughout Reveal Math Integrated. The related Standard for Mathematical Practice for each is also indicated.

- **Talk About It** questions encourage students to engage in mathematical discourse with classmates (SMP3)
- **Alternate Method** shows students another way to solve a problem and asks them to compare and contrast the methods and solutions (SMP1)
- **Avoid a Common Error** shows students a problem similar to an example but with a flaw in reasoning, and students have to find and explain the error (SMP3)
- **State Your Assumptions** requires that student state the assumptions they made to solve a problem (SMP4)
- **Use a Source** asks students to find information using an external source, such as the Internet, and use it to pose or solve a problem (SMP5)
- **Think About It** questions help students make sense of mathematical problems (SMP1)
- **Concept Checks** prompt students to analyze how the Key Concepts of the lesson apply to various use cases (SMP3)

Scott Dunlap/iStockvectors/Getty Images

Reveal Student Readiness with Individualized Learning Tools

Reveal Math Integrated I, II, III (Reveal Math Integrated) incorporate innovative, technology-based tools that are designed to extend the teacher's reach in the classroom to help address a wide range of knowledge gaps, set and align academic goals, and meet student individualized learning needs.

LEARNSMART®

Topic-Mastery

With embedded **LearnSmart**,® students have a built-in study partner for topic practice and review to prepare for multi-module, mid-year, or end-of-year testing.

LearnSmart's revolutionary adaptive technology measures students' awareness of their own learning, time on topic, answer accuracy, and suggests alternative resources to support student learning, confidence, and topic mastery.

ALEKS®

Individualized Learning Pathways

Learners of all levels benefit from the use of **ALEKS'** adaptive, online math technology designed to pinpoint what each student knows, does not know, and most importantly, what each student is ready to learn.

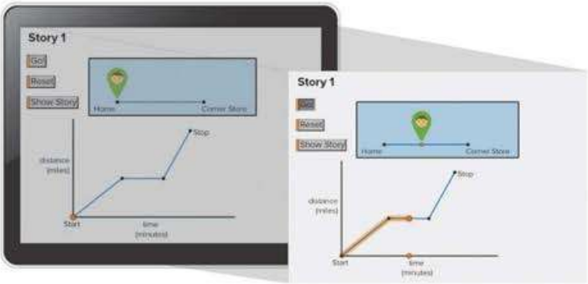
When paired with Reveal Math Integrated, **ALEKS** is a powerful tool designed to provide integrated instructionally actionable data enabling teachers to utilize Reveal Math Integrated resources for individual students, groups, or the entire classroom.

Activity Report

Scott Dunlap/iStockvectors/Getty Images

Powerful Tools for Modeling Mathematics

Reveal Math Integrated I, II, III (Reveal Math Integrated) have been designed with purposeful, embedded digital tools to increase student engagement and provide unique modeling opportunities.



Web Sketchpad® Activities

The leading dynamic mathematics visualization software has now been integrated with **Web Sketchpad Activities** at point of use within Reveal Math Integrated. Student exploration (and practice) using **Web Sketchpad** encourages problem solving and visualization of abstract math concepts.

desmos

The powerful **Desmos** graphing calculator is available in Reveal Math Integrated for students to explore, model, and apply math to the real-world.

eTools

By using a wide variety of digital **eTools** embedded within Reveal Math Integrated, students gain additional hands-on experience while they learn and teachers have the option to create problem-based learning opportunities.

Technology-Enhanced Items

Embedded within the digital lesson, technology-enhanced items—such as drag-and-drop, flashcard flips, or diagram completion—are strategically placed to give students the practice with common computer functions needed to master computer-based testing.

- TYPE
- SWIPE
- DRAG & DROP
- FLASHCARDS
- eTOOLS
- MULTI-SELECT
- WATCH
- EXPAND

McGraw-Hill Education

Standards for Mathematical Content, Reveal Math Integrated III

This correlation shows the alignment of *Reveal Math Integrated III* to the Standards for Mathematical Content from the Common Core State Standards for Mathematics.

Standard		Lesson(s)
S.ID.4	Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.	**8-4**
S.IC.1	Understand that statistics allows inferences to be made about population parameters based on a random sample from that population.	**8-1**
S.IC.2	Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?	**8-2**
S.IC.3	Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.	**8-1**
S.IC.4	Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.	**8-3, 8-5**
S.IC.5	Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.	**8-2**
S.IC.6	Evaluate reports based on data.	**8-1, 8-2, 8-4, 8-5**
S.MD.6 (+)	Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).	**8-2**
S.MD.7 (+)	Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).	**8-2**
N.CN.8 (+)	Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.	**1-3, 1-4, 1-6**
N.CN.9 (+)	Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.	**3-5**
A.SSE.1	Interpret expressions that represent a quantity in terms of its context.★ a. Interpret parts of an expression, such as terms, factors, and coefficients.	**1-1, 1-6, 2-1, 2-2, 5-1, 9-4**
	b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P.*	
A.SSE.2	Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.	**1-4, 1-5, 4-3, 4-5, 5-2, 6-1, 6-3, 6-4, 6-5**
A.SSE.4	Derive the formula for the sum of a geometric series (when the common ratio is not 1) , and use the formula to solve problems. *For example, calculate mortgage payments.* ★	**5-4**
A.APR.1	Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.	**2-3**
A.APR.2	Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.	**3-4**
A.APR.3	Identify zeros of polynomials when suitable factorizations are available and use the zeros to construct a rough graph of the function defined by the polynomial.	**3-5**

★ Mathematical Modeling Standards

Professional Development Support for Continuous Learning

McGraw-Hill Education supports lifelong learning and demonstrates commitment to teachers with a built-in professional learning environment designed for support during planning or extended learning opportunities.

What You Will Find

- Best-practice resources
- Implementation support
- Teaching Strategies
- Classroom Videos
- Math Misconception Videos
- Content and Pedagogy Videos
- Content Progression Information

Why Professional Development Is so Important

- Research-based understanding of student learning
- Improved student performance
- Evidence-based instructional best practices
- Collaborative content strategy planning
- Extended knowledge of program how-to's

Kevin Dodge/Blend Images/Getty Images

Reveal Math Expert Advisors

Cathy Seeley, Ed.D.

Austin, Texas

Mathematics educator, speaker, and writer, former Senior Fellow at the Charles A. Dana Center at The University of Texas at Austin, past President of NCTM, former Director of K-12 Mathematics for the State of Texas

Areas of expertise:
Mathematics Teaching, Equity, Assessment, STEM Learning, Informal Learning, Upside-Down Teaching, Productive Struggling, Mathematical Practices, Mathematical Habits of Mind, Family and Community Outreach, Mathematics Education Policy, Advocacy

> “We want students to believe deeply that mathematics makes sense–in generating answers to problems, discussing their thinking and other students’ thinking, and learning new material.”
>
> **—Seeley, 2016, Making Sense of Math**

Cheryl R. Tobey, M.Ed.

Gardiner, Maine

Senior Mathematics Associate at Education Development Center (EDC)

Areas of expertise:
Formative assessment and professional development for mathematics teachers; tools and strategies to uncovering misconceptions

> “Misunderstandings and partial understandings develop as a normal part of learning mathematics. Our job as educators is to minimize the chances of students’ harboring misconceptions by knowing the potential difficulties students are likely to encounter, using assessments to elicit misconceptions and implementing instruction designed to build new and accurate mathematical ideas.”
>
> **— Tobey, et al 2007, 2009, 2010, 2013, 2104, Uncovering Student Thinking Series**

Nevels Nevels, Ph.D.

Saint Louis, Missouri

PK–12 Mathematics Curriculum Coordinator for Hazelwood School District

Areas of expertise:
Mathematics Teacher Education; Student Agency & Identity; Socio-Cultural Perspective in Mathematics Learning

> “A school building is one setting for learning mathematics. It is understood that all children should be expected to learn meaningful mathematics within its walls. Additionally, teachers should be expected to learn within the walls of this same building. More poignantly, I posit that if teachers are not learning mathematics in their school building, then it is not a school.
>
> **—Nevels, 2018**

Raj Shah, Ph.D.

Columbus, Ohio

Founder of Math Plus Academy, a STEM enrichment program and founding member of The Global Math Project

Areas of expertise:
Sparking student curiosity, promoting productive struggle, and creating math experiences that kids love

> “As teachers, it’s imperative that we start every lesson by getting students to ask more questions because curiosity is the fuel that drives engagement, deeper learning and perseverance.”
>
> **—Shah, 2017**

(l to r, t to b) Photo by Damon Leo, Shannon Larsen, Matthew McClellan, Raj Shah

Walter Secada, Ph.D.

Coral Gables, Florida

Professor of Teaching and Learning at the University of Miami

Areas of expertise:
Improving education for English language learners, equity in education, mathematics education, bilingual education, school restructuring, professional development of teachers, student engagement, Hispanic dropout and prevention, and reform

> "The best lessons take place when teachers have thought about how their individual English language learners will respond not just to the mathematical content of that lesson, but also to its language demands and mathematical practices."
>
> **—Secada, 2018**

Ryan Baker, Ph.D.

Philadelphia, Pennsylvania

Associate Professor and Director of Penn Center for Learning Analytics at the University of Pennsylvania

Areas of expertise:
Interactions between students and educational software; data mining and learning analytics to understand student learning

> "The ultimate goal of the field of Artificial Intelligence in Education is not to promote artificial intelligence, but to promote education... systems that are designed intelligently, and that leverage teachers' intelligence. Modern online learning systems used at scale are leveraging human intelligence to improve their design, and they're bringing human beings into the decision-making loop and trying to inform them."
>
> **—Baker, 2016**

Chris Dede, Ph.D.

Cambridge, Massachusetts

Timothy E. Wirth Professor in Learning Technologies at Harvard Graduate School of Education

Areas of expertise:
Provides leadership in educational innovation; educational improvements using technology

> "People are very diverse in how they prefer to learn. Good instruction is like an ecosystem that has many niches for alternative types of learning: lectures, games, engaging video-based animations, readings, etc. Learners then can navigate to the niche that best fulfills their current needs."
>
> **—Dede, 2017**

Dinah Zike, M.Ed.

Comfort, Texas

President of Dinah.com in San Antonio, Texas and Dinah Zike Academy

Areas of expertise:
Developing educational materials that include three-dimensional graphic organizers; interactive notebook activities for differentiation; and kinesthetic, cross-curricular manipulatives

> "It is education's responsibility to meet the unique needs of students, and not the students' responsibility to meet education's need for uniformity."
>
> **—Zike, 2017, InRIGORating Math Notebooks**

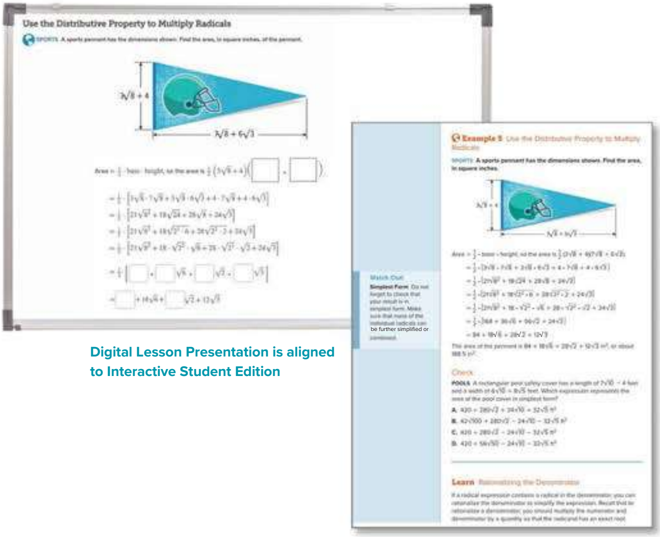
(l to r, t to b) Walter Secada, Elisabeth Reinkordt, Chris Dede, Sergio Rodriguez

Reveal Everything Needed for Effective Instruction

Reveal Math Integrated I, II, III (Reveal Math Integrated) both print and innovative, technology-based tools designed to address a wide range of classrooms. No matter whether you're in a 1:1 district, or have a classroom projector, Reveal Math Integrated provides you with the resources you need for a rich learning experience.

Blended Classrooms

Focused on projection of the **Interactive Presentation**, students follow along, taking notes and working through problems in a notebook during class time. Also included in the Interactive Student Edition is a glossary, selected answers, and a reference sheet.



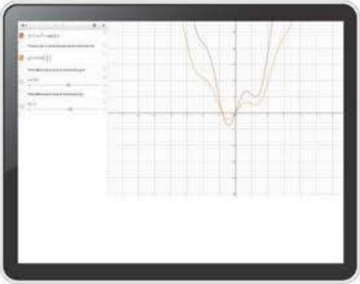
Digital Lesson Presentation is aligned to Interactive Student Edition

Andrey_Kuzmin/iStock/Getty Images

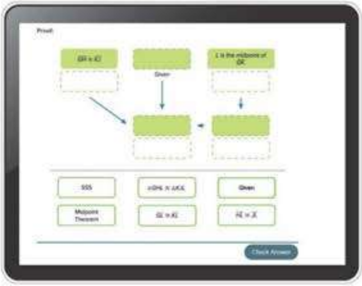
Digital Classrooms

Projection is a focal point for key areas of the course with students interacting with the lesson using their own devices. Each student can access teacher-assigned sections of the lessons for **Explore** activities, **Learn** sections, and **Examples**. Point of use videos, animations, as well as interactive content enable students to experience math in interesting and impactful ways.

Web Sketchpad

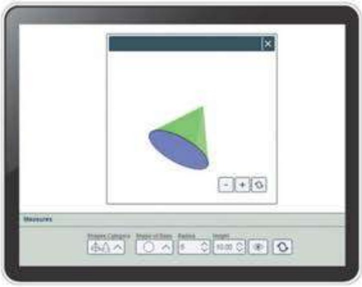


Desmos



Drag-and-Drop

Video



eTools

In Reveal AGA,
R is for—

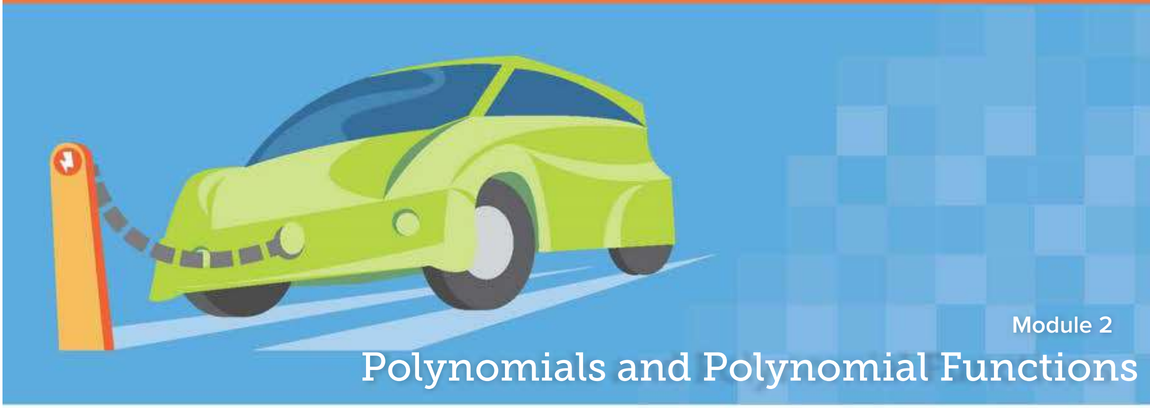
- Research
- Rigor
- Relevant Connections

Are you…
READY to start?

McGraw-Hill Education

Module 1

Quadratic Functions



Module 2

Polynomials and Polynomial Functions

Module 3

Polynomial Equations

Module 4

Inverses and Radical Functions

Module 5

Exponential Functions

Module 6

Logarithmic Functions

Module 7

Rational Functions

Module 8

Inferential Statistics

Module 9

Trigonometric Functions

Standards for Mathematical Content, Reveal Math Integrated III

This correlation shows the alignment of *Reveal Math Integrated III* to the Standards for Mathematical Content, from the Common Core State Standards for Mathematics.

Standard		Lesson(s)
Number and Quantity		
The Complex Number System N.CN		
N.CN.8	(+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.	**1-3, 1-4, 1-6**
N.CN.9	(+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.	**3-5**
Algebra		
Seeing Structures in Expressions A.SSE		
A.SSE.1	Interpret expressions that represent a quantity in terms of its context.★ **a.** Interpret parts of an expression, such as terms, factors, and coefficients. **b.** Interpret complex expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P.*	**1-1, 1-6, 2-1, 2-2, 5-1, 9-4**
A.SSE.2	Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.	**1-4, 1-5, 4-3, 4-5, 5-2, 6-1, 6-3, 6-4, 6-5**
A.SSE.4	Derive the formula for the sum of a geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★	**5-4**
Arithmetic With Polynomials and Rational Expressions A.APR		
A.APR.1	Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.	**2-3**
A.APR.2	Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.	**3-4**
A.APR.3	Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.	**3-5**
A.APR.4	Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.*	**1-4, 3-3**
A.APR.5	(+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.	**2-5**
A.APR.6	Rewrite simple rational expressions in different forms; write $\frac{a(x)}{b(x)}$ in the form $q(x) + \frac{r(x)}{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division or, for the more complicated examples, a computer algebra system.	**2-4**
A.APR.7	(+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.	**7-1, 7-2**

★ Mathematical Modeling Standards

Creating Equations A.CED		
A.CED.1	Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*	**1-2, 1-4, 1-5, 1-6, 1-7, 3-1, 3-2, 3-4, 3-5, 5-2, 6-2, 6-5, 7-5, 7-6**
A.CED.2	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.	**1-1, 1-2, 2-1, 2-2, 3-1, 4-4, 4-6, 5-1, 5-3, 5-5, 6-1, 6-5, 7-3, 7-4, 7-5, 9-3, 9-4, 9-5, 9-6**
A.CED.3	Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*	**1-7, 6-5, 7-6**
A.CED.4	Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.*	**1-6, 4-2, 6-1, 7-5**
Reasoning With Equations and Inequalities A.REI		
A.REI.2	Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.	**4-6, 7-6**
A.REI.11	Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★	**1-8, 3-1, 5-2, 6-3, 7-6**
FUNCTIONS		
Interpreting Functions F.IF		
F.IF.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★	**1-1, 1-2, 2-1, 2-2, 3-5, 4-2, 4-4, 5-1, 6-1, 6-4, 6-5, 7-3, 7-4, 9-4, 9-5, 9-6**
F.IF.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★	**1-1, 2-1, 4-2, 4-4, 5-1, 6-1, 7-3, 7-4, 9-4, 9-5, 9-6**
F.IF.6	Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★	**1-1, 2-2, 5-3**
F.IF.7	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★	**2-1, 2-2, 3-5, 4-4, 5-1, 6-1, 9-4, 9-5, 9-6**
	b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.	
	c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.	
	e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.	
F.IF.8	Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.	**1-4, 1-5, 5-1, 6-5**
F.IF.9	Compare properties of two functions, each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*	**1-1, 2-1, 4-4, 5-1, 6-1, 7-4**

★ Mathematical Modeling Standards

Building Functions F.BF		
F.BF.1	Write a function that describes a relationship between two quantities.★ **b.** Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.*	**4-1**
F.BF.3	Identify the effect on the graph of replacing $f(x)$ *by* $f(x) + k$, $k\,f(x)$, $f(kx)$, *and* $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*	**4-4, 5-1, 6-1, 7-3, 9-6**
F.BF.4	Find inverse functions. **a.** Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2x^3$ *or* $f(x) = (x+1)/(x-1)$ *for* $x \neq 1$.	**4-2**
Linear, Quadratic, and Exponential Models F.LE		
F.LE.4	For exponential models, express as a logarithm the solution to $a\,b^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.	**6-3, 6-4, 6-5**
Trigonometric Functions F.TF		
F.TF.1	Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.	**9-1**
F.TF.2	Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.	**9-3**
F.TF.5	Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★	**9-3, 9-4, 9-5, 9-6**
GEOMETRY		
Similarity, Right Triangles, and Trigonometry G.SRT		
G.SRT.9	(+) Derive the formula $A = \frac{1}{2}\,ab\,\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.	**Standard G.SRT.9 is taught in Integrated Math Course 2, 9-6 Solving Systems of Inequalities**
G.SRT.10	(+) Prove the Laws of Sines and Cosines and use them to solve problems.	**Standard G.SRT.10 is taught in Integrated Math Course 2, 4-7 The Law of Sines and 4-8 The Law of Cosines**
G.SRT.11	(+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).	**Standard G.SRT.11 is taught in Integrated Math Course 2, 4-7 The Law of Sines and 4-8 The Law of Cosines**
Geometric Measurement and Dimension G.GMD		
G.GMD.4	Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.	**Standard G.GMD.4 is taught in Integrated Math Course 2, 6-5 Cross Sections and Solids of Revolution**

★ Mathematical Modeling Standards

Modeling With Geometry G.MG		
G.MG.1	Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★	**2-1**
G.MG.2	Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★	**6-9**
G.MG.3	Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★	**Standard G.MG.3 is taught in Integrated Math Course 2, 6-4 Surface Area**
STATISTICS		
Interpreting Categorical and Quantitative Data S.ID		
S.ID.4	Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.	**8-4**
Making Inferences and Justifying Conclusions S.IC		
S.IC.1	Understand that statistics allow inferences to be made about population parameters based on a random sample from that population.	**8-1**
S.IC.2	Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?	**8-2**
S.IC.3	Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.	**8-1**
S.IC.4	Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.	**8-3, 8-5**
S.IC.5	Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.	**8-2**
S.IC.6	Evaluate reports based on data.	**8-1, 8-2, 8-4, 8-5**
Using Probability to Make Decisions S.MD		
S.MD.6	(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).	**8-2**
S.MD.7	(+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).	**8-2**

★ Mathematical Modeling Standards

Standards for Mathematical Practice

This correlation shows the alignment of *Reveal Math Integrated III* to the Standards for Mathematical Practice, from the Common Core State Standards.

Standard	Lesson(s)
1 Make sense of problems and persevere in solving them. Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.	*Reveal Math Integrated III* requires students to make sense of problems and persevere in solving them in Examples and Practice throughout the program. Some specific lessons for review are: Lessons 1-1, 1-3, 1-4, 1-5, 1-6, 1-7, 1-8, 2-1, 3-2, 3-5, 4-1, 4-4, 5-1, 6-1, 7-2, 7-6, 8-1, 9-3, 9-4
2 Reason abstractly and quantitatively. Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.	*Reveal Math Integrated III* requires students to reason abstractly and quantitatively in Think About It features and Higher Order Thinking Skills throughout the program. Some specific lessons for review are: Lessons 1-1, 2-3, 3-1, 4-2, 4-3, 5-3, 6-3, 6-5, 7-5, 7-6, 8-3, 8-5, 9-1, 9-4, 9-5
3 Construct viable arguments and critique the reasoning of others. Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.	*Reveal Math Integrated III* requires students to construct viable arguments and critique the reasoning of others in Talk About It features and Practice throughout the program. Some specific lessons for review are: Lessons 2-4, 3-4, 4-4, 4-6, 5-4, 6-2, 7-1, 7-3, 8-2, 8-5, 9-2, 9-6
4 Model with mathematics. Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.	*Reveal Math Integrated III* requires students to model with mathematics, collaborate, and discuss mathematics in Examples and Practice throughout the program. Some specific lessons for review are: Lessons 1-2, 1-3, 1-6, 1-7, 1-8, 2-2, 2-5, 3-4, 4-1, 4-5, 5-5, 6-4, 7-4, 7-5, 8-3, 9-3, 9-7

Standard	Lesson(s)
5 Use appropriate tools strategically. Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.	*Reveal Math Integrated III* requires students to use appropriate tools strategically in Explore activities throughout the program. Some specific lessons for review are: Lessons 1-2, 2-2, 3-1, 4-4, 4-6, 5-1, 5-5, 6-1, 6-3, 6-4, 6-5, 7-4, 7-6, 8-4, 9-1, 9-6
6 Attend to precision. Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.	*Reveal Math Integrated III* requires students to attend to precision in Examples and Practice throughout the program. Some specific lessons for review are: Lessons 2-1, 3-3, 4-2, 4-3, 5-2, 6-5, 7-3, 7-5, 8-4, 9-2, 9-3, 9-7
7 Look for and make use of structure. Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.	*Reveal Math Integrated III* requires students to look for and make use of structure in Explore activities and Higher Order Thinking Skills throughout the program. Some specific lessons for review are: Lessons 1-4, 1-5, 2-5, 3-2, 3-5, 4-1, 4-3, 4-5, 5-2, 5-3, 6-3, 6-4, 7-2, 7-3, 8-1, 9-4, 9-6
8 Look for and express regularity in repeated reasoning. Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.	*Reveal Math Integrated III* requires students to look for and express regularity in repeated reasoning in Concept Check and Think About It features and Higher Order Thinking Skills throughout the program. Some specific lessons for review are: Lessons 2-3, 2-4, 3-3, 4-6, 5-4, 6-2, 7-1, 8-2, 9-5

Module 5

Exponential Functions

Module Goals

- Students graph exponential growth and decay functions.
- Students solve exponential equations and inequalities algebraically and by graphing.
- Students analyze expressions and functions involving the natural base *e*.
- Students generate geometric series and find their sums.
- Students choose the best function type to model sets of data by using technology.

Focus

Domain: Algebra, Functions

Standards for Mathematical Content:

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Also addresses A.SSE.4, A.CED.2, A.REI.11, F.IF.4, and F.IF.6.

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this module.

Be Sure to Cover

To completely cover A.SSE.4, go online to assign the following activity:

- Sum of a Finite Geometric Series (Expand, Lesson 5-4)

Coherence

Vertical Alignment

Previous

Students studied linear, exponential, and quadratic functions.

F.IF.7a, F.IF.7e (Course 1, Course 2)

Now

Students graph and analyze exponential functions and determine whether a set of data is best modeled by a linear, quadratic, or exponential function.

F.IF.4, F.IF.7e

Next

Students will graph and analyze logarithmic functions. Students will translate between exponential and logarithmic forms of expressions and solve exponential equations by using logarithms.

A.SSE.2, F.LE.4, F.IF.7e

The Three Pillars of Rigor

To help students meet standards, they need to illustrate their ability to use the three pillars of rigor. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. Once they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Practice.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

EXPLORE → LEARN → EXAMPLE & PRACTICE

Suggested Pacing

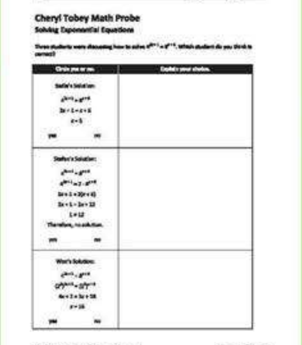
Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
5-1 Graphing Exponential Functions	F.IF.4, F.IF.7e	2	1
5-2 Solving Exponential Equations and Inequalities	A.CED.1, A.REI.11	1	0.5
5-3 Special Exponential Functions	A.CED.2, F.IF.6	1	0.5
5-4 Geometric Sequences and Series	A.SSE.4	3	1.5
5-5 Modeling Data	A.CED.2	1	0.5
Module Review		1	0.5
Module Assessment		1	0.5
	Total Days	11	5.5

CHERYL TOBEY MATH PROBES

Formative Assessment Math Probe

Solving Exponential Equations

Module Resource



Cheryl Tobey Math Probe

Solving Exponential Equations

Sadie's Solution:

Stefan's Solution:

Therefore, no solution.

Wen's Solution:

Correct Answers:
Sadie: no Stefan: no Wen: yes

Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will critique the work of others as they attempt to solve exponential equations.

Targeted Concepts Understand when and how an exponential equation can be solved by rewriting each side of the equation using the same base.

Targeted Misconceptions

- Students may not find a common base before attempting to solve the equation.
- Students may misidentify a common base.

Use the Probe after Lesson 5-2.

Collect and Assess Student Answers

If the student selects these responses...	Then the student likely...
Sadie: yes	does not recognize that the bases are not the same.
Stefan: yes	is using factors of the base that include the same number as the other base, but does not write the 8 as a power with a common base.
Wen: no	does not recognize the process of finding common bases to solve for a variable that is in the exponent.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

- ALEKS Exponential and Logarithmic Equations
- Lesson 5-2, Learn, Example 1

Revisit the Probe at the end of the module to be sure that your students no longer carry these misconceptions.

IGNITE!

The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

How are real-world situations involving quantities that grow or decline rapidly modeled mathematically? Sample answer: Quantities that grow rapidly are modeled by exponential growth functions, and quantities that decline rapidly are modeled by exponential decay functions.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

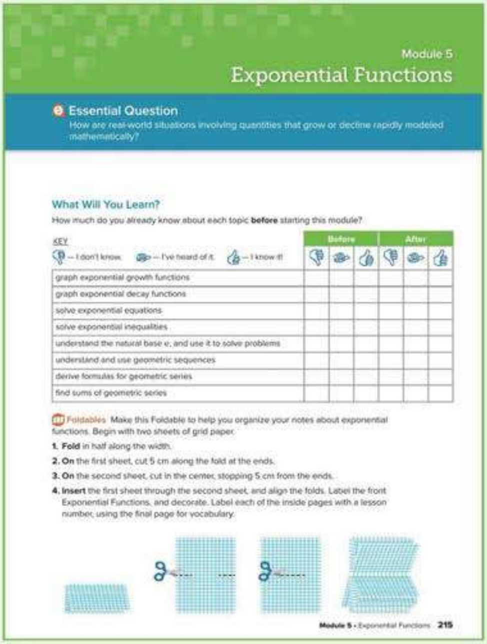
Focus As students work through the lessons in this module, they write notes about exponential functions and relations.

Teach Have students make and label one page of their Foldable for every two lessons in the module and use the appropriate pages as they cover the material. Have students list the Key Concepts and the vocabulary terms and their definitions in their Foldable. Point out that the Foldable can also be used to record positive and negative experiences during learning.

When to Use It Encourage students to add to their Foldable as they work through the module and to use them to review for the module test.

Launch the Module

For this module, the Launch the Module video uses human, insect, and virus populations to show real-world applications of exponential functions. Students learn about using exponential relationships to model the global population over time and viral load over time.



Module 5

Exponential Functions

Essential Question

How are real-world situations involving quantities that grow or decline rapidly modeled mathematically?

What Will You Learn?

How much do you already know about each topic **before** starting this module?

KEY

👎 — I don't know. 👉 — I've heard of it. 👍 — I know it!

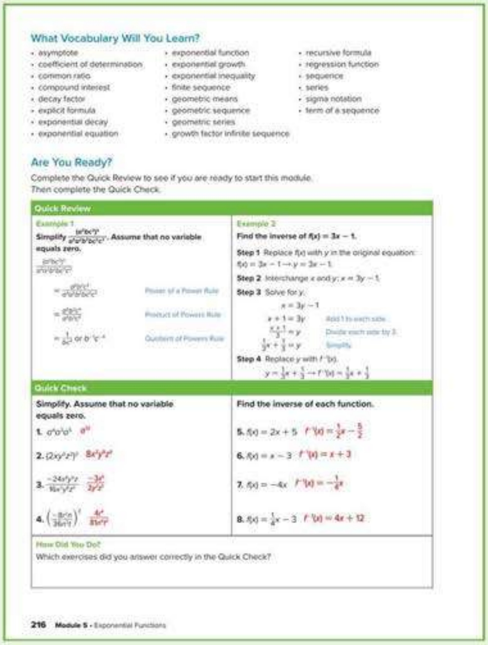
	Before			After		
	👎	👉	👍	👎	👉	👍
graph exponential growth functions						
graph exponential decay functions						
solve exponential equations						
solve exponential inequalities						
understand the natural base e, and use it to solve problems						
understand and use geometric sequences						
derive formulas for geometric series						
find sums of geometric series						

Foldables Make this Foldable to help you organize your notes about exponential functions. Begin with two sheets of grid paper.

1. **Fold** in half along the width.
2. **On** the first sheet, cut 5 cm along the fold at the ends.
3. **On** the second sheet, cut in the center, stopping 5 cm from the ends.
4. **Insert** the first sheet through the second sheet, and align the folds. Label the front Exponential Functions, and decorate. Label each of the inside pages with a lesson number, using the final page for vocabulary.

Module 5 • Exponential Functions 215

Interactive Presentation



What Vocabulary Will You Learn?

- asymptote
- coefficient of determination
- common ratio
- compound interest
- decay factor
- explicit formula
- exponential decay
- exponential equation
- exponential function
- exponential growth
- exponential inequality
- finite sequence
- geometric means
- geometric sequence
- geometric series
- growth factor infinite sequence
- recursive formula
- regression function
- sequence
- series
- sigma notation
- term of a sequence

Are You Ready?

Complete the Quick Review to see if you are ready to start this module. Then complete the Quick Check.

Quick Review

Example 1

Simplify. Assume that no variable equals zero.

Power of a Power Rule

Product of Powers Rule

Quotient of Powers Rule

Example 2

Find the inverse of $f(x) = 3x - 1$.

Step 1 Replace f(x) with y in the original equation.

$f(x) = 3x - 1 \rightarrow y = 3x - 1$

Step 2 Interchange x and y: $x = 3y - 1$

Step 3 Solve for y.

$x = 3y - 1$

$x + 1 = 3y$ Add 1 to each side.

$\frac{x+1}{3} = y$ Divide each side by 3.

$\frac{1}{3}x + \frac{1}{3} = y$ Simplify.

Step 4 Replace y with $f^{-1}(x)$.

$y = \frac{1}{3}x + \frac{1}{3} \rightarrow f^{-1}(x) = \frac{1}{3}x + \frac{1}{3}$

Quick Check

Simplify. Assume that no variable equals zero.

Find the inverse of each function.

5. $f(x) = 2x + 5$ $f^{-1}(x) = \frac{1}{2}x - \frac{5}{2}$

6. $f(x) = x - 3$ $f^{-1}(x) = x + 3$

7. $f(x) = -4x$ $f^{-1}(x) = -\frac{1}{4}x$

8. $f(x) = \frac{1}{4}x - 3$ $f^{-1}(x) = 4x + 12$

How Did You Do?

Which exercises did you answer correctly in the Quick Check?

216 **Module 5** • Exponential Functions

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define A function in form $y = b$;^{x}where b is the base and x is the independent variable, is an exponential function.

Example $y = 90^{x}$

Ask Is $y = x^{35}$ an exponential function? Why or why not? No; the variable must be an exponent in an exponential function.

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- simplifying exponential expressions
- solving equations by taking square roots
- solving exponential equations
- using arithmetic sequences
- using linear regression

ALEKS

ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You may want to use the Exponential and Logarithmic Functions, and Sequences and Probability sections to ensure student success in this module.

Mindset Matters

Attitude Ownership

Part of developing a growth mindset involves acknowledging progress in growth thinking and sharing it with others. It's important for students to own their mindset, be proud of their growth, and see themselves as someone who has a growth mentality—not just in math, but with learning in general.

How Can I Apply It?

Have students complete a **math mindset** project to share how they have grown throughout the year. They might choose the delivery method, such as a poster, blog post, video, or podcast. Students can share their mindset journey with their classmates as part of a class discussion, or they might post their projects for others to see.

Lesson 5-1

Graphing Exponential Functions

F.IF.4, F.IF.7e

LESSON GOAL

Students graph exponential growth and decay functions.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Using Technology to Analyze Graphs of Exponential Functions

Develop:

Graphing Exponential Growth Functions

- Graph Exponential Growth Functions
- Graph Transformations of Exponential Growth Functions
- Analyze Graphs of Exponential Functions
- Use Exponential Growth Functions

Graphing Exponential Decay Functions

- Interpret Exponential Functions
- Graph Exponential Decay Functions
- Graph Transformations of Exponential Decay Functions
- Compare Exponential Functions

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Exponential Functions	●	●		●
Extension: Families of Curves		●	●	●

Language Development Handbook

Assign page 25 of the *Language Development Handbook* to help your students build mathematical language related to graphing exponential growth and decay functions.

ELL You can use the tips and suggestions on page T25 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

5 Use appropriate tools strategically.

Coherence

Vertical Alignment

Previous
Students graphed linear, exponential, and quadratic functions.
F.IF.7a, F.IF.7e (Course 1, Course 2)

Now
Students graph exponential growth and decay functions.
F.IF.4, F.IF.7e

Next
Students will analyze expressions and functions involving natural base *e*.
A.CED.2, F.IF.6

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of exponential functions and use it to build fluency by graphing exponential functions. They apply their understanding of exponential functions by solving real-world problems.

Interactive Presentation

Warm Up

Simplify each expression.

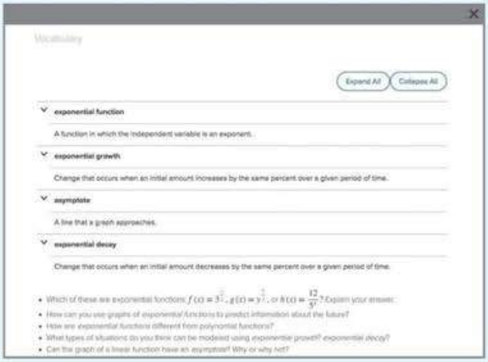
Warm Up

Launch the Lesson

Integrated circuits, or microchips, are in nearly every electronic device we use, from computers to mobile phones. These small chips are a result of integrating a large number of tiny transistors into a small piece of silicon. Since their invention, microchips have been getting smaller and more powerful, making technology like laptops and tablets possible.

This trend, known as Moore's Law, was first predicted by Gordon Moore in 1965. Moore's Law states that the number of transistors on a microchip roughly doubles every two years. This relationship can be represented by the function $T = 2300(2)^{\frac{x}{2}}$, where T is the number of transistors x years after 1971. You can graph this function to analyze its key features and make a prediction about the number of transistors on a chip for a given year.

Launch the Lesson



Vocabulary

exponential function

A function in which the independent variable is an exponent.

exponential growth

Change that occurs when an initial amount increases by the same percent over a given period of time.

asymptote

A line that a graph approaches.

exponential decay

Change that occurs when an initial amount decreases by the same percent over a given period of time.

- Which of these are exponential functions: $f(x) = 3^{\frac{x}{2}}$, $g(x) = y^{\frac{1}{2}}$, or $h(x) = \frac{12}{3^x}$? Explain your answer.
- How can you use graphs of exponential functions to predict information about the future?
- How are exponential functions different from polynomial functions?
- What types of situations do you think can be modeled using exponential growth? exponential decay?
- Can the graph of a linear function have an asymptote? Why or why not?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- simplifying exponential expressions

Answers:

1. 7

2. 5

3. 9

4. 8

5. 64

Launch the Lesson

Teaching the Mathematical Practices

2 Attend Quantities Point out that it is important to note the meaning of the quantities used in the function that represents Moore's Law. Ask students to evaluate the function and explain what the value of the variable and the function represent in the context of the situation.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards*? and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Mathematical Background

The function $f(x) = b^x$, where b is a positive real number and $b \neq 1$, is an exponential function. When $b > 1$, the function has no x-intercepts and one y-intercept. It is an increasing function with a horizontal asymptote (the x-axis). When $0 < b < 1$, the function has no x-intercepts and one y-intercept. It is a decreasing function with a horizontal asymptote (the x-axis).

Explore Using Technology to Analyze Graphs of Exponential Functions

Objective

Students use a graphing calculator to explore transformations and key features of graphs of exponential functions.

Teaching the Mathematical Practices

5 Analyze Graphs Throughout the Explore, students will analyze the graphs they have generated using graphing calculators to determine the effect of performing an operation on an exponential function.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

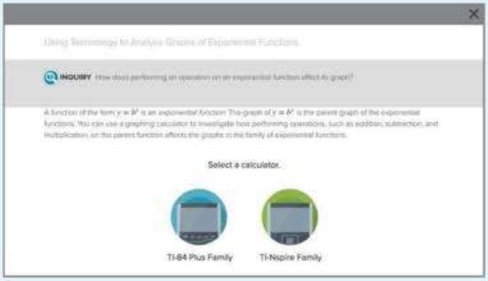
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will be presented with the parent exponential function and analyze the change in key features of transformations on the parent function. Then, students will answer the Inquiry Question.

(*continued on the next page*)

Interactive Presentation



Using Technology to Analyze Graphs of Exponential Functions

INQUIRY How does performing an operation on an exponential function affect its graph?

A function of the form $y = b^x$ is an *exponential function*. The graph of $y = b^x$ is the parent graph of the exponential functions. You can use a graphing calculator to investigate how performing operations, such as addition, subtraction, and multiplication, on the parent function affects the graphs in the family of exponential functions.

Select a calculator.

Explore

Interactive Presentation

How does performing an operation on an exponential function affect its graph?

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Using Technology to Analyze Graphs of Exponential Functions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How does addition affect the graphs of other functions? Sample answer: Addition can move a graph vertically or horizontally.
- Does the transformation from $y = b^x$ to $y = -3b^{x+2}$ change any key features of the graph? Sample answer: Multiplying by a negative number will change the range, *y*-intercept, end behavior and changes the graph from increasing to decreasing.

Inquiry

How does performing an operation on an exponential function affect its graph? Adding or subtracting a constant causes the graph to be shifted horizontally or vertically. Multiplying by a constant stretches or compresses the graph, making the graph steeper or less steep, and can change the *y*-intercept. Multiplying by a negative constant causes the graph to be reflected in the *x*- or *y*-axis.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Graphing Exponential Growth Functions

Objective

Students graph exponential growth functions.

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationship between the value of *b* and an exponential growth function.

DIFFERENTIATE

Language Development Activity AL ELL

Ask students where they have heard the term *exponential* before and what they think it might mean. Students may have heard terms like *exponential growth* on a television news program, and they might think that exponential means "enormous." Use students' answers to introduce the concept of exponential functions.

Example 1 Graph Exponential Growth Functions

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the function, table, and graph used in this example to identify key features of the function.

Questions for Mathematical Discourse

AL What is b^0? 1

OL Why is $0 < y < 1$ for $x < 0$? For $x < 0$, $y = \frac{1}{(2^{|x|})}$, which is always less than 1.

BL Why is $y = 0$ an asymptote of the function? As x approaches negative infinity, the denominator of $\frac{1}{(2^{|x|})}$ becomes infinitely large, making the fraction infinitely small; however, no value of x can make the function equal zero.

Essential Question Follow-Up

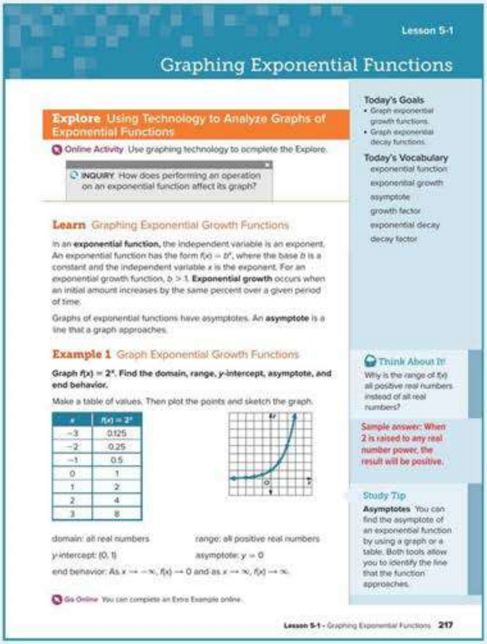
Students have explored exponential functions.

Ask:

How can being financially literate help you to make good decisions? Sample answer: If you are financially literate, you understand financial terms and know how to analyze data and trends. Successfully applying these skills can help you to make good decisions in real-world situations such as opening a bank account, applying for college loans, and buying a house.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 5-1

Graphing Exponential Functions

Explore Using Technology to Analyze Graphs of Exponential Functions

Online Activity Use graphing technology to ocmplete the Explore.

INQUIRY How does performing an operation on an exponential function affect its graph?

Learn Graphing Exponential Growth Functions

In an **exponential function,** the independent variable is an exponent. An exponential function has the form $f(x) = b^x$, where the base b is a constant and the independent variable x is the exponent. For an exponential growth function, $b > 1$. **Exponential growth** occurs when an initial amount increases by the same percent over a given period of time.

Graphs of exponential functions have asymptotes. An **asymptote** is a line that a graph approaches.

Example 1 Graph Exponential Growth Functions

Graph $f(x) = 2^x$. Find the domain, range, y-intercept, asymptote, and end behavior.

Make a table of values. Then plot the points and sketch the graph.

x	$f(x) = 2^x$
−3	0.125
−2	0.25
−1	0.5
0	1
1	2
2	4
3	8

domain: all real numbers

range: all positive real numbers

y-intercept: (0, 1)

asymptote: $y = 0$

end behavior: As $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$.

Go Online You can complete an Extra Example online.

Today's Goals

- Graph exponential growth functions.
- Graph exponential decay functions.

Today's Vocabulary

exponential function
exponential growth
asymptote
growth factor
exponential decay
decay factor

Think About It!

Why is the range of $f(x)$ all positive real numbers instead of all real numbers?

Sample answer: When 2 is raised to any real number power, the result will be positive.

Study Tip

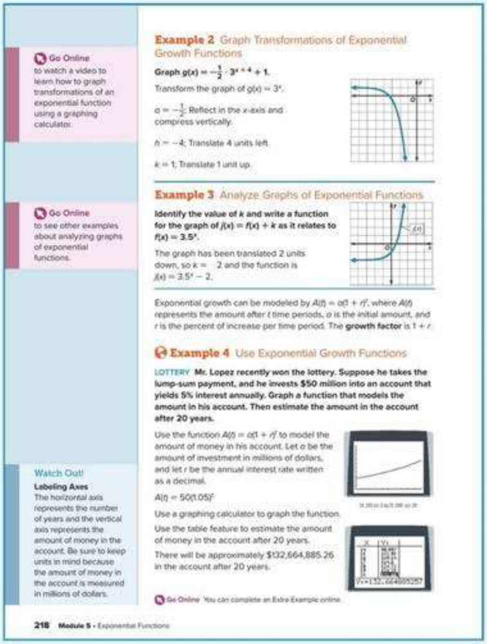
Asymptotes You can find the asymptote of an exponential function by using a graph or a table. Both tools allow you to identify the line that the function approaches.

Lesson 5-1 • Graphing Exponential Functions 217

Interactive Presentation

Learn

TYPE	
a	Students answer a question to show they understand transformations on the parent exponential growth functions.



Go Online to watch a video to learn how to graph transformations of an exponential function using a graphing calculator.

Go Online to see other examples about analyzing graphs of exponential functions.

Example 2 Graph Transformations of Exponential Growth Functions

Graph $g(x) = -\frac{1}{2} \cdot 3^{x+4} + 1$.

Transform the graph of $g(x) = 3^x$.

$a = -\frac{1}{2}$: Reflect in the x-axis and compress vertically.

$h = -4$: Translate 4 units left.

$k = 1$: Translate 1 unit up.

Example 3 Analyze Graphs of Exponential Functions

Identify the value of k and write a function for the graph of $j(x) = f(x) + k$ as it relates to $f(x) = 3.5^x$.

The graph has been translated 2 units down, so $k = -2$ and the function is $j(x) = 3.5^x - 2$.

Exponential growth can be modeled by $A(t) = a(1 + r)^t$, where $A(t)$ represents the amount after t time periods, a is the initial amount, and r is the percent of increase per time period. The **growth factor** is $1 + r$.

Example 4 Use Exponential Growth Functions

LOTTERY **Mr. Lopez recently won the lottery. Suppose he takes the lump-sum payment, and he invests $50 million into an account that yields 5% interest annually. Graph a function that models the amount in his account. Then estimate the amount in the account after 20 years.**

Use the function $A(t) = a(1 + r)^t$ to model the amount of money in his account. Let a be the amount of investment in millions of dollars, and let r be the annual interest rate written as a decimal.

$A(t) = 50(1.05)^t$

Use a graphing calculator to graph the function.

Use the table feature to estimate the amount of money in the account after 20 years.

There will be approximately $132,664,885.26 in the account after 20 years.

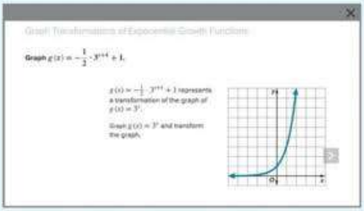
Watch Out!

Labeling Axes The horizontal axis represents the number of years and the vertical axis represents the amount of money in the account. Be sure to keep units in mind because the amount of money in the account is measured in millions of dollars.

Go Online You can complete an Extra Example online.

218 **Module 5** • Exponential Functions

Interactive Presentation



Graph Transformations of Exponential Growth Functions

Graph $g(x) = -\frac{1}{2} \cdot 3^{x+4} + 1$.

$g(x) = -\frac{1}{2} \cdot 3^{x+4} + 1$ represents a transformation of the graph of $f(x) = 3^x$.

Graph $f(x) = 3^x$ and transform the graph.

Example 2

TAP

Students move through steps to learn to graph an exponential function using transformations.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Graph Transformations of Exponential Growth Functions

Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of an exponential growth function to identify the transformations in the function and to graph the function.

Questions for Mathematical Discourse

AL How does a transformation of an exponential function compare to transformations of other types of functions? The transformations follow the same pattern as for the other types of functions.

OL What is the equation that translates $y = 3^x$ to the right 4 units and down 5 units? $y = 3^{(x-4)} - 5$ What is the equation of the asymptote for this function? $y = -5$

BL As $x \to -\infty$, how does the rate of increase of $f(x)$ change? The rate of increase decreases but does not reach zero.

Example 3 Analyze Graphs of Exponential Functions

Teaching the Mathematical Practices

6 Communicate Precisely The Think About It! feature online asks students to write and explain the solution process they used in Example 3.

Questions for Mathematical Discourse

AL How can you tell the graph has been translated down two units? The asymptote of the parent function is $y = 0$, and the asymptote of the graph is $y = -2$.

OL Do the domain and range change based on the value of k in $g(x) = f(x) + k$? The domain does not change, but the range depends on the value of k.

BL What is the equation for the asymptote of $f(x) = ab^x + k$? $y = k$

Example 4 Use Exponential Growth Functions

Teaching the Mathematical Practices

5 Use Mathematical Tools In Example 4, students will need to use a graphing calculator to estimate the balance in the account.

Questions for Mathematical Discourse

AL How could you check the equation to verify the original investment was 50 million? Find $A(0) = 50$.

OL What does it mean in this context for the function to be increasing? The amount of money in the account is increasing as time goes by.

BL In the context of the example, why does the rate of increase increase as $t \to \infty$? The interest earned previously contributes to the interest earned subsequently.

Learn Graphing Exponential Decay Functions

Objective

Students graph exponential decay functions.

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationship between the value of *b* and an exponential decay function.

Example 5 Interpret Exponential Functions

Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of functions to determine whether they are exponential growth or decay functions.

Questions for Mathematical Discourse

AL What must be true about $f(x) = b^x$ for the function to represent exponential decay? $0 < b < 1$

OL Does $f(x) = 5^{-x}$ represent exponential growth or decay? Explain. It represents exponential decay because $5^{-x} = \frac{1}{5^x} = \left(\frac{1}{5}\right)^x$.

BL How are $y = 5^x$ and $y = \left(\frac{1}{5}\right)^x$ related? One is the reflection of the other across the *y*-axis.

Example 6 Graph Exponential Decay Functions

Teaching the Mathematical Practices

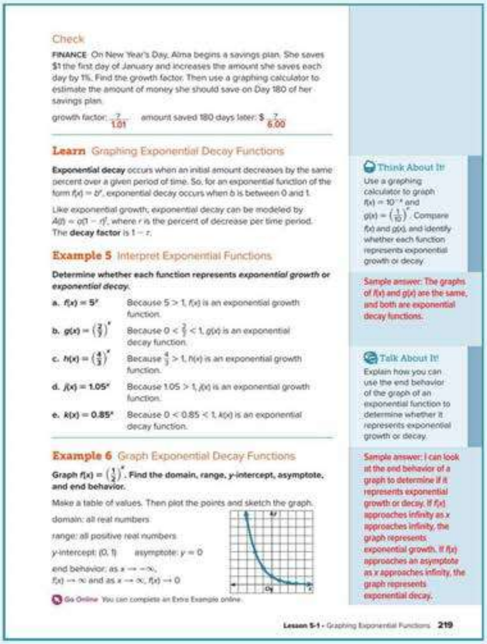
1 Explain Correspondences The Talk About It! asks students to explain the relationship between the end behavior of an exponential function and the type of exponential function.

Questions for Mathematical Discourse

AL Why does $\left(\frac{1}{2}\right)^{-3} = 8$? A term with a negative exponent can be written as the term with the positive version of the exponent in the denominator of a fraction with 1 as the numerator. $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ and $\frac{1}{\left(\frac{1}{8}\right)} = 8$.

OL What transformation is done to the graph of $y = 2^x$ to get the graph of $y = \left(\frac{1}{2}\right)^x$? reflection in the *y*-axis

BL Given the graph of $y = \left(\frac{1}{b}\right)^x$ is a reflection of the graph of $y = b^x$ across the *y*-axis, how does this relate to the form of reflections across the *y*-axis for transformations of other function types? The reflection of $f(x)$ across the *y*-axis is $f(-x)$, and $\left(\frac{1}{b}\right)^x = b^{-x}$.



Check

FINANCE On New Year's Day, Alma begins a savings plan. She saves \$1 the first day of January and increases the amount she saves each day by 1%. Find the growth factor. Then use a graphing calculator to estimate the amount of money she should save on Day 180 of her savings plan.

growth factor: ? 1.01 amount saved 180 days later: \$? 6.00

Learn Graphing Exponential Decay Functions

Exponential decay occurs when an initial amount decreases by the same percent over a given period of time. So, for an exponential function of the form $f(x) = b^x$, exponential decay occurs when b is between 0 and 1.

Like exponential growth, exponential decay can be modeled by $A(t) = a(1 - r)^t$, where r is the percent of decrease per time period. The **decay factor** is $1 - r$.

Example 5 Interpret Exponential Functions

Determine whether each function represents *exponential growth* or *exponential decay*.

a. $f(x) = 5^x$	Because $5 > 1$, $f(x)$ is an exponential growth function.
b. $g(x) = \left(\frac{2}{3}\right)^x$	Because $0 < \frac{2}{3} < 1$, $g(x)$ is an exponential decay function.
c. $h(x) = \left(\frac{4}{3}\right)^x$	Because $\frac{4}{3} > 1$, $h(x)$ is an exponential growth function.
d. $j(x) = 1.05^x$	Because $1.05 > 1$, $j(x)$ is an exponential growth function.
e. $k(x) = 0.85^x$	Because $0 < 0.85 < 1$, $k(x)$ is an exponential decay function.

Example 6 Graph Exponential Decay Functions

Graph $f(x) = \left(\frac{1}{2}\right)^x$. Find the domain, range, *y*-intercept, asymptote, and end behavior.

Make a table of values. Then plot the points and sketch the graph.

domain: all real numbers

range: all positive real numbers

y-intercept: (0, 1) asymptote: $y = 0$

end behavior: as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to 0$

Go Online You can complete an Extra Example online.

Think About It! Use a graphing calculator to graph $f(x) = 10^{-x}$ and $g(x) = \left(\frac{1}{10}\right)^x$. Compare $f(x)$ and $g(x)$, and identify whether each function represents exponential growth or decay.

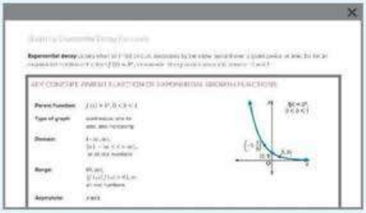
Sample answer: The graphs of $f(x)$ and $g(x)$ are the same, and both are exponential decay functions.

Talk About It! Explain how you can use the end behavior of the graph of an exponential function to determine whether it represents exponential growth or decay.

Sample answer: I can look at the end behavior of a graph to determine if it represents exponential growth or decay. If $f(x)$ approaches infinity as x approaches infinity, the graph represents exponential growth. If $f(x)$ approaches an asymptote as x approaches infinity, the graph represents exponential decay.

Lesson 5-1 • Graphing Exponential Functions 219

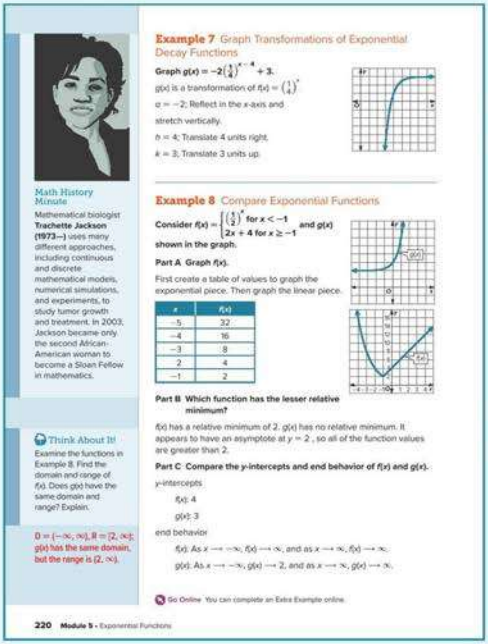
Interactive Presentation



Learn

TYPE

Students are introduced to the effect of transformations on the parent exponential decay functions.



Math History Minute

Mathematical biologist **Trachette Jackson (1973–)** uses many different approaches, including continuous and discrete mathematical models, numerical simulations, and experiments, to study tumor growth and treatment. In 2003, Jackson became only the second African-American woman to become a Sloan Fellow in mathematics.

Think About It!

Examine the functions in Example 8. Find the domain and range of f(x). Does g(x) have the same domain and range? Explain.

$D = (-\infty, \infty)$, $R = [2, \infty)$; g(x) has the same domain, but the range is $(2, \infty)$.

Example 7 Graph Transformations of Exponential Decay Functions

Graph $g(x) = -2\left(\frac{1}{4}\right)^{x-4} + 3$.

g(x) is a transformation of $f(x) = \left(\frac{1}{4}\right)^x$.

$a = -2$; Reflect in the x-axis and stretch vertically.

$h = 4$; Translate 4 units right.

$k = 3$; Translate 3 units up.

Example 8 Compare Exponential Functions

Consider $f(x) = \begin{cases} \left(\frac{1}{2}\right)^x & \text{for } x < -1 \\ 2x + 4 & \text{for } x \geq -1 \end{cases}$ and g(x) shown in the graph.

Part A Graph f(x).

First create a table of values to graph the exponential piece. Then graph the linear piece.

x	f(x)
−5	32
−4	16
−3	8
2	4
−1	2

Part B Which function has the lesser relative minimum?

f(x) has a relative minimum of 2. g(x) has no relative minimum. It appears to have an asymptote at $y = 2$, so all of the function values are greater than 2.

Part C Compare the y-intercepts and end behavior of f(x) and g(x).

y-intercepts

f(x): 4

g(x): 3

end behavior

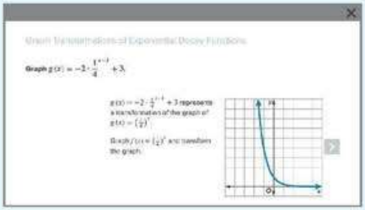
f(x): As $x \longrightarrow -\infty$, $f(x) \longrightarrow \infty$, and as $x \longrightarrow \infty$, $f(x) \longrightarrow \infty$.

g(x): As $x \longrightarrow -\infty$, $g(x) \longrightarrow 2$, and as $x \longrightarrow \infty$, $g(x) \longrightarrow \infty$.

Go Online You can complete an Extra Example online.

220 Module 5 • Exponential Functions

Interactive Presentation



Graph Transformations of Exponential Decay Functions

Graph $g(x) = -2 \cdot \frac{1}{4}^{x-4} + 3$.

Example 7

TYPE

Students move through steps to learn to graph an exponential function using transformations.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 7 Graph T ransformations of Exponential Decay Functions

Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of an exponential decay function to identify the transformations in the function and to graph the function.

Questions for Mathematical Discourse

AL Why does g(x) represent exponential decay? It is exponential decay because $b = \frac{1}{4}$, which is less than 1.

OL Why is k not included in the range of an exponential function of the form $y = ab^{x-h} + k$? No value of x can make the exponential term zero.

BL Why does g(x) represent exponential decay even though the function is increasing as $x \longrightarrow \infty$? The rate at which the function is increasing decreases as $x \longrightarrow \infty$, making it approach an asymptote.

Example 8 Compare Exponential Functions

Teaching the Mathematical Practices

1 Explain Correspondences Students will explain the relationships between the key features of two functions represented in different ways.

Questions for Mathematical Discourse

AL What kind of function is f(x)? piecewise with exponential and linear components

OL What is the difference between the exponential component of f(x) and g(x)? f(x) represents exponential decay and g(x) represents exponential growth.

BL How many intersection points do g(x) and f(x) have? three Could translating g(x) to the right ever result in a different number of intersection points? No, because g(x) never reaches 2.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–34
2	exercises that use a variety of skills from this lesson	35–48
3	exercises that emphasize higher-order and critical-thinking skills	49–53

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice Exercises 1–47 odd, 49–53
- Extension: Families of Curves
- ALEKS® Graphing Exponential and Logarithmic Functions

OL

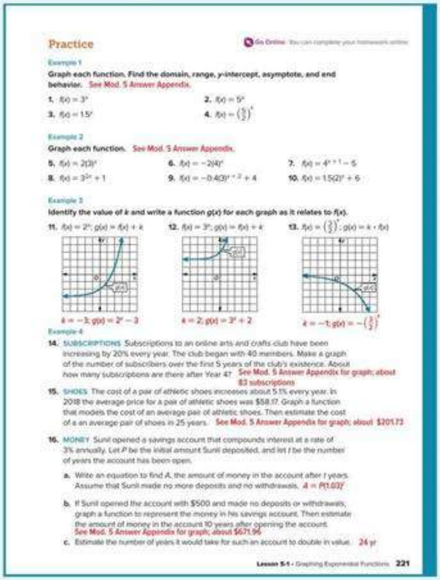
IF students score 66%–89% on the Checks,
THEN assign:

- Practice Exercises 1–53 odd
- Remediation, Review Resources: Exponential Functions
- Personal Tutors
- Extra Examples 1–8
- ALEKS® Exponential Functions

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice Exercises 1–33 odd
- Remediation, Review Resources: Exponential Functions
- *Quick Review Math Handbook*: Graphing Exponential Functions
- ALEKS® Exponential Functions



Practice

Go Online You can complete your homework online.

Example 1

Graph each function. Find the domain, range, *y*-intercept, asymptote, and end behavior. See Mod. 5 Answer Appendix.

1. $f(x) = 3^x$
2. $f(x) = 5^x$
3. $f(x) = 1.5^x$

Example 2

Graph each function. See Mod. 5 Answer Appendix.

5. $f(x) = 2(3)^x$
6. $f(x) = -2(4)^x$
7. $f(x) = 4^{x+1} - 5$
8. $f(x) = 3^{2x} + 1$
9. $f(x) = -0.4(3)^{x+2} + 4$
10. $f(x) = 1.5(2)^x + 6$

Example 3

Identify the value of *k* and write a function *g*(*x*) for each graph as it relates to *f*(*x*).

11. $f(x) = 2^x$; $g(x) = f(x) + k$ — $k = -3$; $g(x) = 2^x - 3$
12. $f(x) = 3^x$; $g(x) = f(x) + k$ — $k = 2$; $g(x) = 3^x + 2$
13. $g(x) = k \cdot f(x)$ — $k = -1$

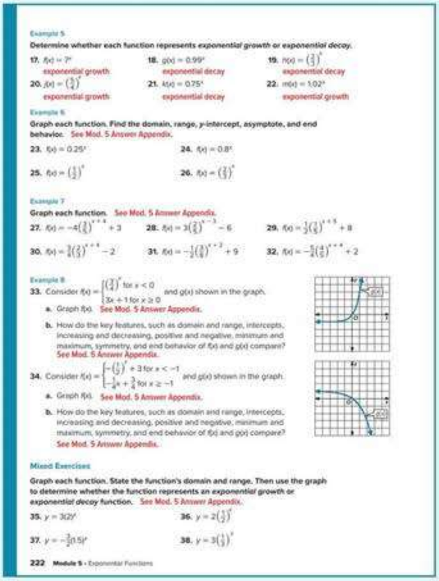
Example 4

14. **SUBSCRIPTIONS** Subscriptions to an online arts and crafts club have been increasing by 20% every year. The club began with 40 members. Make a graph of the number of subscribers over the first 5 years of the club's existence. About how many subscriptions are there after Year 4? See Mod. 5 Answer Appendix for graph; about 83 subscriptions

15. **SHOES** The cost of a pair of athletic shoes increases about 5.1% every year. In 2018 the average price for a pair of athletic shoes was $58.17. Graph a function that models the cost of an average pair of athletic shoes. Then estimate the cost of a an average pair of shoes in 25 years. See Mod. 5 Answer Appendix for graph; about $201.73

16. **MONEY** Sunil opened a savings account that compounds interest at a rate of 3% annually. Let *P* be the initial amount Sunil deposited, and let *t* be the number of years the account has been open.
    a. Write an equation to find *A*, the amount of money in the account after *t* years. Assume that Sunil made no more deposits and no withdrawals. $A = P(1.03)^t$
    b. If Sunil opened the account with $500 and made no deposits or withdrawals, graph a function to represent the money in his savings account. Then estimate the amount of money in the account 10 years after opening the account. See Mod. 5 Answer Appendix for graph; about $671.96
    c. Estimate the number of years it would take for such an account to double in value. 24 yr

Lesson 5-1 • Graphing Exponential Functions 221



Example 5

Determine whether each function represents *exponential growth* or *exponential decay*.

17. $f(x) = 7^x$ exponential growth
18. $g(x) = 0.99^x$ exponential decay
19. exponential decay
20. exponential growth
21. $k(x) = 0.75^x$ exponential decay
22. $m(x) = 1.02^x$ exponential growth

Example 6

Graph each function. Find the domain, range, *y*-intercept, asymptote, and end behavior. See Mod. 5 Answer Appendix.

23. $f(x) = 0.25^x$
24. $f(x) = 0.8^x$

Example 7

Graph each function. See Mod. 5 Answer Appendix.

Example 8

33. Consider $f(x)$ and $g(x)$ shown in the graph.
    a. Graph $f(x)$. See Mod. 5 Answer Appendix.
    b. How do the key features, such as domain and range, intercepts, increasing and decreasing, positive and negative, minimum and maximum, symmetry, and end behavior of $f(x)$ and $g(x)$ compare? See Mod. 5 Answer Appendix.

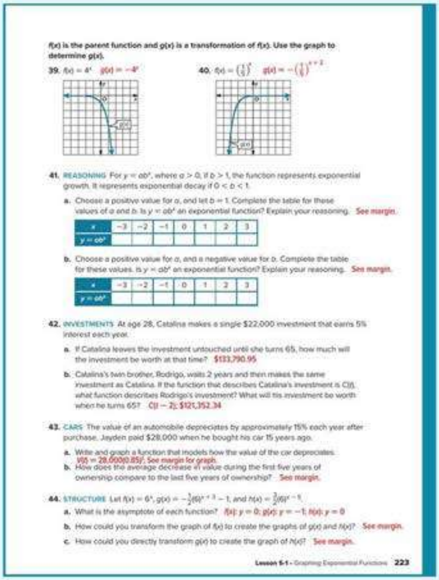
34. Consider $f(x)$ and $g(x)$ shown in the graph.
    a. Graph $f(x)$. See Mod. 5 Answer Appendix.
    b. How do the key features, such as domain and range, intercepts, increasing and decreasing, positive and negative, minimum and maximum, symmetry, and end behavior of $f(x)$ and $g(x)$ compare? See Mod. 5 Answer Appendix.

Mixed Exercises

Graph each function. State the function's domain and range. Then use the graph to determine whether the function represents an *exponential growth* or *exponential decay* function. See Mod. 5 Answer Appendix.

35. $y = 3(2)^x$

222 Module 5 • Exponential Functions



$f(x)$ is the parent function and $g(x)$ is a transformation of $f(x)$. Use the graph to determine $g(x)$.

39. $f(x) = 4^x$ $g(x) = -4^x$

40. $f(x) = \left(\frac{1}{5}\right)^x$ $g(x) = -\left(\frac{1}{5}\right)^{x+2}$

41. REASONING For $y = ab^x$, where $a > 0$, if $b > 1$, the function represents exponential growth. It represents exponential decay if $0 < b < 1$.

a. Choose a positive value for a, and let $b = 1$. Complete the table for these values of a and b. Is $y = ab^x$ an exponential function? Explain your reasoning. See margin.

x	−3	−2	−1	0	1	2	3
$y = ab^x$							

b. Choose a positive value for a, and a negative value for b. Complete the table for these values. Is $y = ab^x$ an exponential function? Explain your reasoning. See margin.

x	−3	−2	−1	0	1	2	3
$y = ab^x$							

42. INVESTMENTS At age 28, Catalina makes a single \$22,000 investment that earns 5% interest each year.

a. If Catalina leaves the investment untouched until she turns 65, how much will the investment be worth at that time? \$133,790.95

b. Catalina's twin brother, Rodrigo, waits 2 years and then makes the same investment as Catalina. If the function that describes Catalina's investment is $C(t)$, what function describes Rodrigo's investment? What will his investment be worth when he turns 65? $C(t - 2)$; \$121,352.34

43. CARS The value of an automobile depreciates by approximately 15% each year after purchase. Jayden paid \$28,000 when he bought his car 15 years ago.

a. Write and graph a function that models how the value of the car depreciates. $V(t) = 28{,}000(0.85)^t$; See margin for graph.

b. How does the average decrease in value during the first five years of ownership compare to the last five years of ownership? See margin.

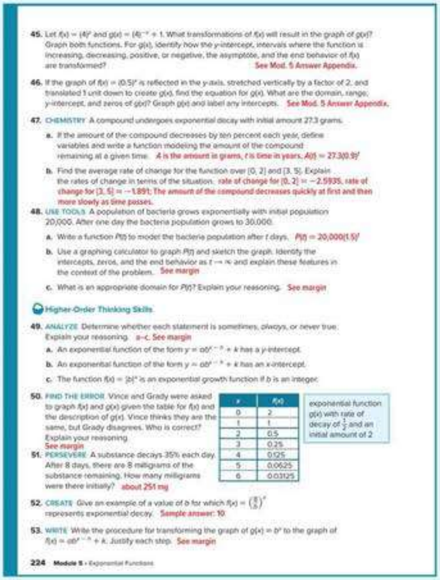
44. STRUCTURE Let $f(x) = 6^x$, $g(x) = -\frac{1}{2}(6)^{x+3} - 1$, and $h(x) = \frac{3}{2}(6)^{x-5}$.

a. What is the asymptote of each function? $f(x)$: $y = 0$; $g(x)$: $y = -1$; $h(x)$: $y = 0$

b. How could you transform the graph of $f(x)$ to create the graphs of $g(x)$ and $h(x)$? See margin.

c. How could you directly transform $g(x)$ to create the graph of $h(x)$? See margin.

Lesson 5-1 • Graphing Exponential Functions 223



45. Let $f(x) = (4)^x$ and $g(x) = (4)^{-x} + 1$. What transformations of $f(x)$ will result in the graph of $g(x)$? Graph both functions. For $g(x)$, identify how the y-intercept, intervals where the function is increasing, decreasing, positive, or negative, the asymptote, and the end behavior of $f(x)$ are transformed? See Mod. 5 Answer Appendix.

46. If the graph of $f(x) = (0.5)^x$ is reflected in the y-axis, stretched vertically by a factor of 2, and translated 1 unit down to create $g(x)$, find the equation for $g(x)$. What are the domain, range, y-intercept, and zeros of $g(x)$? Graph $g(x)$ and label any intercepts. See Mod. 5 Answer Appendix.

47. CHEMISTRY A compound undergoes exponential decay with initial amount 27.3 grams.

a. If the amount of the compound decreases by ten percent each year, define variables and write a function modeling the amount of the compound remaining at a given time. A is the amount in grams, t is time in years. $A(t) = 27.3(0.9)^t$

b. Find the average rate of change for the function over [0, 2] and [3, 5]. Explain the rates of change in terms of the situation. rate of change for [0, 2] = −2.5935, rate of change for [3, 5] = −1.891; The amount of the compound decreases quickly at first and then more slowly as time passes.

48. USE TOOLS A population of bacteria grows exponentially with initial population 20,000. After one day the bacteria population grows to 30,000.

a. Write a function $P(t)$ to model the bacteria population after t days. $P(t) = 20{,}000(1.5)^t$

b. Use a graphing calculator to graph $P(t)$ and sketch the graph. Identify the intercepts, zeros, and the end behavior as $t \to \infty$ and explain these features in the context of the problem. See margin

c. What is an appropriate domain for $P(t)$? Explain your reasoning. See margin

Higher-Order Thinking Skills

49. ANALYZE Determine whether each statement is sometimes, always, or never true. Explain your reasoning. a–c. See margin

a. An exponential function of the form $y = ab^{x-h} + k$ has a y-intercept.

b. An exponential function of the form $y = ab^{x-h} + k$ has an x-intercept.

c. The function $f(x) = |b|^x$ is an exponential growth function if b is an integer.

50. FIND THE ERROR Vince and Grady were asked to graph $f(x)$ and $g(x)$ given the table for $f(x)$ and the description of $g(x)$. Vince thinks they are the same, but Grady disagrees. Who is correct? Explain your reasoning. See margin

x	$f(x)$
0	2
1	1
2	0.5
3	0.25
4	0.125
5	0.0625
6	0.03125

exponential function $g(x)$ with rate of decay of $\frac{1}{2}$ and an initial amount of 2

51. PERSEVERE A substance decays 35% each day. After 8 days, there are 8 milligrams of the substance remaining. How many milligrams were there initially? about 251 mg

52. CREATE Give an example of a value of b for which $f(x) = \left(\frac{8}{b}\right)^x$ represents exponential decay. Sample answer: 10

53. WRITE Write the procedure for transforming the graph of $g(x) = b^x$ to the graph of $f(x) = ab^{x-h} + k$. Justify each step. See margin

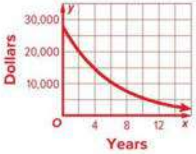
224 Module 5 • Exponential Functions

Answers

41a. Sample answer: Choose $a = 2$, so $y = 2(1)^x$. All values for y are 2. $y = a$ is a constant function, not an exponential one, since the value of y never changes.

41b. Sample answer: Choose $a = 2$ and $b = -2$, so $y = 2(-2)^x$. The table of values for y are $-\frac{1}{4}, \frac{1}{2}, -1, 2, -4, 8, -16$. Students should indicate that this function is not exponential. Since values of y alternate signs, the function does not continuously increase or decrease.

43a.

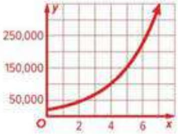


43b. The car depreciates about 5 times as much in the first 5 years as compared the the last 5 years.

44b. To create $g(x)$, $f(x)$ must be reflected in the x-axis, compressed vertically by a factor of 2, and translated 3 units left and 1 unit down. To create $h(x)$, $f(x)$ must be stretched vertically by a factor of $\frac{3}{2}$, and translated 5 units right.

44c. To create $h(x)$, $g(x)$ must be reflected in the x-axis, stretched vertically by a factor of 3, and translated 8 units right and 1 unit up.

48b.



There is an intercept at (0, 20,000) which represents the initial population. There are no zeros. As $t \longrightarrow +\infty$, $P(t) \longrightarrow +\infty$ which means that the population will grow without bound.

48c. Because the population is considered from the time it is initially measured and onward, the appropriate domain is $t \geq 0$.

49a. Always; sample answer: The domain of exponential functions is all real numbers, so $(0, y)$ always exists.

49b. Sometimes; sample answer: The graph of an exponential function crosses the x-axis when $k < 0$.

49c. Sometimes; Sample answer: The function is not exponential if $b = 1$ or -1.

50. Vince; the graphs of the function would be the same.

53. Sample answer: The parent function, $g(x) = b^x$, is stretched if $|a|$ is greater than 1 or compressed if $|a|$ is less than 1. The graph is reflected in the x-axis when a is negative. The parent function is translated up k units if k is positive and down $|k|$ units if k is negative. The parent function is translated h units to the right if h is positive and $|h|$ units to the left if h is negative.

Lesson 5-2

A.CED.1, A.REI.11

Solving Exponential Equations and Inequalities

LESSON GOAL

Students solve exponential equations and inequalities algebraically and by graphing.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Solving Exponential Equations

Develop:

Solving Exponential Equations

- Solve Exponential Equations Algebraically
- Solve an Exponential Equation by Graphing
- Use the Compound Interest Formula

Solving Exponential Inequalities

- Solve Exponential Inequalities Algebraically

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

Formative Assessment Math Probe

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Solving Quadratic Equations by Completing the Square	●	●		●
Extension: Musical Relationships		●	●	●

Language Development Handbook

Assign page 26 of the *Language Development Handbook* to help your students build mathematical language related to solving exponential equations and inequalities.

ELL You can use the tips and suggestions on page T26 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra

Standards for Mathematical Content:

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

A.REI.11 Explain why the *x*-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Standards for Mathematical Practice:

6 Attend to precision.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous

Students solved linear, quadratic, and exponential equations.

A.REI.3, A.REI.4 (Course 1, Course 2)

Now

Students solve exponential equations and inequalities algebraically and by graphing. **A.CED.1, A.REI.11**

Next

Students will analyze expressions and functions involving the natural base *e*. **A.CED.2, F.IF.6**

Rigor

The Three Pillars of Rigor

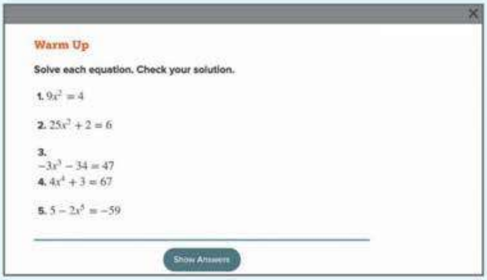
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand on their understanding of exponential functions and build fluency by solving exponential equations. They apply their understanding of exponential equations by solving real-world problems.

Mathematical Background

Simple exponential equations can be solved by rewriting one or both sides of the equation so that the bases are the same. Once that has been achieved, the Property of Equality for Exponential Functions can be used to solve for the variable.

Interactive Presentation



Warm Up

Solve each equation. Check your solution.

1. $9x^2 = 4$

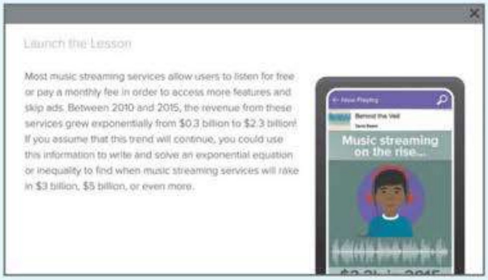
2. $25x^2 + 2 = 6$

3. $-3x^3 - 34 = 47$

4. $4x^4 + 3 = 67$

5. $5 - 2x^5 = -59$

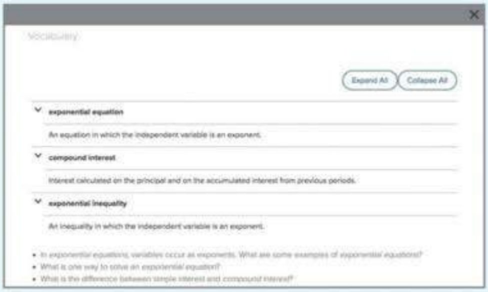
Warm Up



Launch the Lesson

Most music streaming services allow users to listen for free or pay a monthly fee in order to access more features and skip ads. Between 2010 and 2015, the revenue from these services grew exponentially from \$0.3 billion to \$2.3 billion! If you assume that this trend will continue, you could use this information to write and solve an exponential equation or inequality to find when music streaming services will rake in \$3 billion, \$5 billion, or even more.

Launch the Lesson



Vocabulary

exponential equation

An equation in which the independent variable is an exponent.

compound interest

Interest calculated on the principal and on the accumulated interest from previous periods.

exponential inequality

An inequality in which the independent variable is an exponent.

- In exponential equations, variables occur as exponents. What are some examples of exponential equations?
- What is one way to solve an exponential equation?
- What is the difference between simple interest and compound interest?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- solving equations by taking square roots

Answers:

1. $\pm\frac{2}{3}$
2. $\pm\frac{2}{5}$
3. -3
4. ± 2
5. 2

Launch the Lesson

Teaching the Mathematical Practices

4 Make Assumptions In the Launch the Lesson, have students explain why you must assume that the trend in streaming services revenue continues.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Solving Exponential Equations

Objective

Students explore solving exponential equations.

Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. Throughout the Explore, students must rewrite exponential expressions with the same base to solve exponential equations.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

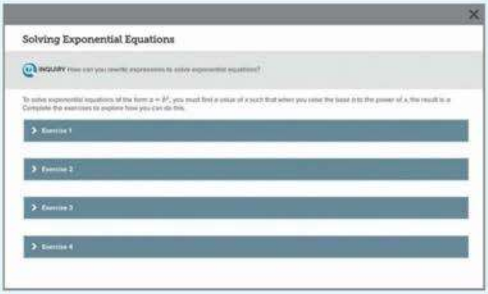
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete four exercises that explore finding the value of a variable that is an exponent in an equation. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Solving Exponential Equations

INQUIRY How can you rewrite expressions to solve exponential equations?

Explore

TAP

Students tap to solve exponential equations.

Interactive Presentation

INQUIRY How can you rewrite expressions to solve exponential equations?

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Solving Exponential Equations (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How can you write 16 in exponential form with a base of 2? Sample answer: You can think of it as 4 • 4, or 2 • 2 • 2 • 2. So, in exponential form $16 = 2^4$.
- Does it help to change $3^{2x+1} = 81$ to have the form $3^{2x+1} = 9^2$? Why or why not? Sample answer: While 9^2 is an exponential form of 81, the base needs to be 3.

Inquiry

How can you rewrite expressions to solve exponential equations? Rewrite the expressions on each side of an exponential equation to have the same base. Then set the exponents equal to each other and solve for *x*.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Solving Exponential Equations

Objective

Students solve exponential equations in one variable algebraically and by graphing.

Teaching the Mathematical Practices

1 Understand Different Approaches Work with students to look at how to solve exponential equations graphically and by using the Property of Equality for Exponential Equations.

DIFFERENTIATE

Enrichment Activity BL

Allow students to develop their sense of consumerism by providing them with an initial deposit amount and having them shop around for the best interest rates. Students should record relevant information, including bank name, account type, interest rate, how often interest is compounded, and restrictions on the account. Ask students to graph the growth of their initial deposit over time.

Example 1 Solve Exponential Equations Algebraically

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. The Think About It! feature online requires students to use clear definitions when they discuss their solution method.

Questions for Mathematical Discourse

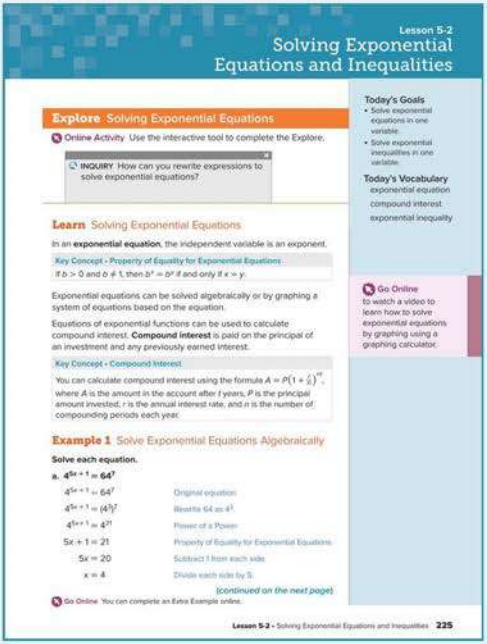
AL In **part a**, why must 64 be written as 4^3? Changing 64 to the equivalent value 4^3 makes both sides of the equation have the same base.

OL What is a method to determine the common base? Sample answer: Take the lesser number to increasing powers until the result is the greater number.

BL Why must the restriction of $b \neq 1$ be put on the Property of Equality for Exponential Equations? Because 1 raised to any power is 1, the property does not hold true for $b = 1$.

Things to Remember

Remind students to distribute to each term in the exponent when using the Power of a Power Property.



Lesson 5-2

Solving Exponential Equations and Inequalities

Explore Solving Exponential Equations

Online Activity Use the interactive tool to complete the Explore.

INQUIRY How can you rewrite expressions to solve exponential equations?

Learn Solving Exponential Equations

In an **exponential equation**, the independent variable is an exponent.

Key Concept • Property of Equality for Exponential Equations

If $b > 0$ and $b \neq 1$, then $b^x = b^y$ if and only if $x = y$.

Exponential equations can be solved algebraically or by graphing a system of equations based on the equation.

Equations of exponential functions can be used to calculate compound interest. **Compound interest** is paid on the principal of an investment and any previously earned interest.

Key Concept • Compound Interest

You can calculate compound interest using the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where A is the amount in the account after t years, P is the principal amount invested, r is the annual interest rate, and n is the number of compounding periods each year.

Example 1 Solve Exponential Equations Algebraically

Solve each equation.

a. $4^{5x+1} = 64^7$

$4^{5x+1} = 64^7$ Original equation

$4^{5x+1} = (4^3)^7$ Rewrite 64 as 4^3.

$4^{5x+1} = 4^{21}$ Power of a Power

$5x + 1 = 21$ Property of Equality for Exponential Equations

$5x = 20$ Subtract 1 from each side.

$x = 4$ Divide each side by 5.

(continued on the next page)

Go Online You can complete an Extra Example online.

Today's Goals
- Solve exponential equations in one variable.
- Solve exponential inequalities in one variable.

Today's Vocabulary

exponential equation

compound interest

exponential inequality

Go Online to watch a video to learn how to solve exponential equations by graphing using a graphing calculator.

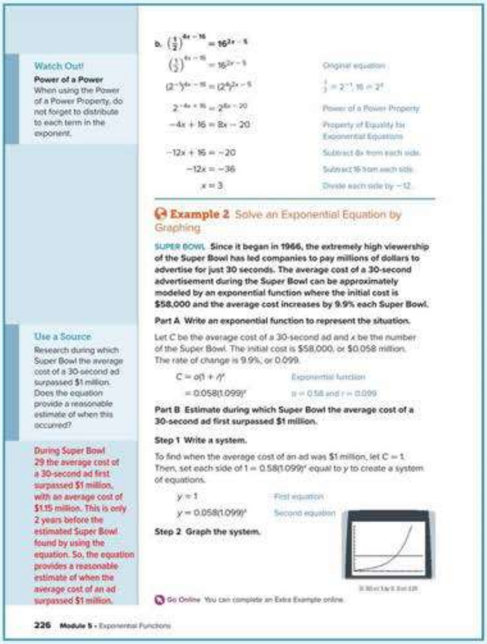
Lesson 5-2 • Solving Exponential Equations and Inequalities 225

Interactive Presentation

Learn

TYPE

Students answer a question to show they understand solving exponential equations.



Watch Out!

Power of a Power When using the Power of a Power Property, do not forget to distribute to each term in the exponent.

b. $\left(\frac{1}{2}\right)^{4x-16} = 16^{2x-5}$

$\left(\frac{1}{2}\right)^{4x-16} = 16^{2x-5}$ Original equation

$(2^{-1})^{4x-16} = (2^4)^{2x-5}$ $\frac{1}{2} = 2^{-1}, 16 = 2^4$

$2^{-4x+16} = 2^{8x-20}$ Power of a Power Property

$-4x + 16 = 8x - 20$ Property of Equality for Exponential Equations

$-12x + 16 = -20$ Subtract 8x from each side.

$-12x = -36$ Subtract 16 from each side.

$x = 3$ Divide each side by −12.

Example 2 Solve an Exponential Equation by Graphing

SUPER BOWL **Since it began in 1966, the extremely high viewership of the Super Bowl has led companies to pay millions of dollars to advertise for just 30 seconds. The average cost of a 30-second advertisement during the Super Bowl can be approximately modeled by an exponential function where the initial cost is $58,000 and the average cost increases by 9.9% each Super Bowl.**

Part A Write an exponential function to represent the situation.

Let C be the average cost of a 30-second ad and x be the number of the Super Bowl. The initial cost is $58,000, or $0.058 million. The rate of change is 9.9%, or 0.099.

$C = a(1 + r)^x$ Exponential function

$= 0.058(1.099)^x$ $a = 0.58$ and $r = 0.099$

Part B Estimate during which Super Bowl the average cost of a 30-second ad first surpassed $1 million.

Use a Source

Research during which Super Bowl the average cost of a 30-second ad surpassed $1 million. Does the equation provide a reasonable estimate of when this occurred?

During Super Bowl 29 the average cost of a 30-second ad first surpassed $1 million, with an average cost of $1.15 million. This is only 2 years before the estimated Super Bowl found by using the equation. So, the equation provides a reasonable estimate of when the average cost of an ad surpassed $1 million.

Step 1 Write a system.

To find when the average cost of an ad was $1 million, let C = 1. Then, set each side of $1 = 0.58(1.099)^x$ equal to y to create a system of equations.

$y = 1$ First equation

$y = 0.058(1.099)^x$ Second equation

Step 2 Graph the system.

Go Online You can complete an Extra Example online.

226 **Module 5** • Exponential Functions

Interactive Presentation

Solve an Exponential Equation by Graphing

SUPER BOWL **Since it began in 1966, the extremely high viewership of the Super Bowl has led companies to pay millions of dollars to advertise for just 30 seconds. The average cost of a 30-second advertisement can be approximately modeled by an exponential function where the initial cost is $58,000 and the average cost increases by 9.9% each Super Bowl.**

Example 2

TYPE

Students complete the calculations to solve an exponential equation by graphing.

TAP

Students tap to choose a calculator to graph an exponential equation.

Example 2 Solve an Exponential Equation by Graphing

Teaching the Mathematical Practices

5 Use a Source Guide students to find external information to answer the questions posed in the Use a Source feature.

Questions for Mathematical Discourse

- **AL** Why is $58,000 written as $0.058? The equation represents money in millions of dollars.
- **OL** Why is this equation only an approximation? Because it is modeling real-world data.
- **BL** How can you use your calculator to estimate at which Super Bowl the cost to advertise doubled? Graph $y = 0.116$ and $y = 0.058(1.099)^x$ and find the intersection.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Example 3 Use the Compound Interest Formula

Teaching the Mathematical Practices

4 Analyze Mathematical Results In Example 3, point out that to solve the problem, students should interpret their mathematical results in the context of the problem.

Questions for Mathematical Discourse

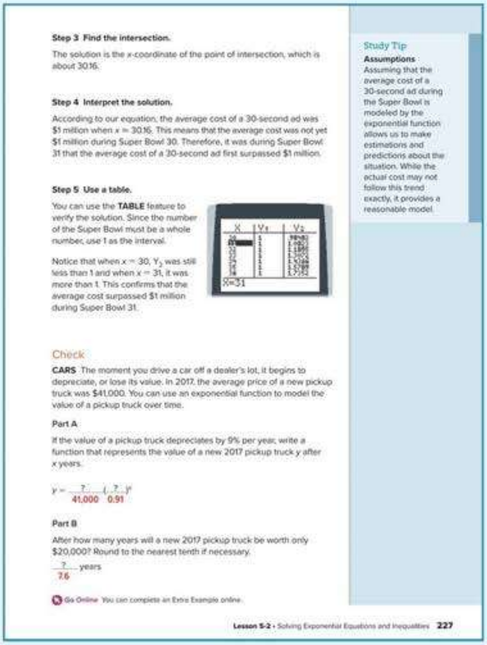
AL Why is $n = 12$? The interest is compounded monthly, which means it will be compounded 12 times a year.

OL What is the y-intercept of the related function $A(t)$, and what does it mean in the context of the example? 1700; It means at $t = 0$, the amount in the account is the initial investment.

BL How does the balance change as you increase the number of times the interest is compounded per year? The balance increases.

Common Error

Students may use the numbers as given in the problem without considering the units. Remind students to convert all percentages to decimal form.



Step 3 Find the intersection.

The solution is the x-coordinate of the point of intersection, which is about 30.16.

Step 4 Interpret the solution.

According to our equation, the average cost of a 30-second ad was $1 million when $x = 30.16$. This means that the average cost was not yet $1 million during Super Bowl 30. Therefore, it was during Super Bowl 31 that the average cost of a 30-second ad first surpassed $1 million.

Step 5 Use a table.

You can use the **TABLE** feature to verify the solution. Since the number of the Super Bowl must be a whole number, use 1 as the interval.

Notice that when $x = 30$, Y_2 was still less than 1 and when $x = 31$, it was more than 1. This confirms that the average cost surpassed $1 million during Super Bowl 31.

Study Tip

Assumptions Assuming that the average cost of a 30-second ad during the Super Bowl is modeled by the exponential function allows us to make estimations and predictions about the situation. While the actual cost may not follow this trend exactly, it provides a reasonable model.

Check

CARS The moment you drive a car off a dealer's lot, it begins to depreciate, or lose its value. In 2017, the average price of a new pickup truck was $41,000. You can use an exponential function to model the value of a pickup truck over time.

Part A

If the value of a pickup truck depreciates by 9% per year, write a function that represents the value of a new 2017 pickup truck y after x years.

$y = \underset{41{,}000}{\underline{\ ?\ }}(\underset{0.91}{\underline{\ ?\ }})^x$

Part B

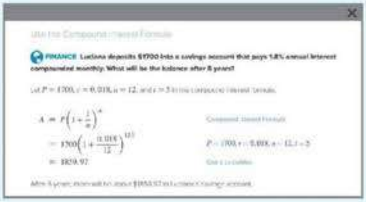
After how many years will a new 2017 pickup truck be worth only $20,000? Round to the nearest tenth if necessary.

$\underset{7.6}{\underline{\ ?\ }}$ years

Go Online You can complete an Extra Example online.

Lesson 5-2 • Solving Exponential Equations and Inequalities 227

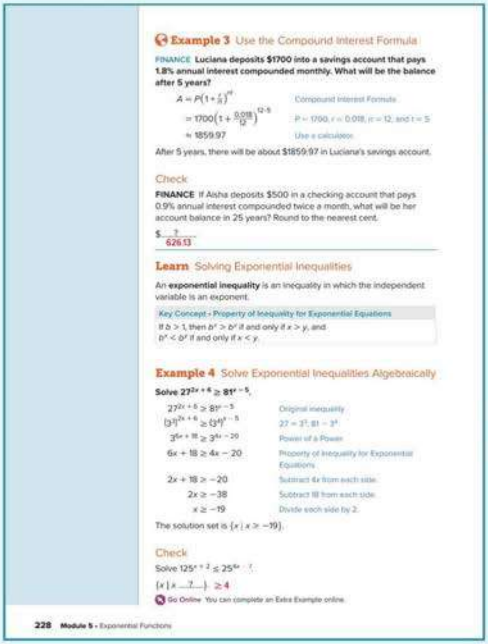
Interactive Presentation



Example 3

CHECK

Students complete the Check online to determine whether they are ready to move on.



Example 3 Use the Compound Interest Formula

FINANCE **Luciana deposits $1700 into a savings account that pays 1.8% annual interest compounded monthly. What will be the balance after 5 years?**

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ Compound Interest Formula

$= 1700\left(1 + \frac{0.018}{12}\right)^{12 \cdot 5}$ P = 1700, r = 0.018, n = 12, and t = 5

≈ 1859.97 Use a calculator.

After 5 years, there will be about $1859.97 in Luciana's savings account.

Check

FINANCE If Aisha deposits $500 in a checking account that pays 0.9% annual interest compounded twice a month, what will be her account balance in 25 years? Round to the nearest cent.

$? 626.13

Learn Solving Exponential Inequalities

An **exponential inequality** is an inequality in which the independent variable is an exponent.

Key Concept • Property of Inequality for Exponential Equations

If $b > 1$, then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.

Example 4 Solve Exponential Inequalities Algebraically

Solve $27^{2x+6} \geq 81^{x-5}$.

$27^{2x+6} \geq 81^{x-5}$ Original inequality

$(3^3)^{2x+6} \geq (3^4)^{x-5}$ $27 = 3^3$, $81 = 3^4$

$3^{6x+18} \geq 3^{4x-20}$ Power of a Power

$6x + 18 \geq 4x - 20$ Property of Inequality for Exponential Equations

$2x + 18 \geq -20$ Subtract 4x from each side.

$2x \geq -38$ Subtract 18 from each side.

$x \geq -19$ Divide each side by 2.

The solution set is $\{x \mid x \geq -19\}$.

Check

Solve $125^{x+2} \leq 25^{4x-7}$.

$\{x \mid x$? $\}$ ≥4

Go Online You can complete an Extra Example online.

228 Module 5 • Exponential Functions

Interactive Presentation

Solve Exponential Inequalities Algebraically

Solve $27^{2x+6} \geq 81^{x-5}$.

$27^{2x+6} \geq 81^{x-5}$ Original inequality

$(3^3)^{2x+6} \geq (3^4)^{x-5}$ $27 = 3^3$, $81 = 3^4$

$3^{6x+18} \geq 3^{4x-20}$ Power of a Power

$6x + 18 \geq 4x - 20$ Property of Inequality for Exponential Equations

$2x + 18 \geq -20$

$2x \geq -38$ Subtract 18 from each side.

$x \geq -19$ Divide each side by 2.

Example 4

CHECK

Students complete the Check online to determine whether they are ready to move on.

Learn Solving Exponential Inequalities

Objective

Students solve exponential inequalities in one variable algebraically.

Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as being composed of several objects. Help student to see how they can use the Property of Inequality for Exponential Equations to more easily solve complicated exponential inequalities.

DIFFERENTIATE

Reteaching Activity AL

IF you want students to check to see if their answer is correct,
THEN remind students to choose any value in the solution interval and see if it satisfies the original inequality.

Example 4 Solve Exponential Inequalities Algebraically

Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. In Example 4, students must rewrite the base of exponential expressions to solve an exponential inequality.

Questions for Mathematical Discourse

AL Why are the expressions written in terms of powers of 3 instead of 9? Both expressions can be written as powers of 3 with integer powers.

OL How can you use your calculator to check your solution? Graph $y = 27^{2x+6}$ and $y = 81^{x-5}$, and the solution will be where the graph of $y = 27^{2x+6}$ is above the graph of $y = 81^{x-5}$.

BL Write a rule similar to the Property of Inequality for Exponential Equations for $0 < b < 1$. $b^x > b$ if and only if $x < y$, and $b^x < b$ if^y and only if $x > y$.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–20
2	exercises that use a variety of skills from this lesson	21–57
3	exercises that emphasize higher-order and critical-thinking skills	58–65

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–57 odd, 58–65
- Extension: Musical Relationships
- **ALEKS®** Exponential and Logarithmic Equations

OL

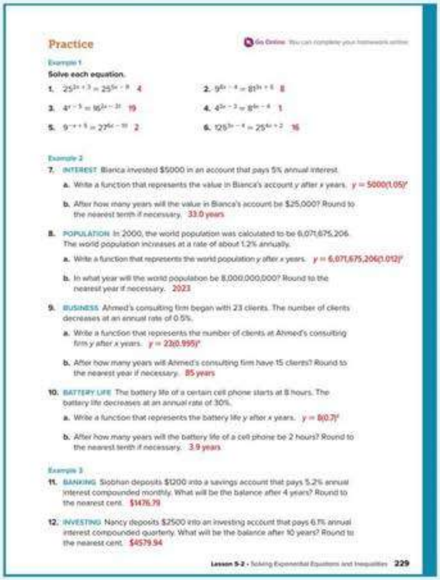
IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–65 odd
- Remediation, Review Resources: Solving Exponential Equations and Inequalities
- Personal Tutors
- Extra Examples 1–4
- **ALEKS®** Quadratic Equations

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–19 odd
- Remediation, Review Resources: Solving Exponential Equations and Inequalities
- *Quick Review Math Handbook*: Solving Exponential Equations and Inequalities
- **ALEKS®** Quadratic Equations



Practice

Go Online You can complete your homework online.

Example 1

Solve each equation.

Example 2

7. INTEREST Bianca invested $5000 in an account that pays 5% annual interest.
   a. Write a function that represents the value in Bianca's account y after x years. $y = 5000(1.05)^x$
   b. After how many years will the value in Bianca's account be $25,000? Round to the nearest tenth if necessary. 33.0 years

8. POPULATION In 2000, the world population was calculated to be 6,071,675,206. The world population increases at a rate of about 1.2% annually.
   a. Write a function that represents the world population y after x years. $y = 6{,}071{,}675{,}206(1.012)^x$
   b. In what year will the world population be 8,000,000,000? Round to the nearest year if necessary. 2023

9. BUSINESS Ahmed's consulting firm began with 23 clients. The number of clients decreases at an annual rate of 0.5%.
   a. Write a function that represents the number of clients at Ahmed's consulting firm y after x years. $y = 23(0.995)^x$
   b. After how many years will Ahmed's consulting firm have 15 clients? Round to the nearest year if necessary. 85 years

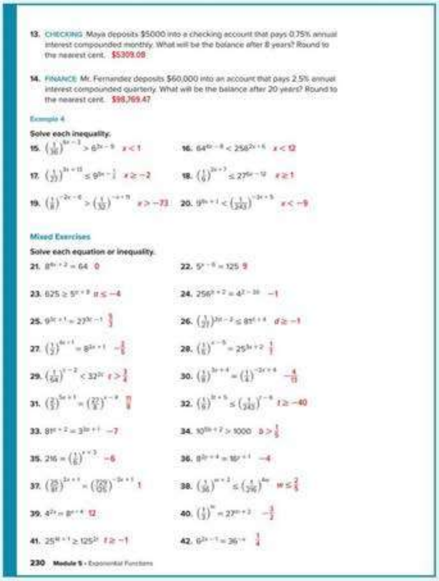
10. BATTERY LIFE The battery life of a certain cell phone starts at 8 hours. The battery life decreases at an annual rate of 30%.
    a. Write a function that represents the battery life y after x years. $y = 8(0.7)^x$
    b. After how many years will the battery life of a cell phone be 2 hours? Round to the nearest tenth if necessary. 3.9 years

Example 3

11. BANKING Siobhan deposits $1200 into a savings account that pays 5.2% annual interest compounded monthly. What will be the balance after 4 years? Round to the nearest cent. $1476.79

12. INVESTING Nancy deposits $2500 into an investing account that pays 6.1% annual interest compounded quarterly. What will be the balance after 10 years? Round to the nearest cent. $4579.94

Lesson 5-2 • Solving Exponential Equations and Inequalities 229



13. CHECKING Maya deposits $5000 into a checking account that pays 0.75% annual interest compounded monthly. What will be the balance after 8 years? Round to the nearest cent. $5309.08

14. FINANCE Mr. Fernandez deposits $60,000 into an account that pays 2.5% annual interest compounded quarterly. What will be the balance after 20 years? Round to the nearest cent. $98,769.47

Example 4

Solve each inequality.

Mixed Exercises

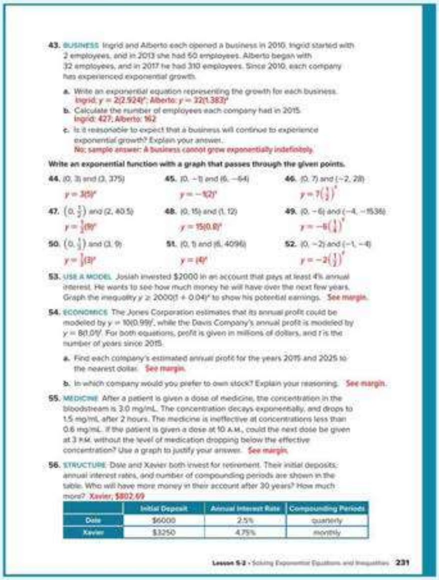
Solve each equation or inequality.

230 Module 5 • Exponential Functions

3 REFLECT AND PRACTICE

A.CED.1, A.REI.11

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION



43. BUSINESS Ingrid and Alberto each opened a business in 2010. Ingrid started with 2 employees, and in 2013 she had 50 employees. Alberto began with 32 employees, and in 2017 he had 310 employees. Since 2010, each company has experienced exponential growth.
   a. Write an exponential equation representing the growth for each business. Ingrid: $y = 2(2.924)^t$; Alberto: $y = 32(1.383)^t$
   b. Calculate the number of employees each company had in 2015. Ingrid: 427; Alberto: 162
   c. Is it reasonable to expect that a business will continue to experience exponential growth? Explain your answer. No; sample answer: A business cannot grow exponentially indefinitely.

Write an exponential function with a graph that passes through the given points.

44. (0, 3) and (3, 375) $y = 3(5)^x$
45. (0, −1) and (6, −64) $y = -1(2)^x$
46. (0, 7) and (−2, 28) $y = 7\left(\frac{1}{2}\right)^x$
47. $\left(0, \frac{1}{2}\right)$ and (2, 40.5) $y = \frac{1}{2}(9)^x$
48. (0, 15) and (1, 12) $y = 15(0.8)^x$
49. (0, −6) and (−4, −1536) $y = -6\left(\frac{1}{4}\right)^x$
50. $\left(0, \frac{1}{3}\right)$ and (3, 9) $y = \frac{1}{3}(3)^x$
51. (0, 1) and (6, 4096) $y = (4)^x$
52. (0, −2) and (−1, −4) $y = -2\left(\frac{1}{2}\right)^x$

53. USE A MODEL Josiah invested \$2000 in an account that pays at least 4% annual interest. He wants to see how much money he will have over the next few years. Graph the inequality $y \geq 2000(1 + 0.04)^x$ to show his potential earnings. See margin.

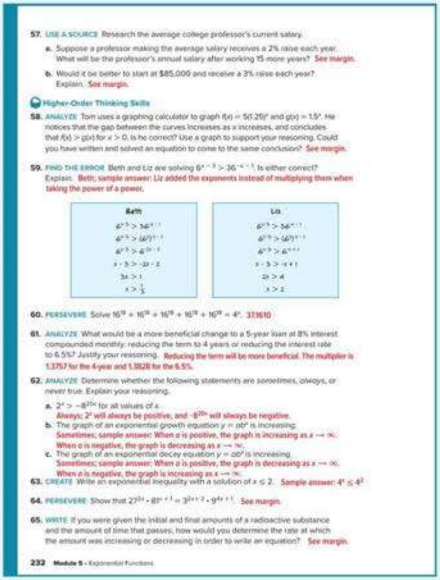
54. ECONOMICS The Jones Corporation estimates that its annual profit could be modeled by $y = 10(0.99)^t$, while the Davis Company's annual profit is modeled by $y = 8(1.01)^t$. For both equations, profit is given in millions of dollars, and t is the number of years since 2015.
   a. Find each company's estimated annual profit for the years 2015 and 2025 to the nearest dollar. See margin.
   b. In which company would you prefer to own stock? Explain your reasoning. See margin.

55. MEDICINE After a patient is given a dose of medicine, the concentration in the bloodstream is 3.0 mg/mL. The concentration decays exponentially, and drops to 1.5 mg/mL after 2 hours. The medicine is ineffective at concentrations less than 0.6 mg/mL. If the patient is given a dose at 10 A.M., could the next dose be given at 3 P.M. without the level of medication dropping below the effective concentration? Use a graph to justify your answer. See margin.

56. STRUCTURE Dale and Xavier both invest for retirement. Their initial deposits, annual interest rates, and number of compounding periods are shown in the table. Who will have more money in their account after 30 years? How much more? Xavier; \$802.69

	Initial Deposit	Annual Interest Rate	Compounding Periods
Dale	\$6000	2.5%	quarterly
Xavier	\$3250	4.75%	monthly

Lesson 5-2 • Solving Exponential Equations and Inequalities 231



57. USE A SOURCE Research the average college professor's current salary.
   a. Suppose a professor making the average salary receives a 2% raise each year. What will be the professor's annual salary after working 15 more years? See margin.
   b. Would it be better to start at \$85,000 and receive a 3% raise each year? Explain. See margin.

Higher-Order Thinking Skills

58. ANALYZE Tom uses a graphing calculator to graph $f(x) = 5(1.25)^x$ and $g(x) = 1.5^x$. He notices that the gap between the curves increases as x increases, and concludes that $f(x) > g(x)$ for $x > 0$. Is he correct? Use a graph to support your reasoning. Could you have written and solved an equation to come to the same conclusion? See margin.

59. FIND THE ERROR Beth and Liz are solving $6^{x-3} > 36^{-x-1}$. Is either correct? Explain. Beth; sample answer: Liz added the exponents instead of multiplying them when taking the power of a power.

Beth	Liz
$6^{x-3} > 36^{-x-1}$	$6^{x-3} > 36^{-x-1}$
$6^{x-3} > (6^2)^{-x-1}$	$6^{x-3} > (6^2)^{-x-1}$
$6^{x-3} > 6^{-2x-2}$	$6^{x-3} > 6^{-x+1}$
$x - 3 > -2x - 2$	$x - 3 > -x + 1$
$3x > 1$	$2x > 4$
$x > \frac{1}{3}$	$x > 2$

60. PERSEVERE Solve $16^{18} + 16^{18} + 16^{18} + 16^{18} + 16^{18} = 4^x$. 37.1610

61. ANALYZE What would be a more beneficial change to a 5-year loan at 8% interest compounded monthly: reducing the term to 4 years or reducing the interest rate to 6.5%? Justify your reasoning. Reducing the term will be more beneficial. The multiplier is 1.3757 for the 4-year and 1.3828 for the 6.5%.

62. ANALYZE Determine whether the following statements are sometimes, always, or never true. Explain your reasoning.
   a. $2^x > -8^{20x}$ for all values of x. Always; 2^x will always be positive, and -8^{20x} will always be negative.
   b. The graph of an exponential growth equation $y = ab^x$ is increasing. Sometimes; sample answer: When a is positive, the graph is increasing as $x \to \infty$. When a is negative, the graph is decreasing as $x \to \infty$.
   c. The graph of an exponential decay equation $y = ab^x$ is increasing. Sometimes; sample answer: When a is positive, the graph is decreasing as $x \to \infty$. When a is negative, the graph is increasing as $x \to \infty$.

63. CREATE Write an exponential inequality with a solution of $x \leq 2$. Sample answer: $4^x \leq 4^2$

64. PERSEVERE Show that $27^{2x} \cdot 81^{x+1} = 3^{2x+2} \cdot 9^{4x+1}$. See margin.

65. WRITE If you were given the initial and final amounts of a radioactive substance and the amount of time that passes, how would you determine the rate at which the amount was increasing or decreasing in order to write an equation? See margin.

232 Module 5 • Exponential Functions

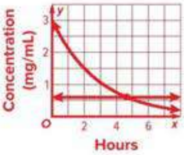
Answers

53.

54a. Jones Corporation: \$10,000,000; \$9,043,821; Davis Company: \$8,000,000; \$8,836,977

54b. Sample answer: Davis Company; Since annual profits have been increasing for the Davis Company, stock in this company would most likely increase in value also. Jones Corporation's profit equation represents exponential decay since $b = 0.99$ (it is less than 1), while the Davis Company's profit equation represents exponential growth since $b = 1.01$ (it is greater than 1).

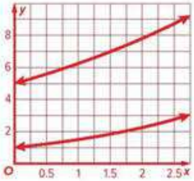
55. No; A function to model the concentration is $f(t) = 3(0.5)^{0.5t}$. The intersection with $g(t) = 0.6$ occurs at $t \approx 4.6$, meaning the concentration drops below the effective level before 3 P.M.

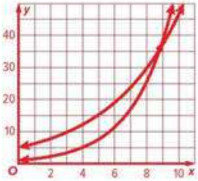


57a. Sample answer: The average salary for a professor in 2017 was \$100,100. After 15 years, the professor's annual salary would be $100,100(1.02)^{15} = \$134,721.42$.

57b. Sample answer: It depends on how long the professor intends to work. For years 1-16, the average salary with a 2% raise earns more. After year 17, the lower starting salary with a 3% raise earns more. The professor should consider the cumulative salary over the time he or she intends to work to determine which is better.

58. He is not correct; Eventually $g(x) > f(x)$ because the base of $g(x)$ is a greater value. Tom should have enlarged the viewing window in order to identify this result. To disprove that $f(x) < g(x)$ for $x > 0$, the equation $f(x) = g(x)$ or $5(1.25)^x = 1.5^x$ could be shown to have a real solution.





64.

$27^{2x} \cdot 81^{x+1} = 3^{2x+2} \cdot 9^{4x+1}$	Original equation
$(3^3)^{2x} \cdot (3^4)^{x+1} = 3^{2x+2} \cdot (3^2)^{4x+1}$	$3^2 = 9$, $3^3 = 27$, and $3^4 = 81$
$3^{6x} \cdot 3^{4x+4} = 3^{2x+2} \cdot 3^{8x+2}$	Power of a Power
$3^{10x+4} = 3^{10x+4}$	Product of Powers
$10x + 4 = 10x + 4$	Property of Equality for Exponential Equations
$10x = 10x$	Subtract 4 from each side.
$x = x$	Divide each side by 10.

65. Sample answer: Divide the final amount by the initial amount. If n is the number of time intervals that pass, take the nth root of the answer.

Lesson 5-3

Special Exponential Functions

A.CED.2, F.IF.6

LESSON GOAL

Students analyze expressions and functions involving the natural base *e*.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Finding the Value of *e*

Develop:

Exponential Functions with Base *e*
- Expressions with Base *e*
- Graph Functions with Base *e*
- Apply Functions with Base *e*
- Solve an Exponential Equation by Using Technology

You may want your students to complete the **Check** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Writing Exponential Functions	●	●		●
Extension: Logistic Growth		●	●	●

Language Development Handbook

Assign page 27 of the *Language Development Handbook* to help your students build mathematical language related to analyzing expressions and functions involving the natural base *e*.

ELL You can use the tips and suggestions on page T27 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra, Functions

Standards for Mathematical Content:

A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students graphed exponential growth and decay functions. **F.IF.4, F.IF.7e**

Now
Students analyze expressions and functions involving the natural base *e*.
A.CED.2, F.IF.6

Next
Students will simplify logarithmic expressions and solve exponential equations by using natural logarithms.
A.SSE.2, F.LE.4

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Conceptual Bridge In this lesson, students extend their understanding of exponential functions to functions with base *e*. They build fluency by solving problems related to these types of exponential functions, and they apply their understanding by solving real-world problems.

Mathematical Background

The number *e* is an irrational number. It is an important number in mathematics.

The first several digits are: 2.7182818284590452353602874713527.

It is called Euler's number, named after Leonhard Euler.

1 LAUNCH

A.CED.2, F.IF.6

Interactive Presentation

Warm Up

Solve each equation.

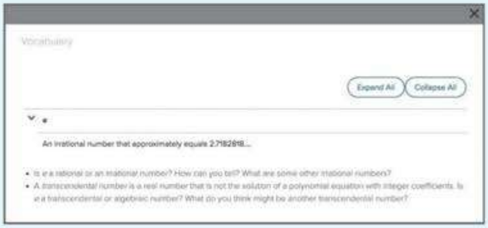
1. $7 = 49^x$
2. $\left(\frac{1}{4}\right)^x = 16$
3. $2^{x+3} = 2^4$
4. $3^{x+1} = 9^{x-2}$
5. $\left(\frac{1}{3}\right)^{2-x} = \left(\frac{1}{27}\right)^{3+x}$

Warm Up

Launch the Lesson

Watch this video to learn about the mathematician Leonhard Euler and the number e.

Launch the Lesson



Vocabulary

e

An irrational number that approximately equals 2.7182818...

- Is e a rational or an irrational number? How can you tell? What are some other irrational numbers?
- A transcendental number is a real number that is not the solution of a polynomial equation with integer coefficients. Is e a transcendental or algebraic number? What do you think might be another transcendental number?

Today’s Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- solving exponential equations

Answers:

1. $\frac{1}{2}$

2. -2

3. 1

4. 5

5. $-\frac{7}{4}$

Launch the Lesson

Teaching the Mathematical Practices

6 State Meaning of Symbols As students watch the Launch the Lesson video, encourage students to familiarize themselves with the meaning of *e* and reiterate that it is a number, not a variable.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today’s Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today’s Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then discuss the questions below with the class.

Explore Finding the Value of e

Objective

Students use algebraic expressions to explore the value of *e*.

Teaching the Mathematical Practices

6 Use Precision Throughout the Explore, students calculate accurately and express numerical answers with an appropriate degree of precision to approximate the value of *e*.

7 Look for a Pattern Help students to see the pattern in the value of *e* as the value of *n* increases.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

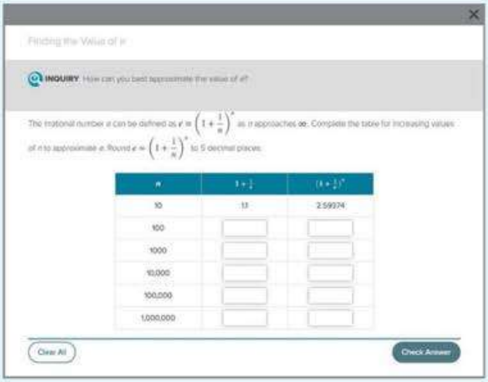
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises to approximate the value of *e* and make conjectures on how to get the best approximation. Then, students will complete the Inquiry Question.

(continued on the next page)

Interactive Presentation



Finding the Value of e

INQUIRY How can you best approximate the value of e?

The irrational number e can be defined as $e = \left(1 + \frac{1}{n}\right)^n$ as n approaches ∞. Complete the table for increasing values of n to approximate e. Round $e \approx \left(1 + \frac{1}{n}\right)^n$ to 5 decimal places.

n	$1 + \frac{1}{n}$	$\left(1 + \frac{1}{n}\right)^n$
10	1.1	2.59374
100	☐	☐
1000	☐	☐
10,000	☐	☐
100,000	☐	☐
1,000,000	☐	☐

Explore

TYPE

Students complete the calculations to find the value of *e*.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Finding the Value of *e* (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What happens to the expression $1 + \frac{1}{n}$ as n increases? Sample answer: The fractional part of the number is approaching 0, so the value is getting closer to 1.
- Would it be better to approximate with $n = 1000$ or $n = 10{,}000$? Sample answer: The greater value is a closer approximation, so it's better to use $n = 10{,}000$.

Inquiry

How can you best approximate the value of *e*? By evaluating greater values of *n* or more terms of the expression, the approximation approaches the actual value of *e*.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Exponential Functions with Base *e*

Objective

Students analyze expressions and functions involving the natural base *e*.

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationship between the function and graph of an exponential function with base *e*. Then, ask students how an exponential function with base *e* relates to an exponential growth function.

Things to Remember

Most calculators have an e^x key for evaluating natural base.

Important to Know

Stress that *e* is a constant, like π, and not a variable, like *x* or *y*.

DIFFERENTIATE

Reteaching Activity AL

IF some students mistakenly think that an equation like $4e^{-2x} - 5 = 3$ contains two variables,

THEN point out that the letter *e* represents a constant, just as π does. Both *e* and π are irrational numbers, which cannot be expressed exactly with numerals. To help students avoid this confusion, have them highlight the variables in the equation with a marker.

Example 1 Expressions with Base *e*

Teaching the Mathematical Practices

5 Decide When to Use Tools Use the Watch Out! feature to help students recognize the insight to be gained from and the limitations of using a calculator to evaluate expressions with base *e*.

Questions for Mathematical Discourse

AL Can the value of *e* be written exactly? Explain. No, *e* is an irrational number, so it is a non-terminating, non-repeating decimal. The exact way to represent the value is *e*.

OL What are the domain and range of $f(x) = e^x$? $D = (-\infty, \infty)$, $R = (0, \infty)$

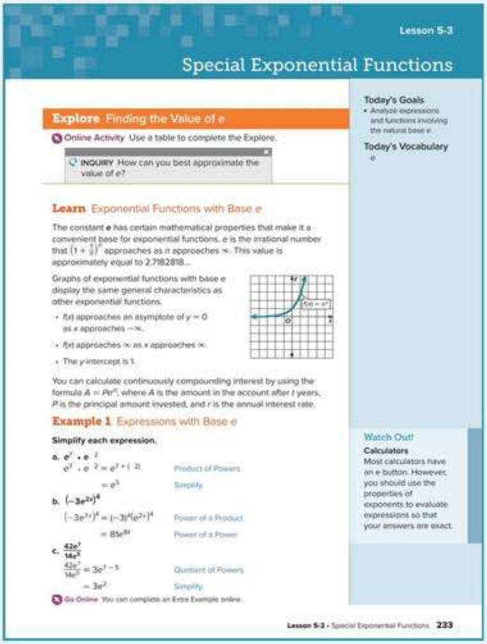
BL The average rate of change for e^x over the interval $[a, a + b]$ approaches the function value e^a as b approaches 0. What does this suggest? The instantaneous rate of change of e^x at $x = a$ is e^a.

Common Error

Most calculators have an *e* button. However, students should use the properties of exponents to evaluate expressions so that their answers are exact.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 5-3

Special Exponential Functions

Explore Finding the Value of *e*

Online Activity Use a table to complete the Explore.

INQUIRY How can you best approximate the value of *e*?

Today's Goals
- Analyze expressions and functions involving the natural base *e*.

Today's Vocabulary
e

Learn Exponential Functions with Base *e*

The constant ***e*** has certain mathematical properties that make it a convenient base for exponential functions. *e* is the irrational number that $\left(1 + \frac{1}{n}\right)^n$ approaches as n approaches ∞. This value is approximately equal to 2.7182818...

Graphs of exponential functions with base *e* display the same general characteristics as other exponential functions.

- $f(x)$ approaches an asymptote of $y = 0$ as x approaches $-\infty$.
- $f(x)$ approaches ∞ as x approaches ∞.
- The y-intercept is 1.

You can calculate continuously compounding interest by using the formula $A = Pe^{rt}$, where A is the amount in the account after t years, P is the principal amount invested, and r is the annual interest rate.

Example 1 Expressions with Base *e*

Simplify each expression.

a. $e^7 \cdot e^{-2}$

$e^7 \cdot e^{-2} = e^{7+(-2)}$ Product of Powers

$= e^5$ Simplify.

b. $(-3e^{2x})^4$

$(-3e^{2x})^4 = (-3)^4(e^{2x})^4$ Power of a Product

$= 81e^{8x}$ Power of a Power

c. $\frac{42e^7}{14e^5}$

$\frac{42e^7}{14e^5} = 3e^{7-5}$ Quotient of Powers

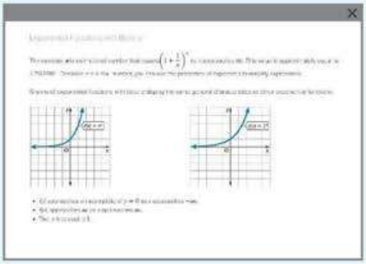
$= 3e^2$ Simplify.

Go Online You can complete an Extra Example online.

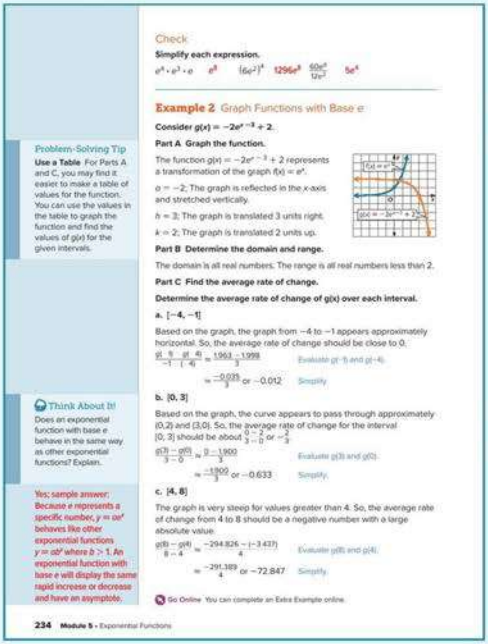
Watch Out!
Calculators
Most calculators have an *e* button. However, you should use the properties of exponents to evaluate expressions so that your answers are exact.

Lesson 5-3 • Special Exponential Functions 233

Interactive Presentation



Learn



Check

Simplify each expression.

$e^4 \cdot e^3 \cdot e$ e^8 $(6e^2)^4$ $1296e^8$ $\frac{60e^6}{12e^2}$ $5e^4$

Example 2 Graph Functions with Base *e*

Consider $g(x) = -2e^{x-3} + 2$.

Part A Graph the function.

The function $g(x) = -2e^{x-3} + 2$ represents a transformation of the graph $f(x) = e^x$.

$a = -2$: The graph is reflected in the *x*-axis and stretched vertically.

$h = 3$: The graph is translated 3 units right.

$k = 2$: The graph is translated 2 units up.

Part B Determine the domain and range.

The domain is all real numbers. The range is all real numbers less than 2.

Part C Find the average rate of change.

Determine the average rate of change of g(x) over each interval.

a. $[-4, -1]$

Based on the graph, the graph from −4 to −1 appears approximately horizontal. So, the average rate of change should be close to 0.

$\frac{g(-1) - g(-4)}{-1-(-4)} \approx \frac{1.963 - 1.999}{3}$ Evaluate g(−1) and g(−4).

$\approx \frac{-0.035}{3}$ or -0.012 Simplify.

b. $[0, 3]$

Based on the graph, the curve appears to pass through approximately (0,2) and (3,0). So, the average rate of change for the interval [0, 3] should be about $\frac{0-2}{3-0}$ or $-\frac{2}{3}$.

$\frac{g(3) - g(0)}{3-0} \approx \frac{0 - 1.900}{3}$ Evaluate g(3) and g(0).

$\approx \frac{-1.900}{3}$ or -0.633 Simplify.

c. $[4, 8]$

The graph is very steep for values greater than 4. So, the average rate of change from 4 to 8 should be a negative number with a large absolute value.

$\frac{g(8) - g(4)}{8-4} \approx \frac{-294.826 - (-3.437)}{4}$ Evaluate g(8) and g(4).

$\approx \frac{-291.389}{4}$ or -72.847 Simplify.

Go Online You can complete an Extra Example online.

Problem-Solving Tip

Use a Table For Parts A and C, you may find it easier to make a table of values for the function. You can use the values in the table to graph the function and find the values of *g*(*x*) for the given intervals.

Think About It!

Does an exponential function with base *e* behave in the same way as other exponential functions? Explain.

Yes; sample answer: Because *e* represents a specific number, $y = ae^x$ behaves like other exponential functions $y = ab^x$ where $b > 1$. An exponential function with base *e* will display the same rapid increase or decrease and have an asymptote.

234 **Module 5** • Exponential Functions

Interactive Presentation

Graph Functions with Base *e*

Example 2

TYPE

Students answer a question to show they understand how to graph and analyze functions with base *e*.

SWIPE

Students move through the slides to graph a function with base *e*.

1 CONCEPTUAL UNDERSTANDING **2 FLUENCY** 3 APPLICATION

Example 2 Graph Functions with Base *e*

Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of the function to identify the transformations in the function.

Questions for Mathematical Discourse

AL Does a function with base *e* represent exponential growth or exponential decay? It is exponential growth because $e > 1$.

OL How does the average rate of change over $[x, x + 1]$ change as $x \longrightarrow \infty$? It decreases more rapidly.

BL What is the instantaneous rate of change at *g*(3)? −2

Important to Know

For Parts A and C, students may find it easier to make a table of values for the function. They can use the values in the table to graph the function and find the values of *g*(*x*) for the given intervals.

Example 3 Apply Functions with Base *e*

Teaching the Mathematical Practices

2 Make Sense of Quantities In Example 3, students must make sense of the quantities and their relationships given in the situation to apply the formula for continuous exponential growth.

Questions for Mathematical Discourse

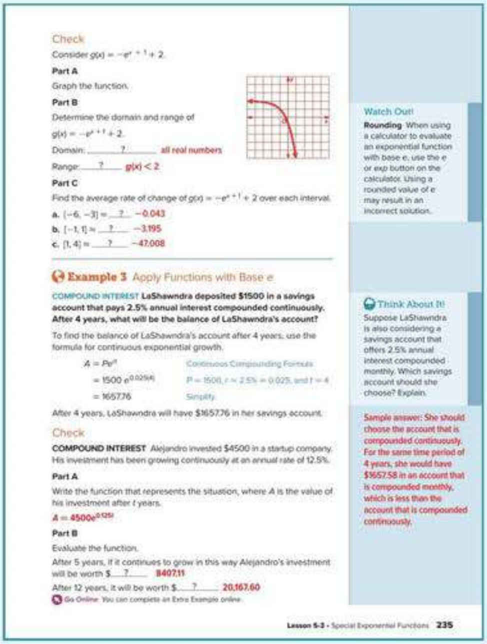
AL What is *A* for $t = 0$? 1500

OL Will the formula work for a monthly interest rate? Y es, if the unit of *t* is months.

BL What is the relationship between the compound interest formula and the continuous compounding formula? The continuous formula is the compound interest formula for $n \rightarrow \infty$.

Common Error

Students often introduce error by rounding during each step of a solution. In order to avoid any errors due to rounding, do not round until the very end of the calculations.



Check

Consider $g(x) = -e^{x+1} + 2$.

Part A

Graph the function.

Part B

Determine the domain and range of $g(x) = -e^{x+1} + 2$.

Domain: ___?___ all real numbers

Range: ___?___ $g(x) < 2$

Part C

Find the average rate of change of $g(x) = -e^{x+1} + 2$ over each interval.

a. $[-6, -3] \approx$ ___?___ -0.043

b. $[-1, 1] \approx$ ___?___ -3.195

c. $[1, 4] \approx$ ___?___ -47.008

Example 3 Apply Functions with Base *e*

COMPOUND INTEREST **LaShawndra deposited $1500 in a savings account that pays 2.5% annual interest compounded continuously. After 4 years, what will be the balance of LaShawndra's account?**

To find the balance of LaShawndra's account after 4 years, use the formula for continuous exponential growth.

$$A = Pe^{rt} \quad \text{Continuous Compounding Formula}$$
$$= 1500\, e^{0.025(4)} \quad P = 1500,\ r = 2.5\% = 0.025, \text{ and } t = 4$$
$$= 1657.76 \quad \text{Simplify.}$$

After 4 years, LaShawndra will have $1657.76 in her savings account.

Check

COMPOUND INTEREST Alejandro invested $4500 in a startup company. His investment has been growing continuously at an annual rate of 12.5%.

Part A

Write the function that represents the situation, where *A* is the value of his investment after *t* years.

$A = 4500e^{0.125t}$

Part B

Evaluate the function.

After 5 years, if it continues to grow in this way Alejandro's investment will be worth $___?___ 8407.11

After 12 years, it will be worth $___?___ 20,167.60

Go Online You can complete an Extra Example online.

Watch Out!

Rounding When using a calculator to evaluate an exponential function with base *e*, use the *e* or exp button on the calculator. Using a rounded value of *e* may result in an incorrect solution.

Think About It!

Suppose LaShawndra is also considering a savings account that offers 2.5% annual interest compounded monthly. Which savings account should she choose? Explain.

Sample answer: She should choose the account that is compounded continuously. For the same time period of 4 years, she would have $1657.58 in an account that is compounded monthly, which is less than the account that is compounded continuously.

Lesson 5-3 • Special Exponential Functions 235

Interactive Presentation

Apply Functions with Base e

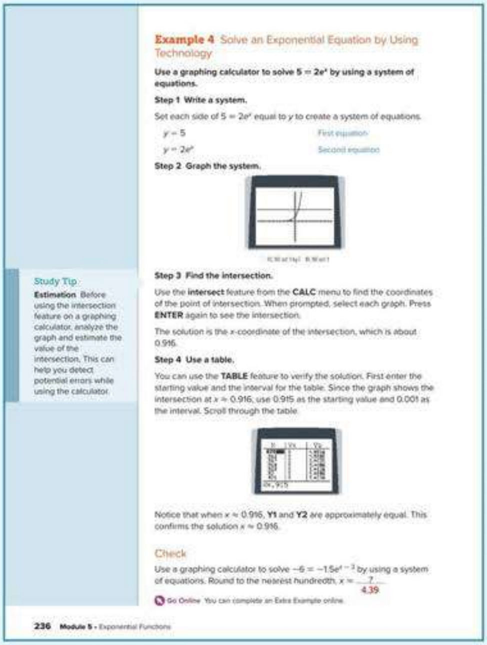
COMPOUND INTEREST LaShawndra deposited $1500 in a savings account that pays 2.5% annual interest compounded continuously. After 4 years, what will be the balance of LaShawndra's account?

Example 3

TYPE

a|

Students answer a question to show they understand continuously compounded interest.



Example 4 Solve an Exponential Equation by Using Technology

Use a graphing calculator to solve $5 = 2e^x$ by using a system of equations.

Step 1 Write a system.

Set each side of $5 = 2e^x$ equal to y to create a system of equations.

$y = 5$ First equation
$y = 2e^x$ Second equation

Step 2 Graph the system.

Step 3 Find the intersection.

Use the **intersect** feature from the **CALC** menu to find the coordinates of the point of intersection. When prompted, select each graph. Press **ENTER** again to see the intersection.

The solution is the x-coordinate of the intersection, which is about 0.916.

Step 4 Use a table.

You can use the **TABLE** feature to verify the solution. First enter the starting value and the interval for the table. Since the graph shows the intersection at $x \approx 0.916$, use 0.915 as the starting value and 0.001 as the interval. Scroll through the table.

Notice that when $x \approx 0.916$, **Y1** and **Y2** are approximately equal. This confirms the solution $x \approx 0.916$.

Study Tip

Estimation Before using the intersection feature on a graphing calculator, analyze the graph and estimate the value of the intersection. This can help you detect potential errors while using the calculator.

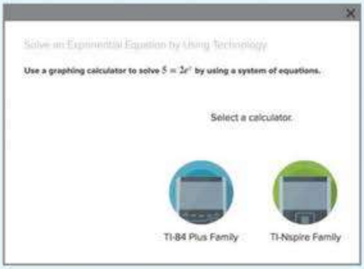
Check

Use a graphing calculator to solve $-6 = -1.5e^{x-2}$ by using a system of equations. Round to the nearest hundredth. $x \approx$ __?__ 4.39

Go Online You can complete an Extra Example online.

236 Module 5 • Exponential Functions

Interactive Presentation



Solve an Exponential Equation by Using Technology

Use a graphing calculator to solve $5 = 2e^x$ by using a system of equations.

Select a calculator.

Example 4

TAP

Students tap to select a calculator to solve an exponential equation.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 4 Solve an Exponential Equation by Using Technology

Teaching the Mathematical Practices

5 Decide When to Use Tools Mathematically proficient students can make sound decisions about when to use mathematical tools such as graphing calculators. In Example 4, help them see why using a graphing calculator will help to solve the problem and the limitations of the tool.

Questions for Mathematical Discourse

AL When using the table to verify a solution, what is the effect of the interval of the table? A smaller interval allows you to estimate solutions to more decimal places.

OL What is another way to solve the equation using a graphing calculator? Identify the solution to $5 - 2e^x = 0$.

BL What approach would you take to develop a way to solve this equation algebraically? Sample answer: You would need to define the inverse function of $y = e^x$ and apply it to $\frac{5}{2}$.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–21
2	exercises that use a variety of skills from this lesson	22–34
3	exercises that emphasize higher-order and critical-thinking skills	35–39

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Check, **BL**
THEN assign:

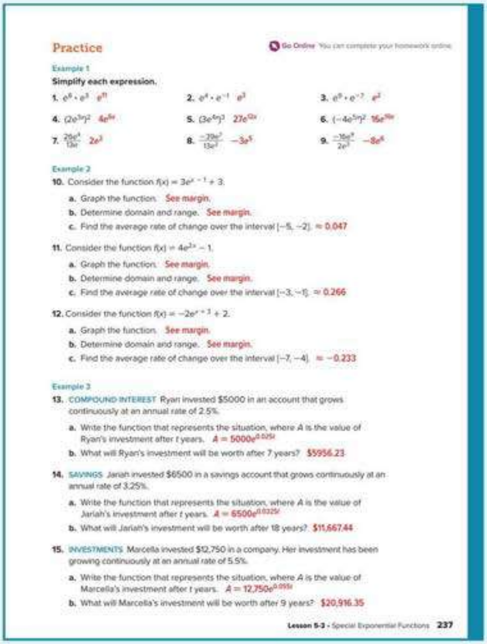
- Practice Exercises 1–33 odd, 35–39
- Extension: Logistic Growth
- ALEKS Graphing Exponential and Logarithmic Functions

IF students score 66%–89% on the Check, **OL**
THEN assign:

- Practice Exercises 1–39 odd
- Remediation, Review Resources: Writing Exponential Functions
- Personal Tutors
- Extra Examples 1–4
- ALEKS Exponential Functions

IF students score 65% or less on the Check, **AL**
THEN assign:

- Practice Exercises 1–21 odd
- Remediation, Review Resources: Writing Exponential Functions
- ALEKS Exponential Functions



Practice

Go Online You can complete your homework online.

Example 1

Simplify each expression.

1. $e^8 \cdot e^3$ e^{11}
2. $e^4 \cdot e^{-1}$ e^3
3. $e^9 \cdot e^{-7}$ e^2
4. $(2e^{3x})^2$ $4e^{6x}$
5. $(3e^{4x})^3$ $27e^{12x}$
6. $(-4e^{5x})^2$ $16e^{10x}$
7. $\frac{26e^4}{13e}$ $2e^3$
8. $\frac{-39e^7}{13e^2}$ $-3e^5$
9. $\frac{-16e^9}{2e^3}$ $-8e^6$

Example 2

10. Consider the function $f(x) = 3e^{x-1} + 3$.
    a. Graph the function. See margin.
    b. Determine domain and range. See margin.
    c. Find the average rate of change over the interval $[-5, -2]$. ≈ 0.047

11. Consider the function $f(x) = 4e^{2x} - 1$.
    a. Graph the function. See margin.
    b. Determine domain and range. See margin.
    c. Find the average rate of change over the interval $[-3, -1]$. ≈ 0.266

12. Consider the function $f(x) = -2e^{x+3} + 2$.
    a. Graph the function. See margin.
    b. Determine domain and range. See margin.
    c. Find the average rate of change over the interval $[-7, -4]$. ≈ -0.233

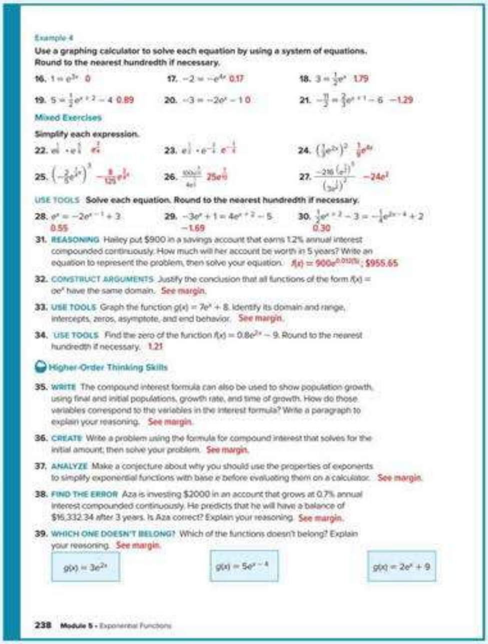
Example 3

13. COMPOUND INTEREST Ryan invested $5000 in an account that grows continuously at an annual rate of 2.5%.
    a. Write the function that represents the situation, where A is the value of Ryan's investment after t years. $A = 5000e^{0.025t}$
    b. What will Ryan's investment will be worth after 7 years? $5956.23

14. SAVINGS Jariah invested $6500 in a savings account that grows continuously at an annual rate of 3.25%.
    a. Write the function that represents the situation, where A is the value of Jariah's investment after t years. $A = 6500e^{0.0325t}$
    b. What will Jariah's investment will be worth after 18 years? $11,667.44

15. INVESTMENTS Marcella invested $12,750 in a company. Her investment has been growing continuously at an annual rate of 5.5%.
    a. Write the function that represents the situation, where A is the value of Marcella's investment after t years. $A = 12,750e^{0.055t}$
    b. What will Marcella's investment will be worth after 9 years? $20,916.35

Lesson 5-3 • Special Exponential Functions **237**



Example 4

Use a graphing calculator to solve each equation by using a system of equations. Round to the nearest hundredth if necessary.

16. $1 = e^{3x}$ 0
17. $-2 = -e^{4x}$ 0.17
18. $3 = \frac{1}{2}e^x$ 1.79
19. $5 = \frac{1}{2}e^{x+2} - 4$ 0.89
20. $-3 = -2e^x - 1$ 0
21. $-\frac{11}{2} = \frac{2}{3}e^{x+1} - 6$ −1.29

Mixed Exercises

Simplify each expression.

22. [illegible]
23. [illegible]
24. $\left(\frac{1}{3}e^{2x}\right)^2$ $\frac{1}{9}e^{4x}$
25. $\left(-\frac{2}{5}e^{\frac{1}{3}x}\right)^3$ [illegible]
26. [illegible]
27. [illegible] $-24e^2$

USE TOOLS **Solve each equation. Round to the nearest hundredth if necessary.**

28. $e^x = -2e^{x-1} + 3$ 0.55
29. $-3e^x + 1 = 4e^{x+2} - 5$ −1.69
30. $\frac{1}{2}e^{x+2} - 3 = -\frac{1}{4}e^{2x-4} + 2$ 0.30

31. REASONING Hailey put $900 in a savings account that earns 1.2% annual interest compounded continuously. How much will her account be worth in 5 years? Write an equation to represent the problem, then solve your equation. $f(x) = 900e^{0.012(5)}$; $955.65

32. CONSTRUCT ARGUMENTS Justify the conclusion that all functions of the form $f(x) = ae^x$ have the same domain. See margin.

33. USE TOOLS Graph the function $g(x) = 7e^x + 8$. Identify its domain and range, intercepts, zeros, asymptote, and end behavior. See margin.

34. USE TOOLS Find the zero of the function $f(x) = 0.8e^{2x} - 9$. Round to the nearest hundredth if necessary. 1.21

Higher-Order Thinking Skills

35. WRITE The compound interest formula can also be used to show population growth, using final and initial populations, growth rate, and time of growth. How do those variables correspond to the variables in the interest formula? Write a paragraph to explain your reasoning. See margin.

36. CREATE Write a problem using the formula for compound interest that solves for the initial amount; then solve your problem. See margin.

37. ANALYZE Make a conjecture about why you should use the properties of exponents to simplify exponential functions with base e before evaluating them on a calculator. See margin.

38. FIND THE ERROR Aza is investing $2000 in an account that grows at 0.7% annual interest compounded continuously. He predicts that he will have a balance of $16,332.34 after 3 years. Is Aza correct? Explain your reasoning. See margin.

39. WHICH ONE DOESN'T BELONG? Which of the functions doesn't belong? Explain your reasoning. See margin.

$g(x) = 3e^{2x}$

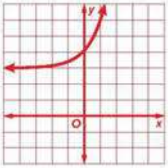
$g(x) = 5e^{x-4}$

$g(x) = 2e^x + 9$

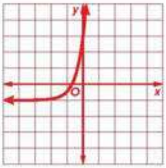
238 Module 5 • Exponential Functions

Answers

10a.



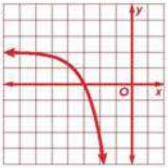
11a.



10b. The domain is all real numbers. The range is all real numbers greater than 3.

11b. The domain is all real numbers. The range is all real numbers greater than −1.

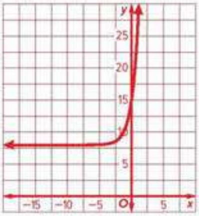
12a.



12b. The domain is all real numbers. The range is all real numbers less than 2.

32. Sample answer: Since x is substituted into the exponent and e^x is defined for x equal to all real numbers, the domain is not affected by a; therefore, the domain of any function of the form $f(x) = ae^x$ is all real numbers.

33.



Domain: all real numbers; Range: all real numbers greater than 8; y-intercept: 15; no zeros; asymptote: $y = 8$; end behavior: as $x \longrightarrow -\infty$, $f(x) \longrightarrow 8$, as $x \longrightarrow \infty$, $f(x) \longrightarrow \infty$

35. Final amount corresponds to final population, initial amount corresponds to initial population, interest rate corresponds to growth rate, and time of growth is the same for both equations.

36. Sample answer: Lee has $2142.87 in a savings account that pays 0.8% interest compounded continuously. If there have been no additional deposits, what was the initial deposit 2 years ago?; $2108.86

37. Sample answer: Since e is irrational, evaluating any expression with e will approximate its value, which introduces error. To minimize the error, do as much work as possible with the exact value (e) before evaluating.

38. No; Aza forgot to change the percent to a decimal; he used 0.7 instead of 0.007 in the calculation.

39. $g(x) = 2e^x + 9$; It has a range of all real numbers greater than 9, and the other two functions have a range of all real numbers greater than 0.

Lesson 5-4

Geometric Sequences and Series

A.SSE.4

LESSON GOAL

Students generate geometric series and find their sums.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Explicit and Recursive Formulas

Develop:

Sequences
- Identify Geometric Sequences
- Graph Geometric Sequences
- Find the *n*th Term
- Write an Equation for the *n*th Term
- Recursive and Explicit Formulas
- Find Geometric Means

Geometric Series
- Find the Sum of a Geometric Series
- Find the First Term in a Series
- Sum in Sigma Notation

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Arithmetic Sequences	●	●		●
Extension: Half the Distance		●	●	●

Language Development Handbook

Assign page 28 of the *Language Development Handbook* to help your students build mathematical language related to geometric sequences and series.

ELL You can use the tips and suggestions on page T28 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1.5 days
45 min	3 days

Focus

Domain: Algebra

Standards for Mathematical Content:

A.SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students related geometric sequences to exponential functions.
F.BF.2, F.LE.1a

Now
Students generate geometric series and find their sums.
A.SSE.4

Next
Students will choose the best function type to model sets of data by using technology.
A.CED.2

Rigor

The Three Pillars of Rigor

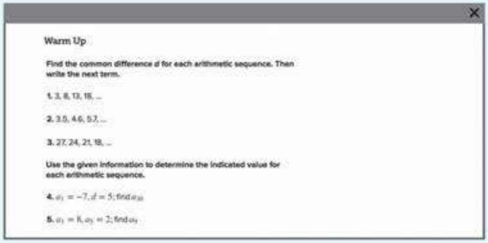
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand on their understanding of geometric sequences and build fluency by deriving the formula for the sum of a geometric series. They apply their understanding by solving real-world problems related to geometric sequences and series.

Mathematical Background

The formula for the sum of a geometric series, $s_n = \frac{a_1 - a_1 r^n}{1 - r}$, is used when the number of terms is not given. If *n* is known, the formula $s_n = \frac{a_1(1 - r^n)}{1 - r}$ can be used, as it is not necessary to calculate a_n.

Interactive Presentation



Warm Up

Find the common difference d for each arithmetic sequence. Then write the next term.

1. 3, 8, 13, 18, ...

2. 3.5, 4.6, 5.7, ...

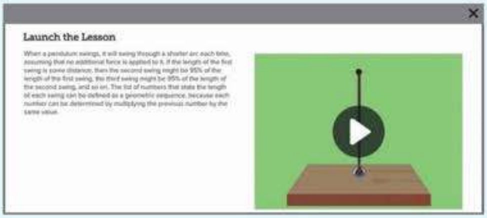
3. 27, 24, 21, 18, ...

Use the given information to determine the indicated value for each arithmetic sequence.

4. $a_1 = -7, d = 5$; find a_{30}

5. $a_1 = 8, a_3 = 2$; find a_5

Warm Up



Launch the Lesson

When a pendulum swings, it will swing through a shorter arc each time, assuming that no additional force is applied to it. If the length of the first swing is some distance, then the second swing might be 95% of the length of the first swing, the third swing might be 95% of the length of the second swing, and so on. The list of numbers that state the length of each swing can be defined as a geometric sequence, because each number can be determined by multiplying the previous number by the same value.

Launch the Lesson

finite sequence

A sequence that contains a limited number of terms.

infinite sequence

A sequence that continues without end.

geometric sequence

A pattern of numbers that begins with a nonzero term and each term after is found by multiplying the previous term by a nonzero constant r.

common ratio

The ratio of consecutive terms of a geometric sequence.

geometric series

The indicated sum of the terms in a geometric sequence.

sigma notation

A notation that uses the Greek uppercase letter Σ to indicate that a sum should be found.

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- using arithmetic sequences

Answers:

1. 5; 23

2. 1.1; 6.8

3. −3; 15

4. 138

5. −4

Launch the Lesson

Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in the length of each swing. Encourage them to explain how to generate the length of the each swing given the length of the first swing.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Explicit and Recursive Formulas

Objective

Students collect data to explore how to represent geometric sequences using explicit and recursive formulas.

Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in the number of pieces of paper after each cut. Students will generalize the pattern they find to write explicit and recursive formulas to represent the situation.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

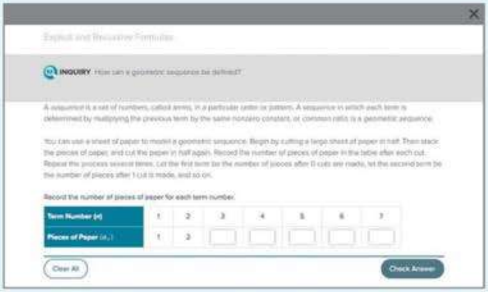
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will explore a geometric pattern by stacking and cutting pieces of paper, and create an exponential function to model the situation. Students will then complete a series of exercises to write the exponential function as a recursive function. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

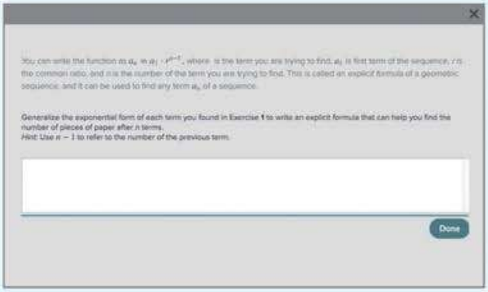


How can a geometric sequence be defined?

Record the number of pieces of paper for each term number.

Term Number (n)	1	2	3	4	5	6	7
Pieces of Paper (a_n)	1	2					

Explore



Generalize the exponential form of each term you found in Exercise **1** to write an explicit formula that can help you find the number of pieces of paper after n terms.
Hint: Use $n - 1$ to refer to the number of the previous term.

Explore

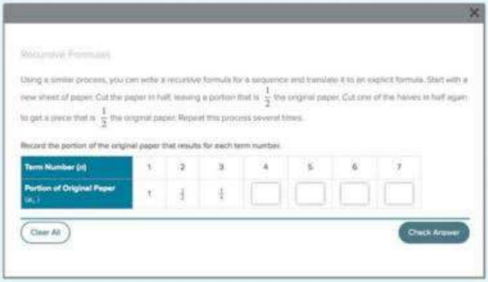
TYPE

Students complete the calculations to generate explicit and recursive formulas.

TYPE

Students answer questions to show they understand the patterns in each type of formula.

Interactive Presentation



Recursive Formulas

Using a similar process, you can write a recursive formula for a sequence and translate it to an explicit formula. Start with a new sheet of paper. Cut the paper in half, leaving a portion that is $\frac{1}{2}$ the original paper. Cut one of the halves in half again to get a piece that is $\frac{1}{4}$ the original paper. Repeat this process several times.

Record the portion of the original paper that results for each term number.

Term Number (n)	1	2	3	4	5	6	7
Portion of Original Paper (a_n)	1	$\frac{1}{2}$	$\frac{1}{4}$	☐	☐	☐	☐

Explore

INQUIRY How can a geometric sequence be defined?

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Explicit and Recursive Formulas (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Why is it useful to have an explicit formula for a geometric sequence? Sample answer: With an explicit formula, you can find any term in a sequence without having to find the previous term a_{n-1}.
- If you are given the first three terms in a sequence, and asked to find the next 4 terms, should you use the recursive formula or the explicit formula? Why? Sample answer: The recursive formula would be better if you have a starting term and just need to find successive terms.

Inquiry

How can a geometric sequence be defined? A geometric sequence can be defined in terms of the number of the term n using an explicit formula to find any term of a sequence or in terms of the previous term.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Sequences

Objective

Students generate geometric sequences by using the common ratio.

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the terms in a sequence and the formulas that can be used to represent the sequence. Help students to see how the recursive and explicit formulas for a sequence are related and when to use each type of formula.

About the Key Concept

Emphasize the importance of writing every step of calculations as an equation so that each numeric value found during the process is clearly identified.

Common Misconception

Tell students that although the graph of a geometric sequence is exponential, its graph is discrete instead of continuous because the domain of a geometric sequence is the set of natural numbers.

Essential Question Follow-Up

Students have explored sequences.

Ask:

What type of patterns can be modeled mathematically? Sample answer: numerical patterns involving real number operations, such as addition and multiplication

Example 1 Identify Geometric Sequences

Teaching the Mathematical Practices

6 Use Definitions In this example, students will use the definition of a geometric sequence to classify sequences.

Questions for Mathematical Discourse

AL What values can a common ratio be? any nonzero real number

OL How many terms do you need to divide to determine that the sequence in the example is not geometric? just the first two pairs

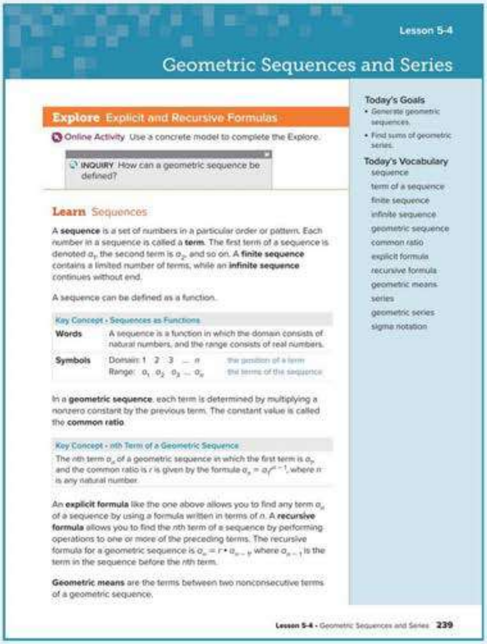
BL What will be true about the signs of a geometric sequence if the common ratio is negative? They will alternate between positive and negative.

Things to Remember

Remind students that if they find the ratio of a term to the previous term, they should set up the remaining ratios the same way.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 5-4

Geometric Sequences and Series

Explore Explicit and Recursive Formulas

Online Activity Use a concrete model to complete the Explore.

INQUIRY How can a geometric sequence be defined?

Learn Sequences

A **sequence** is a set of numbers in a particular order or pattern. Each number in a sequence is called a **term**. The first term of a sequence is denoted a_1, the second term is a_2, and so on. A **finite sequence** contains a limited number of terms, while an **infinite sequence** continues without end.

A sequence can be defined as a function.

Key Concept • Sequences as Functions

Words	A sequence is a function in which the domain consists of natural numbers, and the range consists of real numbers.	
Symbols	Domain: 1 2 3 … n Range: a_1 a_2 a_3 … a_n	the position of a term the terms of the sequence

In a **geometric sequence**, each term is determined by multiplying a nonzero constant by the previous term. The constant value is called the **common ratio**.

Key Concept • *n*th Term of a Geometric Sequence

The *n*th term a_n of a geometric sequence in which the first term is a_1, and the common ratio is r is given by the formula $a_n = a_1r^{n-1}$, where n is any natural number.

An **explicit formula** like the one above allows you to find any term a_n of a sequence by using a formula written in terms of n. A **recursive formula** allows you to find the *n*th term of a sequence by performing operations to one or more of the preceding terms. The recursive formula for a geometric sequence is $a_n = r \cdot a_{n-1}$, where a_{n-1} is the term in the sequence before the *n*th term.

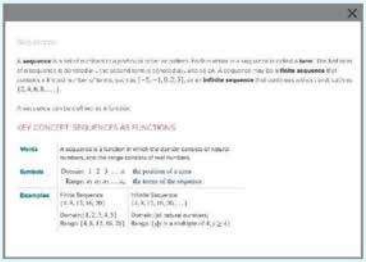
Geometric means are the terms between two nonconsecutive terms of a geometric sequence.

Today's Goals
- Generate geometric sequences.
- Find sums of geometric series.

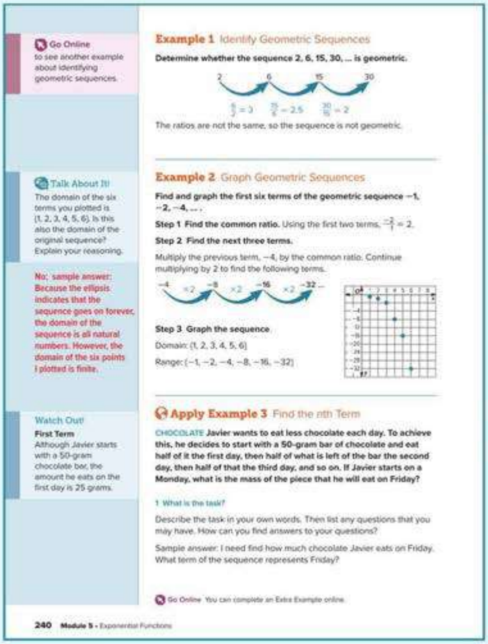
Today's Vocabulary
- sequence
- term of a sequence
- finite sequence
- infinite sequence
- geometric sequence
- common ratio
- explicit formula
- recursive formula
- geometric means
- series
- geometric series
- sigma notation

Lesson 5-4 • Geometric Sequences and Series **239**

Interactive Presentation



Learn



Go Online to see another example about identifying geometric sequences.

Example 1 Identify Geometric Sequences

Determine whether the sequence 2, 6, 15, 30, ... is geometric.

$\frac{6}{2} = 3$ $\frac{15}{6} = 2.5$ $\frac{30}{15} = 2$

The ratios are not the same, so the sequence is not geometric.

Talk About It! The domain of the six terms you plotted is {1, 2, 3, 4, 5, 6}. Is this also the domain of the original sequence? Explain your reasoning.

No; sample answer: Because the ellipsis indicates that the sequence goes on forever, the domain of the sequence is all natural numbers. However, the domain of the six points I plotted is finite.

Example 2 Graph Geometric Sequences

Find and graph the first six terms of the geometric sequence −1, −2, −4, ...

Step 1 Find the common ratio. Using the first two terms, $\frac{-2}{-1} = 2$.

Step 2 Find the next three terms.

Multiply the previous term, −4, by the common ratio. Continue multiplying by 2 to find the following terms.

Step 3 Graph the sequence.

Domain: {1, 2, 3, 4, 5, 6}

Range: {−1, −2, −4, −8, −16, −32}

Watch Out!

First Term

Although Javier starts with a 50-gram chocolate bar, the amount he eats on the first day is 25 grams.

Apply Example 3 Find the nth Term

CHOCOLATE **Javier wants to eat less chocolate each day. To achieve this, he decides to start with a 50-gram bar of chocolate and eat half of it the first day, then half of what is left of the bar the second day, then half of that the third day, and so on. If Javier starts on a Monday, what is the mass of the piece that he will eat on Friday?**

1 What is the task?

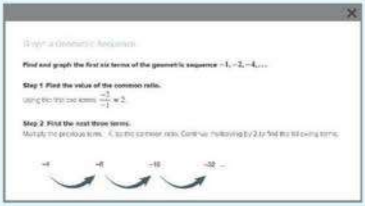
Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need find how much chocolate Javier eats on Friday. What term of the sequence represents Friday?

Go Online You can complete an Extra Example online.

240 Module 5 - Exponential Functions

Interactive Presentation



Find and graph the first six terms of the geometric sequence −1, −2, −4, ...

Step 1 Find the value of the common ratio.

Step 2 Find the next three terms.

Example 2

TYPE

Students answer a question to show they understand the graph of a geometric sequence.

Example 2 Graph Geometric Sequences

Teaching the Mathematical Practices

8 Look for a Pattern Students identify the pattern in the geometric sequence and use it to find and graph terms in the sequence.

Questions for Mathematical Discourse

AL Is the graph of a geometric sequence linear? no; unless $r = 1$

OL Is a geometric sequence a function? yes

BL If you connected the points with a smooth curve, what would a geometric sequence with $r > 1$ resemble? an exponential growth function

Apply Example 3 Find the nth Term

Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them, 4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- How does the amount of chocolate Javier eats on a day compare to how much he ate the day before? How can this be represented with a sequence?
- Could you draw a picture to represent the situation? How could that help you write a formula?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.

Example 4 Write an Equation for the *n*th Term

Teaching the Mathematical Practices

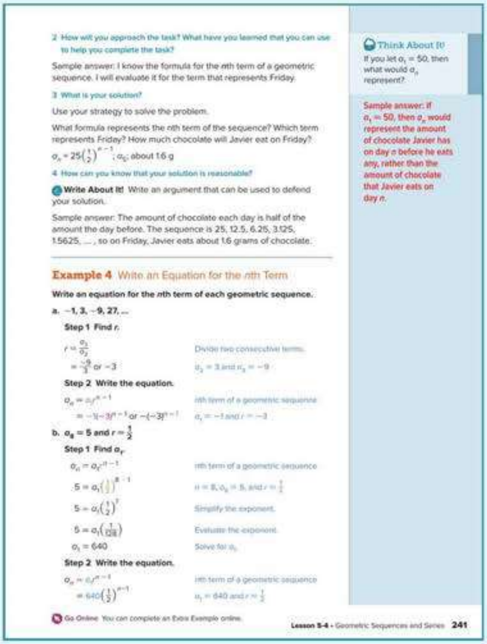
2 Create Representations Guide students to write equations that represent the sequences in Example 4.

Questions for Mathematical Discourse

AL In **part a**, could you use other terms to find *r*? Y es, you could as long as the terms are consecutive.

OL In **part b**, why can't you simplify a_n to 320^{n-1}? The exponent affects the $\frac{1}{2}$ in parentheses, but it does not apply to 640.

BL In **part a**, what is the effect of the negative inside the parentheses? The exponent can be even or odd so the negative in the parentheses will make the terms alternate signs.



2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I know the formula for the *n*th term of a geometric sequence. I will evaluate it for the term that represents Friday.

3 What is your solution?

Use your strategy to solve the problem.

What formula represents the *n*th term of the sequence? Which term represents Friday? How much chocolate will Javier eat on Friday?

$a_n = 25\left(\frac{1}{2}\right)^{n-1}$; a_5; about 1.6 g

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: The amount of chocolate each day is half of the amount the day before. The sequence is 25, 12.5, 6.25, 3.125, 1.5625, ..., so on Friday, Javier eats about 1.6 grams of chocolate.

Think About It!

If you let $a_1 = 50$, then what would a_n represent?

Sample answer: If $a_1 = 50$, then a_n would represent the amount of chocolate Javier has on day *n* before he eats any, rather than the amount of chocolate that Javier eats on day *n*.

Example 4 Write an Equation for the *n*th Term

Write an equation for the *n*th term of each geometric sequence.

a. −1, 3, −9, 27, ...

Step 1 Find *r*.

$r = \frac{a_3}{a_2}$	Divide two consecutive terms.
$= \frac{-9}{3}$ or -3	$a_2 = 3$ and $a_3 = -9$

Step 2 Write the equation.

$a_n = a_1r^{n-1}$	*n*th term of a geometric sequence
$= -1(-3)^{n-1}$ or $-(-3)^{n-1}$	$a_1 = -1$ and $r = -3$

b. $a_8 = 5$ and $r = \frac{1}{2}$

Step 1 Find a_1.

$a_n = a_1r^{n-1}$	*n*th term of a geometric sequence
$5 = a_1\left(\frac{1}{2}\right)^{8-1}$	$n = 8$, $a_8 = 5$, and $r = \frac{1}{2}$
$5 = a_1\left(\frac{1}{2}\right)^7$	Simplify the exponent.
$5 = a_1\left(\frac{1}{128}\right)$	Evaluate the exponent.
$a_1 = 640$	Solve for a_1.

Step 2 Write the equation.

$a_n = a_1r^{n-1}$	*n*th term of a geometric sequence
$= 640\left(\frac{1}{2}\right)^{n-1}$	$a_1 = 640$ and $r = \frac{1}{2}$

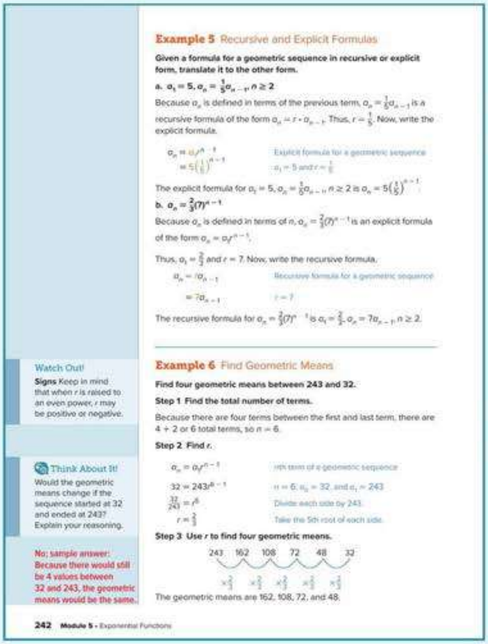
Go Online You can complete an Extra Example online.

Lesson 5-4 • Geometric Sequences and Series 241

Interactive Presentation

Write an Equation for the *n*th Term

Example 4



Example 5 Recursive and Explicit Formulas

Given a formula for a geometric sequence in recursive or explicit form, translate it to the other form.

a. $a_1 = 5, a_n = \frac{1}{5}a_{n-1}, n \geq 2$

Because a_n is defined in terms of the previous term, $a_n = \frac{1}{5}a_{n-1}$ is a recursive formula of the form $a_n = r \cdot a_{n-1}$. Thus, $r = \frac{1}{5}$. Now, write the explicit formula.

$a_n = a_1r^{n-1}$ Explicit formula for a geometric sequence

$= 5\left(\frac{1}{5}\right)^{n-1}$ $a_1 = 5$ and $r = \frac{1}{5}$

The explicit formula for $a_1 = 5, a_n = \frac{1}{5}a_{n-1}, n \geq 2$ is $a_n = 5\left(\frac{1}{5}\right)^{n-1}$.

b. $a_n = \frac{2}{3}(7)^{n-1}$

Because a_n is defined in terms of n, $a_n = \frac{2}{3}(7)^{n-1}$ is an explicit formula of the form $a_n = a_1r^{n-1}$.

Thus, $a_1 = \frac{2}{3}$ and $r = 7$. Now, write the recursive formula.

$a_n = ra_{n-1}$ Recursive formula for a geometric sequence

$= 7a_{n-1}$ $r = 7$

The recursive formula for $a_n = \frac{2}{3}(7)^{n-1}$ is $a_1 = \frac{2}{3}, a_n = 7a_{n-1}, n \geq 2$.

Watch Out!

Signs Keep in mind that when r is raised to an even power, r may be positive or negative.

Example 6 Find Geometric Means

Find four geometric means between 243 and 32.

Step 1 Find the total number of terms.

Because there are four terms between the first and last term, there are 4 + 2 or 6 total terms, so $n = 6$.

Step 2 Find r.

$a_n = a_1r^{n-1}$ nth term of a geometric sequence

$32 = 243r^{6-1}$ $n = 6$, $a_6 = 32$, and $a_1 = 243$

$\frac{32}{243} = r^5$ Divide each side by 243.

$r = \frac{2}{3}$ Take the 5th root of each side.

Step 3 Use r to find four geometric means.

243 162 108 72 48 32

$\times\frac{2}{3}$ $\times\frac{2}{3}$ $\times\frac{2}{3}$ $\times\frac{2}{3}$ $\times\frac{2}{3}$

The geometric means are 162, 108, 72, and 48.

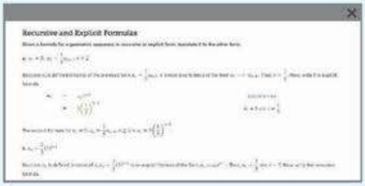
Think About It!

Would the geometric means change if the sequence started at 32 and ended at 243? Explain your reasoning.

No; sample answer: Because there would still be 4 values between 32 and 243, the geometric means would be the same.

242 **Module 5** • Exponential Functions

Interactive Presentation



Recursive and Explicit Formulas

Example 5

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 5 Recursive and Explicit Formulas

Teaching the Mathematical Practices

7 Use Structure Help students use the structure of geometric sequences in Example 5 to translate between recursive and explicit forms.

Questions for Mathematical Discourse

- **AL** What does recursive form mean? Each term is determined by the previous term.
- **OL** How does explicit form differ from recursive form? Explicit form allows you to find any term in the series given the first term and the common ratio, while recursive form allows you to find any term based on the previous term(s).
- **BL** When would each form be useful when working with geometric series? Sample answer: Recursive form is useful for generating multiple terms of the series over a range, while explicit form is useful for finding specific terms without needing to find the terms in between.

Example 6 Find Geometric Means

Teaching the Mathematical Practices

3 Justify Conclusions The Think About It! feature asks students to justify their conclusion about geometric means.

Questions for Mathematical Discourse

- **AL** Why does $a_6 = 32$? The series has six terms: 243, the four geometric means, and 32; thus, 32 is the sixth term.
- **OL** Would it be possible to find 4 geometric means between 243 and −32? If so, find r. yes; $r = -\frac{2}{3}$
- **BL** What would happen if you tried to find more than 4 geometric means between 243 and 32? r would be irrational.

Learn Geometric Series

Objective

Students find sums of geometric series.

Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. The Think About It! feature requires students to see the denominator of the formula for the sum of a geometric series as a single object.

Things to Remember

Encourage students to begin a geometric sequence problem by writing the known values for each of the variables, n, a, and r.

Example 7 Find the Sum of a Geometric Series

Teaching the Mathematical Practices

4 Interpret Mathematical Results Use the Study Tip to reiterate the importance of interpreting results in the context of the situation. Have students explain why S_n cannot be negative.

Questions for Mathematical Discourse

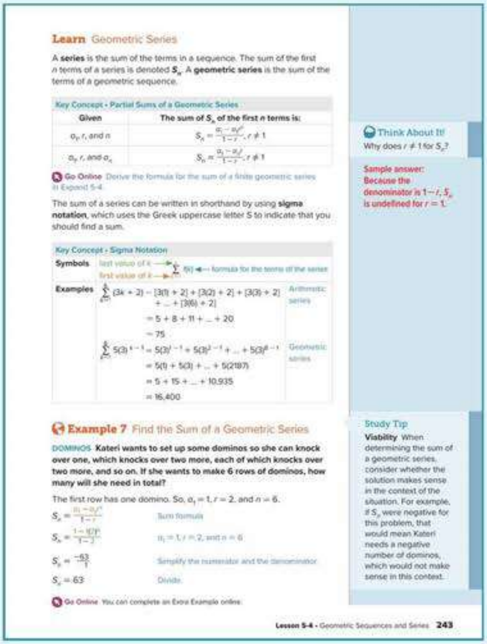
- AL Why is $n = 6$? We are solving for the total number of dominos in 6 rows.
- OL How would you determine how many rows can be made with d dominoes? Set $S_n = d$ and solve for n.
- BL What happens to the sums if r is negative? They alternate between negative and positive as n increments.

DIFFERENTIATE

Language Development Activity ELL

Beginning Help students access text by using an interactive whiteboard to work through finding the sum of a geometric series. Point out each variable in the formula and each step of the solution, using short phrases to explain.

Intermediate Provide students with a study guide to make the text accessible to all students. Paraphrasing content helps students make connections more easily.



Learn Geometric Series

A **series** is the sum of the terms in a sequence. The sum of the first n terms of a series is denoted S_n. A **geometric series** is the sum of the terms of a geometric sequence.

Key Concept • Partial Sums of a Geometric Series

Given	The sum of S_n of the first n terms is:
a_1, r, and n	$S_n = \frac{a_1 - a_1r^n}{1-r}, r \neq 1$
a_1, r, and a_n	$S_n = \frac{a_1 - a_nr}{1-r}, r \neq 1$

Go Online Derive the formula for the sum of a finite geometric series in Expand 5-4.

The sum of a series can be written in shorthand by using **sigma notation**, which uses the Greek uppercase letter S to indicate that you should find a sum.

Key Concept • Sigma Notation

Symbols last value of k → $\sum_{k=1}^{n} f(k)$ ← formula for the terms of the series; first value of k →

Examples

$$\sum_{k=1}^{6}(3k+2) = [3(1)+2]+[3(2)+2]+[3(3)+2]+\dots+[3(6)+2] = 5+8+11+\dots+20 = 75$$

Arithmetic series

$$\sum_{k=1}^{8}5(3)^{k-1} = 5(3)^{1-1}+5(3)^{2-1}+\dots+5(3)^{8-1} = 5(1)+5(3)+\dots+5(2187) = 5+15+\dots+10{,}935 = 16{,}400$$

Geometric series

Example 7 Find the Sum of a Geometric Series

DOMINOS **Kateri wants to set up some dominos so she can knock over one, which knocks over two more, each of which knocks over two more, and so on. If she wants to make 6 rows of dominos, how many will she need in total?**

The first row has one domino. So, $a_1 = 1$, $r = 2$, and $n = 6$.

$S_n = \frac{a_1 - a_1r^n}{1-r}$ Sum formula

$S_6 = \frac{1-1(2)^6}{1-2}$ $a_1 = 1$, $r = 2$, and $n = 6$

$S_6 = \frac{-63}{-1}$ Simplify the numerator and the denominator.

$S_6 = 63$ Divide.

Go Online You can complete an Extra Example online.

Think About It!
Why does $r \neq 1$ for S_n?

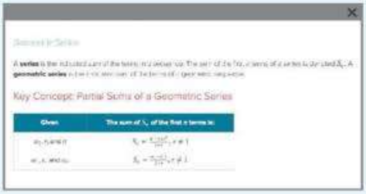
Sample answer: Because the denominator is $1 - r$, S_n is undefined for $r = 1$.

Study Tip

Viability When determining the sum of a geometric series, consider whether the solution makes sense in the context of the situation. For example, if S_n were negative for this problem, that would mean Kateri needs a negative number of dominos, which would not make sense in this context.

Lesson 5-4 • Geometric Sequences and Series 243

Interactive Presentation

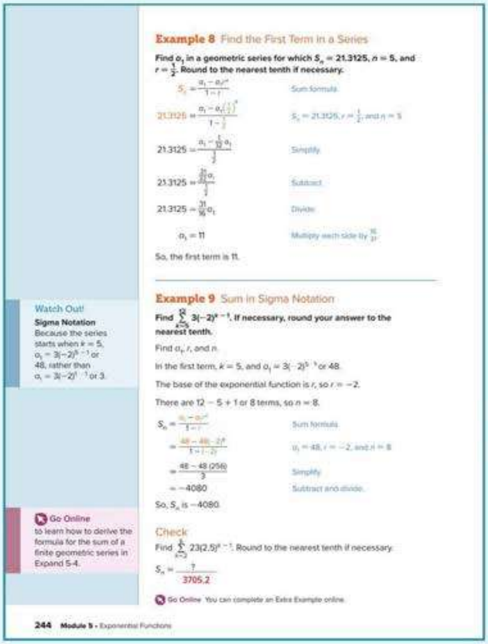


Learn

TYPE

a|

Students answer a question to determine whether they understand the formula for the sum of a geometric series.



Example 8 Find the First Term in a Series

Find a_1 in a geometric series for which $S_n = 21.3125$, $n = 5$, and $r = \frac{1}{2}$. Round to the nearest tenth if necessary.

$S_n = \frac{a_1 - a_1r^n}{1 - r}$ Sum formula

$21.3125 = \frac{a_1 - a_1\left(\frac{1}{2}\right)^5}{1 - \frac{1}{2}}$ $S_n = 21.3125$, $r = \frac{1}{2}$, and $n = 5$

$21.3125 = \frac{a_1 - \frac{1}{32}a_1}{\frac{1}{2}}$ Simplify.

$21.3125 = \frac{\frac{31}{32}a_1}{\frac{1}{2}}$ Subtract.

$21.3125 = \frac{31}{16}a_1$ Divide.

$a_1 = 11$ Multiply each side by $\frac{16}{31}$.

So, the first term is 11.

Example 9 Sum in Sigma Notation

Find $\sum_{k=5}^{12} 3(-2)^{k-1}$. If necessary, round your answer to the nearest tenth.

Find a_1, r, and n.

In the first term, $k = 5$, and $a_1 = 3(-2)^{5-1}$ or 48.

The base of the exponential function is r, so $r = -2$.

There are $12 - 5 + 1$ or 8 terms, so $n = 8$.

$S_n = \frac{a_1 - a_1r^n}{1 - r}$ Sum formula

$= \frac{48 - 48(-2)^8}{1 - (-2)}$ $a_1 = 48$, $r = -2$, and $n = 8$

$= \frac{48 - 48(256)}{3}$ Simplify.

$= -4080$ Subtract and divide.

So, S_n is -4080.

Watch Out!
Sigma Notation Because the series starts when $k = 5$, $a_1 = 3(-2)^{5-1}$ or 48, rather than $a_1 = 3(-2)^{1-1}$ or 3.

Go Online to learn how to derive the formula for the sum of a finite geometric series in Expand 5-4.

Check

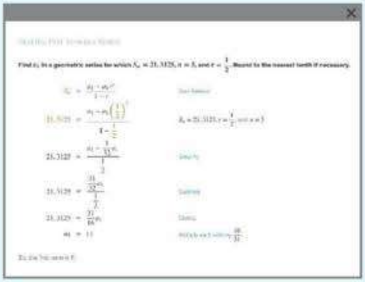
Find $\sum_{k=3}^{8} 23(2.5)^{k-1}$. Round to the nearest tenth if necessary.

$S_n = \underline{\quad ? \quad}$ 3705.2

Go Online You can complete an Extra Example online.

244 Module 5 • Exponential Functions

Interactive Presentation



Example 8

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 8 Find the First Term in a Series

Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. In Example 8, students must solve the sum formula for a given variable.

Questions for Mathematical Discourse

AL What is a step you could have shown in between simplify and subtract? Distributive Property: $a_1 - \frac{1}{32}a_1 = a_1\left(1 - \frac{1}{32}\right)$

OL Which values would we need to use this formula to solve for r? S_n, a_1, and n

BL How could you write a_1 as a function of r with the given S_n and n? Factor a_1 out and then multiply each side by $\frac{1-r}{1-r^n}$.

Example 9 Sum in Sigma Notation

Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. In Example 9, guide students to see what information they can gather about the geometric series just from looking at the sum in sigma notation.

Questions for Mathematical Discourse

AL How can you check your answer? Write out each term for k from 5 to 12 and add them.

OL For k from 1 to ∞, will any term in the series be irrational? no

BL How would you write a sum in sigma notation that would have the same sum but start at $k = 1$? The exponent would need to equal 4 for $k = 1$, so it would be $k + 3$, and the sum would go from 1 to 8.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–37
2	exercises that use a variety of skills from this lesson	38–65
3	exercises that emphasize higher-order and critical-thinking skills	66–73

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice Exercises 1–65 odd, 66–73
- Extension: Half the Distance
- **ALEKS** Sequences and Series

OL

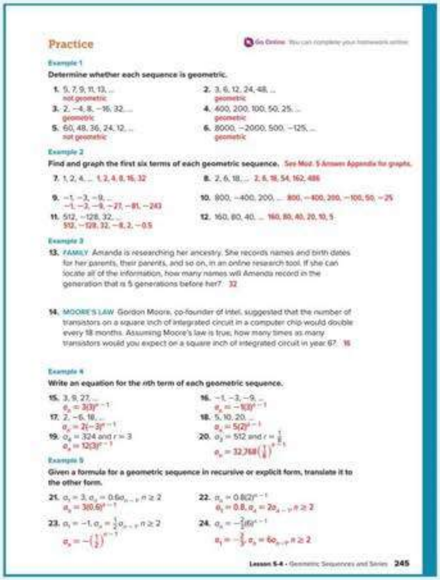
IF students score 66%–89% on the Checks,
THEN assign:

- Practice Exercises 1–73 odd
- Remediation, Review Resources: Arithmetic Sequences
- Personal Tutors
- Extra Examples 1–9
- **ALEKS** Arithmetic Sequences

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice Exercises 1–37 odd
- Remediation, Review Resources: Arithmetic Sequences
- *Quick Review Math Handbook*: Geometric Sequences and Series
- **ALEKS** Arithmetic Sequences



Practice

Go Online You can complete your homework online.

Example 1

Determine whether each sequence is geometric.

1. 5, 7, 9, 11, 13, … not geometric
2. 3, 6, 12, 24, 48, … geometric
3. 2, −4, 8, −16, 32, … geometric
4. 400, 200, 100, 50, 25, … geometric
5. 60, 48, 36, 24, 12, … not geometric
6. 8000, −2000, 500, −125, … geometric

Example 2

Find and graph the first six terms of each geometric sequence. See Mod. 5 Answer Appendix for graphs.

7. 1, 2, 4, … 1, 2, 4, 8, 16, 32
8. 2, 6, 18, … 2, 6, 18, 54, 162, 486
9. −1, −3, −9, … −1, −3, −9, −27, −81, −243
10. 800, −400, 200, … 800, −400, 200, −100, 50, −25
11. 512, −128, 32, … 512, −128, 32, −8, 2, −0.5
12. 160, 80, 40, … 160, 80, 40, 20, 10, 5

Example 3

13. FAMILY Amanda is researching her ancestry. She records names and birth dates for her parents, their parents, and so on, in an online research tool. If she can locate all of the information, how many names will Amanda record in the generation that is 5 generations before her? 32

14. MOORE'S LAW Gordon Moore, co-founder of Intel, suggested that the number of transistors on a square inch of integrated circuit in a computer chip would double every 18 months. Assuming Moore's law is true, how many times as many transistors would you expect on a square inch of integrated circuit in year 6? 16

Example 4

Write an equation for the *n*th term of each geometric sequence.

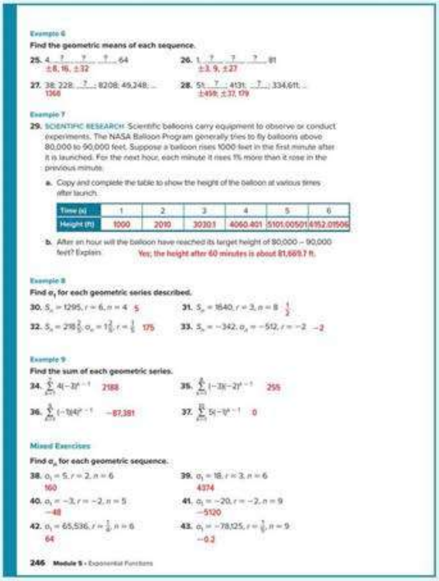
15. 3, 9, 27, … $a_n = 3(3)^{n-1}$
16. −1, −3, −9, … $a_n = -1(3)^{n-1}$
17. 2, −6, 18, … $a_n = 2(-3)^{n-1}$
18. 5, 10, 20, … $a_n = 5(2)^{n-1}$
19. $a_4 = 324$ and $r = 3$ $a_n = 12(3)^{n-1}$
20. $a_3 = 512$ and $r = \frac{1}{8}$ $a_n = 32{,}768\left(\frac{1}{8}\right)^{n-1}$

Example 5

Given a formula for a geometric sequence in recursive or explicit form, translate it to the other form.

21. $a_1 = 3, a_n = 0.6a_{n-1}, n \geq 2$ $a_n = 3(0.6)^{n-1}$
22. $a_n = 0.8(2)^{n-1}$ $a_1 = 0.8, a_n = 2a_{n-1}, n \geq 2$
23. $a_1 = -1, a_n = \frac{1}{2}a_{n-1}, n \geq 2$ $a_n = -\left(\frac{1}{2}\right)^{n-1}$
24. $a_n = -\frac{2}{3}(6)^{n-1}$ $a_1 = -\frac{2}{3}, a_n = 6a_{n-1}, n \geq 2$

Lesson 5-4 • Geometric Sequences and Series 245



Example 6

Find the geometric means of each sequence.

25. 4, __?__, __?__, __?__, 64 ±8, 16, ±32
26. 1, __?__, __?__, __?__, 81 ±3, 9, ±27
27. 38; 228; __?__; 8208; 49,248; … 1368
28. 51; __?__; 4131; __?__; 334,611; … ±459; ±37,179

Example 7

29. SCIENTIFIC RESEARCH Scientific balloons carry equipment to observe or conduct experiments. The NASA Balloon Program generally tries to fly balloons above 80,000 to 90,000 feet. Suppose a balloon rises 1000 feet in the first minute after it is launched. For the next hour, each minute it rises 1% more than it rose in the previous minute.

a. Copy and complete the table to show the height of the balloon at various times after launch.

Time (s)	1	2	3	4	5	6
Height (ft)	1000	2010	3030.1	4060.401	5101.00501	6152.01506

b. After an hour will the balloon have reached its target height of 80,000 – 90,000 feet? Explain. Yes; the height after 60 minutes is about 81,669.7 ft.

Example 8

Find a_1 for each geometric series described.

30. $S_n = 1295, r = 6, n = 4$ 5
31. $S_n = 1640, r = 3, n = 8$ $\frac{1}{2}$
32. $S_n = 218\frac{2}{5}, a_n = 1\frac{2}{5}, r = \frac{1}{5}$ 175
33. $S_n = -342, a_n = -512, r = -2$ −2

Example 9

Find the sum of each geometric series.

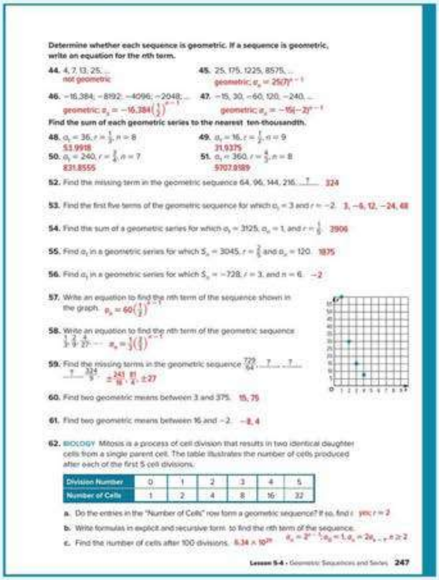
34. $\sum_{k=1}^{7} 4(-3)^{k-1}$ 2188
35. $\sum_{k=1}^{8} (-3)(-2)^{k-1}$ 255
36. $\sum_{k=1}^{9} (-1)(4)^{k-1}$ −87,381
37. $\sum_{k=1}^{10} 5(-1)^{k-1}$ 0

Mixed Exercises

Find a_n for each geometric sequence.

38. $a_1 = 5, r = 2, n = 6$ 160
39. $a_1 = 18, r = 3, n = 6$ 4374
40. $a_1 = -3, r = -2, n = 5$ −48
41. $a_1 = -20, r = -2, n = 9$ −5120
42. $a_1 = 65{,}536, r = \frac{1}{4}, n = 6$ 64
43. $a_1 = -78{,}125, r = \frac{1}{5}, n = 9$ −0.2

246 Module 5 • Exponential Functions



Determine whether each sequence is geometric. If a sequence is geometric, write an equation for the nth term.

44. 4, 7, 13, 25, … not geometric

45. 25, 175, 1225, 8575, … geometric; $a_n = 25(7)^{n-1}$

46. −16,384; −8192; −4096; −2048; … geometric; $a_n = -16{,}384\left(\frac{1}{2}\right)^{n-1}$

47. −15, 30, −60, 120, −240, … geometric; $a_n = -15(-2)^{n-1}$

Find the sum of each geometric series to the nearest ten-thousandth.

48. $a_1 = 36, r = \frac{1}{3}, n = 8$ 53.9918

49. $a_1 = 16, r = \frac{1}{2}, n = 9$ 31.9375

50. $a_1 = 240, r = \frac{3}{4}, n = 7$ 831.8555

51. $a_1 = 360, r = \frac{4}{3}, n = 8$ 5707.8189

52. Find the missing term in the geometric sequence 64, 96, 144, 216, __?__. 324

53. Find the first five terms of the geometric sequence for which $a_1 = 3$ and $r = -2$. 3, −6, 12, −24, 48

54. Find the sum of a geometric series for which $a_1 = 3125$, $a_n = 1$, and $r = \frac{1}{5}$. 3906

55. Find a_1 in a geometric series for which $S_n = 3045$, $r = \frac{2}{5}$ and $a_n = 120$. 1875

56. Find a_1 in a geometric series for which $S_n = -728$, $r = 3$, and $n = 6$. −2

57. Write an equation to find the nth term of the sequence shown in the graph. $a_n = 60\left(\frac{1}{2}\right)^{n-1}$

58. Write an equation to find the nth term of the geometric sequence $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \ldots$ $a_n = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$

59. Find the missing terms in the geometric sequence $\frac{729}{64}$, __?__, __?__, __?__, $\frac{324}{9}$. $\pm\frac{243}{16}, \frac{81}{4}, \pm 27$

60. Find two geometric means between 3 and 375. 15, 75

61. Find two geometric means between 16 and −2. −8, 4

62. BIOLOGY Mitosis is a process of cell division that results in two identical daughter cells from a single parent cell. The table illustrates the number of cells produced after each of the first 5 cell divisions.

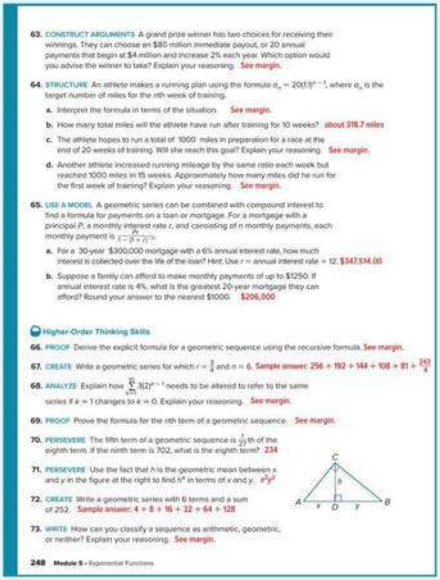
Division Number	0	1	2	3	4	5
Number of Cells	1	2	4	8	16	32

a. Do the entries in the "Number of Cells" row form a geometric sequence? If so, find r. yes; r = 2

b. Write formulas in explicit and recursive form to find the nth term of the sequence. $a_n = 2^{n-1}$; $a_0 = 1, a_n = 2a_{n-1}, n \geq 2$

c. Find the number of cells after 100 divisions. 6.34×10^{29}

Lesson 5-4 • Geometric Sequences and Series 247



63. CONSTRUCT ARGUMENTS A grand prize winner has two choices for receiving their winnings. They can choose an $80 million immediate payout, or 20 annual payments that begin at $4 million and increase 2% each year. Which option would you advise the winner to take? Explain your reasoning. See margin.

64. STRUCTURE An athlete makes a running plan using the formula $a_n = 20(1.1)^{n-1}$, where a_n is the target number of miles for the nth week of training.

a. Interpret the formula in terms of the situation. See margin.

b. How many total miles will the athlete have run after training for 10 weeks? about 318.7 miles

c. The athlete hopes to run a total of 1000 miles in preparation for a race at the end of 20 weeks of training. Will she reach this goal? Explain your reasoning. See margin.

d. Another athlete increased running mileage by the same ratio each week but reached 1000 miles in 15 weeks. Approximately how many miles did he run for the first week of training? Explain your reasoning. See margin.

65. USE A MODEL A geometric series can be combined with compound interest to find a formula for payments on a loan or mortgage. For a mortgage with a principal P, a monthly interest rate r, and consisting of n monthly payments, each monthly payment is $\frac{Pr}{1-(1+r)^{-n}}$.

a. For a 30-year $300,000 mortgage with a 6% annual interest rate, how much interest is collected over the life of the loan? Hint: Use r = annual interest rate ÷ 12. $347,514.00

b. Suppose a family can afford to make monthly payments of up to $1250. If annual interest rate is 4%, what is the greatest 20-year mortgage they can afford? Round your answer to the nearest $1000. $206,000

Higher-Order Thinking Skills

66. PROOF Derive the explicit formula for a geometric sequence using the recursive formula. See margin.

67. CREATE Write a geometric series for which $r = \frac{3}{4}$ and $n = 6$. Sample answer: $256 + 192 + 144 + 108 + 81 + \frac{243}{4}$

68. ANALYZE Explain how $\sum_{k=1}^{10} 3(2)^{k-1}$ needs to be altered to refer to the same series if k = 1 changes to k = 0. Explain your reasoning. See margin.

69. PROOF Prove the formula for the nth term of a geometric sequence. See margin.

70. PERSEVERE The fifth term of a geometric sequence is $\frac{1}{27}$th of the eighth term. If the ninth term is 702, what is the eighth term? 234

71. PERSEVERE Use the fact that h is the geometric mean between x and y in the figure at the right to find h^4 in terms of x and y. x^2y^2

72. CREATE Write a geometric series with 6 terms and a sum of 252. Sample answer: 4 + 8 + 16 + 32 + 64 + 128

73. WRITE How can you classify a sequence as arithmetic, geometric, or neither? Explain your reasoning. See margin.

248 Module 5 • Exponential Functions

Answers

63. Sample answer: The second option; receiving 20 annual payments represents a geometric series, and $S = \frac{4{,}000{,}000[1 - (1.02)^{20}]}{[1 - (1.02)]}$, or $97,189,479.20. By choosing the annual payments, the winner will receive approximately $17 million more than by choosing the single payout.

64a. 20 indicates the athlete will run 20 miles in the first week, 1.1 indicates that the number of miles increases by 0.1 or 10% each week.

64c. Yes; the total number of miles over 20 weeks is $20 + 20(1.1) + \ldots + 20(1.1)^{19} = 20(1 + 1.1 + \ldots + 1.1^{19}) = 20\left(\frac{1.1^{20} - 1}{1.1 - 1}\right) \approx 1145.5$, so the athlete will run over 1000 miles.

64d. about 31.5 miles; Sample answer: Let x = miles ran during the first week.

$$1000 \leq \sum_{n=1}^{15} x(1.1)^{n-1}$$

$$1000 \leq x + x(1.1) + x(1.1)^2 + \ldots + x(1.1)^{14}$$

$$1000 \leq 31.77x$$

$$31.47 \leq x$$

66. $S_n = \frac{a_1 - a_1 r^n}{1 - r} = \frac{a_1 - a_1 r^{n-1} r}{1 - r} = \frac{a_1 - a_n r}{1 - r}$

68. Sample answer: $k - 1$ needs to change to k, and the 10 needs to change to a 9. When this happens, the terms for both series will be identical (a_1 in the first series will equal a_0 in the second series, and so on), and the series will be equal to each other.

69. Sample answer:

Let a_n = the nth term of the sequence and r = the common ratio.

$a_2 = a_1 \cdot r$	Definition of the second term of a geometric sequence
$a_3 = a_2 \cdot r$	Definition of the third term of a geometric sequence
$a_3 = a_1 \cdot r \cdot r$	Substitution
$a_3 = a_1 \cdot r^2$	Associative Property of Multiplication
$a_3 = a_1 \cdot r^{3-1}$	$3 - 1 = 2$
$a_n = a_1 \cdot r^{n-1}$	$n = 3$

73. Sample answer: A series is arithmetic if every pair of consecutive terms shares a common difference. A series is geometric if every pair of consecutive terms shares a common ratio. If the series displays both qualities, then it is both arithmetic and geometric. If the series displays neither quality, then it is neither geometric nor arithmetic.

Lesson 5-5

Modeling Data

A.CED.2

LESSON GOAL

Students choose the best function type to model sets of data by using technology.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Modeling Exponential Decay

Develop:

Choosing the Best Model
- Examine Scatter Plots
- Model Data by Using Technology

You may want your students to complete the **Check** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Linear Regression	●	●		●
Extension: Analyzing Residual Plots		●	●	●

Language Development Handbook

Assign page 29 of the *Language Development Handbook* to help your students build mathematical language related to modeling sets of data by using technology.

ELL You can use the tips and suggestions on page T29 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra

Standards for Mathematical Content:

A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Standards for Mathematical Practice:

4 Model with mathematics.

5 Use appropriate tools strategically.

Coherence

Vertical Alignment

Previous

Students used linear and nonlinear functions to model real-world data and relationships. **F.BF.1 (Course 1, Course 2)**

Now

Students choose the best function type to model sets of data by using technology.

A.CED.2

Next

Students will study logarithmic functions and how they can model real-world relationships. **F.BF.1, F.BF.5**

Rigor

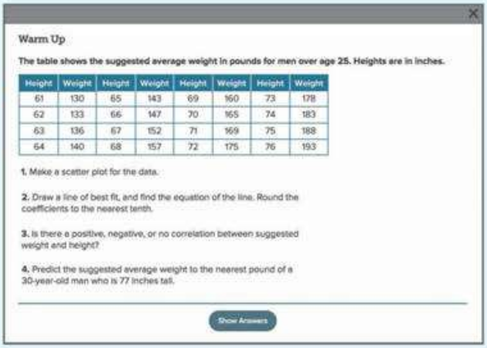
The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION
Conceptual Bridge In this lesson, students develop their understanding of nonlinear associations. They build fluency in fitting functions to data, and they apply their understanding by solving real-world problems.		

Mathematical Background

Functions can be used to model data. Look at the shape formed by the data to determine what type of function best models the data.

Interactive Presentation



Warm Up

The table shows the suggested average weight in pounds for men over age 25. Heights are in inches.

Height	Weight	Height	Weight	Height	Weight	Height	Weight
61	130	65	143	69	160	73	178
62	133	66	147	70	165	74	183
63	136	67	152	71	169	75	188
64	140	68	157	72	175	76	193

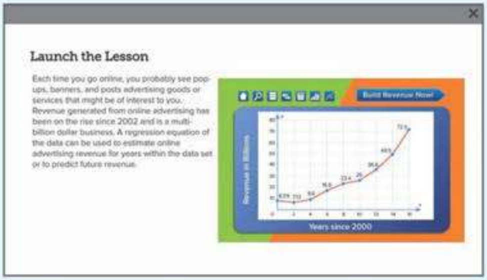
1. Make a scatter plot for the data.

2. Draw a line of best fit, and find the equation of the line. Round the coefficients to the nearest tenth.

3. Is there a positive, negative, or no correlation between suggested weight and height?

4. Predict the suggested average weight to the nearest pound of a 30-year-old man who is 77 inches tall.

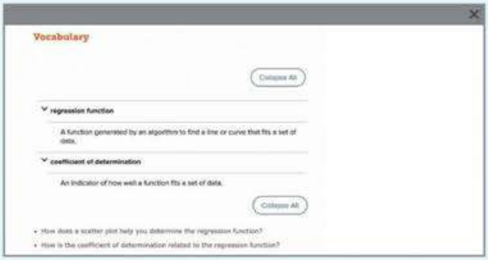
Warm Up



Launch the Lesson

Each time you go online, you probably see pop-ups, banners, and posts advertising goods or services that might be of interest to you. Revenue generated from online advertising has been on the rise since 2002 and is a multi-billion dollar business. A regression equation of the data can be used to estimate online advertising revenue for years within the data set or to predict future revenue.

Launch the Lesson



Vocabulary

regression function

A function generated by an algorithm to find a line or curve that fits a set of data.

coefficient of determination

An indicator of how well a function fits a set of data.

- How does a scatter plot help you determine the regression function?
- How is the coefficient of determination related to the regression function?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- finding and using a linear regression

Answers:

1. See students' graphs.

2. Sample answer: $y = 4.2x + 123$

3. positive

4. about 194 lb

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students observe a situation in which a regression equation can be used to predict data. Ask students to use the graph shown to estimate the revenue for a year within the data set. Once students have completed the lesson, students can use the regression equation to estimate the revenue for the same year and compare their estimates.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Modeling Exponential Decay

Objective

Students use a model to explore exponential decay.

Teaching the Mathematical Practices

2 Make Sense of Quantities Mathematically proficient students need to be able to make sense of quantities and their relationships. Throughout the Explore, students must explain the relationship between the number of trials and the number of coins remaining.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

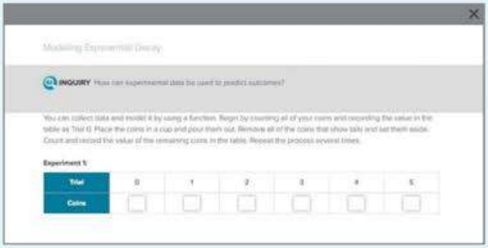
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will perform an experiment by recording the number of coins after each trial. They will use the results to answer guiding exercises to explore how experimental data can be modeled and used to make predictions. Each group of students will need 50–100 coins. Candy coated chocolates can be substituted for coins, where students use the imprinted logo on the candy to determine whether it should be set aside.

(*continued on the next page*)

Interactive Presentation



Modeling Exponential Decay

INQUIRY How can experimental data be used to predict outcomes?

You can collect data and model it by using a function. Begin by counting all of your coins and recording the value in the table as Trial 0. Place the coins in a cup and pour them out. Remove all of the coins that show tails and set them aside. Count and record the value of the remaining coins in the table. Repeat the process several times.

Experiment 1

Trial	0	1	2	3	4	5
Coins	☐	☐	☐	☐	☐	☐

Explore

TYPE

Students record the results of an experiment to model exponential decay.

Interactive Presentation

INQUIRY How can experimental data be used to predict outcomes?

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Modeling Exponential Decay (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What is the probability that a coin will land on heads? On tails? Sample answer: The coin has an even chance of landing on heads or tails, so the probability is 50%.
- When gathering experimental data, why should you perform more than 1 or 2 trials? Sample answer: Using only 2 trials wouldn't give you very accurate data. The more trials you use, the closer you get to theoretical probabilities.

Inquiry

How can experimental data be used to predict outcomes? By first making an equation that models the situation, the equation can be evaluated to make predictions.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Choosing the Best Model

Objective

Students choose the best function type to model sets of data by using technology.

Teaching the Mathematical Practices

4 Interpret Mathematical Results The Think About It! feature asks students to reflect on whether the predictions made using a regression function will always make sense in the context of a real-world situation.

Things to Remember

Remind students to clear lists **L1** and **L2** before starting any activity. They should also enter appropriate settings for the graphing window.

Example 1 Examine Scatter Plots

Teaching the Mathematical Practices

5 Analyze Graphs Help students analyze the graph they have generated using a graphing calculator. Point out that to see the entire graph, students may need to adjust the viewing window.

Questions for Mathematical Discourse

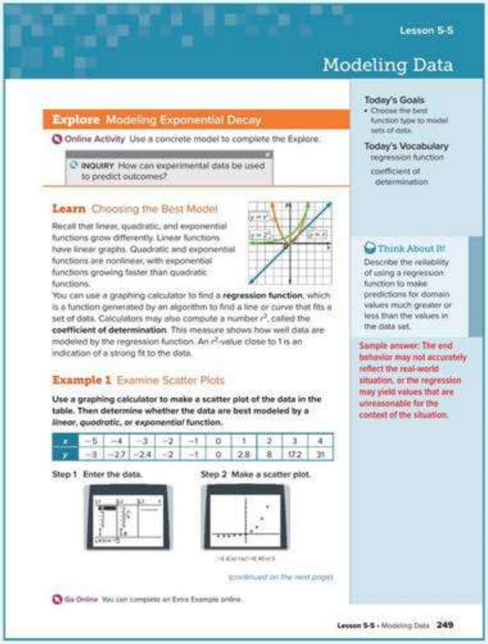
AL Based on the values in the table, why is a linear function likely not a good fit for the data? The y-values are increasing faster than the x-values.

OL If you only had the data over the interval $[-2, 0]$, what conclusion might you make about the function that best models the data? It would appear to be best modeled by a linear function.

BL What do you know about k, the vertical translation of the parent function, for the exponential function that models this data set? $k < -3$ because the asymptote is very close to but less than $k = -3$.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 5-5

Modeling Data

Explore Modeling Exponential Decay

Online Activity Use a concrete model to complete the Explore.

INQUIRY How can experimental data be used to predict outcomes?

Learn Choosing the Best Model

Recall that linear, quadratic, and exponential functions grow differently. Linear functions have linear graphs. Quadratic and exponential functions are nonlinear, with exponential functions growing faster than quadratic functions.

You can use a graphing calculator to find a **regression function**, which is a function generated by an algorithm to find a line or curve that fits a set of data. Calculators may also compute a number r^2, called the **coefficient of determination**. This measure shows how well data are modeled by the regression function. An r^2-value close to 1 is an indication of a strong fit to the data.

Example 1 Examine Scatter Plots

Use a graphing calculator to make a scatter plot of the data in the table. Then determine whether the data are best modeled by a *linear, quadratic, or exponential* function.

x	−5	−4	−3	−2	−1	0	1	2	3	4
y	−3	−2.7	−2.4	−2	−1	0	2.8	8	17.2	31

Step 1 Enter the data.

Step 2 Make a scatter plot.

(continued on the next page)

Go Online You can complete an Extra Example online.

Today's Goals
- Choose the best function type to model sets of data.

Today's Vocabulary
regression function
coefficient of determination

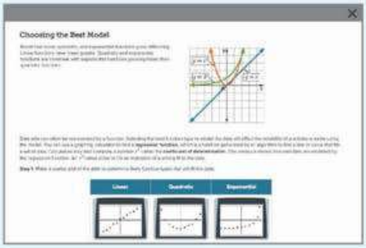
Think About It!

Describe the reliability of using a regression function to make predictions for domain values much greater or less than the values in the data set.

Sample answer: The end behavior may not accurately reflect the real-world situation, or the regression may yield values that are unreasonable for the context of the situation.

Lesson 5-5 • Modeling Data 249

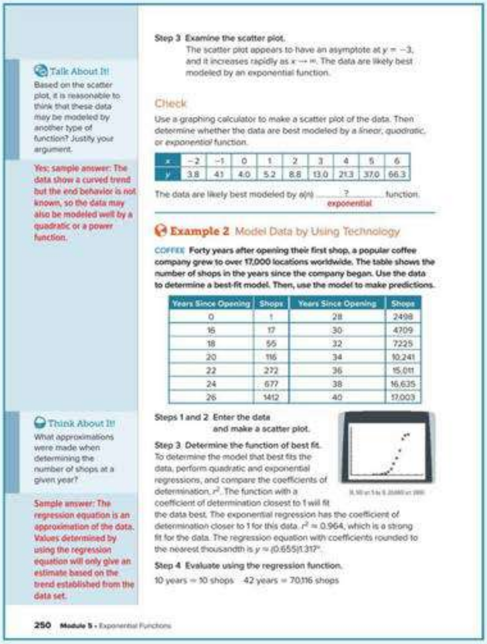
Interactive Presentation



Learn

TYPE

Students answer a question to show they understand the limitations of using regression functions.



Talk About It! Based on the scatter plot, it is reasonable to think that these data may be modeled by another type of function? Justify your argument.

Yes; sample answer: The data show a curved trend but the end behavior is not known, so the data may also be modeled well by a quadratic or a power function.

Step 3 Examine the scatter plot.
The scatter plot appears to have an asymptote at $y = -3$, and it increases rapidly as $x \to \infty$. The data are likely best modeled by an exponential function.

Check

Use a graphing calculator to make a scatter plot of the data. Then determine whether the data are best modeled by a *linear*, *quadratic*, or *exponential* function.

x	−2	−1	0	1	2	3	4	5	6
y	3.8	4.1	4.0	5.2	8.8	13.0	21.3	37.0	66.3

The data are likely best modeled by a(n) ___?___ function.
exponential

Example 2 Model Data by Using Technology

COFFEE **Forty years after opening their first shop, a popular coffee company grew to over 17,000 locations worldwide. The table shows the number of shops in the years since the company began. Use the data to determine a best-fit model. Then, use the model to make predictions.**

Years Since Opening	Shops	Years Since Opening	Shops
0	1	28	2498
16	17	30	4709
18	55	32	7225
20	116	34	10,241
22	272	36	15,011
24	677	38	16,635
26	1412	40	17,003

Steps 1 and 2 Enter the data and make a scatter plot.

Step 3 Determine the function of best fit.
To determine the model that best fits the data, perform quadratic and exponential regressions, and compare the coefficients of determination, r^2. The function with a coefficient of determination closest to 1 will fit the data best. The exponential regression has the coefficient of determination closer to 1 for this data. $r^2 \approx 0.964$, which is a strong fit for the data. The regression equation with coefficients rounded to the nearest thousandth is $y \approx (0.655)(1.317)^x$.

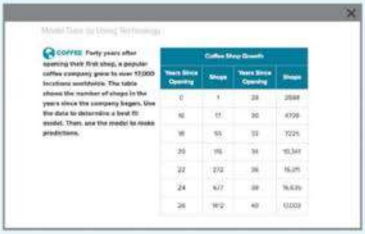
Step 4 Evaluate using the regression function.
10 years ≈ 10 shops 42 years ≈ 70,116 shops

Think About It! What approximations were made when determining the number of shops at a given year?

Sample answer: The regression equation is an approximation of the data. Values determined by using the regression equation will only give an estimate based on the trend established from the data set.

250 **Module 5** • Exponential Functions

Interactive Presentation



Example 2

TAP

Students select a calculator to use to determine the function that best models the data and use it to make predictions.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Model Data by Using Technology

Teaching the Mathematical Practices

4 Make Assumptions In the Think About It! feature, have students explain an approximation that was made to solve the problem.

5 Compare Predictions with Data Point out that in Example 2, students should use a graphing calculator to predict the number of coffee shops in a given year.

Questions for Mathematical Discourse

AL What characteristic of the data indicates an exponential function would be a good model? Sample answer: As x increases, y increases by larger intervals.

OL Would you expect the trend to continue indefinitely? Explain. No; sample answer: The market will eventually reach a saturation point for the coffee shop.

BL A quartic function has an r^2 of 0.988 for the data set. Does this mean it is a better model? Sample answer: Not necessarily. The quartic function may have multiple direction changes in between data points that would not make sense for the model. Graphing the data set and the modeling function can show you whether the model makes sense in the context.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–6
2	exercises that use a variety of skills from this lesson	7–13
3	exercises that emphasize higher-order and critical-thinking skills	14–17

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **BL**
THEN assign:

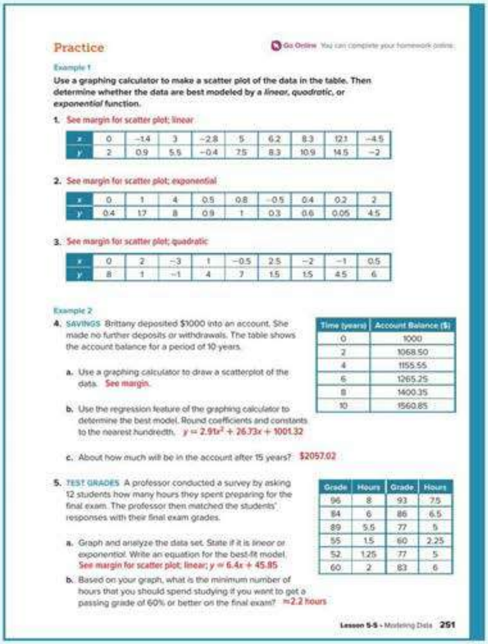
- Practice Exercises 1–13 odd, 14–17
- Extension: Analyzing Residual Plots

IF students score 66%–89% on the Checks, **OL**
THEN assign:

- Practice Exercises 1–17 odd
- Remediation, Review Resources: Linear Regression
- Personal Tutors
- Extra Examples 1, 2
- ALEKS® Scatter Plots and Lines of Best Fit

IF students score 65% or less on the Checks, **AL**
THEN assign:

- Practice Exercises 1–5 odd
- Remediation, Review Resources: Linear Regression
- ALEKS® Scatter Plots and Lines of Best Fit



Practice

Go Online You can complete your homework online.

Example 1

Use a graphing calculator to make a scatter plot of the data in the table. Then determine whether the data are best modeled by a *linear*, *quadratic*, or *exponential* function.

1. See margin for scatter plot; linear

x	0	−1.4	3	−2.8	5	6.2	8.3	12.1	−4.5
y	2	0.9	5.5	−0.4	7.5	8.3	10.9	14.5	−2

2. See margin for scatter plot; exponential

x	0	1	4	0.5	0.8	−0.5	0.4	0.2	2
y	0.4	1.7	8	0.9	1	0.3	0.6	0.05	4.5

3. See margin for scatter plot; quadratic

x	0	2	−3	1	−0.5	2.5	−2	−1	0.5
y	8	1	−1	4	7	1.5	1.5	4.5	6

Example 2

4. SAVINGS Brittany deposited $1000 into an account. She made no further deposits or withdrawals. The table shows the account balance for a period of 10 years.

Time (years)	Account Balance ($)
0	1000
2	1068.50
4	1155.55
6	1265.25
8	1400.35
10	1560.85

a. Use a graphing calculator to draw a scatterplot of the data. See margin.

b. Use the regression feature of the graphing calculator to determine the best model. Round coefficients and constants to the nearest hundredth. $y = 2.91x^2 + 26.73x + 1001.32$

c. About how much will be in the account after 15 years? $2057.02

5. TEST GRADES A professor conducted a survey by asking 12 students how many hours they spent preparing for the final exam. The professor then matched the students' responses with their final exam grades.

Grade	Hours	Grade	Hours
96	8	93	7.5
84	6	86	6.5
89	5.5	77	5
55	1.5	60	2.25
52	1.25	77	5
60	2	83	6

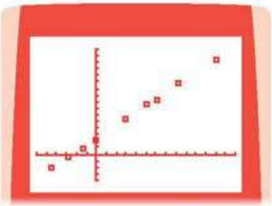
a. Graph and analyze the data set. State if it is linear or exponential. Write an equation for the best-fit model. See margin for scatter plot; linear; $y = 6.4x + 45.85$

b. Based on your graph, what is the minimum number of hours that you should spend studying if you want to get a passing grade of 60% or better on the final exam? ≈2.2 hours

Lesson 5-5 • Modeling Data 251

Answer

1.

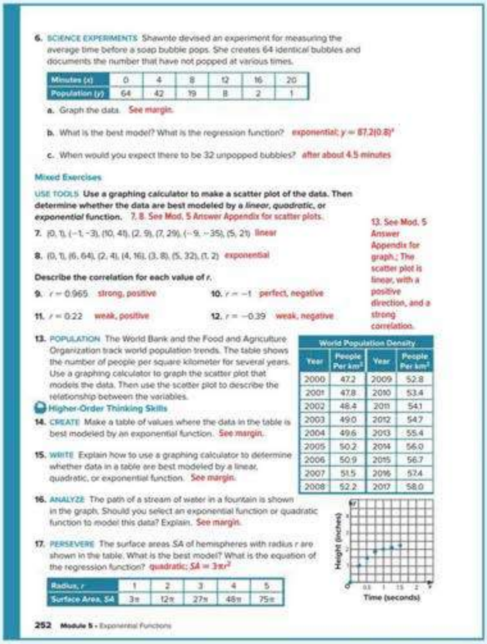


[—6, 14] scl: 1 by [—4, 16] scl: 1

3 REFLECT AND PRACTICE

A.CED.2

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**



6. SCIENCE EXPERIMENTS Shawnte devised an experiment for measuring the average time before a soap bubble pops. She creates 64 identical bubbles and documents the number that have not popped at various times.

Minutes (x)	0	4	8	12	16	20
Population (y)	64	42	19	8	2	1

a. Graph the data. See margin.

b. What is the best model? What is the regression function? exponential; $y = 87.2(0.8)^x$

c. When would you expect there to be 32 unpopped bubbles? after about 4.5 minutes

Mixed Exercises

USE TOOLS **Use a graphing calculator to make a scatter plot of the data. Then determine whether the data are best modeled by a *linear*, *quadratic*, or *exponential* function.** 7, 8. See Mod. 5 Answer Appendix for scatter plots.

7. (0, 1), (−1, −3), (10, 41), (2, 9), (7, 29), (−9, −35), (5, 21) linear

8. (0, 1), (6, 64), (2, 4), (4, 16), (3, 8), (5, 32), (1, 2) exponential

Describe the correlation for each value of *r*.

9. $r = 0.965$ strong, positive

10. $r = -1$ perfect, negative

11. $r = 0.22$ weak, positive

12. $r = -0.39$ weak, negative

13. POPULATION The World Bank and the Food and Agriculture Organization track world population trends. The table shows the number of people per square kilometer for several years. Use a graphing calculator to graph the scatter plot that models the data. Then use the scatter plot to describe the relationship between the variables.

13. See Mod. 5 Answer Appendix for graph.; The scatter plot is linear, with a positive direction, and a strong correlation.

World Population Density

Year	People Per km²	Year	People Per km²
2000	47.2	2009	52.8
2001	47.8	2010	53.4
2002	48.4	2011	54.1
2003	49.0	2012	54.7
2004	49.6	2013	55.4
2005	50.2	2014	56.0
2006	50.9	2015	56.7
2007	51.5	2016	57.4
2008	52.2	2017	58.0

Higher-Order Thinking Skills

14. CREATE Make a table of values where the data in the table is best modeled by an exponential function. See margin.

15. WRITE Explain how to use a graphing calculator to determine whether data in a table are best modeled by a linear, quadratic, or exponential function. See margin.

16. ANALYZE The path of a stream of water in a fountain is shown in the graph. Should you select an exponential function or quadratic function to model this data? Explain. See margin.

17. PERSEVERE The surface areas *SA* of hemispheres with radius *r* are shown in the table. What is the best model? What is the equation of the regression function? quadratic; $SA = 3\pi r^2$

Radius, *r*	1	2	3	4	5
Surface Area, *SA*	3π	12π	27π	48π	75π

252 Module 5 • Exponential Functions

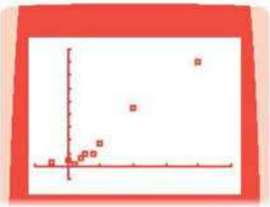
14. Sample answer:

x	−3	−2	−1	0	1	2	3
y	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

15. Enter the *x*-values in the table in L1 in a calculator. Enter the *y*-values in the table in L2 in a calculator. Perform linear, quadratic, and exponential regressions. Then compare the coefficients of determination, r^2. The function with a coefficient of determination closest to 1 will fit the data best.

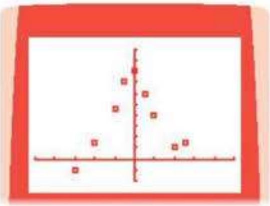
16. Sample answer: Since the data accounts for only the first 1.5 seconds, it appears as if an exponential function should be selected, but that height of the water in the fountain will start to decrease after reaching its maximum height. Therefore, a quadratic function should be selected to model the data.

2.



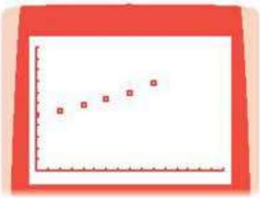
[—1, 5] scl: 1 by [—1, 9] scl: 1

3.



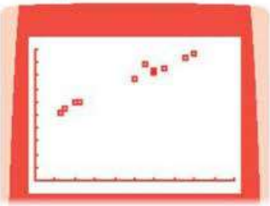
[—5, 5] scl: 1 by [—2, 10] scl: 1

4a.



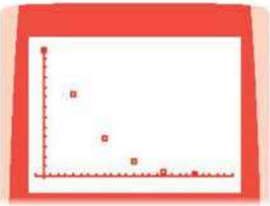
[0, 16] scl: 1 by [0, 2200] scl: 200

5a.



[0, 10] scl: 1 by [0, 100] scl: 10

6a.



[—1, 25] scl: 1 [—1, 65] scl: 5

Module 5 • Exponential Functions

Review

Rate Yourself!

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have them respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- How can being financially literate help you to make good decisions?
- What type of patterns can be modeled mathematically?

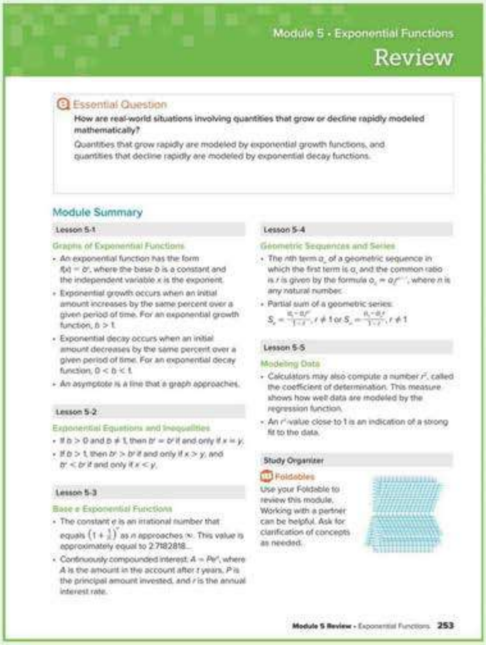
Then have them write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

ELL A completed Foldable for this module should include the key concepts related to exponential functions.

LS **LearnSmart** Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Polynomial, Rational, and Radical Relationships** and **Modeling with Functions**.

- Series Expressions
- Applications of Function Models
- Exponential and Logarithmic Functions



Module 5 • Exponential Functions

Review

Essential Question

How are real-world situations involving quantities that grow or decline rapidly modeled mathematically?

Quantities that grow rapidly are modeled by exponential growth functions, and quantities that decline rapidly are modeled by exponential decay functions.

Module Summary

Lesson 5-1

Graphs of Exponential Functions

- An exponential function has the form $f(x) = b^x$, where the base b is a constant and the independent variable x is the exponent.
- Exponential growth occurs when an initial amount increases by the same percent over a given period of time. For an exponential growth function, $b > 1$.
- Exponential decay occurs when an initial amount decreases by the same percent over a given period of time. For an exponential decay function, $0 < b < 1$.
- An asymptote is a line that a graph approaches.

Lesson 5-2

Exponential Equations and Inequalities

- If $b > 0$ and $b \neq 1$, then $b^x = b^y$ if and only if $x = y$.
- If $b > 1$, then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.

Lesson 5-3

Base e Exponential Functions

- The constant e is an irrational number that equals $\left(1 + \frac{1}{n}\right)^n$ as n approaches ∞. This value is approximately equal to 2.7182818...
- Continuously compounded interest: $A = Pe^{rt}$, where A is the amount in the account after t years, P is the principal amount invested, and r is the annual interest rate.

Lesson 5-4

Geometric Sequences and Series

- The nth term a_n of a geometric sequence in which the first term is a_1 and the common ratio is r is given by the formula $a_n = a_1r^{n-1}$, where n is any natural number.
- Partial sum of a geometric series: $S_n = \frac{a_1 - a_1r^n}{1 - r}, r \neq 1$ or $S_n = \frac{a_1 - a_nr}{1 - r}, r \neq 1$

Lesson 5-5

Modeling Data

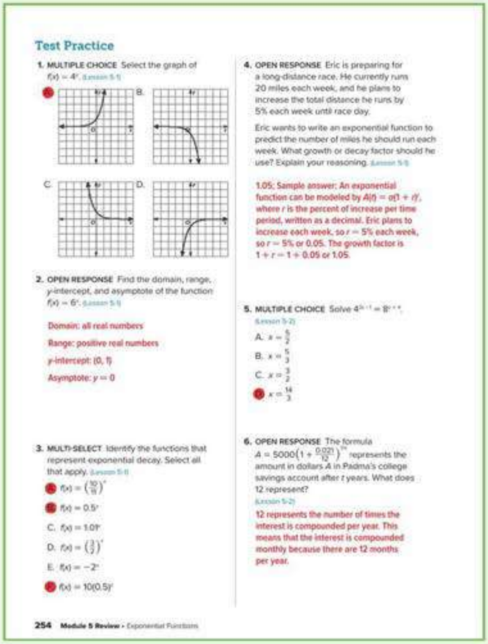
- Calculators may also compute a number r^2, called the coefficient of determination. This measure shows how well data are modeled by the regression function.
- An r^2-value close to 1 is an indication of a strong fit to the data.

Study Organizer

Foldables

Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.

Module 5 Review • Exponential Functions 253



Test Practice

1. **MULTIPLE CHOICE** Select the graph of $f(x) = 4^x$. (Lesson 5-1)

   ● B. C. D.

2. **OPEN RESPONSE** Find the domain, range, y-intercept, and asymptote of the function $f(x) = 6^x$. (Lesson 5-1)

   Domain: all real numbers

   Range: positive real numbers

   y-intercept: (0, 1)

   Asymptote: $y = 0$

3. **MULTI-SELECT** Identify the functions that represent exponential decay. Select all that apply. (Lesson 5-1)

   ● $f(x) = \left(\frac{10}{11}\right)^x$

   ● $f(x) = 0.5^x$

   C. $f(x) = 1.01^x$

   D. $f(x) = \left(\frac{3}{2}\right)^x$

   E. $f(x) = -2^x$

   ● $f(x) = 10(0.5)^x$

4. **OPEN RESPONSE** Eric is preparing for a long-distance race. He currently runs 20 miles each week, and he plans to increase the total distance he runs by 5% each week until race day.

   Eric wants to write an exponential function to predict the number of miles he should run each week. What growth or decay factor should he use? Explain your reasoning. (Lesson 5-1)

   1.05; Sample answer: An exponential function can be modeled by $A(t) = a(1 + r)^t$, where r is the percent of increase per time period, written as a decimal. Eric plans to increase each week, so $r = 5\%$ each week, so $r = 5\%$ or 0.05. The growth factor is $1 + r = 1 + 0.05$ or 1.05.

5. **MULTIPLE CHOICE** Solve 4[illegible] = 8[illegible]. (Lesson 5-2)

   A. $x = \frac{5}{2}$

   B. $x = \frac{5}{3}$

   C. $x = \frac{3}{2}$

   ● $x = \frac{14}{3}$

6. **OPEN RESPONSE** The formula $A = 5000\left(1 + \frac{0.021}{12}\right)^{12t}$ represents the amount in dollars A in Padma's college savings account after t years. What does 12 represent? (Lesson 5-2)

   12 represents the number of times the interest is compounded per year. This means that the interest is compounded monthly because there are 12 months per year.

254 **Module 5 Review** • Exponential Functions

Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Vocabulary Activity
Module Review

Assessment Resources

Vocabulary Test
AL Module Test Form B
OL Module Test Form A
BL Module Test Form C
Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

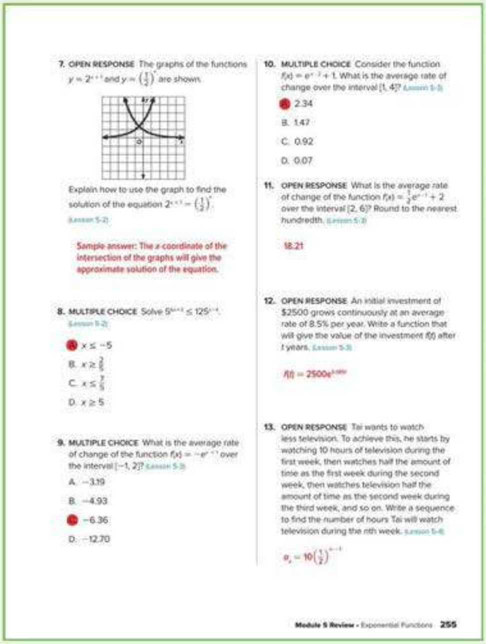
Test Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1–20 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Multiple Choice	Students select one correct answer.	1, 5, 8, 9, 10, 14, 15, 16, 17
Multi-Select	Multiple answers may be correct. Students must select all correct answers.	3
Open Response	Students construct their own response.	2, 4, 6, 7, 11, 12, 13, 18

To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
A.SSE.2	5-2	5, 8
A.SSE.4	5-4	14, 15
A.CED.2	5-1, 5-3, 5-5	4, 12, 16, 17, 18
A.REI.11	5-2	7
F.IF.4	5-1	2
F.IF.6	5-3	9, 10, 11
F.IF.7e	5-1	1
F.IF.8b	5-1	3
F.LE.5	5-2	6
F.BF.1a	5-4	13



7. OPEN RESPONSE The graphs of the functions $y = 2^{x+1}$ and $y = \left(\frac{1}{2}\right)^x$ are shown.

Explain how to use the graph to find the solution of the equation $2^{x+1} = \left(\frac{1}{2}\right)^x$. (Lesson 5-2)

Sample answer: The x-coordinate of the intersection of the graphs will give the approximate solution of the equation.

8. MULTIPLE CHOICE Solve $5^{[illegible]} \leq 125^{[illegible]}$. (Lesson 5-2)

A. $x \leq -5$

B. $x \geq \frac{2}{5}$

C. $x \leq \frac{7}{5}$

D. $x \geq 5$

9. MULTIPLE CHOICE What is the average rate of change of the function $f(x) = -e^{[illegible]}$ over the interval $[-1, 2]$? (Lesson 5-3)

A. −3.19

B. −4.93

C. −6.36

D. −12.70

10. MULTIPLE CHOICE Consider the function $f(x) = e^{[illegible]} + 1$. What is the average rate of change over the interval $[1, 4]$? (Lesson 5-3)

A. 2.34

B. 1.47

C. 0.92

D. 0.07

11. OPEN RESPONSE What is the average rate of change of the function $f(x) = \frac{1}{2}e^{[illegible]} + 2$ over the interval $[2, 6]$? Round to the nearest hundredth. (Lesson 5-3)

18.21

12. OPEN RESPONSE An initial investment of $2500 grows continuously at an average rate of 8.5% per year. Write a function that will give the value of the investment $f(t)$ after t years. (Lesson 5-3)

$f(t) = 2500e^{[illegible]}$

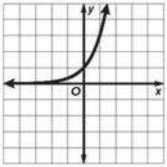
13. OPEN RESPONSE Tai wants to watch less television. To achieve this, he starts by watching 10 hours of television during the first week, then watches half the amount of time as the first week during the second week, then watches television half the amount of time as the second week during the third week, and so on. Write a sequence to find the number of hours Tai will watch television during the nth week. (Lesson 5-4)

$a_n = 10\left(\frac{1}{2}\right)^{n-1}$

Module 5 Review • Exponential Functions 255

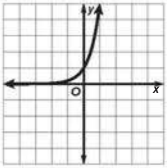
Lesson 5-1

1.



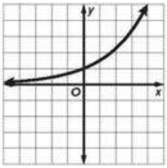
Domain: all real numbers; Range: all positive real numbers; *y*-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$

2.



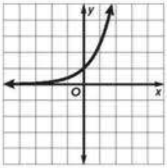
Domain: all real numbers; Range: all positive real numbers; *y*-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$

3.



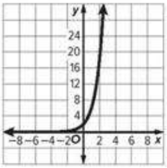
Domain: all real numbers; Range: all positive real numbers; *y*-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$

4.

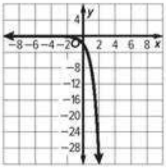


Domain: all real numbers; Range: all positive real numbers; *y*-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$

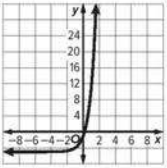
5.



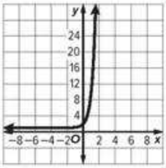
6.



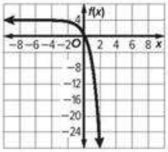
7.



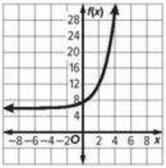
8.



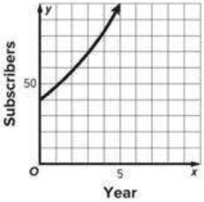
9.



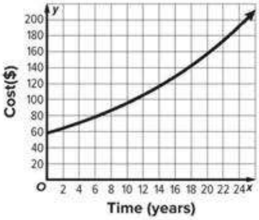
10.



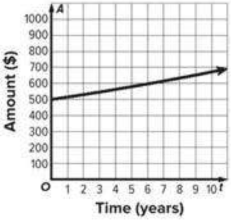
14.



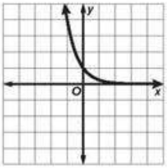
15.



16b.

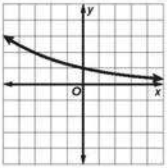


23.



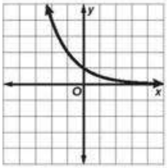
Domain: all real numbers; Range: all positive real numbers; *y*-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to 0$

24.



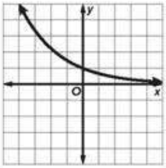
Domain: all real numbers; Range: all positive real numbers; *y*-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to 0$

25.



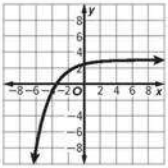
Domain: all real numbers; Range: all positive real numbers; *y*-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to 0$

26.

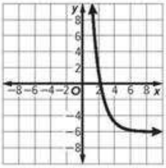


Domain: all real numbers; Range: all positive real numbers; *y*-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to 0$

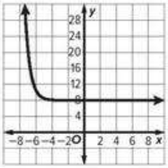
27.



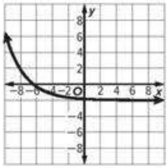
28.



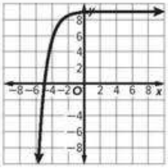
29.



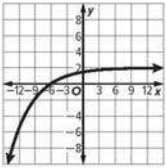
30.



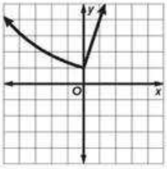
31.



32.

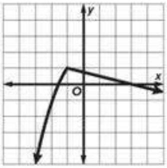


33a.



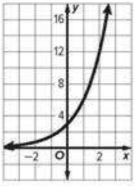
33b. Both $f(x)$ and $g(x)$ have a domain of all real numbers. $f(x)$ has a range of $y \geq 1$ and $g(x)$ has a range of $y > -1$, so the range of $f(x)$ is greater than the range of $g(x)$. $f(x)$ has no *x*-intercept and $g(x)$ has an *x*-intercept at (0, 0). $f(x)$ has a *y*-intercept at (0, 1) and $g(x)$ has a *y*-intercept at (0, 0), so the *y*-intercept of $f(x)$ is greater than the *y*-intercept of $g(x)$. $f(x)$ is increasing when $x > 0$ and $g(x)$ is increasing for all values of *x*. $f(x)$ is decreasing when $x < 0$ and $g(x)$ is never decreasing. $f(x)$ is positive for all values of *x* and $g(x)$ is positive when $x > 0$. $f(x)$ is never negative and $g(x)$ is negative when $x < 0$. $f(x)$ has a minimum at (0, 1) and $g(x)$ has a minimum slightly greater than –1, so the minimum of $f(x)$ is greater than the minimum of $g(x)$. Neither $f(x)$ nor $g(x)$ are symmetric. The end behavior of $f(x)$ is: As $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to \infty$. The end behavior of $g(x)$ is: As $x \to -\infty$, $g(x) \to -1$ and as $x \to \infty$, $g(x) \to \infty$.

34a.



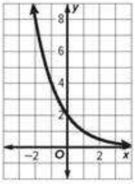
34b. Both $f(x)$ and $g(x)$ have a domain of all real numbers. $f(x)$ has a range of $y \leq 1$ and $g(x)$ has a range of $y < 1$, so the range of $f(x)$ is greater than the range of $g(x)$. $f(x)$ has *x*-intercepts at about (−1.5, 0) and (3, 0) and $g(x)$ has an *x*-intercept at (0, 0). So, the *x*-intercept of $g(x)$ is between the *x*-intercepts of $f(x)$. $f(x)$ has a *y*-intercept at about (0, 0.75) and $g(x)$ has a *y*-intercept at (0, 0), so the *y*-intercept of $f(x)$ is greater than the *y*-intercept of $g(x)$. $f(x)$ is increasing when $x < -1$ and $g(x)$ is never increasing. $f(x)$ is decreasing when $x > -1$ and $g(x)$ is decreasing for all values of *x*. $f(x)$ is positive when $-1.5 < x < 3$ and $g(x)$ is positive when $x < 0$. $f(x)$ is negative when $x < -1.5$ and when $x > 3$ and $g(x)$ is negative when $x > 0$. $f(x)$ has a maximum at (−1, 1) and $g(x)$ has a maximum slightly less than 1, so the maximum of $f(x)$ is greater than the maximum of $g(x)$. Neither $f(x)$ nor $g(x)$ are symmetric. The end behavior of $f(x)$ is: As $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$. The end behavior of $g(x)$ is: As $x \to -\infty$, $g(x) \to 1$ and as $x \to \infty$, $g(x) \to -\infty$.

35.



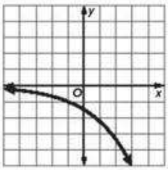
D = {all real numbers}, R = $\{y \mid y > 0\}$; exponential growth

36.



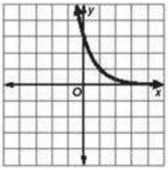
D = {all real numbers}, R = $\{y \mid y > 0\}$; exponential decay

37.



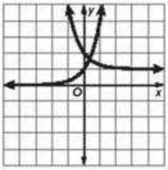
D = {all real numbers}, R = {$y \mid y < 0$}; exponential growth

38.



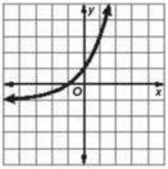
D = {all real numbers}, R = {$y \mid y > 0$}; exponential decay

45.



Reflect $f(x)$ in the y-axis and translate the result one unit up to obtain the graph of $g(x)$. The y-intercept of $g(x)$ is (0, 2), which was translated up 1 unit from the y-intercept of $f(x)$, (0, 1). $g(x)$ is decreasing on its entire domain, $f(x)$ is increasing on its entire domain. Both $g(x)$ and $f(x)$ are positive on its entire domain. The asymptote of $g(x)$is $y = 1$, which was translated up one unit from the asymptote of $f(x)$, $y = 0$. For $g(x)$ the end behavior is as $x \rightarrow -\infty$, $g(x) \rightarrow \infty$, and as $x \rightarrow \infty$, $g(x) \rightarrow 1$. The end behavior of $f(x)$ is as $x \rightarrow -\infty$, $f(x) \rightarrow 0$, and as $x \rightarrow \infty$, $f(x) \rightarrow \infty$.

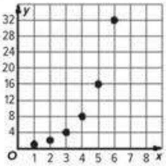
46.



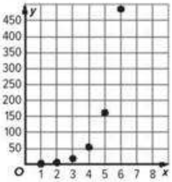
$g(x) = 2(0.5)^{-x} - 1$. The domain is the set of real numbers. Because $f(x)$ is translated 1 unit down, the range is the set of real numbers greater than −1. The y-intercept is (0, 1). The zero is −1.

Lesson 5-4

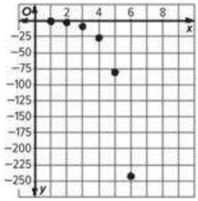
7.



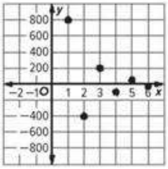
8.



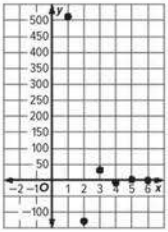
9.



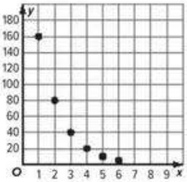
10.



11.

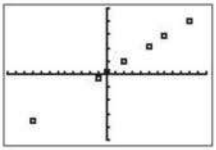


12.



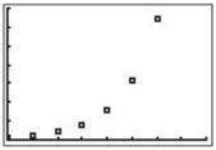
Lesson 5-5

7.



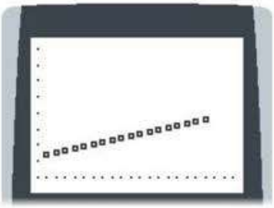
[−12, 12] scl: 1 by [−50, 50] scl: 10

8.

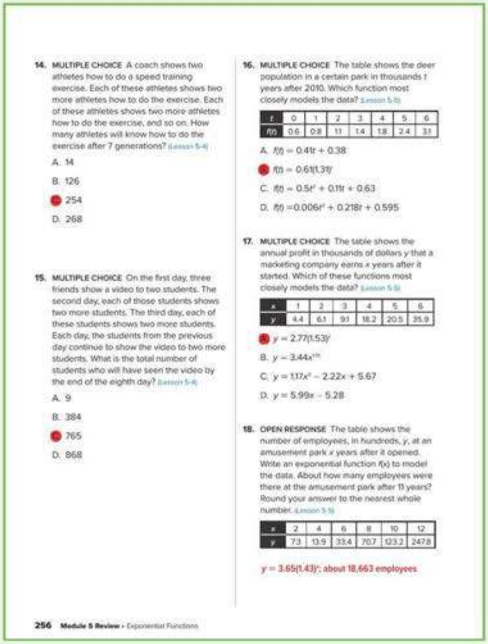


[0, 8] scl: 1 by [0, 70] scl: 10

13.



[1999, 2020] scl: 1 [40, 80] scl: 5



14. **MULTIPLE CHOICE** A coach shows two athletes how to do a speed training exercise. Each of these athletes shows two more athletes how to do the exercise. Each of these athletes shows two more athletes how to do the exercise, and so on. How many athletes will know how to do the exercise after 7 generations? (Lesson 5-4)

A. 14

B. 126

[✓ C.] 254

D. 268

15. **MULTIPLE CHOICE** On the first day, three friends show a video to two students. The second day, each of those students shows two more students. The third day, each of these students shows two more students. Each day, the students from the previous day continue to show the video to two more students. What is the total number of students who will have seen the video by the end of the eighth day? (Lesson 5-4)

A. 9

B. 384

[✓ C.] 765

D. 868

16. **MULTIPLE CHOICE** The table shows the deer population in a certain park in thousands t years after 2010. Which function most closely models the data? (Lesson 5-5)

t	0	1	2	3	4	5	6
$f(t)$	0.6	0.8	1.1	1.4	1.8	2.4	3.1

A. $f(t) = 0.41t + 0.38$

[✓ B.] $f(t) = 0.61(1.31)^t$

C. $f(t) = 0.5t^2 + 0.11t + 0.63$

D. $f(t) = 0.006t^2 + 0.218t + 0.595$

17. **MULTIPLE CHOICE** The table shows the annual profit in thousands of dollars y that a marketing company earns x years after it started. Which of these functions most closely models the data? (Lesson 5-5)

x	1	2	3	4	5	6
y	4.4	6.1	9.1	18.2	20.5	35.9

[✓ A.] $y = 2.77(1.53)^x$

B. $y = 3.44x^{1.11}$

C. $y = 1.17x^2 - 2.22x + 5.67$

D. $y = 5.99x - 5.28$

18. **OPEN RESPONSE** The table shows the number of employees, in hundreds, y, at an amusement park x years after it opened. Write an exponential function $f(x)$ to model the data. About how many employees were there at the amusement park after 11 years? Round your answer to the nearest whole number. (Lesson 5-5)

x	2	4	6	8	10	12
y	7.3	13.9	33.4	70.7	123.2	247.8

$y = 3.65(1.43)^x$; about 18,663 employees

256 **Module 5 Review** • Exponential Functions

Module 6

Logarithmic Functions

Module Goals

- Students write and evaluate logarithms and graph logarithmic functions.
- Students simplify logarithmic expressions and solve logarithmic equations.
- Students simplify logarithmic expressions and solve exponential equations by using common logarithms.
- Students simplify logarithmic expressions and solve exponential equations by using natural logarithms.
- Students write exponential growth and decay equations and solve them by using logarithms.

Focus

Domain: Algebra, Functions

Standards for Mathematical Content:

A.SSE.2 Use the structure of an expression to identify ways to rewrite it.

F.LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$, where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Also addresses A.CED.1, A.REI.11, and F.IF.7e

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this module.

Coherence

Vertical Alignment

Previous
Students graphed and analyzed exponential functions and solved exponential equations.
A.CED.2, F.IF.7e

Now
Students graph and analyze logarithmic functions. Students translate between exponential and logarithmic forms of expressions and solve exponential equations by using logarithms.
A.SSE.2, F.IF.7e, F.LE.4

Next
Students will graph and analyze rational functions.
F.IF.4, F.IF.5

Rigor

The Three Pillars of Rigor

Students will use the three pillars of rigor to help them meet standards. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. Once they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Practice.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

EXPLORE → LEARN → EXAMPLE & PRACTICE

Suggested Pacing

Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
6-1 Logarithms and Logarithmic Functions	A.SSE.2, F.IF.7e	2	1
6-2 Properties of Logarithms	A.CED.1	2	1
6-3 Common Logarithms	A.REI.11, F.LE.4	2	1
6-4 Natural Logarithms	A.SSE.2, F.LE.4	2	1
6-5 Using Exponential and Logarithmic Functions	A.CED.1, F.LE.4	1	0.5
Module Review		1	0.5
Module Assessment		1	0.5
	Total Days	12	6

CHERYL TOBEY MATH PROBES

Formative Assessment Math Probe

Properties of Logarithms

Module Resource

Correct Answers: 1. B, D; 2. B, D; 3. A, C; 4. E

Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will determine whether various expressions are equivalent to a given expression.

Targeted Concepts Understand that logarithms have properties that allow them to be rewritten in different forms.

Targeted Misconceptions

- Students may incorrectly treat log as a variable and:
  - factor it out of an expression: $\log a + \log b \neq \log (a + b)$,
  - distribute it into an expression: $\log (a + b) \neq \log a + \log b$, or
  - cancel the logs when dividing.
- Students may not understand how to manipulate the logarithm of a power and may square the log: $\log a^2 \neq (\log a)^2$.

Use the Probe after Lesson 6-2.

Collect and Assess Student Answers

If the student selects these responses...	**Then** the student likely...
1. A, C **2.** A, C **4.** C, D	incorrectly factored out the log in each expression.
4. A, B	incorrectly canceled the log in the numerator with the log in the denominator.
3. B	is confusing $\log 6^2$ with $(\log 6)^2$.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

- ALEKS® Properties of Logarithms
- Lesson 6-2, Learn, Examples 3–5

Revisit the Probe at the end of the module to be sure that your students no longer carry these misconceptions.

IGNITE!

The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

How are logarithms defined and used to model situations in the real world? Sample answer: A logarithm is the power to which a base must be raised to equal the given number. Logarithmic functions model slow-growing functions.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

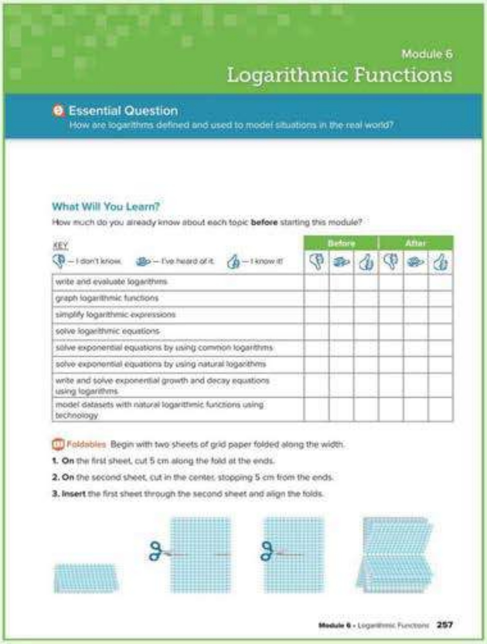
Focus Students write notes about new terms and concepts as they are presented in each lesson of this module.

Teach Have students construct their Foldable as illustrated. Have students write an explanation of each term or concept on the appropriate section of their Foldable while working through each lesson. Encourage students to record examples of each term or concept on the back of each flap.

When to Use It Encourage students to add to their Foldable as they work through the module and to use it to review for the module test.

Launch the Module

The Launch the Module video shows students that logarithmic functions are the inverses of exponential functions. Students learn that the Richter scale and mathematical representation of the musical scale are real-world applications of logarithmic functions. Students also learn that logarithms can help with calculations involving large numbers.



Module 6

Logarithmic Functions

Essential Question

How are logarithms defined and used to model situations in the real world?

What Will You Learn?

How much do you already know about each topic **before** starting this module?

KEY

👎 — I don't know. 👋 — I've heard of it. 👍 — I know it!

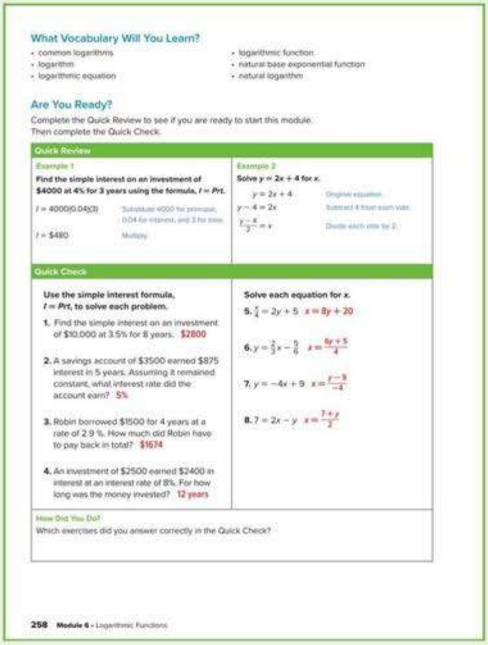
	Before			After		
	👎	👋	👍	👎	👋	👍
write and evaluate logarithms						
graph logarithmic functions						
simplify logarithmic expressions						
solve logarithmic equations						
solve exponential equations by using common logarithms						
solve exponential equations by using natural logarithms						
write and solve exponential growth and decay equations using logarithms						
model datasets with natural logarithmic functions using technology						

Foldables Begin with two sheets of grid paper folded along the width.

1. **On** the first sheet, cut 5 cm along the fold at the ends.
2. **On** the second sheet, cut in the center, stopping 5 cm from the ends.
3. **Insert** the first sheet through the second sheet and align the folds.

Module 6 • Logarithmic Functions 257

Interactive Presentation



What Vocabulary Will You Learn?

- common logarithms
- logarithm
- logarithmic equation
- logarithmic function
- natural base exponential function
- natural logarithm

Are You Ready?

Complete the Quick Review to see if you are ready to start this module. Then complete the Quick Check.

Quick Review

Example 1

Find the simple interest on an investment of $4000 at 4% for 3 years using the formula, $I = Prt$.

$I = 4000(0.04)(3)$ Substitute 4000 for principal, 0.04 for interest, and 3 for time.

$I = \$480$ Multiply.

Example 2

Solve $y = 2x + 4$ for x.

$y = 2x + 4$ Original equation.

$y - 4 = 2x$ Subtract 4 from each side.

$\frac{y-4}{2} = x$ Divide each side by 2.

Quick Check

Use the simple interest formula, $I = Prt$, to solve each problem.

1. Find the simple interest on an investment of $10,000 at 3.5% for 8 years. $2800
2. A savings account of $3500 earned $875 interest in 5 years. Assuming it remained constant, what interest rate did the account earn? 5%
3. Robin borrowed $1500 for 4 years at a rate of 2.9 %. How much did Robin have to pay back in total? $1674
4. An investment of $2500 earned $2400 in interest at an interest rate of 8%. For how long was the money invested? 12 years

Solve each equation for x.

5. $\frac{x}{4} = 2y + 5$ $x = 8y + 20$
6. $y = \frac{2}{3}x - \frac{5}{6}$ $x = \frac{6y+5}{4}$
7. $y = -4x + 9$ $x = \frac{y-9}{-4}$
8. $7 = 2x - y$ $x = \frac{7+y}{2}$

How Did You Do?

Which exercises did you answer correctly in the Quick Check?

258 **Module 6** • Logarithmic Functions

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define A logarithmic function is a function of the form $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$.

Example $g(x) = \log_7 x$

Ask Is the function a logarithmic function? Explain. Yes; the base of the logarithm is 7, which is greater than 1.

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- finding inverses of functions
- using properties of exponents
- using powers of 10
- using the Change of Base formula
- solving exponential equations

ALEKS

ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You can use the ALEKS pie report to see which students know the topics in the Logarithmic Functions module—who is ready to learn these topics and who isn't quite ready to learn them yet—in order to adjust your instruction as appropriate.

Mindset Matters

Growth vs. Fixed Mindset

Everyone has a core belief or mindset about how they learn. People with a growth mindset believe that hard work will make them smarter. Those with a fixed mindset believe they can learn new things, but can't become smarter. When students change their mindset, they are more likely to work through challenging problems, learn from their mistakes, and ultimately learn more deeply.

How Can I Apply It?

Assign students tasks, celebrate mistakes, and provide opportunities for critique, revision, and reflection. The **Explore** activities and discussion prompts are a great tool to begin this journey!

Lesson 6-1

Logarithms and Logarithmic Functions

A.SSE.2, F.IF.7e

LESSON GOAL

Students write and evaluate logarithms and graph logarithmic functions.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Logarithmic Functions

- Logarithmic to Exponential Form
- Exponential to Logarithmic Form
- Evaluate Logarithmic Expressions
- Find Inverses of Exponential Functions

Explore: Transforming Logarithmic Functions

Develop:

Graphing Logarithmic Functions

- Graph Logarithmic Functions
- Graph Transformations of Logarithmic Functions
- Compare Logarithmic Functions
- Write Logarithmic Functions From Graphs

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Inverses of Linear Functions	●	●		●
Extension: Comparing Logarithmic Graphs		●	●	●

Language Development Handbook

Assign page 30 of the *Language Development Handbook* to help your students build mathematical language related to writing and evaluating logarithms and graphing logarithmic functions.

ELL You can use the tips and suggestions on page T30 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Algebra, Functions

Standards for Mathematical Content:

A.SSE.2 Use the structure of an expression to identify ways to rewrite it.

F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions showing period, midline, and amplitude.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

5 Use appropriate tools strategically.

Coherence

Vertical Alignment

Previous
Students understood the relationship between functions and their inverses.
F.B.4

Now
Students write and evaluate logarithms and graph logarithmic functions.
A.SSE.2, F.IF.7e

Next
Students will simplify logarithmic expressions and solve logarithmic equations.
A.CED.1

Rigor

The Three Pillars of Rigor

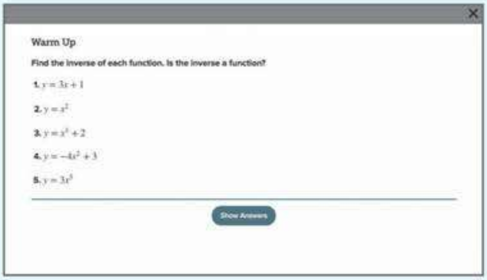
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students extend their understanding of exponential functions and inverses to logarithmic functions. They build fluency by writing equations in exponential and logarithmic forms, and they apply their understanding by solving real-world problems related to logarithmic functions.

Mathematical Background

The equation $y = \log_b x$ is read "y equals the logarithm to the base b of the number x." The base b is always positive and $b \neq 1$. Because the equation $y = \log_b x$ is equivalent to the exponential equation $x = b^y$, a logarithm is an exponent. It is the exponent that the base b requires in order to equal the number x.

Interactive Presentation



Warm Up

Find the inverse of each function. Is the inverse a function?

1. $y = 3x + 1$
2. $y = x^2$
3. $y = x^3 + 2$
4. $y = -4x^2 + 3$
5. $y = 3x^5$

Warm Up

Launch the Lesson

According to a report by the World Health Organization, 1.1 billion teens and young adults are risking hearing damage from listening to music at high volumes. Sound level is measured in decibels. The decibel measure of the loudness of a sound is the exponent, or logarithm, of its relative intensity multiplied by 10. So an increase of 10 decibels indicates that a sound is 10 times as intense.

Launch the Lesson

Vocabulary

logarithm

In $x = b^y$, y is called the logarithm, base b, of x.

logarithmic function

A function of the form $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$.

- Suppose the statement $4^3 = 64$ can be stated as a logarithm in this way: $\log_4 64 = 3$. How could you state $\log_2 128 = 7$ in exponential form?
- Which equations are equivalent: $2^3 = 8$, $3^2 = 9$, $\log_3 9 = 2$, $\log_2 8 = 3$, $\sqrt[3]{8} = 2$?
- If the graph of the exponential function $y = 2^x$ starts near the x-axis and trends upward to the right, what do you think the graph of its inverse, $y = \log_2 x$, would look like?

Today’s Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- finding the inverse of functions

Answers:

1. $y = \frac{x-1}{3}$; yes
2. $y = \pm\sqrt{x}$; no
3. $y = \sqrt[3]{x-2}$; yes
4. $y = \pm\sqrt{\frac{x-3}{-4}}$; no
5. $y = \sqrt[5]{\frac{x}{3}}$; yes

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will apply the concept of logarithms to a real-world situation. Encourage students to explain how decibel measures are calculated.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today’s Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today’s Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Transforming Logarithmic Functions

Objective

Students use a sketch to explore how changing the parameters changes the graphs of logarithmic functions.

Teaching the Mathematical Practices

3 Analyze Cases Work with students to analyze transformations of logarithmic functions for various values of each parameter. Encourage students to familiarize themselves with all of the cases.

5 Use Mathematical Tools Work with students to explore and deepen their understanding of transformations of logarithmic functions.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

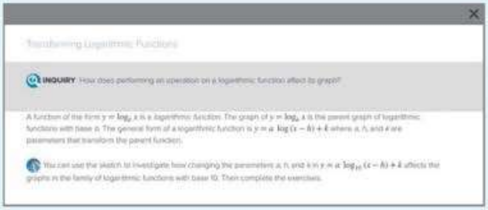
Summary of the Activity

Students will be presented with an Inquiry Question to answer at the end of the activity. They will use a sketch to explore the transformations of the graph of a logarithmic function. Students will work through three guiding exercises, then answer the Inquiry Question.

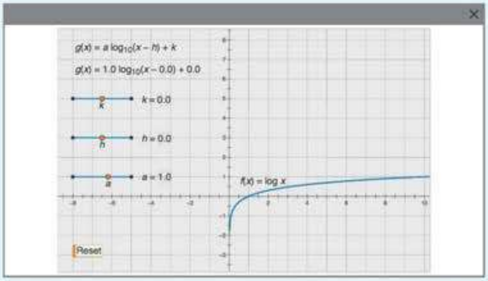
Go Online to find additional teaching notes and sample answers for the guiding exercises.

(continued on the next page)

Interactive Presentation



Explore



Explore

WEB SKETCHPAD

Students use the sketch to graph the transformations of a logarithmic function.

TYPE

Students move through the exercises and answer questions pertaining to the transformations.

SELECT

Students select the correct word or phrase for each transformation performed.

Interactive Presentation

INQUIRY How does performing an operation on a logarithmic function affect its graph?

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Transforming Logarithmic Functions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Suppose your friend graphed the transformation $f(x) = \log(x - h)$ and told you the graph crosses the x-axis at -3. What could be the value of h? Explain your reasoning. -3; Sample answer: Since the graph of the parent function crosses the x-axis at 1, if the new graph crosses at -2, the parent graph has been shifted left 3 units.
- Why does the value of a in $f(x) = a \log x$ not affect the x-intercept of the graph? Sample answer: Because a stretches or compresses the function values, it does not affect the x-intercept since the function value is zero. Zero cannot be stretched or compressed.

Inquiry

How does performing an operation on a logarithmic function affect its graph? Sample answer: Adding or subtracting a constant before or after the function is evaluated causes the graph to be shifted horizontally or vertically, respectively. Multiplying by a constant stretches or compresses the graph vertically. Multiplying by a negative constant reflects the graph in the x-axis, causing the end behavior to change.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Logarithmic Functions

Objective

Students write logarithmic expressions in exponential form and exponential expressions in logarithmic form.

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the tables, equations, and graphs of exponential and logarithmic functions.

What Students Are Learning

A logarithmic function is the inverse of an exponential function. The base of logarithmic and exponential functions must be a positive, real number not equal to one. Logarithmic and exponential expressions can be rewritten into the opposite form.

DIFFERENTIATE

Language Development Activity ELL

Intermediate/Advanced The word *logarithm* derives from the Greek *logos*, which means reason or proportion and *arithmos* which means number. Have students discuss how this knowledge can help them remember the definition of *logarithm*.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students are struggling with converting forms of logarithmic and exponential expressions,

THEN have students write $\log_b n = p$ and $b^p = n$. Students can highlight corresponding variables with different colors to help when converting.

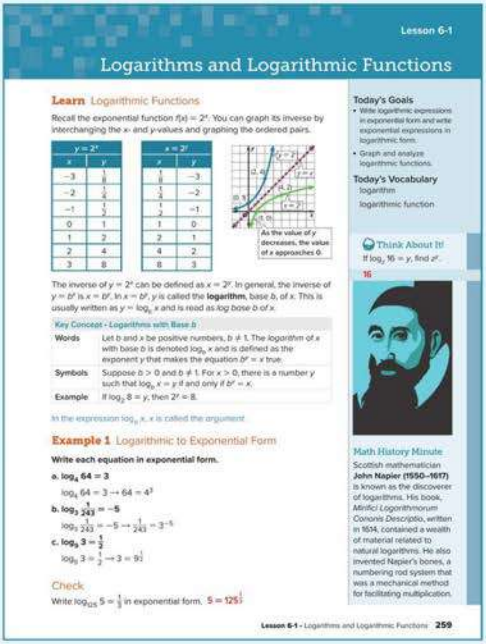
Example 1 Logarithmic to Exponential Form

Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of the equations to rewrite them in exponential form.

Questions for Mathematical Discourse

- **AL** How can you identify the base of a logarithmic equation? The base is written as a subscript.
- **OL** How does the value to which a logarithmic expression is equal relate to an exponential expression? Sample answer: The logarithmic expression is always equal to the exponent of the related exponential expression.
- **BL** When a logarithmic equation contains only numerical values, how can you check the equivalent exponential equation for accuracy? Sample answer: Evaluate the exponential side of the equation and see if the result is a true statement.



Lesson 6-1

Logarithms and Logarithmic Functions

Learn Logarithmic Functions

Recall the exponential function $f(x) = 2^x$. You can graph its inverse by interchanging the x- and y-values and graphing the ordered pairs.

$y = 2^x$	
x	y
−3	$\frac{1}{8}$
−2	$\frac{1}{4}$
−1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

$x = 2^y$	
x	y
$\frac{1}{8}$	−3
$\frac{1}{4}$	−2
$\frac{1}{2}$	−1
1	0
2	1
4	2
8	3

As the value of y decreases, the value of x approaches 0.

The inverse of $y = 2^x$ can be defined as $x = 2^y$. In general, the inverse of $y = b^x$ is $x = b^y$. In $x = b^y$, y is called the **logarithm**, base b, of x. This is usually written as $y = \log_b x$ and is read as *log base b of x*.

Key Concept • Logarithms with Base b

Words	Let b and x be positive numbers, $b \neq 1$. The logarithm of x with base b is denoted $\log_b x$ and is defined as the exponent y that makes the equation $b^y = x$ true.
Symbols	Suppose $b > 0$ and $b \neq 1$. For $x > 0$, there is a number y such that $\log_b x = y$ if and only if $b^y = x$.
Example	If $\log_2 8 = y$, then $2^y = 8$.

In the expression $\log_b x$, x is called the argument.

Example 1 Logarithmic to Exponential Form

Write each equation in exponential form.

a. $\log_4 64 = 3$

$\log_4 64 = 3 \rightarrow 64 = 4^3$

b. $\log_3 \frac{1}{243} = -5$

$\log_3 \frac{1}{243} = -5 \rightarrow \frac{1}{243} = 3^{-5}$

c. $\log_9 3 = \frac{1}{2}$

$\log_9 3 = \frac{1}{2} \rightarrow 3 = 9^{\frac{1}{2}}$

Check

Write $\log_{125} 5 = \frac{1}{3}$ in exponential form. $5 = 125^{\frac{1}{3}}$

Today's Goals
- Write logarithmic expressions in exponential form and write exponential expressions in logarithmic form.
- Graph and analyze logarithmic functions.

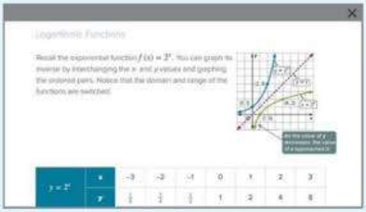
Today's Vocabulary
logarithm
logarithmic function

Think About It!
If $\log_2 16 = y$, find 2^y.
16

Math History Minute
Scottish mathematician **John Napier (1550–1617)** is known as the discoverer of logarithms. His book, *Mirifici Logarithmorum Canonis Descriptio*, written in 1614, contained a wealth of material related to natural logarithms. He also invented Napier's bones, a numbering rod system that was a mechanical method for facilitating multiplication.

Lesson 6-1 • Logarithms and Logarithmic Functions 259

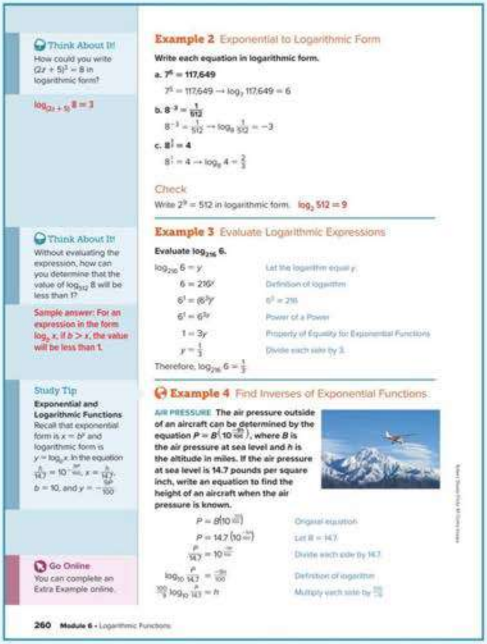
Interactive Presentation



Learn

TYPE

Students answer a question to determine whether they understand the value of an exponential expression using a logarithmic equation.



Think About It! How could you write $(2x + 5)^3 = 8$ in logarithmic form?

$\log_{(2x+5)} 8 = 3$

Example 2 Exponential to Logarithmic Form

Write each equation in logarithmic form.

a. $7^6 = 117{,}649$

$7^6 = 117{,}649 \rightarrow \log_7 117{,}649 = 6$

b. $8^{-3} = \frac{1}{512}$

$8^{-3} = \frac{1}{512} \rightarrow \log_8 \frac{1}{512} = -3$

c. $8^{\frac{2}{3}} = 4$

$8^{\frac{2}{3}} = 4 \rightarrow \log_8 4 = \frac{2}{3}$

Check

Write $2^9 = 512$ in logarithmic form. $\log_2 512 = 9$

Think About It! Without evaluating the expression, how can you determine that the value of $\log_{512} 8$ will be less than 1?

Sample answer: For an expression in the form $\log_b x$, if $b > x$, the value will be less than 1.

Example 3 Evaluate Logarithmic Expressions

Evaluate $\log_{216} 6$.

$\log_{216} 6 = y$ Let the logarithm equal y.

$6 = 216^y$ Definition of logarithm

$6^1 = (6^3)^y$ $6^3 = 216$

$6^1 = 6^{3y}$ Power of a Power

$1 = 3y$ Property of Equality for Exponential Functions

$y = \frac{1}{3}$ Divide each side by 3.

Therefore, $\log_{216} 6 = \frac{1}{3}$

Study Tip

Exponential and Logarithmic Functions Recall that exponential form is $x = b^y$ and logarithmic form is $y = \log_b x$. In the equation $\frac{P}{14.7} = 10^{-\frac{3h}{100}}$, $x = \frac{P}{14.7}$, $b = 10$, and $y = -\frac{3h}{100}$.

Example 4 Find Inverses of Exponential Functions

AIR PRESSURE **The air pressure outside of an aircraft can be determined by the equation $P = B\left(10^{-\frac{3h}{100}}\right)$, where B is the air pressure at sea level and h is the altitude in miles. If the air pressure at sea level is 14.7 pounds per square inch, write an equation to find the height of an aircraft when the air pressure is known.**

$P = B\left(10^{-\frac{3h}{100}}\right)$ Original equation

$P = 14.7\left(10^{-\frac{3h}{100}}\right)$ Let $B = 14.7$.

$\frac{P}{14.7} = 10^{-\frac{3h}{100}}$ Divide each side by 14.7.

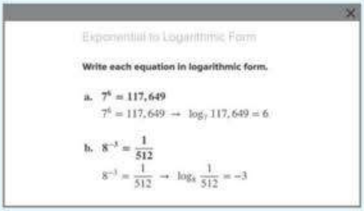
$\log_{10} \frac{P}{14.7} = -\frac{3h}{100}$ Definition of logarithm

$-\frac{100}{3} \log_{10} \frac{P}{14.7} = h$ Multiply each side by $-\frac{100}{3}$.

Go Online You can complete an Extra Example online.

260 Module 6 • Logarithmic Functions

Interactive Presentation



Exponential to Logarithmic Form

Write each equation in logarithmic form.

a. $7^6 = 117,649$

$7^6 = 117,649 \rightarrow \log_7 117,649 = 6$

b. $8^{-3} = \frac{1}{512}$

$8^{-3} = \frac{1}{512} \rightarrow \log_8 \frac{1}{512} = -3$

Example 2

TYPE

Students write an exponential equation involving a variable in logarithmic form.

Example 2 Exponential to Logarithmic Form

Teaching the Mathematical Practices

6 Use Definitions Students will use the definition of a logarithm to rewrite equations in logarithmic form.

Questions for Mathematical Discourse

AL In each example, where does the exponential base in the equation get placed in the logarithmic form? It is written as the subscript because it is the base of the logarithm.

OL In each example, where does the exponent in the equation get placed in the logarithmic form? The logarithmic expression is set equal to the exponent.

BL Why is the base restricted to positive numbers? Negative bases for logarithms or exponential function yield functions ranges that span the complex plane.

Example 3 Evaluate Logarithmic Expressions

Teaching the Mathematical Practices

1 Seek Information Students must transform the logarithmic expression using the definition of a logarithm and properties of exponents to reach the solution.

Questions for Mathematical Discourse

AL Why is $6^3 = 216$ used? This makes each side of the equation have the same base.

OL Without using a calculator, how can you determine that $216 = 6^3$? Sample answer: You can multiply 6 by 6 to get 36, then multiply 36 by 6 to get 216. This means you multiplied $6 \times 6 \times 6$, which is 6^3.

BL When is $\log_b x$ negative? Sample answer: when $0 < x < 1$.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Example 4 Find Inverses of Exponential Functions

Teaching the Mathematical Practices

4 Apply Mathematics Students will use what they have learned about logarithms to solve a real-world problem.

Questions for Mathematical Discourse

AL Why does the inverse have a logarithm with base 10? Because the original exponential equation has a base of 10, its inverse is a logarithm with base 10.

OL Why would solving for another variable of a real-world exponential equation be useful? It allows you to find values of the equation given values for the other variable.

BL What are the domain and range for the original equation in the context of the example? The domain is all real positive numbers, and the range is $0 < P < 14.7$

Learn Graphing Logarithmic Functions

Objective

Students graph and analyze logarithmic functions.

Teaching the Mathematical Practices

5 Decide When to Use Tools Use the Watch Out! feature to have students point out the limitations of using the **LOG** button on a graphing calculator to evaluate and graph logarithmic functions.

DIFFERENTIATE

Reteaching Activity **AL** **ELL**

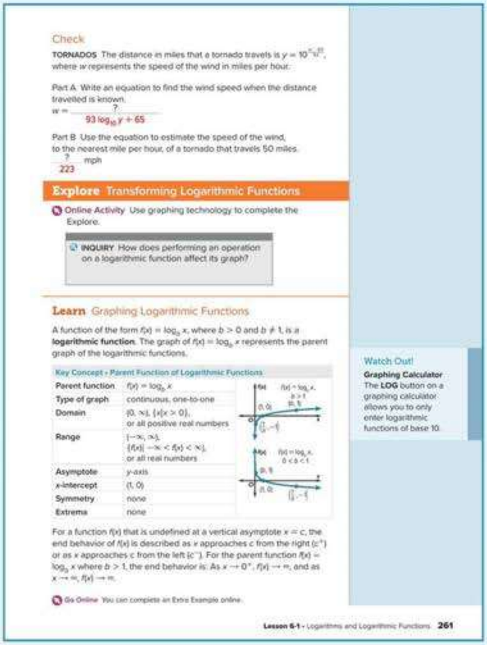
IF students are struggling with transforming logarithmic equations, **THEN** remind students the rules are exactly the same as for other functions they have studied. Coefficients of the function stretch or compress the y-values, while numbers added or subtracted to the function shift the graph up or down. Values being added to or subtracted from x shift the graph left or right, but the effect on x is opposite of what the value implies.

Essential Question Follow-Up

Students have begun learning about the graphs of logarithmic functions.

Ask:

How are the graphs of exponential functions and logarithmic functions related? Sample answer: The key features of an exponential function are swapped when graphing the logarithmic inverse. Domain becomes range, range becomes domain, the y-intercept becomes the x-intercept, and the asymptote switches from a horizontal to a vertical line.



Check

TORNADOS The distance in miles that a tornado travels is $y = 10^{[illegible]}$, where w represents the speed of the wind in miles per hour.

Part A Write an equation to find the wind speed when the distance travelled is known.

$w =$? $93 \log_{10} y + 65$

Part B Use the equation to estimate the speed of the wind, to the nearest mile per hour, of a tornado that travels 50 miles.

? 223 mph

Explore Transforming Logarithmic Functions

Online Activity Use graphing technology to complete the Explore.

INQUIRY How does performing an operation on a logarithmic function affect its graph?

Learn Graphing Logarithmic Functions

A function of the form $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$, is a **logarithmic function**. The graph of $f(x) = \log_b x$ represents the parent graph of the logarithmic functions.

Key Concept • Parent Function of Logarithmic Functions

Parent function	$f(x) = \log_b x$
Type of graph	continuous, one-to-one
Domain	$(0, \infty)$, $\{x \mid x > 0\}$, or all positive real numbers
Range	$(-\infty, \infty)$, $\{f(x) \mid -\infty < f(x) < \infty\}$, or all real numbers
Asymptote	y-axis
x-intercept	$(1, 0)$
Symmetry	none
Extrema	none

For a function $f(x)$ that is undefined at a vertical asymptote $x = c$, the end behavior of $f(x)$ is described as x approaches c from the right (c^+) or as x approaches c from the left (c^-). For the parent function $f(x) = \log_b x$ where $b > 1$, the end behavior is: As $x \to 0^+$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$.

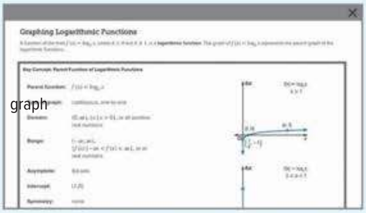
Go Online You can complete an Extra Example online.

Watch Out!

Graphing Calculator The **LOG** button on a graphing calculator allows you to only enter logarithmic functions of base 10.

Lesson 6-1 • Logarithms and Logarithmic Functions 261

Interactive Presentation



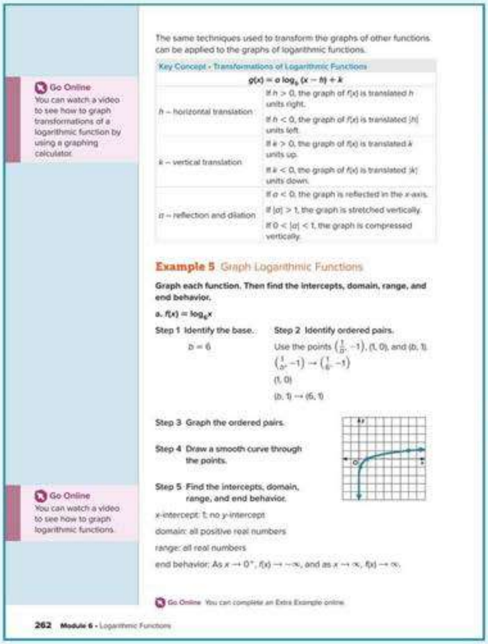
Learn

WATCH

Students watch a video to learn how to graph transformations of a logarithmic function by using a graphing calculator.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Go Online You can watch a video to see how to graph transformations of a logarithmic function by using a graphing calculator.

The same techniques used to transform the graphs of other functions can be applied to the graphs of logarithmic functions.

Key Concept • Transformations of Logarithmic Functions

$g(x) = a \log_b (x - h) + k$	
h – horizontal translation	If $h > 0$, the graph of $f(x)$ is translated h units right. If $h < 0$, the graph of $f(x)$ is translated $\lvert h \rvert$ units left.
k – vertical translation	If $k > 0$, the graph of $f(x)$ is translated k units up. If $k < 0$, the graph of $f(x)$ is translated $\lvert k \rvert$ units down.
a – reflection and dilation	If $a < 0$, the graph is reflected in the x-axis. If $\lvert a \rvert > 1$, the graph is stretched vertically. If $0 < \lvert a \rvert < 1$, the graph is compressed vertically.

Example 5 Graph Logarithmic Functions

Graph each function. Then find the intercepts, domain, range, and end behavior.

a. $f(x) = \log_6 x$

Step 1 Identify the base.

$b = 6$

Step 2 Identify ordered pairs.

Use the points $\left(\frac{1}{b}, -1\right)$, $(1, 0)$, and $(b, 1)$.

$\left(\frac{1}{b}, -1\right) \to \left(\frac{1}{6}, -1\right)$

$(1, 0)$

$(b, 1) \to (6, 1)$

Step 3 Graph the ordered pairs.

Step 4 Draw a smooth curve through the points.

Step 5 Find the intercepts, domain, range, and end behavior.

x-intercept: 1; no y-intercept

domain: all positive real numbers

range: all real numbers

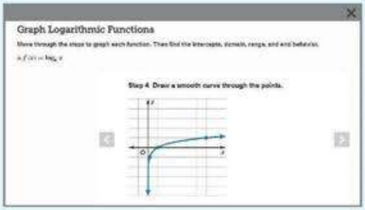
end behavior: As $x \to 0^+$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$.

Go Online You can watch a video to see how to graph logarithmic functions.

Go Online You can complete an Extra Example online.

262 Module 6 • Logarithmic Functions

Interactive Presentation



Graph Logarithmic Functions

Move through the steps to graph each function. Then find the intercepts, domain, range, and end behavior.

a. $f(x) = \log_6 x$

Step 4 Draw a smooth curve through the points.

Example 5

TAP

Students move through the steps to see how to graph logarithmic functions.

Example 5 Graph Logarithmic Functions

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the functions, the points used to graph the functions, and the graphs of the functions.

Questions for Mathematical Discourse

AL Why were the points in **Step 2** selected? These are points that are easy to evaluate and graph because any value of b will yield the same y-value for a logarithmic function in the form $y = \log_b x$.

OL What is the equation of the vertical asymptote for a logarithmic function in the form $y = \log_b x$? $x = 0$

BL What is the x-intercept of the graph $y = \log_b x$? $(1, 0)$

Common Error

Many students will struggle to calculate ordered pairs on a logarithmic graph. Encourage students to write $\left(\frac{1}{b}, -1\right)$, $(1, 0)$, and $(b, 1)$ in their Foldable as these are the best choices when determining ordered pairs. Remind students that b is the base of the logarithm.

Example 6 Graph Transformations of Logarithmic Functions

Teaching the Mathematical Practices

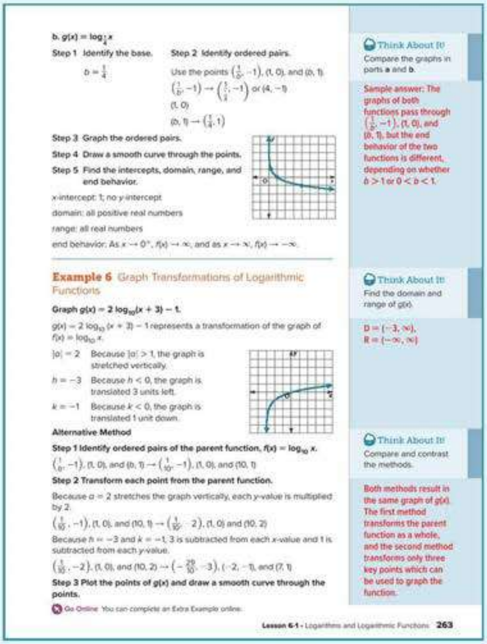
7 Use Structure Help students to use the structure of the function to identify the transformations and graph the function.

Questions for Mathematical Discourse

AL What three points should you graph for the equation $y = \log_{10} x$? Sample answer: $\left(\frac{1}{10}, -1\right)$, (1, 0) and (10, 1)

OL What are the coordinates for the new location of the *x*-intercept of the parent function after being transformed to the new function? $(1, 0) \longrightarrow (1 - 3, 0 - 1) \longrightarrow (-2, -1)$

BL What is the equation of the vertical asymptote for the transformation? $x = -3$



b. $g(x) = \log_{\frac{1}{4}} x$

Step 1 Identify the base.

$b = \frac{1}{4}$

Step 2 Identify ordered pairs.

Use the points $\left(\frac{1}{b}, -1\right)$, (1, 0), and (b, 1)

$\left(\frac{1}{b}, -1\right) \rightarrow \left(\frac{1}{\frac{1}{4}}, -1\right)$ or (4, −1)

(1, 0)

$(b, 1) \rightarrow \left(\frac{1}{4}, 1\right)$

Step 3 Graph the ordered pairs.

Step 4 Draw a smooth curve through the points.

Step 5 Find the intercepts, domain, range, and end behavior.

x-intercept: 1; no *y*-intercept

domain: all positive real numbers

range: all real numbers

end behavior: As $x \rightarrow 0^+$, $f(x) \rightarrow \infty$, and as $x \rightarrow \infty$, $f(x) \rightarrow -\infty$.

Think About It! Compare the graphs in parts **a** and **b**.

Sample answer: The graphs of both functions pass through $\left(\frac{1}{b}, -1\right)$, (1, 0), and (b, 1), but the end behavior of the two functions is different, depending on whether $b > 1$ or $0 < b < 1$.

Example 6 Graph Transformations of Logarithmic Functions

Graph $g(x) = 2 \log_{10}(x + 3) - 1$.

$g(x) = 2 \log_{10}(x + 3) - 1$ represents a transformation of the graph of $f(x) = \log_{10} x$.

$|a| = 2$ Because $|a| > 1$, the graph is stretched vertically.

$h = -3$ Because $h < 0$, the graph is translated 3 units left.

$k = -1$ Because $k < 0$, the graph is translated 1 unit down.

Think About It! Find the domain and range of g(x).

$D = (-3, \infty)$, $R = (-\infty, \infty)$

Alternative Method

Step 1 Identify ordered pairs of the parent function, $f(x) = \log_{10} x$.

$\left(\frac{1}{b}, -1\right)$, (1, 0), and $(b, 1) \rightarrow \left(\frac{1}{10}, -1\right)$, (1, 0), and (10, 1)

Step 2 Transform each point from the parent function.

Because $a = 2$ stretches the graph vertically, each *y*-value is multiplied by 2.

$\left(\frac{1}{10}, -1\right)$, (1, 0), and $(10, 1) \rightarrow \left(\frac{1}{10}, -2\right)$, (1, 0) and (10, 2)

Because $h = -3$ and $k = -1$, 3 is subtracted from each *x*-value and 1 is subtracted from each *y*-value.

$\left(\frac{1}{10}, -2\right)$, (1, 0), and $(10, 2) \rightarrow \left(-\frac{29}{10}, -3\right)$, (−2, −1), and (7, 1)

Step 3 Plot the points of g(x) and draw a smooth curve through the points.

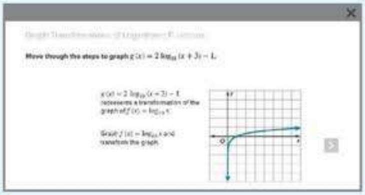
Think About It! Compare and contrast the methods.

Both methods result in the same graph of g(x). The first method transforms the parent function as a whole, and the second method transforms only three key points which can be used to graph the function.

Go Online You can complete an Extra Example online.

Lesson 6-1 • Logarithms and Logarithmic Functions 263

Interactive Presentation



Move through the steps to graph $g(x) = 2 \log_{10}(x + 3) - 1$.

$g(x) = 2 \log_{10}(x + 3) - 1$ represents a transformation of the graph of $f(x) = \log_{10} x$.

Graph $f(x) = \log_{10} x$ and transform the graph.

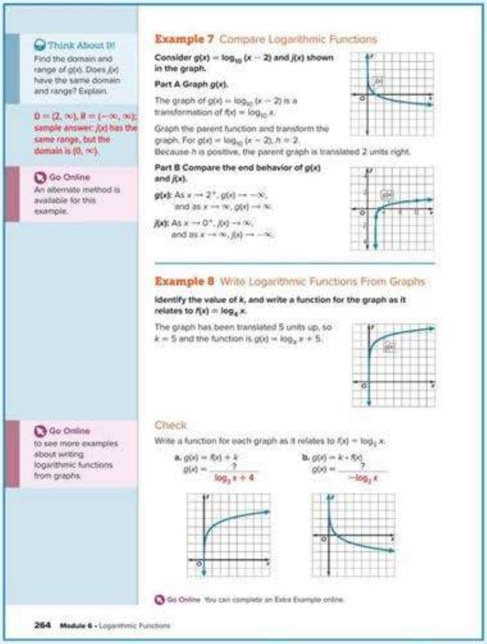
Example 6

TAP

Students move through the steps of graphing and analyzing a transformation of a logarithmic function.

TYPE

Students state the domain and range for the function.



Think About It!

Find the domain and range of $g(x)$. Does $j(x)$ have the same domain and range? Explain.

$D = (2, \infty)$, $R = (-\infty, \infty)$; sample answer: $j(x)$ has the same range, but the domain is $(0, \infty)$.

Go Online

An alternate method is available for this example.

Example 7 Compare Logarithmic Functions

Consider $g(x) = \log_{10}(x - 2)$ and $j(x)$ shown in the graph.

Part A Graph $g(x)$.

The graph of $g(x) = \log_{10}(x - 2)$ is a transformation of $f(x) = \log_{10} x$.

Graph the parent function and transform the graph. For $g(x) = \log_{10}(x - 2)$, $h = 2$. Because h is positive, the parent graph is translated 2 units right.

Part B Compare the end behavior of $g(x)$ and $j(x)$.

$g(x)$: As $x \to 2^+$, $g(x) \to -\infty$, and as $x \to \infty$, $g(x) \to \infty$.

$j(x)$: As $x \to 0^+$, $j(x) \to \infty$, and as $x \to \infty$, $j(x) \to -\infty$.

Example 8 Write Logarithmic Functions From Graphs

Identify the value of k, and write a function for the graph as it relates to $f(x) = \log_4 x$.

The graph has been translated 5 units up, so $k = 5$ and the function is $g(x) = \log_4 x + 5$.

Go Online

to see more examples about writing logarithmic functions from graphs.

Check

Write a function for each graph as it relates to $f(x) = \log_3 x$.

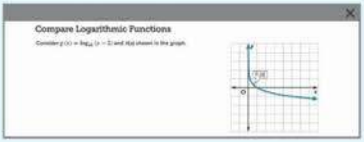
a. $g(x) = f(x) + k$
$g(x) =$? $\log_3 x + 4$

b. $g(x) = k \cdot f(x)$
$g(x) =$? $-\log_3 x$

Go Online You can complete an Extra Example online.

264 Module 6 • Logarithmic Functions

Interactive Presentation



Compare Logarithmic Functions

Consider $g(x) = \log_{10}(x - 2)$ and $j(x)$ shown in the graph.

Example 7

TYPE

Students answer questions to determine whether they understand how to graph transformations of logarithmic functions.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 7 Compare Logarithmic Functions

Teaching the Mathematical Practices

6 Communicate Precisely The Think About It! feature requires students to use clear mathematical language and definitions to state the domain and range of the functions.

Questions for Mathematical Discourse

AL Why does the graph of the parent function not shift up or down when graphing the transformation $g(x)$? There is no value being added to the parent function, so there is no vertical shift in the graph.

OL How does a horizontal translation affect the vertical asymptote of a logarithmic function? The vertical asymptote also shifts $|h|$ units right or left.

BL How could you further transform $g(x)$ so that it has the same end behavior as $j(x)$ as $x \longrightarrow \infty$? Reflect the function in the x-axis by multiplying $g(x)$ by a negative value.

Example 8 Write Logarithmic Functions from Graphs

Teaching the Mathematical Practices

1 Explain Correspondences In Example 8, encourage students to explain the relationship between the graph of $g(x)$ and the graph of the parent function. Students will use this relationship to write an equation for the transformed function.

Questions for Mathematical Discourse

AL What are the coordinates for the x-intercept of the parent graph $y = \log_b x$? (1, 0) What is the new y-coordinate for $x = 1$? $y = 5$

OL If a parent logarithmic function has been reflected in the x-axis, what will the end behavior of the function be as $x \longrightarrow 0$? $f(x) \longrightarrow \infty$

BL How can you tell if the graph of a parent logarithmic function has been translated horizontally? Sample answer: The vertical asymptote will no longer be $x = 0$.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–36
2	exercises that use a variety of skills from this lesson	37–47
3	exercises that emphasize higher-order and critical-thinking skills	48–53

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice Exercises 1–47 odd, 48–53
- Extension: Comparing Logarithmic Graphs
- **ALEKS** Logarithmic Functions

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice Exercises 1–53 odd
- Remediation, Review Resources: Inverses of Linear Functions
- Personal Tutors
- Extra Examples 1–8
- **ALEKS** Composition and Inverse Functions

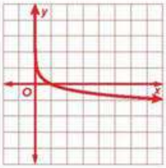
AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice Exercises 1–35 odd
- Remediation, Review Resources: Inverses of Linear Functions
- *Quick Review Math Handbook*: Logarithms and Logarithmic Functions
- **ALEKS** Composition and Inverse Functions

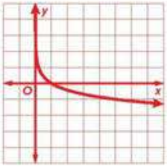
Answers

24.

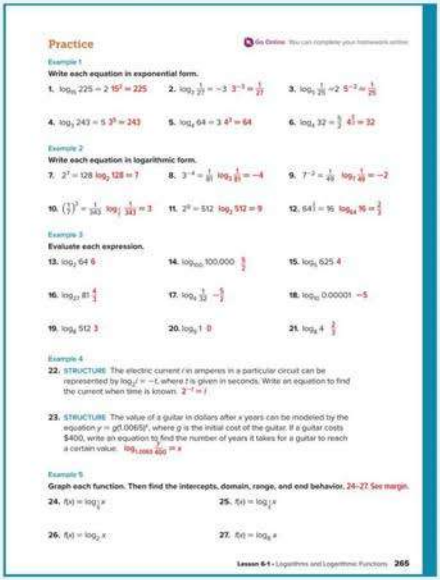


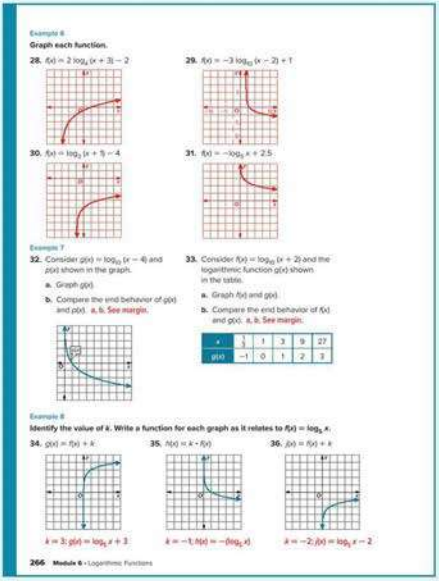
x-intercept: 1; no y-intercept
D: $(0, \infty)$ R: $(-\infty, \infty)$
end behavior: As $x \to 0^+$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to -\infty$.

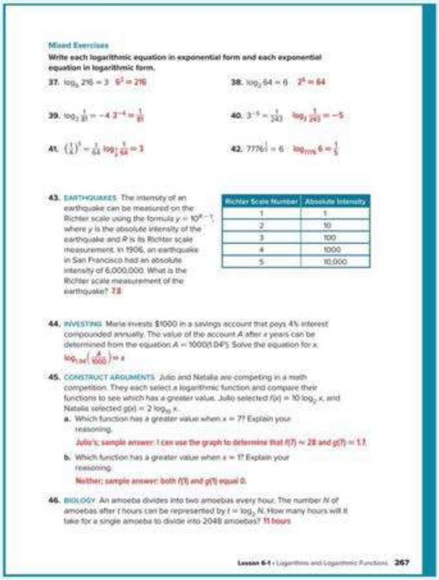
25.



x-intercept: 1; no y-intercept
D: $(0, \infty)$ R: $(-\infty, \infty)$
end behavior: As $x \to 0^+$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to -\infty$.







Mixed Exercises

Write each logarithmic equation in exponential form and each exponential equation in logarithmic form.

37. $\log_6 216 = 3$ $6^3 = 216$

38. $\log_2 64 = 6$ $2^6 = 64$

39. $\log_3 \frac{1}{81} = -4$ $3^{-4} = \frac{1}{81}$

40. $3^{-5} = \frac{1}{243}$ $\log_3 \frac{1}{243} = -5$

41. $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$ $\log_{\frac{1}{4}} \frac{1}{64} = 3$

42. $7776^{\frac{1}{5}} = 6$ $\log_{7776} 6 = \frac{1}{5}$

43. EARTHQUAKES The intensity of an earthquake can be measured on the Richter scale using the formula $y = 10^{R-1}$, where y is the absolute intensity of the earthquake and R is its Richter scale measurement. In 1906, an earthquake in San Francisco had an absolute intensity of 6,000,000. What is the Richter scale measurement of the earthquake? 7.8

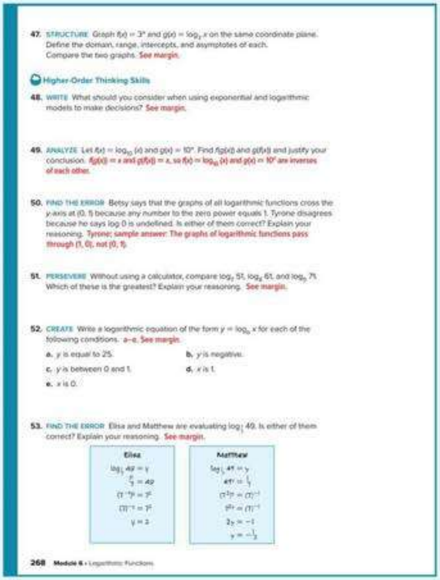
Richter Scale Number	Absolute Intensity
1	1
2	10
3	100
4	1000
5	10,000

44. INVESTING Maria invests $1000 in a savings account that pays 4% interest compounded annually. The value of the account A after x years can be determined from the equation $A = 1000(1.04^x)$. Solve the equation for x. $\log_{1.04}\left(\frac{A}{1000}\right) = x$

45. CONSTRUCT ARGUMENTS Julio and Natalia are competing in a math competition. They each select a logarithmic function and compare their functions to see which has a greater value. Julio selected $f(x) = 10 \log_2 x$, and Natalia selected $g(x) = 2 \log_{10} x$.
   a. Which function has a greater value when $x = 7$? Explain your reasoning.
   Julio's; sample answer: I can use the graph to determine that $f(7) \approx 28$ and $g(7) \approx 1.7$.
   b. Which function has a greater value when $x = 1$? Explain your reasoning.
   Neither; sample answer: both $f(1)$ and $g(1)$ equal 0.

46. BIOLOGY An amoeba divides into two amoebas every hour. The number N of amoebas after t hours can be represented by $t = \log_2 N$. How many hours will it take for a single amoeba to divide into 2048 amoebas? 11 hours

Lesson 6-1 • Logarithms and Logarithmic Functions 267



47. STRUCTURE Graph $f(x) = 3^x$ and $g(x) = \log_3 x$ on the same coordinate plane. Define the domain, range, intercepts, and asymptotes of each. Compare the two graphs. See margin.

Higher-Order Thinking Skills

48. WRITE What should you consider when using exponential and logarithmic models to make decisions? See margin.

49. ANALYZE Let $f(x) = \log_{10}(x)$ and $g(x) = 10^x$. Find $f(g(x))$ and $g(f(x))$ and justify your conclusion. $f(g(x)) = x$ and $g(f(x)) = x$, so $f(x) = \log_{10}(x)$ and $g(x) = 10^x$ are inverses of each other.

50. FIND THE ERROR Betsy says that the graphs of all logarithmic functions cross the y-axis at (0, 1) because any number to the zero power equals 1. Tyrone disagrees because he says log 0 is undefined. Is either of them correct? Explain your reasoning. Tyrone; sample answer: The graphs of logarithmic functions pass through (1, 0), not (0, 1).

51. PERSEVERE Without using a calculator, compare $\log_7 51$, $\log_8 61$, and $\log_9 71$. Which of these is the greatest? Explain your reasoning. See margin.

52. CREATE Write a logarithmic equation of the form $y = \log_b x$ for each of the following conditions. a–e. See margin.
   a. y is equal to 25.
   b. y is negative.
   c. y is between 0 and 1.
   d. x is 1.
   e. x is 0.

53. FIND THE ERROR Elisa and Matthew are evaluating $\log_{\frac{1}{7}} 49$. Is either of them correct? Explain your reasoning. See margin.

Elisa

$\log_{\frac{1}{7}} 49 = y$

$\left(\frac{1}{7}\right)^y = 49$

$(7^{-1})^y = 7^2$

$(7)^{-y} = 7^2$

$y = 2$

Matthew

$\log_{\frac{1}{7}} 49 = y$

$49^y = \frac{1}{7}$

$(7^2)^y = (7)^{-1}$

$7^{2y} = (7)^{-1}$

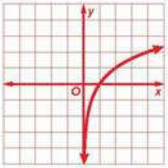
$2y = -1$

$y = -\frac{1}{2}$

268 Module 6 • Logarithmic Functions

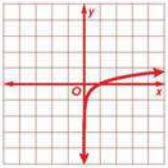
Answers

26.



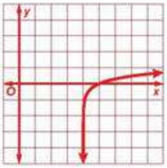
x-intercept: 1; no y-intercept
D: $(0, \infty)$ R: $(-\infty, \infty)$
end behavior: As $x \to 0^+$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$.

27.



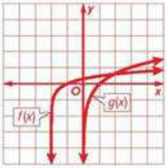
x-intercept: 1; no y-intercept
D: $(0, \infty)$ R: $(-\infty, \infty)$
end behavior: As $x \to 0^+$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$.

32a.



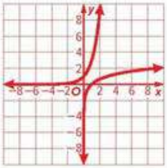
32b. $g(x)$: As $x \to 4^+$, $g(x) \to -\infty$, and as $x \to \infty$, $g(x) \to \infty$. $p(x)$: As $x \to 0^+$, $p(x) \to \infty$, and as $x \to \infty$, $p(x) \to -\infty$.

33a.



33b. $f(x)$: As $x \to -2^+$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$. $g(x)$: As $x \to {}^{+}0$, $g(x) \to -\infty$, and as $x \to \infty$, $g(x) \to \infty$.

47.



$f(x)$: D: $(-\infty, \infty)$ R: $(0, \infty)$; y-int: $(0, 1)$; asymptote: $y = 0$; $g(x)$: D: $(0, \infty)$ R: $(-\infty, \infty)$; x-int: $(1, 0)$; asymptote: $x = 0$. The graphs have an inverse relationship with one another. They are symmetric about the line $y = x$.

48. Sample answer: Exponential and logarithmic models can grow without bound, which is usually not the case of the situation that is being modeled. For instance, a population cannot grow without bound due to space and food constraints. Therefore, when using a model to make decisions, the situation that is being modeled should be carefully considered.

51. $\log_7 51$; sample answer: $\log_7 51$ is more than 2. $\log_8 61$ is less than 2. $\log_9 71$ equals a little less than 2. Therefore, $\log_7 51$ is the greatest.

52. Sample answers:
   a. $\log_2 33{,}554{,}432 = 25$
   b. $\log_4 \frac{1}{64} = -3$
   c. $\log_2 \sqrt{2} = \frac{1}{2}$
   d. $\log_7 1 = 0$
   e. There is no possible solution; this is the empty set.

53. No; Elisa was closer. She should have written $-y = 2$ or $y = -2$ instead of $y = 2$. Matthew used the definition of logarithms incorrectly.

Lesson 6-2

Properties of Logarithms

A.CED.1

LESSON GOAL

Students simplify logarithmic expressions and solve logarithmic equations.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Logarithmic Expressions and Equations

Develop:

Logarithmic Equations
- Solve a Logarithmic Equation by Using Definitions
- Solve a Logarithmic Equation by Using Properties of Equality

Properties of Logarithms
- Product Property of Logarithms
- Quotient Property of Logarithms
- Power Property of Logarithms
- Solve a Logarithmic Equation by Using Properties

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

Formative Assessment Math Probe

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Multiplication Properties of Exponents	●	●		●
Extension: Logarithmic Spirals		●	●	●

Language Development Handbook

Assign page 31 of the *Language Development Handbook* to help your students build mathematical language related to simplifying logarithmic expressions and solving logarithmic equations.

ELL You can use the tips and suggestions on page T31 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Algebra

Standards for Mathematical Content:

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous

Students wrote and evaluated logarithms and graphed logarithmic functions.

A.SSE.2, F.IF.7e

Now

Students simplify logarithmic expressions and solve logarithmic equations.

A.CED.1

Next

Students will simplify logarithmic expressions and solve logarithmic equations by using common logarithms.

A.REI.11, F.LE.4

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students build on their knowledge of the rules for exponents to develop an understanding of the properties of logarithms. They build fluency by using the properties to approximate logarithmic values, and they apply their understanding by solving real-world problems.

Mathematical Background

In addition to the Product, Quotient, and Power Properties of Logarithms, there are other properties such as the four listed below that can be helpful.

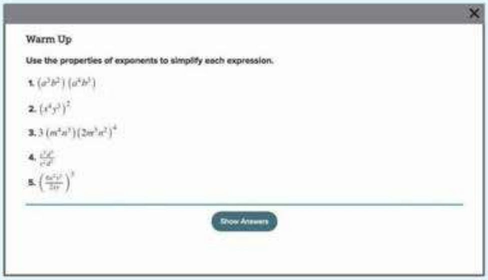
$\log_b 1 = 0$

$\log_b b = 1$

$\log_b b^x = x$

$b^{\log_b x} = x$

Interactive Presentation



Warm Up

Use the properties of exponents to simplify each expression.

1. $(a^3b^2)(a^4b^3)$
2. $(x^4y^3)^2$
3. $3(m^4n^3)(2m^3n^2)^4$
4. [illegible]
5. [illegible]

Warm Up


Launch the Lesson

The acidity of shampoo can affect how your hair reacts to it. Although the pH of shampoos tend to range from 5 to 7, the most acidic shampoo is not simply $\frac{7}{5} = 1.4$ times more acidic than the least acidic shampoo. Because pH is measured using logarithms of base 10, the strongest shampoo is $10^{7-5} = 10^2$ or 100 times as acidic.

Launch the Lesson

Vocabulary

logarithmic equation

- You use subtraction to solve an equation involving addition. How do you think you will solve a logarithmic equation?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- using properties of exponents

Answers:

1. a^7b^5
2. x^8y^6
3. $48m^{16}n^{11}$
4. c^3d
5. $27x^3y^6$

Launch the Lesson

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to use the properties of exponents and logarithms to determine the acidity of shampoo.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using the vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then discuss the question below with the class.

Explore Logarithmic Expressions and Equations

Objective

Students use a sketch to explore the logarithm of a product.

Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in the way logarithms of products and quotients are calculated to write a general method for determining $\log_{10} xy$ and $\log_{10} \frac{x}{y}$.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

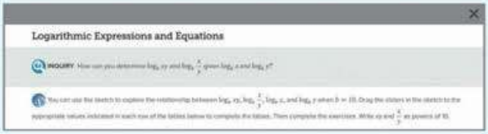
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will be presented with an Inquiry Question to answer at the end of the activity. They will use a sketch to explore the properties of logarithms. Students will work through guiding exercises, then answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

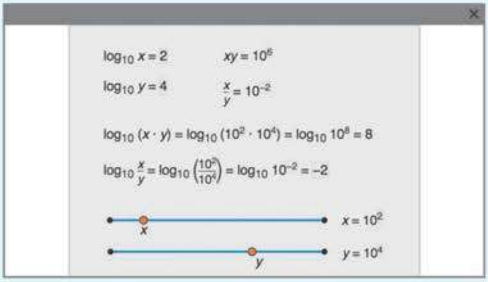


Logarithmic Expressions and Equations

INQUIRY How can you determine $\log_b xy$ and $\log_b \frac{x}{y}$ given $\log_b x$ and $\log_b y$?

You can use the sketch to explore the relationship between $\log_b xy$, $\log_b \frac{x}{y}$, $\log_b x$, and $\log_b y$ when $b = 10$. Drag the sliders in the sketch to the appropriate values indicated in each row of the tables below to complete the tables. Then complete the exercises. Write xy and $\frac{x}{y}$ as powers of 10.

Explore



$\log_{10} x = 2$

$\log_{10} y = 4$

$xy = 10^6$

$\frac{x}{y} = 10^{-2}$

$\log_{10} (x \cdot y) = \log_{10} (10^2 \cdot 10^4) = \log_{10} 10^8 = 8$

$\log_{10} \frac{x}{y} = \log_{10} \left(\frac{10^2}{10^4}\right) = \log_{10} 10^{-2} = -2$

Explore

SKETCHPAD

Students use the sketch to investigate the properties of logarithms.

TYPE

Students move through the exercises and answer questions pertaining to the properties of logarithms.

SELECT

Students select the correct property represented in the exercises.

Interactive Presentation

INQUIRY Write rules for how you can determine $\log_b xy$ and $\log_b \frac{x}{y}$ given $\log_b x$ and $\log_b y$.

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Logarithmic Expressions and Equations (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What will be the value of the expression $\log_5 9 + \log_5 10$? Explain. $\log_5 90$; Sample answer: The sum of the logarithms of x and y is equivalent to the logarithm of the product of x and y, so $\log_5 9 + \log_5 10 = \log_5 90$.
- What is the value of the expression $\log_3 18 - \log_3 9$? Explain. $\log_3 2$; Sample answer: The difference of the logarithms of x and y is equivalent to the logarithm of the quotient of x and y, so $\log_3 18 - \log_3 9 = \log_3 (18 \div 9)$.

Inquiry

How can you determine $\log_b xy$ and $\log_b \frac{x}{y}$ given $\log_b x$ and $\log_b y$? Sample answer: $\log_b xy = \log_b x + \log_b y$ and $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Logarithmic Equations

Objective

Students solve logarithmic equations using properties of equality.

Teaching the Mathematical Practices

3 Construct Arguments In the Think About It! feature, students will use the definition of the Property of Equality for Logarithmic Equations to construct an argument.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students are having a hard time solving logarithmic equations, **THEN** have students highlight the logarithmic bases given in the equation. If the equation has a common logarithmic base on each side, students can use the Property of Equality for Logarithmic Functions. If the equation contains only one logarithmic function, they must rewrite it as an exponential equation.

Example 1 Solve a Logarithmic Equation by Using Definitions

Teaching the Mathematical Practices

6 Use Definitions Students will use the definition of the Property of Equality for Logarithmic Functions to solve an equation.

Questions for Mathematical Discourse

AL Why is $4 = 2^2$ used in the second step? This simplifies the exponent because a power of a power is simplified by multiplying the powers, and $2 \cdot \frac{5}{2} = 5$.

OL In the equation $\log_b x = n$, will x ever be a negative number? Explain. No; sample answer: Because the base of the logarithm must be a positive number other than 1, the argument of a logarithm x can never be negative.

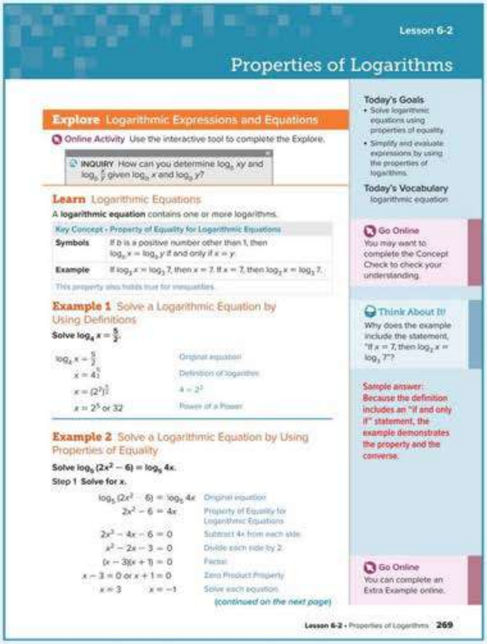
BL Without rewriting 4 as 2^2, how else could you evaluate $4^{\frac{5}{2}}$? Sample answer: I can use the definition of a rational exponent: $4^{\frac{5}{2}} = \sqrt{4^5} = 2^5$.

Example 2 Solve a Logarithmic Equation by Using Properties of Equality

Teaching the Mathematical Practices

1 Check Answers Students should check their answers to identify extraneous solutions.

(continued on the next page)



Lesson 6-2

Properties of Logarithms

Explore Logarithmic Expressions and Equations

Online Activity Use the interactive tool to complete the Explore.

INQUIRY How can you determine $\log_b xy$ and $\log_b \frac{x}{y}$ given $\log_b x$ and $\log_b y$?

Learn Logarithmic Equations

A **logarithmic equation** contains one or more logarithms.

Key Concept • Property of Equality for Logarithmic Equations

Symbols	If b is a positive number other than 1, then $\log_b x = \log_b y$ if and only if $x = y$.
Example	If $\log_3 x = \log_3 7$, then $x = 7$. If $x = 7$, then $\log_3 x = \log_3 7$.

This property also holds true for inequalities.

Example 1 Solve a Logarithmic Equation by Using Definitions

Solve $\log_4 x = \frac{5}{2}$.

$\log_4 x = \frac{5}{2}$ Original equation

$x = 4^{\frac{5}{2}}$ Definition of logarithm

$x = (2^2)^{\frac{5}{2}}$ $4 = 2^2$

$x = 2^5$ or 32 Power of a Power

Example 2 Solve a Logarithmic Equation by Using Properties of Equality

Solve $\log_5 (2x^2 - 6) = \log_5 4x$.

Step 1 Solve for x.

$\log_5 (2x^2 - 6) = \log_5 4x$ Original equation

$2x^2 - 6 = 4x$ Property of Equality for Logarithmic Equations

$2x^2 - 4x - 6 = 0$ Subtract $4x$ from each side.

$x^2 - 2x - 3 = 0$ Divide each side by 2.

$(x - 3)(x + 1) = 0$ Factor.

$x - 3 = 0$ or $x + 1 = 0$ Zero Product Property

$x = 3$ $x = -1$ Solve each equation.

(continued on the next page)

Today's Goals
- Solve logarithmic equations using properties of equality.
- Simplify and evaluate expressions by using the properties of logarithms.

Today's Vocabulary
logarithmic equation

Go Online You may want to complete the Concept Check to check your understanding.

Think About It! Why does the example include the statement, "If $x = 7$, then $\log_3 x = \log_3 7$"?

Sample answer: Because the definition includes an "if and only if" statement, the example demonstrates the property and the converse.

Go Online You can complete an Extra Example online.

Lesson 6-2 • Properties of Logarithms 269

Interactive Presentation

Logarithmic Equations

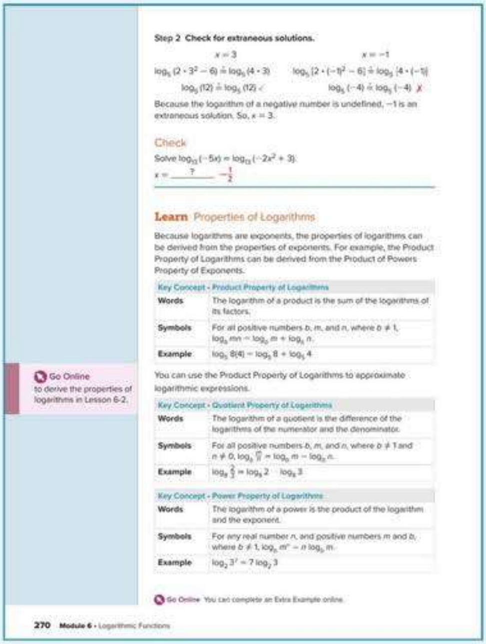
Learn

TYPE

Students explain why the given example includes an “if, then” statement.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Step 2 Check for extraneous solutions.

$x = 3$

$\log_5 (2 \cdot 3^2 - 6) \stackrel{?}{=} \log_5 (4 \cdot 3)$

$\log_5 (12) \stackrel{?}{=} \log_5 (12)$ ✓

$x = -1$

$\log_5 [2 \cdot (-1)^2 - 6] \stackrel{?}{=} \log_5 [4 \cdot (-1)]$

$\log_5 (-4) \stackrel{?}{=} \log_5 (-4)$ ✗

Because the logarithm of a negative number is undefined, −1 is an extraneous solution. So, $x = 3$.

Check

Solve $\log_{13} (-5x) = \log_{13} (-2x^2 + 3)$.

$x =$ ___?___ $-\frac{1}{2}$

Learn Properties of Logarithms

Because logarithms are exponents, the properties of logarithms can be derived from the properties of exponents. For example, the Product Property of Logarithms can be derived from the Product of Powers Property of Exponents.

Key Concept • Product Property of Logarithms	
Words	The logarithm of a product is the sum of the logarithms of its factors.
Symbols	For all positive numbers b, m, and n, where $b \neq 1$, $\log_b mn = \log_b m + \log_b n$.
Example	$\log_5 8(4) = \log_5 8 + \log_5 4$

Go Online to derive the properties of logarithms in Lesson 6-2.

You can use the Product Property of Logarithms to approximate logarithmic expressions.

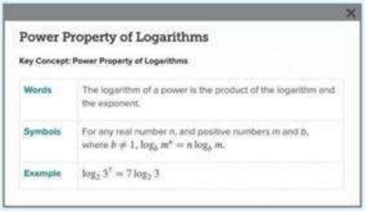
Key Concept • Quotient Property of Logarithms	
Words	The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator.
Symbols	For all positive numbers b, m, and n, where $b \neq 1$ and $n \neq 0$, $\log_b \frac{m}{n} = \log_b m - \log_b n$.
Example	$\log_8 \frac{2}{3} = \log_8 2 - \log_8 3$

Key Concept • Power Property of Logarithms	
Words	The logarithm of a power is the product of the logarithm and the exponent.
Symbols	For any real number n, and positive numbers m and b, where $b \neq 1$, $\log_b m^n = n \log_b m$.
Example	$\log_2 3^7 = 7 \log_2 3$

Go Online You can complete an Extra Example online.

270 Module 6 • Logarithmic Functions

Interactive Presentation



Power Property of Logarithms

Key Concept: Power Property of Logarithms

Words	The logarithm of a power is the product of the logarithm and the exponent.
Symbols	For any real number n, and positive numbers m and b, where $b \neq 1$, $\log_b m^n = n \log_b m$.
Example	$\log_2 3^7 = 7 \log_2 3$

Learn

Questions for Mathematical Discourse

AL What does the Property of Equality for Logarithmic Functions say? Sample answer: If two logarithmic expressions are equal and have the same base, then the arguments of the logarithms are equal.

OL How do we know there may be two solutions to the resulting equation after the property of equality for logarithmic functions is applied? Sample answer: The resulting equation is a quadratic, which means there are two, one, or no solutions.

BL Why can we not evaluate a logarithmic function for a negative value? Sample answer: The base of a logarithm must be a positive number other than 1 and a positive number raised to a power is always positive.

Learn Properties of Logarithms

Objective

Students simplify and evaluate expressions by using the properties of logarithms.

Teaching the Mathematical Practices

3 Make Conjectures Help students to see how the properties of logarithms can be derived from the properties of exponents. Encourage students to make conjectures about how to derive each property and build a logical progression of statements to validate the conjectures.

About the Key Concepts

The properties of logarithms are derived from the properties of exponents. The logarithm of a product is the sum of the logarithms of its factors. The logarithm of a quotient is the difference of the logarithms of the dividend and the divisor. The logarithm of a power is the product of the logarithm and the exponent.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students are having a hard time applying the properties of logarithms,

THEN have students write the exponential properties and logarithmic properties together. Often, seeing the two sets of properties side by side helps students with the similarities and differences.

Logarithms	Exponentials
$\log_b m + \log_b n = \log_b mn$	$b^m \cdot b^n = b^{m+n}$
$\log_b m - \log_b n = \log_b \frac{m}{n}$	$\frac{b^m}{b^n} = b^{m-n}$
$\log_b m^p = p \log_b m$	$(b^m)^n = b^{mn}$

Example 3 Product Property of Logarithms

Teaching the Mathematical Practices

7 Use Structure Students will use the structure of the expression to rewrite and evaluate it.

Questions for Mathematical Discourse

AL Why should we rewrite 405 using a power of 3? The base of the logarithm is 3, so that portion of the logarithm will be able to simplify to an integer.

OL What method can you use to determine the power of 3 and the remaining factor that multiply to 405? Sample answer: Divide 405 by 3 and continue dividing the result by 3 until you cannot divide evenly anymore. The number of times 3 divides evenly is the power of 3, and the last value that cannot be divided by 3 is the other factor.

BL If you know the approximate values for log 5 and log 7, how could you find $\log_3 35$? Y ou could add the values for $\log_3 5$ and $\log_3 7$ because $\log_3 35 = \log_3(5 \cdot 7) = \log_3 5 + \log_3 7$.

Example 4 Quotient Property of Logarithms

Teaching the Mathematical Practices

6 Use Precision Students must calculate accurately and express the numerical answer with a degree of precision appropriate to the problem.

Questions for Mathematical Discourse

AL How can the logarithm of a quotient be written as two logarithms? Subtract the logarithm of the divisor from the logarithm of the dividend.

OL Does the order in which the two logarithms are subtracted matter? Explain. Yes; sample answer: Subtraction and division are not commutative, so the order of the difference must be the same as the quotient.

BL Could $\log_3 5 \approx 1.465$ be used to approximate $\log_3 \frac{18}{10}$? Explain. Y es; sample answer: Because the quotient can be rewritten as $\frac{9}{5}$, the approximation can be found as $\log_3 9 - \log_3 5 = 2 - 1.465$ or 0.535.

Example 5 Power Property of Logarithms

Teaching the Mathematical Practices

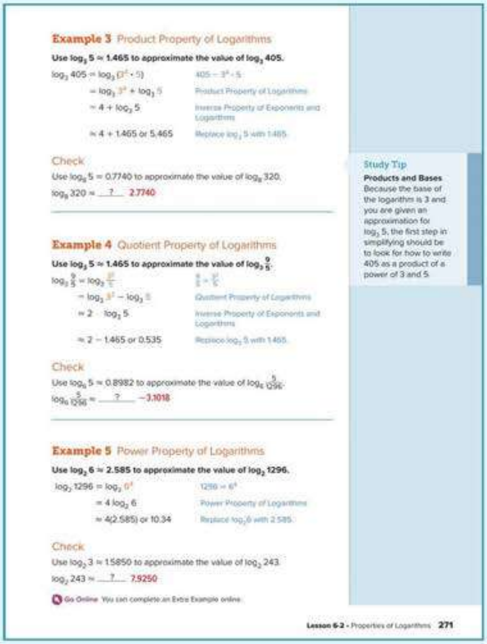
1 Seek Information In Example 5, students must transform the logarithmic expression to reach a solution.

Questions for Mathematical Discourse

AL How can the coefficient of a logarithm be rewritten? as the exponent of the argument of the logarithm

OL Would the approximation be the same for $\log_3 1296$? Explain. No; sample answer: Because the base of the logarithm is different from the one given, we cannot approximate it using the same value.

BL How else could the logarithm be approximated without using the Power Property? Sample answer: The Product Property could be used, which would be the sum of 4 logarithms.



Example 3 Product Property of Logarithms

Use $\log_3 5 \approx 1.465$ to approximate the value of $\log_3 405$.

$\log_3 405 = \log_3 (3^4 \cdot 5)$ $405 = 3^4 \cdot 5$

$= \log_3 3^4 + \log_3 5$ Product Property of Logarithms

$= 4 + \log_3 5$ Inverse Property of Exponents and Logarithms

$\approx 4 + 1.465$ or 5.465 Replace $\log_3 5$ with 1.465

Check

Use $\log_8 5 \approx 0.7740$ to approximate the value of $\log_8 320$.

$\log_8 320 \approx$ __?__ 2.7740

Example 4 Quotient Property of Logarithms

Use $\log_3 5 \approx 1.465$ to approximate the value of $\log_3 \frac{9}{5}$.

$\log_3 \frac{9}{5} = \log_3 \frac{3^2}{5}$ $\frac{9}{5} = \frac{3^2}{5}$

$= \log_3 3^2 - \log_3 5$ Quotient Property of Logarithms

$= 2 - \log_3 5$ Inverse Property of Exponents and Logarithms

$\approx 2 - 1.465$ or 0.535 Replace $\log_3 5$ with 1.465

Check

Use $\log_6 5 \approx 0.8982$ to approximate the value of $\log_6 \frac{5}{1296}$.

$\log_6 \frac{5}{1296} \approx$ __?__ −3.1018

Example 5 Power Property of Logarithms

Use $\log_2 6 \approx 2.585$ to approximate the value of $\log_2 1296$.

$\log_2 1296 = \log_2 6^4$ $1296 = 6^4$

$= 4 \log_2 6$ Power Property of Logarithms

$\approx 4(2.585)$ or 10.34 Replace $\log_2 6$ with 2.585

Check

Use $\log_2 3 \approx 1.5850$ to approximate the value of $\log_2 243$.

$\log_2 243 \approx$ __?__ 7.9250

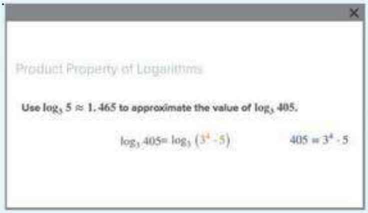
Go Online You can complete an Extra Example online.

Study Tip

Products and Bases Because the base of the logarithm is 3 and you are given an approximation for $\log_3 5$, the first step in simplifying should be to look for how to write 405 as a product of a power of 3 and 5.

Lesson 6-2 • Properties of Logarithms 271

Interactive Presentation



Product Property of Logarithms

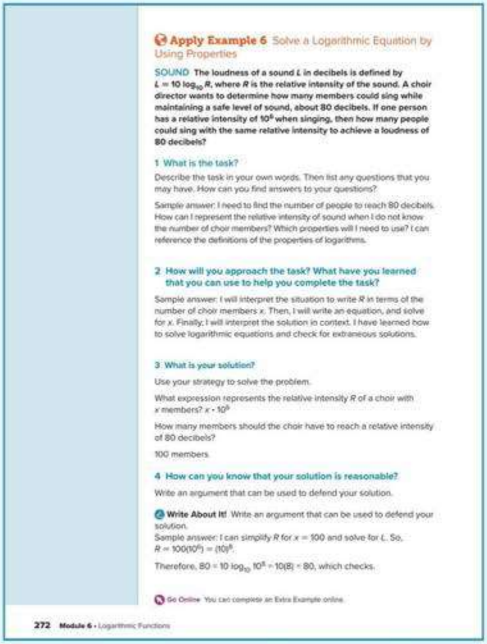
Use $\log_3 5 \approx 1.465$ to approximate the value of $\log_3 405$.

$\log_3 405 = \log_3 (3^4 \cdot 5)$ $405 = 3^4 \cdot 5$

Example 3

TAP

Students move through the steps for applying the Product Property of Logarithms.



Apply Example 6 Solve a Logarithmic Equation by Using Properties

SOUND **The loudness of a sound L in decibels is defined by $L = 10 \log_{10} R$, where R is the relative intensity of the sound. A choir director wants to determine how many members could sing while maintaining a safe level of sound, about 80 decibels. If one person has a relative intensity of 10^6 when singing, then how many people could sing with the same relative intensity to achieve a loudness of 80 decibels?**

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to find the number of people to reach 80 decibels. How can I represent the relative intensity of sound when I do not know the number of choir members? Which properties will I need to use? I can reference the definitions of the properties of logarithms.

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I will interpret the situation to write R in terms of the number of choir members x. Then, I will write an equation, and solve for x. Finally, I will interpret the solution in context. I have learned how to solve logarithmic equations and check for extraneous solutions.

3 What is your solution?

Use your strategy to solve the problem.

What expression represents the relative intensity R of a choir with x members? $x \cdot 10^6$

How many members should the choir have to reach a relative intensity of 80 decibels?

100 members

4 How can you know that your solution is reasonable?

Write an argument that can be used to defend your solution.

Write About It! Write an argument that can be used to defend your solution.
Sample answer: I can simplify R for $x = 100$ and solve for L. So, $R = 100(10^6) = (10)^8$.

Therefore, $80 = 10 \log_{10} 10^8 = 10(8) = 80$, which checks.

Go Online You can complete an Extra Example online.

272 Module 6 • Logarithmic Functions

Interactive Presentation

Solve a Logarithmic Equation by Using Properties

SOUND The loudness of a sound L in decibels is defined by $L = 10 \log_{10} R$, where R is the relative intensity of the sound. A choir director wants to determine how many members could sing while maintaining a safe level of sound, about 80 decibels. If one person has a relative intensity of 10^6 when singing, then how many people could sing with the same relative intensity to achieve a loudness of 80 decibels?

Apply Example 6

TYPE

a|

Students solve a logarithmic equation to answer a real-world question.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Common Error

Many students will raise the given logarithmic value to the power found instead of multiplying by the power. Remind students that exponents can be rewritten as coefficients, and the approximated value should never be raised to the power.

Apply Example 6 Solve a Logarithmic Equation by Using Properties

Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them, 4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- What is the relative intensity of two people singing compared to one?
- What is the loudness of one person singing?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

3 REFLECT AND PRACTICE

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–26
2	exercises that use a variety of skills from this lesson	27–57
3	exercises that emphasize higher-order and critical-thinking skills	58–62

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice Exercises 1–57 odd, 58–62
- Extension: Logarithmic Spirals
- ALEKS Properties of Logarithms

OL

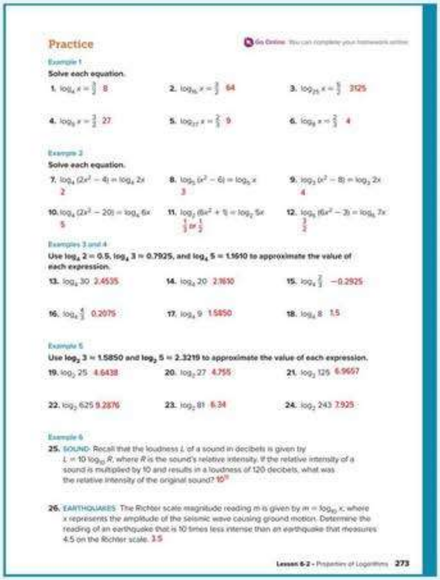
IF students score 66%–89% on the Checks,
THEN assign:

- Practice Exercises 1–61 odd
- Remediation, Review Resources: Multiplication Properties of Exponents
- Personal Tutors
- Extra Examples 1–6
- ALEKS Product, Power, and Quotient Rules

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice Exercises 1–25 odd
- Remediation, Review Resources: Multiplication Properties of Exponents
- *Quick Review Math Handbook:* Properties of Logarithms
- ALEKS Product, Power, and Quotient Rules



Practice

Go Online You can complete your homework online.

Example 1

Solve each equation.

1. $\log_4 x = \frac{3}{2}$ 8
2. $\log_{16} x = \frac{3}{2}$ 64
3. $\log_{25} x = \frac{5}{2}$ 3125
4. $\log_9 x = \frac{3}{2}$ 27
5. $\log_{27} x = \frac{2}{3}$ 9
6. $\log_8 x = \frac{2}{3}$ 4

Example 2

Solve each equation.

7. $\log_4 (2x^2 - 4) = \log_4 2x$ 2
8. $\log_5 (x^2 - 6) = \log_5 x$ 3
9. $\log_3 (x^2 - 8) = \log_3 2x$ 4
10. $\log_4 (2x^2 - 20) = \log_4 6x$ 5
11. $\log_7 (6x^2 + 1) = \log_7 5x$ $\frac{1}{3}$ or $\frac{1}{2}$
12. $\log_9 (6x^2 - 3) = \log_9 7x$ $\frac{3}{2}$

Examples 3 and 4

Use $\log_4 2 = 0.5$, $\log_4 3 \approx 0.7925$, and $\log_4 5 \approx 1.1610$ to approximate the value of each expression.

13. $\log_4 30$ 2.4535
14. $\log_4 20$ 2.1610
15. $\log_4 \frac{2}{3}$ −0.2925
16. $\log_4 \frac{4}{3}$ 0.2075
17. $\log_4 9$ 1.5850
18. $\log_4 8$ 1.5

Example 5

Use $\log_2 3 \approx 1.5850$ and $\log_2 5 \approx 2.3219$ to approximate the value of each expression.

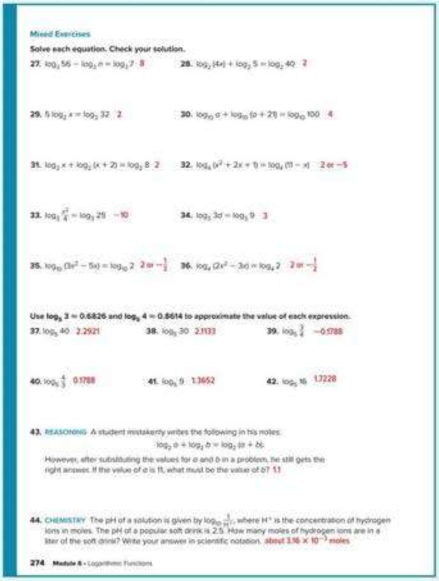
19. $\log_2 25$ 4.6438
20. $\log_2 27$ 4.755
21. $\log_2 125$ 6.9657
22. $\log_2 625$ 9.2876
23. $\log_2 81$ 6.34
24. $\log_2 243$ 7.925

Example 6

25. SOUND Recall that the loudness L of a sound in decibels is given by $L = 10 \log_{10} R$, where R is the sound's relative intensity. If the relative intensity of a sound is multiplied by 10 and results in a loudness of 120 decibels, what was the relative intensity of the original sound? 10^{11}

26. EARTHQUAKES The Richter scale magnitude reading m is given by $m = \log_{10} x$, where x represents the amplitude of the seismic wave causing ground motion. Determine the reading of an earthquake that is 10 times less intense than an earthquake that measures 4.5 on the Richter scale. 3.5

Lesson 6-2 • Properties of Logarithms 273



Mixed Exercises

Solve each equation. Check your solution.

27. $\log_3 56 - \log_3 n = \log_3 7$ 8
28. $\log_2 (4x) + \log_2 5 = \log_2 40$ 2
29. $5 \log_2 x = \log_2 32$ 2
30. $\log_{10} a + \log_{10} (a + 21) = \log_{10} 100$ 4
31. $\log_2 x + \log_2 (x + 2) = \log_2 8$ 2
32. $\log_6 (x^2 + 2x + 1) = \log_6 (11 - x)$ 2 or −5
33. $\log_3 \frac{x^2}{4} = \log_3 25$ −10
34. $\log_3 3d = \log_3 9$ 3
35. $\log_{10} (3x^2 - 5x) = \log_{10} 2$ 2 or $-\frac{1}{3}$
36. $\log_4 (2x^2 - 3x) = \log_4 2$ 2 or $-\frac{1}{2}$

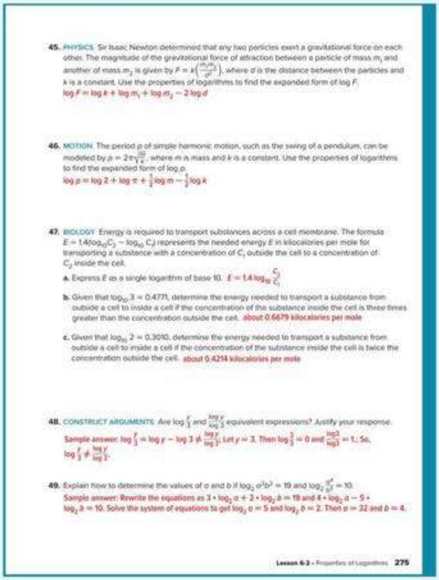
Use $\log_5 3 \approx 0.6826$ and $\log_5 4 \approx 0.8614$ to approximate the value of each expression.

37. $\log_5 40$ 2.2921
38. $\log_5 30$ 2.1133
39. $\log_5 \frac{3}{4}$ −0.1788
40. $\log_5 \frac{4}{3}$ 0.1788
41. $\log_5 9$ 1.3652
42. $\log_5 16$ 1.7228

43. REASONING A student mistakenly writes the following in his notes:
$$\log_2 a + \log_2 b = \log_2 (a + b).$$
However, after substituting the values for a and b in a problem, he still gets the right answer. If the value of a is 11, what must be the value of b? 1.1

44. CHEMISTRY The pH of a solution is given by $\log_{10} \frac{1}{H^+}$, where H^+ is the concentration of hydrogen ions in moles. The pH of a popular soft drink is 2.5. How many moles of hydrogen ions are in a liter of the soft drink? Write your answer in scientific notation. about 3.16×10^{-3} moles

274 Module 6 • Logarithmic Functions



45. PHYSICS Sir Isaac Newton determined that any two particles exert a gravitational force on each other. The magnitude of the gravitational force of attraction between a particle of mass m_1 and another of mass m_2 is given by $F = k\left(\frac{m_1 m_2}{d^2}\right)$, where d is the distance between the particles and k is a constant. Use the properties of logarithms to find the expanded form of log F.
$\log F = \log k + \log m_1 + \log m_2 - 2 \log d$

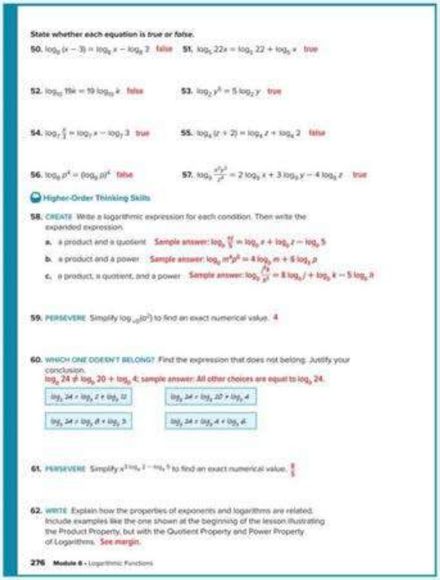
46. MOTION The period p of simple harmonic motion, such as the swing of a pendulum, can be modeled by $p = 2\pi\sqrt{\frac{m}{k}}$, where m is mass and k is a constant. Use the properties of logarithms to find the expanded form of log p.
$\log p = \log 2 + \log \pi + \frac{1}{2}\log m - \frac{1}{2}\log k$

47. BIOLOGY Energy is required to transport substances across a cell membrane. The formula $E = 1.4(\log_{10} C_2 - \log_{10} C_1)$ represents the needed energy E in kilocalories per mole for transporting a substance with a concentration of C_1 outside the cell to a concentration of C_2 inside the cell.
a. Express E as a single logarithm of base 10. $E = 1.4 \log_{10}\frac{C_2}{C_1}$
b. Given that $\log_{10} 3 \approx 0.4771$, determine the energy needed to transport a substance from outside a cell to inside a cell if the concentration of the substance inside the cell is three times greater than the concentration outside the cell. about 0.6679 kilocalories per mole
c. Given that $\log_{10} 2 \approx 0.3010$, determine the energy needed to transport a substance from outside a cell to inside a cell if the concentration of the substance inside the cell is twice the concentration outside the cell. about 0.4214 kilocalories per mole

48. CONSTRUCT ARGUMENTS Are $\log \frac{y}{3}$ and $\frac{\log y}{\log 3}$ equivalent expressions? Justify your response.
Sample answer: $\log \frac{y}{3} = \log y - \log 3 \neq \frac{\log y}{\log 3}$; Let $y = 3$. Then $\log \frac{3}{3} = 0$ and $\frac{\log 3}{\log 3} = 1$; So, $\log \frac{y}{3} \neq \frac{\log y}{\log 3}$.

49. Explain how to determine the values of a and b if $\log_2 a^3b^2 = 19$ and $\log_2 \frac{a^4}{b^5} = 10$.
Sample answer: Rewrite the equations as $3 \cdot \log_2 a + 2 \cdot \log_2 b = 19$ and $4 \cdot \log_2 a - 5 \cdot \log_2 b = 10$. Solve the system of equations to get $\log_2 a = 5$ and $\log_2 b = 2$. Then $a = 32$ and $b = 4$.

Lesson 6-2 • Properties of Logarithms 275



State whether each equation is *true* or *false*.

50. $\log_8 (x - 3) = \log_8 x - \log_8 3$ false
51. $\log_5 22x = \log_5 22 + \log_5 x$ true
52. $\log_{10} 19k = 19 \log_{10} k$ false
53. $\log_2 y^5 = 5 \log_2 y$ true
54. $\log_7 \frac{x}{3} = \log_7 x - \log_7 3$ true
55. $\log_4 (z + 2) = \log_4 z + \log_4 2$ false
56. $\log_8 p^4 = (\log_8 p)^4$ false
57. $\log_9 \frac{x^2y^3}{z^4} = 2 \log_9 x + 3 \log_9 y - 4 \log_9 z$ true

Higher-Order Thinking Skills

58. CREATE Write a logarithmic expression for each condition. Then write the expanded expression.
a. a product and a quotient Sample answer: $\log_b \frac{xz}{5} = \log_b x + \log_b z - \log_b 5$
b. a product and a power Sample answer: $\log_b m^4p^6 = 4 \log_b m + 6 \log_b p$
c. a product, a quotient, and a power Sample answer: $\log_b \frac{j^8k}{h^5} = 8 \log_b j + \log_b k - 5 \log_b h$

59. PERSEVERE Simplify $\log_{\sqrt{a}} (a^2)$ to find an exact numerical value. 4

60. WHICH ONE DOESN'T BELONG? Find the expression that does not belong. Justify your conclusion.
$\log_2 24 \neq \log_2 20 + \log_2 4$; sample answer: All other choices are equal to $\log_2 24$.

$\log_2 24 = \log_2 2 + \log_2 12$	$\log_2 24 = \log_2 20 + \log_2 4$
$\log_2 24 = \log_2 8 + \log_2 3$	$\log_2 24 = \log_2 4 + \log_2 6$

61. PERSEVERE Simplify $x^{3\log_x 2 - \log_x 5}$ to find an exact numerical value. $\frac{8}{5}$

62. WRITE Explain how the properties of exponents and logarithms are related. Include examples like the one shown at the beginning of the lesson illustrating the Product Property, but with the Quotient Property and Power Property of Logarithms. See margin.

276 Module 6 • Logarithmic Functions

Answers

62. Since logarithms are exponents, the properties of logarithms are similar to the properties of exponents. The Product Property states that to multiply two powers that have the same base, add the exponents. Similarly, the logarithm of a product is the sum of the logarithms of its factors. The Quotient Property states that to divide two powers that have the same base, subtract their exponents. Similarly the logarithm of a quotient is the difference of the logarithms of the numerator and the denominator. The Power Property states that to find the power of a power, multiply the exponents. Similarly, the logarithm of a power is the product of the logarithm and the exponent. Answers should include the following.

- Quotient Property:

$\log_2\left(\frac{32}{8}\right) = \log_2\left(\frac{2^5}{2^3}\right)$ Replace 32 with 2^5 and 8 with 2^3.
$= \log_2 2^{(5-3)}$ Quotient of Powers
$= 5 - 3$, or 2 Inverse Property of Exponents and Logarithms
$\log_2 32 - \log_2 8 = \log_2 2^5 - \log_2 2^3$ Replace 32 with 2^5 and 8 with 2^3.
$= 5 - 3$, or 2 Inverse Property of Exponents and Logarithms

So, $\log_2\left(\frac{32}{8}\right) = \log_2 32 - \log_2 8$.

- Power Property:

$\log_3 9^4 = \log_3 (3^2)^4$ Replace 9 with 3^2.
$= \log_3 3^{(2 \cdot 4)}$ Power of a Power
$= 2 \cdot 4$, or 8 Inverse Property of Exponents and Logarithms
$4 \log_3 9 = \log_3 9 \cdot 4$ Commutative Property of Multiplication
$= \log_3 (3^2) \cdot 4$ Replace 9 with 3^2.
$= 2 \cdot 4$, or 8 Inverse Property of Exponents and Logarithms

So, $\log_3 9^4 = 4 \log_3 9$.

- The Product of Powers Property and Product Property of Logarithms both involve the addition of exponents, since logarithms are exponents.

Lesson 6-3

Common Logarithms

A.REI.11, F.LE.4

LESSON GOAL

Students simplify logarithmic expressions and solve exponential equations by using common logarithms.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Common Logarithms

- Find Common Logarithms by Using Technology
- Solve a Logarithmic Equation by Using Exponential Form
- Solve an Exponential Equation by Using Logarithms
- Solve an Exponential Inequality by Using Logarithms

Change of Base Formula

- Change of Base Formula
- Use the Change of Base Formula

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Division Properties of Exponents	●	●		●

Language Development Handbook

Assign page 32 of the *Language Development Handbook* to help your students build mathematical language related to using common logarithms.

ELL You can use the tips and suggestions on page T32 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Algebra, Functions

Standards for Mathematical Content:

A.REI.11 Explain why the x-coordinates of the points where the graphs of the equation $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases were $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

F.LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.
5 Use appropriate tools strategically.
7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students simplified logarithmic expressions and solved logarithmic equations.
A.CED.1

Now
Students simplify logarithmic expressions and solve exponential equations by using common logarithms.
A.REI.11, F.LE.4

Next
Students will simplify logarithmic expressions and solve exponential equations by using natural logarithms.
A.SSE.2, F.LE.4

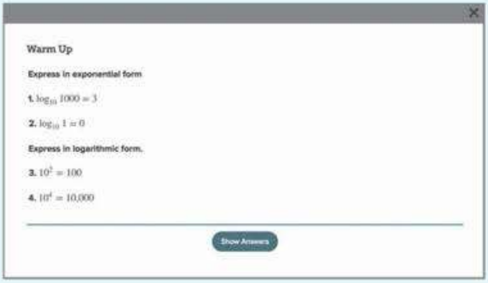
Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students extend their understanding of logarithms to include common logarithms. They build fluency and apply their understanding by solving real-world problems related to common logarithms.

Interactive Presentation



Warm Up

Express in exponential form

1. $\log_{10} 1000 = 3$

2. $\log_{10} 1 = 0$

Express in logarithmic form.

3. $10^2 = 100$

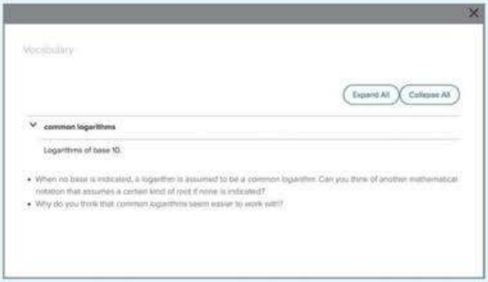
4. $10^4 = 10,000$

Warm Up

Launch the Lesson

Earth's atmosphere affects how well stars can be seen in the night sky. To find the brightness, or magnitude, of a star without atmospheric effects, astronomers use the formula $M = m + 5 + 5 \log_{10} p$, where M is the star's magnitude without Earth's atmosphere, m is the star's apparent magnitude as seen from Earth, and p is parallax, which is the difference in direction of the star as seen from two separate points. Logarithms like this one with base 10, called *common logarithms*, are programmed in calculators and computers for easy use.

Launch the Lesson



Vocabulary

common logarithms

Logarithms of base 10.

- When no base is indicated, a logarithm is assumed to be a common logarithm. Can you think of another mathematical notation that assumes a certain kind of root if none is indicated?
- Why do you think that common logarithms seem easier to work with?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- using properties of exponents

Answers:

1. $10^3 = 1000$
2. $10^0 = 1$
3. $\log_{10} 100 = 2$
4. $\log_{10} 10,000 = 4$

Launch the Lesson

Teaching the Mathematical Practices

2 Attend to Quantities Point out that it is important to note the meaning for the quantities used in the formula for the magnitude of a star without atmospheric effects. When there are several variables in a formula, it is important to be aware of what each one represents so that you can substitute and solve for the correct variables.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?,* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using the vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then discuss the questions below with the class.

Mathematical Background

Base 10 logarithms are called common logarithms. When the base of a logarithm is not shown, the base is assumed to be 10. When you have a logarithmic expression of any base, evaluate it using the Change of Base Formula to translate the expression into one that involves common logarithms.

Learn Common Logarithms

Objective

Students solve exponential equations by using common logarithms.

Teaching the Mathematical Practices

7 Use Structure Students will use structure of logarithms and how they relate to exponential expressions to complete the table.

DIFFERENTIATE

Enrichment Activity BL

Without a calculator, evaluate the following:

log 100,000 5   log 0.001 −3   log 0.0000001 −7

Example 1 Find Common Logarithms by Using Technology

Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in Example 1, students will need to use a calculator.

Questions for Mathematical Discourse

AL How do you know this is a common logarithm? Sample answer: There is no base written on the log.

OL Which is closer to 8, $10^{0.9}$ or $10^{0.9031}$? $10^{0.9031}$ is closer to 8 than $10^{0.9}$.

BL The approximation of log 8 is three times the approximation of what common logarithm? Explain. log 2; Sample answer: Because $2^3 = 8$, the approximation of log 8 will be three times the approximation of log 2.

Example 2 Solve a Logarithmic Equation by Using Exponential Form

Teaching the Mathematical Practices

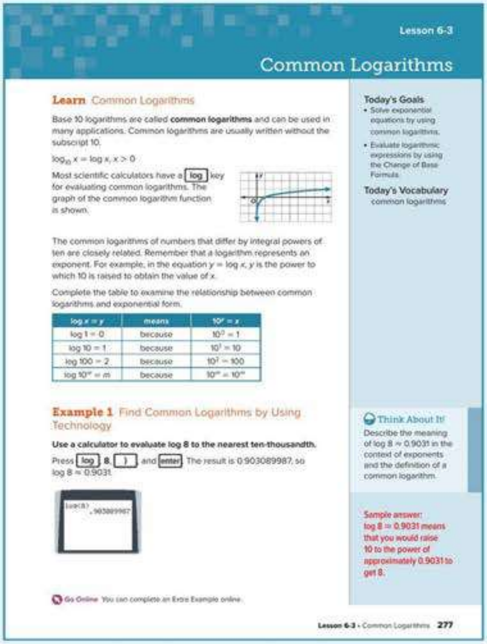
4 Make Assumptions In the Talk About It! feature, have students explain an assumption or approximation that was made to find the amount the energy released.

Questions for Mathematical Discourse

AL Why is the answer approximated as 2×10^{23}? Sample answer: This form allows you to write approximations for very large or small numbers. Writing this approximation in standard form would result in writing 23 places.

OL What is the horizontal shift to the graph of the parent function for the original equation? $\frac{11.8}{1.5} \approx 7.867$ to the left

BL How would you solve the example if you wanted to find the energy released for multiple earthquakes? Sample answer: Solve for *E* as a function of *M* before substituting values of *M*.



Lesson 6-3

Common Logarithms

Learn Common Logarithms

Base 10 logarithms are called **common logarithms** and can be used in many applications. Common logarithms are usually written without the subscript 10.

$\log_{10} x = \log x, x > 0$

Most scientific calculators have a log key for evaluating common logarithms. The graph of the common logarithm function is shown.

The common logarithms of numbers that differ by integral powers of ten are closely related. Remember that a logarithm represents an exponent. For example, in the equation $y = \log x$, y is the power to which 10 is raised to obtain the value of x.

Complete the table to examine the relationship between common logarithms and exponential form.

$\log x = y$	means	$10^y = x$
$\log 1 = 0$	because	$10^0 = 1$
$\log 10 = 1$	because	$10^1 = 10$
$\log 100 = 2$	because	$10^2 = 100$
$\log 10^m = m$	because	$10^m = 10^m$

Example 1 Find Common Logarithms by Using Technology

Use a calculator to evaluate log 8 to the nearest ten-thousandth.

Press log, 8,), and enter. The result is 0.903089987, so log 8 ≈ 0.9031.

Go Online You can complete an Extra Example online.

Today's Goals
- Solve exponential equations by using common logarithms.
- Evaluate logarithmic expressions by using the Change of Base Formula.

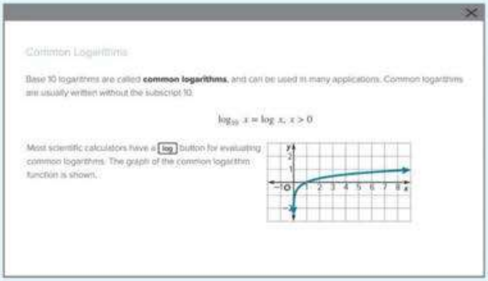
Today's Vocabulary
common logarithms

Think About It!
Describe the meaning of log 8 ≈ 0.9031 in the context of exponents and the definition of a common logarithm.

Sample answer: log 8 ≈ 0.9031 means that you would raise 10 to the power of approximately 0.9031 to get 8.

Lesson 6-3 • Common Logarithms 277

Interactive Presentation



Common Logarithms

Base 10 logarithms are called **common logarithms**, and can be used in many applications. Common logarithms are usually written without the subscript 10.

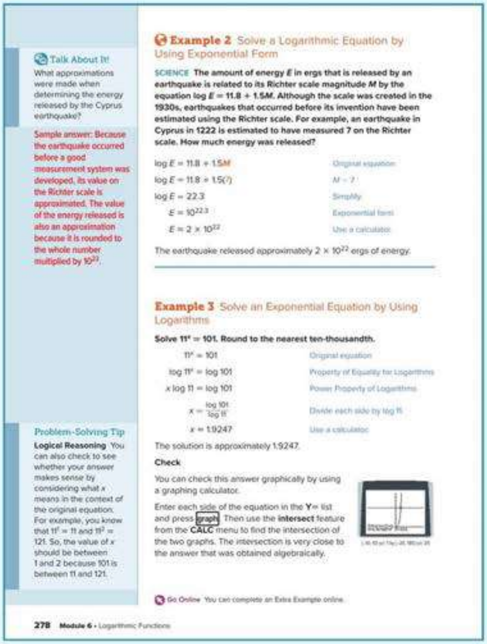
$\log_{10} x = \log x, x > 0$

Most scientific calculators have a log button for evaluating common logarithms. The graph of the common logarithm function is shown.

Learn

TYPE

Students complete the table by evaluating certain common logarithms.



Talk About It!

What approximations were made when determining the energy released by the Cyprus earthquake?

Sample answer: Because the earthquake occurred before a good measurement system was developed, its value on the Richter scale is approximated. The value of the energy released is also an approximation because it is rounded to the whole number multiplied by 10^{22}.

Example 2 Solve a Logarithmic Equation by Using Exponential Form

SCIENCE **The amount of energy E in ergs that is released by an earthquake is related to its Richter scale magnitude M by the equation $\log E = 11.8 + 1.5M$. Although the scale was created in the 1930s, earthquakes that occurred before its invention have been estimated using the Richter scale. For example, an earthquake in Cyprus in 1222 is estimated to have measured 7 on the Richter scale. How much energy was released?**

$\log E = 11.8 + 1.5M$ Original equation

$\log E = 11.8 + 1.5(7)$ $M = 7$

$\log E = 22.3$ Simplify.

$E = 10^{22.3}$ Exponential form

$E \approx 2 \times 10^{22}$ Use a calculator.

The earthquake released approximately 2×10^{22} ergs of energy.

Example 3 Solve an Exponential Equation by Using Logarithms

Solve $11^x = 101$. Round to the nearest ten-thousandth.

$11^x = 101$ Original equation

$\log 11^x = \log 101$ Property of Equality for Logarithms

$x \log 11 = \log 101$ Power Property of Logarithms

$x = \frac{\log 101}{\log 11}$ Divide each side by log 11.

$x \approx 1.9247$ Use a calculator.

The solution is approximately 1.9247.

Check

You can check this answer graphically by using a graphing calculator.

Enter each side of the equation in the **Y=** list and press graph. Then use the **intersect** feature from the **CALC** menu to find the intersection of the two graphs. The intersection is very close to the answer that was obtained algebraically.

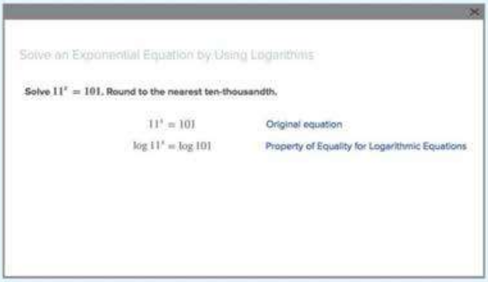
Problem-Solving Tip

Logical Reasoning You can also check to see whether your answer makes sense by considering what x means in the context of the original equation. For example, you know that $11^1 = 11$ and $11^2 = 121$. So, the value of x should be between 1 and 2 because 101 is between 11 and 121.

Go Online You can complete an Extra Example online.

278 Module 6 • Logarithmic Functions

Interactive Presentation



Solve an Exponential Equation by Using Logarithms

Solve $11^x = 101$. Round to the nearest ten-thousandth.

$11^x = 101$ Original equation

$\log 11^x = \log 101$ Property of Equality for Logarithmic Equations

Example 3

TAP

Students move through the steps to solve the exponential equation by using logarithms.

Example 3 Solve an Exponential Equation by Using Logarithms

Teaching the Mathematical Practices

2 Different Properties Mathematically proficient students look for different ways to solve problems. In Example 3, students check their solution by solving the equation by graphing. Guide students to work through both methods.

Questions for Mathematical Discourse

AL Why is the Power Property of Logarithms applied? This property is used to write x as the coefficient of the logarithm so that it can be isolated in the next step.

OL Why is the Property of Equality for Logarithms used? This allows you to take the logarithm of each side of the equation so that you can use the Power Property of Logarithms to rewrite x as the coefficient.

BL Is $\frac{\log 101}{\log 11}$ equivalent to $\log \frac{101}{11}$? Explain. No; sample answer: The logarithm of a quotient is the difference of the logarithms.

Things to Remember

Students may forget to apply the Power Property of Logarithms when solving an exponential equation. Remind students that inverse operations are used to solve equations and that logarithms and exponential functions are inverses.

Example 4 Solve an Exponential Inequality by Using Logarithms

Teaching the Mathematical Practices

1 Check Answers Students need to check their answer and should ask themselves whether their answer makes sense.

Questions for Mathematical Discourse

AL Why is the Distributive Property used on $(2y - 5) \log 6$? The terms are separated so that the one with y can be grouped with the other y-term.

OL When dividing both sides by the logarithmic expression $3\log 5 - 2\log 6$, how can we use the properties of logarithms to determine of the related equation whether the value is negative so that the inequality symbol should be reversed? Sample answer: You can rewrite the coefficients as exponents in the logarithms and rewrite the difference of logarithms as one logarithm of a quotient. 5^3 is greater than 6^2, making the quotient greater than 1, thus the logarithm is positive.

BL How do you know the solution will be a decimal before even solving the inequality? Sample answer: Because the exponential bases cannot be written as the same power, the exponents will not be integers.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | 3 APPLICATION

Learn Change of Base Formula

Objective

Students evaluate logarithmic expressions by using the Change of Base Formula.

Teaching the Mathematical Practices

1 Seek Information Guide students to see how the Change of Base Formula can be used to transform logarithmic expressions. Point out that this skill is important when evaluating and graphing logarithms by using a calculator.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students are struggling to apply the Change of Base Formula, **THEN** tell students that they can rewrite any base logarithm as the log of the argument divided by the log of the base. Give students the hint that the base of the log goes in the denominator.

Example 5 Change of Base Formula

Teaching the Mathematical Practices

7 Use Structure Help students use the structure of the Change of Base Formula in Example 5 to evaluate the logarithmic expression.

Questions for Mathematical Discourse

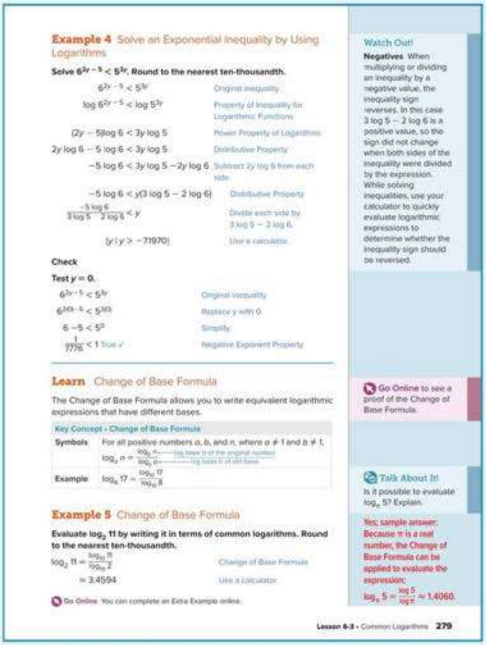
AL What is a way to remember which number goes in the denominator when applying the Change of Base Formula? Sample answer: The base is subscript and thus lower, so it goes in the denominator.

OL Why do we use a common log when using the Change of Base Formula? Sample answer: Using a common logarithm makes it possible to evaluate the expression by using a calculator.

BL Given $\log_3 11 \approx 2.1827$ and $\log_3 2 \approx 0.6309$, how could the Change of Base Formula be used to evaluate the expression without using common logarithms? You can use the formula to write $\log_2 11$ as $\frac{\log_3 11}{\log_3 2} \approx \frac{2.1827}{0.6309} \approx 3.4594$.

Common Error

Some students may not understand the difference between the quotient of two logarithms, like the Change of Base formula, and the logarithm of a quotient. They may try to evaluate an expression like $\log_2 11$ as $\log \frac{11}{2}$, which will always be incorrect. Remind students that the formula takes the logarithm of both the argument and the base, not the quotient of them.



Example 4 Solve an Exponential Inequality by Using Logarithms

Solve $6^{2y-5} < 5^{3y}$. Round to the nearest ten-thousandth.

$6^{2y-5} < 5^{3y}$ Original inequality

$\log 6^{2y-5} < \log 5^{3y}$ Property of Inequality for Logarithmic Functions

$(2y-5)\log 6 < 3y \log 5$ Power Property of Logarithms

$2y \log 6 - 5 \log 6 < 3y \log 5$ Distributive Property

$-5 \log 6 < 3y \log 5 - 2y \log 6$ Subtract 2y log 6 from each side.

$-5 \log 6 < y(3 \log 5 - 2 \log 6)$ Distributive Property

$\frac{-5 \log 6}{3 \log 5 - 2 \log 6} < y$ Divide each side by $3 \log 5 - 2 \log 6$.

$\{y \mid y > -7.1970\}$ Use a calculator.

Check

Test $y = 0$.

$6^{2y-5} < 5^{3y}$ Original inequality

$6^{2(0)-5} < 5^{3(0)}$ Replace y with 0.

$6^{-5} < 5^{0}$ Simplify.

$\frac{1}{7776} < 1$ True ✓ Negative Exponent Property

Learn Change of Base Formula

The Change of Base Formula allows you to write equivalent logarithmic expressions that have different bases.

Key Concept • Change of Base Formula

Symbols	For all positive numbers a, b, and n, where $a \neq 1$ and $b \neq 1$, $\log_a n = \frac{\log_b n}{\log_b a}$ ← log base b of the original number ← log base b of old base
Example	$\log_8 17 = \frac{\log_{10} 17}{\log_{10} 8}$

Example 5 Change of Base Formula

Evaluate $\log_2 11$ by writing it in terms of common logarithms. Round to the nearest ten-thousandth.

$\log_2 11 = \frac{\log_{10} 11}{\log_{10} 2}$ Change of Base Formula

≈ 3.4594 Use a calculator.

Go Online You can complete an Extra Example online.

Watch Out!

Negatives When multiplying or dividing an inequality by a negative value, the inequality sign reverses. In this case $3 \log 5 - 2 \log 6$ is a positive value, so the sign did not change when both sides of the inequality were divided by the expression. While solving inequalities, use your calculator to quickly evaluate logarithmic expressions to determine whether the inequality sign should be reversed.

Go Online to see a proof of the Change of Base Formula.

Talk About It!

Is it possible to evaluate $\log_\pi 5$? Explain.

Yes; sample answer: Because π is a real number, the Change of Base Formula can be applied to evaluate the expression: $\log_\pi 5 = \frac{\log 5}{\log \pi} \approx 1.4060$.

Lesson 6-3 • Common Logarithms 279

Interactive Presentation

Change of Base Formula

The Change of Base Formula allows you to write equivalent logarithmic expressions that have different bases. Because the common logarithm can be evaluated using a calculator, the Change of Base Formula is often used to rewrite logarithmic expressions to have base 10.

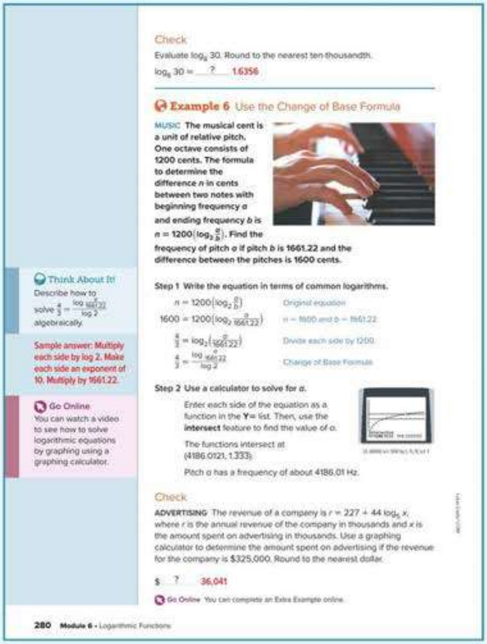
Key Concept: Change of Base Formula

Symbols	For all positive numbers a, b, and n, where $a \neq 1$ and $b \neq 1$, $\log_a n = \frac{\log_b n}{\log_b a}$ ← log base b of the original number ← log base b of old base
Example	$\log_8 17 = \frac{\log_{10} 17}{\log_{10} 8}$

Learn

CHECK

Students complete the Check online to determine whether they are ready to move on.



Check

Evaluate $\log_8 30$. Round to the nearest ten-thousandth.

$\log_8 30 =$ ___?___ 1.6356

Example 6 Use the Change of Base Formula

MUSIC **The musical cent is a unit of relative pitch. One octave consists of 1200 cents. The formula to determine the difference n in cents between two notes with beginning frequency a and ending frequency b is $n = 1200\left(\log_2 \frac{a}{b}\right)$. Find the frequency of pitch a if pitch b is 1661.22 and the difference between the pitches is 1600 cents.**

Step 1 Write the equation in terms of common logarithms.

$n = 1200\left(\log_2 \frac{a}{b}\right)$ Original equation

$1600 = 1200\left(\log_2 \frac{a}{1661.22}\right)$ $n = 1600$ and $b = 1661.22$

$\frac{4}{3} = \log_2\left(\frac{a}{1661.22}\right)$ Divide each side by 1200.

$\frac{4}{3} = \frac{\log \frac{a}{1661.22}}{\log 2}$ Change of Base Formula

Step 2 Use a calculator to solve for a.

Enter each side of the equation as a function in the **Y=** list. Then, use the **intersect** feature to find the value of a.

The functions intersect at (4186.0121, 1.333).

Pitch a has a frequency of about 4186.01 Hz.

Think About It!

Describe how to solve $\frac{4}{3} = \frac{\log \frac{a}{1661.22}}{\log 2}$ algebraically.

Sample answer: Multiply each side by log 2. Make each side an exponent of 10. Multiply by 1661.22.

Go Online

You can watch a video to see how to solve logarithmic equations by graphing using a graphing calculator.

Check

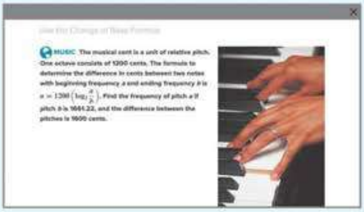
ADVERTISING The revenue of a company is $r = 227 + 44 \log_5 x$, where r is the annual revenue of the company in thousands and x is the amount spent on advertising in thousands. Use a graphing calculator to determine the amount spent on advertising if the revenue for the company is \$325,000. Round to the nearest dollar.

\$ ___?___ 36,041

Go Online You can complete an Extra Example online.

280 **Module 6** • Logarithmic Functions

Interactive Presentation



MUSIC The musical cent is a unit of relative pitch. One octave consists of 1200 cents. The formula to determine the difference in cents between two notes with beginning frequency a and ending frequency b is $n = 1200\left(\log_2 \frac{a}{b}\right)$. Find the frequency of pitch a if pitch b is 1661.22, and the difference between the pitches is 1600 cents.

Example 6

TYPE

Students describe how to solve a given logarithmic equation algebraically.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 6 Use the Change of Base Formula

Teaching the Mathematical Practices

5 Analyze Graphs Guide students to analyze the graph they have generated using a graphing calculator. Point out that to see the entire graph and the point of intersection, students may need to adjust the viewing window.

Questions for Mathematical Discourse

AL Why must you use the Change of Base Formula to solve the equation? You need logarithms with base 10 to be able to use a calculator.

OL When solving the equation, does the coefficient of 1200 have to be divided out first? Explain. The Change of Base Formula can still be applied; the *y*-scale of the graph will change, but the solution will be the same.

BL How could you use the Power Property of Logarithms to solve the example algebraically? Starting at the end of Step 1, multiply each side by log 2. $\frac{4}{3} \log 2 = \log 2^{\frac{4}{3}}$. Both logarithms have the same base, so $2^{\frac{4}{3}} = \frac{a}{1661.22}$. Multiply both sides by 1661.22 to solve for a.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Essential Question Follow-Up

Students have begun learning about common logarithms.

Ask:

Why are common logarithms useful in the real world? Sample answer: Because common logarithms allow the use of technology to evaluate or solve, and they are prevalent in science such as chemistry.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–28
2	exercises that use a variety of skills from this lesson	29–50
3	exercises that emphasize higher-order and critical-thinking skills	51–54

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **BL**
THEN assign:

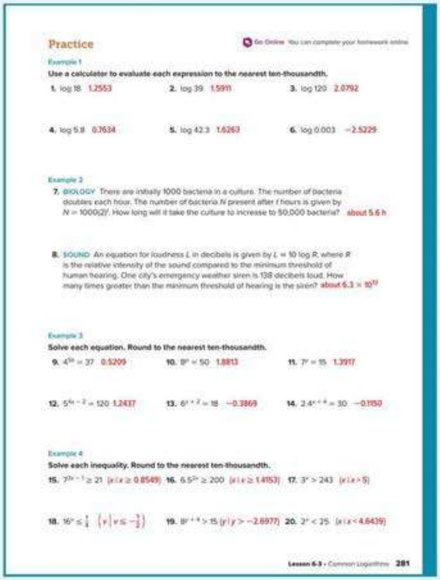
- Practice Exercises 1–49 odd, 51–54
- Extension: The Slide Rule
- ALEKS® Logarithmic and Exponential Equations

IF students score 66%–89% on the Checks, **OL**
THEN assign:

- Practice Exercises 1–53 odd
- Remediation, Review Resources: Division Properties of Exponents
- Personal Tutors
- Extra Examples 1–6
- ALEKS® Product, Power, and Quotient Rules

IF students score 65% or less on the Checks, **AL**
THEN assign:

- Practice Exercises 1–27 odd
- Remediation, Review Resources: Division Properties of Exponents
- *Quick Review Math Handbook*: Common Logarithms
- ALEKS® Product, Power, and Quotient Rules



Practice

Go Online You can complete your homework online.

Example 1

Use a calculator to evaluate each expression to the nearest ten-thousandth.

1. log 18 1.2553
2. log 39 1.5911
3. log 120 2.0792
4. log 5.8 0.7634
5. log 42.3 1.6263
6. log 0.003 −2.5229

Example 2

7. BIOLOGY There are initially 1000 bacteria in a culture. The number of bacteria doubles each hour. The number of bacteria N present after t hours is given by $N = 1000(2)^t$. How long will it take the culture to increase to 50,000 bacteria? about 5.6 h

8. SOUND An equation for loudness L in decibels is given by $L = 10 \log R$, where R is the relative intensity of the sound compared to the minimum threshold of human hearing. One city's emergency weather siren is 138 decibels loud. How many times greater than the minimum threshold of hearing is the siren? about 6.3×10^{13}

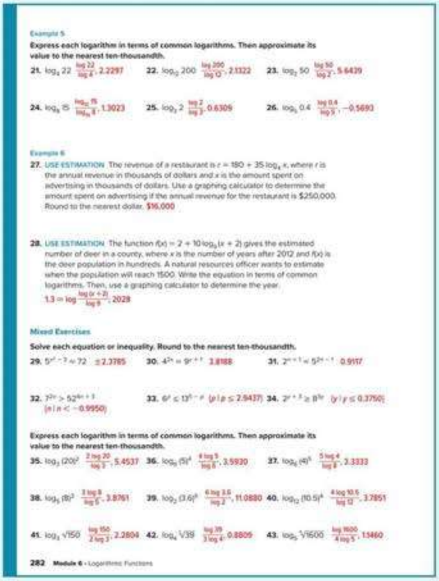
Example 3

Solve each equation. Round to the nearest ten-thousandth.

Example 4

Solve each inequality. Round to the nearest ten-thousandth.

Lesson 6-3 • Common Logarithms 281



Example 5

Express each logarithm in terms of common logarithms. Then approximate its value to the nearest ten-thousandth.

Example 6

27. USE ESTIMATION The revenue of a restaurant is $r = 180 + 35 \log_4 x$, where r is the annual revenue in thousands of dollars and x is the amount spent on advertising in thousands of dollars. Use a graphing calculator to determine the amount spent on advertising if the annual revenue for the restaurant is $250,000. Round to the nearest dollar. $16,000

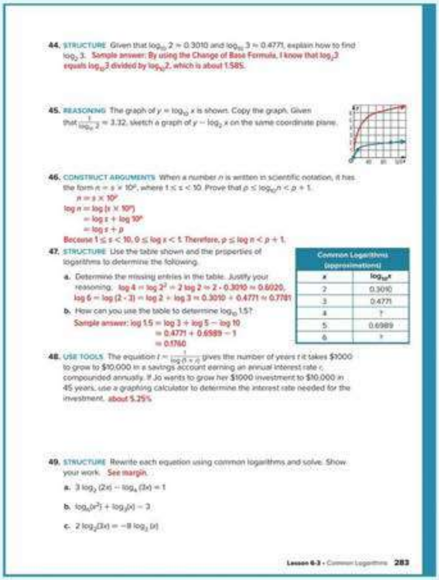
28. USE ESTIMATION The function $f(x) = 2 + 10 \log_8 (x + 2)$ gives the estimated number of deer in a county, where x is the number of years after 2012 and $f(x)$ is the deer population in hundreds. A natural resources officer wants to estimate when the population will reach 1500. Write the equation in terms of common logarithms. Then, use a graphing calculator to determine the year.

Mixed Exercises

Solve each equation or inequality. Round to the nearest ten-thousandth.

Express each logarithm in terms of common logarithms. Then approximate its value to the nearest ten-thousandth.

282 Module 6 • Logarithmic Functions



44. STRUCTURE Given that $\log_{10} 2 \approx 0.3010$ and $\log_{10} 3 \approx 0.4771$, explain how to find $\log_2 3$. Sample answer: By using the Change of Base Formula, I know that $\log_2 3$ equals $\log_{10} 3$ divided by $\log_{10} 2$, which is about 1.585.

45. REASONING The graph of $y = \log_{10} x$ is shown. Copy the graph. Given that $\frac{1}{\log_{10} 2} \approx 3.32$, sketch a graph of $y = \log_2 x$ on the same coordinate plane.

46. CONSTRUCT ARGUMENTS When a number n is written in scientific notation, it has the form $n = s \times 10^p$, where $1 \leq s < 10$. Prove that $p \leq \log_{10} n < p + 1$.

$n = s \times 10^p$

$\log n = \log (s \times 10^p)$

$= \log s + \log 10^p$

$= \log s + p$

Because $1 \leq s < 10$, $0 \leq \log s < 1$. Therefore, $p \leq \log n < p + 1$.

47. STRUCTURE Use the table shown and the properties of logarithms to determine the following.

a. Determine the missing entries in the table. Justify your reasoning. $\log 4 = \log 2^2 = 2 \log 2 \approx 2 \cdot 0.3010 \approx 0.6020$, $\log 6 = \log (2 \cdot 3) = \log 2 + \log 3 \approx 0.3010 + 0.4771 \approx 0.7781$

b. How can you use the table to determine $\log_{10} 1.5$? Sample answer: $\log 1.5 = \log 3 + \log 5 - \log 10$
$= 0.4771 + 0.6989 - 1$
$= 0.1760$

Common Logarithms (approximations)

x	$\log_{10} x$
2	0.3010
3	0.4771
4	?
5	0.6989
6	?

48. USE TOOLS The equation $t = \frac{1}{\log (1 + r)}$ gives the number of years t it takes \$1000 to grow to \$10,000 in a savings account earning an annual interest rate r, compounded annually. If Jo wants to grow her \$1000 investment to \$10,000 in 45 years, use a graphing calculator to determine the interest rate needed for the investment. about 5.25%

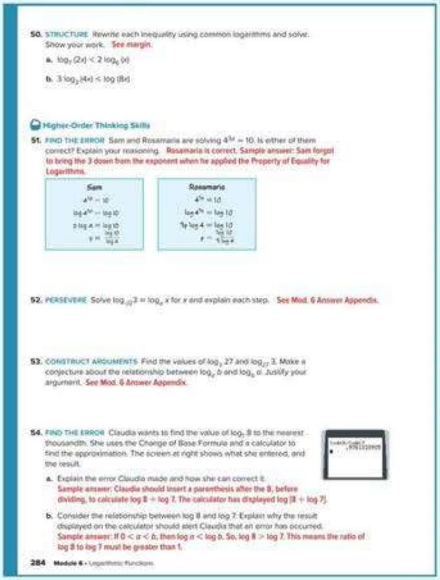
49. STRUCTURE Rewrite each equation using common logarithms and solve. Show your work. See margin.

a. $3 \log_2 (2x) - \log_4 (3x) = 1$

b. $\log_6(x^2) + \log_3(x) = 3$

c. $2 \log_2(3x) = -8 \log_3 (x)$

Lesson 6-3 • Common Logarithms 283



50. STRUCTURE Rewrite each inequality using common logarithms and solve. Show your work. See margin.

a. $\log_7 (2x) < 2 \log_6 (x)$

b. $3 \log_2 (4x) \leq \log (8x)$

Higher-Order Thinking Skills

51. FIND THE ERROR Sam and Rosamaria are solving $4^{3p} = 10$. Is either of them correct? Explain your reasoning. Rosamaria is correct. Sample answer: Sam forgot to bring the 3 down from the exponent when he applied the Property of Equality for Logarithms.

Sam

$4^{3p} = 10$

$\log 4^{3p} = \log 10$

$p \log 4 = \log 10$

$p = \frac{\log 10}{\log 4}$

Rosamaria

$4^{3p} = 10$

$\log 4^{3p} = \log 10$

$3p \log 4 = \log 10$

$p = \frac{\log 10}{3 \log 4}$

52. PERSEVERE Solve $\log_{\sqrt{2}} 3 = \log_a x$ for x and explain each step. See Mod. 6 Answer Appendix.

53. CONSTRUCT ARGUMENTS Find the values of $\log_3 27$ and $\log_{27} 3$. Make a conjecture about the relationship between $\log_a b$ and $\log_b a$. Justify your argument. See Mod. 6 Answer Appendix.

54. FIND THE ERROR Claudia wants to find the value of $\log_7 8$ to the nearest thousandth. She uses the Change of Base Formula and a calculator to find the approximation. The screen at right shows what she entered, and the result.

a. Explain the error Claudia made and how she can correct it. Sample answer: Claudia should insert a parenthesis after the 8, before dividing, to calculate $\log 8 \div \log 7$. The calculator has displayed $\log (8 \div \log 7)$.

b. Consider the relationship between log 8 and log 7. Explain why the result displayed on the calculator should alert Claudia that an error has occurred. Sample answer: If $0 < a < b$, then $\log a < \log b$. So, $\log 8 > \log 7$. This means the ratio of log 8 to log 7 must be greater than 1.

284 Module 6 • Logarithmic Functions

Answers

49a. $3 \log_2 (2x) - \log_4 (3x) = 1$

$$3\left[\frac{\log (2x)}{\log (2)}\right] - \frac{\log (3x)}{\log (4)} = 1$$

$$\frac{3 \log (2x)}{\log (2)} - \frac{\log (3x)}{2 \log (2)} = 1$$

$$\frac{6 \log (2x) - \log (3x)}{2 \log (2)} = 1$$

$6 (\log (2) + \log (x)) - \log (3) - \log (x) = 2 \log (2)$

$5 \log (x) = \log (3) - 4 \log (2)$

So, $x \approx 0.715$.

49b. $\log_6 (x^2) + \log_3 (x) = 3$

$$\frac{\log (x^2)}{\log (6)} + \frac{\log (x)}{\log (3)} = 3$$

$$\frac{\log (3) \log (x^2) + \log (6) \log (x)}{\log (3) \log (6)} = 3$$

$2 \log (3) \log (x) + \log (6) \log (x) = 3 \log (3) \log (6)$

$\log (x) [2 \log (3) + \log (6)] = 3 \log (3) \log (6)$

So, $x \approx 4.39$.

49c. $2 \log_2 (3x) = -8 \log_3 (x)$

$$2\left[\frac{\log (3x)}{\log (2)}\right] = -8\left[\frac{\log (x)}{\log (3)}\right]$$

$2 \log (3) \log (3x) = -8 \log (2) \log (x)$

$2 \log (3) [\log (3) + \log (x)] = -8 \log (2) \log (x)$

$\log (x) [2 \log (3) + 8 \log (2)] = -2[\log (3)]^2$

So, $x \approx 0.732$.

50a. $\log_7 (2x) < 2 \log_6 (x)$

$$\frac{\log (2x)}{\log (7)} < \frac{2 \log (x)}{\log (6)}$$

$$\frac{\log (2x)}{\log (7)} - \frac{2 \log (x)}{\log (6)} < 0$$

$$\frac{\log (6) \log (2x) - 2 \log (7) \log (x)}{\log (6) \log (7)} < 0$$

$\log (6) \log (2x) - 2 \log (7) \log (x) < 0$

$\log (6) [\log (2) + \log (x)] - 2 \log (7) \log (x) < 0$

$\log (6) \log (x) - 2 \log (7) \log (x) < -\log (2) \log (6)$

$\log (x) [\log (6) - 2 \log (7)] < -\log (2) \log (6)$

$$\log (x) > \frac{-\log (2) \log (6)}{\log (6) - 2 \log (7)}$$

$$x > 10^{\frac{-\log (2) \log (6)}{\log (6) - 2 \log (7)}}$$

So, x is approximately greater than 1.81.

50b. $3 \log_2 (4x) \leq \log (8x)$

$$\frac{3 \log (4x)}{\log (2)} \leq \log (8x)$$

$$\frac{3 \log (4x)}{\log (2)} - \log (8x) \leq 0$$

$3 \log (4x) - \log (2) \log (8x) \leq 0$

$3 [\log (4) + \log (x)] - \log (2) [\log (8) + \log (x)] \leq 0$

$3 \log (x) - \log (2) \log (x) \leq \log (2) \log (8) - 3 \log (4)$

$\log (x)[3 - \log (2)] \leq \log (2) \log (8) - 3 \log (4)$

$$\log (x) \leq \frac{\log (2) \log (8) - 3 \log (4)}{3 - \log (2)}$$

$$x \leq 10^{\frac{\log (2) \log (8) - 3 \log (4)}{3 - \log (2)}}$$

So, x is approximately less than or equal to 0.27.

Lesson 6-4

Natural Logarithms

A.SSE.2, F.LE.4

LESSON GOAL

Students simplify logarithmic expressions and solve exponential equations by using natural logarithms.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Simplifying Expressions with Natural Logarithms

- Write Equivalent Logarithmic Equations
- Write Equivalent Exponential Equations
- Simplify Logarithmic Expressions

Solving Exponential Equations by Using Natural Logarithms

- Solve Exponential Equations with Base *e*
- Solve Natural Logarithmic Equations
- Apply Functions with Base *e*

Explore: Using a Scatter Plot to Analyze Data

Develop:

- Examine Logarithmic Data

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Common Logarithms	●	●		●
Extension: Approximations for π and *e*		●	●	●

Language Development Handbook

Assign page 33 of the *Language Development Handbook* to help your students build mathematical language related to using natural logarithms.

ELL You can use the tips and suggestions on page T33 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Algebra, Functions

Standards for Mathematical Content:

A.SSE.2 Use the structure of an expression to identify ways to rewrite it.

F.LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where *a*, *c*, and *d* are numbers and the base *b* is 2, 10, or *e*; evaluate the logarithm using technology.

Standards for Mathematical Practice:

4 Model with mathematics.

5 Use appropriate tools strategically.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous

Students simplified logarithmic expressions and solved exponential equations by using common logarithms.

A.REI.11, F.LE.4

Now

Students simplify logarithmic expressions and solve exponential equations by using natural logarithms.

A.SSE.2, F.LE.4

Next

Students will write exponential growth and decay equations and solve them by using logarithms.

A.CED.1, F.LE.4

Rigor

The Three Pillars of Rigor

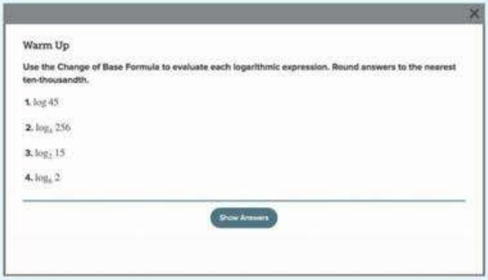
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students extend their understanding of logarithms to include natural logarithms. They build fluency and apply their understanding by solving real-world problems related to natural logarithms.

Mathematical Background

Logarithm with base *e* is called a natural logarithm, written as $\log_e x$ or $\ln x$. The natural logarithmic function, $y = \ln x$, is the inverse of the natural base exponential function $y = e^x$.

Interactive Presentation



Warm Up

Use the Change of Base Formula to evaluate each logarithmic expression. Round answers to the nearest ten-thousandth.

1. $\log 45$
2. $\log_4 256$
3. $\log_2 15$
4. $\log_6 2$

Warm Up

Launch the Lesson

You may not think about it, but new words or phrases are constantly being added to your language, while others are no longer used. In 2017, the Merriam-Webster Dictionary added *macaron, binge-watch, airball,* and *photobomb,* just to name a few. Language is constantly evolving.

Launch the Lesson

Vocabulary

natural base exponential function

An exponential function with base e, written as $y = e^x$.

natural logarithm

The inverse of the natural base exponential function, most often abbreviated as $\ln x$.

- What is the difference between a common logarithms and a natural logarithm?
- What is the relationship between $f(x) = e^x$ and $f(x) = \ln x$?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- using the Change of Base Formula

Answers:

1. 1.6532
2. 4
3. 3.9069
4. 0.3869

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will read about a real-world problem that can be represented by a natural logarithmic function. Once students have completed the lesson, encourage them to use the formula to find the percentage of common words that would exist between two languages given some number of years since they split.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Using a Scatter Plot to Analyze Data

Objective

Students collect and use data to explore logarithmic functions.

Teaching the Mathematical Practices

4 Use Tools Point out that to complete the Explore, students will need to use a table and graph to represent the important quantities and relationships in the situation.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will be presented with an Inquiry Question to answer at the end of the activity. They will watch a time-lapse video of cooling coffee and record data. Students will work through different exercises, including generating a scatter plot. Then students will answer the Inquiry Question.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

(continued on the next page)

Interactive Presentation

Using a Scatter Plot to Analyze Data

Explore

Explore

TYPE	Students enter different time/temperature pairs from the time-lapse video.
eTOOLS	Students graph the data and use the function tool to determine the function that best models the data.
TYPE	Students discuss which function they chose and why.

Interactive Presentation

INQUIRY How can you use a scatter plot to determine the type of function that best models a set of data?

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Using a Scatter Plot to Analyze Data (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- When collecting data, is it best to get all of the coordinates from the beginning or end of the video or across the entire time interval? Explain. across the entire time interval; Sample answer: To best see what function models the data, the data should represent the entire time interval, not just one part.
- Based on this situation, what real-world situation do you think would follow an exponential model? Explain. heating coffee; Sample answer: Heating coffee would be the inverse of cooling coffee and an exponential function is the inverse of a logarithmic function.

Inquiry

How can you use a scatter plot to determine the type of function that best models a set of data? Sample answer: By plotting the data, you can determine which functions approximate the curve made by the points.

Learn Simplifying Expressions with Natural Logarithms

Objective

Students simplify expressions with natural logarithms.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write and explain their reasoning. Point out that they should use clear definitions when answering the question in the Think About It! feature.

Common Misconception

Some students may think that “ln” is just a regular logarithm and may use the common log button on the calculator. Remind students that a natural logarithm is base e, not base 10.

DIFFERENTIATE

Enrichment Activity BL

To check if students understand the inverse relationship of e and ln, have them evaluate the following without a calculator:

$\ln e^{-5}$ -5 $e^{\ln 2x}$ $2x$ $3 \ln e$ 3

Example 1 Write Equivalent Logarithmic Equations

Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of the exponential equations to rewrite them in logarithmic form.

Questions for Mathematical Discourse

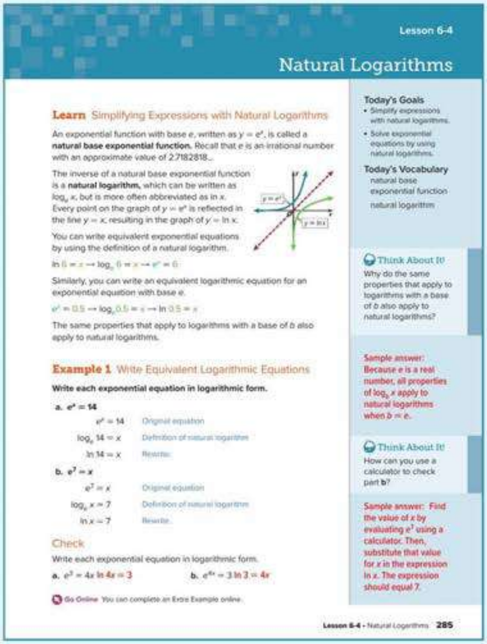
AL What is the difference between $\log_e x$ and $\ln x$? They are the same logarithm, $\ln x$ is the typical abbreviation for a logarithm of base e.

OL What is ln 1? Explain. $\ln 1 = 0$ because $e^0 = 1$.

BL What is another approach you could take to write the equation in logarithmic form? Sample answer: Take the natural logarithm of each side then simplify the natural logarithm of the exponential term, which brings down the exponent.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 6-4

Natural Logarithms

Learn Simplifying Expressions with Natural Logarithms

An exponential function with base e, written as $y = e^x$, is called a **natural base exponential function.** Recall that e is an irrational number with an approximate value of 2.7182818...

The inverse of a natural base exponential function is a **natural logarithm,** which can be written as $\log_e x$, but is more often abbreviated as $\ln x$. Every point on the graph of $y = e^x$ is reflected in the line $y = x$, resulting in the graph of $y = \ln x$.

You can write equivalent exponential equations by using the definition of a natural logarithm.

$\ln 6 = x \rightarrow \log_e 6 = x \rightarrow e^x = 6$

Similarly, you can write an equivalent logarithmic equation for an exponential equation with base e.

$e^x = 0.5 \rightarrow \log_e 0.5 = x \rightarrow \ln 0.5 = x$

The same properties that apply to logarithms with a base of b also apply to natural logarithms.

Example 1 Write Equivalent Logarithmic Equations

Write each exponential equation in logarithmic form.

a. $e^x = 14$

$e^x = 14$ Original equation

$\log_e 14 = x$ Definition of natural logarithm

$\ln 14 = x$ Rewrite.

b. $e^7 = x$

$e^7 = x$ Original equation

$\log_e x = 7$ Definition of natural logarithm

$\ln x = 7$ Rewrite.

Check

Write each exponential equation in logarithmic form.

a. $e^3 = 4x$ $\ln 4x = 3$ **b.** $e^{4x} = 3$ $\ln 3 = 4x$

Go Online You can complete an Extra Example online.

Today's Goals

- Simplify expressions with natural logarithms.
- Solve exponential equations by using natural logarithms.

Today's Vocabulary

natural base exponential function

natural logarithm

Think About It!

Why do the same properties that apply to logarithms with a base of b also apply to natural logarithms?

Sample answer: Because e is a real number, all properties of $\log_b x$ apply to natural logarithms when $b = e$.

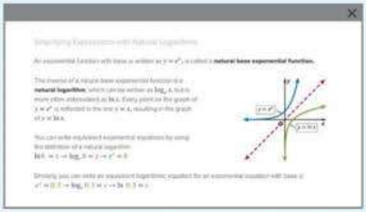
Think About It!

How can you use a calculator to check part **b**?

Sample answer: Find the value of x by evaluating e^7 using a calculator. Then, substitute that value for x in the expression $\ln x$. The expression should equal 7.

Lesson 6-4 • Natural Logarithms 285

Interactive Presentation

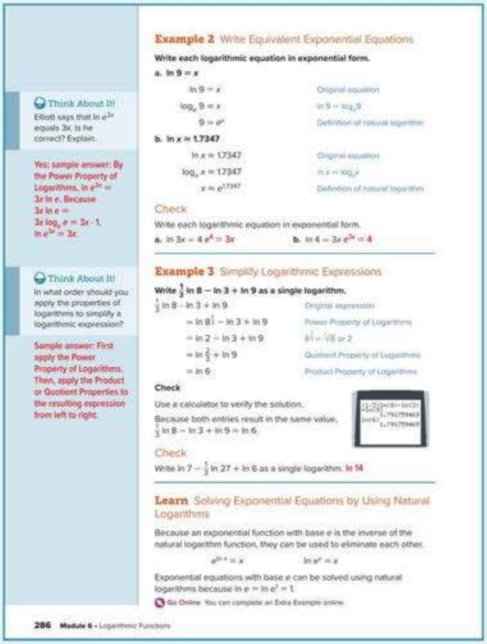


Learn

TYPE

a|

Students explain why the properties that apply to logarithms with a base of b also apply to natural logarithms.



Think About It! Elliott says that $\ln e^{3x}$ equals $3x$. Is he correct? Explain.

Yes; sample answer: By the Power Property of Logarithms, $\ln e^{3x} = 3x \ln e$. Because $3x \ln e = 3x \log_e e = 3x \cdot 1$, $\ln e^{3x} = 3x$.

Example 2 Write Equivalent Exponential Equations

Write each logarithmic equation in exponential form.

a. $\ln 9 = x$

$\ln 9 = x$	Original equation
$\log_e 9 = x$	$\ln 9 = \log_e 9$
$9 = e^x$	Definition of natural logarithm

b. $\ln x \approx 1.7347$

$\ln x \approx 1.7347$	Original equation
$\log_e x \approx 1.7347$	$\ln x = \log_e x$
$x \approx e^{1.7347}$	Definition of natural logarithm

Check

Write each logarithmic equation in exponential form.

a. $\ln 3x = 4$ $e^4 = 3x$ b. $\ln 4 = 3x$ $e^{3x} = 4$

Think About It! In what order should you apply the properties of logarithms to simplify a logarithmic expression?

Sample answer: First apply the Power Property of Logarithms. Then, apply the Product or Quotient Properties to the resulting expression from left to right.

Example 3 Simplify Logarithmic Expressions

Write $\frac{1}{3} \ln 8 - \ln 3 + \ln 9$ as a single logarithm.

$\frac{1}{3} \ln 8 - \ln 3 + \ln 9$	Original expression
$= \ln 8^{\frac{1}{3}} - \ln 3 + \ln 9$	Power Property of Logarithms
$= \ln 2 - \ln 3 + \ln 9$	$8^{\frac{1}{3}} = \sqrt[3]{8}$ or 2
$= \ln \frac{2}{3} + \ln 9$	Quotient Property of Logarithms
$= \ln 6$	Product Property of Logarithms

Check

Use a calculator to verify the solution.

Because both entries result in the same value, $\frac{1}{3} \ln 8 - \ln 3 + \ln 9 = \ln 6$.

Check

Write $\ln 7 - \frac{1}{3} \ln 27 + \ln 6$ as a single logarithm. ln 14

Learn Solving Exponential Equations by Using Natural Logarithms

Because an exponential function with base e is the inverse of the natural logarithm function, they can be used to eliminate each other.

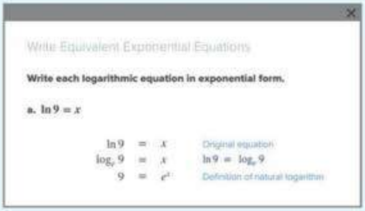
$e^{\ln x} = x$ $\ln e^x = x$

Exponential equations with base e can be solved using natural logarithms because $\ln e = \ln e^1 = 1$.

Go Online You can complete an Extra Example online.

286 Module 6 • Logarithmic Functions

Interactive Presentation



Write Equivalent Exponential Equations

Write each logarithmic equation in exponential form.

a. $\ln 9 = x$

$\ln 9 = x$		Original equation
$\log_e 9 = x$		$\ln 9 = \log_e 9$
$9 = e^x$		Definition of natural logarithm

Example 2

TYPE

Students enter the values necessary to rewrite a logarithmic equation as an exponential equation.

TYPE

Students determine if a student's claim is correct.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Write Equivalent Exponential Equations

Teaching the Mathematical Practices

1 Understand the Approaches of Others The Think About It! feature asks students to justify a student's reasoning.

Questions for Mathematical Discourse

AL In **part b**, what happens if you substitute $e^{1.7346}$ for x in the original equation? You get a true statement 1.7347 ≈ 1.7347.

OL What is $e^{\ln x}$? x

BL What is the base of the logarithm of x that is equivalent to $\ln \frac{x}{2}$? e^{2}

Example 3 Simplify Logarithmic Expressions

Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to check their solution in Example 3, students will need to use a calculator.

Questions for Mathematical Discourse

AL What step could you show in between $\ln \frac{2}{3}$ ln 9 and ln 6? $\ln \left(\frac{2}{3} \times 9\right)$

OL Why must the Power Property be used first when simplifying logarithmic expressions? Sample answer: In the order of operations, evaluating exponents comes before multiplying and dividing. The Power Property deals with exponents.

BL Suppose a student first simplified the given expression as $\ln 2 + \ln \frac{1}{3} + \ln 9$. What did the student do? Sample answer: The student used the subtraction as a coefficient of −1 and rewrote −ln 3 as $\ln 3^{-1}$.

Learn Solving Exponential Equations by Using Natural Logarithms

Objective

Students solve exponential equations by using natural logarithms.

Teaching the Mathematical Practices

2 Make Sense of Quantities Mathematically proficient students need to be able to make sense of quantities and their relationships. Guide students to see the relationship between the number e and a natural logarithm.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students are having difficulty solving exponential equations using natural logarithms,

THEN work through several simple equations using inverse functions like $x^2 = 4$, $\sqrt{x} = 3$, and $\log x = -1$. This will help students recognize the use of the inverse function and they will be more successful when solving equations involving *e*.

Example 4 Solve Exponential Equations with Base e

Teaching the Mathematical Practices

5 Use Mathematical Tools To solve the problem in Example, students will need to use a calculator to evaluate a natural logarithm.

Questions for Mathematical Discourse

- AL What is the exact solution? ln (11) − 3
- OL How can we check our solution using a calculator? Sample answer: Substitute −0.6021 into the original equation for *x* and see if the expression is approximately −14.
- BL How could you check the solution algebraically? Sample answer: substitute ln (11) − 3 for *x* in the original equation and simplify to see if you get −14 = −14.

Example 5 Solve Natural Logarithmic Equations

Teaching the Mathematical Practices

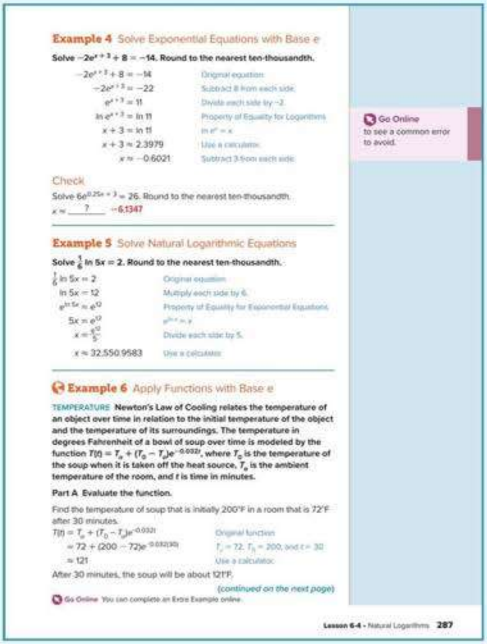
7 Use Structure Help students use the structure of logarithms to solve a natural logarithmic equation.

Questions for Mathematical Discourse

- AL What is the inverse function of a natural logarithm? an exponential function with base *e*
- OL Could you have used the Power Property of Logarithms for the $\frac{1}{6}$ coefficient and arrived at the correct answer? Yes, but the method shown in the example is simpler.
- BL Why do we not need to check for extraneous solutions after solving the natural logarithmic equation? Sample answer: An extraneous solution occurs when the argument of a logarithm is negative. Because *x* is positive, the argument will be positive and the solution will not be extraneous.

Common Error

Students tend to forget ln is the logarithm with base *e*. They may try to use a different base with the Property of Equality for Exponential Equations. No other base is ever the inverse of ln, so reinforce to students that when dealing with ln they should always use *e*.



Example 4 Solve Exponential Equations with Base e

Solve $-2e^{x+3} + 8 = -14$. Round to the nearest ten-thousandth.

$-2e^{x+3} + 8 = -14$	Original equation
$-2e^{x+3} = -22$	Subtract 8 from each side.
$e^{x+3} = 11$	Divide each side by −2.
$\ln e^{x+3} = \ln 11$	Property of Equality for Logarithms
$x + 3 = \ln 11$	$\ln e^x = x$
$x + 3 \approx 2.3979$	Use a calculator.
$x \approx -0.6021$	Subtract 3 from each side.

Check

Solve $6e^{0.25x+3} = 26$. Round to the nearest ten-thousandth.

$x \approx$ ___?___ −6.1347

Example 5 Solve Natural Logarithmic Equations

Solve $\frac{1}{6}\ln 5x = 2$. Round to the nearest ten-thousandth.

$\frac{1}{6}\ln 5x = 2$	Original equation
$\ln 5x = 12$	Multiply each side by 6.
$e^{\ln 5x} = e^{12}$	Property of Equality for Exponential Equations
$5x = e^{12}$	$e^{\ln x} = x$
$x = \frac{e^{12}}{5}$	Divide each side by 5.
$x \approx 32{,}550.9583$	Use a calculator.

Example 6 Apply Functions with Base e

TEMPERATURE **Newton's Law of Cooling relates the temperature of an object over time in relation to the initial temperature of the object and the temperature of its surroundings. The temperature in degrees Fahrenheit of a bowl of soup over time is modeled by the function $T(t) = T_a + (T_0 - T_a)e^{-0.032t}$, where T_0 is the temperature of the soup when it is taken off the heat source, T_a is the ambient temperature of the room, and t is time in minutes.**

Part A **Evaluate the function.**

Find the temperature of soup that is initially 200°F in a room that is 72°F after 30 minutes.

$T(t) = T_a + (T_0 - T_a)e^{-0.032t}$	Original function
$= 72 + (200 - 72)e^{-0.032(30)}$	$T_a = 72$, $T_0 = 200$, and $t = 30$
≈ 121	Use a calculator.

After 30 minutes, the soup will be about 121°F.

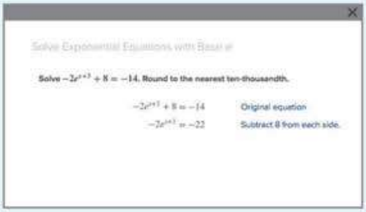
(continued on the next page)

Go Online You can complete an Extra Example online.

Go Online to see a common error to avoid.

Lesson 6-4 • Natural Logarithms 287

Interactive Presentation



Solve Exponential Equations with Base e

Solve $-2e^{x+3} + 8 = -14$. Round to the nearest ten-thousandth.

$-2e^{x+3} + 8 = -14$	Original equation
$-2e^{x+3} = -22$	Subtract 8 from each side.

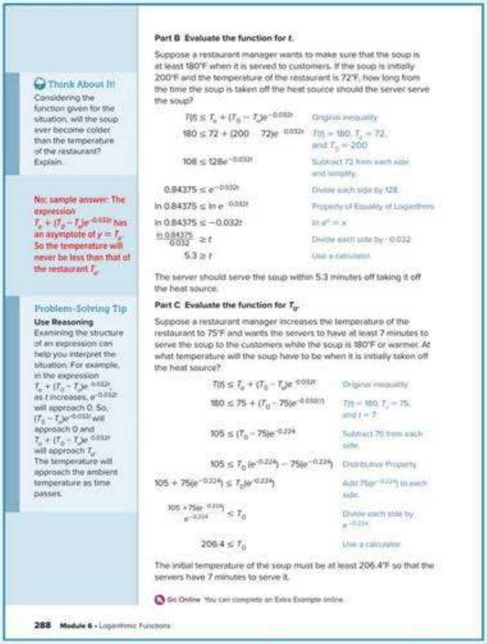
Example 4

TAP

Students move through the steps to solve an exponential equation with base *e*.

TYPE

Students describe the error that was made when the equation was solved.



Think About It! Considering the function given for the situation, will the soup ever become colder than the temperature of the restaurant? Explain.

No; sample answer: The expression $T_a + (T_0 - T_a)e^{-0.032t}$ has an asymptote of $y = T_a$. So the temperature will never be less than that of the restaurant T_a.

Problem-Solving Tip

Use Reasoning Examining the structure of an expression can help you interpret the situation. For example, in the expression $T_a + (T_0 - T_a)e^{-0.032t}$, as t increases, $e^{-0.032t}$ will approach 0. So, $(T_0 - T_a)e^{-0.032t}$ will approach 0 and $T_a + (T_0 - T_a)e^{-0.032t}$ will approach T_a. The temperature will approach the ambient temperature as time passes.

Part B Evaluate the function for *t*.

Suppose a restaurant manager wants to make sure that the soup is at least 180°F when it is served to customers. If the soup is initially 200°F and the temperature of the restaurant is 72°F, how long from the time the soup is taken off the heat source should the server serve the soup?

$T(t) \le T_a + (T_0 - T_a)e^{-0.032t}$ Original inequality

$180 \le 72 + (200 - 72)e^{-0.032t}$ $T(t) = 180$, $T_a = 72$, and $T_0 = 200$

$108 \le 128e^{-0.032t}$ Subtract 72 from each side and simplify.

$0.84375 \le e^{-0.032t}$ Divide each side by 128.

$\ln 0.84375 \le \ln e^{-0.032t}$ Property of Equality of Logarithms

$\ln 0.84375 \le -0.032t$ $\ln e^x = x$

$\frac{\ln 0.84375}{-0.032} \ge t$ Divide each side by −0.032.

$5.3 \ge t$ Use a calculator.

The server should serve the soup within 5.3 minutes off taking it off the heat source.

Part C Evaluate the function for T_0.

Suppose a restaurant manager increases the temperature of the restaurant to 75°F and wants the servers to have at least 7 minutes to serve the soup to the customers while the soup is 180°F or warmer. At what temperature will the soup have to be when it is initially taken off the heat source?

$T(t) \le T_a + (T_0 - T_a)e^{-0.032t}$ Original inequality

$180 \le 75 + (T_0 - 75)e^{-0.032(7)}$ $T(t) = 180$, $T_a = 75$, and $t = 7$.

$105 \le (T_0 - 75)e^{-0.224}$ Subtract 75 from each side.

$105 \le T_0(e^{-0.224}) - 75(e^{-0.224})$ Distributive Property

$105 + 75(e^{-0.224}) \le T_0(e^{-0.224})$ Add $75(e^{-0.224})$ to each side.

$\frac{105 + 75(e^{-0.224})}{e^{-0.224}} \le T_0$ Divide each side by $e^{-0.224}$.

$206.4 \le T_0$ Use a calculator.

The initial temperature of the soup must be at least 206.4°F so that the servers have 7 minutes to serve it.

Go Online You can complete an Extra Example online.

288 Module 6 • Logarithmic Functions

Interactive Presentation

Apply Functions with Base e

TEMPERATURE Newton's Law of Cooling relates the temperature of an object over time in relation to the initial temperature of the object and the temperature of its surroundings. The temperature in degrees Fahrenheit of a bowl of soup over time is modeled by the function where $T(t) = T_a + (T_0 - T_a)e^{-0.032t}$, T_0 is the temperature of the soup when it is taken off the heat source, T_a is the ambient temperature of the room, and t is time in minutes.

Example 6

TAP

Students move through the steps to evaluate the function for different variables.

TYPE

Students answer a question to determine whether they can interpret an asymptote in context.

Example 6 Apply Functions with Base *e*

Teaching the Mathematical Practices

4 Interpret Mathematical Results. In Example 6, students should interpret their mathematical results in the context of the problem. Point out that the solution to each part of the problem may have a different unit and meaning.

Questions for Mathematical Discourse

AL In **Part B**, why do you subtract 72 from each side rather than add the 72 to the coefficient 128? Sample answer: Y ou cannot add the 72 to 128 because the 128 has an exponential term, which means 72 is not a like term.

OL In **Part C**, how do you know you do not need to reverse the inequality symbol when you divide by $e^{-0.224}$? *e* is a positive number, and the exponent does not make the term negative.

BL What would happen if you tried to find a value of *t* for a serving temperature greater than the initial temperature? The result would be a negative value for *t*, which does not make sense in the context of the situation, just as a serving temperature greater than the initial temperature does not make sense.

Example 7 Examine Logarithmic Data

Teaching the Mathematical Practices

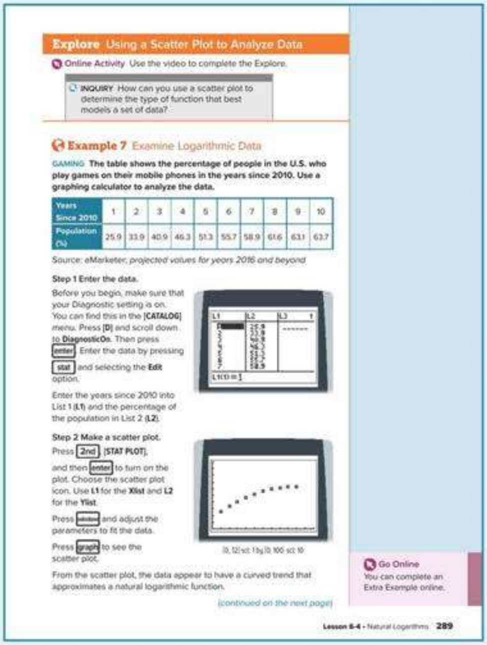
5 Compare Predictions with Data Point out that in Example 7 students should use a graphing calculator to enable them to visualize the situation. Help students to use a graphing calculator to compare their predictions about the percentage of the population who play games on their phones in a given year with the data.

Questions for Mathematical Discourse

AL Why do you need to evaluate the function for $x = 15$, not $x = 25$? *x* represents the years since 2010.

OL What is the value of the function at $x = 1$, and how does it compare to the dataset? $f(1) = 23.3237$, which is close to (1, 25.9).

BL What point is the upper limit of the domain and range of the function in the context of the example? (70.2, 100)



Explore Using a Scatter Plot to Analyze Data

Online Activity Use the video to complete the Explore.

INQUIRY How can you use a scatter plot to determine the type of function that best models a set of data?

Example 7 Examine Logarithmic Data

GAMING **The table shows the percentage of people in the U.S. who play games on their mobile phones in the years since 2010. Use a graphing calculator to analyze the data.**

Years Since 2010	1	2	3	4	5	6	7	8	9	10
Population (%)	25.9	33.9	40.9	46.3	51.3	55.7	58.9	61.6	63.1	63.7

Source: eMarketer; projected values for years 2016 and beyond

Step 1 Enter the data.

Before you begin, make sure that your Diagnostic setting is on. You can find this in the **[CATALOG]** menu. Press **[D]** and scroll down to **DiagnosticOn**. Then press enter. Enter the data by pressing stat and selecting the **Edit** option.

Enter the years since 2010 into List 1 (**L1**) and the percentage of the population in List 2 (**L2**).

Step 2 Make a scatter plot.

Press 2nd, **[STAT PLOT]**, and then enter to turn on the plot. Choose the scatter plot icon. Use **L1** for the **Xlist** and **L2** for the **Ylist**.

Press window and adjust the parameters to fit the data.

Press graph to see the scatter plot.

[0, 12] scl: 1 by [0, 100] scl: 10

From the scatter plot, the data appear to have a curved trend that approximates a natural logarithmic function.

(continued on the next page)

Go Online You can complete an Extra Example online.

Lesson 6-4 • Natural Logarithms 289

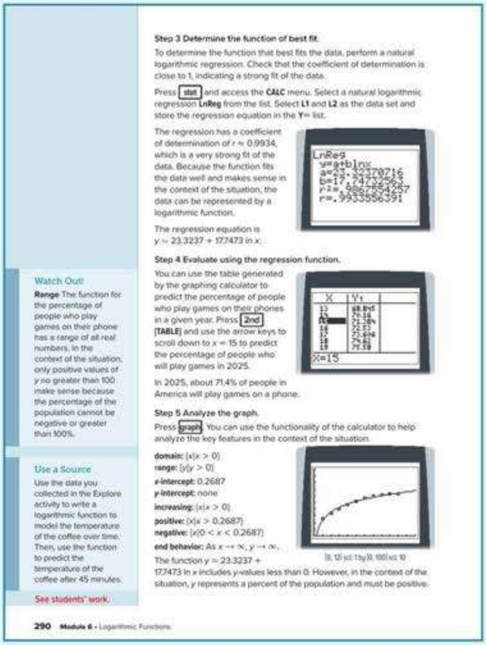
Interactive Presentation

Examine Logarithmic Data

Example 7

SELECT

Students select a calculator to analyze logarithmic data in a real-world context.



Watch Out!

Range The function for the percentage of people who play games on their phone has a range of all real numbers. In the context of the situation, only positive values of y no greater than 100 make sense because the percentage of the population cannot be negative or greater than 100%.

Use a Source

Use the data you collected in the Explore activity to write a logarithmic function to model the temperature of the coffee over time. Then, use the function to predict the temperature of the coffee after 45 minutes.

See students' work.

Step 3 Determine the function of best fit.

To determine the function that best fits the data, perform a natural logarithmic regression. Check that the coefficient of determination is close to 1, indicating a strong fit of the data.

Press stat and access the **CALC** menu. Select a natural logarithmic regression **LnReg** from the list. Select **L1** and **L2** as the data set and store the regression equation in the **Y=** list.

The regression has a coefficient of determination of $r \approx 0.9934$, which is a very strong fit of the data. Because the function fits the data well and makes sense in the context of the situation, the data can be represented by a logarithmic function.

```
LnReg
 y=a+blnx
 a=23.32370716
 b=17.74732563
 r²=.9867554257
 r=.9933556391
```

The regression equation is $y \approx 23.3237 + 17.7473 \ln x$.

Step 4 Evaluate using the regression function.

You can use the table generated by the graphing calculator to predict the percentage of people who play games on their phones in a given year. Press 2nd [TABLE] and use the arrow keys to scroll down to $x = 15$ to predict the percentage of people who will play games in 2025.

```
 X    | Y₁
 13   | 68.845
 14   | 70.16
 15   | 71.384
 16   | 72.53
 17   | 73.606
 18   | 74.62
 19   | 75.58
X=15
```

In 2025, about 71.4% of people in America will play games on a phone.

Step 5 Analyze the graph.

Press graph. You can use the functionality of the calculator to help analyze the key features in the context of the situation.

domain: $\{x \mid x > 0\}$
range: $\{y \mid y > 0\}$
x-intercept: 0.2687
y-intercept: none
increasing: $\{x \mid x > 0\}$
positive: $\{x \mid x > 0.2687\}$
negative: $\{x \mid 0 < x < 0.2687\}$
end behavior: As $x \to \infty$, $y \to \infty$.

[0, 12] scl: 1 by [0, 100] scl: 10

The function $y = 23.3237 + 17.7473 \ln x$ includes y-values less than 0. However, in the context of the situation, y represents a percent of the population and must be positive.

290 Module 6 • Logarithmic Functions

CHECK

Students complete the Check online to determine whether they are ready to move on.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–45
2	exercises that use a variety of skills from this lesson	46–56
3	exercises that emphasize higher-order and critical-thinking skills	57–60

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **BL**
THEN assign:

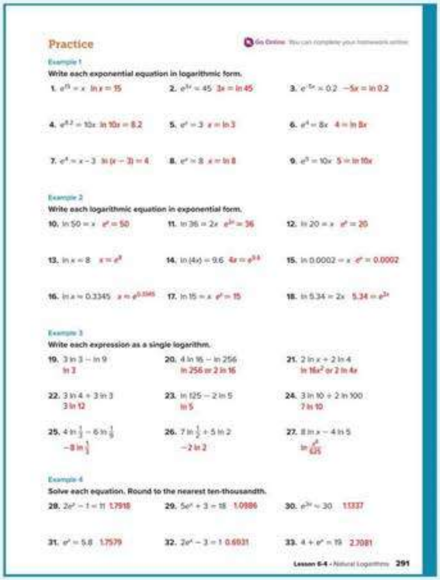
- Practice Exercises 1–55 odd, 57–60
- Extension: Approximations for π and e
- ALEKS Logarithmic and Exponential Equations

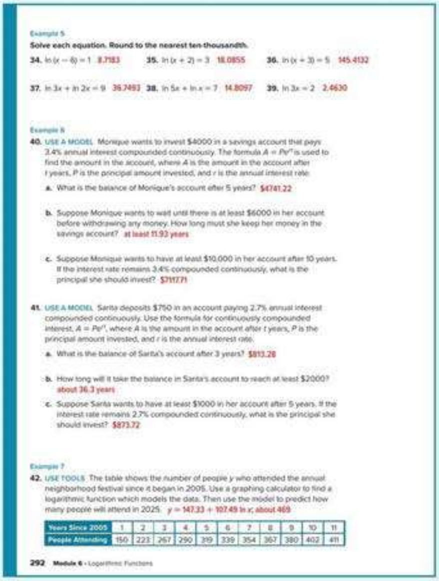
IF students score 66%–89% on the Checks, **OL**
THEN assign:

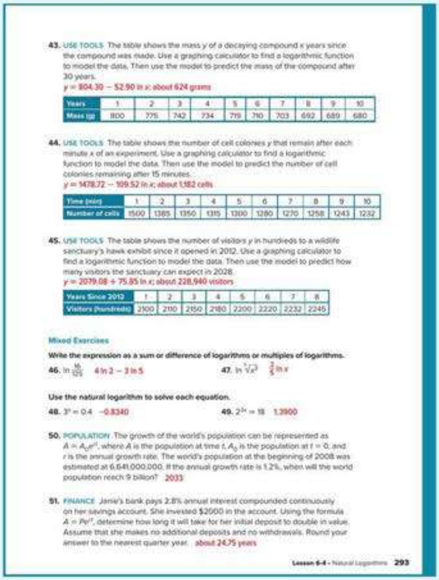
- Practice Exercises 1–59 odd
- Remediation, Review Resources: Common Logarithms
- Personal Tutors
- Extra Examples 1–7
- ALEKS Logarithmic and Exponential Equations

IF students score 65% or less on the Checks, **AL**
THEN assign:

- Practice Exercises 1–45 odd
- Remediation, Review Resources: Common Logarithms
- *Quick Review Math Handbook*: Base e and Natural Logarithms
- ALEKS Logarithmic and Exponential Equations







43. USE TOOLS The table shows the mass y of a decaying compound x years since the compound was made. Use a graphing calculator to find a logarithmic function to model the data. Then use the model to predict the mass of the compound after 30 years.
$y = 804.30 - 52.90 \ln x$; about 624 grams

Years	1	2	3	4	5	6	7	8	9	10
Mass (g)	800	775	742	734	719	710	703	692	689	680

44. USE TOOLS The table shows the number of cell colonies y that remain after each minute x of an experiment. Use a graphing calculator to find a logarithmic function to model the data. Then use the model to predict the number of cell colonies remaining after 15 minutes.
$y = 1478.72 - 109.52 \ln x$; about 1,182 cells

Time (min)	1	2	3	4	5	6	7	8	9	10
Number of cells	1500	1385	1350	1315	1300	1280	1270	1258	1243	1232

45. USE TOOLS The table shows the number of visitors y in hundreds to a wildlife sanctuary's hawk exhibit since it opened in 2012. Use a graphing calculator to find a logarithmic function to model the data. Then use the model to predict how many visitors the sanctuary can expect in 2028.
$y = 2079.08 + 75.85 \ln x$; about 228,940 visitors

Years Since 2012	1	2	3	4	5	6	7	8
Visitors (hundreds)	2100	2110	2150	2180	2200	2220	2232	2245

Mixed Exercises

Write the expression as a sum or difference of logarithms or multiples of logarithms.

46. $\ln \frac{16}{125}$ $4 \ln 2 - 3 \ln 5$

47. $\ln \sqrt[5]{x^2}$ $\frac{2}{5} \ln x$

Use the natural logarithm to solve each equation.

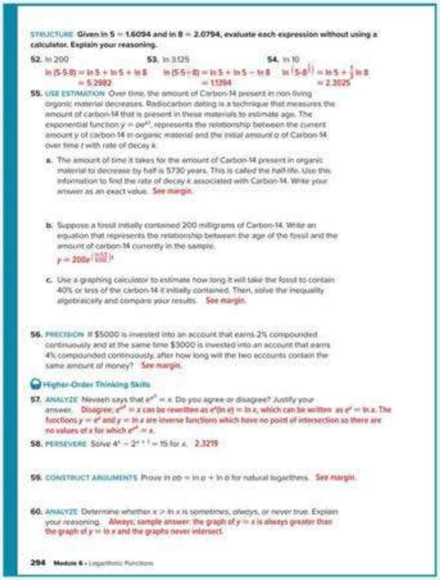
48. $3^x = 0.4$ -0.8340

49. $2^{3x} = 18$ 1.3900

50. POPULATION The growth of the world's population can be represented as $A = A_0e^{rt}$, where A is the population at time t, A_0 is the population at $t = 0$, and r is the annual growth rate. The world's population at the beginning of 2008 was estimated at 6,641,000,000. If the annual growth rate is 1.2%, when will the world population reach 9 billion? 2033

51. FINANCE Jamie's bank pays 2.8% annual interest compounded continuously on her savings account. She invested $2000 in the account. Using the formula $A = Pe^{rT}$, determine how long it will take for her initial deposit to double in value. Assume that she makes no additional deposits and no withdrawals. Round your answer to the nearest quarter year. about 24.75 years

Lesson 6-4 • Natural Logarithms 293



STRUCTURE **Given ln 5 = 1.6094 and ln 8 = 2.0794, evaluate each expression without using a calculator. Explain your reasoning.**

52. ln 200
$\ln(5 \cdot 5 \cdot 8) = \ln 5 + \ln 5 + \ln 8$
$= 5.2982$

53. ln 3.125
$\ln(5 \cdot 5 \div 8) = \ln 5 + \ln 5 - \ln 8$
$= 1.1394$

54. ln 10
$\ln\left(5 \cdot 8^{\frac{1}{3}}\right) = \ln 5 + \frac{1}{3} \ln 8$
$= 2.3025$

55. USE ESTIMATION Over time, the amount of Carbon-14 present in non-living organic material decreases. Radiocarbon dating is a technique that measures the amount of carbon-14 that is present in these materials to estimate age. The exponential function $y = ae^{kt}$, represents the relationship between the current amount y of carbon-14 in organic material and the initial amount a of Carbon-14 over time t with rate of decay k.

a. The amount of time it takes for the amount of Carbon-14 present in organic material to decrease by half is 5730 years. This is called the half-life. Use this information to find the rate of decay k associated with Carbon-14. Write your answer as an exact value. See margin.

b. Suppose a fossil initially contained 200 milligrams of Carbon-14. Write an equation that represents the relationship between the age of the fossil and the amount of carbon-14 currently in the sample.
$y = 200e^{\left(\frac{\ln 0.5}{5730}\right)t}$

c. Use a graphing calculator to estimate how long it will take the fossil to contain 40% or less of the carbon-14 it initially contained. Then, solve the inequality algebraically and compare your results. See margin.

56. PRECISION If $5000 is invested into an account that earns 2% compounded continuously and at the same time $3000 is invested into an account that earns 4% compounded continuously, after how long will the two accounts contain the same amount of money? See margin.

Higher-Order Thinking Skills

57. ANALYZE Nevaeh says that $e^{e^x} = x$. Do you agree or disagree? Justify your answer. Disagree; $e^{e^x} = x$ can be rewritten as $e^x(\ln e) = \ln x$, which can be written as $e^x = \ln x$. The functions $y = e^x$ and $y = \ln x$ are inverse functions which have no point of intersection so there are no values of x for which $e^{e^x} = x$.

58. PERSEVERE Solve $4^x - 2^{x+1} = 15$ for x. 2.3219

59. CONSTRUCT ARGUMENTS Prove $\ln ab = \ln a + \ln b$ for natural logarithms. See margin.

60. ANALYZE Determine whether $x > \ln x$ is sometimes, always, or never true. Explain your reasoning. Always; sample answer: the graph of $y = x$ is always greater than the graph of $y = \ln x$ and the graphs never intersect.

294 Module 6 • Logarithmic Functions

Answers

55a. $P = P_0e^{kt}$; $0.5P_0 = P_0e^{5730k}$; $0.5 = e^{5730k}$; $\ln(0.5) = 5730k\,(\ln e)$; $k = \frac{\text{h } 0.5}{5730}$

55c. Sample answer: at least about 7500 years; Solve inequality:

$P \le 200e^{\frac{\ln 0.5}{5730}t}$; $80 \le 200e^{\frac{\ln 0.5}{5730}t}$; $0.4 \le e^{\frac{\ln 0.5}{5730}t}$; $\ln(0.4) \le \frac{\ln 0.5}{5730}\, t \cdot \ln(e)$;

$t \ge \frac{5730 \cdot \ln 0.4}{\ln 0.5}$; $\approx$ 7575 years

56. approximately 25.5 years; the accounts will contain the same amount at the solution to $5000e^{0.02t} = 3000e^{0.04t}$. Rewrite as $\frac{5000}{3000} = \frac{e^{0.04t}}{e^{0.02t}}$, which means $\frac{5}{3} = e^{(0.04t-0.02t)}$. This means $\ln\left(\frac{5}{3}\right) = 0.02t$, so $t \approx 25.5$.

59. Let $p = \ln a$ and $q = \ln b$.

That means that $e^p = a$ and $e^q = b$.

$ab = e^p \times e^q$

$ab = e^{p+q}$

$\ln(ab) = (p + q)$

$\ln(ab) = \ln a + \ln b$

Lesson 6-5

A.CED.1, F.LE.4

Using Exponential and Logarithmic Functions

LESSON GOAL

Students write exponential growth and decay equations and solve them by using logarithms.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Using Logarithms to Solve Exponential Growth Problems
- Continuous Exponential Growth

Using Logarithms to Solve Exponential Decay Problems
- Continuous Exponential Decay
- Radiocarbon Dating

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Solving Exponential Equations and Inequalities	●	●		●
Extension: Effective Annual Yield		●	●	●

Language Development Handbook

Assign page 34 of the *Language Development Handbook* to help your students build mathematical language related to using exponential and logarithmic functions.

ELL You can use the tips and suggestions on page T34 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra, Functions

Standards for Mathematical Content:

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

F.LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where *a, c,* and *d* are numbers and the base *b* is 2, 10, or *e*; evaluate the logarithm using technology.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

5 Use appropriate tools strategically.

6 Attend to precision.

Coherence

Vertical Alignment

Previous

Students simplified logarithmic expressions and solved exponential equations by using natural logarithms.

A.SSE.2, F.LE.4

Now

Students write exponential growth and decay equations and solve them by using logarithms.

A.CED.1, F.LE.4

Next

Students will study rational functions and equations.

A.CED.1, F.IF.4, F.IF.7d

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION
Conceptual Bridge In this lesson, students build fluency and apply their understanding by solving problems that involve exponential and logarithmic functions.		

Mathematical Background

The exponential decay formulas are of the form $y = a(1 - r)^t$, or $y = ae^{-kt}$. The exponential growth formulas are of the form $y = a(1 + r)^t$ or $y = ae^{kt}$. Logarithms can be used to solve problems involving exponential growth and decay.

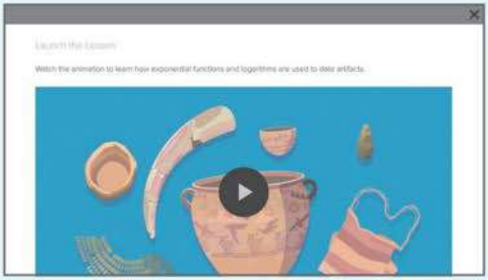
Interactive Presentation

Warm Up

Solve each equation. Check your solution.

1. $25^x = 5$
2. $4^x = \frac{1}{32}$
3. $3^{6x} = 9^{4x+2}$
4. $25^{2x+3} = 5^{x+4}$
5. $\left(\frac{1}{8}\right)^x = 2^{3x-1}$

Warm Up



Launch the Lesson

Watch the animation to learn how exponential functions and logarithms are used to date artifacts.

Launch the Lesson

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- solving exponential equations

Answers:

1. $\frac{1}{2}$
2. $-\frac{5}{2}$
3. -2
4. $-\frac{2}{3}$
5. $\frac{1}{6}$

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics Encourage students to consider how exponential functions and logarithms are used in dating artifacts. Ask students to explain the mathematical relationship between time and the amount of Carbon-14 in an object.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Learn Using Logarithms to Solve Exponential Growth Problems

Objective

Students write exponential growth equations and inequalities and solve them by using logarithms.

Teaching the Mathematical Practices

2 Attend to Quantities Point out that it is important to note the meaning of the variables used in the formulas for Continuously Compounded Interest and Continuous Exponential Growth.

6 Use Precision In this lesson, students learn how to calculate accurately and to express numerical answers with a degree of precision appropriate to the problem context.

About the Key Concept

Exponential growth can be modeled by the function $f(x) = ae^{kt}$, where a is the initial value, t is time in years, and k is a constant representing the rate of continuous growth.

Common Misconception

Some students may think that they should use the rate of growth as is, without converting to a decimal. Reinforce to students that a percent must be written in decimal form before being used in a formula.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students are struggling to identify the parts of an exponential growth formula,

THEN have them highlight each variable with a different color in $y = ae$Then they can use the same colors to highlight corresponding pieces in any given problem.

Example 1 Continuous Exponential Growth

Teaching the Mathematical Practices

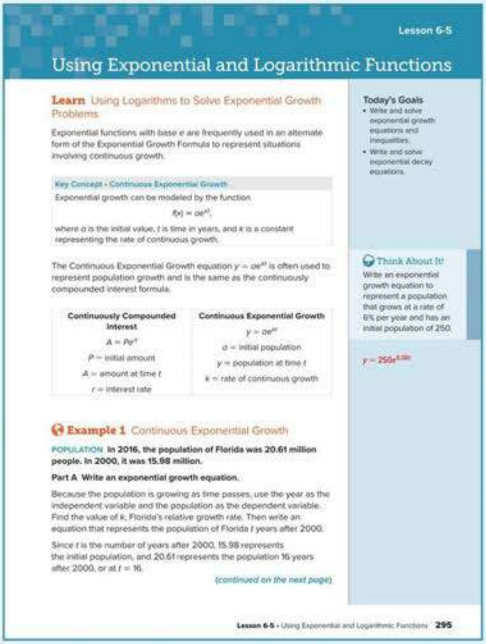
2 Represent a Situation Symbolically Guide students to write an equation that represents the exponential growth and then use the equation to solve the problem.

Questions for Mathematical Discourse

AL What is another way you can determine when Florida's population will surpass California's population? Graph both equations and find the intersection point of the functions.

OL If a state has a lesser population in 2016 compared to 2000, what do you know about k? k is negative.

BL In **part b**, what would happen if you tried to find the time when the population would be 14 million? Explain.You would get a negative value for t because you would be taking the natural logarithm of a number between 0 and 1.



Lesson 6-5

Using Exponential and Logarithmic Functions

Learn Using Logarithms to Solve Exponential Growth Problems

Exponential functions with base e are frequently used in an alternate form of the Exponential Growth Formula to represent situations involving continuous growth.

Key Concept • Continuous Exponential Growth

Exponential growth can be modeled by the function

$$f(x) = ae^{kt},$$

where a is the initial value, t is time in years, and k is a constant representing the rate of continuous growth.

The Continuous Exponential Growth equation $y = ae^{kt}$ is often used to represent population growth and is the same as the continuously compounded interest formula.

Continuously Compounded Interest	Continuous Exponential Growth
$A = Pe^{rt}$	$y = ae^{kt}$
P = initial amount	a = initial population
A = amount at time t	y = population at time t
r = interest rate	k = rate of continuous growth

Example 1 Continuous Exponential Growth

POPULATION **In 2016, the population of Florida was 20.61 million people. In 2000, it was 15.98 million.**

Part A Write an exponential growth equation.

Because the population is growing as time passes, use the year as the independent variable and the population as the dependent variable. Find the value of k, Florida's relative growth rate. Then write an equation that represents the population of Florida t years after 2000.

Since t is the number of years after 2000, 15.98 represents the initial population, and 20.61 represents the population 16 years after 2000, or at $t = 16$.

(continued on the next page)

Today's Goals
- Write and solve exponential growth equations and inequalities.
- Write and solve exponential decay equations.

Think About It!

Write an exponential growth equation to represent a population that grows at a rate of 6% per year and has an initial population of 250.

$y = 250e^{0.06t}$

Lesson 6-5 • Using Exponential and Logarithmic Functions 295

Interactive Presentation

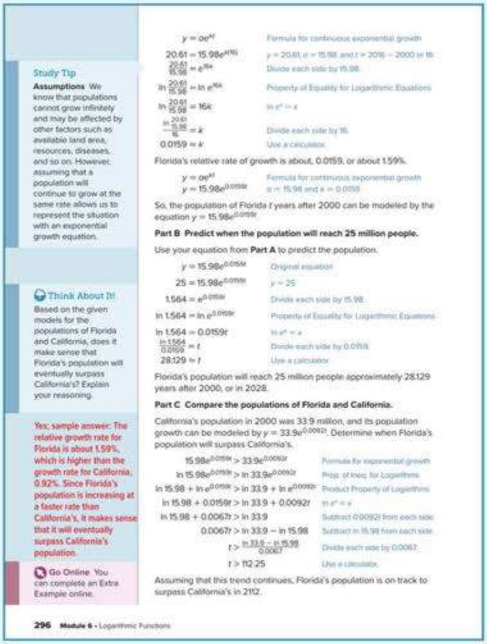
Using Logarithms to Solve Exponential Growth Problems

Key Concept: Continuous Exponential Growth

Learn

TYPE

Students answer a question to determine whether they understand how to write an exponential growth equation in context.



Study Tip

Assumptions We know that populations cannot grow infinitely and may be affected by other factors such as available land area, resources, diseases, and so on. However, assuming that a population will continue to grow at the same rate allows us to represent the situation with an exponential growth equation.

$y = ae^{kt}$ Formula for continuous exponential growth

$20.61 = 15.98e^{k(16)}$ $y = 20.61$, $a = 15.98$, and $t = 2016 - 2000$ or 16

$\frac{20.61}{15.98} = e^{16k}$ Divide each side by 15.98.

$\ln \frac{20.61}{15.98} = \ln e^{16k}$ Property of Equality for Logarithmic Equations

$\ln \frac{20.61}{15.98} = 16k$ $\ln e^x = x$

$\frac{\ln \frac{20.61}{15.98}}{16} = k$ Divide each side by 16.

$0.0159 \approx k$ Use a calculator.

Florida's relative rate of growth is about, 0.0159, or about 1.59%.

$y = ae^{kt}$ Formula for continuous exponential growth

$y = 15.98e^{0.0159t}$ $a = 15.98$ and $k = 0.0159$

So, the population of Florida t years after 2000 can be modeled by the equation $y = 15.98e^{0.0159t}$.

Part B Predict when the population will reach 25 million people.

Use your equation from **Part A** to predict the population.

$y = 15.98e^{0.0159t}$ Original equation

$25 = 15.98e^{0.0159t}$ $y = 25$

$1.564 = e^{0.0159t}$ Divide each side by 15.98.

$\ln 1.564 = \ln e^{0.0159t}$ Property of Equality for Logarithmic Equations

$\ln 1.564 = 0.0159t$ $\ln e^x = x$

$\frac{\ln 1.564}{0.0159} = t$ Divide each side by 0.0159.

$28.129 \approx t$ Use a calculator.

Florida's population will reach 25 million people approximately 28.129 years after 2000, or in 2028.

Think About It!

Based on the given models for the populations of Florida and California, does it make sense that Florida's population will eventually surpass California's? Explain your reasoning.

Yes; sample answer: The relative growth rate for Florida is about 1.59%, which is higher than the growth rate for California, 0.92%. Since Florida's population is increasing at a faster rate than California's, it makes sense that it will eventually surpass California's population.

Part C Compare the populations of Florida and California.

California's population in 2000 was 33.9 million, and its population growth can be modeled by $y = 33.9e^{0.0092t}$. Determine when Florida's population will surpass California's.

$15.98e^{0.0159t} > 33.9e^{0.0092t}$ Formula for exponential growth

$\ln 15.98e^{0.0159t} > \ln 33.9e^{0.0092t}$ Prop. of Ineq. for Logarithms

$\ln 15.98 + \ln e^{0.0159t} > \ln 33.9 + \ln e^{0.0092t}$ Product Property of Logarithms

$\ln 15.98 + 0.0159t > \ln 33.9 + 0.0092t$ $\ln e^x = x$

$\ln 15.98 + 0.0067t > \ln 33.9$ Subtract 0.0092t from each side.

$0.0067t > \ln 33.9 - \ln 15.98$ Subtract ln 15.98 from each side.

$t > \frac{\ln 33.9 - \ln 15.98}{0.0067}$ Divide each side by 0.0067.

$t > 112.25$ Use a calculator.

Assuming that this trend continues, Florida's population is on track to surpass California's in 2112.

Go Online You can complete an Extra Example online.

296 Module 6 • Logarithmic Functions

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Interactive Presentation


Continuous Exponential Growth

Example 1

TAP

Students move through three parts in which the exponential growth function is written, used to predict, or used to compare.

TYPE

Students determine whether it makes sense that Florida's population will eventually surpass California's.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Learn Using Logarithms to Solve Exponential Decay Problems

Objective

Students write exponential decay equations and solve them by using logarithms.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write and explain their solution methods. Point out that they should use clear mathematical language when answering the question in the Think About It! feature.

About the Key Concept

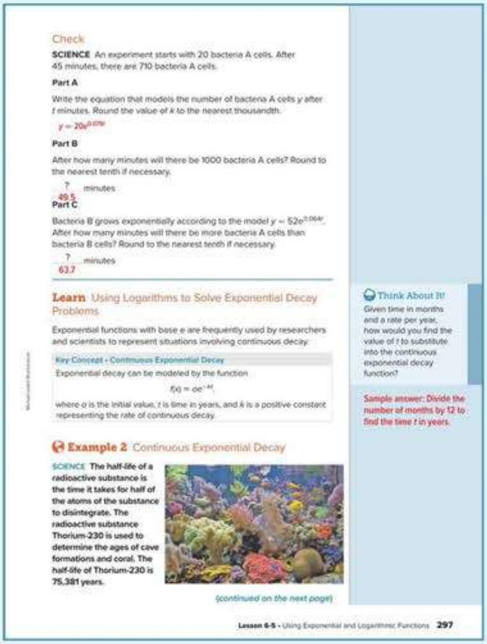
Exponential decay can be modeled by the function $f(x) = ae^{-kt}$, where a is the initial value, t is time in years, and k is a positive constant representing the rate of continuous growth.

Essential Question Follow-Up

Students have begun learning about exponential growth and decay.

Ask:

Why are exponential growth and decay functions useful in the real world? Sample answer: Many real-world situations follow an exponential models such as investments, populations, bacteria growth or decay, and temperature of heated or cooled liquids.



Check

SCIENCE An experiment starts with 20 bacteria A cells. After 45 minutes, there are 710 bacteria A cells.

Part A

Write the equation that models the number of bacteria A cells y after t minutes. Round the value of k to the nearest thousandth.

$y = 20e^{0.079t}$

Part B

After how many minutes will there be 1000 bacteria A cells? Round to the nearest tenth if necessary.

? minutes 49.5

Part C

Bacteria B grows exponentially according to the model $y = 52e^{0.064t}$. After how many minutes will there be more bacteria A cells than bacteria B cells? Round to the nearest tenth if necessary.

? minutes 63.7

Learn Using Logarithms to Solve Exponential Decay Problems

Exponential functions with base e are frequently used by researchers and scientists to represent situations involving continuous decay.

Key Concept • Continuous Exponential Decay

Exponential decay can be modeled by the function

$$f(x) = ae^{-kt},$$

where a is the initial value, t is time in years, and k is a positive constant representing the rate of continuous decay.

Example 2 Continuous Exponential Decay

SCIENCE **The half-life of a radioactive substance is the time it takes for half of the atoms of the substance to disintegrate. The radioactive substance Thorium-230 is used to determine the ages of cave formations and coral. The half-life of Thorium-230 is 75,381 years.**

(continued on the next page)

Think About It!

Given time in months and a rate per year, how would you find the value of t to substitute into the continuous exponential decay function?

Sample answer: Divide the number of months by 12 to find the time t in years.

Lesson 6-5 • Using Exponential and Logarithmic Functions 297

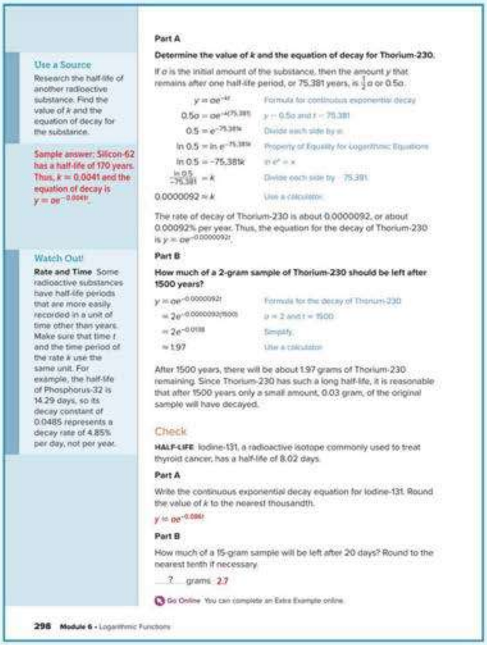
Interactive Presentation

Using Logarithms to Solve Exponential Decay Problems

Learn

TYPE

Students explain how to find the value of t when given months and a rate per year.



Use a Source
Research the half-life of another radioactive substance. Find the value of k and the equation of decay for the substance.

Sample answer: Silicon-62 has a half-life of 170 years. Thus, $k = 0.0041$ and the equation of decay is $y = ae^{-0.0041t}$.

Watch Out!
Rate and Time Some radioactive substances have half-life periods that are more easily recorded in a unit of time other than years. Make sure that time t and the time period of the rate k use the same unit. For example, the half-life of Phosphorus-32 is 14.29 days, so its decay constant of 0.0485 represents a decay rate of 4.85% per day, not per year.

Part A

Determine the value of k and the equation of decay for Thorium-230.

If a is the initial amount of the substance, then the amount y that remains after one half-life period, or 75,381 years, is $\frac{1}{2}a$ or $0.5a$.

$$\begin{aligned} y &= ae^{-kt} && \text{Formula for continuous exponential decay} \\ 0.5a &= ae^{-k(75,381)} && y = 0.5a \text{ and } t = 75,381 \\ 0.5 &= e^{-75,381k} && \text{Divide each side by } a. \\ \ln 0.5 &= \ln e^{-75,381k} && \text{Property of Equality for Logarithmic Equations} \\ \ln 0.5 &= -75,381k && \ln e^{x} = x \\ \frac{\ln 0.5}{-75,381} &= k && \text{Divide each side by } -75,381. \\ 0.0000092 &\approx k && \text{Use a calculator.} \end{aligned}$$

The rate of decay of Thorium-230 is about 0.0000092, or about 0.00092% per year. Thus, the equation for the decay of Thorium-230 is $y = ae^{-0.0000092t}$.

Part B

How much of a 2-gram sample of Thorium-230 should be left after 1500 years?

$$\begin{aligned} y &= ae^{-0.0000092t} && \text{Formula for the decay of Thorium-230} \\ &= 2e^{-0.0000092(1500)} && a = 2 \text{ and } t = 1500 \\ &= 2e^{-0.0138} && \text{Simplify.} \\ &\approx 1.97 && \text{Use a calculator.} \end{aligned}$$

After 1500 years, there will be about 1.97 grams of Thorium-230 remaining. Since Thorium-230 has such a long half-life, it is reasonable that after 1500 years only a small amount, 0.03 gram, of the original sample will have decayed.

Check

HALF-LIFE Iodine-131, a radioactive isotope commonly used to treat thyroid cancer, has a half-life of 8.02 days.

Part A

Write the continuous exponential decay equation for Iodine-131. Round the value of k to the nearest thousandth.

$y = ae^{-0.086t}$

Part B

How much of a 15-gram sample will be left after 20 days? Round to the nearest tenth if necessary.

___?___ grams 2.7

Go Online You can complete an Extra Example online.

298 Module 6 • Logarithmic Functions

Interactive Presentation

Continuous Exponential Decay

SCIENCE The half-life of a radioactive substance is the time it takes for half of the atoms of the substance to disintegrate. The radioactive substance Thorium-230 is used to determine the ages of cave formations and coral. The half-life of Thorium-230 is 75,381 years.

Example 2

TYPE

Students enter values to complete the exponential decay formula.

TYPE

Students research the half-life of another radioactive substance and then write the equation of decay.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Example 2 Continuous Exponential Decay

Teaching the Mathematical Practices

5 Use a Source In Example 2, guide students to find external information to answer the questions posed in the Use a Source feature.

Questions for Mathematical Discourse

AL What does half-life represent? the time it takes a substance to decay to half its original mass

OL If the half-life is decreased, what is the effect on k? k increases. What does this mean in the context? A substance that takes less time to decay to half the initial amount has a faster rate of decay.

BL The exponential decay function approaches but never reaches zero. In the context, how could you calculate when the mass of a 2-gram sample of Thorium decays to as close to zero as possible? Sample answer: Look up the approximate mass of one thorium atom in grams and use that as y. Solve for t.

Apply Example 3 Radiocarbon Dating

Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them, 4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

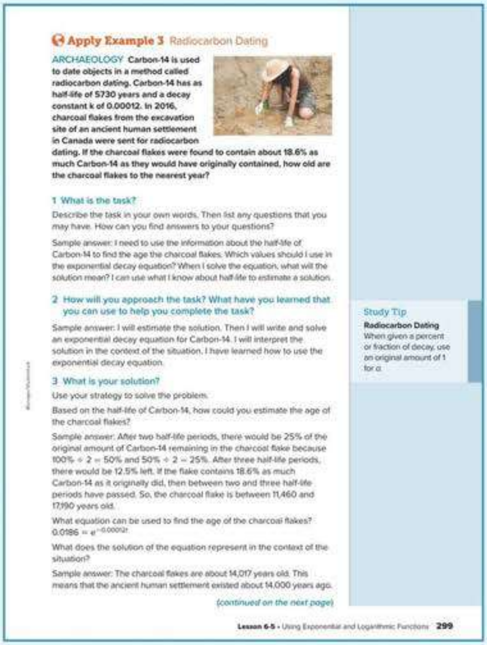
Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- I s the number 14 in Carbon-14 important for answering the questions?
- H ow many years would it take for charcoal flakes to contain one quarter of the original amount of Carbon-14?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.



Apply Example 3 Radiocarbon Dating

ARCHAEOLOGY **Carbon-14 is used to date objects in a method called radiocarbon dating. Carbon-14 has as half-life of 5730 years and a decay constant k of 0.00012. In 2016, charcoal flakes from the excavation site of an ancient human settlement in Canada were sent for radiocarbon dating. If the charcoal flakes were found to contain about 18.6% as much Carbon-14 as they would have originally contained, how old are the charcoal flakes to the nearest year?**

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to use the information about the half-life of Carbon-14 to find the age the charcoal flakes. Which values should I use in the exponential decay equation? When I solve the equation, what will the solution mean? I can use what I know about half-life to estimate a solution.

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I will estimate the solution. Then I will write and solve an exponential decay equation for Carbon-14. I will interpret the solution in the context of the situation. I have learned how to use the exponential decay equation.

3 What is your solution?

Use your strategy to solve the problem.

Based on the half-life of Carbon-14, how could you estimate the age of the charcoal flakes?

Sample answer: After two half-life periods, there would be 25% of the original amount of Carbon-14 remaining in the charcoal flake because 100% ÷ 2 = 50% and 50% ÷ 2 = 25%. After three half-life periods, there would be 12.5% left. If the flake contains 18.6% as much Carbon-14 as it originally did, then between two and three half-life periods have passed. So, the charcoal flake is between 11,460 and 17,190 years old.

What equation can be used to find the age of the charcoal flakes?
$0.0186 = e^{-0.00012t}$

What does the solution of the equation represent in the context of the situation?

Sample answer: The charcoal flakes are about 14,017 years old. This means that the ancient human settlement existed about 14,000 years ago.

(continued on the next page)

Study Tip
Radiocarbon Dating
When given a percent or fraction of decay, use an original amount of 1 for a.

Lesson 6-5 • Using Exponential and Logarithmic Functions 299

Interactive Presentation

Radiocarbon Dating

Apply Example 3

TYPE

Students answer a question to determine whether they understand continuous exponential decay.

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: The solution falls within the initial estimate of 11,460 to 17,190 years. I can also substitute 14,017 into the formula for the decay of Carbon-14 to check that the remaining amount is about 18.6%. For $y = ae^{-0.00012t}$, let $a = 1$ and $t = 14{,}017$. Then $y = 1 \cdot e^{-0.00012(14{,}017)}$ ≈ 0.186 or 18.6%.

Check

ARCHAEOLOGY An archaeologist examining a prehistoric painting estimates that it contains about 2.34% as much Carbon-14 as it would have contained when it was painted. How long ago was the painting created? Round to the nearest year.

[Hint: The formula for the decay of Carbon-14 is $y = ae^{-0.00012t}$.]

? years ago 31,292

Pause and Reflect

Did you struggle with anything in this lesson? If so, how did you deal with it?

See students' observations.

300 Module 6 • Logarithmic Functions

CHECK

Students complete the Check online to determine whether they are ready to move on.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–5
2	exercises that use a variety of skills from this lesson	6–18
3	exercises that emphasize higher-order and critical-thinking skills	19–21

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice Exercises 1–17 odd, 19–21
- Extension: Effective Annual Yield
- ALEKS Logarithmic and Exponential Equations

OL

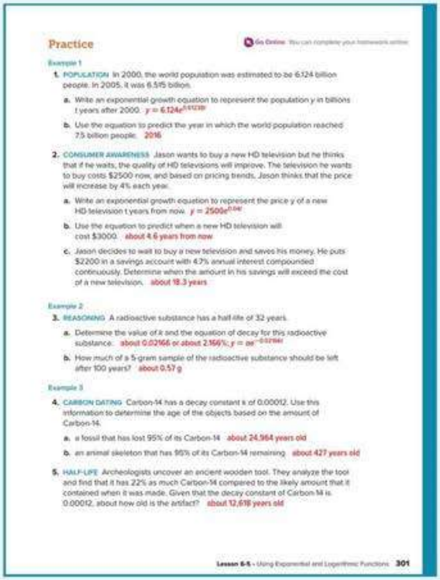
IF students score 66%–89% on the Checks,
THEN assign:

- Practice Exercises 1–21 odd
- Remediation, Review Resources: Solving Exponential Equations and Inequalities
- Personal Tutors
- Extra Examples 1–3
- ALEKS Logarithmic and Exponential Equations

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice Exercises 1–5 odd
- Remediation, Review Resources: Solving Exponential Equations and Inequalities
- *Quick Review Math Handbook:* Using Exponential and Logarithmic Functions
- ALEKS Logarithmic and Exponential Equations



Practice

Go Online You can complete your homework online.

Example 1

1. POPULATION In 2000, the world population was estimated to be 6.124 billion people. In 2005, it was 6.515 billion.
   a. Write an exponential growth equation to represent the population y in billions t years after 2000. $y = 6.124e^{0.0124t}$
   b. Use the equation to predict the year in which the world population reached 7.5 billion people. 2016

2. CONSUMER AWARENESS Jason wants to buy a new HD television but he thinks that if he waits, the quality of HD televisions will improve. The television he wants to buy costs $2500 now, and based on pricing trends, Jason thinks that the price will increase by 4% each year.
   a. Write an exponential growth equation to represent the price y of a new HD television t years from now. $y = 2500e^{0.04t}$
   b. Use the equation to predict when a new HD television will cost $3000. about 4.6 years from now
   c. Jason decides to wait to buy a new television and saves his money. He puts $2200 in a savings account with 4.7% annual interest compounded continuously. Determine when the amount in his savings will exceed the cost of a new television. about 18.3 years

Example 2

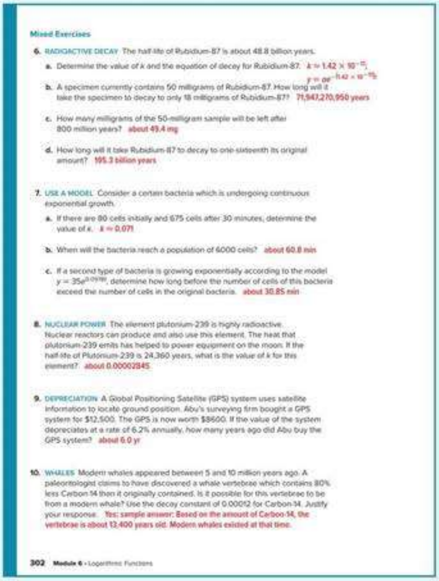
3. REASONING A radioactive substance has a half-life of 32 years.
   a. Determine the value of k and the equation of decay for this radioactive substance. about 0.02166 or about 2.166%; $y = ae^{-0.02166t}$
   b. How much of a 5-gram sample of the radioactive substance should be left after 100 years? about 0.57 g

Example 3

4. CARBON DATING Carbon-14 has a decay constant k of 0.00012. Use this information to determine the age of the objects based on the amount of Carbon-14.
   a. a fossil that has lost 95% of its Carbon-14 about 24,964 years old
   b. an animal skeleton that has 95% of its Carbon-14 remaining about 427 years old

5. HALF-LIFE Archeologists uncover an ancient wooden tool. They analyze the tool and find that it has 22% as much Carbon-14 compared to the likely amount that it contained when it was made. Given that the decay constant of Carbon-14 is 0.00012, about how old is the artifact? about 12,618 years old

Lesson 6-5 • Using Exponential and Logarithmic Functions 301



Mixed Exercises

6. RADIOACTIVE DECAY The half-life of Rubidium-87 is about 48.8 billion years.
   a. Determine the value of k and the equation of decay for Rubidium-87. $k \approx 1.42 \times 10^{-11}$; $y = ae^{-(1.42 \times 10^{-11})t}$
   b. A specimen currently contains 50 milligrams of Rubidium-87. How long will it take the specimen to decay to only 18 milligrams of Rubidium-87? 71,947,270,950 years
   c. How many milligrams of the 50-milligram sample will be left after 800 million years? about 49.4 mg
   d. How long will it take Rubidium-87 to decay to one-sixteenth its original amount? 195.3 billion years

7. USE A MODEL Consider a certain bacteria which is undergoing continuous exponential growth.
   a. If there are 80 cells initially and 675 cells after 30 minutes, determine the value of k. $k \approx 0.071$
   b. When will the bacteria reach a population of 6000 cells? about 60.8 min
   c. If a second type of bacteria is growing exponentially according to the model $y = 35e^{0.0978t}$, determine how long before the number of cells of this bacteria exceed the number of cells in the original bacteria. about 30.85 min

8. NUCLEAR POWER The element plutonium-239 is highly radioactive. Nuclear reactors can produce and also use this element. The heat that plutonium-239 emits has helped to power equipment on the moon. If the half-life of Plutonium-239 is 24,360 years, what is the value of k for this element? about 0.00002845

9. DEPRECIATION A Global Positioning Satellite (GPS) system uses satellite information to locate ground position. Abu's surveying firm bought a GPS system for $12,500. The GPS is now worth $8600. If the value of the system depreciates at a rate of 6.2% annually, how many years ago did Abu buy the GPS system? about 6.0 yr

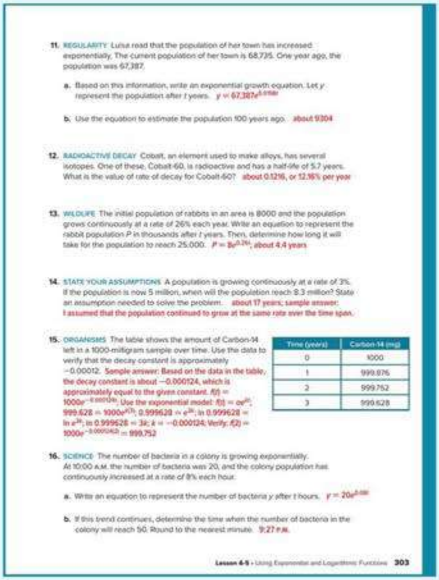
10. WHALES Modern whales appeared between 5 and 10 million years ago. A paleontologist claims to have discovered a whale vertebrae which contains 80% less Carbon-14 than it originally contained. Is it possible for this vertebrae to be from a modern whale? Use the decay constant of 0.00012 for Carbon-14. Justify your response. Yes; sample answer: Based on the amount of Carbon-14, the vertebrae is about 13,400 years old. Modern whales existed at that time.

302 Module 6 • Logarithmic Functions

3 REFLECT AND PRACTICE

A.CED.1, F.LE.4

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION



11. **REGULARITY** Luisa read that the population of her town has increased exponentially. The current population of her town is 68,735. One year ago, the population was 67,387.
    a. Based on this information, write an exponential growth equation. Let y represent the population after t years. $y = 67{,}387e^{[illegible]}$
    b. Use the equation to estimate the population 100 years ago. about 9304

12. **RADIOACTIVE DECAY** Cobalt, an element used to make alloys, has several isotopes. One of these, Cobalt-60, is radioactive and has a half-life of 5.7 years. What is the value of rate of decay for Cobalt-60? about 0.1216, or 12.16% per year

13. **WILDLIFE** The initial population of rabbits in an area is 8000 and the population grows continuously at a rate of 26% each year. Write an equation to represent the rabbit population P in thousands after t years. Then, determine how long it will take for the population to reach 25,000. $P = 8e^{0.26t}$; about 4.4 years

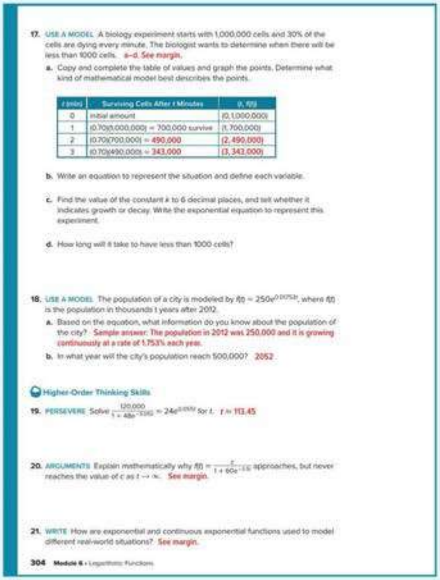
14. **STATE YOUR ASSUMPTIONS** A population is growing continuously at a rate of 3%. If the population is now 5 million, when will the population reach 8.3 million? State an assumption needed to solve the problem. about 17 years; sample answer: I assumed that the population continued to grow at the same rate over the time span.

15. **ORGANISMS** The table shows the amount of Carbon-14 left in a 1000-milligram sample over time. Use the data to verify that the decay constant is approximately −0.00012. Sample answer: Based on the data in the table, the decay constant is about −0.000124, which is approximately equal to the given constant. $f(t) = 1000e^{[illegible]}$; Use the exponential model: $f(t) = ae^{kt}$; $999.628 = 1000e^{k(3)}$; $0.999628 = e^{3k}$; $\ln 0.999628 = \ln e^{3k}$; $\ln 0.999628 = 3k$; $k = -0.000124$; Verify: $f(2) = 1000e^{[illegible]} = 999.752$

Time (years)	Carbon-14 (mg)
0	1000
1	999.876
2	999.752
3	999.628

16. **SCIENCE** The number of bacteria in a colony is growing exponentially. At 10:00 A.M. the number of bacteria was 20, and the colony population has continuously increased at a rate of 8% each hour.
    a. Write an equation to represent the number of bacteria y after t hours. $y = 20e^{[illegible]}$
    b. If this trend continues, determine the time when the number of bacteria in the colony will reach 50. Round to the nearest minute. 9:27 P.M.

Lesson 4-5 • Using Exponential and Logarithmic Functions 303



17. **USE A MODEL** A biology experiment starts with 1,000,000 cells and 30% of the cells are dying every minute. The biologist wants to determine when there will be less than 1000 cells. a–d. See margin.
    a. Copy and complete the table of values and graph the points. Determine what kind of mathematical model best describes the points.

t (min)	Surviving Cells After t Minutes	$(t, f(t))$
0	initial amount	(0, 1,000,000)
1	(0.70)(1,000,000) = 700,000 survive	(1, 700,000)
2	(0.70)(700,000) = 490,000	(2, 490,000)
3	(0.70)(490,000) = 343,000	(3, 343,000)

    b. Write an equation to represent the situation and define each variable.
    c. Find the value of the constant k to 6 decimal places, and tell whether it indicates growth or decay. Write the exponential equation to represent this experiment.
    d. How long will it take to have less than 1000 cells?

18. **USE A MODEL** The population of a city is modeled by $f(t) = 250e^{[illegible]}$, where $f(t)$ is the population in thousands t years after 2012.
    a. Based on the equation, what information do you know about the population of the city? Sample answer: The population in 2012 was 250,000 and it is growing continuously at a rate of 1.753% each year.
    b. In what year will the city's population reach 500,000? 2052

Higher-Order Thinking Skills

19. **PERSEVERE** Solve $\frac{120{,}000}{1 + 48e^{[illegible]}} = 24e^{[illegible]}$ for t. $t \approx 113.45$

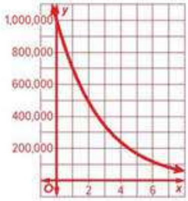
20. **ARGUMENTS** Explain mathematically why $f(t) = \frac{c}{1 + 60e^{[illegible]}}$ approaches, but never reaches the value of c as $t \to \infty$. See margin.

21. **WRITE** How are exponential and continuous exponential functions used to model different real-world situations? See margin.

304 Module 6 • Logarithmic Functions

Answers

17a. Sample answer: An exponential model best describes the points because the number of cells decreases by the same percentage every minute.



17b. In the model $f(t) = ae^{kt}$; $f(t)$ represents the number of cells remaining after t minutes, a represents the number of cells at the start of the experiment, t represents the number of minutes since the experiment began, k is the growth or decay constant. $F(t) = 1{,}000{,}000e^{kt}$

17c. $k \approx 0.356675$; It is exponential decay. Substitute known values into the formula: $700{,}000 = 1{,}000{,}000e^{-k(1)}$; $0.7 = e^{-k}$; $\ln 0.7 = -k$; $k \approx 0.356675$; The exponential decay model for this experiment is $f(t) = 1{,}000{,}000e^{-0.356675t}$.

17d. It will take approximately 19.367 minutes to have less than 1000 cells. Write the model and solve for t; $1000 = 1{,}000{,}000e^{-0.356675t}$; $0.001 = e^{-0.356675t}$; $\ln 0.001 = -0.356675t$; $t \approx 19.367$.

20. Sample answer: As $t \longrightarrow +\infty$, $e^{-t} \longrightarrow 0$. So, the denominator approaches $1 + 0$, or 1. As the denominator approaches 1, $f(t) \longrightarrow \frac{c}{1}$, or c. However, since e^{-t} never reaches 0, $f(t)$ can never reach c.

21. Sample answer: Exponential functions can be used to model situations that incorporate a percentage of growth or decay for a specific number of times per year. Continuous exponential functions can be used to model situations that incorporate a percentage of growth or decay continuously.

Module 6 • Logarithmic Functions

Review

Rate Yourself!

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have students respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- How are the graphs of exponential functions and logarithmic functions related?
- Why are common logarithms useful in the real world?
- Why are exponential growth and decay functions useful in the real world?

Then have students write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

ELL A completed Foldable for this module should include the key concepts related to creating logarithmic functions.

LS **LearnSmart** Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Modeling with Functions**.

- Applications of Function Models
- Exponential and Logarithmic Functions
- Building Functions

Module 6 • Logarithmic Functions

Review

Essential Question

How are logarithms defined and used to model situations in the real world?

A logarithm is the power to which a base must be raised to equal the given number. Logarithmic functions model slow-growing functions.

Module Summary

Lesson 6-1

Logarithmic Functions

- Let b and x be positive numbers, $b \neq 1$. The logarithm of x with base b is denoted $\log_b x$ and is defined as the exponent y that makes the equation $b^y = x$ true.
- The same techniques used to transform the graphs of other functions can be applied to the graphs of logarithmic functions in the form $g(x) = a \log_b (x - h) + k$.

Lesson 6-2

Properties of Logarithms

- If b is a positive number other than 1, then $\log_b x = \log_b y$ if and only if $x = y$.
- For all positive numbers b, m, and n, where $b \neq 1$, $\log_b mn = \log_b m + \log_b n$.
- For all positive numbers b, m, and n, where $b \neq 1$ and $n \neq 0$, $\log_b \frac{m}{n} = \log_b m - \log_b n$.
- For any real number n, and positive numbers m and b, where $b \neq 1$, $\log_b m^n = n \log_b m$.

Lessons 6-3 and 6-4

Common Logarithms and Natural Logarithms

- Base 10 logarithms are called common logarithms.
- For all positive numbers a, b, and n, where $a \neq 1$ and $b \neq 1$, $\log_b n = \frac{\log_a n}{\log_a b}$.
- An exponential function with base e, written as $y = e^x$, is called a natural base exponential function.
- The inverse of a natural base exponential function is a natural logarithm, which can be written as $\log_e x$, but is more often abbreviated as $\ln x$.
- The same properties that apply to logarithms with a base of b also apply to natural logarithms.

Lesson 6-5

Using Exponential and Logarithmic Functions

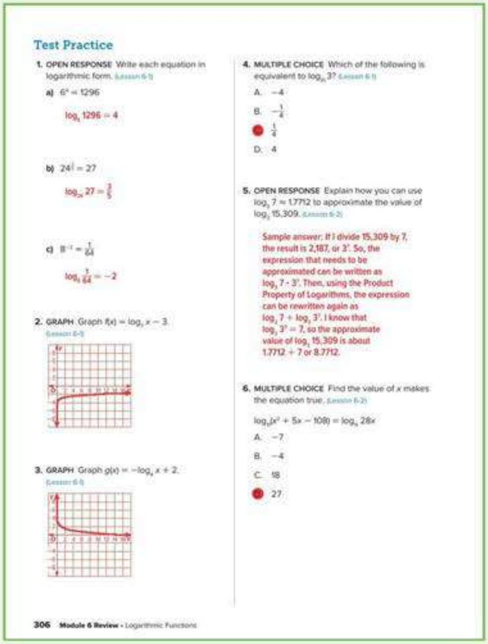
- Exponential growth can be modeled by the function $f(x) = ae^{kt}$, where a is the initial value, t is time in years, and k is a constant representing the rate of continuous growth.
- The Continuous Exponential Growth equation $y = ae^{kt}$ is often used to represent population growth and is the same as the continuously compounded interest formula.
- Exponential decay can be modeled by the function $f(x) = ae^{-kt}$, where a is the initial value, t is time in years, and k is a positive constant representing the rate of continuous decay.

Study Organizer

Foldables

Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.

Module 6 Review • Logarithmic Functions 305



Test Practice

1. **OPEN RESPONSE** Write each equation in logarithmic form. (Lesson 6-1)

a) $6^4 = 1296$

$\log_6 1296 = 4$

b) $243^{\frac{3}{5}} = 27$

$\log_{243} 27 = \frac{3}{5}$

c) $8^{-2} = \frac{1}{64}$

$\log_8 \frac{1}{64} = -2$

2. **GRAPH** Graph $f(x) = \log_3 x - 3$. (Lesson 6-1)

3. **GRAPH** Graph $g(x) = -\log_4 x + 2$. (Lesson 6-1)

4. **MULTIPLE CHOICE** Which of the following is equivalent to $\log_{81} 3$? (Lesson 6-1)

A. -4

B. $-\frac{1}{4}$

● $\frac{1}{4}$

D. 4

5. **OPEN RESPONSE** Explain how you can use $\log_3 7 \approx 1.7712$ to approximate the value of $\log_3 15{,}309$. (Lesson 6-2)

Sample answer: If I divide 15,309 by 7, the result is 2,187, or 3^7. So, the expression that needs to be approximated can be written as $\log_3 7 \cdot 3^7$. Then, using the Product Property of Logarithms, the expression can be rewritten again as $\log_3 7 + \log_3 3^7$. I know that $\log_3 3^7 = 7$, so the approximate value of $\log_3 15{,}309$ is about $1.7712 + 7$ or 8.7712.

6. **MULTIPLE CHOICE** Find the value of x makes the equation true. (Lesson 6-2)

$\log_4(x^2 + 5x - 108) = \log_4 28x$

A. -7

B. -4

C. 18

● 27

306 **Module 6 Review** • Logarithmic Functions

Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Vocabulary Activity

Module Review

Assessment Resources

Vocabulary Test

AL Module Test Form B

OL Module Test Form A

BL Module Test Form C

Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

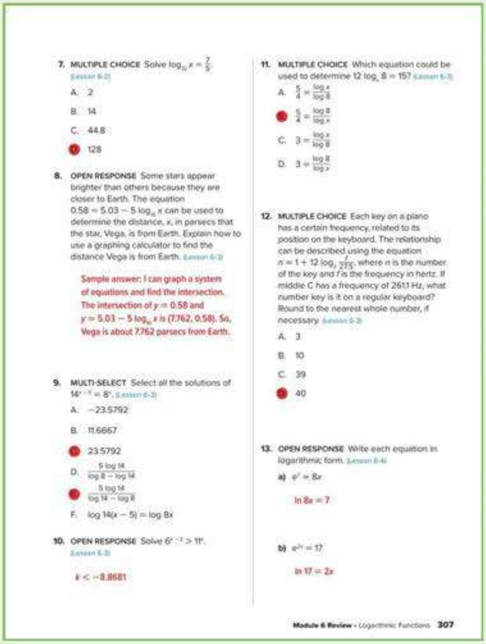
Test Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1–19 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Multiple Choice	Students select one correct answer.	4, 6, 7, 11, 12, 15, 17
Multi-Select	Multiple answers may be correct. Students must select all correct answers.	9, 14
Graph	Students create a graph on an online coordinate plane.	2, 3
Open Response	Students construct their own response.	1, 5, 8, 10, 13, 16, 18, 19

To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
A.SSE.2	6-1, 6-2, 6-4	4–7, 14, 16
A.REI.11	6-3	8
A.CED.2	6-5	17
A.CED.4	6-1	1
F.IF.7e	6-1	2, 3
F.BF.a	6-3	11, 12
F.LE.4	6-3, 6-4, 6-5	9, 10, 13, 15, 18, 19



7. **MULTIPLE CHOICE** Solve $\log_{32} x = \frac{7}{5}$. (Lesson 6-2)
A. 2
B. 14
C. 44.8
● 128

8. **OPEN RESPONSE** Some stars appear brighter than others because they are closer to Earth. The equation $0.58 = 5.03 - 5 \log_{10} x$ can be used to determine the distance, x, in parsecs that the star, Vega, is from Earth. Explain how to use a graphing calculator to find the distance Vega is from Earth. (Lesson 6-3)

Sample answer: I can graph a system of equations and find the intersection. The intersection of $y = 0.58$ and $y = 5.03 - 5 \log_{10} x$ is (7.762, 0.58). So, Vega is about 7.762 parsecs from Earth.

9. **MULTI-SELECT** Select all the solutions of $14^{x-5} = 8^x$. (Lesson 6-3)
A. −23.5792
B. 11.6667
● 23.5792
D. $\frac{5 \log 14}{\log 8 - \log 14}$
● $\frac{5 \log 14}{\log 14 - \log 8}$
F. $\log 14(x - 5) = \log 8x$

10. **OPEN RESPONSE** Solve $6^{x-1} > 11^x$. (Lesson 6-3)

$x < -8.8681$

11. **MULTIPLE CHOICE** Which equation could be used to determine $12 \log_x 8 = 15$? (Lesson 6-3)
A. $\frac{5}{4} = \frac{\log x}{\log 8}$
● $\frac{5}{4} = \frac{\log 8}{\log x}$
C. $3 = \frac{\log x}{\log 8}$
D. $3 = \frac{\log 8}{\log x}$

12. **MULTIPLE CHOICE** Each key on a piano has a certain frequency, related to its position on the keyboard. The relationship can be described using the equation $n = 1 + 12 \log_2 \frac{f}{27.5}$, where n is the number of the key and f is the frequency in hertz. If middle C has a frequency of 261.1 Hz, what number key is it on a regular keyboard? Round to the nearest whole number, if necessary. (Lesson 6-3)
A. 3
B. 10
C. 39
● 40

13. **OPEN RESPONSE** Write each equation in logarithmic form. (Lesson 6-4)

a) $e^7 = 8x$

$\ln 8x = 7$

b) $e^{2x} = 17$

$\ln 17 = 2x$

Module 6 Review • Logarithmic Functions **307**

Lesson 6-3

52. $\log_{\sqrt{a}} 3 = \log_a x$ Original equation

$\dfrac{\log_a 3}{\log_a \sqrt{a}} = \log_a x$ Change of Base Formula

$\dfrac{\log_a 3}{\frac{1}{2}} = \log_a x \quad \sqrt{a} = a^{\frac{1}{2}}$

$2 \log_a 3 = \log_a x$ Multiply numerator and denominator by 2.

$\log_a 3^2 = \log_a x$ Power Property of Logarithms

$3^2 = x$ Property of Equality for Logarithmic Functions

$9 = x$ Simplify.

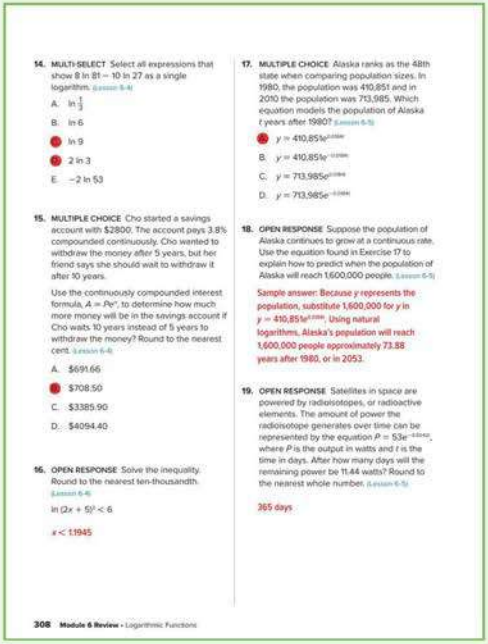
53. $\log_3 27 = 3$ and $\log_{27} 3 = \frac{1}{3}$; Conjecture: $\log_a b = \dfrac{1}{\log_b a}$

Proof:

$\log_a b \overset{?}{=} \dfrac{1}{\log_b a}$ Original statement

$\dfrac{\log_b b}{\log_b a} \overset{?}{=} \dfrac{1}{\log_b a}$ Change of Base Formula

$\dfrac{1}{\log_b a} = \dfrac{1}{\log_b a}$ Inverse Property of Exponents and Logarithms



14. **MULTI-SELECT** Select all expressions that show 8 ln 81 − 10 ln 27 as a single logarithm. (Lesson 6-4)

A. $\ln \frac{1}{3}$

B. ln 6

● ln 9

● 2 ln 3

E. −2 ln 53

15. **MULTIPLE CHOICE** Cho started a savings account with $2800. The account pays 3.8% compounded continuously. Cho wanted to withdraw the money after 5 years, but her friend says she should wait to withdraw it after 10 years.

Use the continuously compounded interest formula, $A = Pe^{rt}$, to determine how much more money will be in the savings account if Cho waits 10 years instead of 5 years to withdraw the money? Round to the nearest cent. (Lesson 6-4)

A. $691.66

● $708.50

C. $3385.90

D. $4094.40

16. **OPEN RESPONSE** Solve the inequality. Round to the nearest ten-thousandth. (Lesson 6-4)

$\ln (2x + 5)^2 < 6$

$x < 1.1945$

17. **MULTIPLE CHOICE** Alaska ranks as the 48th state when comparing population sizes. In 1980, the population was 410,851 and in 2010 the population was 713,985. Which equation models the population of Alaska t years after 1980? (Lesson 6-5)

● $y = 410{,}851e^{[illegible]}$

B. $y = 410{,}851e^{-[illegible]}$

C. $y = 713{,}985e^{[illegible]}$

D. $y = 713{,}985e^{-[illegible]}$

18. **OPEN RESPONSE** Suppose the population of Alaska continues to grow at a continuous rate. Use the equation found in Exercise 17 to explain how to predict when the population of Alaska will reach 1,600,000 people. (Lesson 6-5)

Sample answer: Because y represents the population, substitute 1,600,000 for y in $y = 410{,}851e^{[illegible]}$. Using natural logarithms, Alaska's population will reach 1,600,000 people approximately 73.88 years after 1980, or in 2053.

19. **OPEN RESPONSE** Satellites in space are powered by radioisotopes, or radioactive elements. The amount of power the radioisotope generates over time can be represented by the equation $P = 53e^{-[illegible]}$, where P is the output in watts and t is the time in days. After how many days will the remaining power be 11.44 watts? Round to the nearest whole number. (Lesson 6-5)

365 days

308 **Module 6 Review** • Logarithmic Functions

Module 7

Rational Functions

Module Goals

- Students multiply and divide rational expressions.
- Students add and subtract rational expressions.
- Students graph and analyze reciprocal functions.
- Students graph and analyze rational functions.
- Students recognize and solve direct, joint, inverse, and combined variation equations.

Focus

Domain: Algebra, Functions

Standards for Mathematical Content:

A.APR.7 Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Also addresses A.CED.1, A.CED.2, A.REI.2, A.REI.11, F.IF.4, and F.BF.3.

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this module.

Coherence

Vertical Alignment

Previous

Students simplified expressions with rational exponents.

N.RN.1, N.RN.2 (Course 2)

Now

Students graph and analyze rational functions with vertical, horizontal, and/or oblique asymptotes. Students write and solve rational equations.

A.CED.1, F.IF.4, A.REI.2

Next

Students will graph and analyze trigonometric functions, identifying key features and aymptotes.

F.IF.4, F.IF.7e

The Three Pillars of Rigor

Students will use the three pillars of rigor to help them meet standards. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. Once they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Practice.

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION
EXPLORE	LEARN	EXAMPLE & PRACTICE

Suggested Pacing

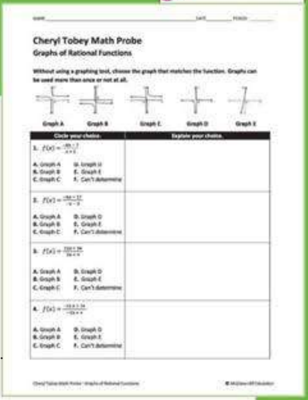
Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
7-1 Multiplying and Dividing Rational Expressions	A.APR.7	1	0.5
7-2 Adding and Subtracting Rational Expressions	A.APR.7	1	0.5
Put It All Together: Lessons 7-1 through 7-2		1	0.5
7-3 Graphing Reciprocal Functions	F.IF.5, F.BF.3	1	0.5
7-4 Graphing Rational Functions	F.IF.4, F.IF.5	1	0.5
7-5 Variation	A.CED.1, A.CED.2	1	0.5
7-6 Solving Rational Equations and Inequalities	A.CED.1, A.REI.2, A.REI.11	2	1
Module Review		1	0.5
Module Assessment		1	0.5
	Total Days	11	5.5

CHERYL TOBEY MATH PROBES

Formative Assessment Math Probe

Graphs of Rational Functions

Module Resource



Cheryl Tobey Math Probe

Graphs of Rational Functions

Without using a graphing tool, choose the graph that matches the function. Graphs can be used more than once or not at all.

Correct Answers: 1. B 2. A 3. A 4. C

Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will analyze algebraic representations of rational functions to identify their graphical representations.

Targeted Concepts Understand how the algebraic representation of a function can be analyzed to find intercepts and asymptotes of the graphical representation.

Targeted Misconceptions

- Students may determine the intercepts by incorrectly swapping the process (for example, finding the x-intercept by substituting zero for x) or by making a computation error when solving.
- Students may not understand that the horizontal asymptotes are found by analyzing the degrees of the numerator and the denominator.
- Students may not understand that vertical asymptotes are associated with values of x that are undefined (analyzing the denominator).
- Instead of dividing the entire numerator and denominator by common factors to simplify or analyze a rational function, students may incorrectly "cancel" terms or parts of terms.

Use the Probe after Lesson 7-4.

Collect and Assess Student Answers

If the student selects these responses...	**Then** the student likely...
1. A, C, D **2.** B, C, D **3.** B, C, D **4.** A, B, D	is making one or more of the following errors: • is not analyzing the power of the numerator and denominator and/or not correctly using the coefficients of the numerator and denominator to determine the horizontal asymptotes, • does not understand how to evaluate the denominator (set it equal to 0 and solve for x) to determine vertical asymptotes, or • is not finding $f(0)$ to find y-intercepts and/or is not setting the numerator equal to 0 and solving for x to find x-intercepts.
E	incorrectly "cancels" terms or parts of terms in the numerator and denominator instead of dividing each by a common factor (For example, reducing the expression in item 4 to $6x + 7$.)
F	is reasoning incorrectly by: • not recognizing that the equation is written in a form that gives important information about what the graphical representation looks like, without a provided scale, or • only using one identifying feature to determine what the graph looks like. For example, if only using y-intercepts, the student may not be able to choose between Graph A or Graph D.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

- ALEKS® Graphing Rational Functions
- Lesson 7-3 and 7-4, Learn, Examples 1-2

Revisit the Probe at the end of the module to be sure that your students no longer carry these misconceptions.

IGNITE!

The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

How are the rules for operations with rational numbers applied to operations with rational expressions and equations? Sample answer: Similar to adding and subtracting rational numbers, you can add and subtract rational expressions by using a common denominator. Sums, differences, products, and quotients can be simplified by removing common factors from the numerator and denominator of the rational expression. Also, rational expressions can be set equal to one another to solve for unknowns.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

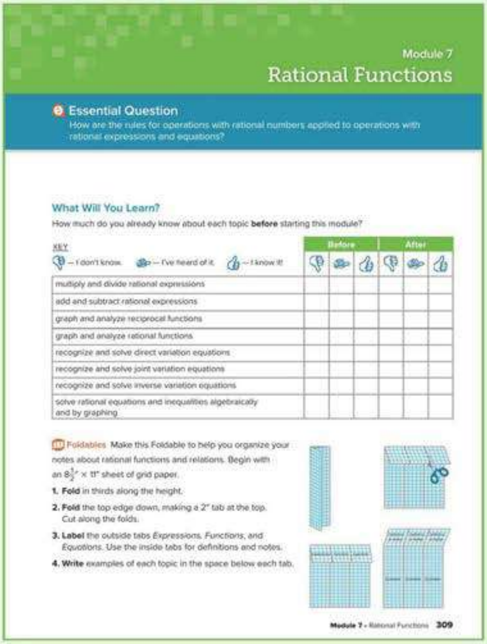
Focus Students write notes about rational functions in this module.

Teach Have students make and label their Foldable as illustrated. Students should use the appropriate tab to record their notes and examples for the concepts in each lesson of this module.

When to Use It Encourage students to add to their Foldable as they work through the module and to use them to review for the module test.

Launch the Module

For this module, the Launch the Module video uses the Doppler effect, scuba diving, and personal goals to show real-world applications of rational equations. Students learn about using rational equations to model the amount of time a scuba diver can safely spend at various depths.



Module 7

Rational Functions

Essential Question

How are the rules for operations with rational numbers applied to operations with rational expressions and equations?

What Will You Learn?

How much do you already know about each topic **before** starting this module?

KEY

👎 — I don't know. 👋 — I've heard of it. 👍 — I know it!

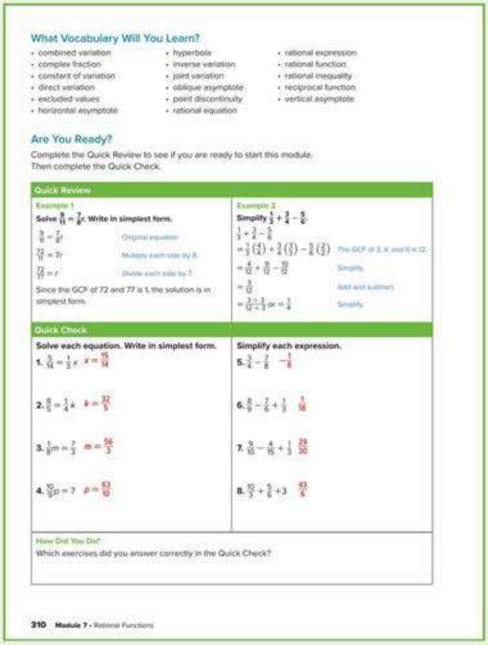
	Before			After		
	👎	👋	👍	👎	👋	👍
multiply and divide rational expressions						
add and subtract rational expressions						
graph and analyze reciprocal functions						
graph and analyze rational functions						
recognize and solve direct variation equations						
recognize and solve joint variation equations						
recognize and solve inverse variation equations						
solve rational equations and inequalities algebraically and by graphing						

Foldables Make this Foldable to help you organize your notes about rational functions and relations. Begin with an $8\frac{1}{2}'' \times 11''$ sheet of grid paper.

1. **Fold** in thirds along the height.
2. **Fold** the top edge down, making a 2" tab at the top. Cut along the folds.
3. **Label** the outside tabs *Expressions, Functions,* and *Equations.* Use the inside tabs for definitions and notes.
4. **Write** examples of each topic in the space below each tab.

Module 7 • Rational Functions 309

Interactive Presentation



What Vocabulary Will You Learn?

- combined variation
- complex fraction
- constant of variation
- direct variation
- excluded values
- horizontal asymptote
- hyperbola
- inverse variation
- joint variation
- oblique asymptote
- point discontinuity
- rational equation
- rational expression
- rational function
- rational inequality
- reciprocal function
- vertical asymptote

Are You Ready?

Complete the Quick Review to see if you are ready to start this module. Then complete the Quick Check.

Quick Review

Example 1

Solve $\frac{9}{11} = \frac{7}{8}r$. Write in simplest form.

$\frac{9}{11} = \frac{7}{8}r$ Original equation

$\frac{72}{11} = 7r$ Multiply each side by 8.

$\frac{72}{77} = r$ Divide each side by 7.

Since the GCF of 72 and 77 is 1, the solution is in simplest form.

Example 2

Simplify $\frac{1}{3} + \frac{3}{4} - \frac{5}{6}$.

$\frac{1}{3} + \frac{3}{4} - \frac{5}{6}$

$= \frac{1}{3}\left(\frac{4}{4}\right) + \frac{3}{4}\left(\frac{3}{3}\right) - \frac{5}{6}\left(\frac{2}{2}\right)$ The GCF of 3, 4, and 6 is 12.

$= \frac{4}{12} + \frac{9}{12} - \frac{10}{12}$ Simplify.

$= \frac{3}{12}$ Add and subtract.

$= \frac{3 \div 3}{12 \div 3}$ or $= \frac{1}{4}$ Simplify.

Quick Check

Solve each equation. Write in simplest form.

1. $\frac{5}{14} = \frac{1}{3}x$ $x = \frac{15}{14}$
2. $\frac{8}{5} = \frac{1}{4}k$ $k = \frac{32}{5}$
3. $\frac{1}{8}m = \frac{7}{3}$ $m = \frac{56}{3}$
4. $\frac{10}{9}p = 7$ $p = \frac{63}{10}$

Simplify each expression.

5. $\frac{3}{4} - \frac{7}{8}$ $-\frac{1}{8}$
6. $\frac{8}{9} - \frac{7}{6} + \frac{1}{3}$ $\frac{1}{18}$
7. $\frac{9}{10} - \frac{4}{15} + \frac{1}{3}$ $\frac{29}{30}$
8. $\frac{10}{3} + \frac{5}{6} + 3$ $\frac{43}{6}$

How Did You Do?

Which exercises did you answer correctly in the Quick Check?

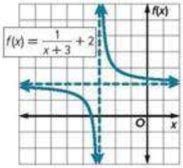
310 **Module 7** • Rational Functions

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define An asymptote is a line that a graph approaches.

Example The graph shows asymptotes at $x = -3$ and $f(x) = 2$.



Ask What does a vertical asymptote show? where a function is undefined

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- factoring
- multiplying polynomials
- evaluating expressions
- identifying domains of functions
- finding function values
- solving proportions

ALEKS

ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You may want to use the **Factoring Quadratic Trinomials** section to ensure student success in this module.

Mindset Matters

View Challenges as Opportunities

Part of cultivating a growth mindset in math involves viewing challenging problems or tasks as an opportunity to learn and make new connections in your brain.

How Can I Apply It?

Encourage students to embrace challenges by trying problems that are thought-provoking, such as the **Higher Order Thinking Problems** in the practice section of each lesson. Remember to regularly remind students that each new challenge is an opportunity to grow!

Lesson 7-1

A.APR.7

Multiplying and Dividing Rational Expressions

LESSON GOAL

Students multiply and divide rational expressions.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Simplifying Rational Expressions

- Simplify a Rational Expression
- Simplify by Using −1

Explore: Simplifying Complex Fractions

Develop:

Multiplying and Dividing Rational Expressions

- Multiply and Divide Rational Expressions
- Multiply and Divide Polynomial Expressions
- Simplify Complex Fractions

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Factoring Quadratic Trinomials	●	●		●
Extension: Dimensional Analysis		●	●	●

Language Development Handbook

Assign page 35 of the *Language Development Handbook* to help your students build mathematical language related to multiplying and dividing rational expressions.

ELL You can use the tips and suggestions on page T35 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra

Standards for Mathematical Content:

A.APR.7 Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous

Students simplified expressions with rational exponents.

N.RN.1, N.RN.2 (Course 2)

Now

Students multiply and divide rational expressions.

A.APR.7

Next

Students will add and subtract rational expressions.

A.APR.7

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of rational expressions. They build fluency by validating that rational expressions are closed under multiplication and division, and they apply their understanding by solving real-world problems related to multiplying and dividing rational expressions.

Mathematical Background

The primary skill needed to multiply and divide rational expressions is simplifying. After division is changed to multiplication by the reciprocal of the divisor, and numerators and denominators are multiplied. Complete the problem by dividing by the common factors.

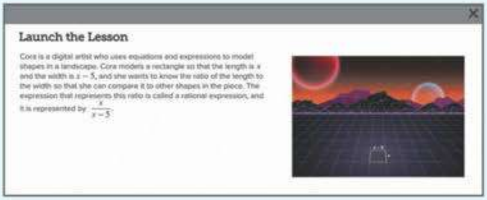
Interactive Presentation

Warm Up

Factor each expression.

1. $45(y^8 + 30y^5)$
2. $9b^2 - 1$
3. $4x^2 + 12x + 9$
4. $4d^2 + 12d - 16$
5. $-6xy^2 + 21xy - 18x$

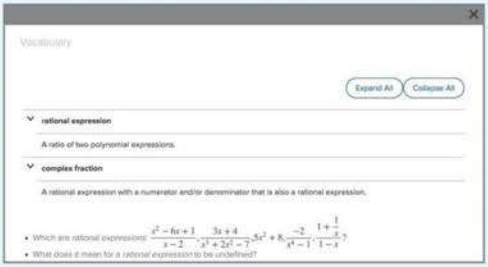
Warm Up



Launch the Lesson

Cora is a digital artist who uses equations and expressions to model shapes in a landscape. Cora models a rectangle so that the length is x and the width is $x - 5$, and she wants to know the ratio of the length to the width so that she can compare it to other shapes in the piece. The expression that represents this ratio is called a rational expression, and it is represented by $\frac{x}{x-5}$.

Launch the Lesson



Vocabulary

rational expression

A ratio of two polynomial expressions.

complex fraction

A rational expression with a numerator and/or denominator that is also a rational expression.

- Which are rational expressions: $\frac{x^2 - 6x + 1}{x - 2}$, $\frac{3x + 4}{x^3 + 2x^2 - 7}$, $5x^2 + 8$, $\frac{-2}{x^4 - 1}$, $\frac{1 + \frac{1}{x}}{1 - x}$?
- What does it mean for a rational expression to be undefined?

Today’s Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- factoring

Answers:

1. $45y^5(3y + 30)$
2. $(3b + 1)(3b - 1)$
3. $(2x + 3)^2$
4. $4(d - 1)(d + 4)$
5. $-3x(2y - 3)(y - 2)$

Launch the Lesson

Teaching the Mathematical Practices

2 Create Representations Guide students to describe and write other ratios using the given information, such as the ratio of the area to the perimeter.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today’s Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today’s Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Simplifying Complex Fractions

Objective

Students use a real-world model to simplify complex fractions.

Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions, like complex fractions, as single objects or as being composed of several objects. Throughout the Explore, guide students to see how complex fractions containing polynomials may be composed of a polynomial and its factors. This skill will help students simplify complex fractions that contain polynomials.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will answer a series of questions about the relationship between numerators and denominators in complex fractions. Then, students will answer the Inquiry Question.

(*continued on the next page*)

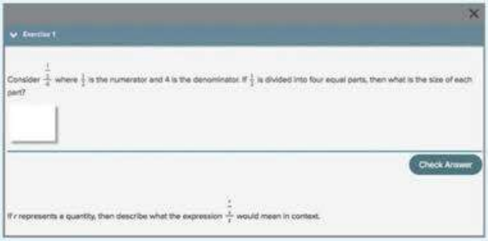
Interactive Presentation

Simplifying Complex Fractions

INQUIRY Can you simplify complex fractions that contain polynomials in the numerator or denominator?

A complex fraction has a numerator and/or a denominator that is also a fraction. Consider a complex fraction of the form $\frac{\frac{r}{s}}{t}$, where the numerator is $\frac{r}{s}$ and the denominator is t.

Explore



Consider $\frac{\frac{1}{2}}{4}$ where $\frac{1}{2}$ is the numerator and 4 is the denominator. If $\frac{1}{2}$ is divided into four equal parts, then what is the size of each part?

If r represents a quantity, then describe what the expression [illegible] would mean in context.

Explore

TYPE

Students answer questions to determine how to simplify complex fractions.

Interactive Presentation

INQUIRY Can you simplify complex fractions that contain polynomials in the numerator or denominator?

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Simplifying Complex Fractions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How do you divide fractions? Sample answer: You can multiply by the reciprocal.
- What happens if both the numerator and denominator have the same terms? Sample answer: The fraction will simplify to 1.

Inquiry

Can you simplify complex fractions that contain polynomials in the numerator or denominator? Yes; sample answer: If the numerators or denominators have common factors, then those factors can be eliminated.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Simplifying Rational Expressions

Objective

Students simplify rational expressions.

Teaching the Mathematical Practices

1 Explain Correspondences Guide students to see the relationships between rational numbers and rational expressions and how they can be simplified.

Things to Remember

Point out that rational expressions are usually used without specifically excluding those values that make the expression undefined. It is understood that only those values for which the expression has meaning are included.

Example 1 Simplify a Rational Expression

Teaching the Mathematical Practices

7 Use the Distributive Property Point out that the Distributive Property is one of the most-used properties in algebra. In Example 1, students must use the Distributive Property to identify common factors and simplify the rational expression.

Questions for Mathematical Discourse

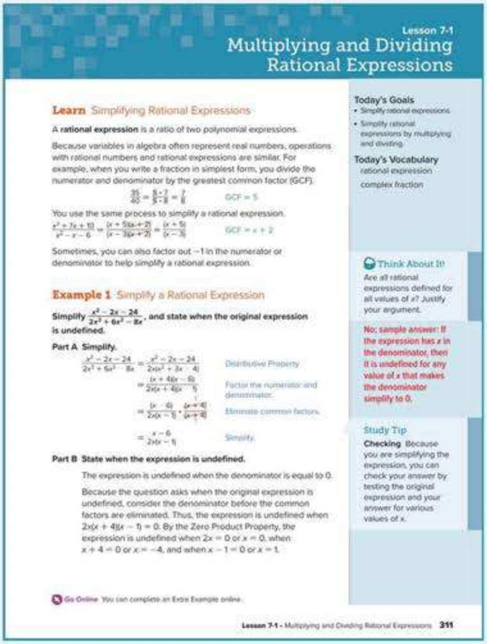
AL What does it mean to simplify a rational expression? Eliminate any common factors in the numerator and denominator.

OL Why can you eliminate common factors and still have an equivalent expression? Sample answer: The common factors form an expression that is equal to 1, and anything multiplied by 1 is itself.

BL Why is the expression undefined when $x = -4$ even though the denominator of the simplified expression is only equal to zero when $x = 0$ and $x = 2$? Sample answer: Because you are finding when the original expression is undefined, you must consider the denominator of the original expression.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 7-1

Multiplying and Dividing Rational Expressions

Learn Simplifying Rational Expressions

A **rational expression** is a ratio of two polynomial expressions.

Because variables in algebra often represent real numbers, operations with rational numbers and rational expressions are similar. For example, when you write a fraction in simplest form, you divide the numerator and denominator by the greatest common factor (GCF).

$$\frac{35}{40} = \frac{5 \cdot 7}{5 \cdot 8} = \frac{7}{8} \qquad \text{GCF} = 5$$

You use the same process to simplify a rational expression.

$$\frac{x^2 + 7x + 10}{x^2 - x - 6} = \frac{(x + 5)(x + 2)}{(x - 3)(x + 2)} = \frac{(x + 5)}{(x - 3)} \qquad \text{GCF} = x + 2$$

Sometimes, you can also factor out −1 in the numerator or denominator to help simplify a rational expression.

Example 1 Simplify a Rational Expression

Simplify $\frac{x^2 - 2x - 24}{2x^3 + 6x^2 - 8x}$, and state when the original expression is undefined.

Part A Simplify.

$$\begin{aligned}\frac{x^2 - 2x - 24}{2x^3 + 6x^2 - 8x} &= \frac{x^2 - 2x - 24}{2x(x^2 + 3x - 4)} && \text{Distributive Property}\\ &= \frac{(x + 4)(x - 6)}{2x(x + 4)(x - 1)} && \text{Factor the numerator and denominator.}\\ &= \frac{(x - 6)}{2x(x - 1)} \cdot \frac{\cancel{(x + 4)}}{\cancel{(x + 4)}} && \text{Eliminate common factors.}\\ &= \frac{x - 6}{2x(x - 1)} && \text{Simplify.}\end{aligned}$$

Part B State when the expression is undefined.

The expression is undefined when the denominator is equal to 0.

Because the question asks when the original expression is undefined, consider the denominator before the common factors are eliminated. Thus, the expression is undefined when $2x(x + 4)(x - 1) = 0$. By the Zero Product Property, the expression is undefined when $2x = 0$ or $x = 0$, when $x + 4 = 0$ or $x = -4$, and when $x - 1 = 0$ or $x = 1$.

Go Online You can complete an Extra Example online.

Today's Goals
- Simplify rational expressions.
- Simplify rational expressions by multiplying and dividing.

Today's Vocabulary
rational expression
complex fraction

Think About It!
Are all rational expressions defined for all values of x? Justify your argument.

No; sample answer: If the expression has x in the denominator, then it is undefined for any value of x that makes the denominator simplify to 0.

Study Tip
Checking Because you are simplifying the expression, you can check your answer by testing the original expression and your answer for various values of x.

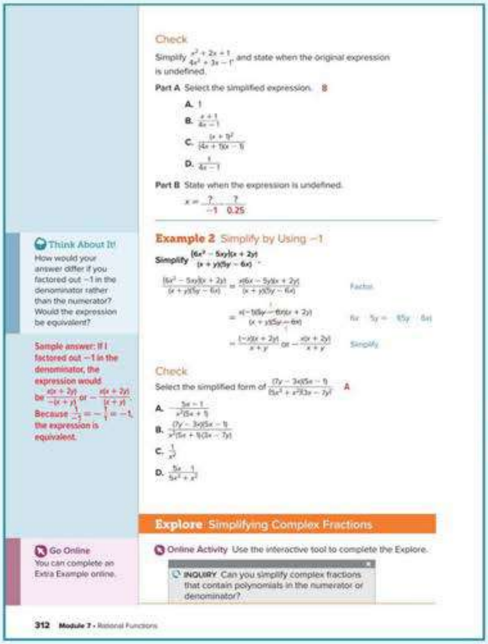
Lesson 7-1 • Multiplying and Dividing Rational Expressions 311

Interactive Presentation

Learn

WATCH

Students watch videos to see how to simplify rational expressions.



Check

Simplify $\frac{x^2+2x+1}{4x^2+3x-1}$, and state when the original expression is undefined.

Part A Select the simplified expression. B

A. 1

B. $\frac{x+1}{4x-1}$

C. $\frac{(x+1)^2}{(4x+1)(x-1)}$

D. $\frac{1}{4x-1}$

Part B State when the expression is undefined.

$x = \underline{\ ?\ }, \underline{\ ?\ }$ −1 0.25

Example 2 Simplify by Using −1

Simplify $\frac{(6x^2-5xy)(x+2y)}{(x+y)(5y-6x)}$.

$$\frac{(6x^2-5xy)(x+2y)}{(x+y)(5y-6x)} = \frac{x(6x-5y)(x+2y)}{(x+y)(5y-6x)}$$ Factor.

$$= \frac{x(-1)(5y-6x)(x+2y)}{(x+y)(5y-6x)}$$ $6x - 5y = -1(5y - 6x)$

$$= \frac{(-x)(x+2y)}{x+y} \text{ or } -\frac{x(x+2y)}{x+y}$$ Simplify.

Check

Select the simplified form of $\frac{(7y-3x)(5x-1)}{(5x^3+x^2)(3x-7y)}$. A

A. $\frac{5x-1}{x^2(5x+1)}$

B. $\frac{(7y-3x)(5x-1)}{x^2(5x+1)(3x-7y)}$

C. $\frac{1}{x^2}$

D. $\frac{5x-1}{5x^3+x^2}$

Think About It! How would your answer differ if you factored out −1 in the denominator rather than the numerator? Would the expression be equivalent?

Sample answer: If I factored out −1 in the denominator, the expression would be $\frac{x(x+2y)}{-(x+y)}$ or $-\frac{x(x+2y)}{(x+y)}$. Because $\frac{1}{-1} = -\frac{1}{1} = -1$, the expression is equivalent.

Explore Simplifying Complex Fractions

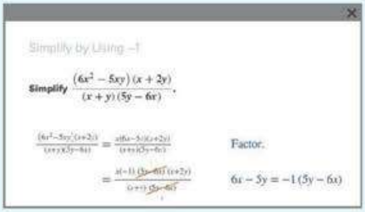
Go Online You can complete an Extra Example online.

Online Activity Use the interactive tool to complete the Explore.

INQUIRY Can you simplify complex fractions that contain polynomials in the numerator or denominator?

312 **Module 7** • Rational Functions

Interactive Presentation



Simplify by Using −1

Simplify $\frac{(6x^2-5xy)(x+2y)}{(x+y)(5y-6x)}$.

$$\frac{(6x^2-5xy)(x+2y)}{(x+y)(5y-6x)} = \frac{x(6x-5y)(x+2y)}{(x+y)(5y-6x)}$$ Factor.

$$= \frac{x(-1)\cancel{(5y-6x)}(x+2y)}{(x+y)\cancel{(5y-6x)}}$$ $6x - 5y = -1(5y - 6x)$

Example 2

TYPE

Students answer a question to determine whether they understand how to simplify rational expressions by using −1.

CHECK

Students complete the Check online to determine whether they are ready to move on.

DIFFERENTIATE

Reteaching Activity AL

IF students are having difficulty simplifying rational expressions, **THEN** encourage them to first view the numerator and denominator as two polynomial expressions that can be factored. Once they have factored each polynomial, they can rewrite the rational expression.

Example 2 Simplify by Using −1

Teaching the Mathematical Practices

1 Understand Different Approaches Ask students to explain the method described in the Think About It! feature and compare it to the method shown in Example 2.

Questions for Mathematical Discourse

AL How can you determine when you will be able to simplify a rational expression using −1? You can simplify using −1 when the numerator contains a term in the form $(a - b)$ and the denominator contains a term in the form $(b - a)$.

OL For what values of x is this expression undefined? $x = -y$ and $x = \frac{5}{6}y$

BL What two properties would you use to prove that $(a - b) = -1(b - a)$? The Commutative Property and the Distributive Property

Learn Multiplying and Dividing Rational Expressions

Objective

Students simplify rational expressions by multiplying and dividing.

Teaching the Mathematical Practices

3 Construct Arguments In the Think About It! feature, students must use definitions and previously established results to construct an argument about the restrictions that apply when multiplying and dividing rational expressions.

DIFFERENTIATE

Enrichment Activity BL

To prepare students for the next lesson and to build a strong base for future work with rational expressions, give them an expression like $\frac{5x^2(x+3)}{5x(x+3)}$. Ask them to explain in detail, citing fundamentals from arithmetic, why five can be divided out but three cannot. Also explain why the first x^2 and x can be divided by x to simplify, but not those within parentheses. Students' explanations should mention that common factors of both the numerator and denominator can be divided out, but not terms that are parts of polynomials. Substitution of a number, like 2, for x may help some students reach this realization.

Example 3 Multiply and Divide Rational Expressions

Teaching the Mathematical Practices

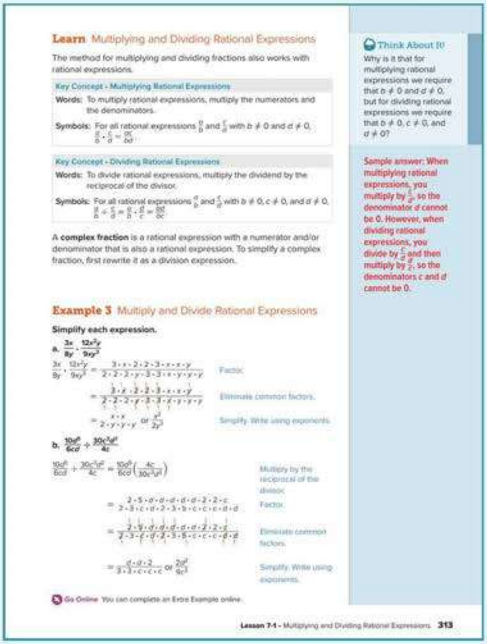
8 Notice Regularity Help students see the regularity in the way that the expressions in the numerator and denominator are factored and in the way that common factors are eliminated.

Questions for Mathematical Discourse

AL How do you know when you have completely factored a number? Each factor is a prime number.

OL How could you simplify the expression in **part a** without writing out the prime factorization of the numerator and denominator? Sample answer: Multiply the numerators to get $36x^3y$ and the denominators to get $72xy^4$. Then, eliminate the greatest common factor of the two expressions, $36xy$.

BL Why is dividing the same as multiplying by the reciprocal? Sample answer: Consider $\frac{\frac{a}{b}}{\frac{c}{d}}$ as a complex fraction. Multiplying the numerator and denominator both by $\frac{d}{c}$ yields $\frac{\left(\frac{a}{b}\right)\cdot\left(\frac{d}{c}\right)}{1}$, which equals $\left(\frac{a}{b}\right)\cdot\left(\frac{d}{c}\right)$.



Learn Multiplying and Dividing Rational Expressions

The method for multiplying and dividing fractions also works with rational expressions.

Key Concept • Multiplying Rational Expressions

Words: To multiply rational expressions, multiply the numerators and the denominators.

Symbols: For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$ and $d \neq 0$, $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$.

Key Concept • Dividing Rational Expressions

Words: To divide rational expressions, multiply the dividend by the reciprocal of the divisor.

Symbols: For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$, $c \neq 0$, and $d \neq 0$, $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$.

A **complex fraction** is a rational expression with a numerator and/or denominator that is also a rational expression. To simplify a complex fraction, first rewrite it as a division expression.

Think About It!

Why is it that for multiplying rational expressions we require that $b \neq 0$ and $d \neq 0$, but for dividing rational expressions we require that $b \neq 0$, $c \neq 0$, and $d \neq 0$?

Sample answer: When multiplying rational expressions, you multiply by $\frac{c}{d}$, so the denominator d cannot be 0. However, when dividing rational expressions, you divide by $\frac{c}{d}$ and then multiply by $\frac{d}{c}$, so the denominators c and d cannot be 0.

Example 3 Multiply and Divide Rational Expressions

Simplify each expression.

a. $\frac{3x}{8y}\cdot\frac{12x^2y}{9xy^3}$

$\frac{3x}{8y}\cdot\frac{12x^2y}{9xy^3}=\frac{3\cdot x\cdot 2\cdot 2\cdot 3\cdot x\cdot x\cdot y}{2\cdot 2\cdot 2\cdot y\cdot 3\cdot 3\cdot x\cdot y\cdot y\cdot y}$ Factor.

Eliminate common factors.

$=\frac{x\cdot x}{2\cdot y\cdot y\cdot y}$ or $\frac{x^2}{2y^3}$ Simplify. Write using exponents.

b. $\frac{10d^5}{6cd}\div\frac{30c^2d^2}{4c}$

$\frac{10d^5}{6cd}\div\frac{30c^2d^2}{4c}=\frac{10d^5}{6cd}\left(\frac{4c}{30c^2d^2}\right)$ Multiply by the reciprocal of the divisor.

$=\frac{2\cdot 5\cdot d\cdot d\cdot d\cdot d\cdot d\cdot 2\cdot 2\cdot c}{2\cdot 3\cdot c\cdot d\cdot 2\cdot 3\cdot 5\cdot c\cdot c\cdot c\cdot d\cdot d}$ Factor.

Eliminate common factors.

$=\frac{d\cdot d\cdot 2}{3\cdot 3\cdot c\cdot c\cdot c}$ or $\frac{2d^2}{9c^3}$ Simplify. Write using exponents.

Go Online You can complete an Extra Example online.

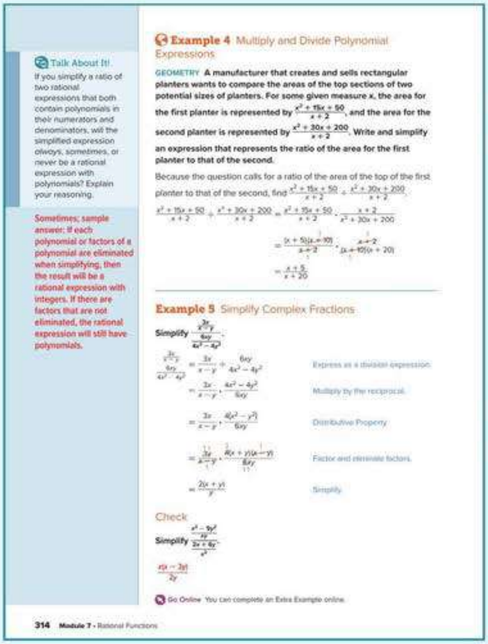
Lesson 7-1 • Multiplying and Dividing Rational Expressions 313

Interactive Presentation

Learn

SWIPE

Students move through slides to see how to multiply and divide rational expressions.



Talk About It!

If you simplify a ratio of two rational expressions that both contain polynomials in their numerators and denominators, will the simplified expression always, sometimes, or never be a rational expression with polynomials? Explain your reasoning.

Sometimes; sample answer: If each polynomial or factors of a polynomial are eliminated when simplifying, then the result will be a rational expression with integers. If there are factors that are not eliminated, the rational expression will still have polynomials.

Example 4 Multiply and Divide Polynomial Expressions

GEOMETRY A manufacturer that creates and sells rectangular planters wants to compare the areas of the top sections of two potential sizes of planters. For some given measure x, the area for the first planter is represented by $\frac{x^2+15x+50}{x+2}$, and the area for the second planter is represented by $\frac{x^2+30x+200}{x+2}$. Write and simplify an expression that represents the ratio of the area for the first planter to that of the second.

Because the question calls for a ratio of the area of the top of the first planter to that of the second, find $\frac{x^2+15x+50}{x+2} \div \frac{x^2+30x+200}{x+2}$.

$$\frac{x^2+15x+50}{x+2} \div \frac{x^2+30x+200}{x+2} = \frac{x^2+15x+50}{x+2} \cdot \frac{x+2}{x^2+30x+200}$$
$$= \frac{(x+5)\cancel{(x+10)}}{\cancel{x+2}} \cdot \frac{\cancel{x+2}}{\cancel{(x+10)}(x+20)}$$
$$= \frac{x+5}{x+20}$$

Example 5 Simplify Complex Fractions

Simplify $\frac{\frac{3x}{x-y}}{\frac{6xy}{4x^2-4y^2}}$.

$\frac{\frac{3x}{x-y}}{\frac{6xy}{4x^2-4y^2}} = \frac{3x}{x-y} \div \frac{6xy}{4x^2-4y^2}$ Express as a division expression.

$= \frac{3x}{x-y} \cdot \frac{4x^2-4y^2}{6xy}$ Multiply by the reciprocal.

$= \frac{3x}{x-y} \cdot \frac{4(x^2-y^2)}{6xy}$ Distributive Property

$= \frac{\cancel{3x}}{\cancel{x-y}} \cdot \frac{\cancel{4}(x+y)\cancel{(x-y)}}{\cancel{6xy}}$ Factor and eliminate factors.

$= \frac{2(x+y)}{y}$ Simplify.

Check

Simplify $\frac{\frac{x^2-9y^2}{xy}}{\frac{2x+6y}{x^2}}$

$\frac{x(x-3y)}{2y}$

Go Online You can complete an Extra Example online.

314 Module 7 • Rational Functions

Interactive Presentation

Example 4

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 4 Multiply and Divide Polynomial Expressions

Teaching the Mathematical Practices

3 Make Conjectures In the Talk About It! feature, students will make a conjecture about the truth of a given statement and then build a logical progression of statements to validate the conjecture. Once students have made their conjectures, guide students to validate them.

Questions for Mathematical Discourse

AL What operation does *ratio* indicate? division

OL Why is it helpful to factor the numerator and denominator of each expression before multiplying the numerators and denominators together? Sample answer: Factoring the quadratic expressions allows for a common factor to be eliminated before multiplying the expressions, simplifying the process.

BL For which values of x is the expression undefined? -2, -10, -20 What values of x make sense in the context of the situation? Explain. $x > -2$; Sample answer: For $x \leq -2$, the expression would be undefined or result in a negative value, which is not valid in the context of the situation.

Example 5 Simplify Complex Fractions

Teaching the Mathematical Practices

7 Use Structure Help students use the structure of the complex fraction to rewrite it as a the product of two rational expressions.

Questions for Mathematical Discourse

AL How do you know when you have completely simplified a complex fraction? The result is one fraction with no common factors in the numerator and denominator.

OL What are the restrictions on x and y? $x \neq -y, x \neq y, x \neq 0, y \neq 0$

BL If $\frac{a}{b} \div \frac{c}{d} = \frac{d}{b}$, what must be true? $a = c$

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–23
2	exercises that use a variety of skills from this lesson	24–45
3	exercises that emphasize higher-order and critical-thinking skills	46–53

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–45 odd, 46–53
- Extension: Dimensional Analysis
- ALEKS Simplifying Expressions

OL

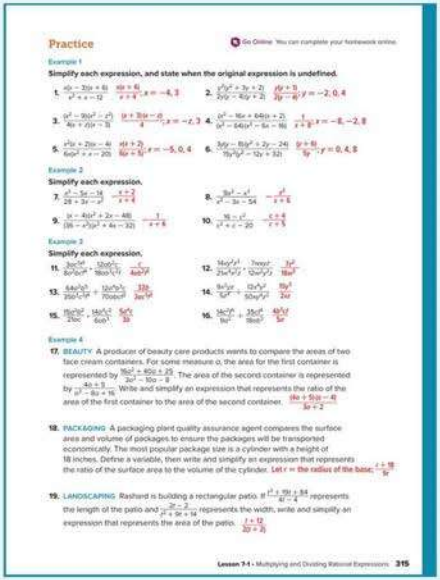
IF students score 66%–89% on the Checks,
THEN assign:

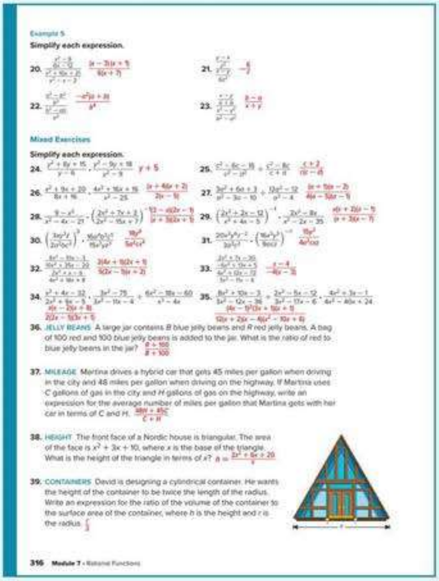
- Practice, Exercises 1–53 odd
- Remediation, Review Resources: Factoring Quadratic Trinomials
- Personal Tutors
- Extra Examples 1–5
- ALEKS Factoring Quadratic Trinomials

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–23 odd
- Remediation, Review Resources: Factoring Quadratic Trinomials
- *Quick Review Math Handbook*: Multiplying and Dividing Rational Expressions
- ALEKS Factoring Quadratic Trinomials

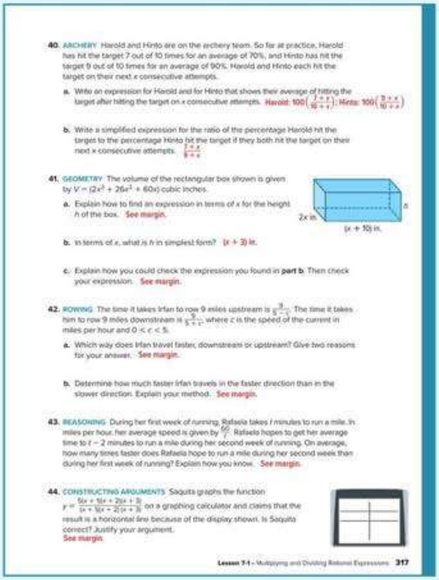




3 REFLECT AND PRACTICE

A.APR.7

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION



40. ARCHERY Harold and Hinto are on the archery team. So far at practice, Harold has hit the target 7 out of 10 times for an average of 70%, and Hinto has hit the target 9 out of 10 times for an average of 90%. Harold and Hinto each hit the target on their next x consecutive attempts.

a. Write an expression for Harold and for Hinto that shows their average of hitting the target after hitting the target on x consecutive attempts. Harold: $100\left(\frac{7+x}{10+x}\right)$; Hinto: $100\left(\frac{9+x}{10+x}\right)$

b. Write a simplified expression for the ratio of the percentage Harold hit the target to the percentage Hinto hit the target if they both hit the target on their next x consecutive attempts. $\frac{7+x}{9+x}$

41. GEOMETRY The volume of the rectangular box shown is given by $V = (2x^3 + 26x^2 + 60x)$ cubic inches.

2x in.

(x + 10) in.

h

a. Explain how to find an expression in terms of x for the height h of the box. See margin.

b. In terms of x, what is h in simplest form? $(x + 3)$ in.

c. Explain how you could check the expression you found in **part b**. Then check your expression. See margin.

42. ROWING The time it takes Irfan to row 9 miles upstream is $\frac{9}{5-c}$. The time it takes him to row 9 miles downstream is $\frac{9}{5+c}$, where c is the speed of the current in miles per hour and $0 < c < 5$.

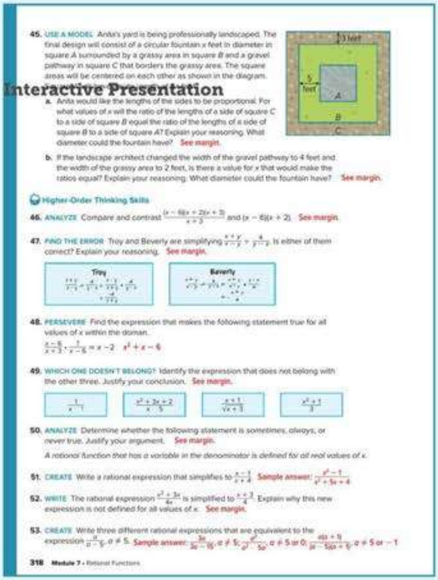
a. Which way does Irfan travel faster, downstream or upstream? Give two reasons for your answer. See margin.

b. Determine how much faster Irfan travels in the faster direction than in the slower direction. Explain your method. See margin.

43. REASONING During her first week of running, Rafaela takes t minutes to run a mile. In miles per hour, her average speed is given by $\frac{60}{t}$. Rafaela hopes to get her average time to $t - 2$ minutes to run a mile during her second week of running. On average, how many times faster does Rafaela hope to run a mile during her second week than during her first week of running? Explain how you know. See margin.

44. CONSTRUCTING ARGUMENTS Saquita graphs the function $y = \frac{5(x+1)(x+2)(x+3)}{(x+1)(x+2)(x+3)}$ on a graphing calculator and claims that the result is a horizontal line because of the display shown. Is Saquita correct? Justify your argument. See margin.

Lesson 7-1 • Multiplying and Dividing Rational Expressions 317



45. USE A MODEL Anita's yard is being professionally landscaped. The final design will consist of a circular fountain x feet in diameter in square A surrounded by a grassy area in square B and a gravel pathway in square C that borders the grassy area. The square areas will be centered on each other as shown in the diagram.

3 feet

5 feet

A

B

C

Interactive Presentation

a. Anita would like the lengths of the sides to be proportional. For what values of x will the ratio of the lengths of a side of square C to a side of square B equal the ratio of the lengths of a side of square B to a side of square A? Explain your reasoning. What diameter could the fountain have? See margin.

b. If the landscape architect changed the width of the gravel pathway to 4 feet and the width of the grassy area to 2 feet, is there a value for x that would make the ratios equal? Explain your reasoning. What diameter could the fountain have? See margin.

Higher-Order Thinking Skills

46. ANALYZE Compare and contrast $\frac{(x-6)(x+2)(x+3)}{x+3}$ and $(x - 6)(x + 2)$. See margin.

47. FIND THE ERROR Troy and Beverly are simplifying $\frac{x+y}{x-y} \div \frac{4}{y-x}$. Is either of them correct? Explain your reasoning. See margin.

Troy

Beverly

48. PERSEVERE Find the expression that makes the following statement true for all values of x within the domain.

$\frac{x-6}{x+3} \cdot \frac{?}{x-6} = x - 2$ $x^2 + x - 6$

49. WHICH ONE DOESN'T BELONG? Identify the expression that does not belong with the other three. Justify your conclusion. See margin.

$\frac{1}{x-1}$ | $\frac{x^2+3x+2}{x-5}$ | $\frac{x+1}{\sqrt{x+3}}$ | $\frac{x^2+1}{3}$

50. ANALYZE Determine whether the following statement is sometimes, always, or never true. Justify your argument. See margin.

A rational function that has a variable in the denominator is defined for all real values of x.

51. CREATE Write a rational expression that simplifies to $\frac{x-1}{x+4}$. Sample answer: $\frac{x^2-1}{x^2+5x+4}$

52. WRITE The rational expression $\frac{x^2+3x}{4x}$ is simplified to $\frac{x+3}{4}$. Explain why this new expression is not defined for all values of x. See margin.

53. CREATE Write three different rational expressions that are equivalent to the expression $\frac{a}{a-5}$, $a \neq 5$. Sample answer: $\frac{3a}{3a-15}$, $a \neq 5$; $\frac{a^2}{a^2-5a}$, $a \neq 5$ or 0; [illegible], $a \neq 5$ or -1

318 Module 7 • Rational Functions

Answers

41a. Sample answer: Divide the volume by the product of the length and width of the box.

41c. Sample answer: Substitute a value for x in each of the given expressions for the length, width, and volume, and the same value for x in the expression found for h, and then check that $V = \ell wh$.

CHECK: For $x = 5$,

length $= (5) + 10 = 15$ in.

width $= 2(5) = 10$ in.

volume $= 2(5)^3 + 26(5)^2 + 60(5)$

$= 1200$ in^3

height $= (5) + 3 = 8$ in.

Verify $V = \ell wh$: $1200 = (15)(10)(8)$

42a. Downstream; sample answer: He is moving with the current and all values for c make his upstream time greater than his downstream time.

42b. Sample answer: Divide the faster (downstream) speed by the slower (upstream) speed to find how many times faster the downstream speed is than the upstream speed. $\frac{9}{5+c} \div \frac{9}{5-c} = \frac{9}{5+c} \cdot \frac{5-c}{9} = \frac{5-c}{5+c}$, so Irfan travels downstream $\frac{5-c}{5+c}$ times faster than he does upstream.

43. Sample answer: By dividing the second week's speed by the first week's speed, you can determine how much faster she hopes to run on average. $\frac{60}{t-2} \div \frac{60}{t} = \frac{60}{t-2} \cdot \frac{t}{60} = \frac{t}{t-2}$. She hopes to run $\frac{t}{t-2}$ faster during her second week than during her first week.

44. No; sample answer: Because of the common factors, the function simplifies to $y = 5$, which is a horizontal line. However, the domain of $y = 5$ is all real numbers, whereas the original function is undefined at $x = -1, -2$, and -3. If Saquita were to use the Trace function, at these values of x she would see that the function is undefined. The result is almost a horizontal line.

45a. None; sample answer: $C : B = \frac{(x+8)}{(x+5)}$, $B : A = \frac{(x+5)}{x}$; For these two ratios to be equal, $\frac{C}{B} \cdot \frac{A}{B}$ must equal 1. To equal 1, their product must have the numerator equal to the denominator. So, $\frac{(x+8)}{(x+5)} \cdot \frac{x}{(x+5)} = \frac{x^2+8x}{(x+5)^2}$ and $x^2 + 8x = (x + 5)(x + 5)$. Solving for x results in $x = -12.5$. Since x cannot be a negative value, there are no values of x that will make these ratios equal.

45b. Yes; sample answer: $C : B = \frac{(x+6)}{(x+2)}$, $B : A = \frac{(x+2)}{x}$ and $\frac{(x+6)}{(x+2)} \cdot \frac{x}{(x+2)} = \frac{x^2+6x}{(x+2)^2}$. When $x^2 + 6x = (x + 2)(x + 2)$, $x = 2$. The length of a side of square A would need to be 4 feet, and the fountain would need to be 2 feet in diameter.

46. Sample answer: The two expressions are equivalent except that the rational expression is undefined at $x = -3$.

47. Beverly; sample answer: Troy's mistake was multiplying by the reciprocal of the dividend instead of the divisor.

49. $\frac{x+1}{\sqrt{x+3}}$; sample answer: The other three expressions are rational expressions. Since the denominator of $\frac{x+1}{\sqrt{x+3}}$ is not a polynomial, $\frac{x+1}{\sqrt{x+3}}$ is not a rational expression.

50. Sometimes; sample answer: With a denominator like $x^2 + 2$, in which the denominator cannot equal 0, the rational expression can be defined for all values of x.

52. Sample answer: When the original expression was simplified, a factor of x was taken out of the denominator. If x were to equal 0, then this expression would be undefined. So, the simplified expression is also undefined for $x = 0$.

Lesson 7-2

Adding and Subtracting Rational Expressions

A.APR.7

LESSON GOAL

Students add and subtract rational expressions.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Closure of Rational Expressions

Develop:

Adding and Subtracting Rational Expressions

- Add and Subtract Rational Expressions with Monomial Denominators
- Add and Subtract Rational Expressions with Polynomial Denominators
- Use Addition and Subtraction of Rational Expressions

Simplifying Complex Fractions

- Simplify Complex Fractions by Using Different LCDs
- Simplify Complex Fractions by Using the Same LCD

You may want your students to complete the Checks online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Multiplying Polynomials by Monomials	●	●		●
Extension: Zeno's Paradox		●	●	●

Language Development Handbook

Assign page 36 of the *Language Development Handbook* to help your students build mathematical language related to adding and subtracting rational expressions.

ELL You can use the tips and suggestions on page T36 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra

Standards for Mathematical Content:

A.APR.7 Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students added, subtracted, and multiplied polynomials.
A.APR.1

Now
Students add and subtract rational expressions.
A.APR.7

Next
Students will solve rational equations and inequalities algebraically and by graphing.
A.CED.1, A.REI.2. A.REI.11

Rigor

The Three Pillars of Rigor

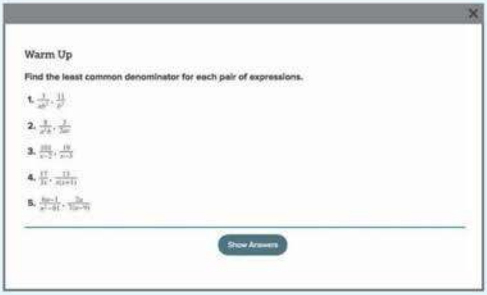
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of rational expressions. They build fluency by validating that rational expressions are closed under addition and subtraction, and they apply their understanding by solving real-world problems related to adding and subtracting rational expressions.

Mathematical Background

When rational expressions are given the same denominators in preparation for addition or subtraction, they are equivalent to the original expressions. Multiplying a rational expression by a form of 1 such as $\frac{6x}{6x}$ or $\frac{y-3}{y-3}$ does not change the value of the expression.

Interactive Presentation



Warm Up

Find the least common denominator for each pair of expressions.

Warm Up

Launch the Lesson

The acidity A of an object is measured using the pH scale, which goes from 0 to 14. After eating food, the acid content remains in the mouth. The acidity can be determined t minutes after eating by using the formula $A = \frac{20.4t}{t^2+36} + 6.5$, which can be simplified by adding and dividing rational expressions.

Launch the Lesson

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- multiplying polynomials

Answers:

1. ab^3

2. $2a^2bc$

3. $(x - 2)(x - 3)$

4. $3x(x + 1)$

5. $7(u + 9)(u - 9)$ or $7(u^2 - 81)$

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will learn how performing operations with rational expressions can be used to solve a real-world problem. Once students have completed the lesson, encourage them to simplify the formula for acidity.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Explore Closure of Rational Expressions

Objective

Students use the rules of closure to determine if the set of rational expressions is closed under all operations.

Teaching the Mathematical Practices

3 Construct Arguments Throughout the Explore, students will use stated assumptions, definitions, and previously established results to construct arguments about the closure of rational expressions.

7 Use Structure Guide students to explore the structure of the sums, differences, products, quotients of rational expressions. Help them to see why rational expressions are closed under addition, subtraction, multiplication, and division.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

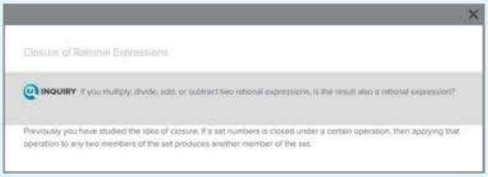
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will answer a series of questions about adding, subtracting, multiplying, and dividing rational expressions to identify patterns in the results of these operations. Then students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

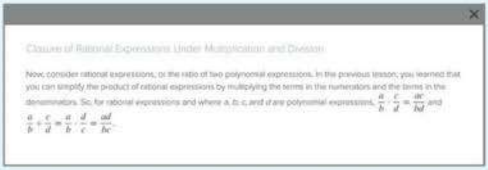


Closure of Rational Expressions

INQUIRY If you multiply, divide, add, or subtract two rational expressions, is the result also a rational expression?

Previously you have studied the idea of closure. If a set numbers is closed under a certain operation, then applying that operation to any two members of the set produces another member of the set.

Explore



Closure of Rational Expressions Under Multiplication and Division

Now, consider rational expressions, or the ratio of two polynomial expressions. In the previous lesson, you learned that you can simplify the product of rational expressions by multiplying the terms in the numerators and the terms in the denominators. So, for rational expressions and where a, b, c, and d are polynomial expressions, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ and $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

Explore

TYPE

Students answer questions to determine whether rational expressions are closed under multiplication, division, addition, and subtraction.

Interactive Presentation

INQUIRY If you multiply, divide, add, or subtract two rational expressions, is the result also a rational expression?

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Closure of Rational Expressions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Consider the integers 4 and 8. For which operations is the result an integer? a rational number? Sample answer: The results of addition, subtraction and multiplication are integers, but for division, $4 \div 8 = \frac{1}{2}$. This is not an integer, but it is a rational number. All operations result in a rational number.
- Is the result of $\frac{3}{x-4} + \frac{6}{x+4}$ a rational expression? Why or why not? Yes; sample answer: Both terms are rational expressions and the set of all rational expressions is closed under addition.

Inquiry

If you multiply, divide, add, or subtract two rational expressions, is the result also a rational expression? Yes; sample answer: The set of rational expressions is closed under all operations.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Adding and Subtracting Rational Expressions

Objective

Students simplify rational expressions by adding and subtracting.

Teaching the Mathematical Practices

7 Use Structure Help students to see how the structure of rational expressions can be used to find the LCD and then add or subtract the expressions.

About the Key Concept

Have students discuss the differences in procedure for adding and multiplying fractions. It is important that they see why common denominators are required for addition but not for multiplication.

Common Misconception

The same misconceptions that cause errors in adding and subtracting rational numbers often cause errors when adding and subtracting rational expressions. When adding and subtracting rational expressions, students often add or subtract both the numerators and denominators, without finding the common denominator. Review addition of rational numbers and fractions in order to reinforce students' understanding of the correct procedure.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students have difficulty adding and subtracting rational expressions, **THEN** have students work as partners, one in the role of coach and one in the role of athlete. The athlete works a problem, following the correct steps and explaining the thinking, while the coach listens and watches for errors, correcting as necessary. Then the partners switch roles.

Example 1 Add and Subtract Rational Expressions with Monomial Denominators

Teaching the Mathematical Practices

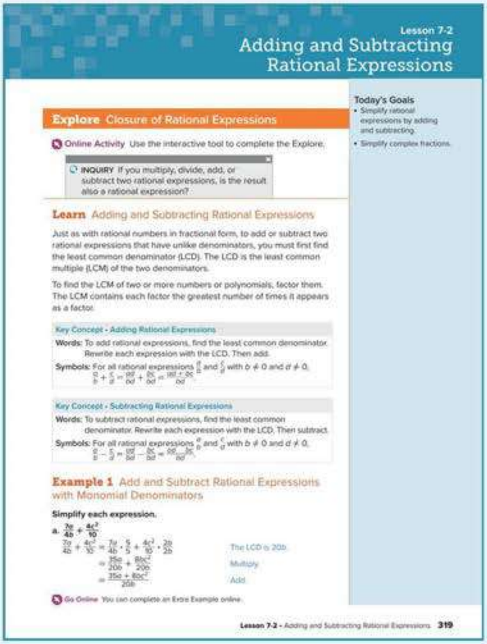
1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions.

Questions for Mathematical Discourse

AL In **part a**, how do you determine that 20*b* is the LCD? Sample answer: Factor 4*b* to 2 • 2 • *b* and 10 to 5 • 2. The greatest number of times 2 appears as a factor is twice, and both *b* and 5 appear once; thus, the LCD is 2 • 2 • 5 • *b* = 20b.

OL What do you multiply each term by once you have determined the LCD? Multiply each term by a fraction with the same numerator and denominator that will make the denominator equal to the LCD.

BL If the LCD of $\frac{1}{a}$ and $\frac{1}{b}$ is *b*, what must be true about a? *a* is a factor of *b*.



Lesson 7-2

Adding and Subtracting Rational Expressions

Today's Goals

- Simplify rational expressions by adding and subtracting.
- Simplify complex fractions.

Explore Closure of Rational Expressions

Online Activity Use the interactive tool to complete the Explore.

INQUIRY If you multiply, divide, add, or subtract two rational expressions, is the result also a rational expression?

Learn Adding and Subtracting Rational Expressions

Just as with rational numbers in fractional form, to add or subtract two rational expressions that have unlike denominators, you must first find the least common denominator (LCD). The LCD is the least common multiple (LCM) of the two denominators.

To find the LCM of two or more numbers or polynomials, factor them. The LCM contains each factor the greatest number of times it appears as a factor.

Key Concept • Adding Rational Expressions

Words: To add rational expressions, find the least common denominator. Rewrite each expression with the LCD. Then add.

Symbols: For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$ and $d \neq 0$,
$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$

Key Concept • Subtracting Rational Expressions

Words: To subtract rational expressions, find the least common denominator. Rewrite each expression with the LCD. Then subtract.

Symbols: For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$ and $d \neq 0$,
$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}$

Example 1 Add and Subtract Rational Expressions with Monomial Denominators

Simplify each expression.

a. $\frac{7a}{4b} + \frac{4c^2}{10}$

$\frac{7a}{4b} + \frac{4c^2}{10} = \frac{7a}{4b} \cdot \frac{5}{5} + \frac{4c^2}{10} \cdot \frac{2b}{2b}$ The LCD is 20b.

$= \frac{35a}{20b} + \frac{8bc^2}{20b}$ Multiply.

$= \frac{35a + 8bc^2}{20b}$ Add.

Go Online You can complete an Extra Example online.

Lesson 7-2 • Adding and Subtracting Rational Expressions 319

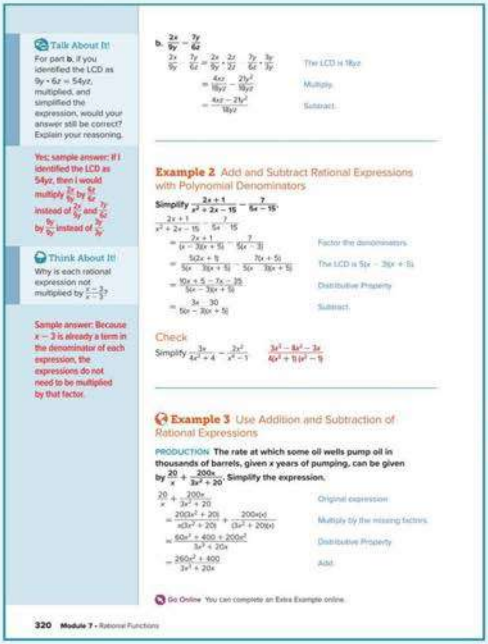
Interactive Presentation

Learn

TYPE

a|

Students answer a question to determine whether they understand how to find common denominators.



Talk About It!

For part **b**, if you identified the LCD as $9y \cdot 6z = 54yz$, multiplied, and simplified the expression, would your answer still be correct? Explain your reasoning.

Yes; sample answer: If I identified the LCD as 54yz, then I would multiply $\frac{2x}{9y}$ by $\frac{6z}{6z}$ instead of $\frac{2z}{2z}$ and $\frac{7y}{6z}$ by $\frac{9y}{9y}$ instead of $\frac{3y}{3y}$.

Think About It!

Why is each rational expression not multiplied by $\frac{x-3}{x-3}$?

Sample answer: Because $x - 3$ is already a term in the denominator of each expression, the expressions do not need to be multiplied by that factor.

b. $\frac{2x}{9y} - \frac{7y}{6z}$

$\frac{2x}{9y} - \frac{7y}{6z} = \frac{2x}{9y} \cdot \frac{2z}{2z} - \frac{7y}{6z} \cdot \frac{3y}{3y}$ The LCD is 18yz.

$= \frac{4xz}{18yz} - \frac{21y^2}{18yz}$ Multiply.

$= \frac{4xz - 21y^2}{18yz}$ Subtract.

Example 2 Add and Subtract Rational Expressions with Polynomial Denominators

Simplify $\frac{2x+1}{x^2+2x-15} - \frac{7}{5x-15}$.

$\frac{2x+1}{x^2+2x-15} - \frac{7}{5x-15}$

$= \frac{2x+1}{(x-3)(x+5)} - \frac{7}{5(x-3)}$ Factor the denominators.

$= \frac{5(2x+1)}{5(x-3)(x+5)} - \frac{7(x+5)}{5(x-3)(x+5)}$ The LCD is $5(x-3)(x+5)$.

$= \frac{10x+5-7x-35}{5(x-3)(x+5)}$ Distributive Property

$= \frac{3x-30}{5(x-3)(x+5)}$ Subtract.

Check

Simplify $\frac{3x}{4x^2+4} - \frac{2x^2}{x^4-1}$. $\frac{3x^3-8x^2-3x}{4(x^2+1)(x^2-1)}$

Example 3 Use Addition and Subtraction of Rational Expressions

PRODUCTION **The rate at which some oil wells pump oil in thousands of barrels, given x years of pumping, can be given by $\frac{20}{x} + \frac{200x}{3x^2+20}$. Simplify the expression.**

$\frac{20}{x} + \frac{200x}{3x^2+20}$ Original expression

$= \frac{20(3x^2+20)}{x(3x^2+20)} + \frac{200x(x)}{(3x^2+20)(x)}$ Multiply by the missing factors.

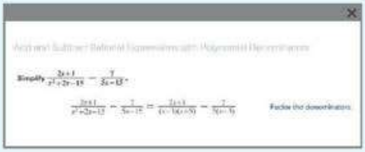
$= \frac{60x^2+400+200x^2}{3x^3+20x}$ Distributive Property

$= \frac{260x^2+400}{3x^3+20x}$ Add.

Go Online You can complete an Extra Example online.

320 **Module 7** • Rational Functions

Interactive Presentation



Example 2

SWIPE

Students move through the steps to subtract rational expressions with polynomial denominators.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Add and Subtract Rational Expressions with Polynomial Denominators

Teaching the Mathematical Practices

7 Use the Distributive Property In Example 2, students must use the Distributive Property to factor the polynomial denominators and identify the LCM of the expressions.

Questions for Mathematical Discourse

AL How does multiplying by the missing factors relate to finding the LCD? Sample answer: Multiplying by the missing factors is what makes the denominators equal the LCD.

OL For what values of *x* is the difference undefined? 3 and —5

BL What would happen if you used a common denominator that was not the least common denominator? Sample answer: After subtracting the rational expressions, you would need to eliminate common factors in the numerator and denominator.

Example 3 Use Addition and Subtraction of Rational Expressions

Teaching the Mathematical Practices

4 Apply Mathematics In Example 3, students apply what they have learned about adding and subtracting rational expressions to solving a real-world problem.

Questions for Mathematical Discourse

AL Why do you not factor the denominators in the first step? The denominators cannot be factored; each denominator is in the simplest form.

OL Do the restrictions on *x* change when you add or subtract rational expressions? Explain. No; sample answer: Because the LCD is composed of each factor of the original denominators.

BL What do you know about the denominator of the combined rational expression if a common factor can be eliminated from the numerator and denominator? The common denominator used was not the least common denominator.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Learn Simplifying Complex Fractions

Objective

Students simplify complex fractions by using the least common denominator.

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the equations used in these examples. Have students identify how the algebraic expressions transform as the complex fractions are simplified.

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. Guide students to see how complex fractions can be simplified by first simplifying the numerator and denominator separately.

Example 4 Simplify Complex Fractions by Using Different LCDs

Teaching the Mathematical Practices

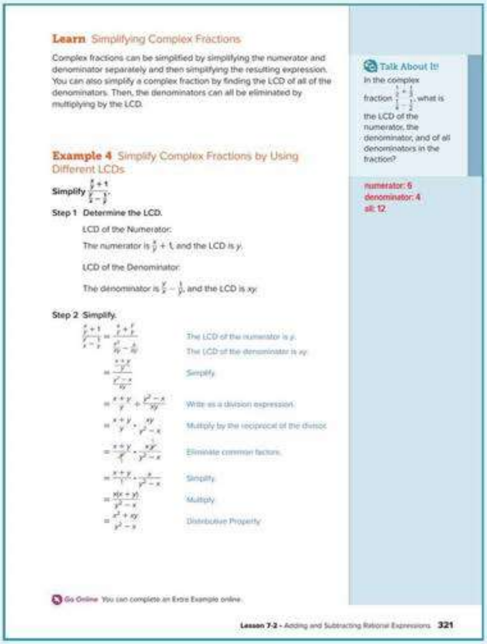
1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach a solution. Point out that in Example 4, students need to transform the expressions in the numerator and denominator to simplify the complex fraction.

Questions for Mathematical Discourse

AL Why do you need to find two different LCDs in this example? Sample answer: Both the numerator and denominator involve addition or subtraction of rational expressions.

OL At what point in the example can you tell that a factor of *y* will be eliminated? Sample answer: when the LCD for the numerator and the LCD for the denominator are found, and each contain a factor of *y*

BL What would happen if the numerator and denominator had the same LCD? Sample answer: The denominators of the expressions in the numerator and denominator would be eliminated when you multiply by the reciprocal.



Learn Simplifying Complex Fractions

Complex fractions can be simplified by simplifying the numerator and denominator separately and then simplifying the resulting expression. You can also simplify a complex fraction by finding the LCD of all of the denominators. Then, the denominators can all be eliminated by multiplying by the LCD.

Example 4 Simplify Complex Fractions by Using Different LCDs

Simplify $\frac{\frac{x}{y}+1}{\frac{y}{x}-\frac{1}{y}}$.

Step 1 **Determine the LCD.**

LCD of the Numerator:

The numerator is $\frac{x}{y}+1$, and the LCD is y.

LCD of the Denominator:

The denominator is $\frac{y}{x}-\frac{1}{y}$, and the LCD is xy.

Step 2 **Simplify.**

$$\frac{\frac{x}{y}+1}{\frac{y}{x}-\frac{1}{y}}=\frac{\frac{x}{y}+\frac{y}{y}}{\frac{y^2}{xy}-\frac{x}{xy}}$$ The LCD of the numerator is y. The LCD of the denominator is xy.

$$=\frac{\frac{x+y}{y}}{\frac{y^2-x}{xy}}$$ Simplify.

$$=\frac{x+y}{y}\div\frac{y^2-x}{xy}$$ Write as a division expression.

$$=\frac{x+y}{y}\cdot\frac{xy}{y^2-x}$$ Multiply by the reciprocal of the divisor.

$$=\frac{x+y}{\cancel{y}}\cdot\frac{x\cancel{y}}{y^2-x}$$ Eliminate common factors.

$$=\frac{x+y}{1}\cdot\frac{x}{y^2-x}$$ Simplify.

$$=\frac{x(x+y)}{y^2-x}$$ Multiply.

$$=\frac{x^2+xy}{y^2-x}$$ Distributive Property

Go Online You can complete an Extra Example online.

Talk About It!

In the complex fraction $\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}-\frac{1}{2}}$, what is the LCD of the numerator, the denominator, and of all denominators in the fraction?

numerator: 6
denominator: 4
all: 12

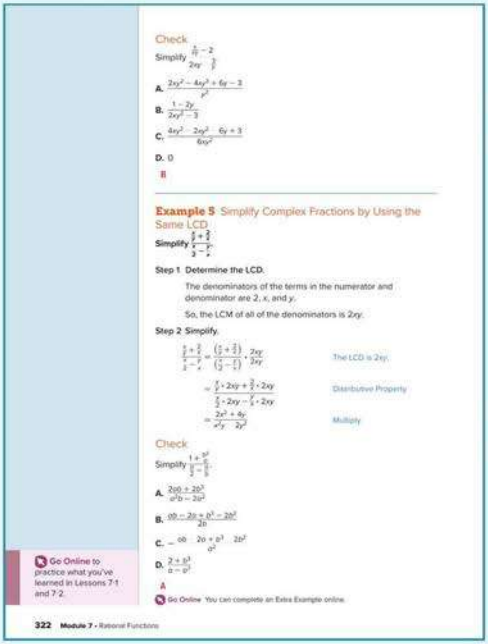
Lesson 7-2 • Adding and Subtracting Rational Expressions 321

Interactive Presentation

Learn

TYPE

Students answer questions to determine the LCDs of parts of a complex fraction.



Check

Simplify

D. 0

B

Example 5 Simplify Complex Fractions by Using the Same LCD

Simplify

Step 1 Determine the LCD.

The denominators of the terms in the numerator and denominator are 2, x, and y.

So, the LCM of all of the denominators is 2xy.

Step 2 Simplify.

The LCD is 2xy.

Distributive Property

Multiply

Check

Simplify

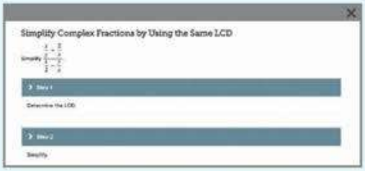
A

Go Online You can complete an Extra Example online.

Go Online to practice what you've learned in Lessons 7-1 and 7-2.

322 Module 7 • Rational Functions

Interactive Presentation



Simplify Complex Fractions by Using the Same LCD

Simplify

Step 1

Determine the LCD

Step 2

Simplify

Example 5

EXPAND

Students tap to see the steps of simplifying a complex fraction.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 5 Simplify Complex Fractions by Using the Same LCD

Teaching the Mathematical Practices

2 Different Properties Mathematically proficient students look for different ways to solve problems. Encourage students to work through the methods shown in Example 4 and Example 5 and choose the method that works best for them.

Questions for Mathematical Discourse

AL Two of the denominators in the original expression are *x*; does this mean the LCD should be $2x^2y$? No; sample answer: You would only need x^2 to be part of the LCD if x^2 was in one of the denominators.

OL What would happen if you converted each term to a fraction with the LCD first? Sample answer: The LCD would be eliminated, and the remaining fraction would be the same as the result from multiplying the entire expression by $\frac{2xy}{2xy}$.

BL What is the advantage of using the same LCD to simplify a complex fraction? Sample answer: The LCD cancels all of the denominators of the terms, yielding one fraction in one step.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–19
2	exercises that use a variety of skills from this lesson	20–52
2	exercises that extend concepts learned in this lesson to new contexts	53–55
3	exercises that emphasize higher-order and critical-thinking skills	56–60

ASSESS AND DIFFERENTIATE

Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–55 odd, 56–60
- Extension: Zeno's Paradox
- ALEKS Simplifying Expressions

OL

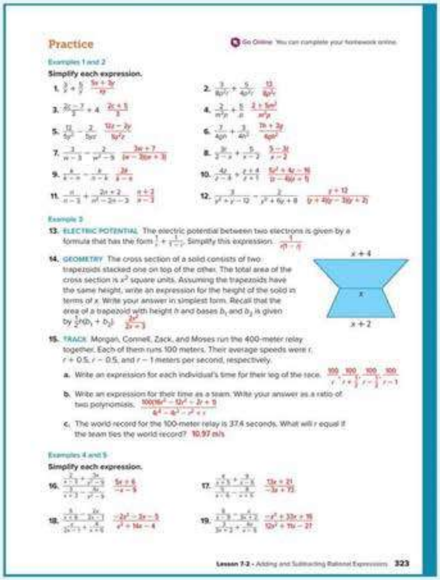
IF students score 66%–89% on the Checks,
THEN assign:

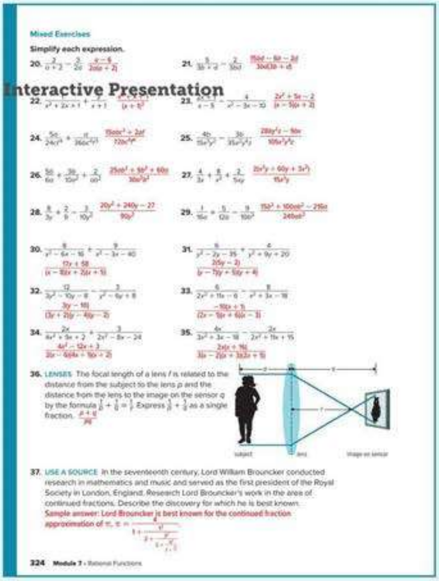
- Practice, Exercises 1–59 odd
- Remediation, Review Resources: Multiplying Polynomials by Monomials
- Personal Tutors
- Extra Examples 1–5
- ALEKS Polynomial Expressions

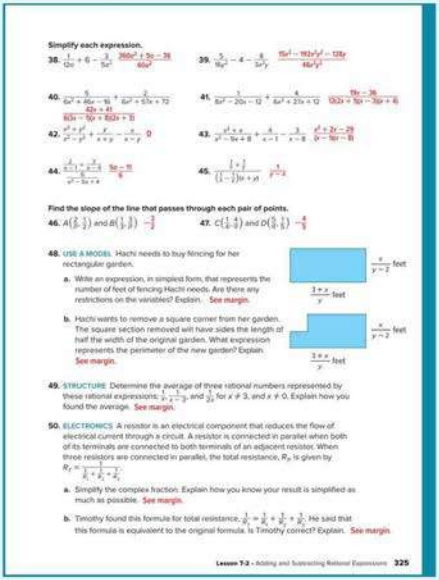
AL

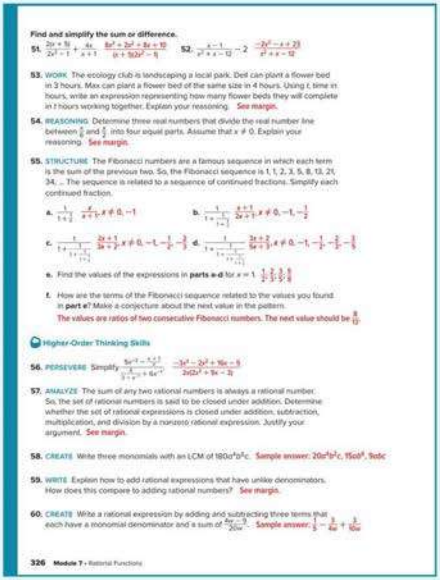
IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–19 odd
- Remediation, Review Resources: Multiplying Polynomials by Monomials
- *Quick Review Math Handbook*: Adding and Subtracting Rational Expressions
- ALEKS Polynomial Expressions









Answers

48a. Perimeter: $2\left(\frac{3+x}{y}\right) + 2\left(\frac{x}{y-2}\right) = \frac{(6+2x)(y-2)+2xy}{y(y-2)}$; sample answer: x and y must be positive rational numbers, $y \geq 0$, $y \neq 2$, and, $x \geq 0$. When $x = 0$, the garden's width would be 0 feet. When x or y is a negative rational number, the $\frac{\text{length}}{\text{width}}$ would result in a negative value. When $y = 0$ or $y = 2$, the length is undefined.

48b. $\left(\frac{3+x}{y}\right) + \left(\frac{3+x}{y} - \frac{x}{2(y-2)}\right) + \left(\frac{x}{y-2}\right) + 3\left(\frac{x}{2(y-2)}\right)$
$= 2\left(\frac{3+x}{y}\right) + \left(\frac{x}{y-2}\right) + \left(\frac{2x}{2(y-2)}\right) = \frac{(6+2x)(y-2)+2xy}{y(y-2)}$;

sample answer: The expression is the same as the perimeter of the original garden.

49. Sample answer: The average is the sum of the given numbers divided by the number of given numbers. So, the average of $\frac{1}{x}$, $\frac{1}{x-3}$, and $\frac{1}{2x}$ is $\frac{\frac{1}{x}+\frac{1}{x-3}+\frac{1}{2x}}{3} = \frac{\frac{1(2)(x-3)}{x(2)(x-3)}+\frac{1(2)(x)}{(x-3)(2)(x)}+\frac{1(x-3)}{2x(x-3)}}{3}$
$= \frac{\frac{2(x-3)}{2x(x-3)}+\frac{2x}{2x(x-3)}+\frac{x-3}{2x(x-3)}}{3} = \frac{\frac{2(x-3)+2x+x-3}{2x(x-3)}}{3} = \frac{2(x-3)+2x+x-3}{6x(x-3)}$
$= \frac{5x-9}{6x(x-3)}$ for $x \neq 0$ and $x \neq 3$.

50a. $\frac{R_1R_2R_3}{R_2R_3+R_1R_3+R_1R_2}$; sample answer: Because there are no factors common to both the numerator and the denominator that can be factored out, my result is simplified.

50b. Yes; sample answer: Given $R_T = \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}} = \frac{R_1R_2R_3}{R_2R_3+R_1R_3+R_1R_2}$, $\frac{1}{R_T}$
$= \frac{R_2R_3+R_1R_3+R_1R_2}{R_1R_2R_3} = \frac{R_2R_3}{R_1R_2R_3} + \frac{R_1R_3}{R_1R_2R_3} + \frac{R_1R_2}{R_1R_2R_3} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$.

53. $\frac{7t}{12}$; Sample answer: In 1 hour, Dell will plant $\frac{1}{3}$ of a flower bed, or $\frac{t}{3}$ flower beds in t hours. In 1 hour, Max will plant $\frac{1}{4}$ of a flower bed, or $\frac{t}{4}$ flower beds in t hours. Adding the two expressions, $\frac{t}{3} + \frac{t}{4} = \frac{7t}{12}$.

54. $\frac{x}{4}$, $\frac{x}{3}$, and $\frac{5x}{12}$ for $x \neq 0$; sample answer: If m is the middle number between $\frac{x}{6}$ and $\frac{x}{2}$, then $m = \left(\frac{x}{6} + \frac{x}{2}\right) \cdot \frac{1}{2} = \frac{4x}{12} = \frac{x}{3}$. If n is the middle number between $\frac{x}{6}$ and $\frac{x}{3}$, then $n = \left(\frac{x}{6} + \frac{x}{3}\right) \cdot \frac{1}{2} = \frac{3x}{12} = \frac{x}{4}$. If p is the middle number between $\frac{x}{3}$ and $\frac{x}{2}$, then $p = \left(\frac{x}{3} + \frac{x}{2}\right) \cdot \frac{1}{2} = \frac{5x}{12}$.

57. Sample answer: The set of rational expressions is closed under all of these operations because the sum, difference, product, and quotient of two rational expressions is a rational expression.

59. Sample answer: First, factor the denominators of all of the expressions. Find the LCD of the denominators. Convert each expression so they all have the LCD. Add or subtract the numerators. Then simplify. It is the same as adding rational numbers.

Lesson 7-3

Graphing Reciprocal Functions

F.IF.5, F.BF.3

LESSON GOAL

Students graph and analyze reciprocal functions.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Graphing Reciprocal Functions

- Limitations on the Domains of Reciprocal Functions
- Graph a Reciprocal Function by Using a Table
- Analyze a Reciprocal Function

Explore: Transforming Reciprocal Functions

Develop:

Transformations of Reciprocal Functions

- Graph a Transformation of a Reciprocal Function
- Write a Reciprocal Function from a Graph

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Algebraic Expressions	●	●		●
Extension: Queue Lengths		●	●	●

Language Development Handbook

Assign page 37 of the *Language Development Handbook* to help your students build mathematical language related to graphing and analyzing reciprocal functions.

ELL You can use the tips and suggestions on page T37 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

6 Attend to precision.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students solved linear equations and inequalities in one variable.
A.CED.1, A.CED.2

Now
Students graph and analyze reciprocal functions.
F.IF.5, F.BF.3

Next
Students will graph and analyze rational functions.
F.IF.4, F.IF.5

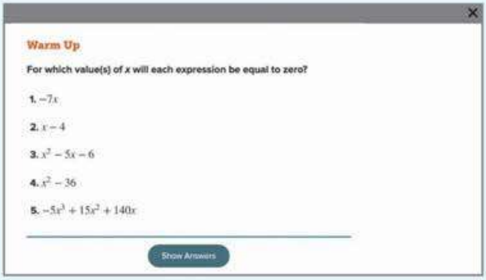
Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of reciprocal functions and build fluency by graphing them. They apply their understanding of graphing reciprocal functions by solving real-world problems.

Interactive Presentation

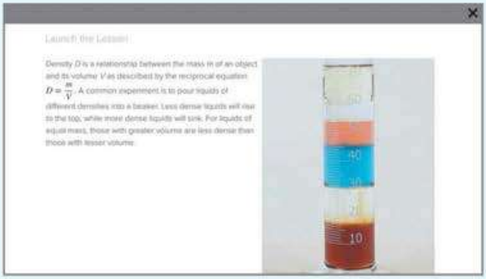


Warm Up

For which value(s) of x will each expression be equal to zero?

1. $-7x$
2. $x - 4$
3. $x^2 - 5x - 6$
4. $x^2 - 36$
5. $-5x^3 + 15x^2 + 140x$

Warm Up



Launch the Lesson

Density D is a relationship between the mass m of an object and its volume V as described by the reciprocal equation $D = \frac{m}{V}$. A common experiment is to pour liquids of different densities into a beaker. Less dense liquids will rise to the top, while more dense liquids will sink. For liquids of equal mass, those with greater volume are less dense than those with lesser volume.

Launch the Lesson

Vocabulary

- reciprocal function
- vertical asymptote
- horizontal asymptote
- hyperbola
- excluded values

- Are the identity function $f(x) = x$ and the reciprocal function $f(x) = \frac{1}{x}$ inverse functions? Why or why not?
- Do you think the graphs of $f(x) = x$ and $f(x) = \frac{1}{x}$ will ever intersect? If so, where?
- How could you determine the excluded values for a reciprocal function?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skills for this lesson:

- finding function values
- factoring

Answers:

1. 0
2. 4
3. −1, 6
4. −6, 6
5. −4, 0, 7

Launch the Lesson

Teaching the Mathematical Practices

2 Make Sense of Quantities Mathematically proficient students need to be able to make sense of quantities and their relationships. In the Launch the Lesson, ask students to describe the relationship between the density, mass, and volume of a liquid. Have students explain how the volume affects the density for liquids with the same mass.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Mathematical Background

A reciprocal function is a function of the form $f(x) = \frac{1}{a(x)}$, where $a(x)$ is a linear function and $a(x) \neq 0$. Reciprocal functions may have breaks in continuity for values that are excluded from the domain, and some may have an asymptote, or a line that the graph approaches.

Explore Transforming Reciprocal Functions

Objective

Students use a sketch to explore how changing the parameters changes the graphs of reciprocal functions.

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the graphs and equations used in this Explore activity.

7 Use Structure Help students explore the structure of reciprocal equations as their parameters are transformed.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

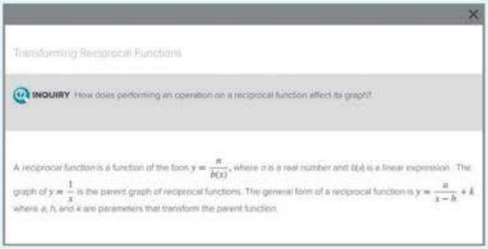
What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will adjust parameters on a sketch and explore the changes to a graph of a reciprocal function. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Transforming Reciprocal Functions

INQUIRY How does performing an operation on a reciprocal function affect its graph?

A *reciprocal function* is a function of the form $y = \frac{a}{b(x)}$, where a is a real number and $b(x)$ is a linear expression. The graph of $y = \frac{1}{x}$ is the parent graph of reciprocal functions. The general form of a reciprocal function is $y = \frac{a}{x-h} + k$ where a, h, and k are parameters that transform the parent function.

Explore

You can use the sketch to investigate how changing the parameters a, h, and k in $y = \frac{a}{x-h} + k$ affects the graphs in the family of reciprocal functions. Then complete the exercises.

Explore

WEB SKETCHPAD

Students will use a sketch to explore transformations of reciprocal functions.

Interactive Presentation

INQUIRY How does performing an operation on a reciprocal function affect its graph?

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Transforming Reciprocal Functions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Why does changing the value of h change the vertical asymptote? Sample answer: The vertical asymptote is based on which values make the denominator equal to zero. If $x - h \neq 0$, then $x \neq h$. So, as you change the value of h, you are changing the vertical asymptote.
- What transformations have been performed on $g(x) = \frac{-3}{x - 2} + 5$? Sample answer: The graph is being shifted to the right 2, up 5, has been reflected across the x-axis and stretched vertically.

Inquiry

How does performing an operation on a reciprocal function affect its graph? Sample answer: Adding or subtracting a constant before or after the function is evaluated causes the graph to be shifted horizontally or vertically, respectively. Multiplying by a constant stretches or compresses the graph vertically. Multiplying by a negative constant reflects the graph in the x-axis, causing the end behavior to change.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Graphing Reciprocal Functions

Objective

Students graph reciprocal functions by examining properties and making tables of values.

MP Teaching the Mathematical Practices

3 Construct Arguments In the Talk About It! feature, students will use definitions and previously established results to construct an argument about the number of vertical asymptotes of a reciprocal function.

Example 1 Limitations on the Domains of Reciprocal Functions

MP Teaching the Mathematical Practices

6 Communicate Precisely Students should write and explain their reasoning when answering the question in the Think About It! feature. Encourage students to use clear mathematical language when explaining why domains are restricted.

7 Use Structure Help students to use the structure of the reciprocal functions to determine the values for which each function is not defined.

Questions for Mathematical Discourse

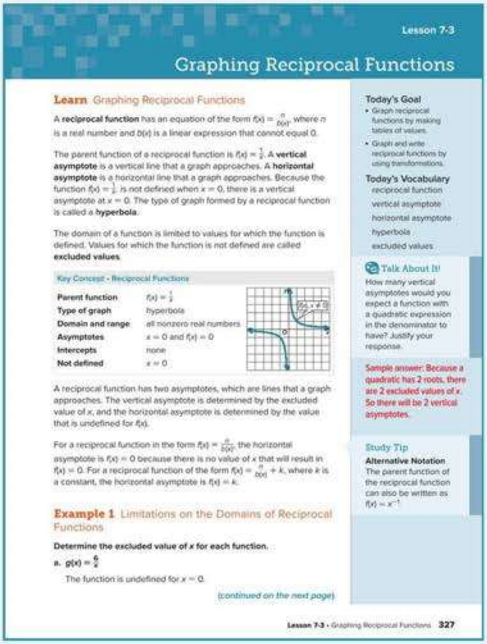
AL Why is a function undefined when the denominator is zero? Sample answer: When the denominator is equal to 0, you are dividing by 0, which is undefined.

OL In **part c**, what is the domain of $g(x)$? $\left(-\infty, -\frac{4}{3}\right)$ $\left(-\frac{4}{3}, \infty\right)$

BL Describe the end behavior of $f(x) = \frac{6}{x}$ as x approaches zero. As x approaches zero from the left, $f(x)$ approaches negative infinity. As x approaches zero from the right, $f(x)$ approaches infinity.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 7-3

Graphing Reciprocal Functions

Learn Graphing Reciprocal Functions

A **reciprocal function** has an equation of the form $f(x) = \frac{n}{b(x)}$, where n is a real number and $b(x)$ is a linear expression that cannot equal 0.

The parent function of a reciprocal function is $f(x) = \frac{1}{x}$. A **vertical asymptote** is a vertical line that a graph approaches. A **horizontal asymptote** is a horizontal line that a graph approaches. Because the function $f(x) = \frac{1}{x}$ is not defined when $x = 0$, there is a vertical asymptote at $x = 0$. The type of graph formed by a reciprocal function is called a **hyperbola**.

The domain of a function is limited to values for which the function is defined. Values for which the function is not defined are called **excluded values**.

Key Concept • Reciprocal Functions

Parent function	$f(x) = \frac{1}{x}$
Type of graph	hyperbola
Domain and range	all nonzero real numbers
Asymptotes	$x = 0$ and $f(x) = 0$
Intercepts	none
Not defined	$x = 0$

A reciprocal function has two asymptotes, which are lines that a graph approaches. The vertical asymptote is determined by the excluded value of x, and the horizontal asymptote is determined by the value that is undefined for $f(x)$.

For a reciprocal function in the form $f(x) = \frac{n}{b(x)}$, the horizontal asymptote is $f(x) = 0$ because there is no value of x that will result in $f(x) = 0$. For a reciprocal function of the form $f(x) = \frac{n}{b(x)} + k$, where k is a constant, the horizontal asymptote is $f(x) = k$.

Example 1 Limitations on the Domains of Reciprocal Functions

Determine the excluded value of x for each function.

a. $g(x) = \frac{6}{x}$

The function is undefined for $x = 0$.

(continued on the next page)

Today's Goal
- Graph reciprocal functions by making tables of values.
- Graph and write reciprocal functions by using transformations.

Today's Vocabulary

reciprocal function
vertical asymptote
horizontal asymptote
hyperbola
excluded values

Talk About It!

How many vertical asymptotes would you expect a function with a quadratic expression in the denominator to have? Justify your response.

Sample answer: Because a quadratic has 2 roots, there are 2 excluded values of x. So there will be 2 vertical asymptotes.

Study Tip

Alternative Notation The parent function of the reciprocal function can also be written as $f(x) = x^{-1}$.

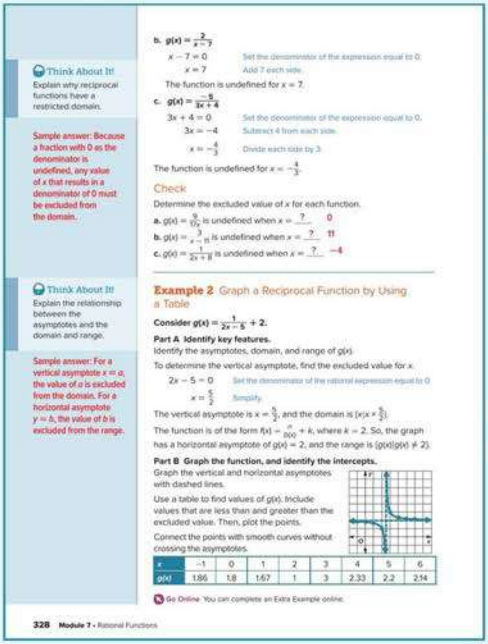
Lesson 7-3 • Graphing Reciprocal Functions 327

Interactive Presentation

Learn

WATCH

Students watch a video to see how to graph reciprocal functions on a graphing calculator.



Think About It!
Explain why reciprocal functions have a restricted domain.

Sample answer: Because a fraction with 0 as the denominator is undefined, any value of x that results in a denominator of 0 must be excluded from the domain.

Think About It!
Explain the relationship between the asymptotes and the domain and range.

Sample answer: For a vertical asymptote $x = a$, the value of a is excluded from the domain. For a horizontal asymptote $y = b$, the value of b is excluded from the range.

b. $g(x) = \frac{2}{x-7}$

$x - 7 = 0$ Set the denominator of the expression equal to 0.

$x = 7$ Add 7 each side.

The function is undefined for $x = 7$.

c. $g(x) = \frac{-5}{3x+4}$

$3x + 4 = 0$ Set the denominator of the expression equal to 0.

$3x = -4$ Subtract 4 from each side.

$x = -\frac{4}{3}$ Divide each side by 3.

The function is undefined for $x = -\frac{4}{3}$.

Check

Determine the excluded value of x for each function.

a. $g(x) = \frac{9}{17x}$ is undefined when $x =$ ___?___ 0

b. $g(x) = \frac{3}{x-11}$ is undefined when $x =$ ___?___ 11

c. $g(x) = \frac{1}{2x+8}$ is undefined when $x =$ ___?___ −4

Example 2 Graph a Reciprocal Function by Using a Table

Consider $g(x) = \frac{1}{2x-5} + 2$.

Part A Identify key features.

Identify the asymptotes, domain, and range of $g(x)$.

To determine the vertical asymptote, find the excluded value for x.

$2x - 5 = 0$ Set the denominator of the rational expression equal to 0.

$x = \frac{5}{2}$ Simplify.

The vertical asymptote is $x = \frac{5}{2}$, and the domain is $\{x \mid x \neq \frac{5}{2}\}$.

The function is of the form $f(x) = \frac{a}{b(x)} + k$, where $k = 2$. So, the graph has a horizontal asymptote of $g(x) = 2$, and the range is $\{g(x) \mid g(x) \neq 2\}$.

Part B Graph the function, and identify the intercepts.

Graph the vertical and horizontal asymptotes with dashed lines.

Use a table to find values of $g(x)$. Include values that are less than and greater than the excluded value. Then, plot the points.

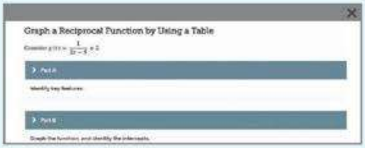
Connect the points with smooth curves without crossing the asymptotes.

x	−1	0	1	2	3	4	5	6
$g(x)$	1.86	1.8	1.67	1	3	2.33	2.2	2.14

Go Online You can complete an Extra Example online.

328 Module 7 • Rational Functions

Interactive Presentation



Graph a Reciprocal Function by Using a Table

Consider $g(x) = \frac{1}{2x-5} + 2$

Part A

Identify key features.

Part B

Graph the function, and identify the intercepts.

Example 2

TYPE

Students answer a question to explain the relationship between asymptotes and the domain and range of reciprocal functions.

Example 2 Graph a Reciprocal Function by Using a Table

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the equations, tables, and graphs used in this example.

Questions for Mathematical Discourse

AL What happens to the value of y as x approaches infinity or negative infinity? It approaches 2. How does this relate to the asymptotes? Sample answer: The function approaches the horizontal asymptote, $y = 2$.
As $x \longrightarrow -\infty$, $y \longrightarrow 2$ and as $x \longrightarrow \infty$, $y \longrightarrow 2$.

OL Why do you need to include x-values that are less than and greater than the excluded value? Sample answer: The reciprocal function exists on each side of the excluded value.

BL If two points on $g(x)$ are equidistant from the vertical asymptote, what do you know about the distance of the points from the horizontal asymptote? The points are equidistant from the horizontal asymptote.

Example 3 Analyze a Reciprocal Function

Teaching the Mathematical Practices

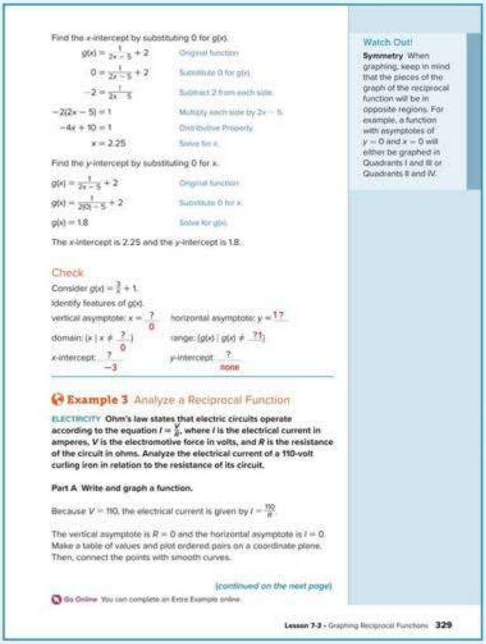
5 Use a Source Guide students to find external information about the standard voltage in another country to answer the question posed in the Use a Source feature.

Questions for Mathematical Discourse

AL For what value(s) of R is the function undefined? $R = 0$ What does this mean in the context of the situation? It is not possible to have a resistance of 0 ohms.

OL What is the valid domain of the function? Explain. $R > 0$; sample answer: Resistance must be positive.

BL Do the asymptotes make sense in the context of the example? Y es; sample answer: As the resistance approaches zero, the current approaches infinity, and as the resistance approaches infinity, the current approaches zero.



Find the x-intercept by substituting 0 for $g(x)$.

$g(x) = \frac{1}{2x-5} + 2$	Original function
$0 = \frac{1}{2x-5} + 2$	Substitute 0 for $g(x)$.
$-2 = \frac{1}{2x-5}$	Subtract 2 from each side.
$-2(2x-5) = 1$	Multiply each side by $2x-5$.
$-4x + 10 = 1$	Distributive Property
$x = 2.25$	Solve for x.

Find the y-intercept by substituting 0 for x.

$g(x) = \frac{1}{2x-5} + 2$	Original function
$g(x) = \frac{1}{2(0)-5} + 2$	Substitute 0 for x.
$g(x) = 1.8$	Solve for $g(x)$.

The x-intercept is 2.25 and the y-intercept is 1.8.

Check

Consider $g(x) = \frac{3}{x} + 1$.

Identify features of $g(x)$.

vertical asymptote: $x = \underline{\ ?\ }$ 0

horizontal asymptote: $y = \underline{\ 1 ?\ }$

domain: $\{x \mid x \neq \underline{\ ?\ }\}$ 0

range: $\{g(x) \mid g(x) \neq \underline{\ ?1\ }\}$

x-intercept: $\underline{\ ?\ }$ −3

y-intercept: $\underline{\ ?\ }$ none

Example 3 Analyze a Reciprocal Function

ELECTRICITY **Ohm's law states that electric circuits operate according to the equation $I = \frac{V}{R}$, where I is the electrical current in amperes, V is the electromotive force in volts, and R is the resistance of the circuit in ohms. Analyze the electrical current of a 110-volt curling iron in relation to the resistance of its circuit.**

Part A Write and graph a function.

Because $V = 110$, the electrical current is given by $I = \frac{110}{R}$.

The vertical asymptote is $R = 0$ and the horizontal asymptote is $I = 0$. Make a table of values and plot ordered pairs on a coordinate plane. Then, connect the points with smooth curves.

(continued on the next page)

Go Online You can complete an Extra Example online.

Watch Out!

Symmetry When graphing, keep in mind that the pieces of the graph of the reciprocal function will be in opposite regions. For example, a function with asymptotes of $y = 0$ and $x = 0$ will either be graphed in Quadrants I and III or Quadrants II and IV.

Lesson 7-3 • Graphing Reciprocal Functions 329

Interactive Presentation

Analyze a Reciprocal Function

Write and graph a function.

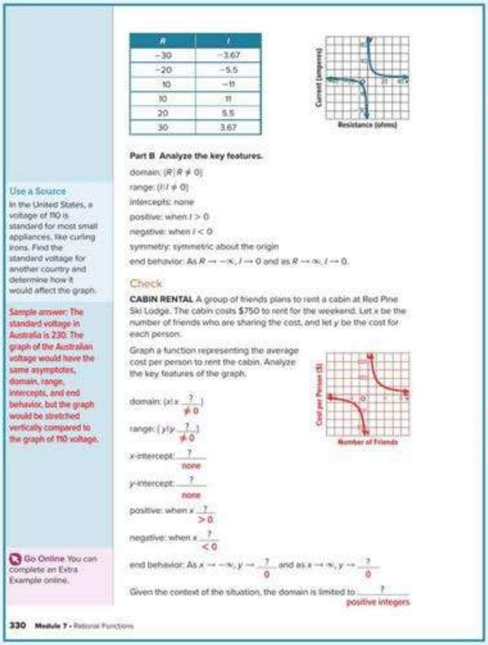
Example 3

EXPAND

Students tap see the steps to analyze a reciprocal function in a real-world context.

CHECK

Students complete the Check online to determine whether they are ready to move on.



R	I
−30	−3.67
−20	−5.5
10	−11
10	11
20	5.5
30	3.67

Part B Analyze the key features.

domain: $\{R \mid R \neq 0\}$

range: $\{I \mid I \neq 0\}$

intercepts: none

positive: when $I > 0$

negative: when $I < 0$

symmetry: symmetric about the origin

end behavior: As $R \to -\infty$, $I \to 0$ and as $R \to \infty$, $I \to 0$.

Use a Source
In the United States, a voltage of 110 is standard for most small appliances, like curling irons. Find the standard voltage for another country and determine how it would affect the graph.

Sample answer: The standard voltage in Australia is 230. The graph of the Australian voltage would have the same asymptotes, domain, range, intercepts, and end behavior, but the graph would be stretched vertically compared to the graph of 110 voltage.

Check

CABIN RENTAL A group of friends plans to rent a cabin at Red Pine Ski Lodge. The cabin costs $750 to rent for the weekend. Let x be the number of friends who are sharing the cost, and let y be the cost for each person.

Graph a function representing the average cost per person to rent the cabin. Analyze the key features of the graph.

domain: $\{x \mid x$ _?_ $\}$ $\neq 0$

range: $\{y \mid y$ _?_ $\}$ $\neq 0$

x-intercept: _?_ none

y-intercept: _?_ none

positive: when x _?_ > 0

negative: when x _?_ < 0

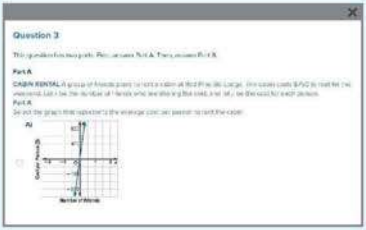
end behavior: As $x \to -\infty$, $y \to$ _?_ 0 and as $x \to \infty$, $y \to$ _?_ 0

Given the context of the situation, the domain is limited to _?_ positive integers

Go Online You can complete an Extra Example online.

330 Module 7 • Rational Functions

Interactive Presentation



Question 3

Part A

CABIN RENTAL

Part A

Check

SELECT

Students select the graph that represents the situation.

Essential Question Follow-Up

Students have begun graphing reciprocal functions.

Ask:

Why are graphs useful? Sample answer: Graphs are useful because they can help you visualize relationships between real-world quantities. They can also be used to estimate function values.

DIFFERENTIATE

Reteaching Activity AL ELL

Visual/Spatial Learners Have students graph one of the functions from the lesson on a large sheet of poster board to clearly show how the graph approaches, but never reaches, an asymptote. Students should plot several points near the asymptotes to help them see the end behavior of the function. Encourage students to use different colors for the asymptotes and function.

Learn Transformations of Reciprocal Functions

Objective

Students graph and write reciprocal functions by using transformations.

MP Teaching the Mathematical Practices

7 Use Structure Guide students to explore how the structure of a transformed reciprocal function can be used to identify the transformations in the function as it relates to the parent function.

Common Misconception

Students may believe that a horizontal translation affects the *horizontal asymptote* and a vertical translation affects the *vertical asymptote*. Encourage students to visualize what happens to the asymptotes as the graph of a rational function is translated horizontally and vertically.

Example 4 Graph a Transformation of a Reciprocal Function

MP Teaching the Mathematical Practices

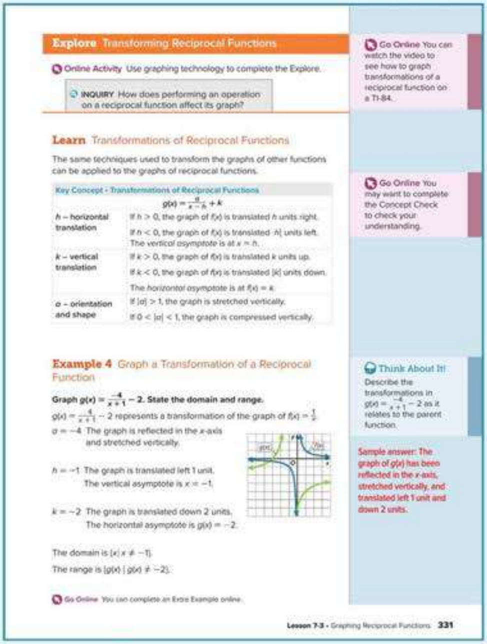
7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. In Example 4, guide students to see what information they can gather about the equation and its related graph just by looking at the equation.

Questions for Mathematical Discourse

AL What is the value of $g(h)$? undefined

OL Why is the vertical asymptote $x = -1$ instead of $x = 1$? Sample answer: $-\frac{4}{x} + 1$ is undefined for $x = -1$.

BL For a linear function, a vertical stretch and a horizontal compression have the same result. Is this the case for the reciprocal function? Explain. Yes; sample answer: For a reciprocal function $f(x) = \frac{n}{x}$, the function $g(x) = a \cdot f(x) = \frac{an}{x}$ where $a > 1$ is a vertical stretch by a factor of a. If you perform a horizontal compression $h(x)$ by a factor of $\frac{1}{a}$, then $h(x) = f(\frac{1}{a} \cdot x) = \frac{n}{\frac{1}{a}} \cdot x$ or $\frac{an}{x}$. The graphs of $g(x)$ and $h(x)$ are the same.



Explore Transforming Reciprocal Functions

Online Activity Use graphing technology to complete the Explore.

INQUIRY How does performing an operation on a reciprocal function affect its graph?

Go Online You can watch the video to see how to graph transformations of a reciprocal function on a TI-84.

Learn Transformations of Reciprocal Functions

The same techniques used to transform the graphs of other functions can be applied to the graphs of reciprocal functions.

Go Online You may want to complete the Concept Check to check your understanding.

Key Concept • Transformations of Reciprocal Functions

$g(x) = \frac{a}{x-h} + k$

h – horizontal translation	If $h > 0$, the graph of $f(x)$ is translated h units right. If $h < 0$, the graph of $f(x)$ is translated $\lvert h \rvert$ units left. The vertical asymptote is at $x = h$.
k – vertical translation	If $k > 0$, the graph of $f(x)$ is translated k units up. If $k < 0$, the graph of $f(x)$ is translated $\lvert k \rvert$ units down. The *horizontal asymptote* is at $f(x) = k$.
a – orientation and shape	If $\lvert a \rvert > 1$, the graph is stretched vertically. If $0 < \lvert a \rvert < 1$, the graph is compressed vertically.

Example 4 Graph a Transformation of a Reciprocal Function

Graph $g(x) = \frac{-4}{x+1} - 2$. State the domain and range.

$g(x) = \frac{-4}{x+1} - 2$ represents a transformation of the graph of $f(x) = \frac{1}{x}$.

$a = -4$ The graph is reflected in the x-axis and stretched vertically.

$h = -1$ The graph is translated left 1 unit. The vertical asymptote is $x = -1$.

$k = -2$ The graph is translated down 2 units. The horizontal asymptote is $g(x) = -2$.

The domain is $\{x \mid x \neq -1\}$.

The range is $\{g(x) \mid g(x) \neq -2\}$.

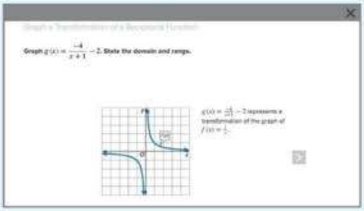
Think About It! Describe the transformations in $g(x) = \frac{-4}{x+1} - 2$ as it relates to the parent function.

Sample answer: The graph of $g(x)$ has been reflected in the x-axis, stretched vertically, and translated left 1 unit and down 2 units.

Go Online You can complete an Extra Example online.

Lesson 7-3 • Graphing Reciprocal Functions 331

Interactive Presentation

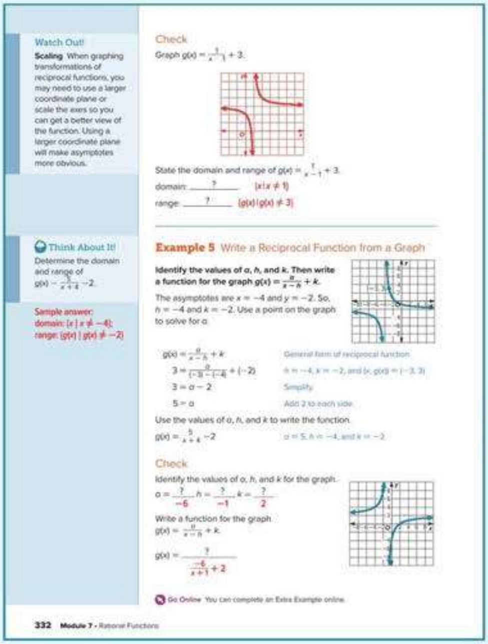


Graph $g(x) = \frac{-4}{x+1} - 2$. State the domain and range.

Example 4

SWIPE

Students move through slides to see how to graph a transformation of a reciprocal function.



Watch Out!

Scaling When graphing transformations of reciprocal functions, you may need to use a larger coordinate plane or scale the axes so you can get a better view of the function. Using a larger coordinate plane will make asymptotes more obvious.

Think About It!

Determine the domain and range of $g(x) = \frac{5}{x+4} - 2$.

Sample answer: domain: $\{x \mid x \neq -4\}$; range: $\{g(x) \mid g(x) \neq -2\}$

Check

Graph $g(x) = \frac{1}{x-1} + 3$.

State the domain and range of $g(x) = \frac{1}{x-1} + 3$.

domain: ___?___ $\{x \mid x \neq 1\}$

range: ___?___ $\{g(x) \mid g(x) \neq 3\}$

Example 5 Write a Reciprocal Function from a Graph

Identify the values of a, h, and k. Then write a function for the graph $g(x) = \frac{a}{x-h} + k$.

The asymptotes are $x = -4$ and $y = -2$. So, $h = -4$ and $k = -2$. Use a point on the graph to solve for a.

$g(x) = \frac{a}{x-h} + k$ General form of reciprocal function

$3 = \frac{a}{(-3)-(-4)} + (-2)$ $h = -4$, $k = -2$, and $(x, g(x)) = (-3, 3)$

$3 = a - 2$ Simplify.

$5 = a$ Add 2 to each side.

Use the values of a, h, and k to write the function.

$g(x) = \frac{5}{x+4} - 2$ $a = 5$, $h = -4$, and $k = -2$

Check

Identify the values of a, h, and k for the graph.

$a =$ ___?___ -6, $h =$ ___?___ -1, $k =$ ___?___ 2

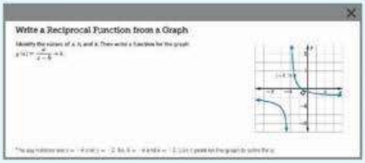
Write a function for the graph $g(x) = \frac{a}{x-h} + k$.

$g(x) =$ ___?___ $\frac{-6}{x+1} + 2$

Go Online You can complete an Extra Example online.

332 Module 7 • Rational Functions

Interactive Presentation



Write a Reciprocal Function from a Graph

Identify the values of a, h, and k. Then write a function for the graph $g(x) = \frac{a}{x-h} + k$.

Example 5

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 5 Write a Reciprocal Function from a Graph

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationship between the function and graph used in this example.

Questions for Mathematical Discourse

AL How do you identify the asymptotes from the graph? Look for the horizontal and vertical lines that the graph approaches but does not reach.

OL How does the vertical asymptote relate to the reciprocal function? The vertical asymptote is the value of *x* that makes the denominator equal to 0.

BL Can you find a reciprocal function given just the two asymptotes? Explain. No; sample answer: Y ou also need to know one of the points. The asymptotes tell you the value of *h* and *k* in the equation, but to solve for the unknown value of *a* you need to substitute in given values of *x* and *g*(*x*).

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–22
2	exercises that use a variety of skills from this lesson	23–48
2	exercises that extend concepts learned in this lesson to new contexts	49, 50
3	exercises that emphasize higher-order and critical-thinking skills	51–56

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–49 odd, 51–56
- Extension: Queue Lengths
- **ALEKS**® Graphing Rational Functions

OL

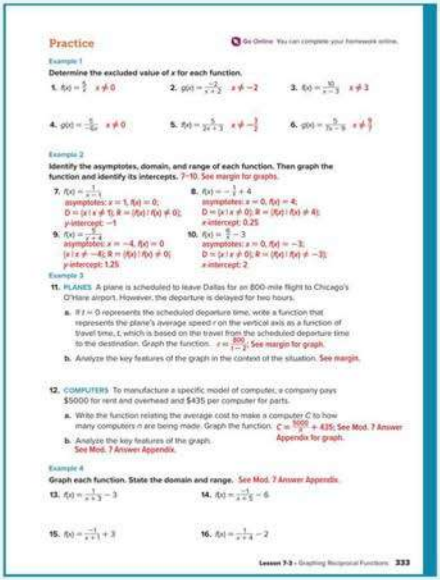
IF students score 66%–89% on the Checks,
THEN assign:

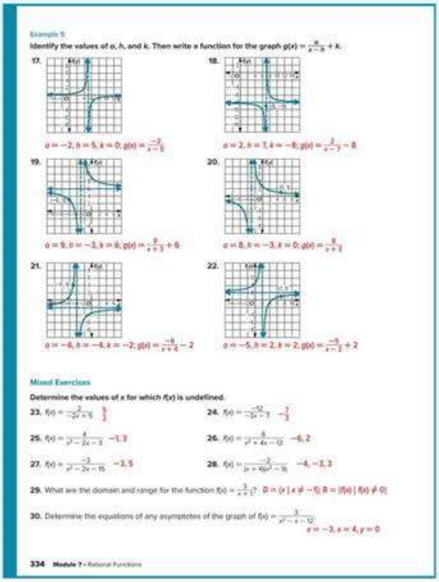
- Practice, Exercises 1–55 odd
- Remediation, Review Resources: Algebraic Expressions
- Personal Tutors
- Extra Examples 1–5
- **ALEKS**® Writing Expressions and Equations

AL

IF students score 65% or less on the Checks,
THEN assign:

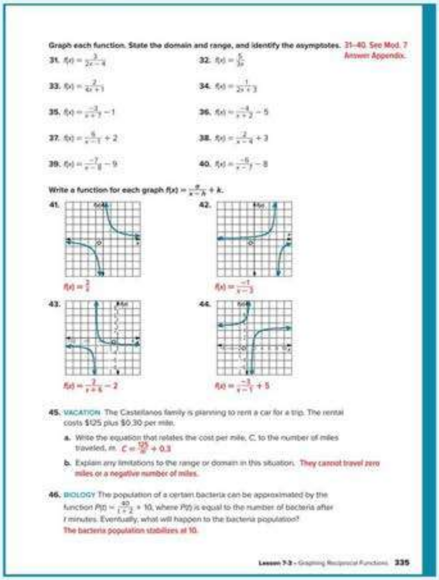
- Practice, Exercises 1–21 odd
- Remediation, Review Resources: Algebraic Expressions
- *Quick Review Math Handbook*: Graphing Reciprocal Functions
- **ALEKS**® Writing Expressions and Equations





3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**



Graph each function. State the domain and range, and identify the asymptotes. 31–40. See Mod. 7 Answer Appendix.

31. $f(x) = \frac{3}{2x-4}$
32. $f(x) = \frac{5}{3x}$
33. $f(x) = \frac{2}{4x+1}$
34. $f(x) = \frac{1}{2x+3}$
35. $f(x) = \frac{-3}{x+7} - 1$
36. $f(x) = \frac{-4}{x+2} - 5$
37. $f(x) = \frac{6}{x-1} + 2$
38. $f(x) = \frac{-2}{x-4} + 3$
39. $f(x) = \frac{-7}{x-8} - 9$
40. $f(x) = \frac{-6}{x-7} - 8$

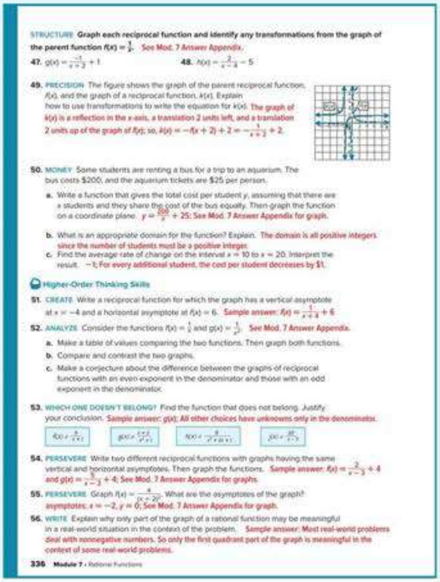
Write a function for each graph $f(x) = \frac{a}{x-h} + k$.

41. $f(x) = \frac{3}{x}$
42. $f(x) = \frac{-1}{x-3}$
43. $f(x) = \frac{-2}{x+6} - 2$
44. $f(x) = \frac{-3}{x-1} + 5$

45. VACATION The Castellanos family is planning to rent a car for a trip. The rental costs \$125 plus \$0.30 per mile.
   a. Write the equation that relates the cost per mile, C, to the number of miles traveled, m. $C = \frac{125}{m} + 0.3$
   b. Explain any limitations to the range or domain in this situation. They cannot travel zero miles or a negative number of miles.

46. BIOLOGY The population of a certain bacteria can be approximated by the function $P(t) = \frac{40}{t+2} + 10$, where $P(t)$ is equal to the number of bacteria after t minutes. Eventually, what will happen to the bacteria population?
The bacteria population stabilizes at 10.

Lesson 7-2 • Graphing Reciprocal Functions 335



STRUCTURE **Graph each reciprocal function and identify any transformations from the graph of the parent function $f(x) = \frac{1}{x}$.** See Mod. 7 Answer Appendix.

47. $g(x) = \frac{-1}{x+2} + 1$
48. $h(x) = \frac{2}{x-4} - 5$

49. PRECISION The figure shows the graph of the parent reciprocal function, $f(x)$, and the graph of a reciprocal function, $k(x)$. Explain how to use transformations to write the equation for $k(x)$. The graph of $k(x)$ is a reflection in the x-axis, a translation 2 units left, and a translation 2 units up of the graph of $f(x)$; so, $k(x) = -f(x + 2) + 2 = -\frac{1}{x+2} + 2$.

50. MONEY Some students are renting a bus for a trip to an aquarium. The bus costs \$200, and the aquarium tickets are \$25 per person.
   a. Write a function that gives the total cost per student y, assuming that there are x students and they share the cost of the bus equally. Then graph the function on a coordinate plane. $y = \frac{200}{x} + 25$; See Mod. 7 Answer Appendix for graph.
   b. What is an appropriate domain for the function? Explain. The domain is all positive integers since the number of students must be a positive integer.
   c. Find the average rate of change on the interval $x = 10$ to $x = 20$. Interpret the result. −1; For every additional student, the cost per student decreases by \$1.

Higher-Order Thinking Skills

51. CREATE Write a reciprocal function for which the graph has a vertical asymptote at $x = -4$ and a horizontal asymptote at $f(x) = 6$. Sample answer: $f(x) = \frac{1}{x+4} + 6$

52. ANALYZE Consider the functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$. See Mod. 7 Answer Appendix.
   a. Make a table of values comparing the two functions. Then graph both functions.
   b. Compare and contrast the two graphs.
   c. Make a conjecture about the difference between the graphs of reciprocal functions with an even exponent in the denominator and those with an odd exponent in the denominator.

53. WHICH ONE DOESN'T BELONG? Find the function that does not belong. Justify your conclusion. Sample answer: g(x); All other choices have unknowns only in the denominator.

$f(x) = \frac{5}{x+1}$ | $g(x) = \frac{x+2}{x^2+1}$ | $h(x) = \frac{4}{x^2+4x+1}$ | $j(x) = \frac{10}{x-3}$

54. PERSEVERE Write two different reciprocal functions with graphs having the same vertical and horizontal asymptotes. Then graph the functions. Sample answer: $f(x) = \frac{2}{x-3} + 4$ and $g(x) = \frac{5}{x-3} + 4$; See Mod. 7 Answer Appendix for graphs.

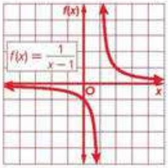
55. PERSEVERE Graph $f(x) = \frac{4}{(x+2)^2}$. What are the asymptotes of the graph? asymptotes: $x = -2$, $y = 0$; See Mod. 7 Answer Appendix for graph.

56. WRITE Explain why only part of the graph of a rational function may be meaningful in a real-world situation in the context of the problem. Sample answer: Most real-world problems deal with nonnegative numbers. So only the first quadrant part of the graph is meaningful in the context of some real-world problems.

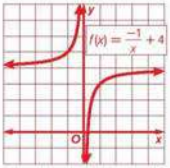
336 Module 7 • Rational Functions

Answers

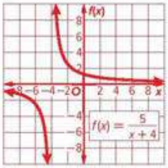
7.



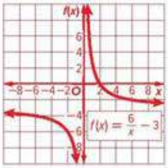
8.



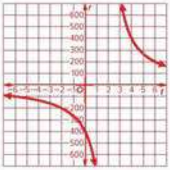
9.



10.



11a.



11b. Sample answer: D = $\{t \mid t > 2\}$; R = $\{r(t) \mid r(t) > 0\}$; intercepts: none; positive: when $t > 2$; negative: when $t < 2$; symmetry: symmetric about (2, 0); end behavior: As $t \longrightarrow -\infty$, $r \longrightarrow 0$, and as $t \longrightarrow \infty$, $r \longrightarrow 0$. Because the plane's speed and travel time cannot be negative, only values in the domain $\{t \mid t > 2\}$ makes sense in the context of this situation.

Lesson 7-4

Graphing Rational Functions

F.IF.4, F.IF.5

LESSON GOAL

Students graph and analyze rational functions.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Analyzing Rational Functions

Develop:

Graphing Rational Functions with Vertical and Horizontal Asymptotes
- Graph with No Horizontal Asymptotes
- Use Graphs of Rational Functions
- Compare Rational Functions

Graphing Rational Functions with Oblique Asymptotes
- Graph with Oblique Asymptotes
- Graph with Point Discontinuity

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Graphing Reciprocal Functions	●	●		●
Extension: Sign Charts for Rational Functions		●	●	●

Language Development Handbook

Assign page 38 of the *Language Development Handbook* to help your students build mathematical language related to graphing and analyzing rational functions.

ELL You can use the tips and suggestions on page T38 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Standards for Mathematical Practice:

4 Model with mathematics.

5 Use appropriate tools strategically.

Coherence

Vertical Alignment

Previous
Students graphed and analyzed reciprocal functions.
F.IF.5, F.BF.3

Now
Students graph and analyze rational functions.
F.IF.4, F.IF.5

Next
Students will solve rational equations and inequalities algebraically and by graphing.
A.CED.1, A.REI.2, A.REI.11

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students extend their understanding of reciprocal functions to rational functions and build fluency by graphing them. They apply their understanding of graphing rational functions by solving real-world problems.

Mathematical Background

A rational function has an equation of the form $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial functions and $b(x) \neq 0$. Some graphs of rational functions have breaks in continuity.

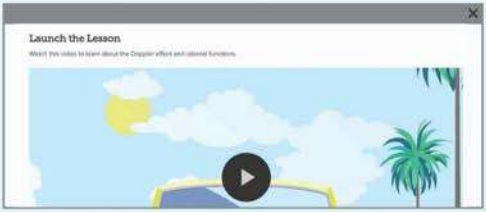
Interactive Presentation

Warm Up

State the domain of each function.

1. $f(x) = -3x + 5$
2. $f(x) = \sqrt{x+2}$
3. $f(x) =$ [illegible]
4. $f(x) =$ [illegible]
5. $f(x) =$ [illegible]

Warm Up



Launch the Lesson

Watch this video to learn about the Doppler effect and rational functions.

Launch the Lesson

Vocabulary

rational function

An equation of the form $f(x) = \frac{a(x)}{b(x)}$, where a(x) and b(x) are polynomial expressions and $b(x) \neq 0$.

oblique asymptote

An asymptote that is neither horizontal nor vertical.

point discontinuity

An area that appears to be a hole in a graph.

- Why are rational functions said to be discontinuous?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- identifying domains of functions

Answers:

1. all real numbers
2. $x \geq -2$
3. all real numbers
4. $x \neq 0$
5. $x \neq -1$

Launch the Lesson

Teaching the Mathematical Practices

2 Attend to Quantities Point out that it is important to note the meaning of the quantities used in the frequency function.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Analyzing Rational Functions

Objective

Students use a graphing calculator to explore graphing rational functions.

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the equations and graphs used in this Explore activity.

5 Analyze Graphs Help students analyze the graph they have generated using a graphing calculator. Point out that to see the entire graph, students may need to adjust the viewing window.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have students' volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They use graphing calculators to graph and analyze features of rational functions. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Analyzing Rational Functions

INQUIRY How can you use a graphing calculator to analyze a rational function?

You can use a graphing calculator to analyze key features of a rational function. Then complete the exercises.

Select a calculator.

Explore

What are the intercepts of the function?

Set the numerator of the function equal to zero and solve for x. How does this compare to the intercepts of the original function?

Explore

TYPE

Students answer questions to describe rational functions.

Interactive Presentation

Explore

TYPE

a|

Students respond to the Inquiry and can view a sample answer.

Explore Analyzing Rational Functions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How is the graph of $g(x) = \frac{x+1}{x+2}$ different than $h(x) = \frac{1}{x+2}$? Sample answer: The function $g(x)$ has x in the numerator and the denominator, while $h(x)$ only has x in the denominator. This changes the x-intercept and the end behavior of the graph.
- What are the x- and y-intercepts of $j(x) = \frac{x-4}{x+8}$? Use your calculator as necessary. The x-intercept is 4, the y-intercept is $-\frac{1}{2}$.

Inquiry

How can you use a graphing calculator to analyze a rational function? Sample answer: A graphing calculator can be used to analyze key features such as the intercepts, end behavior, and asymptotes of rational functions.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Graphing Rational Functions with Vertical and Horizontal Asymptotes

Objective

Students graph and analyze rational functions with vertical and horizontal asymptotes.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write and explain their solution methods. Point out that they should use clear definitions when answering the question in the Think About It feature.

Example 1 Graph with No Horizontal Asymptotes

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the function, table, and graph used in this example.

Questions for Mathematical Discourse

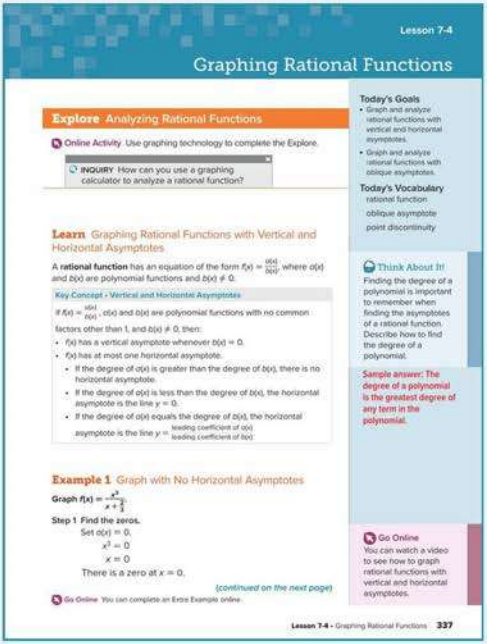
AL Does this function have a horizontal asymptote? Why or why not? No; sample answer: The degree of the numerator is greater than the degree of the denominator.

OL State the domain and range of the function. D: $\left\{x \mid x \neq -\frac{2}{3}\right\}$; R: all real numbers

BL Why does the end behavior of this function resemble the end behavior of a quadratic function? As $|x|$ increases, the constant $\frac{2}{3}$ becomes increasingly insignificant. Therefore, the only terms that have a significant impact on the function as $|x|$ increases are x^3 and x, which simplify to x^2.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 7-4

Graphing Rational Functions

Explore Analyzing Rational Functions

Online Activity Use graphing technology to complete the Explore.

INQUIRY How can you use a graphing calculator to analyze a rational function?

Learn Graphing Rational Functions with Vertical and Horizontal Asymptotes

A **rational function** has an equation of the form $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial functions and $b(x) \neq 0$.

Key Concept • Vertical and Horizontal Asymptotes

If $f(x) = \frac{a(x)}{b(x)}$, $a(x)$ and $b(x)$ are polynomial functions with no common factors other than 1, and $b(x) \neq 0$, then:

- $f(x)$ has a vertical asymptote whenever $b(x) = 0$.
- $f(x)$ has at most one horizontal asymptote.
  - If the degree of $a(x)$ is greater than the degree of $b(x)$, there is no horizontal asymptote.
  - If the degree of $a(x)$ is less than the degree of $b(x)$, the horizontal asymptote is the line $y = 0$.
  - If the degree of $a(x)$ equals the degree of $b(x)$, the horizontal asymptote is the line $y = \frac{\text{leading coefficient of } a(x)}{\text{leading coefficient of } b(x)}$.

Example 1 Graph with No Horizontal Asymptotes

Graph $f(x) = \frac{x^3}{x + \frac{2}{3}}$.

Step 1 Find the zeros.

Set $a(x) = 0$.

$x^3 = 0$

$x = 0$

There is a zero at $x = 0$.

(continued on the next page)

Go Online You can complete an Extra Example online.

Today's Goals
- Graph and analyze rational functions with vertical and horizontal asymptotes.
- Graph and analyze rational functions with oblique asymptotes.

Today's Vocabulary

rational function

oblique asymptote

point discontinuity

Think About It!

Finding the degree of a polynomial is important to remember when finding the asymptotes of a rational function. Describe how to find the degree of a polynomial.

Sample answer: The degree of a polynomial is the greatest degree of any term in the polynomial.

Go Online

You can watch a video to see how to graph rational functions with vertical and horizontal asymptotes.

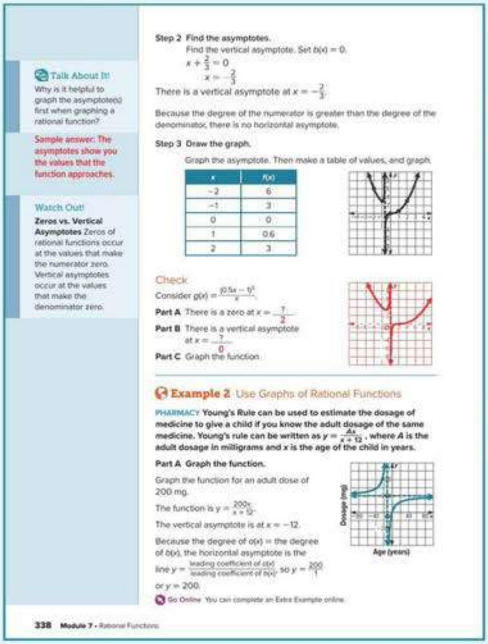
Lesson 7-4 • Graphing Rational Functions 337

Interactive Presentation

Learn

WATCH

Students watch a video to see how to graph rational functions with vertical and horizontal asymptotes.



Talk About It! Why is it helpful to graph the asymptote(s) first when graphing a rational function?

Sample answer: The asymptotes show you the values that the function approaches.

Watch Out!

Zeros vs. Vertical Asymptotes Zeros of rational functions occur at the values that make the numerator zero. Vertical asymptotes occur at the values that make the denominator zero.

Step 2 **Find the asymptotes.**

Find the vertical asymptote. Set $b(x) = 0$.

$x + \frac{2}{3} = 0$

$x = -\frac{2}{3}$

There is a vertical asymptote at $x = -\frac{2}{3}$.

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

Step 3 **Draw the graph.**

Graph the asymptote. Then make a table of values, and graph.

x	$f(x)$
−2	6
−1	3
0	0
1	0.6
2	3

Check

Consider $g(x) = \frac{(0.5x - 1)^3}{x}$.

Part A There is a zero at $x = \frac{1}{2}$.

Part B There is a vertical asymptote at $x = \frac{?}{0}$.

Part C Graph the function.

Example 2 Use Graphs of Rational Functions

PHARMACY **Young's Rule can be used to estimate the dosage of medicine to give a child if you know the adult dosage of the same medicine. Young's rule can be written as $y = \frac{Ax}{x + 12}$, where A is the adult dosage in milligrams and x is the age of the child in years.**

Part A **Graph the function.**

Graph the function for an adult dose of 200 mg.

The function is $y = \frac{200x}{x + 12}$.

The vertical asymptote is at $x = -12$.

Because the degree of $a(x)$ = the degree of $b(x)$, the horizontal asymptote is the line $y = \frac{\text{leading coefficient of } a(x)}{\text{leading coefficient of } b(x)}$, so $y = \frac{200}{1}$ or $y = 200$.

Go Online You can complete an Extra Example online.

338 Module 7 • Rational Functions

Interactive Presentation

Use Graphs of Rational Functions

Example 2

TYPE

Students answer a question about the domain of the function in the context of the situation.

Example 2 Use Graphs of Rational Functions

Teaching the Mathematical Practices

1 Check Answers Mathematically proficient students continually ask themselves, "Does this make sense?" Point out that in this example students need to check their answer. Point out that they should ask themselves whether their answer makes sense and whether they have answered the problem question.

4 Analyze Relationships Mathematically Point out that to solve the problem in Example 2, students will need to analyze the mathematical relationships in the problem to draw conclusions.

Questions for Mathematical Discourse

AL What does the horizontal asymptote represent in the context of the model? the adult dosage

OL Does the end behavior as $x \to \infty$ make sense in the context of the situation? Explain. Yes; sample answer: It makes sense that as a child gets older, and closer to adulthood, they should receive a dosage closer and closer to the adult dosage.

BL Why is the horizontal asymptote the ratio of the leading coefficients? Sample answer: As $|x|$ increases, the constant has less of an effect on the function than the terms with x, so the function looks more like $y = \frac{200x}{x}$ or $y = 200$.

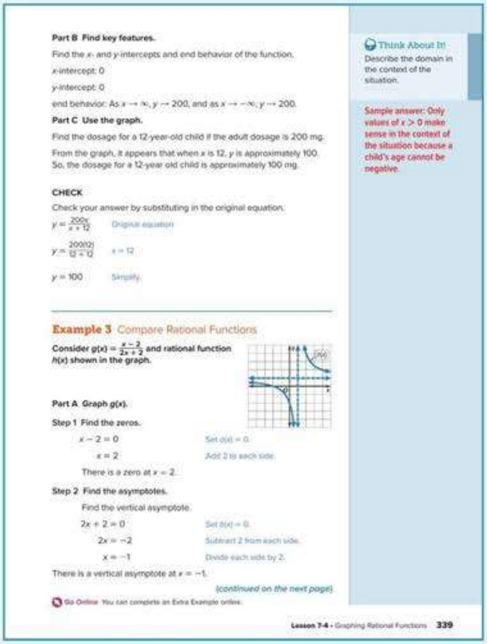
Example 3 Compare Rational Functions

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the key features of the function and graph in Example 3.

Questions for Mathematical Discourse

- **AL** Why do you only set the numerator equal to zero to find the *x*-intercept? When the numerator is zero, the entire expression equals zero.
- **OL** Why is it necessary to determine the asymptotes of the function before making a table of values and graphing? Sample answer: So you know which values to include in the table. If you did not know the asymptotes, you may only choose values of x that fall on one side of the asymptote, giving you only one half of the graph.
- **BL** Compare the end behavior of the functions. $g(x)$: As $x \longrightarrow -\infty$, $y \longrightarrow \frac{1}{2}$ and as $x \longrightarrow \infty$, $y \longrightarrow \frac{1}{2}$; $h(x)$: As $x \longrightarrow -\infty$, $y \longrightarrow 1$ and as $x \longrightarrow \infty$, $y \longrightarrow 1$



Part B **Find key features.**

Find the *x*- and *y*-intercepts and end behavior of the function.

x-intercept: 0

y-intercept: 0

end behavior: As $x \rightarrow \infty$, $y \rightarrow 200$, and as $x \rightarrow -\infty$, $y \rightarrow 200$.

Part C **Use the graph.**

Find the dosage for a 12-year-old child if the adult dosage is 200 mg.

From the graph, it appears that when x is 12, y is approximately 100. So, the dosage for a 12-year old child is approximately 100 mg.

CHECK

Check your answer by substituting in the original equation.

$y = \frac{200x}{x + 12}$ Original equation

$y = \frac{200(12)}{12 + 12}$ $x = 12$

$y = 100$ Simplify.

Example 3 Compare Rational Functions

Consider $g(x) = \frac{x - 2}{2x + 2}$ and rational function $h(x)$ shown in the graph.

Part A **Graph $g(x)$.**

Step 1 **Find the zeros.**

$x - 2 = 0$ Set $a(x) = 0$.

$x = 2$ Add 2 to each side.

There is a zero at $x = 2$.

Step 2 **Find the asymptotes.**

Find the vertical asymptote.

$2x + 2 = 0$ Set $b(x) = 0$.

$2x = -2$ Subtract 2 from each side.

$x = -1$ Divide each side by 2.

There is a vertical asymptote at $x = -1$.

(continued on the next page)

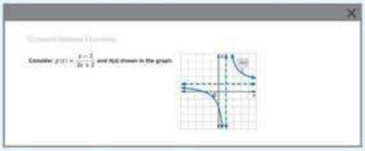
Go Online You can complete an Extra Example online.

Think About It! Describe the domain in the context of the situation.

Sample answer: Only values of $x > 0$ make sense in the context of the situation because a child's age cannot be negative.

Lesson 7-4 • Graphing Rational Functions 339

Interactive Presentation



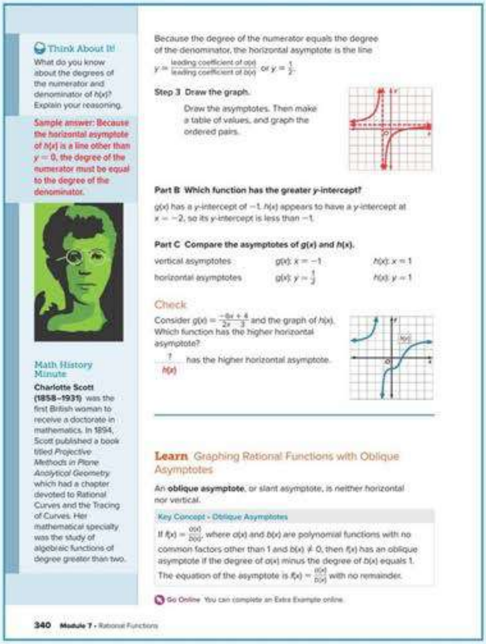
Example 3

EXPAND

Students tap to see how to graph, analyze, and compare rational functions.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Think About It! What do you know about the degrees of the numerator and denominator of $h(x)$? Explain your reasoning.

Sample answer: Because the horizontal asymptote of $h(x)$ is a line other than $y = 0$, the degree of the numerator must be equal to the degree of the denominator.

Because the degree of the numerator equals the degree of the denominator, the horizontal asymptote is the line $y = \frac{\text{leading coefficient of } a(x)}{\text{leading coefficient of } b(x)}$ or $y = \frac{1}{2}$.

Step 3 **Draw the graph.**

Draw the asymptotes. Then make a table of values, and graph the ordered pairs.

Part B **Which function has the greater *y*-intercept?**

$g(x)$ has a y-intercept of -1. $h(x)$ appears to have a y-intercept at $x = -2$, so its y-intercept is less than -1.

Part C **Compare the asymptotes of $g(x)$ and $h(x)$.**

vertical asymptotes	$g(x)$: $x = -1$	$h(x)$: $x = 1$
horizontal asymptotes	$g(x)$: $y = \frac{1}{2}$	$h(x)$: $y = 1$

Check

Consider $g(x) = \frac{-6x + 4}{2x - 3}$ and the graph of $h(x)$. Which function has the higher horizontal asymptote?

$h(x)$ has the higher horizontal asymptote.

Math History Minute

Charlotte Scott (1858–1931) was the first British woman to receive a doctorate in mathematics. In 1894, Scott published a book titled *Projective Methods in Plane Analytical Geometry* which had a chapter devoted to Rational Curves and the Tracing of Curves. Her mathematical specialty was the study of algebraic functions of degree greater than two.

Learn Graphing Rational Functions with Oblique Asymptotes

An **oblique asymptote**, or slant asymptote, is neither horizontal nor vertical.

Key Concept • Oblique Asymptotes

If $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial functions with no common factors other than 1 and $b(x) \neq 0$, then $f(x)$ has an oblique asymptote if the degree of $a(x)$ minus the degree of $b(x)$ equals 1. The equation of the asymptote is $f(x) = \frac{a(x)}{b(x)}$ with no remainder.

Go Online You can complete an Extra Example online.

340 Module 7 • Rational Functions

Interactive Presentation

Graphing Rational Functions with Oblique Asymptotes

Learn

WATCH

Students watch a video to see how to graph rational functions with oblique asymptotes.

Learn Graphing Rational Functions with Oblique Asymptotes

Objective

Students graph and analyze rational functions with oblique asymptotes.

Teaching the Mathematical Practices

1 Special Cases Work with students to look at the cases of discontinuity for rational functions. Encourage students to familiarize themselves with vertical, horizontal, and oblique asymptotes as well as point discontinuities and to know when each one applies.

About the Key Concepts

Students should be familiar with the concept of thinking about asymptotes in terms of how the function approaches that value from each side but never reaches it. Encourage students to consider point discontinuities in the same way, as approaching the same value from either side, but never quite reaching it.

DIFFERENTIATE

Language Development Activity ELL

Have students write a list of tips to help someone draw the graphs of rational functions.

Essential Question Follow-Up

Students have explored rational expressions and functions.

Ask:

How are the properties of a rational function reflected in its graph? Sample answer: Vertical asymptotes occur at values that make the denominator 0; horizontal asymptotes occur when the degree of the numerator is less than or equal to the degree of the denominator; oblique asymptotes occur when the degrees of the numerator and denominator differ by 1; holes occur when the numerator and denominator have a common factor that has a zero.

Example 4 Graph with Oblique Asymptotes

Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. Use the Study Tip to guide students to see how the numerator and denominator of a rational function can be viewed as single objects to determine whether an oblique asymptote exists.

Questions for Mathematical Discourse

AL When finding the oblique asymptote, how can you tell there will be a remainder before you divide the numerator by the denominator? Sample answer: The numerator factors as $(x + 1)^2$and the denominator is $x + 4$, so there are no common factors.

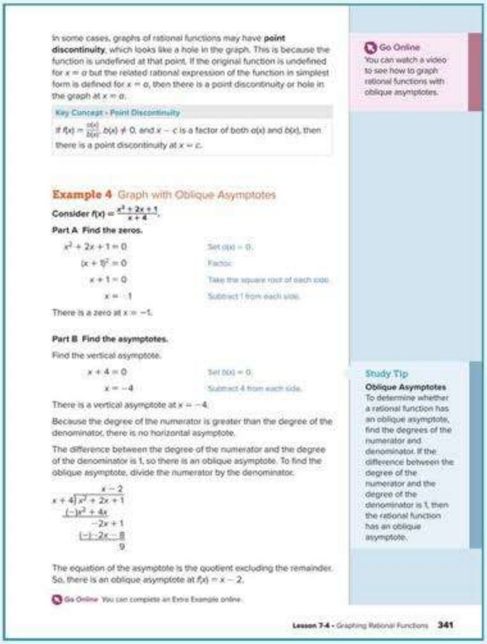
OL How does the end behavior of $f(x)$ compare to the end behavior of the asymptote? The end behavior is the same.

BL Why is the equation of the oblique asymptote $\frac{a(x)}{b(x)}$ with no remainder? Sample answer: No value of x can make the remainder zero, but as $|x|$ approaches infinity, the remainder approaches zero, so the function approaches the quotient.

DIFFERENTIATE

Enrichment Activity BL

Challenge students to explain the rules for finding horizontal and oblique asymptotes. While the module shows students how to find them, encourage students to provide an explanation of why the rules work. Scaffold the task by having students examine graphs of varying degrees in the numerator and denominator and look for general patterns in the asymptotes.



In some cases, graphs of rational functions may have **point discontinuity**, which looks like a hole in the graph. This is because the function is undefined at that point. If the original function is undefined for $x = a$ but the related rational expression of the function in simplest form is defined for $x = a$, then there is a point discontinuity or hole in the graph at $x = a$.

Key Concept • Point Discontinuity

If $f(x) = \frac{a(x)}{b(x)}$, $b(x) \neq 0$, and $x - c$ is a factor of both $a(x)$ and $b(x)$, then there is a point discontinuity at $x = c$.

Go Online You can watch a video to see how to graph rational functions with oblique asymptotes.

Example 4 Graph with Oblique Asymptotes

Consider $f(x) = \frac{x^2 + 2x + 1}{x + 4}$.

Part A Find the zeros.

$x^2 + 2x + 1 = 0$ Set $a(x) = 0$.

$(x + 1)^2 = 0$ Factor.

$x + 1 = 0$ Take the square root of each side.

$x = -1$ Subtract 1 from each side.

There is a zero at $x = -1$.

Part B Find the asymptotes.

Find the vertical asymptote.

$x + 4 = 0$ Set $b(x) = 0$.

$x = -4$ Subtract 4 from each side.

There is a vertical asymptote at $x = -4$.

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

The difference between the degree of the numerator and the degree of the denominator is 1, so there is an oblique asymptote. To find the oblique asymptote, divide the numerator by the denominator.

$$\begin{array}{r} x - 2 \\ x + 4 \overline{) x^2 + 2x + 1} \\ \underline{(-) x^2 + 4x} \\ -2x + 1 \\ \underline{(-) -2x - 8} \\ 9 \end{array}$$

The equation of the asymptote is the quotient excluding the remainder. So, there is an oblique asymptote at $f(x) = x - 2$.

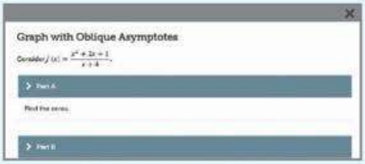
Go Online You can complete an Extra Example online.

Study Tip

Oblique Asymptotes To determine whether a rational function has an oblique asymptote, find the degrees of the numerator and denominator. If the difference between the degree of the numerator and the degree of the denominator is 1, then the rational function has an oblique asymptote.

Lesson 7-4 • Graphing Rational Functions 341

Interactive Presentation



Graph with Oblique Asymptotes

Consider $f(x) = \frac{x^2 + 2x + 1}{x + 4}$.

Part A

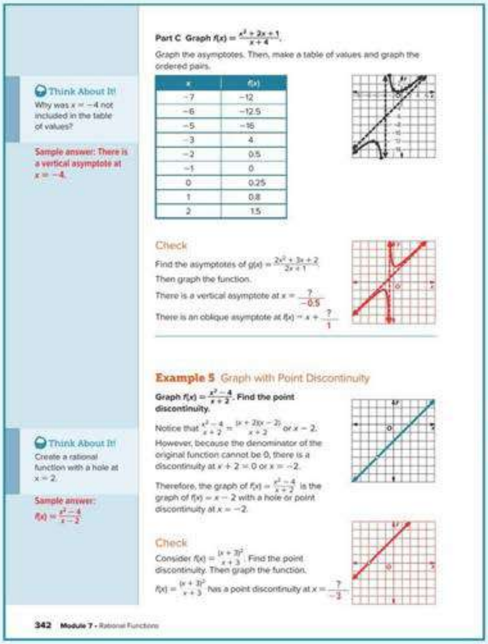
Find the zeros.

Part B

Example 4

TYPE

Students answer a question to determine the effect of undefined values on graphs of rational functions.



Think About It!

Why was $x = -4$ not included in the table of values?

Sample answer: There is a vertical asymptote at $x = -4$.

Part C **Graph $f(x) = \frac{x^2 + 2x + 1}{x + 4}$.**

Graph the asymptotes. Then, make a table of values and graph the ordered pairs.

x	$f(x)$
−7	−12
−6	−12.5
−5	−16
−3	4
−2	0.5
−1	0
0	0.25
1	0.8
2	1.5

Check

Find the asymptotes of $g(x) = \frac{2x^2 + 3x + 2}{2x + 1}$. Then graph the function.

There is a vertical asymptote at $x =$ ___?___ −0.5

There is an oblique asymptote at $f(x) = x +$ ___?___ 1

Example 5 Graph with Point Discontinuity

Graph $f(x) = \frac{x^2 - 4}{x + 2}$. Find the point discontinuity.

Notice that $\frac{x^2 - 4}{x + 2} = \frac{(x + 2)(x - 2)}{x + 2}$ or $x - 2$. However, because the denominator of the original function cannot be 0, there is a discontinuity at $x + 2 = 0$ or $x = -2$.

Therefore, the graph of $f(x) = \frac{x^2 - 4}{x + 2}$ is the graph of $f(x) = x - 2$ with a hole or point discontinuity at $x = -2$.

Think About It!

Create a rational function with a hole at $x = 2$.

Sample answer: $f(x) = \frac{x^2 - 4}{x - 2}$

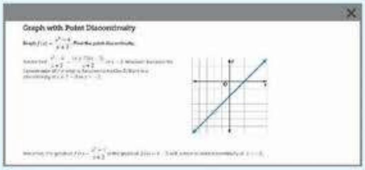
Check

Consider $f(x) = \frac{(x + 3)^2}{x + 3}$. Find the point discontinuity. Then graph the function.

$f(x) = \frac{(x + 3)^2}{x + 3}$ has a point discontinuity at $x =$ ___?___ −3

342 Module 7 • Rational Functions

Interactive Presentation



Graph with Point Discontinuity

Graph $f(x) = \frac{x^2 - 4}{x + 2}$. Find the point discontinuity.

Example 5

TYPE

Students answer a question to determine whether they understand

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 5 Graph with Point Discontinuity

Teaching the Mathematical Practices

1 Explain Correspondences Guide students as they use the information in the Think About It! feature to create a function with the given feature.

Questions for Mathematical Discourse

AL When does a point discontinuity occur? A point discontinuity occurs when the numerator and denominator of a rational function have a common factor.

OL How can you find the *y*-coordinate of a point discontinuity? Sample answer: Substitute the *x*-coordinate of the hole into the simplified form of the rational expression.

BL Why does a common factor in the numerator and the denominator result in a point discontinuity? Sample answer: For any value of *x* that makes the common factor not equal to zero, the common factor can be eliminated, resulting in a continuous function for those values of *x*. When an *x*-value makes the common factor zero, $\frac{0}{0}$ is undefined rather than 1, so the common factor cannot cancel.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–22
2	exercises that use a variety of skills from this lesson	23–53
3	exercises that emphasize higher-order and critical-thinking skills	54–59

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–53 odd, 54–59
- Extension: Sign Charts for Rational Functions
- ALEKS® Graphing Rational Functions

OL

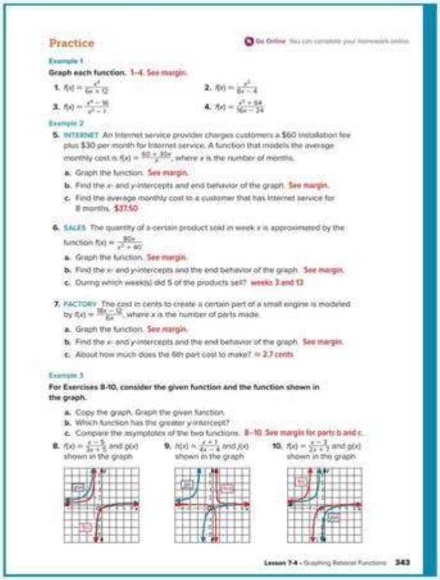
IF students score 66%–89% on the Checks,
THEN assign:

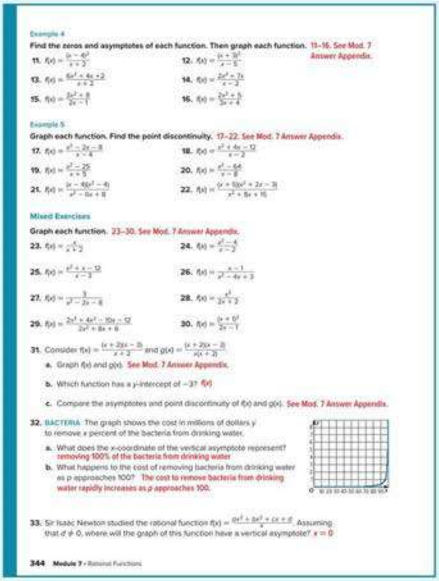
- Practice, Exercises 1–59 odd
- Remediation, Review Resources: Graphing Reciprocal Functions
- Personal Tutors
- Extra Examples 1–5
- ALEKS® Graphing Rational Functions

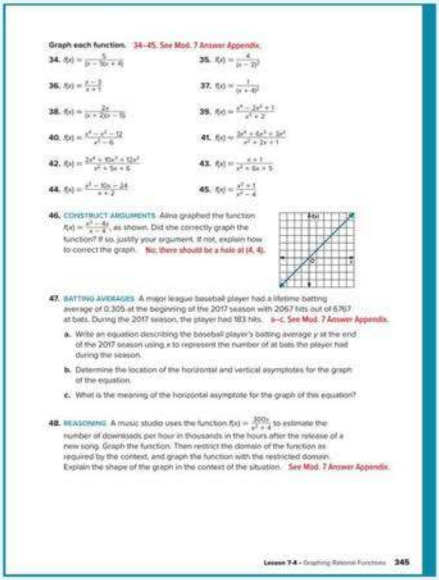
AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–21 odd
- Remediation, Review Resources: Graphing Reciprocal Functions
- *Quick Review Math Handbook*: Graphing Rational Functions
- ALEKS® Graphing Rational Functions







Graph each function. 34–45. See Mod. 7 Answer Appendix.

34. $f(x)=\frac{5}{(x-3)(x+4)}$
35. $f(x)=\frac{4}{(x-2)^2}$
36. $f(x)=\frac{x-3}{x+1}$
37. $f(x)=\frac{1}{(x+4)^2}$
38. $f(x)=\frac{2x}{(x+2)(x-5)}$
39. $f(x)=\frac{x^4-2x^2+1}{x^3+2}$
40. $f(x)=\frac{x^4-x^2-12}{x^3-6}$
41. $f(x)=\frac{3x^4+6x^3+3x^2}{x^2+2x+1}$
42. $f(x)=\frac{2x^4+10x^3+12x^2}{x^2+5x+6}$
43. $f(x)=\frac{x+1}{x^2+6x+5}$
44. $f(x)=\frac{x^2-10x-24}{x+2}$
45. $f(x)=\frac{x^2+1}{x^2-4}$

46. CONSTRUCT ARGUMENTS Alina graphed the function $f(x)=\frac{x^2-4x}{x-4}$, as shown. Did she correctly graph the function? If so, justify your argument. If not, explain how to correct the graph. No; there should be a hole at (4, 4).

47. BATTING AVERAGES A major league baseball player had a lifetime batting average of 0.306 at the beginning of the 2017 season with 2067 hits out of 6767 at bats. During the 2017 season, the player had 183 hits. a–c. See Mod. 7 Answer Appendix.

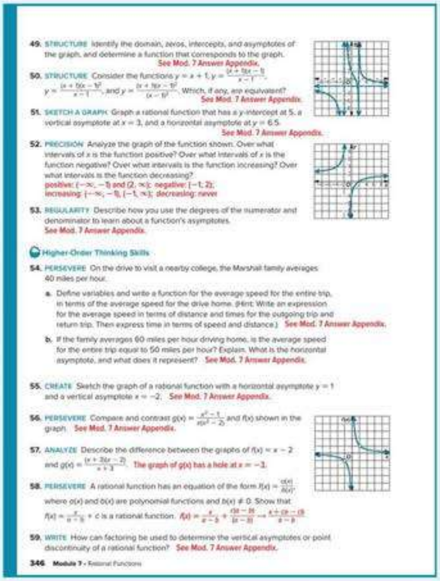
a. Write an equation describing the baseball player's batting average y at the end of the 2017 season using x to represent the number of at bats the player had during the season.

b. Determine the location of the horizontal and vertical asymptotes for the graph of the equation.

c. What is the meaning of the horizontal asymptote for the graph of this equation?

48. REASONING A music studio uses the function $f(x)=\frac{300x}{x^2+4}$ to estimate the number of downloads per hour in thousands in the hours after the release of a new song. Graph the function. Then restrict the domain of the function as required by the context, and graph the function with the restricted domain. Explain the shape of the graph in the context of the situation. See Mod. 7 Answer Appendix.

Lesson 7-4 • Graphing Rational Functions 345



49. STRUCTURE Identify the domain, zeros, intercepts, and asymptotes of the graph, and determine a function that corresponds to the graph. See Mod. 7 Answer Appendix.

50. STRUCTURE Consider the functions $y=x+1$, $y=\frac{(x+1)(x-1)}{x-1}$, $y=\frac{(x+1)(x-1)^2}{x-1}$, and $y=\frac{(x+1)(x-1)^2}{(x-1)^2}$. Which, if any, are equivalent? See Mod. 7 Answer Appendix.

51. SKETCH A GRAPH Graph a rational function that has a y-intercept at 5, a vertical asymptote at $x=3$, and a horizontal asymptote at $y=6.5$. See Mod. 7 Answer Appendix.

52. PRECISION Analyze the graph of the function shown. Over what intervals of x is the function positive? Over what intervals of x is the function negative? Over what intervals is the function increasing? Over what intervals is the function decreasing?
positive: $(-\infty,-1)$ and $(2,\infty)$; negative: $(-1,2)$;
increasing: $(-\infty,-1)$, $(-1,\infty)$; decreasing: never

53. REGULARITY Describe how you use the degrees of the numerator and denominator to learn about a function's asymptotes. See Mod. 7 Answer Appendix.

Higher-Order Thinking Skills

54. PERSEVERE On the drive to visit a nearby college, the Marshall family averages 40 miles per hour.

a. Define variables and write a function for the average speed for the entire trip in terms of the average speed for the drive home. (Hint: Write an expression for the average speed in terms of distance and times for the outgoing trip and return trip. Then express time in terms of speed and distance.) See Mod. 7 Answer Appendix.

b. If the family averages 60 miles per hour driving home, is the average speed for the entire trip equal to 50 miles per hour? Explain. What is the horizontal asymptote, and what does it represent? See Mod. 7 Answer Appendix.

55. CREATE Sketch the graph of a rational function with a horizontal asymptote $y=1$ and a vertical asymptote $x=-2$. See Mod. 7 Answer Appendix.

56. PERSEVERE Compare and contrast $g(x)=\frac{x^2-1}{x(x^2-2)}$ and f(x) shown in the graph. See Mod. 7 Answer Appendix.

57. ANALYZE Describe the difference between the graphs of $f(x)=x-2$ and $g(x)=\frac{(x+3)(x-2)}{x+3}$. The graph of g(x) has a hole at $x=-3$.

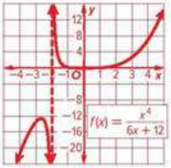
58. PERSEVERE A rational function has an equation of the form $f(x)=\frac{a(x)}{b(x)}$, where a(x) and b(x) are polynomial functions and $b(x)\neq 0$. Show that $f(x)=\frac{a}{x-b}+c$ is a rational function. $f(x)=\frac{a}{x-b}+\frac{c(x-b)}{(x-b)}\rightarrow\frac{a+c(x-b)}{x-b}$

59. WRITE How can factoring be used to determine the vertical asymptotes or point discontinuity of a rational function? See Mod. 7 Answer Appendix.

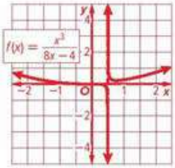
346 Module 7 • Rational Functions

Answers

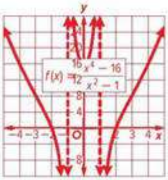
1.



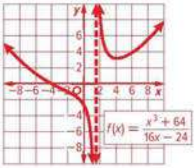
2.



3.



4.



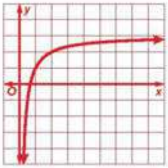
5a.

5b. x-intercept: −2; y-intercept: none; end behavior: As $x\longrightarrow-\infty$, $f(x)\longrightarrow 30$ and as $x\longrightarrow\infty$, $f(x)\longrightarrow 30$.

6a.

6b. x-intercept: 0; y-intercept: 0; end behavior: As $x\longrightarrow-\infty$, $f(x)\longrightarrow 0$ and as $x\longrightarrow\infty$, $f(x)\longrightarrow 0$.

7a.



7b. x-intercept: $\frac{2}{3}$; y-intercept: none; end behavior: As $x\longrightarrow-\infty$, $f(x)\longrightarrow 3$ and as $x\longrightarrow\infty$, $f(x)\longrightarrow 3$.

8b. The intercepts are the same.

8c. Vertical asymptotes $f(x)$: $x=-\frac{5}{3}$ $g(x)$: $x=-2$
Horizontal asymptotes $f(x)$: $y=\frac{1}{3}$ $g(x)$: $y=1$

9b. h(x)

9c. Vertical asymptotes $h(x)$: $x=1$ $j(x)$: $x=-3$
Horizontal asymptotes $h(x)$: $y=\frac{1}{4}$ $j(x)$: $y=\frac{1}{2}$

10b. f(x)

10c. Vertical asymptotes $f(x)$: $x=-\frac{7}{2}$ $g(x)$: $x=-1$
Horizontal asymptotes $f(x)$: $y=\frac{1}{2}$ $g(x)$: $y=\frac{1}{3}$

Lesson 7-5

Variation

A.CED.1, A.CED.2

LESSON GOAL

Students recognize and solve direct, joint, inverse, and combined variation equations.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Variation

Develop:

Direct Variation and Joint Variation
- Direct Variation
- Joint Variation

Inverse Variation and Combined Variation
- Inverse Variation
- Combined Variation
- Write and Solve a Combined Variation

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

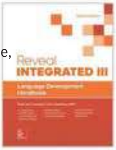
DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Functions	●	●		●
Extension: Geosynchronous Satellites		●	●	●

Language Development Handbook

Assign page 39 of the *Language Development Handbook* to help your students build mathematical language related to direct, joint, inverse, and combined variation equations.



ELL You can use the tips and suggestions on page T39 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra

Standards for Mathematical Content:

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

4 Model with mathematics.

6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students solved equations involving proportions.
A.CED.1, A.SSE.3

Now
Students recognize and solve direct, joint, inverse, and combined variation equations.
A.CED.1, A.CED.2

Next
Students will solve rational equations and inequalities algebraically and by graphing.
A.CED.1, A.REI.2, A.REI.11

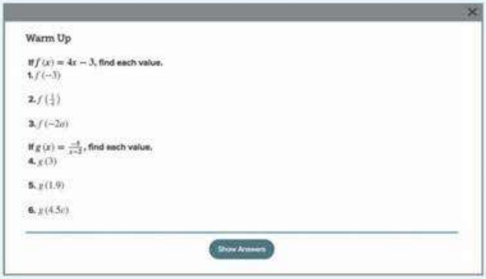
Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand on their knowledge of proportional reasoning to develop an understanding of the different types of variations. They build fluency and apply their understanding by solving problems that reflect the different types of variations.

Interactive Presentation



Warm Up

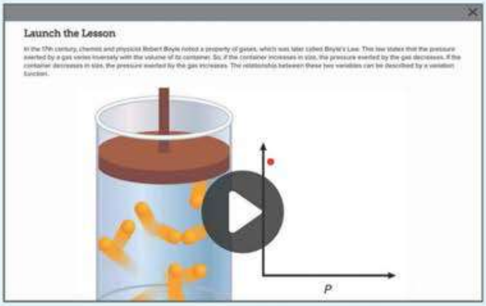
If $f(x) = 4x - 3$, find each value.

1. $f(-3)$
2. $f\left(\frac{1}{4}\right)$
3. $f(-2a)$

If $g(x) = \frac{-5}{x-2}$, find each value.

4. $g(3)$
5. $g(1.9)$
6. $g(4.5c)$

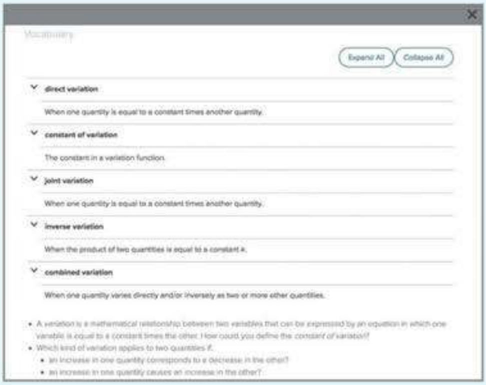
Warm Up



Launch the Lesson

In the 17th century, chemist and physicist Robert Boyle noted a property of gases, which was later called Boyle's Law. This law states that the pressure exerted by a gas varies inversely with the volume of its container. So, if the container increases in size, the pressure exerted by the gas decreases. If the container decreases in size, the pressure exerted by the gas increases. The relationship between these two variables can be described by a variation function.

Launch the Lesson



Vocabulary

direct variation

When one quantity is equal to a constant times another quantity.

constant of variation

The constant in a variation function.

joint variation

When one quantity is equal to a constant times another quantity.

inverse variation

When the product of two quantities is equal to a constant *k*.

combined variation

When one quantity varies directly and/or inversely as two or more other quantities.

- A variation is a mathematical relationship between two variables that can be expressed by an equation in which one variable is equal to a constant times the other. How could you define the *constant of variation*?
- Which kind of variation applies to two quantities if:
  - an increase in one quantity corresponds to a decrease in the other?
  - an increase in one quantity causes an increase in the other?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- finding function values

Answers:

1. −15

2. −2

3. $-8a - 3$

4. −5

5. 50 **6.** $\frac{-5}{4.5c-2}$

Launch the Lesson

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationship between the verbal description of Boyle's Law, the diagram, and the graph in the Launch the Lesson.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Mathematical Background

The type of variation present can sometimes be identified from a table of values for x and y. If the quotient $\frac{x}{y}$ has a constant value, y varies directly as x. If the product xy has a constant value, y varies inversely as x.

Explore Variation

Objective

Students use a sketch to explore variation functions.

Teaching the Mathematical Practices

3 Reason Inductively Throughout the Explore, students must use data they have collected and inductive reasoning to make plausible arguments about the relationships between the length, width, and area of a rectangle.

5 Compare Predictions with Data Point out that in the Explore, students should use sketches to visualize the relationships between the length, width, and area of a rectangle. Guide students to compare their predictions about the relationships with the data and graph.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

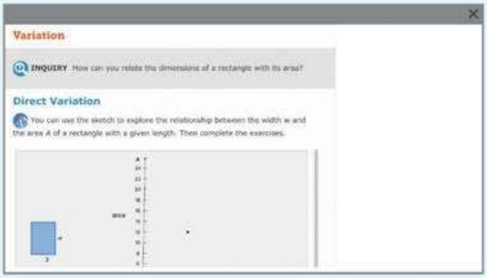
What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will use a sketch to explore the relationship between the area and the dimensions of a rectangle. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



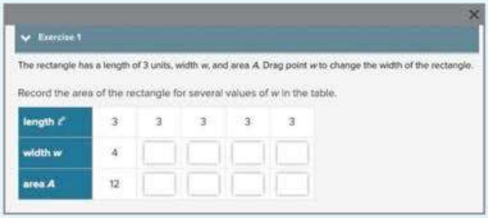
Variation

INQUIRY How can you relate the dimensions of a rectangle with its area?

Direct Variation

You can use the sketch to explore the relationship between the width w and the area A of a rectangle with a given length. Then complete the exercises.

Explore



Exercise 1

The rectangle has a length of 3 units, width w, and area A. Drag point w to change the width of the rectangle.

Record the area of the rectangle for several values of w in the table.

length ℓ	3	3	3	3	3
width w	4				
area A	12				

Explore

WEB SKETCHPAD

Students use a sketch to explore direct and inverse variation.

Interactive Presentation

INQUIRY How can you relate the dimensions of a rectangle with its area?

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Variation (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Is $y = 3x^2$ an example of direct variation? Why or why not? No; sample answer: The x-term is squared, which will make the value of y increase at a faster rate as x increases instead of at a constant rate.
- What types of functions model direct and inverse variation? Sample answer: Linear functions in the family of $y = ax$ model direct variation and rational functions in the family $y = \frac{a}{x}$ model inverse variation.

Inquiry

How can you relate the dimensions of a rectangle with its area? Sample answer: The area of a rectangle is directly related to the side lengths of the rectangle. The side lengths of a rectangle are inversely related to the area of the rectangle.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | 3 APPLICATION

Learn Direct Variation and Joint Variation

Objective

Students recognize and solve direct and joint variation equations.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write and explain their solution methods. Students must use clear definitions and mathematical language when answering the question in the Talk About It feature.

2 Make Sense of Quantities Mathematically proficient students need to be able to make sense of quantities and their relationships. In the Think About It features, students must make sense of the relationships between the variables to answer the question.

Example 1 Direct Variation

MP Teaching the Mathematical Practices

2 Make Sense of Quantities Mathematically proficient students need to be able to make sense of quantities and their relationships. In Example 1, students should notice the relationships between the problem variables.

Questions for Mathematical Discourse

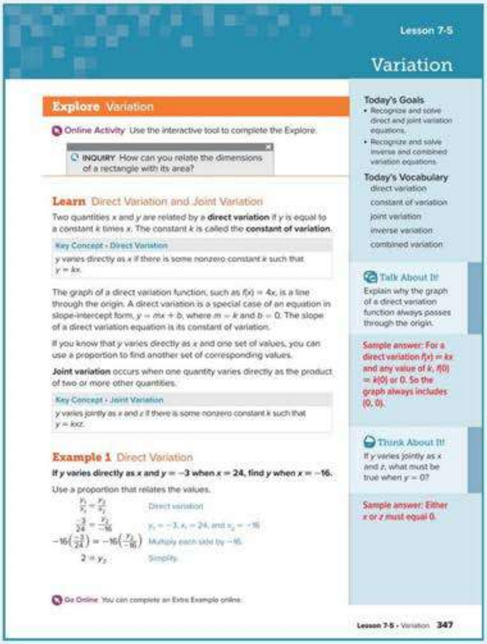
AL What type of function do you use to model direct variation? linear

OL How do you determine the equation that models this relationship? Sample answer: Direct variation is of the form $y = kx$. You can find k from $\frac{y_1}{x_1}$, which is $-\frac{1}{8}$. Thus, the equation is $y = -\frac{1}{8}x$.

BL Why can you use a proportion to solve for y_2? You know y varies directly as x, so $y_1 = kx_1$ and $y_2 = kx_2$. Solving for k shows that each ratio of $\frac{y}{x}$ equals k, thus the ratios can be set as equal.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 7-5

Variation

Explore Variation

Online Activity Use the interactive tool to complete the Explore.

INQUIRY How can you relate the dimensions of a rectangle with its area?

Learn Direct Variation and Joint Variation

Two quantities x and y are related by a **direct variation** if y is equal to a constant k times x. The constant k is called the **constant of variation**.

Key Concept • Direct Variation

y varies directly as x if there is some nonzero constant k such that $y = kx$.

The graph of a direct variation function, such as $f(x) = 4x$, is a line through the origin. A direct variation is a special case of an equation in slope-intercept form, $y = mx + b$, where $m = k$ and $b = 0$. The slope of a direct variation equation is its constant of variation.

If you know that y varies directly as x and one set of values, you can use a proportion to find another set of corresponding values.

Joint variation occurs when one quantity varies directly as the product of two or more other quantities.

Key Concept • Joint Variation

y varies jointly as x and z if there is some nonzero constant k such that $y = kxz$.

Example 1 Direct Variation

If y varies directly as x and $y = -3$ when $x = 24$, find y when $x = -16$.

Use a proportion that relates the values.

$\frac{y_1}{x_1} = \frac{y_2}{x_2}$ Direct variation

$\frac{-3}{24} = \frac{y_2}{-16}$ $y_1 = -3$, $x_1 = 24$, and $x_2 = -16$

$-16\left(\frac{-3}{24}\right) = -16\left(\frac{y_2}{-16}\right)$ Multiply each side by −16.

$2 = y_2$ Simplify.

Go Online You can complete an Extra Example online.

Today's Goals
- Recognize and solve direct and joint variation equations.
- Recognize and solve inverse and combined variation equations.

Today's Vocabulary
direct variation
constant of variation
joint variation
inverse variation
combined variation

Talk About It!
Explain why the graph of a direct variation function always passes through the origin.

Sample answer: For a direct variation $f(x) = kx$ and any value of k, $f(0) = k(0)$ or 0. So the graph always includes (0, 0).

Think About It!
If y varies jointly as x and z, what must be true when $y = 0$?

Sample answer: Either x or z must equal 0.

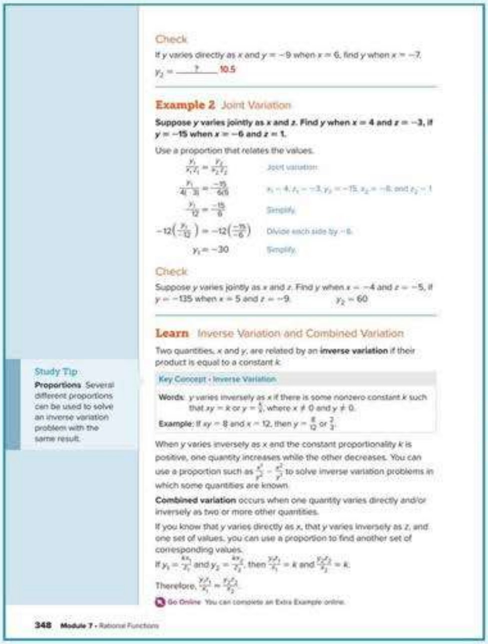
Lesson 7-5 • Variation 347

Interactive Presentation

Learn

TYPE

Students answer a question about the graph of direct variation functions.



Check

If y varies directly as x and $y = -9$ when $x = 6$, find y when $x = -7$.

$y_2 =$ ___?___ 10.5

Example 2 Joint Variation

Suppose y varies jointly as x and z. Find y when $x = 4$ and $z = -3$, if $y = -15$ when $x = -6$ and $z = 1$.

Use a proportion that relates the values.

$\frac{y_1}{x_1z_1} = \frac{y_2}{x_2z_2}$ Joint variation

$\frac{y_1}{4(-3)} = \frac{-15}{-6(1)}$ $x_1 = 4, z_1 = -3, y_2 = -15, x_2 = -6$, and $z_2 = 1$

$\frac{y_1}{-12} = \frac{-15}{-6}$ Simplify.

$-12\left(\frac{y_1}{-12}\right) = -12\left(\frac{-15}{-6}\right)$ Divide each side by −6.

$y_1 = -30$ Simplify.

Check

Suppose y varies jointly as x and z. Find y when $x = -4$ and $z = -5$, if $y = -135$ when $x = 5$ and $z = -9$. $y_2 = 60$

Learn Inverse Variation and Combined Variation

Two quantities, x and y, are related by an **inverse variation** if their product is equal to a constant k.

Key Concept • Inverse Variation

Words: y varies inversely as x if there is some nonzero constant k such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$.

Example: If $xy = 8$ and $x = 12$, then $y = \frac{8}{12}$ or $\frac{2}{3}$.

When y varies inversely as x and the constant proportionality k is positive, one quantity increases while the other decreases. You can use a proportion such as $\frac{x_1}{y_2} = \frac{x_2}{y_1}$ to solve inverse variation problems in which some quantities are known.

Combined variation occurs when one quantity varies directly and/or inversely as two or more other quantities.

If you know that y varies directly as x, that y varies inversely as z, and one set of values, you can use a proportion to find another set of corresponding values.

If $y_1 = \frac{kx_1}{z_1}$ and $y_2 = \frac{kx_2}{z_2}$, then $\frac{y_1z_1}{x_1} = k$ and $\frac{y_2z_2}{x_2} = k$.

Therefore, $\frac{y_1z_1}{x_1} = \frac{y_2z_2}{x_2}$.

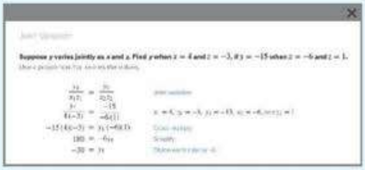
Study Tip

Proportions Several different proportions can be used to solve an inverse variation problem with the same result.

Go Online You can complete an Extra Example online.

348 Module 7 • Rational Functions

Interactive Presentation



Suppose y varies jointly as x and z. Find y when $x = 4$ and $z = -3$, if $y = -15$ when $z = -6$ and $z = 1$.

Example 2

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Joint Variation

Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of joint variation to write and solve a proportion that represents the relationships in the problem.

Questions for Mathematical Discourse

AL Explain why the denominators of the proportion contain the product of x and z. Sample answer: Because y varies jointly as x and z, by the definition of joint variation, y varies directly as the product of x and z.

OL If x increases by a factor of a and z increases by a factor of b, by what factor does y increase? ab

BL What is the value of k? Explain your reasoning. $-\frac{5}{2}$; Sample answer: Substituting the given values into the equation for joint variation, $y = kxz$, gives $-15 = k(-6)(1)$. Solving for k gives $k = -\frac{5}{2}$.

Learn Inverse Variation and Combined Variation

Objective

Students recognize and solve inverse and combined variation equations.

Teaching the Mathematical Practices

2 Different Properties Students should look for different ways to solve problems. In the Study Tip, have students describe how different proportions could be used to represent the same inverse variation problem.

7 Use Structure Help students to determine the structure of the variation relationships throughout this lesson.

Essential Question Follow-Up

Students have explored variation functions.

Ask:

Why can analyzing a rational function algebraically and graphically help you to see the "whole picture?" Sample answer: An algebraic analysis can help you to determine points of discontinuity that may not be clear or noticeable when viewing the graph of the function. A graphical analysis can help you to see the asymptotes and end behavior of the function.

Example 3 Inverse Variation

MP Teaching the Mathematical Practices

2 Create Representations Guide students to write an equation that models the relationships from Example 3. Then use the equation to solve the problem.

Questions for Mathematical Discourse

AL What is the result if you cross multiply before substituting for the variables in the example? $m_1n_1 = m_2n_2$

OL What is the value of k? Explain your reasoning. −24; sample answer: The equation for inverse variation is $xy = k$, so $(-4)6 = -24$ and $2.4(-10) = -24$.

BL How could you use the value of k to determine m when $n = -10$? Sample answer: Substitute −24 and −10 for k and n, respectively, in the equation for inverse variation, $nm = k$ and solve.

Common Error

Remind students that in an inverse variation relationship, the proportion will include different subscripts on the same side of the equals sign. Review the different proportions that can be used to solve an inverse variation problem.

Example 4 Combined Variation

MP Teaching the Mathematical Practices

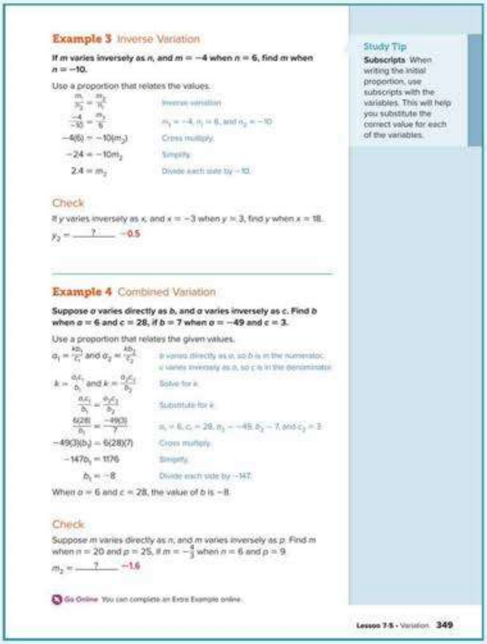
6 State Meanings of Symbols In Example 4, ask students to state the meanings of the variables used in the proportions. Point out that it is important to clearly define and label the variables to avoid making errors when substituting for values.

Questions for Mathematical Discourse

AL How does solving the example relate to solving systems of equations? Sample answer: There are two unknown variables, k and b_1, and there are two equations. You are looking for b_1, so you need to substitute for k.

OL What is another way of calculating the unknown value of b? Sample answer: Use the three known values to calculate the value of k, then set the proportion with the unknown variable equal to k.

BL Write a general formula for finding any unknown b when given values for the corresponding a and c and another set of a, b, and c. $b_1 = \frac{a_1c_1b_2}{a_2c_2}$



Example 3 Inverse Variation

If m varies inversely as n, and $m = -4$ when $n = 6$, find m when $n = -10$.

Use a proportion that relates the values.

$\frac{m_1}{n_2} = \frac{m_2}{n_1}$ Inverse variation

$\frac{-4}{-10} = \frac{m_2}{6}$ $m_1 = -4$, $n_1 = 6$, and $n_2 = -10$

$-4(6) = -10(m_2)$ Cross multiply.

$-24 = -10m_2$ Simplify.

$2.4 = m_2$ Divide each side by −10.

Check

If y varies inversely as x, and $x = -3$ when $y = 3$, find y when $x = 18$.

$y_2 =$ ___?___ −0.5

Example 4 Combined Variation

Suppose a varies directly as b, and a varies inversely as c. Find b when $a = 6$ and $c = 28$, if $b = 7$ when $a = -49$ and $c = 3$.

Use a proportion that relates the given values.

$a_1 = \frac{kb_1}{c_1}$ and $a_2 = \frac{kb_2}{c_2}$ a varies directly as b, so b is in the numerator; a varies inversely as c, so c is in the denominator.

$k = \frac{a_1c_1}{b_1}$ and $k = \frac{a_2c_2}{b_2}$ Solve for k.

$\frac{a_1c_1}{b_1} = \frac{a_2c_2}{b_2}$ Substitute for k.

$\frac{6(28)}{b_1} = \frac{-49(3)}{7}$ $a_1 = 6$, $c_1 = 28$, $a_2 = -49$, $b_2 = 7$, and $c_2 = 3$

$-49(3)(b_1) = 6(28)(7)$ Cross multiply.

$-147b_1 = 1176$ Simplify.

$b_1 = -8$ Divide each side by −147.

When $a = 6$ and $c = 28$, the value of b is −8.

Check

Suppose m varies directly as n, and m varies inversely as p. Find m when $n = 20$ and $p = 25$, if $m = -\frac{4}{3}$ when $n = 6$ and $p = 9$.

$m_2 =$ ___?___ −1.6

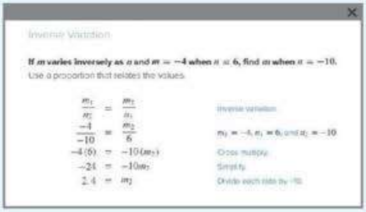
Go Online You can complete an Extra Example online.

Study Tip

Subscripts When writing the initial proportion, use subscripts with the variables. This will help you substitute the correct value for each of the variables.

Lesson 7-5 • Variation 349

Interactive Presentation



Inverse Variation

If m varies inversely as n and $m = -4$ when $n = 6$, find m when $n = -10$.

Use a proportion that relates the values.

$\frac{m_1}{n_2} = \frac{m_2}{n_1}$ Inverse variation

$\frac{-4}{-10} = \frac{m_2}{6}$ $m_1 = -4$, $n_1 = 6$, and $n_2 = -10$

$-4(6) = -10(m_2)$ Cross multiply.

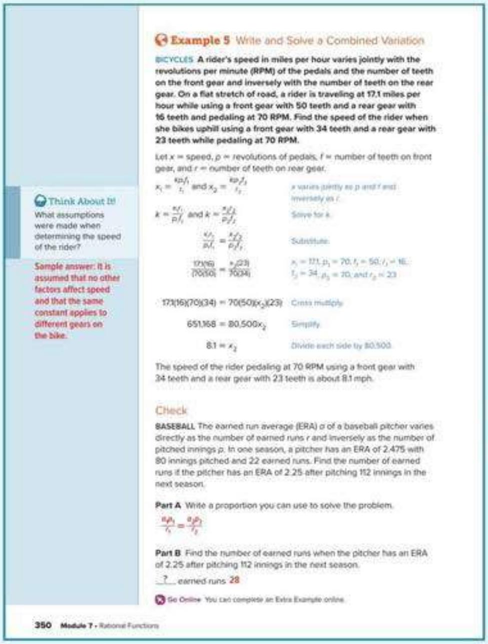
$-24 = -10m_2$ Simplify.

$2.4 = m_2$ Divide each side by −10.

Example 3

EXPAND

Students tap to see a Study Tip about using subscripts.



Example 5 Write and Solve a Combined Variation

BICYCLES **A rider's speed in miles per hour varies jointly with the revolutions per minute (RPM) of the pedals and the number of teeth on the front gear and inversely with the number of teeth on the rear gear. On a flat stretch of road, a rider is traveling at 17.1 miles per hour while using a front gear with 50 teeth and a rear gear with 16 teeth and pedaling at 70 RPM. Find the speed of the rider when she bikes uphill using a front gear with 34 teeth and a rear gear with 23 teeth while pedaling at 70 RPM.**

Let x = speed, p = revolutions of pedals, f = number of teeth on front gear, and r = number of teeth on rear gear.

$x_1 = \frac{kp_1f_1}{r_1}$ and $x_2 = \frac{kp_2f_2}{r_2}$ — x varies jointly as p and f and inversely as r.

$k = \frac{x_1r_1}{p_1f_1}$ and $k = \frac{x_2r_2}{p_2f_2}$ — Solve for k.

$\frac{x_1r_1}{p_1f_1} = \frac{x_2r_2}{p_2f_2}$ — Substitute.

$\frac{17.1(16)}{70(50)} = \frac{x_2(23)}{70(34)}$ — $x_1 = 17.1$, $p_1 = 70$, $f_1 = 50$, $r_1 = 16$, $f_2 = 34$, $p_2 = 70$, and $r_2 = 23$

$17.1(16)(70)(34) = 70(50)(x_2)(23)$ — Cross multiply.

$651{,}168 = 80{,}500x_2$ — Simplify.

$8.1 \approx x_2$ — Divide each side by 80,500.

The speed of the rider pedaling at 70 RPM using a front gear with 34 teeth and a rear gear with 23 teeth is about 8.1 mph.

Think About It!
What assumptions were made when determining the speed of the rider?

Sample answer: It is assumed that no other factors affect speed and that the same constant applies to different gears on the bike.

Check

BASEBALL The earned run average (ERA) a of a baseball pitcher varies directly as the number of earned runs r and inversely as the number of pitched innings p. In one season, a pitcher has an ERA of 2.475 with 80 innings pitched and 22 earned runs. Find the number of earned runs if the pitcher has an ERA of 2.25 after pitching 112 innings in the next season.

Part A Write a proportion you can use to solve the problem.

$\frac{a_1p_1}{r_1} = \frac{a_2p_2}{r_2}$

Part B Find the number of earned runs when the pitcher has an ERA of 2.25 after pitching 112 innings in the next season.

___?___ earned runs 28

Go Online You can complete an Extra Example online.

350 **Module 7** • Rational Functions

Interactive Presentation

Example 5

SWIPE

Students move through the slides to see how to solve a combined variation equation.

CHECK

Students complete the Check online to determine whethar they are ready to move on.

Example 5 Write and Solve a Combined Variation

Teaching the Mathematical Practices

2 Represent a Situation Symbolically Guide students in defining variables to solve the problem in Example 5. Help students to identify the relationships among the various quantities in the problem.

4 Make Assumptions The Think About It! feature asks students to explain assumptions that were made to solve the problem.

Questions for Mathematical Discourse

AL If the front and rear gear do not change, what type of variation relates the rider's speed and the revolutions per minute (RPM) of the pedals? direct variation

OL If the rear gear switches into a gear with more teeth, what happens to the speed of the bicycle? It decreases.

BL What changes in gears would result in an increase in speed without changing the pedaling speed? changing the front gear to a gear with more teeth or changing the rear gear to a gear with fewer teeth

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

3 REFLECT AND PRACTICE

A.CED.1, A.CED.2

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–25
2	exercises that use a variety of skills from this lesson	26–61
2	exercises that extend concepts learned in this lesson to new contexts	62–63
3	exercises that emphasize higher-order and critical-thinking skills	64–68

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–63 odd, 64–68
- Extension: Geosynchronous Satellites
- **ALEKS**® Applications

OL

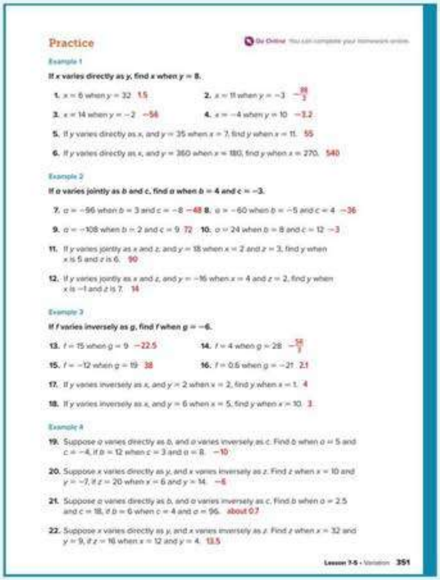
IF students score 66%–89% on the Checks,
THEN assign:

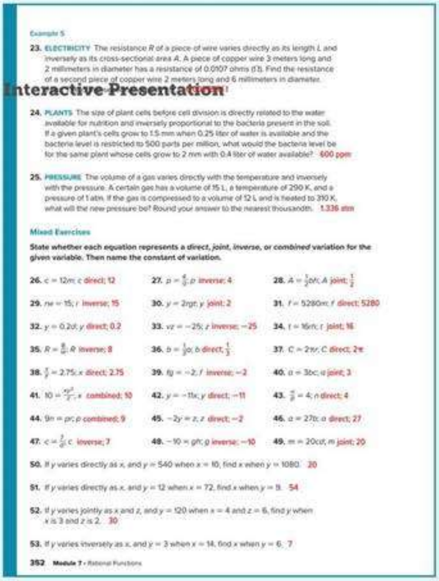
- Practice, Exercises 1–67 odd
- Remediation, Review Resources: Functions
- Personal Tutors
- Extra Examples 1–5
- **ALEKS**® Sets, Relations, and Functions

AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–25 odd
- Remediation, Review Resources: Functions
- *Quick Review Math Handbook*: Variation Functions
- **ALEKS**® Sets, Relations, and Functions



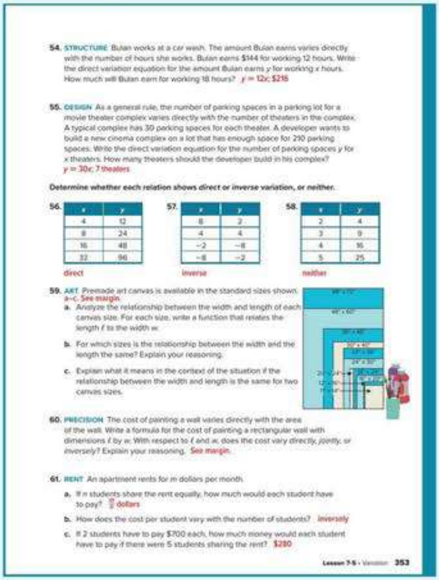


Interactive Presentation

3 REFLECT AND PRACTICE

A.CED.1, A.CED.2

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**



54. **STRUCTURE** Bulan works at a car wash. The amount Bulan earns varies directly with the number of hours she works. Bulan earns $144 for working 12 hours. Write the direct variation equation for the amount Bulan earns y for working x hours. How much will Bulan earn for working 18 hours? $y = 12x$; $216

55. **DESIGN** As a general rule, the number of parking spaces in a parking lot for a movie theater complex varies directly with the number of theaters in the complex. A typical complex has 30 parking spaces for each theater. A developer wants to build a new cinema complex on a lot that has enough space for 210 parking spaces. Write the direct variation equation for the number of parking spaces y for x theaters. How many theaters should the developer build in his complex? $y = 30x$; 7 theaters

Determine whether each relation shows direct or inverse variation, or neither.

56.

x	y
4	12
8	24
16	48
32	96

direct

57.

x	y
8	2
4	4
−2	−8
−8	−2

inverse

58.

x	y
2	4
3	9
4	16
5	25

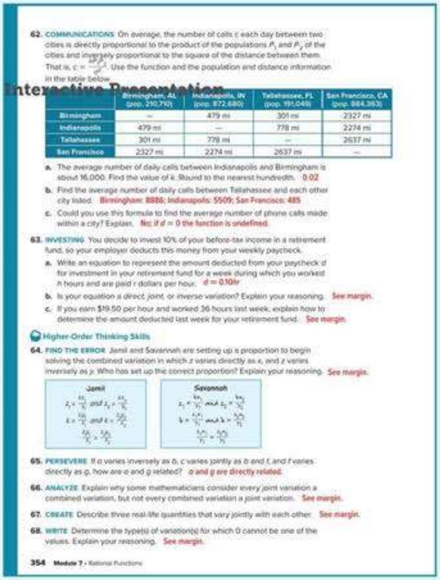
neither

59. **ART** Premade art canvas is available in the standard sizes shown. a–c. See margin.
   a. Analyze the relationship between the width and length of each canvas size. For each size, write a function that relates the length ℓ to the width w.
   b. For which sizes is the relationship between the width and the length the same? Explain your reasoning.
   c. Explain what it means in the context of the situation if the relationship between the width and length is the same for two canvas sizes.

60. **PRECISION** The cost of painting a wall varies directly with the area of the wall. Write a formula for the cost of painting a rectangular wall with dimensions ℓ by w. With respect to ℓ and w, does the cost vary directly, jointly, or inversely? Explain your reasoning. See margin.

61. **RENT** An apartment rents for m dollars per month.
   a. If n students share the rent equally, how much would each student have to pay? $\frac{m}{n}$ dollars
   b. How does the cost per student vary with the number of students? inversely
   c. If 2 students have to pay $700 each, how much money would each student have to pay if there were 5 students sharing the rent? $280

Lesson 7-5 • Variation 353



62. **COMMUNICATIONS** On average, the number of calls c each day between two cities is directly proportional to the product of the populations P_1 and P_2 of the cities and inversely proportional to the square of the distance between them. That is, $c = \frac{kP_1P_2}{d^2}$. Use the function and the population and distance information in the table below.

Interactive Presentation

	Birmingham, AL (pop. 210,710)	Indianapolis, IN (pop. 872,680)	Tallahassee, FL (pop. 191,049)	San Francisco, CA (pop. 884,363)
Birmingham	—	479 mi	301 mi	2327 mi
Indianapolis	479 mi	—	778 mi	2274 mi
Tallahassee	301 mi	778 mi	—	2637 mi
San Francisco	2327 mi	2274 mi	2637 mi	—

   a. The average number of daily calls between Indianapolis and Birmingham is about 16,000. Find the value of k. Round to the nearest hundredth. 0.02
   b. Find the average number of daily calls between Tallahassee and each other city listed. Birmingham: 8886; Indianapolis: 5509; San Francisco: 485
   c. Could you use this formula to find the average number of phone calls made within a city? Explain. No; if $d = 0$ the function is undefined.

63. **INVESTING** You decide to invest 10% of your before-tax income in a retirement fund, so your employer deducts this money from your weekly paycheck.
   a. Write an equation to represent the amount deducted from your paycheck d for investment in your retirement fund for a week during which you worked h hours and are paid r dollars per hour. $d = 0.10hr$
   b. Is your equation a direct, joint, or inverse variation? Explain your reasoning. See margin.
   c. If you earn $19.50 per hour and worked 36 hours last week, explain how to determine the amount deducted last week for your retirement fund. See margin.

Higher-Order Thinking Skills

64. **FIND THE ERROR** Jamil and Savannah are setting up a proportion to begin solving the combined variation in which z varies directly as x, and z varies inversely as y. Who has set up the correct proportion? Explain your reasoning. See margin.

Jamil

$z_1 = \frac{kx_1}{y_1}$ and $z_2 = \frac{kx_2}{y_2}$

$k = \frac{z_1y_1}{x_1}$ and $k = \frac{z_2y_2}{x_2}$

$\frac{z_1y_1}{x_1} = \frac{z_2y_2}{x_2}$

Savannah

$z_1 = \frac{kx_1}{y_1}$ and $z_2 = \frac{kx_2}{y_2}$

$k = \frac{z_1x_1}{y_1}$ and $k = \frac{z_2x_2}{y_2}$

$\frac{z_1x_1}{y_1} = \frac{z_2x_2}{y_2}$

65. **PERSEVERE** If a varies inversely as b, c varies jointly as b and f, and f varies directly as g, how are a and g related? a and g are directly related.

66. **ANALYZE** Explain why some mathematicians consider every joint variation a combined variation, but not every combined variation a joint variation. See margin.

67. **CREATE** Describe three real-life quantities that vary jointly with each other. See margin.

68. **WRITE** Determine the type(s) of variation(s) for which 0 cannot be one of the values. Explain your reasoning. See margin.

354 Module 7 • Rational Functions

Answers

59a. 11 × 14: $\ell = \frac{14}{11}w$; 12 × 16: $\ell = \frac{4}{3}w$; 16 × 20: $\ell = \frac{5}{4}w$; 18 × 24: $\ell = \frac{4}{3}w$; 20 × 24: $\ell = \frac{6}{5}w$; 24 × 30: $\ell = \frac{5}{4}w$; 24 × 36: $\ell = \frac{3}{2}w$; 30 × 40: $\ell = \frac{4}{3}w$; 36 × 48: $\ell = \frac{4}{3}w$; 48 × 60: $\ell = \frac{5}{4}w$; 48 × 72: $\ell = \frac{3}{2}w$

59b. 12 × 16, 18 × 24, 30 × 40, 36 × 48 all have the relationship $\ell = \frac{4}{3}w$. 16 × 20, 24 × 30, and 48 × 60 have the relationship $\ell = \frac{5}{4}w$. 24 × 36 and 48 × 72 have the relationship $\ell = \frac{3}{2}w$.

59c. Sample answer: One canvas is an enlargement of the other.

60. Sample answer: $C = k\ell w$, where C is the cost and k is a constant. C varies jointly with ℓ and w because one quantity (C) varies directly as the product of two other quantities (ℓ and w).

63b. Joint variation; sample answer: The amount deducted varies directly as the product of two quantities: the hourly wage and the number of hours worked.

63c. Sample answer: Substitute $r = 19.50$ and $h = 36$ in the formula from part a.

The amount deducted was $70.20.

64. Jamil; sample answer: Savannah multiplied when she should have divided and divided when she should have multiplied.

66. Sample answer: Every joint variation is a combined variation because there are two combined direct variations. However, a combined variation can have a combination of a direct and an inverse variation, so it cannot be considered a joint variation.

67. Sample answer: The force of an object varies jointly as its mass and acceleration.

68. Sample answer: Inverse and some types of combined variation functions cannot have a value of 0 in the domain because division by zero is undefined.

Lesson 7-6

A.CED.1, A.REI.2, A.REI.11

Solving Rational Equations and Inequalities

LESSON GOAL

Students solve rational equations and inequalities algebraically and by graphing.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Solving Rational Equations

Develop:

Solving Rational Equations

- Solve a Rational Equation
- Solve a Rational Equation with an Extraneous Solution
- Mixture Problem
- Distance Problem
- Work Problem

Solving Rational Inequalities

- Solve a Rational Inequality
- Write and Solve a Rational Inequality

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Solving Proportions	●	●		●
Extension: Asymptotes in Three Dimensions		●	●	●

Language Development Handbook

Assign page 40 of the *Language Development Handbook* to help your students build mathematical language related to solving rational equations and inequalities.

ELL You can use the tips and suggestions on page T40 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Algebra

Standards for Mathematical Content:

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

A.REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

A.REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, making tables of values, or finding successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

2 Reason abstractly and quantitatively.

5 Use appropriate tools strategically.

Coherence

Vertical Alignment

Previous

Students added and subtracted rational expressions.

A.APR.7

Now

Students solve rational equations and inequalities algebraically and by graphing.

A.CED.1, A.REI.2, A.REI.11

Next

Students will determine populations and samples and run simulations to determine the probabilities of outcomes. Students will analyze the distributions of data sets.

S.IC.1, S.IC.6, S.ID.4

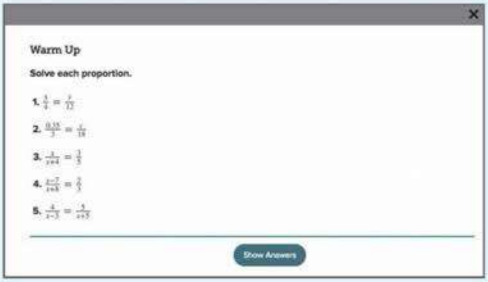
Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand on their understanding of rational functions to build fluency by solving rational equations. They apply their understanding of rational equations by solving real-world problems.

Interactive Presentation



Warm Up

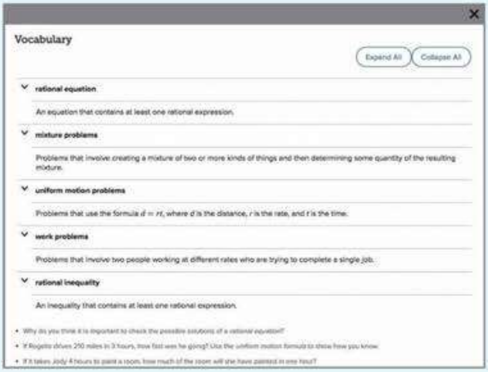
Solve each proportion.

Warm Up

Launch the Lesson

Every year, on June 2, many celebrate National Donut Day with a trip to their local bakery. Others take their love of donuts to the extreme, participating in the National Donut Day World Donut Eating Championship.

Launch the Lesson



Vocabulary

rational equation

An equation that contains at least one rational expression.

mixture problems

Problems that involve creating a mixture of two or more kinds of things and then determining some quantity of the resulting mixture.

uniform motion problems

Problems that use the formula $d = rt$, where d is the distance, r is the rate, and t is the time.

work problems

Problems that involve two people working at different rates who are trying to complete a single job.

rational inequality

An inequality that contains at least one rational expression.

- Why do you think it is important to check the possible solutions of a rational equation?
- If Rogelio drives 250 miles in 3 hours, how fast was he going? Use the uniform motion formula to show how you know.
- If it takes Jody 4 hours to paint a room, how much of the room will she have painted in one hour?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- solving proportions

Answers:

1. 15
2. 2.1
3. 6
4. 37
5. 35

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will read about a real-world problem that can be solved using a rational equation. Once students have completed the lesson, encourage them to answer the question posed in the Launch the Lesson.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Mathematical Background

During the solution process, a rational equation is usually transformed into another type of equation. Solving a rational equation may require solving a related linear, quadratic, or other type of equation.

Explore Solving Rational Equations

Objective

Students use a graphing calculator to explore solving rational equations.

MP Teaching the Mathematical Practices

2 Represent a Situation Symbolically Guide students to define the variables to solve the problem in the Explore. Help students to identify the relationships in the situation and represent them with an equation.

4 Analyze Graphs In Exercises 3 and 4, students must analyze the graph they have generated using graphing calculators. Point out that to see the point of intersection, students may need to adjust the viewing window.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor students' progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will use rational equations to create expressions representing a problem situation, then use that equation to answer questions about the situation. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Solving Rational Equations

INQUIRY How can you solve rational equations by graphing?

Studying abroad is a popular way for college students to experience other cultures while earning credit toward a degree. However, participating in a study abroad program does not come cheap. One university estimates that the cost of tuition, travel, and other fees to study in Paris, France, for a semester is about $17,750. In addition, a student can expect to spend about $40 per day on other personal expenses. A rational equation, which is an equation that contains at least one rational expression, can be used to calculate the cost per day of studying abroad.

Explore

Complete the exercises to explore the costs of studying abroad in Paris.

Exercise 1

Write an expression for the total cost of studying abroad after x days.

Explore

TYPE

Students answer questions to determine how to solve rational equations by graphing.

Interactive Presentation

INQUIRY How can you solve rational equations by graphing?

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Solving Rational Equations (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What type of graph is $y = 750$? How does this help you solve the rational equation? Sample answer: The graph is a horizontal line. As long as this is not equal to the horizontal asymptote, you can find where the graphs will intersect.
- How else could you use a graph to solve $750 = \frac{40x + 17{,}750}{x}$? Sample answer: You find the related equation by setting this equal to zero, $0 = \frac{40x + 17{,}750}{x} - 750$. Then graph $y = \frac{40x + 17{,}750}{x} - 750$ and find the zero of the graph. The solution is still 25 days for the cost to be \$750 per day.

Inquiry

How can you solve rational equations by graphing? Sample answer: I can graph the sides of the rational equation as a system of equations. Then I can find the x-value of the intersection of the graph of the system, which is the solution of the rational equation.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

A.CED.1, A.REI.2, A.REI.11

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Solving Rational Equations

Objective

Students solve rational equations in one variable algebraically.

Teaching the Mathematical Practices

2 Different Properties Mathematically proficient students look for different ways to solve problems. Encourage students to consider why it is easier to first eliminate the fractions when solving a rational equation. Ask students what the alternate solution method would be.

Things to Remember

Remind students that a possible solution must always be checked in the original equation rather than in any of the steps of the solution.

DIFFERENTIATE

Enrichment Activity BL ELL

Logical Learners Have students think about the difference between "pure" mathematics, such as solving an equation, and "applied" mathematics, such as solving a real-world problem. Ask them to list some ways in which these types of mathematics are alike and some ways in which they are different.

Example 1 Solve a Rational Equation

Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. In Example 1, students must use the LCD to transform the rational expressions and solve the problem.

Questions for Mathematical Discourse

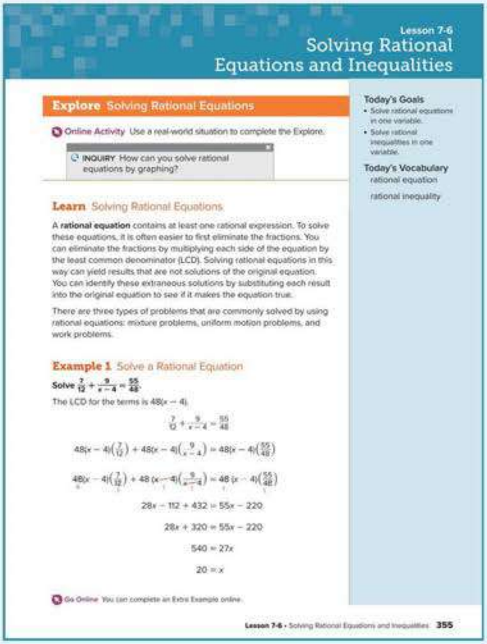
AL For what value(s) of x is the equation undefined? 4

OL Why is the LCD $48(x - 4)$? Sample answer: 12 is a factor of 48, so 48 is the least common multiple of the constant denominators. $48(x - 4)$ is a multiple of $x - 4$ and 48, so it is the simplest expression that is a multiple of all of the denominators.

BL Explain how you could use a calculator to solve the equation. Sample answer: Let $Y_1 = \frac{7}{12} + \frac{9}{x - 4}$ and $Y_2 = \frac{55}{48}$. Graph and find the intersection point.

Go Online

- F ind additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



A.CED.1, A.REI.2, A.REI.11

Lesson 7-6

Solving Rational Equations and Inequalities

Explore Solving Rational Equations

Online Activity Use a real-world situation to complete the Explore.

INQUIRY How can you solve rational equations by graphing?

Learn Solving Rational Equations

A **rational equation** contains at least one rational expression. To solve these equations, it is often easier to first eliminate the fractions. You can eliminate the fractions by multiplying each side of the equation by the least common denominator (LCD). Solving rational equations in this way can yield results that are not solutions of the original equation. You can identify these extraneous solutions by substituting each result into the original equation to see if it makes the equation true.

There are three types of problems that are commonly solved by using rational equations: mixture problems, uniform motion problems, and work problems.

Example 1 Solve a Rational Equation

Solve $\frac{7}{12} + \frac{9}{x-4} = \frac{55}{48}$.

The LCD for the terms is $48(x - 4)$.

$$\frac{7}{12} + \frac{9}{x-4} = \frac{55}{48}$$

$$48(x-4)\left(\frac{7}{12}\right) + 48(x-4)\left(\frac{9}{x-4}\right) = 48(x-4)\left(\frac{55}{48}\right)$$

$$48(x-4)\left(\frac{7}{12}\right) + 48\,\cancel{(x-4)}\left(\frac{9}{\cancel{x-4}}\right) = 48\,(x-4)\left(\frac{55}{48}\right)$$

$$28x - 112 + 432 = 55x - 220$$

$$28x + 320 = 55x - 220$$

$$540 = 27x$$

$$20 = x$$

Go Online You can complete an Extra Example online.

Today's Goals
- Solve rational equations in one variable.
- Solve rational inequalities in one variable.

Today's Vocabulary
rational equation
rational inequality

Lesson 7-6 • Solving Rational Equations and Inequalities 355

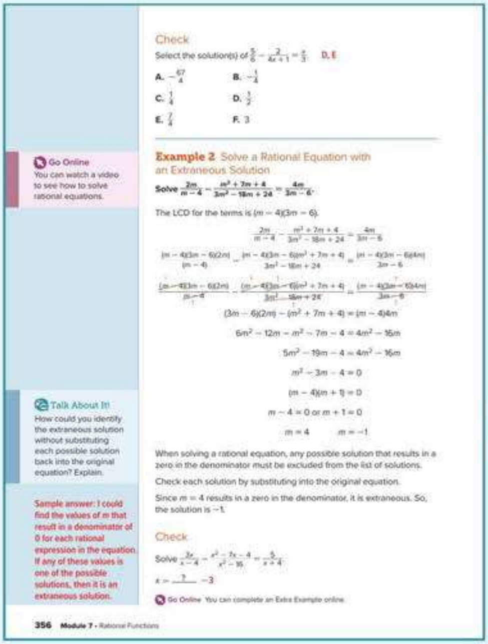
Interactive Presentation

Learn

TYPE

a

Students answer a question to determine how to simplify rational equations.



Check

Select the solution(s) of $\frac{5}{6} - \frac{2}{4x+1} = \frac{x}{3}$. D, E

A. $-\frac{67}{4}$ B. $-\frac{1}{4}$

C. $\frac{1}{4}$ D. $\frac{1}{2}$

E. $\frac{7}{4}$ F. 3

Go Online You can watch a video to see how to solve rational equations.

Example 2 Solve a Rational Equation with an Extraneous Solution

Solve $\frac{2m}{m-4} - \frac{m^2+7m+4}{3m^2-18m+24} = \frac{4m}{3m-6}$.

The LCD for the terms is $(m-4)(3m-6)$.

$$\frac{2m}{m-4} - \frac{m^2+7m+4}{3m^2-18m+24} = \frac{4m}{3m-6}$$

$$\frac{(m-4)(3m-6)(2m)}{(m-4)} - \frac{(m-4)(3m-6)(m^2+7m+4)}{3m^2-18m+24} = \frac{(m-4)(3m-6)(4m)}{3m-6}$$

$$\frac{\cancel{(m-4)}(3m-6)(2m)}{\cancel{m-4}} - \frac{\cancel{(m-4)(3m-6)}(m^2+7m+4)}{\cancel{3m^2-18m+24}} = \frac{(m-4)\cancel{(3m-6)}(4m)}{\cancel{3m-6}}$$

$$(3m-6)(2m) - (m^2+7m+4) = (m-4)4m$$

$$6m^2 - 12m - m^2 - 7m - 4 = 4m^2 - 16m$$

$$5m^2 - 19m - 4 = 4m^2 - 16m$$

$$m^2 - 3m - 4 = 0$$

$$(m-4)(m+1) = 0$$

$$m - 4 = 0 \text{ or } m + 1 = 0$$

$$m = 4 \qquad m = -1$$

When solving a rational equation, any possible solution that results in a zero in the denominator must be excluded from the list of solutions.

Check each solution by substituting into the original equation.

Since $m = 4$ results in a zero in the denominator, it is extraneous. So, the solution is -1.

Talk About It! How could you identify the extraneous solution without substituting each possible solution back into the original equation? Explain.

Sample answer: I could find the values of m that result in a denominator of 0 for each rational expression in the equation. If any of these values is one of the possible solutions, then it is an extraneous solution.

Check

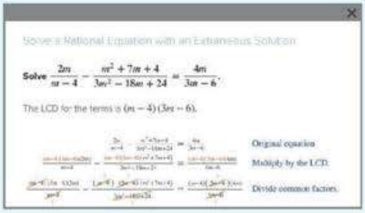
Solve $\frac{3x}{x-4} - \frac{x^2-7x-4}{x^2-16} = \frac{5}{x+4}$.

$x =$ ___?___ -3

Go Online You can complete an Extra Example online.

356 Module 7 • Rational Functions

Interactive Presentation



Solve a Rational Equation with an Extraneous Solution

Solve $\frac{2m}{m-4} - \frac{m^2+7m+4}{3m^2-18m+24} = \frac{4m}{3m-6}$.

The LCD for the terms is $(m-4)(3m-6)$.

Example 2

TYPE

a|

Students answer a question to determine how to identify extraneous solutions.

Example 2 Solve a Rational Equation with an Extraneous Solution

Teaching the Mathematical Practices

1 Check Answers Mathematically proficient students continually ask themselves, “Does this make sense?” Point out that in Example 2, students need to check their answers to identify possible extraneous solutions. The Talk About It! feature asks students to describe how they could check their answer using a different method than is used in the example.

Questions for Mathematical Discourse

AL How do you determine the LCD for the equation? The denominator of the second rational expression factors as $(m - 4)(3m - 6)$, and each of these factors is a denominator of the other rational expressions, so the LCD is $(m - 4)(3m - 6)$.

OL Why can you eliminate $(m - 4)$ and $(3m - 6)$ in the numerator and $3m^2 - 18m + 24$ in the denominator of the second expression? $(m - 4)(3m - 6) = m^2 - 18m + 24$

BL The left side of the equation can be simplified to $\frac{(5m + 1)}{(3m - 6)}$. What happens when you solve the equation by simplifying this way first? What does this imply? Sample answer: You only get the solution $m = -1$, indicating that the extraneous solution is introduced during the process of multiplying the entire expression by the LCD.

Example 3 Mixture Problem

Teaching the Mathematical Practices

1 Analyze Givens and Constraints In this example, guide students through the steps to identify the meaning of the problem and look for entry points to its solution.

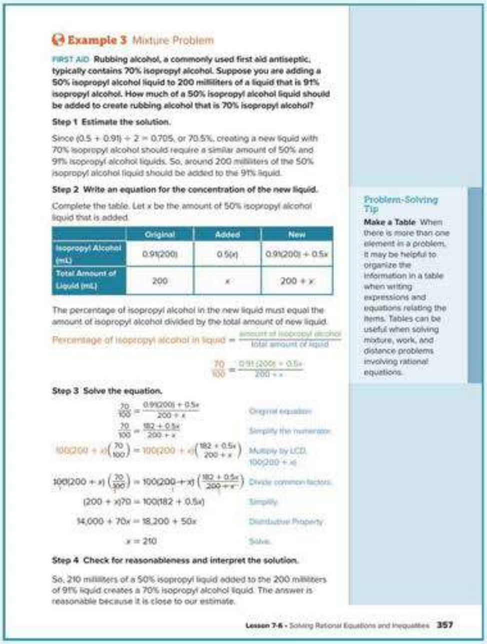
5 Use Estimation Point out that in Example 3, students need to include an estimate of the amount of isopropyl alcohol liquid needed and check against the estimate at the end.

Questions for Mathematical Discourse

AL How does $\frac{70}{100}$ represent 70%? A percent is "part of 100," so 70% is 70 out of 100 and can be represented as $\frac{70}{100}$.

OL Explain why 0.91(200) and 0.5x represent the amount of isopropyl alcohol. Sample answer: Multiplying the percent of isopropyl alcohol in each liquid by the number of milliliters of the liquid gives can find the number of milliliters of isopropyl alcohol.

BL Suppose you wanted to create a liquid that has a higher percentage of isopropyl alcohol. How would this affect your solution? A smaller amount of a 50% isopropyl liquid would be added.



Example 3 Mixture Problem

FIRST AID **Rubbing alcohol, a commonly used first aid antiseptic, typically contains 70% isopropyl alcohol. Suppose you are adding a 50% isopropyl alcohol liquid to 200 milliliters of a liquid that is 91% isopropyl alcohol. How much of a 50% isopropyl alcohol liquid should be added to create rubbing alcohol that is 70% isopropyl alcohol?**

Step 1 Estimate the solution.

Since (0.5 + 0.91) ÷ 2 = 0.705, or 70.5%, creating a new liquid with 70% isopropyl alcohol should require a similar amount of 50% and 91% isopropyl alcohol liquids. So, around 200 milliliters of the 50% isopropyl alcohol liquid should be added to the 91% liquid.

Step 2 Write an equation for the concentration of the new liquid.

Complete the table. Let x be the amount of 50% isopropyl alcohol liquid that is added.

	Original	Added	New
Isopropyl Alcohol (mL)	0.91(200)	0.5(x)	0.91(200) + 0.5x
Total Amount of Liquid (mL)	200	x	200 + x

The percentage of isopropyl alcohol in the new liquid must equal the amount of isopropyl alcohol divided by the total amount of new liquid.

$$\text{Percentage of isopropyl alcohol in liquid} = \frac{\text{amount of isopropyl alcohol}}{\text{total amount of liquid}}$$

$$\frac{70}{100} = \frac{0.91(200) + 0.5x}{200 + x}$$

Step 3 Solve the equation.

$$\frac{70}{100} = \frac{0.91(200) + 0.5x}{200 + x} \quad \text{Original equation}$$

$$\frac{70}{100} = \frac{182 + 0.5x}{200 + x} \quad \text{Simplify the numerator.}$$

$$100(200 + x)\left(\frac{70}{100}\right) = 100(200 + x)\left(\frac{182 + 0.5x}{200 + x}\right) \quad \text{Multiply by LCD, } 100(200 + x).$$

$$\cancel{100}(200 + x)\left(\frac{70}{\cancel{100}}\right) = 100\cancel{(200 + x)}\left(\frac{182 + 0.5x}{\cancel{200 + x}}\right) \quad \text{Divide common factors.}$$

$$(200 + x)70 = 100(182 + 0.5x) \quad \text{Simplify.}$$

$$14{,}000 + 70x = 18{,}200 + 50x \quad \text{Distributive Property}$$

$$x = 210 \quad \text{Solve.}$$

Step 4 Check for reasonableness and interpret the solution.

So, 210 milliliters of a 50% isopropyl liquid added to the 200 milliliters of 91% liquid creates a 70% isopropyl alcohol liquid. The answer is reasonable because it is close to our estimate.

Problem-Solving Tip

Make a Table When there is more than one element in a problem, it may be helpful to organize the information in a table when writing expressions and equations relating the items. Tables can be useful when solving mixture, work, and distance problems involving rational equations.

Lesson 7-6 • Solving Rational Equations and Inequalities **357**

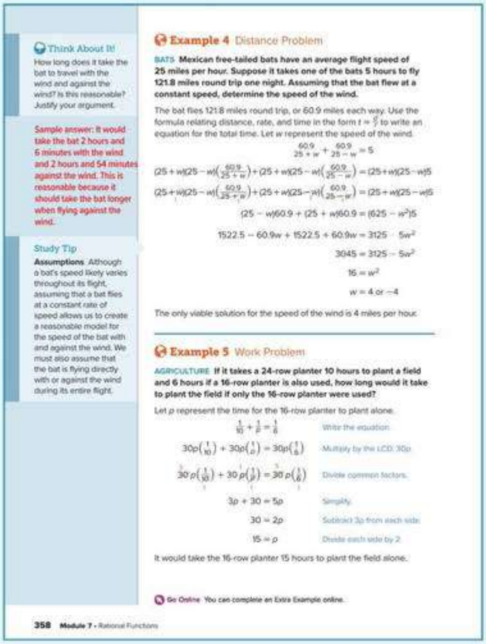
Interactive Presentation

Example 3

TYPE

a|

Students answer a question about checking their solution.



Think About It!

How long does it take the bat to travel with the wind and against the wind? Is this reasonable? Justify your argument.

Sample answer: It would take the bat 2 hours and 6 minutes with the wind and 2 hours and 54 minutes against the wind. This is reasonable because it should take the bat longer when flying against the wind.

Study Tip

Assumptions Although a bat's speed likely varies throughout its flight, assuming that a bat flies at a constant rate of speed allows us to create a reasonable model for the speed of the bat with and against the wind. We must also assume that the bat is flying directly with or against the wind during its entire flight.

Example 4 Distance Problem

BATS **Mexican free-tailed bats have an average flight speed of 25 miles per hour. Suppose it takes one of the bats 5 hours to fly 121.8 miles round trip one night. Assuming that the bat flew at a constant speed, determine the speed of the wind.**

The bat flies 121.8 miles round trip, or 60.9 miles each way. Use the formula relating distance, rate, and time in the form $t = \frac{d}{r}$ to write an equation for the total time. Let w represent the speed of the wind.

$$\frac{60.9}{25+w} + \frac{60.9}{25-w} = 5$$

$$(25+w)(25-w)\left(\frac{60.9}{25+w}\right) + (25+w)(25-w)\left(\frac{60.9}{25-w}\right) = (25+w)(25-w)5$$

$$(25+w)(25-w)\left(\frac{60.9}{25+w}\right) + (25+w)(25-w)\left(\frac{60.9}{25-w}\right) = (25+w)(25-w)5$$

$$(25-w)60.9 + (25+w)60.9 = (625-w^2)5$$

$$1522.5 - 60.9w + 1522.5 + 60.9w = 3125 - 5w^2$$

$$3045 = 3125 - 5w^2$$

$$16 = w^2$$

$$w = 4 \text{ or } -4$$

The only viable solution for the speed of the wind is 4 miles per hour.

Example 5 Work Problem

AGRICULTURE **If it takes a 24-row planter 10 hours to plant a field and 6 hours if a 16-row planter is also used, how long would it take to plant the field if only the 16-row planter were used?**

Let p represent the time for the 16-row planter to plant alone.

$\frac{1}{10} + \frac{1}{p} = \frac{1}{6}$ Write the equation.

$30p\left(\frac{1}{10}\right) + 30p\left(\frac{1}{p}\right) = 30p\left(\frac{1}{6}\right)$ Multiply by the LCD, 30p.

$\overset{3}{\cancel{30}}p\left(\frac{1}{\underset{1}{\cancel{10}}}\right) + 30\overset{1}{\cancel{p}}\left(\frac{1}{\underset{1}{\cancel{p}}}\right) = \overset{5}{\cancel{30}}p\left(\frac{1}{\underset{1}{\cancel{6}}}\right)$ Divide common factors.

$3p + 30 = 5p$ Simplify.

$30 = 2p$ Subtract 3p from each side.

$15 = p$ Divide each side by 2.

It would take the 16-row planter 15 hours to plant the field alone.

Go Online You can complete an Extra Example online.

358 Module 7 • Rational Functions

Interactive Presentation

Example 4

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 4 Distance Problem

Teaching the Mathematical Practices

4 Interpret Mathematical Results Point out that to solve the problem in Example 4 and to answer the question in the Think About It! feature, students should interpret their mathematical results in the context of the problem and reflect on whether the results makes sense.

Questions for Mathematical Discourse

AL Why is the rate against the wind 25 —w? Sample answer: The wind slows the bat down, so to find the bat's average speed against the wind, take its average speed and subtract the speed of the wind, w.

OL Why is —4 not a viable solution? Speed cannot be negative.

BL What assumptions are being made about the wind? Sample answer: The speed of the wind is constant and in one direction aligned with the direction the bat travels.

Common Error

When solving rational equations, students often forget that all terms in the equation must be multiplied by the LCD. Remind them to do so in order to obtain equivalent equations.

Example 5 Work Problem

Teaching the Mathematical Practices

2 Consider Units Point out that it is important to note the units involved in this problem. To solve the problem in Example 5, students should clearly label their solution with the appropriate unit.

Questions for Mathematical Discourse

AL What is the unknown in this work problem? how long it would take the 16-row planter to plant the entire field by itself

OL Before solving, what values of p would make sense as an answer? values that are greater than 10 hours

BL How would you write a general formula for how long it takes two people to complete a job together? Sample answer: Define variables for the time it takes each person to complete the job alone, such as x and y, and define a variable for the time it takes to do the job together, such as z. Then use the same form as in the example, but with variables: $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$

Learn Solving Rational Inequalities

Objective

Students solve rational inequalities in one variable algebraically.

MP Teaching the Mathematical Practices

1 Analyze Givens and Constraints When solving rational inequalities, encourage students to use the solution pathway as outlined by the steps in the Key Concept box.

Example 6 Solve a Rational Inequality

MP Teaching the Mathematical Practices

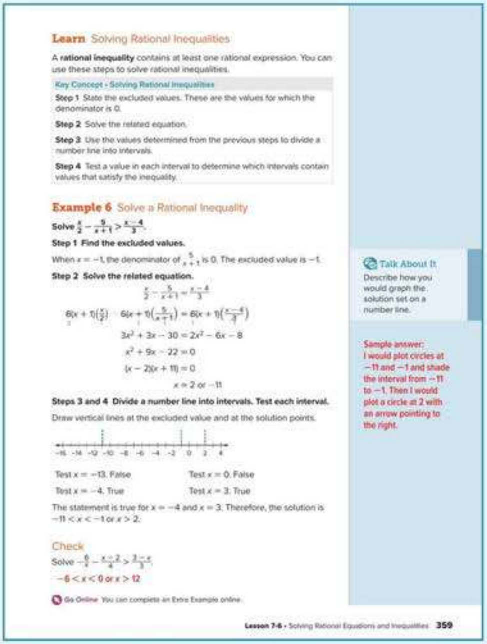
7 Use Structure Help students to use the structure of the rational inequality to find the excluded values and solve the inequality.

Questions for Mathematical Discourse

AL How do you choose what values to test in each interval? Sample answer: Look for values in each interval that are easy to plug into the inequality.

OL How do you determine the solution after testing values in each interval? The solution is the union of all the intervals that satisfy the inequality.

BL What would the solution be if the inequality was *greater than or equal to*? $-11 \leq x < -1$ or $x \geq 2$



Learn Solving Rational Inequalities

A **rational inequality** contains at least one rational expression. You can use these steps to solve rational inequalities.

Key Concept • Solving Rational Inequalities

Step 1 State the excluded values. These are the values for which the denominator is 0.

Step 2 Solve the related equation.

Step 3 Use the values determined from the previous steps to divide a number line into intervals.

Step 4 Test a value in each interval to determine which intervals contain values that satisfy the inequality.

Example 6 Solve a Rational Inequality

Solve $\frac{x}{2} - \frac{5}{x+1} > \frac{x-4}{3}$.

Step 1 Find the excluded values.

When $x = -1$, the denominator of $\frac{5}{x+1}$ is 0. The excluded value is -1.

Step 2 Solve the related equation.

$$\frac{x}{2} - \frac{5}{x+1} = \frac{x-4}{3}$$

$$6(x+1)\left(\frac{x}{2}\right) - 6(x+1)\left(\frac{5}{x+1}\right) = 6(x+1)\left(\frac{x-4}{3}\right)$$

$$3x^2 + 3x - 30 = 2x^2 - 6x - 8$$

$$x^2 + 9x - 22 = 0$$

$$(x-2)(x+11) = 0$$

$$x = 2 \text{ or } -11$$

Steps 3 and 4 Divide a number line into intervals. Test each interval.

Draw vertical lines at the excluded value and at the solution points.

Test $x = -13$. False

Test $x = -4$. True

Test $x = 0$. False

Test $x = 3$. True

The statement is true for $x = -4$ and $x = 3$. Therefore, the solution is $-11 < x < -1$ or $x > 2$.

Check

Solve $-\frac{6}{x} - \frac{x+2}{4} > \frac{3-x}{3}$.

$-6 < x < 0$ or $x > 12$

Go Online You can complete an Extra Example online.

Talk About It

Describe how you would graph the solution set on a number line.

Sample answer: I would plot circles at −11 and −1 and shade the interval from −11 to −1. Then I would plot a circle at 2 with an arrow pointing to the right.

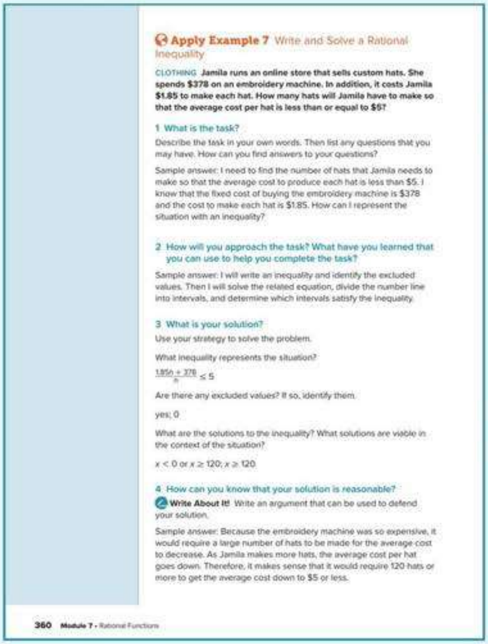
Lesson 7-6 • Solving Rational Equations and Inequalities 359

Interactive Presentation

Learn

EXPAND

Students tap to see the steps to solve a rational inequality.



Apply Example 7 Write and Solve a Rational Inequality

CLOTHING **Jamila runs an online store that sells custom hats. She spends $378 on an embroidery machine. In addition, it costs Jamila $1.85 to make each hat. How many hats will Jamila have to make so that the average cost per hat is less than or equal to $5?**

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to find the number of hats that Jamila needs to make so that the average cost to produce each hat is less than $5. I know that the fixed cost of buying the embroidery machine is $378 and the cost to make each hat is $1.85. How can I represent the situation with an inequality?

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I will write an inequality and identify the excluded values. Then I will solve the related equation, divide the number line into intervals, and determine which intervals satisfy the inequality.

3 What is your solution?

Use your strategy to solve the problem.

What inequality represents the situation?

$\frac{1.85n + 378}{n} \leq 5$

Are there any excluded values? If so, identify them.

yes; 0

What are the solutions to the inequality? What solutions are viable in the context of the situation?

$x < 0$ or $x \geq 120$; $x \geq 120$

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: Because the embroidery machine was so expensive, it would require a large number of hats to be made for the average cost to decrease. As Jamila makes more hats, the average cost per hat goes down. Therefore, it makes sense that it would require 120 hats or more to get the average cost down to $5 or less.

360 Module 7 • Rational Functions

Interactive Presentation

Write and Solve a Rational Inequality

Example 7

TAP

Students move through the steps to solve a rational inequality problem.

DRAG & DROP

Students drag and drop the steps to solve the problem in the correct order.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Apply Example 7 Write and Solve a Rational Inequality

Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them, 4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- What expression represents that total cost of the hats?
- How do you choose the values to test?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–26
2	exercises that use a variety of skills from this lesson	27–43
2	exercises that extend concepts learned in this lesson to new contexts	44
3	exercises that emphasize higher-order and critical-thinking skills	45–51

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **BL**
THEN assign:

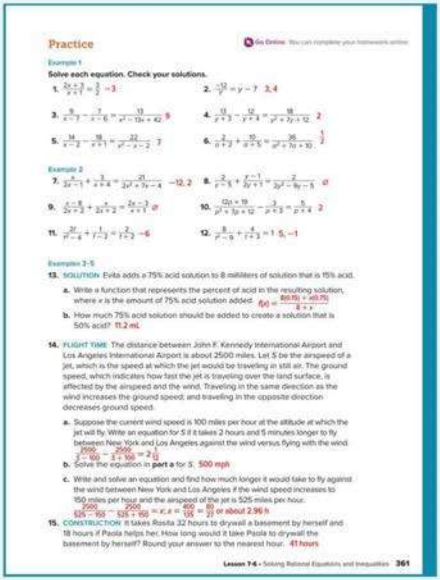
- Practice, Exercises 1–43 odd, 45–51
- Extension: Asymptotes in Three Dimensions
- **ALEKS** Rational Equations and Inequalities

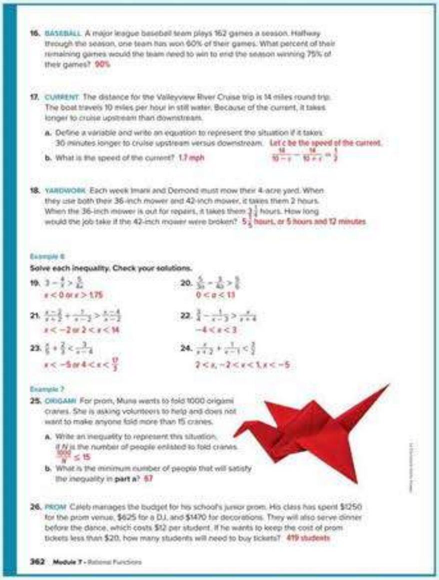
IF students score 66%–89% on the Checks, **OL**
THEN assign:

- Practice, Exercises 1–51 odd
- Remediation, Review Resources: Solving Proportions
- Personal Tutors
- Extra Examples 1–7
- **ALEKS** Proportions

IF students score 65% or less on the Checks, **AL**
THEN assign:

- Practice, Exercises 1–25 odd
- Remediation, Review Resources: Solving Proportions
- *Quick Review Math Handbook*: Solving Rational Equations
- **ALEKS** Proportions

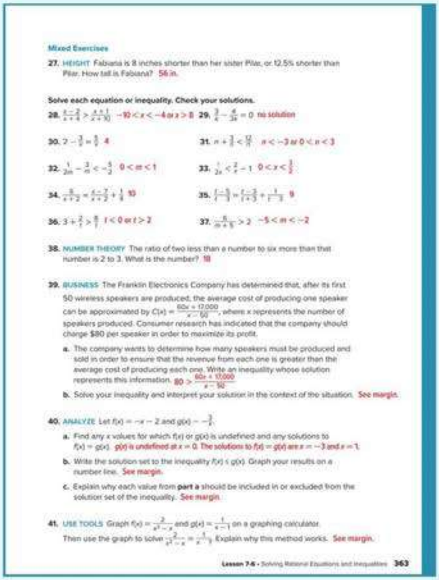


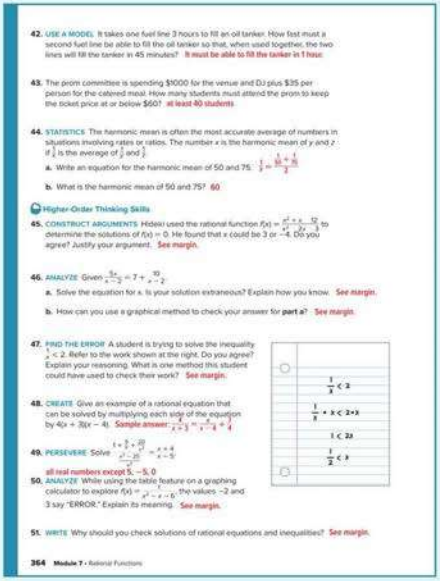


3 REFLECT AND PRACTICE

A.CED.1, A.REI.2, A.REI.11

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION





Answers

39b. $x > 1050$; sample answer: The company must produce and sell at least 1050 audio players in order to ensure that the revenue from each one is greater than the average cost of producing each one.

40b. $-3 \le x < 0$ or $x \ge 1$

40c. Sample answer: The values $x = -3$ and $x = 1$ are included because they satisfy $f(x) \le g(x)$. The value $x = 0$ is excluded because it makes $g(x)$ undefined.

41. $x = 2$; sample answer: This method works because the x-coordinate of the point of intersection will make both sides of the equation equal to the same value, and no extraneous roots are shown.

45. No; sample answer: His solution of 3 must be excluded, as it will make the denominator of $f(x)$ equal to 0. Only -4 is a solution.

46a. $\frac{5x}{x-2} \cdot \frac{x-2}{1} = 7(x-2) + \frac{10}{x-2} \cdot \frac{x-2}{1} \Rightarrow 5x = 7x - 14 + 10 \Rightarrow$ $-2x = -4 \Rightarrow x = 2$; This value is extraneous because it will make the denominators in the original equation equal to 0.

46b. Sample answer: I can graph each side of the equation to see that the two graphs do not intersect and, therefore, I would know there is no solution.

47. No; sample answer: Multiplying both sides of the inequality by x does not result in an equivalent inequality if x is negative. The work is only valid for positive x-values. The student could have graphed the two equations $f(x) = \frac{1}{x}$ and $g(x) = 2$ and looked at x-values for which the graph of $f(x)$ is below the graph of $g(x)$.

50. Sample answer: The values 3 and -2 are undefined values. On the graph of $f(x)$ there would be vertical asymptotes at these values.

51. Sample answer: Multiplying each side of a rational equation or inequality by the LCD can result in extraneous solutions. Therefore, you should check all solutions to make sure that they satisfy the original equation or inequality.

Module 7 • Rational Functions

Review

Rate Yourself!

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have students respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- Why are graphs useful?
- How are the properties of a rational function reflected in its graph?
- Why can analyzing a rational function algebraically and graphically help you to see the "whole picture?"

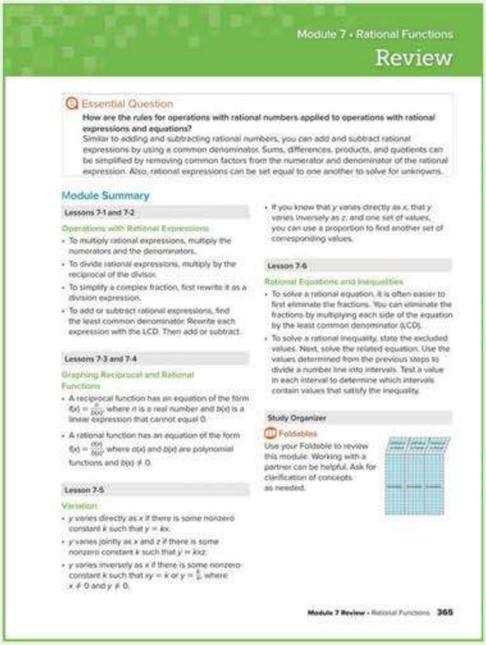
Then have students write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

ELL A completed Foldable for this module should include the key concepts related to rational functions.

LS **LearnSmart** Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Polynomial, Rational, and Radical Relationships.**

- Rational Expressions
- Solving Rational and Radical Equations
- Graphing Rational, Radical, and Polynomial Functions



Module 7 • Rational Functions

Review

Essential Question

How are the rules for operations with rational numbers applied to operations with rational expressions and equations?

Similar to adding and subtracting rational numbers, you can add and subtract rational expressions by using a common denominator. Sums, differences, products, and quotients can be simplified by removing common factors from the numerator and denominator of the rational expression. Also, rational expressions can be set equal to one another to solve for unknowns.

Module Summary

Lessons 7-1 and 7-2

Operations with Rational Expressions

- To multiply rational expressions, multiply the numerators and the denominators.
- To divide rational expressions, multiply by the reciprocal of the divisor.
- To simplify a complex fraction, first rewrite it as a division expression.
- To add or subtract rational expressions, find the least common denominator. Rewrite each expression with the LCD. Then add or subtract.

Lessons 7-3 and 7-4

Graphing Reciprocal and Rational Functions

- A reciprocal function has an equation of the form $f(x) = \frac{n}{b(x)}$, where n is a real number and $b(x)$ is a linear expression that cannot equal 0.
- A rational function has an equation of the form $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial functions and $b(x) \neq 0$.

Lesson 7-5

Variation

- y varies directly as x if there is some nonzero constant k such that $y = kx$.
- y varies jointly as x and z if there is some nonzero constant k such that $y = kxz$.
- y varies inversely as x if there is some nonzero constant k such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$.
- If you know that y varies directly as x, that y varies inversely as z, and one set of values, you can use a proportion to find another set of corresponding values.

Lesson 7-6

Rational Equations and Inequalities

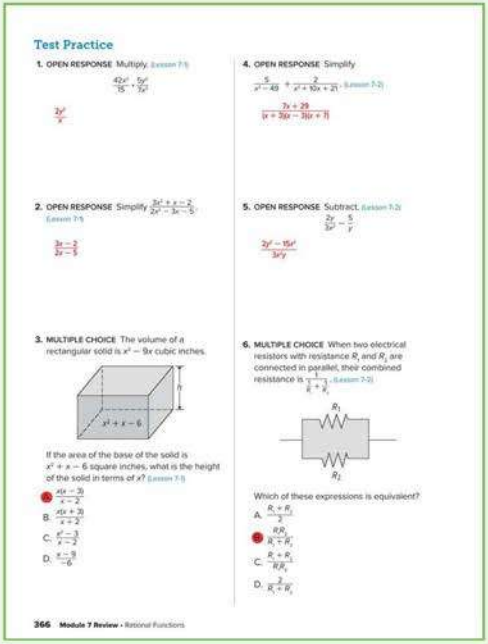
- To solve a rational equation, it is often easier to first eliminate the fractions. You can eliminate the fractions by multiplying each side of the equation by the least common denominator (LCD).
- To solve a rational inequality, state the excluded values. Next, solve the related equation. Use the values determined from the previous steps to divide a number line into intervals. Test a value in each interval to determine which intervals contain values that satisfy the inequality.

Study Organizer

Foldables

Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.

Module 7 Review • Rational Functions **365**



Test Practice

1. **OPEN RESPONSE** Multiply. (Lesson 7-1)

$$\frac{42x^2}{15} \cdot \frac{5y^4}{7x^3}$$

$\frac{2y^4}{x}$

2. **OPEN RESPONSE** Simplify $\frac{3x^2+x-2}{2x^2-3x-5}$. (Lesson 7-1)

$\frac{3x-2}{2x-5}$

3. **MULTIPLE CHOICE** The volume of a rectangular solid is $x^3 - 9x$ cubic inches.

If the area of the base of the solid is $x^2 + x - 6$ square inches, what is the height of the solid in terms of x? (Lesson 7-1)

● A. $\frac{x(x-3)}{x-2}$
B. $\frac{x(x+3)}{x+2}$
C. $\frac{x^2-3}{x-2}$
D. $\frac{x-9}{-6}$

4. **OPEN RESPONSE** Simplify

$$\frac{5}{x^2-49} + \frac{2}{x^2+10x+21}.$$ (Lesson 7-2)

$\frac{7x+29}{(x+3)(x-3)(x+7)}$

5. **OPEN RESPONSE** Subtract. (Lesson 7-2)

$$\frac{2y}{3x^2} - \frac{5}{y}$$

$\frac{2y^2-15x^2}{3x^2y}$

6. **MULTIPLE CHOICE** When two electrical resistors with resistance R_1 and R_2 are connected in parallel, their combined resistance is $\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$. (Lesson 7-2)

Which of these expressions is equivalent?

A. $\frac{R_1+R_2}{2}$
● B. $\frac{R_1R_2}{R_1+R_2}$
C. $\frac{R_1+R_2}{R_1R_2}$
D. $\frac{2}{R_1+R_2}$

366 **Module 7 Review** • Rational Functions

Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Put It All Together: Lessons 7-1 and 7-2
Vocabulary Activity
Module Review

Assessment Resources

Vocabulary Test
AL Module Test Form B
OL Module Test Form A
BL Module Test Form C
Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

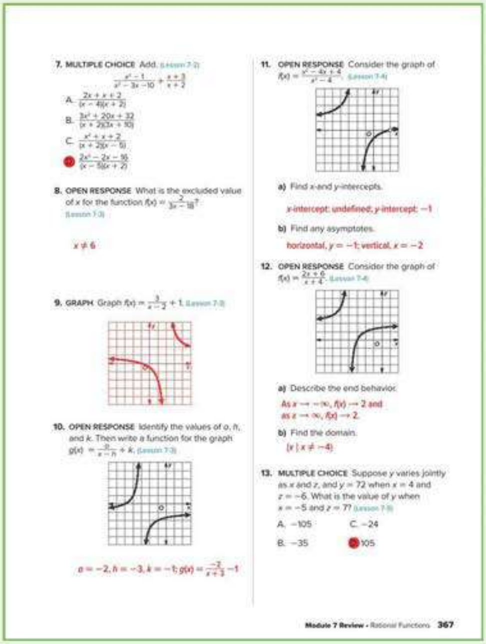
Test Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1–18 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Multiple Choice	Students select one correct answer.	3, 6, 7, 13, 14, 17, 18
Multi-Select	Multiple answers may be correct. Students must select all correct answers.	16
Graph	Students create a graph on an online coordinate plane.	9
Open Response	Students construct their own response.	1, 2, 4, 5, 8, 10, 11, 12, 15

To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
A.APR.7	7-1, 7-2	1-7
A.CED.1	7-6	17, 18
A.CED.2	7-4	10, 13, 14
A.CED.4	7-5	13, 14
F.IF.4	7-3, 7-4	9, 11, 12
F.IF.5	7-3, 7-4	8, 12
F.BF.3	7-3	9, 10
A.REI.2	7-6	16, 18



7. MULTIPLE CHOICE Add. (Lesson 7-2)

$$\frac{x^2-1}{x^2-3x-10}+\frac{x+3}{x+2}$$

A. $\frac{2x+x+2}{(x-4)(x+2)}$

B. $\frac{3x^2+20x+32}{(x+2)(3x+10)}$

C. $\frac{x^2+x+2}{(x+2)(x-5)}$

[✓ D.] $\frac{2x^2-2x-16}{(x-5)(x+2)}$

8. OPEN RESPONSE What is the excluded value of x for the function $f(x)=\frac{2}{3x-18}$? (Lesson 7-3)

$x \neq 6$

9. GRAPH Graph $f(x)=\frac{3}{x-2}+1$. (Lesson 7-3)

10. OPEN RESPONSE Identify the values of a, h, and k. Then write a function for the graph $g(x)=\frac{a}{x-h}+k$. (Lesson 7-3)

$a=-2, h=-3, k=-1; g(x)=\frac{-2}{x+3}-1$

11. OPEN RESPONSE Consider the graph of $f(x)=\frac{x^2-4x+4}{x^2-4}$. (Lesson 7-4)

a) Find x-and y-intercepts.

x-intercept: undefined; y-intercept: -1

b) Find any asymptotes.

horizontal, $y=-1$; vertical, $x=-2$

12. OPEN RESPONSE Consider the graph of $f(x)=\frac{2x+6}{x+4}$. (Lesson 7-4)

a) Describe the end behavior.

As $x \to -\infty$, $f(x) \to 2$ and as $x \to \infty$, $f(x) \to 2$.

b) Find the domain.

$\{x \mid x \neq -4\}$

13. MULTIPLE CHOICE Suppose y varies jointly as x and z, and $y=72$ when $x=4$ and $z=-6$. What is the value of y when $x=-5$ and $z=7$? (Lesson 7-5)

A. -105

B. -35

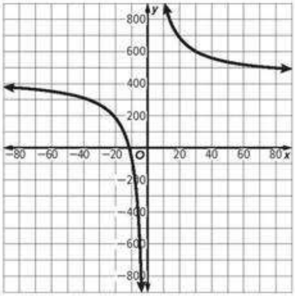
C. -24

[✓ D.] 105

Module 7 Review • Rational Functions 367

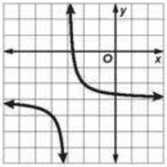
Lesson 7-3

12a.

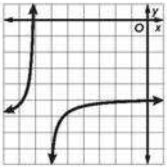


12b. Sample answer: D = $\{n \mid n \neq 0\}$; R = $\{C(n) \mid C(n) \neq 435\}$; intercepts: about $(-11.49, 0)$; positive: when $-\infty < n < -11.49$ and when $n > 0$; negative: when $-11.49 < n < 0$; symmetry: symmetric about $(0, 435)$; end behavior: As $n \rightarrow -\infty$, $C \rightarrow 435$, and as $n \rightarrow \infty$, $C \rightarrow 435$. Because the cost and number of computers cannot be negative, only values in the domain $\{n \mid n > 0\}$ make sense in the context of this situation.

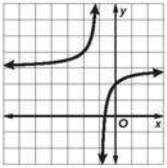
13. D = $\{x \mid x \neq -3\}$; R = $\{f(x) \mid f(x) \neq -3\}$



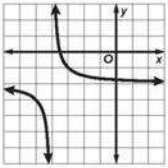
14. D = $\{x \mid x \neq -5\}$; R = $\{f(x) \mid f(x) \neq -6\}$



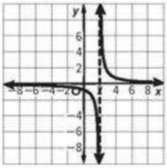
15. D = $\{x \mid x \neq -1\}$; R = $\{f(x) \mid f(x) \neq 3\}$



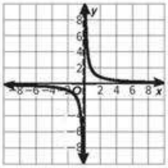
16. D = $\{x \mid x \neq -4\}$; R = $\{f(x) \mid f(x) \neq -2\}$



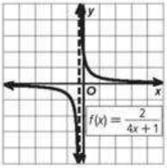
31. asymptotes: $x = 2$, $f(x) = 0$; D = $\{x \mid x \neq 2\}$; R = $\{f(x) \mid f(x) \neq 0\}$



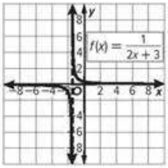
32. asymptotes: $x = 0$, $f(x) = 0$; D = $\{x \mid x \neq 0\}$; R = $\{f(x) \mid f(x) \neq 0\}$



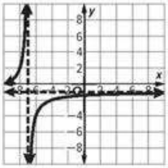
33. asymptotes: $x = -\frac{1}{4}$, $f(x) = 0$; D = $\{x \mid x \neq -\frac{1}{4}\}$; R = $\{f(x) \mid f(x) \neq 0\}$



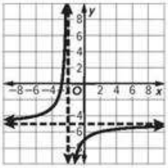
34. asymptotes: $x = -\frac{3}{2}$, $f(x) = 0$; D = $\{x \mid x \neq -\frac{3}{2}\}$; R = $\{f(x) \mid f(x) \neq 0\}$



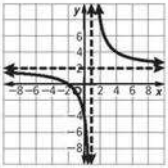
35. asymptotes: $x = -7$, $f(x) = -1$; D = $\{x \mid x \neq -7\}$; R = $\{f(x) \mid f(x) \neq -1\}$



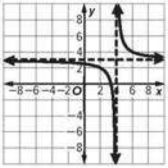
36. asymptotes: $x = -2$, $f(x) = -5$; D = $\{x \mid x \neq -2\}$; R = $\{f(x) \mid f(x) \neq -5\}$



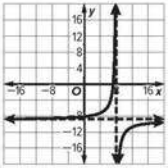
37. asymptotes: $x = 1$, $f(x) = 2$; D = $\{x \mid x \neq 1\}$; R = $\{f(x) \mid f(x) \neq 2\}$



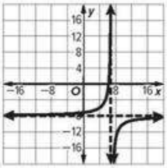
38. asymptotes: $x = 4$, $f(x) = 3$; D = $\{x \mid x \neq 4\}$; R = $\{f(x) \mid f(x) \neq 3\}$



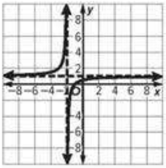
39. asymptotes: $x = 8$, $f(x) = -9$; D = $\{x \mid x \neq 8\}$; R = $\{f(x) \mid f(x) \neq -9\}$



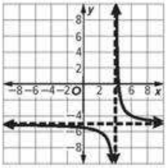
40. asymptotes: $x = 7$, $f(x) = -8$; D = $\{x \mid x \neq 7\}$; R = $\{f(x) \mid f(x) \neq -8\}$



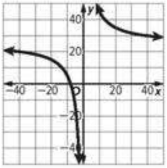
47. The graph of $g(x)$ is a reflection in the x-axis, a translation 2 units left, and a translation 1 unit up of the graph of $f(x)$.



48. The graph of $h(x)$ is a translation 4 units right, and a translation 5 units down of the graph of $f(x)$. The graph is stretched vertically.

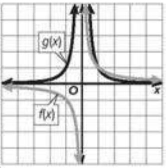


50a.



52a.

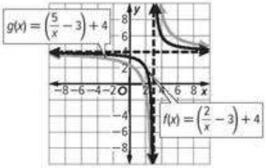
$f(x) = \frac{1}{x}$		$g(x) = \frac{1}{x^2}$	
x	$f(x)$	x	$g(x)$
−3	$-\frac{1}{3}$	−3	$\frac{1}{9}$
−2	$-\frac{1}{2}$	−2	$\frac{1}{4}$
−1	−1	−1	1
0	undefined	0	undefined
1	1	1	1
2	$\frac{1}{2}$	2	$\frac{1}{4}$
3	$\frac{1}{3}$	3	$\frac{1}{9}$



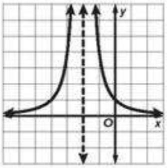
52b. The positive portion of $f(x) = \frac{1}{x^2}$ is similar to the graph of $f(x) = \frac{1}{x}$. Positive values of x produce positive values of $f(x)$. The negative portion of $f(x)$ is a reflection of the positive portion in the line $y = -x$. The negative portion of $g(x)$ is a reflection of the positive portion reflected in the y-axis.

52c. Sample answer: When the exponent is even, one portion of the graph is a reflection of the other portion over the vertical asymptote. When the exponent is odd, one portion of the graph is a rotation of the other 180 degrees about (h, k).

54.

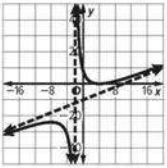


55.

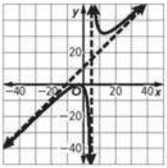


Lesson 7-4

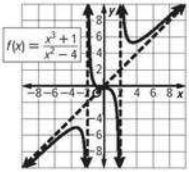
11. zero: $x = 4$; vertical asymptote: $x = -2$; oblique asymptote: $f(x) = x - 10$



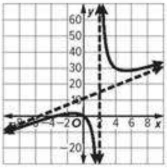
12. zero: $x = -3$; vertical asymptote: $x = 5$; oblique asymptote: $f(x) = x + 11$



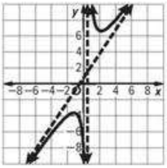
13. zero: none; vertical asymptote: $x = -2$; oblique asymptote: $f(x) = 6x - 8$



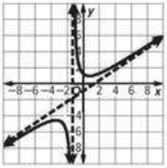
14. zero: $x =$ and $x = -3.5$; vertical asymptote: $x = 2$; oblique asymptote: $f(x) = 2x + 11$



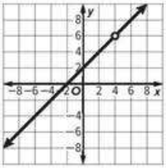
15. zero: none; vertical asymptote: $x = \frac{1}{2}$; oblique asymptote: $f(x) = \frac{3}{2}x + \frac{3}{4}$



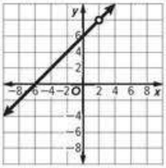
16. zero: none; vertical asymptote: $x = -\frac{4}{3}$; oblique asymptote: $f(x) = \frac{2}{3}x - \frac{8}{9}$



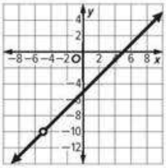
17. point discontinuity at $x = 4$



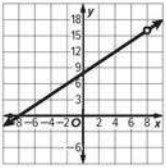
18. point discontinuity at $x = 2$



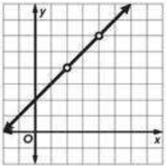
19. point discontinuity at $x = -5$



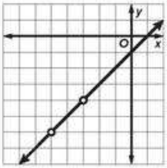
20. point discontinuity at $x = 8$



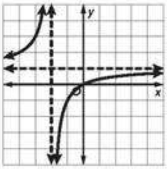
21. point discontinuity at $x = 2$ and $x = 4$



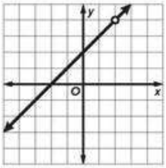
22. point discontinuity at $x = -5$ and $x = -3$



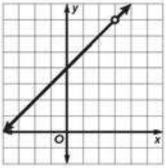
23.



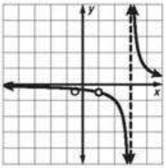
24.



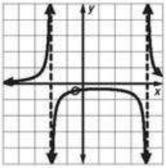
25.



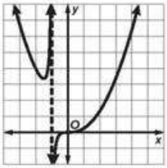
26.



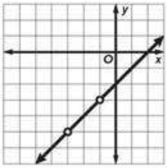
27.



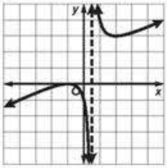
28.



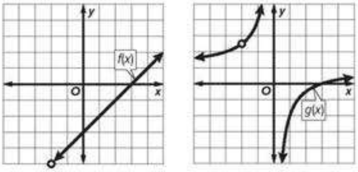
29.



30.

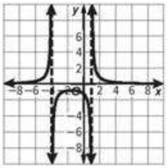


31a.

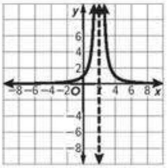


31c. Sample answer: The graph of $f(x)$ has a hole at $x = -2$, but no vertical asymptote. Its graph is a straight line with a hole in it at $(-2, -5)$. The graph of $g(x)$ also has a hole at $x = -2$, but has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 1$. Its graph is not a straight line, but two curves having a hole in the graph at $\left(-2, \frac{5}{2}\right)$.

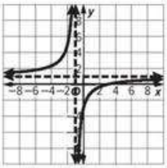
34.



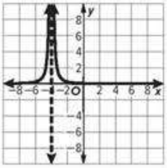
35.



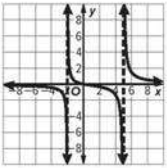
36.



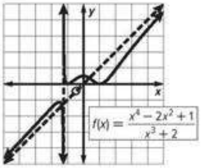
37.



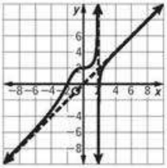
38.



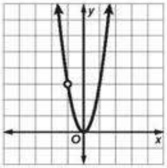
39.



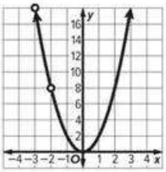
40.



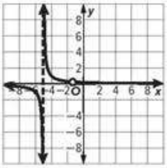
41.



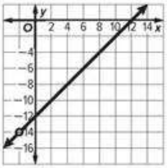
42.



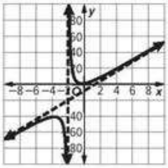
43.



44.



45.

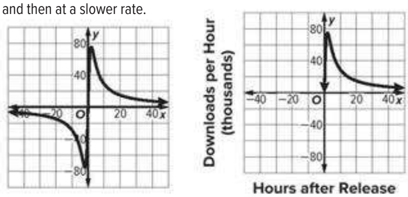


47a. $y = \frac{2067 + 183}{6767 + x}$

47b. vertical asymptote at $x = -6767$; horizontal asymptote at $y = 0$

47c. Sample answer: As the number of at bats increases, the ratio grows closer and closer to zero, since the number of hits remains constant.

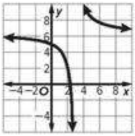
48. Sample answer: The graph shows a spike in demand in the first hours after the release; the rate of downloads then begins to decrease, rapidly at first and then at a slower rate.



49. Sample answer: The domain is $\{x \mid x \neq -2 \text{ and } x \neq 2\}$. The graph has zeros at $x = 1$ and $x = 4$ and a y-intercept at (0, 1). The graph has vertical asymptotes at $x = -2$ and $x = 2$ and a horizontal asymptote at $f(x) = -1$ for $2 \leq x$ or $x \leq -2$. A function with these features is $f(x) = \frac{(x-1)(x-4)}{(x+2)(x-2)} = \frac{x^2 - 5x + 4}{x^2 - 4}$.

50. The graphs of $y = \frac{(x+1)(x-1)}{x-1}$ and $y = \frac{(x+1)(x-1)^2}{(x-1)^2}$ are equivalent; sample answer: Both are the line $y = x + 1$ with a hole at $x = 1$. $y = x + 1$ is almost equivalent to those, but it lacks the hole. $y = \frac{(x+1)(x-1)^2}{x-1}$ is different altogether because the difference in exponents is no longer 1.

51. Sample answer:

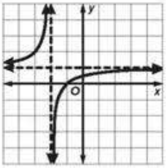


53. Sample answer: If the degrees of the numerator and denominator are equal, there is a horizontal asymptote determined by the coefficients of the numerator and denominator. If the degree of the numerator is greater, there is no horizontal asymptote. If the degree of the numerator is exactly 1 greater than the degree of the denominator, there is an oblique asymptote. If the degree of the denominator is greater, the horizontal asymptote is the line $y = 0$.

54a. Average speed $= \frac{2d}{t_1 + t_2} = \frac{2d}{\frac{d}{r_1} + \frac{d}{r_2}} = \frac{2}{\frac{1}{r_1} + \frac{1}{r_2}}$, where r_1 and t_1 are average speed and time for one direction and r_2 and t_2 are average speed and time for the return trip. Simplifying and substituting 40 for r_1 and x for r_2 gives $f(x) = \frac{80x}{x + 40}$.

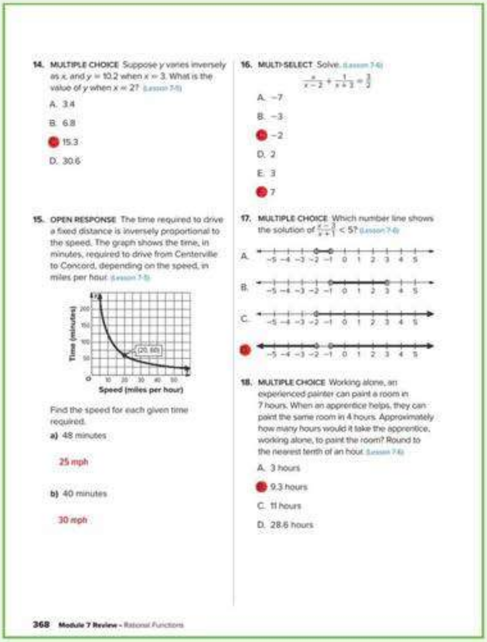
54b. When $x = 60$, $f(x) = 48$, so the average speed for the entire trip is less than 50 miles per hour. This is because more time is spent driving at 40 miles per hour than at 60 miles per hour. The horizontal asymptote is $f(x) = 80$. It represents an overall average speed the family cannot achieve; they would have to travel home at an infinite speed and return in no elapsed time.

55. Sample answer:



56. Sample answer: Both have vertical asymptotes at $x = 0$. Both approach 0 as x approaches $-\infty$ and approach 0 as x approaches ∞. $f(x)$ has holes at $x = 1$ and $x = -1$, while $g(x)$ has vertical asymptotes at $x = \sqrt{2}$ and $x = -\sqrt{2}$. $f(x)$ has no zeros, but $g(x)$ has zeros at $x = 1$ and $x = -1$

59. Sample answer: By factoring the denominator of the expression in a rational function and determining the the values that cause each factor to equal zero, you can find the asymptotes or discontinuity of the function. If the denominator has a factor $x - c$ that does not appear in the numerator, then there is a vertical asymptote at $x = c$. If the numerator and denominator have a common factor of $x - c$, then there is point continuity at $x = c$.



14. MULTIPLE CHOICE Suppose y varies inversely as x, and $y = 10.2$ when $x = 3$. What is the value of y when $x = 2$? (Lesson 7-5)

A. 3.4

B. 6.8

[✓ C.] 15.3

D. 30.6

15. OPEN RESPONSE The time required to drive a fixed distance is inversely proportional to the speed. The graph shows the time, in minutes, required to drive from Centerville to Concord, depending on the speed, in miles per hour. (Lesson 7-5)

Find the speed for each given time required.

a) 48 minutes

25 mph

b) 40 minutes

30 mph

16. MULTI-SELECT Solve. (Lesson 7-6)

$$\frac{x}{x-2}+\frac{1}{x+3}=\frac{3}{2}$$

A. −7

B. −3

[✓ C.] −2

D. 2

E. 3

[✓ F.] 7

17. MULTIPLE CHOICE Which number line shows the solution of $\frac{x-3}{x+1} < 5$? (Lesson 7-6)

A.

B.

C.

[✓ D.]

18. MULTIPLE CHOICE Working alone, an experienced painter can paint a room in 7 hours. When an apprentice helps, they can paint the same room in 4 hours. Approximately how many hours would it take the apprentice, working alone, to paint the room? Round to the nearest tenth of an hour. (Lesson 7-6)

A. 3 hours

[✓ B.] 9.3 hours

C. 11 hours

D. 28.6 hours

368 **Module 7 Review** • Rational Functions

Module 8

Inferential Statistics

Module Goals

- Students classify and identify bias in surveys and studies.
- Students collect, analyze, and use data to compare theoretical and experimental probabilities.
- Students use statistics to summarize, represent, and compare sets of data.
- Students use statistics and normal distributions to analyze data.

Focus

Domain: Statistics and Probability

Standards for Mathematical Content:

S.IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.

S.IC.6 Evaluate reports based on data.

Also addresses S.IC.1, S.IC.2, S.IC.3, S.IC.5, and S.ID.4

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this module.

Be Sure to Cover

To completely cover S.ID.4, go online to assign the following spreadsheet activity:

- Normal Distribution (Lesson 8-4)

Coherence

Vertical Alignment

Previous

Students represented data using numerical statistics and graphical methods, analyzed the shapes of distributions, and interpreted categorical data. **S.ID.1, S.ID.2, S.ID.3**

Now

Students determine populations and samples and run simulations to determine the probabilities of outcomes. Students analyze the distributions of data sets. **S.IC.1, S.IC.6, S.ID.4**

Next

Students will evaluate and graph trigonometric functions. **F.IF.7e, F.TF.5**

Rigor

The Three Pillars of Rigor

To help students meet standards, they need to illustrate their ability to use the three pillars of rigor. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. Once they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Practice.

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

EXPLORE → LEARN → EXAMPLE & PRACTICE

Suggested Pacing

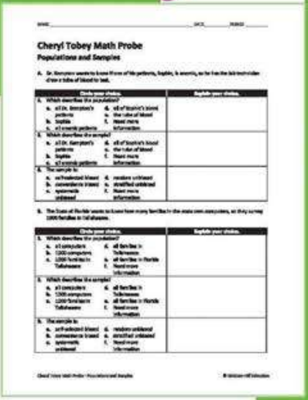
Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
8-1 Random Sampling	S.IC.1, S.IC.3	2	1
8-2 Using Statistical Experiments	S.IC.2, S.IC.5	2	1
8-3 Analyzing Population Data	S.IC.4	1	0.5
8-4 Normal Distributions	S.ID.4, S.IC.6	2	1
8-5 Estimating Population Parameters	S.IC.4, S.IC.6	1	0.5
Module Review		1	0.5
Module Assessment		1	0.5
	Total Days	11	5.5

CHERYL TOBEY MATH PROBES

Formative Assessment Math Probe

Populations and Samples

Module Resource



Cheryl Tobey Math Probe
Populations and Samples

Correct Answers:
A1. d A2. e A3. d
B1. e B2. c B3. b

Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will determine the differences between populations and samples.

Targeted Concepts Understand that inferences can be made about a population based on a representative sample.

Targeted Misconceptions

- Students may not recognize the population and/or the sample if it is not explicit.
- Students may not recognize the difference between a biased and an unbiased sample.
- Students may confuse the different types of biased and unbiased samples.

Use the Probe after Lesson 8-1.

Collect and Assess Student Answers

If the student selects these responses...	**Then** the student likely...
A1. a, b, c	is confusing the topic of the study with the population.
B1. a, b	is confusing the topic of the study with the sample.
A2. a, b, c	is interchanging the sample with the population.
B2. a, b	does not understand how to tell whether a sample is biased or unbiased and/or is basing their answer on an incorrect choice of A2 and/or B2.
A1. e	does not understand the differences between the various biased or unbiased types.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

- Lesson 8-1, Learn, Examples 1–3

Revisit the Probe at the end of the module to be sure that your students no longer carry these misconceptions.

IGNITE!

The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

How can data be collected and interpreted so that it is useful to a specific audience? Sample answer: Data can be collected through surveys, observations, or experiments. Often a simulation can be conducted to generate experimental results. You can find measures of center and measures of variation to interpret the data.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

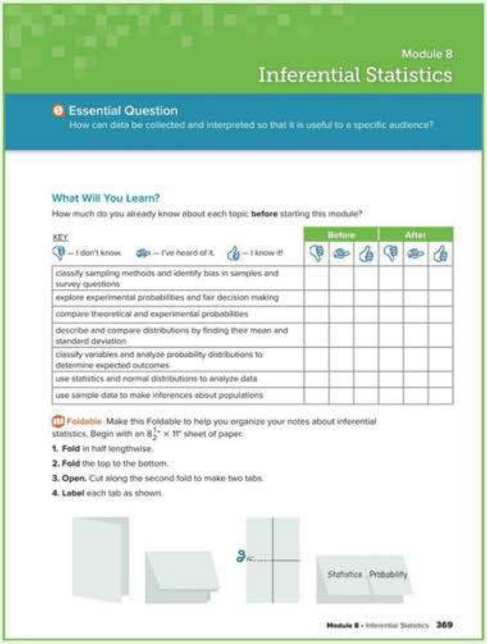
Focus Students write notes about inferential statistics.

Teach Have students make and label their Foldables as illustrated. At the end of each lesson, ask students to write about their experiences with the statistics topics that were presented in the lesson. Encourage students to explain what they found interesting or challenging.

When to Use It Encourage students to add to their Foldables as they work through the module and to use it to review for the module test.

Launch the Module

For this module, the Launch the Module video uses business, research, and politics to show real-world applications of statistics. Students learn about choosing samples and using statistics to analyze voter populations and predict election results.



Module 8

Inferential Statistics

Essential Question

How can data be collected and interpreted so that it is useful to a specific audience?

What Will You Learn?

How much do you already know about each topic **before** starting this module?

KEY

👎 — I don't know. 🤚 — I've heard of it. 👍 — I know it!

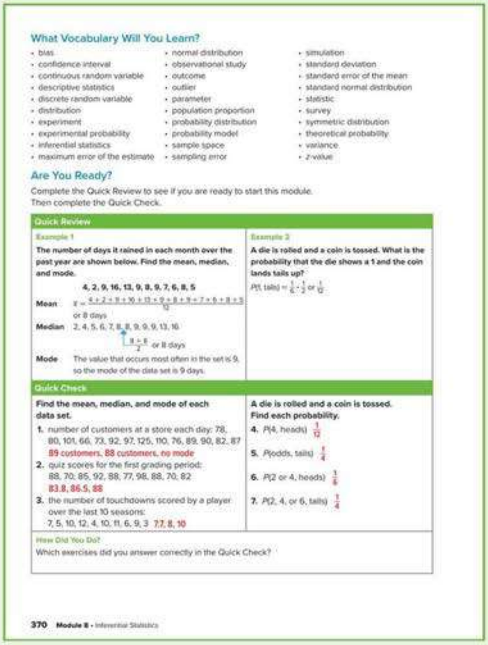
	Before			After		
	👎	🤚	👍	👎	🤚	👍
classify sampling methods and identify bias in samples and survey questions						
explore experimental probabilities and fair decision making						
compare theoretical and experimental probabilities						
describe and compare distributions by finding their mean and standard deviation						
classify variables and analyze probability distributions to determine expected outcomes						
use statistics and normal distributions to analyze data						
use sample data to make inferences about populations						

Foldable Make this Foldable to help you organize your notes about inferential statistics. Begin with an $8\frac{1}{2}"\times 11"$ sheet of paper.

1. **Fold** in half lengthwise.
2. **Fold** the top to the bottom.
3. **Open.** Cut along the second fold to make two tabs.
4. **Label** each tab as shown.

Module 8 • Inferential Statistics 369

Interactive Presentation



What Vocabulary Will You Learn?

- bias
- confidence interval
- continuous random variable
- descriptive statistics
- discrete random variable
- distribution
- experiment
- experimental probability
- inferential statistics
- maximum error of the estimate
- normal distribution
- observational study
- outcome
- outlier
- parameter
- population proportion
- probability distribution
- probability model
- sample space
- sampling error
- simulation
- standard deviation
- standard error of the mean
- standard normal distribution
- statistic
- survey
- symmetric distribution
- theoretical probability
- variance
- z-value

Are You Ready?

Complete the Quick Review to see if you are ready to start this module.
Then complete the Quick Check.

Quick Review

Example 1

The number of days it rained in each month over the past year are shown below. Find the mean, median, and mode.

4, 2, 9, 16, 13, 9, 8, 9, 7, 6, 8, 5

Mean $\bar{x} = \frac{4 + 2 + 9 + 16 + 13 + 9 + 8 + 9 + 7 + 6 + 8 + 5}{12}$ or 8 days

Median 2, 4, 5, 6, 7, 8, 8, 9, 9, 9, 13, 16

$\frac{8 + 8}{2}$ or 8 days

Mode The value that occurs most often in the set is 9, so the mode of the data set is 9 days.

Example 2

A die is rolled and a coin is tossed. What is the probability that the die shows a 1 and the coin lands tails up?

$P(1, \text{tails}) = \frac{1}{6} \cdot \frac{1}{2}$ or $\frac{1}{12}$

Quick Check

Find the mean, median, and mode of each data set.

1. number of customers at a store each day: 78, 80, 101, 66, 73, 92, 97, 125, 110, 76, 89, 90, 82, 87
89 customers, 88 customers, no mode
2. quiz scores for the first grading period: 88, 70, 85, 92, 88, 77, 98, 88, 70, 82
83.8, 86.5, 88
3. the number of touchdowns scored by a player over the last 10 seasons: 7, 5, 10, 12, 4, 10, 11, 6, 9, 3 7.7, 8, 10

A die is rolled and a coin is tossed. Find each probability.

4. $P(4, \text{heads})$ $\frac{1}{12}$
5. $P(\text{odds}, \text{tails})$ $\frac{1}{4}$
6. $P(2 \text{ or } 4, \text{heads})$ $\frac{1}{6}$
7. $P(2, 4, \text{or } 6, \text{tails})$ $\frac{1}{4}$

How Did You Do?

Which exercises did you answer correctly in the Quick Check?

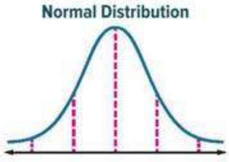
370 **Module 8** • Inferential Statistics

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define A normal distribution is a continuous probability distribution of a random variable, and occurs as a sample size increases due to the Law of Large Numbers.

Example



Ask What do you notice about the normal distribution curve? It is shaped like a bell and is symmetrical.

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- using frequency tables
- determining sample spaces
- finding the mean, median, mode, and range

ALEKS

ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You may want to use the **Sequences and Probability** section to ensure student success in this module.

Mindset Matters

Foster Grit

Grit is defined as a student's perseverance and passion for achieving long-term goals. A student's ability to work hard, endure struggle, remain committed to their goals, make mistakes, and try again are important factors in learning.

How Can I Apply It?

Assign students the **Put It All Together** activity for each module and allow them an opportunity to work through the problems, make mistakes, share their strategies and receive feedback, and then work on the problems again to try new strategies.

Lesson 8-1

Random Sampling

S.IC.1, S.IC.3

LESSON GOAL

Students classify and identify bias in surveys and studies.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Randomness and Bias
- Classify a Random Sample
- Identify Sample Bias
- Identify Bias Questions

Types of Studies
- Classify Study Types
- Design a Survey
- Draw Conclusions from a Study
- Identify Bias in Studies

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

Formative Assessment Math Probe

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Summarizing Categorical Data	●	●		●
Extension: Stratified Surveys		●	●	●

Language Development Handbook

Assign page 41 of the *Language Development Handbook* to help your students build mathematical language related to classifying and identifying bias in surveys and studies.

ELL You can use the tips and suggestions on page T41 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Statistics and Probability

Standards for Mathematical Content:

S.IC.1 Understand statistics as a process for making inferences about population parameters based on a random sample from that population.

S.IC.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students understood that random sampling tends to produce representative samples and support valid inferences. **7.SP.1**

Now
Students classify and identify bias in surveys and studies. **S.IC.1, S.IC.3**

Next
Students will collect, analyze, and use data to compare theoretical and experimental probabilities. **S.IC.2, S.IC.5**

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION
Conceptual Bridge In this lesson, students develop an understanding of inferential statistics, and they build fluency by making inferences about populations based on random samples. They apply their understanding by solving real-world problems related to surveys, experiments, and observational studies.		

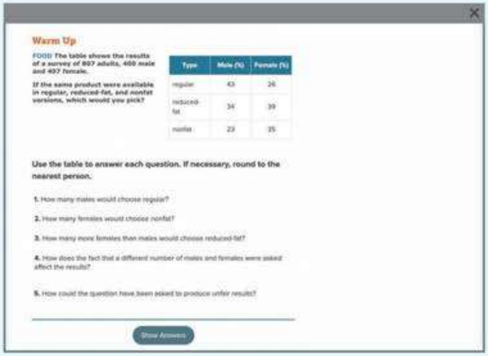
Mathematical Background

When a census is conducted, data are collected from every member of a population. Therefore, the results are known to be correct. Since a survey investigates only part of a population, the results always involve some uncertainty.

1 LAUNCH

S.IC.1, S.IC.3

Interactive Presentation



Warm Up

FOOD The table shows the results of a survey of 807 adults, 400 male and 407 female.

If the same product were available in regular, reduced-fat, and nonfat versions, which would you pick?

Type	Male (%)	Female (%)
regular	43	26
reduced-fat	34	39
nonfat	23	35

Use the table to answer each question. If necessary, round to the nearest person.

1. How many males would choose regular?
2. How many females would choose nonfat?
3. How many more females than males would choose reduced-fat?
4. How does the fact that a different number of males and females were asked affect the results?
5. How could the question have been asked to produce unfair results?

Warm Up

Launch the Lesson

To determine how well candidates are doing in an election campaign, many groups survey likely voters to create a poll. However, how these polls are conducted can result in different results if the survey does not capture a random sample of the intended population, or likely voters. For example, if a survey is done exclusively via landline phones, then the sample will skew older. If a survey is done exclusively online, then the sample will skew younger. This can lead to data that are misleading.

Launch the Lesson

Vocabulary

- parameter
- statistic
- population
- bias
- survey
- experiment
- observational study

- What is the difference between a statistic and a parameter?
- Of the three types of studies, which type places the participants in groups? Explain your answer.

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skills for this lesson:

- using frequency tables
- analyzing sampling methods

Answers:

1. 172 males
2. 142 females
3. 23 more
4. Sample answer: Because the answers were given in percentages, the results were not affected.
5. Sample answer: The question could have been asked at a meeting of dieters.

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will learn about how surveys are used to create polls and predict which candidate will win an election. Point out that the results of surveys may be biased or misleading.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Learn Randomness and Bias

Objective

Students classify sampling methods and identify bias in samples and survey questions.

MP Teaching the Mathematical Practices

1 Special Cases Work with students to examine the types of samples shown. Encourage them to familiarize themselves with the types, and to know when it is appropriate to use each one.

Example 1 Classify a Random Sample

MP Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of the situation in Example 1 to identify the type of sample.

Questions for Mathematical Discourse

- AL Is the sample random? No
- OL Are the survey results reliable only if every student answers the online survey questions? Explain. No; sample answer: A sample of the population can be enough for the survey be reliable.
- BL What is a possible source of bias in this study? Sample answer: Students who choose to answer the online survey are more likely to have strong positive or negative feelings about the app.

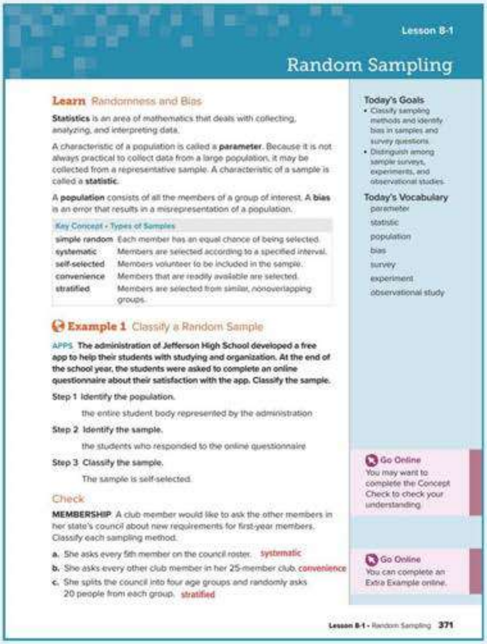
DIFFERENTIATE

Language Development Activity ELL

Verbal/Linguistic Learners Divide students into small groups. Have each group design a survey question and practice asking it in such a way that there is bias in the tone of voice and facial expression of the questioner. Then have them ask other groups the questions and record their answers. As a class, discuss whether the answers correspond to the bias that the question was designed to elicit.

Go Online

- F ind additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 8-1

Random Sampling

Learn Randomness and Bias

Statistics is an area of mathematics that deals with collecting, analyzing, and interpreting data.

A characteristic of a population is called a **parameter**. Because it is not always practical to collect data from a large population, it may be collected from a representative sample. A characteristic of a sample is called a **statistic**.

A **population** consists of all the members of a group of interest. A **bias** is an error that results in a misrepresentation of a population.

Key Concept • Types of Samples

simple random	Each member has an equal chance of being selected.
systematic	Members are selected according to a specified interval.
self-selected	Members volunteer to be included in the sample.
convenience	Members that are readily available are selected.
stratified	Members are selected from similar, nonoverlapping groups.

Example 1 Classify a Random Sample

APPS **The administration of Jefferson High School developed a free app to help their students with studying and organization. At the end of the school year, the students were asked to complete an online questionnaire about their satisfaction with the app. Classify the sample.**

Step 1 Identify the population.

the entire student body represented by the administration

Step 2 Identify the sample.

the students who responded to the online questionnaire

Step 3 Classify the sample.

The sample is self-selected.

Check

MEMBERSHIP A club member would like to ask the other members in her state's council about new requirements for first-year members. Classify each sampling method.

a. She asks every 5th member on the council roster. systematic

b. She asks every other club member in her 25-member club. convenience

c. She splits the council into four age groups and randomly asks 20 people from each group. stratified

Today's Goals
- Classify sampling methods and identify bias in samples and survey questions.
- Distinguish among sample surveys, experiments, and observational studies.

Today's Vocabulary
parameter
statistic
population
bias
survey
experiment
observational study

Go Online You may want to complete the Concept Check to check your understanding.

Go Online You can complete an Extra Example online.

Lesson 8-1 • Random Sampling 371

Interactive Presentation

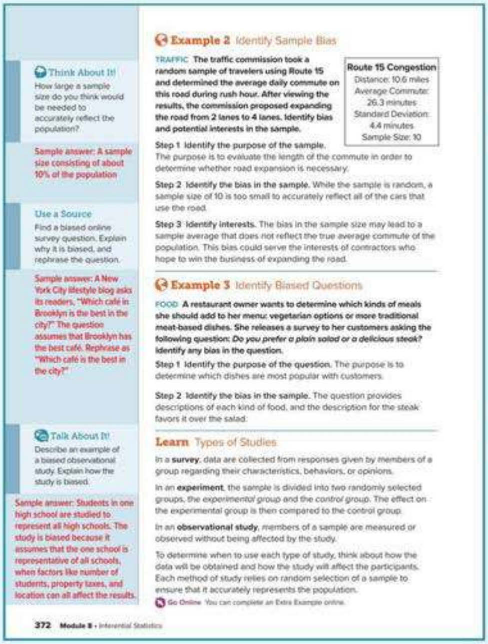
Learn

WATCH

Students can watch an animation about the different types of samples.

DRAG AND DROP

Students drag to sort sampling methods into the correct bins.



Think About It!

How large a sample size do you think would be needed to accurately reflect the population?

Sample answer: A sample size consisting of about 10% of the population

Use a Source

Find a biased online survey question. Explain why it is biased, and rephrase the question.

Sample answer: A New York City lifestyle blog asks its readers, "Which café in Brooklyn is the best in the city?" The question assumes that Brooklyn has the best café. Rephrase as "Which café is the best in the city?"

Talk About It!

Describe an example of a biased observational study. Explain how the study is biased.

Sample answer: Students in one high school are studied to represent all high schools. The study is biased because it assumes that the one school is representative of all schools, when factors like number of students, property taxes, and location can all affect the results.

Example 2 Identify Sample Bias

TRAFFIC **The traffic commission took a random sample of travelers using Route 15 and determined the average daily commute on this road during rush hour. After viewing the results, the commission proposed expanding the road from 2 lanes to 4 lanes. Identify bias and potential interests in the sample.**

Route 15 Congestion
Distance: 10.6 miles
Average Commute: 26.3 minutes
Standard Deviation: 4.4 minutes
Sample Size: 10

Step 1 Identify the purpose of the sample.
The purpose is to evaluate the length of the commute in order to determine whether road expansion is necessary.

Step 2 Identify the bias in the sample. While the sample is random, a sample size of 10 is too small to accurately reflect all of the cars that use the road.

Step 3 Identify interests. The bias in the sample size may lead to a sample average that does not reflect the true average commute of the population. This bias could serve the interests of contractors who hope to win the business of expanding the road.

Example 3 Identify Biased Questions

FOOD **A restaurant owner wants to determine which kinds of meals she should add to her menu: vegetarian options or more traditional meat-based dishes. She releases a survey to her customers asking the following question:** ***Do you prefer a plain salad or a delicious steak?*** **Identify any bias in the question.**

Step 1 Identify the purpose of the question. The purpose is to determine which dishes are most popular with customers.

Step 2 Identify the bias in the sample. The question provides descriptions of each kind of food, and the description for the steak favors it over the salad.

Learn Types of Studies

In a **survey**, data are collected from responses given by members of a group regarding their characteristics, behaviors, or opinions.

In an **experiment**, the sample is divided into two randomly selected groups, the *experimental group* and the *control group*. The effect on the experimental group is then compared to the control group.

In an **observational study**, members of a sample are measured or observed without being affected by the study.

To determine when to use each type of study, think about how the data will be obtained and how the study will affect the participants. Each method of study relies on random selection of a sample to ensure that it accurately represents the population.

Go Online You can complete an Extra Example online.

372 Module 8 • Inferential Statistics

Interactive Presentation

Example 3

TYPE

Students use online resources to find and analyze biased questions.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Identify Sample Bias

Teaching the Mathematical Practices

3 Make Conjectures In Example 2, students will make a conjecture about the sample size needed to eliminate bias in the sample.

Questions for Mathematical Discourse

AL How are the distance and average commute related to each other? The average commute is 26.3 minutes to travel 10.6 miles along Route 15.

OL What additional information do you need to make a reasonable decision about expanding the road based on the survey? Sample answer: Knowing the average number of travelers during rush hour each day would make it possible to determine a reasonable sample size. Knowing the speed limit of the road is relevant when comparing the distance and commute time.

BL Does a source of bias exist only if someone is benefitting from the bias? Explain. No; sample answer: The bias can come from accidentally surveying a set of the population that does not represent the full population, even if it is not intentional.

Example 3 Identify Biased Questions

Teaching the Mathematical Practices

5 Use a Source Guide students to find external information to answer the question posed in the Use a Source feature.

Questions for Mathematical Discourse

AL Why do you want questions to be unbiased? Sample answer: Biased questions could give inaccurate results.

OL How could you rewrite the question to avoid bias? Sample answer: Do you prefer a salad or a steak?

BL How might a question that addresses more than one issue be biased? Sample answer: When a person answers, they may be answering only part of the question.

Learn Types of Studies

Objective

Students distinguish among sample surveys, experiments, and observational studies.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions and mathematical language when they discuss their solution to the Talk About It! feature.

Example 4 Classify Study Types

Teaching the Mathematical Practices

4 Apply Mathematics In Example 4, students will apply what they have learned about types of studies to analyzing a real-world situation.

Questions for Mathematical Discourse

AL Does this study need a control group? No; it is an observational study.

OL How could you change the study to be a survey? Sample answer: Ask the 100 young adults a series of multiple-choice questions about their uniform preferences.

BL Why might an observational study be used in some situations rather than an experiment? Sample answer: It might not be practical or moral to conduct an experiment.

Example 5 Design a Survey

Teaching the Mathematical Practices

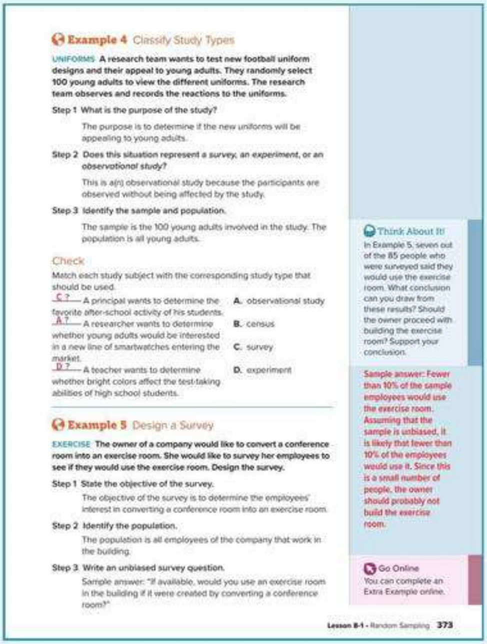
3 Justify Conclusions Mathematically proficient students can explain conclusions drawn when solving a problem. The Think About It! feature asks students to support the conclusion they have drawn from the results of the survey.

Questions for Mathematical Discourse

AL What is the ideal sample size for this survey? All the employees of the company. Why might a smaller sample size be used? Sample answer: It may not be practical or cost effective to survey the entire company.

OL Why is a survey best for this situation? The owner is trying to gather opinions of employees.

BL What bias might be introduced if the owner conducts the survey by asking only the supervisors in the building? Sample answer: The supervisors might feel differently than the rest of the employees in the building.



Example 4 Classify Study Types

UNIFORMS **A research team wants to test new football uniform designs and their appeal to young adults. They randomly select 100 young adults to view the different uniforms. The research team observes and records the reactions to the uniforms.**

Step 1 What is the purpose of the study?

The purpose is to determine if the new uniforms will be appealing to young adults.

Step 2 Does this situation represent a survey, an experiment, or an observational study?

This is a(n) observational study because the participants are observed without being affected by the study.

Step 3 Identify the sample and population.

The sample is the 100 young adults involved in the study. The population is all young adults.

Check

Match each study subject with the corresponding study type that should be used.

C ? A principal wants to determine the favorite after-school activity of his students.

A ? A researcher wants to determine whether young adults would be interested in a new line of smartwatches entering the market.

D ? A teacher wants to determine whether bright colors affect the test-taking abilities of high school students.

A. observational study

B. census

C. survey

D. experiment

Example 5 Design a Survey

EXERCISE **The owner of a company would like to convert a conference room into an exercise room. She would like to survey her employees to see if they would use the exercise room. Design the survey.**

Step 1 State the objective of the survey.

The objective of the survey is to determine the employees' interest in converting a conference room into an exercise room.

Step 2 Identify the population.

The population is all employees of the company that work in the building.

Step 3 Write an unbiased survey question.

Sample answer: "If available, would you use an exercise room in the building if it were created by converting a conference room?"

Think About It!

In Example 5, seven out of the 85 people who were surveyed said they would use the exercise room. What conclusion can you draw from these results? Should the owner proceed with building the exercise room? Support your conclusion.

Sample answer: Fewer than 10% of the sample employees would use the exercise room. Assuming that the sample is unbiased, it is likely that fewer than 10% of the employees would use it. Since this is a small number of people, the owner should probably not build the exercise room.

Go Online

You can complete an Extra Example online.

Lesson 8-1 • Random Sampling 373

Interactive Presentation

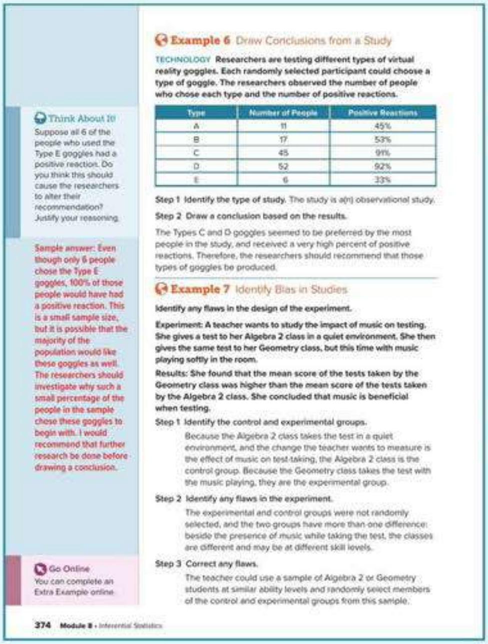
Example 5

EXPAND

Students tap to see the steps of writing an unbiased survey question.

TYPE

Students support a conclusion based on results of a survey.



Think About It!
Suppose all 6 of the people who used the Type E goggles had a positive reaction. Do you think this should cause the researchers to alter their recommendation? Justify your reasoning.

Sample answer: Even though only 6 people chose the Type E goggles, 100% of those people would have had a positive reaction. This is a small sample size, but it is possible that the majority of the population would like these goggles as well. The researchers should investigate why such a small percentage of the people in the sample chose these goggles to begin with. I would recommend that further research be done before drawing a conclusion.

Go Online You can complete an Extra Example online.

Example 6 Draw Conclusions from a Study

TECHNOLOGY **Researchers are testing different types of virtual reality goggles. Each randomly selected participant could choose a type of goggle. The researchers observed the number of people who chose each type and the number of positive reactions.**

Type	Number of People	Positive Reactions
A	11	45%
B	17	53%
C	45	91%
D	52	92%
E	6	33%

Step 1 Identify the type of study. The study is a(n) observational study.

Step 2 Draw a conclusion based on the results.

The Types C and D goggles seemed to be preferred by the most people in the study, and received a very high percent of positive reactions. Therefore, the researchers should recommend that those types of goggles be produced.

Example 7 Identify Bias in Studies

Identify any flaws in the design of the experiment.

Experiment: A teacher wants to study the impact of music on testing. She gives a test to her Algebra 2 class in a quiet environment. She then gives the same test to her Geometry class, but this time with music playing softly in the room.

Results: She found that the mean score of the tests taken by the Geometry class was higher than the mean score of the tests taken by the Algebra 2 class. She concluded that music is beneficial when testing.

Step 1 Identify the control and experimental groups.

Because the Algebra 2 class takes the test in a quiet environment, and the change the teacher wants to measure is the effect of music on test-taking, the Algebra 2 class is the control group. Because the Geometry class takes the test with the music playing, they are the experimental group.

Step 2 Identify any flaws in the experiment.

The experimental and control groups were not randomly selected, and the two groups have more than one difference: beside the presence of music while taking the test, the classes are different and may be at different skill levels.

Step 3 Correct any flaws.

The teacher could use a sample of Algebra 2 or Geometry students at similar ability levels and randomly select members of the control and experimental groups from this sample.

374 **Module 8** • Inferential Statistics

Interactive Presentation

Identify Bias in Studies

Step 1 Identify the control and experimental groups.

Step 2 Identify any flaws in the experiment.

Step 3 Correct any flaws.

Example 7

EXPAND

Students can tap to see the steps to identify flaws in the design of an experiment.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 6 Draw Conclusions from a Study

Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to complete Example 6, students will need to analyze the mathematical relationships in the problem to draw a conclusion about which goggles should be produced.

Questions for Mathematical Discourse

AL How can you find the number of people who had positive reactions for each type? Multiply the number of people by the percent of positive reactions.

OL Why is it possible in this study for the values in the “Percent of Positive Reactions” column to add up to more than 100%? Sample answer: Percent of positive reactions is related to the group for each type of goggles, not the entire group in the study. Participants may have positive reactions to more than one type of goggle. Within each group, the percent of positive reactions and the percent of not-positive reactions would add to 100%.

BL What is a potential flaw in this study? Sample answer: Each participant is only providing a reaction to the goggles they selected, so the variables that affect initial choice may influence the reaction to usage.

Example 7 Identify Bias in Studies

Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of the experiment in Example 7 to identify and correct flaws in the experiment.

Questions for Mathematical Discourse

AL Other than the music, what is a possible reason the Geometry test scores were higher than the Algebra 2 test scores? Sample answer: The Geometry test was easier.

OL In an experimental design, how many differences should there be between the control and experimental groups? Explain. One; the groups should be identical except for the variable being tested by the experiment.

BL Instead of giving both groups different tests, would the flaw be fixed if both groups are given an Algebra 2 test? Explain. No; sample answer: The experimental group has been studying Algebra 2, while the control group has been studying Geometry. Their different performance on the Algebra 2 test would more likely be due to this than the presence or absence of music.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–20
2	exercises that use a variety of skills from this lesson	21–25
2	exercises that extend concepts learned in this lesson to new contexts	26–29
3	exercises that emphasize higher-order and critical-thinking skills	30–36

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice, Exercises 1–29 odd, 30–36
- Extension: Stratified Surveys

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice, Exercises 1–35 odd
- Remediation, Review Resources: Summarizing Categorical Data
- Personal Tutors
- Extra Examples 1–7

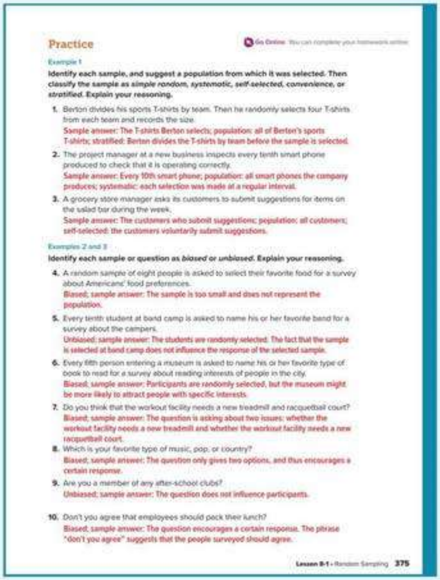
AL

IF students score 65% or less on the Checks,
THEN assign:

- Practice, Exercises 1–19 odd
- Remediation, Review Resources: Summarizing Categorical Data

Answers

15. objective: to determine the number of juniors and seniors planning to attend prom; population: all juniors and seniors at the school; sample survey questions: What grade are you in? Do you plan on attending prom?
16. objective: to determine a popular food item that can be added to the menu; population: all patrons of the restaurant; sample survey questions: What food item would you like to see added to the menu? What item not on our current menu do you like to order at a restaurant?
18. observational study; Sample answer: the teacher can engage his or her students early in the day, but his or her students tend to lose interest later in the day.
19. control group: students who do not receive gift cards; experimental group: students receiving gift cards; Sample answer: The students may not have any interest in the gift cards they are receiving. The principal should conduct a survey to find what gifts or activities might motivate the students.



Practice

Go Online You can complete your homework online.

Example 1

Identify each sample, and suggest a population from which it was selected. Then classify the sample as *simple random, systematic, self-selected, convenience,* or *stratified.* Explain your reasoning.

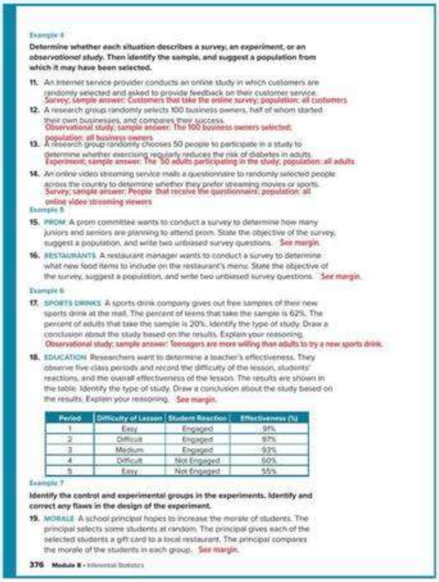
1. Berton divides his sports T-shirts by team. Then he randomly selects four T-shirts from each team and records the size.
   Sample answer: The T-shirts Berton selects; population: all of Berton's sports T-shirts; stratified: Berton divides the T-shirts by team before the sample is selected.
2. The project manager at a new business inspects every tenth smart phone produced to check that it is operating correctly.
   Sample answer: Every 10th smart phone; population: all smart phones the company produces; systematic: each selection was made at a regular interval.
3. A grocery store manager asks its customers to submit suggestions for items on the salad bar during the week.
   Sample answer: The customers who submit suggestions; population: all customers; self-selected: the customers voluntarily submit suggestions.

Examples 2 and 3

Identify each sample or question as *biased* or *unbiased.* Explain your reasoning.

4. A random sample of eight people is asked to select their favorite food for a survey about Americans' food preferences.
   Biased; sample answer: The sample is too small and does not represent the population.
5. Every tenth student at band camp is asked to name his or her favorite band for a survey about the campers.
   Unbiased; sample answer: The students are randomly selected. The fact that the sample is selected at band camp does not influence the response of the selected sample.
6. Every fifth person entering a museum is asked to name his or her favorite type of book to read for a survey about reading interests of people in the city.
   Biased; sample answer: Participants are randomly selected, but the museum might be more likely to attract people with specific interests.
7. Do you think that the workout facility needs a new treadmill and racquetball court?
   Biased; sample answer: The question is asking about two issues: whether the workout facility needs a new treadmill and whether the workout facility needs a new racquetball court.
8. Which is your favorite type of music, pop, or country?
   Biased; sample answer: The question only gives two options, and thus encourages a certain response.
9. Are you a member of any after-school clubs?
   Unbiased; sample answer: The question does not influence participants.
10. Don't you agree that employees should pack their lunch?
   Biased; sample answer: The question encourages a certain response. The phrase "don't you agree" suggests that the people surveyed should agree.

Lesson 8-1 • Random Sampling **375**



Example 4

Determine whether each situation describes a survey, an experiment, or an observational study. Then identify the sample, and suggest a population from which it may have been selected.

11. An Internet service provider conducts an online study in which customers are randomly selected and asked to provide feedback on their customer service.
   Survey; sample answer: Customers that take the online survey; population: all customers
12. A research group randomly selects 100 business owners, half of whom started their own businesses, and compares their success.
   Observational study; sample answer: The 100 business owners selected; population: all business owners
13. A research group randomly chooses 50 people to participate in a study to determine whether exercising regularly reduces the risk of diabetes in adults.
   Experiment; sample answer: The 50 adults participating in the study; population: all adults
14. An online video streaming service mails a questionnaire to randomly selected people across the country to determine whether they prefer streaming movies or sports.
   Survey; sample answer: People that receive the questionnaire; population: all online video streaming viewers

Example 5

15. PROM A prom committee wants to conduct a survey to determine how many juniors and seniors are planning to attend prom. State the objective of the survey, suggest a population, and write two unbiased survey questions. See margin.
16. RESTAURANTS A restaurant manager wants to conduct a survey to determine what new food items to include on the restaurant's menu. State the objective of the survey, suggest a population, and write two unbiased survey questions. See margin.

Example 6

17. SPORTS DRINKS A sports drink company gives out free samples of their new sports drink at the mall. The percent of teens that take the sample is 62%. The percent of adults that take the sample is 20%. Identify the type of study. Draw a conclusion about the study based on the results. Explain your reasoning.
   Observational study; sample answer: Teenagers are more willing than adults to try a new sports drink.
18. EDUCATION Researchers want to determine a teacher's effectiveness. They observe five class periods and record the difficulty of the lesson, students' reactions, and the overall effectiveness of the lesson. The results are shown in the table. Identify the type of study. Draw a conclusion about the study based on the results. Explain your reasoning. See margin.

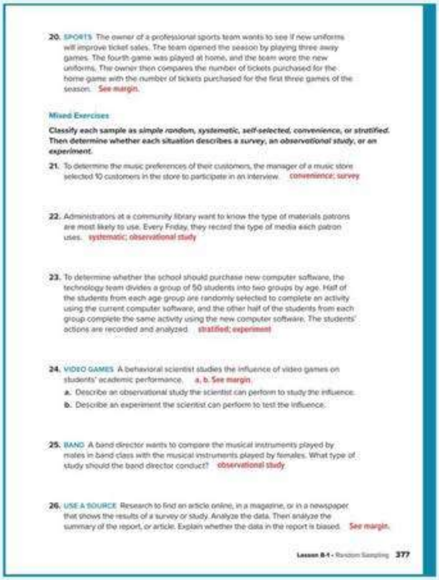
Period	Difficulty of Lesson	Student Reaction	Effectiveness (%)
1	Easy	Engaged	91%
2	Difficult	Engaged	97%
3	Medium	Engaged	93%
4	Difficult	Not Engaged	60%
5	Easy	Not Engaged	55%

Example 7

Identify the control and experimental groups in the experiments. Identify and correct any flaws in the design of the experiment.

19. MORALE A school principal hopes to increase the morale of students. The principal selects some students at random. The principal gives each of the selected students a gift card to a local restaurant. The principal compares the morale of the students in each group. See margin.

376 **Module 8** • Inferential Statistics



20. SPORTS The owner of a professional sports team wants to see if new uniforms will improve ticket sales. The team opened the season by playing three away games. The fourth game was played at home, and the team wore the new uniforms. The owner then compares the number of tickets purchased for the home game with the number of tickets purchased for the first three games of the season. See margin.

Mixed Exercises

Classify each sample as *simple random, systematic, self-selected, convenience,* or *stratified.* Then determine whether each situation describes a *survey,* an *observational study,* or an *experiment.*

21. To determine the music preferences of their customers, the manager of a music store selected 10 customers in the store to participate in an interview. convenience; survey

22. Administrators at a community library want to know the type of materials patrons are most likely to use. Every Friday, they record the type of media each patron uses. systematic; observational study

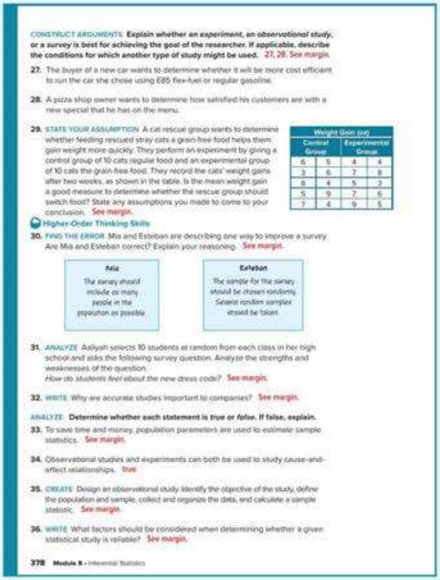
23. To determine whether the school should purchase new computer software, the technology team divides a group of 50 students into two groups by age. Half of the students from each age group are randomly selected to complete an activity using the current computer software, and the other half of the students from each group complete the same activity using the new computer software. The students' actions are recorded and analyzed. stratified; experiment

24. VIDEO GAMES A behavioral scientist studies the influence of video games on students' academic performance. a, b. See margin.
   a. Describe an observational study the scientist can perform to study the influence.
   b. Describe an experiment the scientist can perform to test the influence.

25. BAND A band director wants to compare the musical instruments played by males in band class with the musical instruments played by females. What type of study should the band director conduct? observational study

26. USE A SOURCE Research to find an article online, in a magazine, or in a newspaper that shows the results of a survey or study. Analyze the data. Then analyze the summary of the report, or article. Explain whether the data in the report is biased. See margin.

Lesson 8-1 • Random Sampling 377



CONSTRUCT ARGUMENTS **Explain whether an *experiment*, an *observational study*, or a *survey* is best for achieving the goal of the researcher. If applicable, describe the conditions for which another type of study might be used.** 27, 28. See margin.

27. The buyer of a new car wants to determine whether it will be more cost efficient to run the car she chose using E85 flex-fuel or regular gasoline.

28. A pizza shop owner wants to determine how satisfied his customers are with a new special that he has on the menu.

29. STATE YOUR ASSUMPTION A cat rescue group wants to determine whether feeding rescued stray cats a grain-free food helps them gain weight more quickly. They perform an experiment by giving a control group of 10 cats regular food and an experimental group of 10 cats the grain-free food. They record the cats' weight gains after two weeks, as shown in the table. Is the mean weight gain a good measure to determine whether the rescue group should switch food? State any assumptions you made to come to your conclusion. See margin.

Weight Gain (oz)			
Control Group		Experimental Group	
6	5	4	4
3	6	7	8
8	4	5	3
5	9	7	6
7	4	9	5

Higher-Order Thinking Skills

30. FIND THE ERROR Mia and Esteban are describing one way to improve a survey. Are Mia and Esteban correct? Explain your reasoning. See margin.

Mia
The survey should include as many people in the population as possible.

Esteban
The sample for the survey should be chosen randomly. Several random samples should be taken.

31. ANALYZE Aaliyah selects 10 students at random from each class in her high school and asks the following survey question. Analyze the strengths and weaknesses of the question.
How do students feel about the new dress code? See margin.

32. WRITE Why are accurate studies important to companies? See margin.

ANALYZE **Determine whether each statement is *true* or *false*. If false, explain.**

33. To save time and money, population parameters are used to estimate sample statistics. See margin.

34. Observational studies and experiments can both be used to study cause-and-effect relationships. true

35. CREATE Design an observational study. Identify the objective of the study, define the population and sample, collect and organize the data, and calculate a sample statistic. See margin.

36. WRITE What factors should be considered when determining whether a given statistical study is reliable? See margin.

378 Module 8 • Inferential Statistics

Answers

20. control group: people attending the away games; experimental group: people attending the home game; Sample answer: Because the sports team likely has more fans in their home city, comparing tickets sales in this way may not accurately reflect the impact of the new uniforms. The owner should compare ticket sales of an entire season with the new uniforms to a season with the old uniforms.

24a. Sample answer: The scientist can observe the grades of teens who play video games versus the grades of teens who do not play video games.

24b. Sample answer: Have an experimental group play 8 hours of video games and give them a test. Then give the same test to a control group who do not play video games.

26. Sample answer: A magazine survey asked respondents whether the lead guitarist of a certain band was the best guitarist ever, and 46% agreed that he was. The data is biased because the survey question is biased. The survey leads the respondents to agree. The survey question would be better as, “Who is the best lead guitarist?”

27. Sample answer: She should use an observational study to determine the average miles per gallon that identical cars are able to go on each type of fuel. She can then divide the current price of each fuel by the number of miles to find the average cost per mile. The group with the lower cost per mile would be the one that is more cost efficient. She might also look for the results of a survey by a motor club or magazine stating the average gas mileage for cars similar to her new car.

28. Sample answer: The owner should conduct a survey. He could have all customers in a given month who order the new special rate their satisfaction on a 1–5 scale.

29. No; sample answer: the difference between the means of the experimental group and control group is only10ounce. Both groups had cats that gained significant weight and others that gained little. To compare the groups, I assumed that the cats in each group were similar in age, starting weight, and health. I also assumed that each cat was given the same amount of food.

30. Both are correct; Mia is correct that they should try to collect a large sample, and Esteban is correct that the sample should be random.

31. Sample answer: This method of selecting a sample is valid. Each student has an equally likely chance of being selected for the sample. A weakness may be that this would not reflect that one grade may feel more strongly about the dress code than another.

32. Sample answer: They need accurate surveys to make decisions about how to market and sell products that will earn the company the most profit. They also make decisions about marketing and advertising and how to reach their target audience. Finally, they make decisions about the types of products they will develop or continue to sell.

33. False; sample answer: A sample statistic is used to estimate a population parameter.

35. Sample answer: objective: Determine the average amount of time that students spend studying at the library.

population: All students that study at the library.

sample: 25 randomly selected students studying at the library during a given week.

mean: ≈ 31.3 min

Study Time (minutes)				
38	16	45	41	63
18	20	17	8	15
41	28	55	19	15
30	11	20	79	24
78	24	26	32	19

36. Sample answer: the sampling method used, the type of sample that was selected, the type of study performed, the survey question(s) that were asked or procedures that were used

Lesson 8-2

Using Statistical Experiments

S.IC.2, S.IC.5

LESSON GOAL

Students collect, analyze, and use data to compare theoretical and experimental probabilities.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Fair Decisions

Develop:

Theoretical and Experimental Probability
- Find Probabilities

Explore: Simulations and Experiments

Develop:

Simulations
- Design and Run a Simulation
- Run and Evaluate a Simulation

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Sample Spaces	●	●		●
Extension: Game Shows		●	●	●

Language Development Handbook

Assign page 42 of the *Language Development Handbook* to help your students build mathematical language related to comparing theoretical and experimental probabilities.

ELL You can use the tips and suggestions on page T42 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Statistics and Probability

Standards for Mathematical Content:

S.IC.2 Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation.

S.IC.5 Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students found experimental and theoretical probabilities and designed simulations. **7.SP.7, 7.SP.8**

Now
Students collect, analyze, and use data to compare theoretical and experimental probabilities. **S.IC.2, S.IC.5**

Next
Students will use statistics to summarize, represent, and compare sets of data. **S.IC.4**

Rigor

The Three Pillars of Rigor

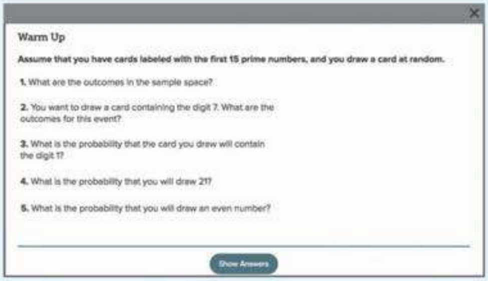
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of theoretical and experimental probabilities. They build fluency and apply their understanding by running and evaluating simulations related to real-world situations.

Mathematical Background

Theoretical probability describes what should occur given the number of events or trials and the sample space. Experimental probability describes the actual results of repeated trials or events. As the number of trials increases, experimental probability more closely approximates theoretical probability. A simulation provides a way to find experimental probability by “acting out” an event that might be difficult to perform.

Interactive Presentation



Warm Up

Assume that you have cards labeled with the first 15 prime numbers, and you draw a card at random.

1. What are the outcomes in the sample space?
2. You want to draw a card containing the digit 7. What are the outcomes for this event?
3. What is the probability that the card you draw will contain the digit 7?
4. What is the probability that you will draw 21?
5. What is the probability that you will draw an even number?

Warm Up

Launch the Lesson

Some companies gather data on consumer preferences using experiments in the form of single blind taste tests. For example, a beverage company might set up a table with two white cups: one containing their drink and the other containing a leading competitor's. People are asked to taste both drinks and select the drink they preferred. Then the representative reveals the two bottles so the taster could see the one that they preferred.

Launch the Lesson

Vocabulary

theoretical probability

Probability based on what is expected to happen.

experimental probability

Probability calculated by using data from an actual experiment.

probability model

A mathematical representation of a random event that consists of the sample space and the probability of each outcome.

simulation

The use of a probability model to imitate a process or situation so it can be studied.

- If theoretical probability is based on what we expect to happen, how is that different from experimental probability?
- How are probability models and simulations related?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skills for this lesson:

- determining sample spaces
- determining probabilities

Answers:

1. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47
2. 7, 17, 37, 47
3. $\frac{2}{5}$
4. 0
5. $\frac{1}{15}$

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will learn about how experiments are used by companies to gather information about consumer preferences. This information can then be used to make decisions.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms inthis lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Fair Decisions

Objective

Students explore probability and fair decision making by using tools.

Teaching the Mathematical Practices

3 Compare Arguments Mathematically proficient students can compare arguments, determine which one is flawed, and explain the flaw. In Exercise 5, students have to identified the flawed argument and choose the correct one.

4 Use Tools Point out that to complete the exercises in the Explore, students will need to use a tree diagram to map the outcomes of a decision-making method.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will be provided with an Inquiry Question to answer at the end of the activity. They will answer a series of guiding exercises about how to use probability to make fair decisions. Then students will answer the Inquiry Question.

(*continued on the next page*)

Interactive Presentation

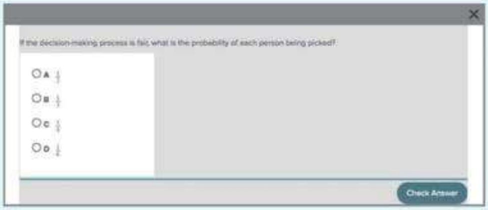
Fair Decisions

INQUIRY How can you use probability to make a fair decision?

A decision is **fair** when each possible outcome has an equal chance of occurring. For example, a coin flip is a fair way to choose one of two options because heads and tails are equally likely.

Suppose you and two friends, Aisha and Jeff, want to see a movie, but each of you wants to see something different. Aisha grabs a die and says that you can assign two numbers between 1 and 6 to each person. A target number represents

Explore



If the decision-making process is fair, what is the probability of each person being picked?

Explore

SELECT Students select the correct answers to questions about the probability of events.

TYPE Students complete the table to show the number of favorable outcomes.

Interactive Presentation

INQUIRY How can you use probability to make a fair decision?

Explore

TYPE

Students will respond to the Inquiry Question and can view a sample answer.

Explore Fair Decisions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How does the number of people affect the method of choosing a winner? Sample answer: In order to be fair, you want to make sure that each person has the same chance of winning. For three people, you need to have at least three options.
- Describe another method that could to choose the movie. Sample answers: A spinner with 3 even sections, a random number generator choosing 1-3, placing the three names into a bag and selecting one.

Inquiry

How can you use probability to make a fair decision? Sample answer: I can make a fair decision by using a tool that has at least n distinct options when I need to represent a probability of $\frac{1}{n}$.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Explore Simulations and Experiments

Objective

Students explore experimental probabilities by using a sketch to simulate experiments.

MP Teaching the Mathematical Practices

3 Reason Inductively In the Explore, students will examine the data and use inductive reasoning to make plausible arguments about the experiment.

5 Use Mathematical Tools Point out that to analyze the results of the experiment, students must use the sketch to run simulations. Work with students to deepen their understanding of simulations and experiments.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

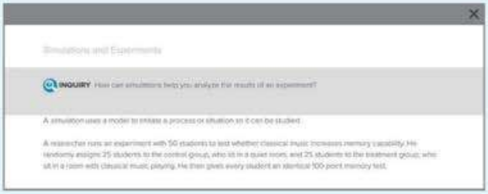
What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will be provided with an Inquiry Question to answer at the end of the activity. They will work with a simulation to analyze the outcome of an experiment. Then students will answer the Inquiry Question.

(*continued on the next page*)

Interactive Presentation



Simulations and Experiments

INQUIRY How can simulations help you analyze the results of an experiment?

A simulation uses a model to imitate a process or situation so it can be studied.

A researcher runs an experiment with 50 students to test whether classical music increases memory capability. He randomly assigns 25 students to the control group, who sit in a quiet room, and 25 students to the treatment group, who sit in a room with classical music playing. He then gives every student an identical 100-point memory test.

Explore

To the nearest hundredth, the results of the experiment showed that the mean score of the treatment group was ☐ points ☐ the mean score of the control group.

Explore

WEB SKETCHPAD

Students will use a sketch to explore the results of an experiment.

SELECT

Students select the correct answers to analyze the results of an experiment.

TYPE

Students record the results of trials in a table.

Interactive Presentation

INQUIRY How can simulations help analyze the results of an experiment?

Explore

Explore Simulations and Experiments *(continued)*

Questions

Have students complete the Explore activity.

Ask:

- Is it reasonable to try to conduct 1000 trials of an experiment? Why or why not? No; sample answer: It may take too much time, cost too much, or there may not be enough participants for 1000 trials. However, it is possible to run 1000 simulations.
- Do you think performing more than 1000 trials in the simulation could prove the results happened purely by chance? Why or why not? No; sample answer: the probability was already less than 1% to have a mean score 4.6 points greater. Performing more trials in the simulation would possibly lessen the probability even more.

Inquiry

How can simulations help you analyze the results of an experiment? Sample answer: They can help determine the probability that the results of the experiment could have happened purely by chance.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Theoretical and Experimental Probability

Objective

Students compare theoretical and experimental probabilities.

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their reasoning. Point out that they should use clear definitions and mathematical language to answer the question in the Talk About It! feature.

Common Misconception

Students often confuse the idea of the Law of Large Numbers with the idea that the probability of independent events are not entirely independent. If a coin flip results in five heads in a row, some students may think that a tails is "due," and so tails have a higher than 50% probability, even for a fair coin. This way of thinking has a name: the gambler's fallacy.

Example 1 Find Probabilities

Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to find the probabilities in Example 1, students will need to analyze the mathematical relationships in the problem to draw a conclusion.

Questions for Mathematical Discourse

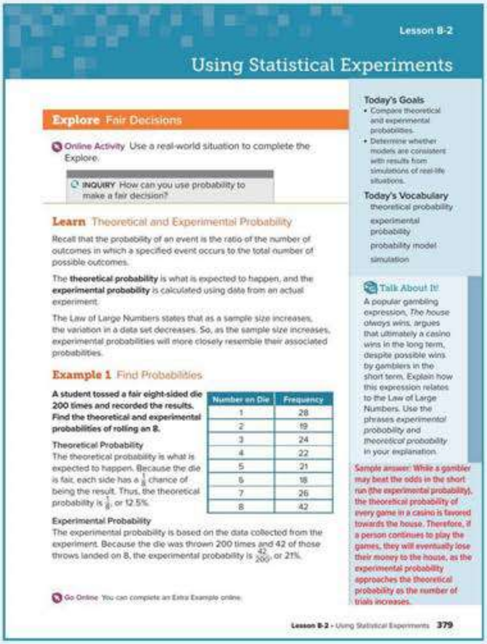
AL What is the sum of the numbers in the Frequency column? 200

OL If a third column contained the frequency numbers divided by 200 and converted to a percent, what would the column represent? the relative frequencies

BL A student says that if the die is fair, then on the 201st roll, 8 is the least likely number to occur. Is this true? Explain. No; sample answer: The die has rolled mostly 8s, but that does not mean the next die roll is less likely to be an 8. If the die is fair, then all sides have the same probability of being rolled on each roll, regardless of the previous results.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 8-2

Using Statistical Experiments

Explore Fair Decisions

Online Activity Use a real-world situation to complete the Explore.

INQUIRY How can you use probability to make a fair decision?

Learn Theoretical and Experimental Probability

Recall that the probability of an event is the ratio of the number of outcomes in which a specified event occurs to the total number of possible outcomes.

The **theoretical probability** is what is expected to happen, and the **experimental probability** is calculated using data from an actual experiment.

The Law of Large Numbers states that as a sample size increases, the variation in a data set decreases. So, as the sample size increases, experimental probabilities will more closely resemble their associated probabilities.

Example 1 Find Probabilities

A student tossed a fair eight-sided die 200 times and recorded the results. Find the theoretical and experimental probabilities of rolling an 8.

Number on Die	Frequency
1	28
2	19
3	24
4	22
5	21
6	18
7	26
8	42

Theoretical Probability

The theoretical probability is what is expected to happen. Because the die is fair, each side has a $\frac{1}{8}$ chance of being the result. Thus, the theoretical probability is $\frac{1}{8}$, or 12.5%.

Experimental Probability

The experimental probability is based on the data collected from the experiment. Because the die was thrown 200 times and 42 of those throws landed on 8, the experimental probability is $\frac{42}{200}$, or 21%.

Go Online You can complete an Extra Example online.

Today's Goals
- Compare theoretical and experimental probabilities.
- Determine whether models are consistent with results from simulations of real-life situations.

Today's Vocabulary
theoretical probability
experimental probability
probability model
simulation

Talk About It!
A popular gambling expression, *The house always wins*, argues that ultimately a casino wins in the long term, despite possible wins by gamblers in the short term. Explain how this expression relates to the Law of Large Numbers. Use the phrases *experimental probability* and *theoretical probability* in your explanation.

Sample answer: While a gambler may beat the odds in the short run (the experimental probability), the theoretical probability of every game in a casino is favored towards the house. Therefore, if a person continues to play the games, they will eventually lose their money to the house, as the experimental probability approaches the theoretical probability as the number of trials increases.

Lesson 8-2 • Using Statistical Experiments 379

Interactive Presentation

Find Probabilities

A student rolled a fair eight-sided die 200 times and recorded the results. Find the theoretical and experimental probabilities of rolling an 8.

Number on Die	Frequency
1	28
2	19
3	24

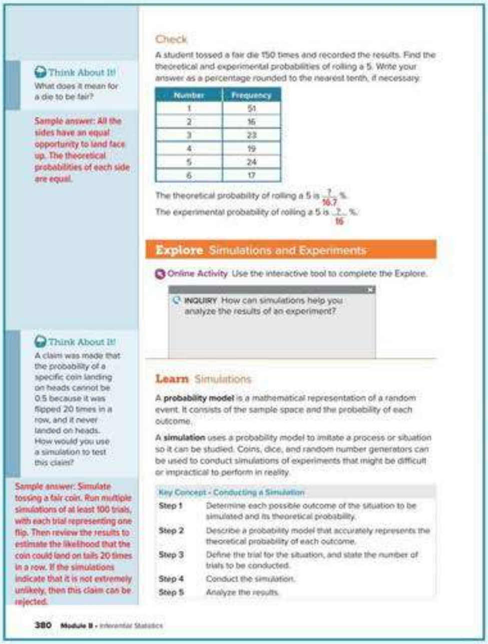
Example 1

TYPE

Students answer a question about what it means for a die to be fair.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Think About It!
What does it mean for a die to be fair?

Sample answer: All the sides have an equal opportunity to land face up. The theoretical probabilities of each side are equal.

Check

A student tossed a fair die 150 times and recorded the results. Find the theoretical and experimental probabilities of rolling a 5. Write your answer as a percentage rounded to the nearest tenth, if necessary.

Number	Frequency
1	51
2	16
3	23
4	19
5	24
6	17

The theoretical probability of rolling a 5 is ? 16.7 %.

The experimental probability of rolling a 5 is ? 16 %.

Explore Simulations and Experiments

Online Activity Use the interactive tool to complete the Explore.

INQUIRY How can simulations help you analyze the results of an experiment?

Think About It!
A claim was made that the probability of a specific coin landing on heads cannot be 0.5 because it was flipped 20 times in a row, and it never landed on heads. How would you use a simulation to test this claim?

Sample answer: Simulate tossing a fair coin. Run multiple simulations of at least 100 trials, with each trial representing one flip. Then review the results to estimate the likelihood that the coin could land on tails 20 times in a row. If the simulations indicate that it is not extremely unlikely, then this claim can be rejected.

Learn Simulations

A **probability model** is a mathematical representation of a random event. It consists of the sample space and the probability of each outcome.

A **simulation** uses a probability model to imitate a process or situation so it can be studied. Coins, dice, and random number generators can be used to conduct simulations of experiments that might be difficult or impractical to perform in reality.

Key Concept • Conducting a Simulation

Step 1	Determine each possible outcome of the situation to be simulated and its theoretical probability.
Step 2	Describe a probability model that accurately represents the theoretical probability of each outcome.
Step 3	Define the trial for the situation, and state the number of trials to be conducted.
Step 4	Conduct the simulation.
Step 5	Analyze the results.

380 Module 8 • Inferential Statistics

Interactive Presentation

Learn

TYPE

Students answer a question to determine how to simulate and test a claim.

Learn Simulations

Objective

Students determine whether models are consistent with results from simulations of real-life situations.

Teaching the Mathematical Practices

5 Use Mathematical Tools Mathematically proficient students consider the available tools when solving a problem. Point out that to run simulations, students must use concrete models or a random number generator.

Essential Question Follow-Up

Students have studied how to gather information with surveys and use probabilities to create simulations.

Ask:

How can you use information to make decisions? Sample answer: You can look for trends and then make a decision based on what has happened in the past and/or is reflected in the information.

DIFFERENTIATE

Enrichment Activity BL

Write the numbers 1, 1, 1, 2, 2, 3, 4, 4 on eight slips of paper, and put them in a paper bag. Have students construct a probability distribution for the number that results when one slip is drawn at random from the bag. Then have them take turns drawing slips of paper, with replacement, and compare the relative frequencies with the theoretical probability distribution. Sample answer: The theoretical probabilities are $P(1) = \frac{3}{8}$, $P(2) = \frac{1}{4}$, $P(3) = \frac{1}{8}$, and $P(4) = \frac{1}{4}$.

Example 2 Design and Run a Simulation

Teaching the Mathematical Practices

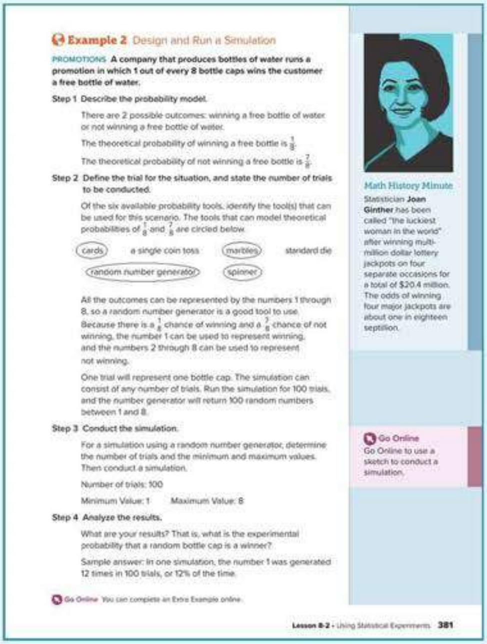
5 Compare Predictions with Data Point out that in Example 2, students use the random number generator tool to run a simulation. Students then analyze the results and compare the theoretical and experimental probabilities.

Questions for Mathematical Discourse

AL Given there are only two outcomes, winning and not winning, why does the probability tool need to have eight possible results? Sample answer: You need eight results to simulate the $\frac{1}{8}$ probability of winning by having one number represent winning and the other seven represent losing.

OL Is it a certainty that 1 will occur once in every 8 trials of the simulation? No; sample answer: over many simulations the occurrence of 1 should approach $\frac{1}{8}$.

BL Assuming that a limited number of bottles were made for the promotion, does buying a non-winning bottle change the probability that the next bottle purchased is a winner? Explain. No; sample answer: The probability of the second purchase being a winning bottle given that the first was a non-winning bottle is different from the probability of the first purchase being a winning bottle. However, on a large scale, the difference becomes less significant.



Example 2 Design and Run a Simulation

PROMOTIONS **A company that produces bottles of water runs a promotion in which 1 out of every 8 bottle caps wins the customer a free bottle of water.**

Step 1 Describe the probability model.

There are 2 possible outcomes: winning a free bottle of water or not winning a free bottle of water.

The theoretical probability of winning a free bottle is $\frac{1}{8}$.

The theoretical probability of not winning a free bottle is $\frac{7}{8}$.

Step 2 Define the trial for the situation, and state the number of trials to be conducted.

Of the six available probability tools, identify the tool(s) that can be used for this scenario. The tools that can model theoretical probabilities of $\frac{1}{8}$ and $\frac{7}{8}$ are circled below.

[✓ cards] a single coin toss [✓ marbles] standard die [✓ random number generator] [✓ spinner]

All the outcomes can be represented by the numbers 1 through 8, so a random number generator is a good tool to use. Because there is a $\frac{1}{8}$ chance of winning and a $\frac{7}{8}$ chance of not winning, the number 1 can be used to represent winning, and the numbers 2 through 8 can be used to represent not winning.

One trial will represent one bottle cap. The simulation can consist of any number of trials. Run the simulation for 100 trials, and the number generator will return 100 random numbers between 1 and 8.

Step 3 Conduct the simulation.

For a simulation using a random number generator, determine the number of trials and the minimum and maximum values. Then conduct a simulation.

Number of trials: 100

Minimum Value: 1 Maximum Value: 8

Step 4 Analyze the results.

What are your results? That is, what is the experimental probability that a random bottle cap is a winner?

Sample answer: In one simulation, the number 1 was generated 12 times in 100 trials, or 12% of the time.

Go Online You can complete an Extra Example online.

Math History Minute

Statistician **Joan Ginther** has been called "the luckiest woman in the world" after winning multi-million dollar lottery jackpots on four separate occasions for a total of $20.4 million. The odds of winning four major jackpots are about one in eighteen septillion.

Go Online

Go Online to use a sketch to conduct a simulation.

Lesson 8-2 • Using Statistical Experiments 381

Interactive Presentation

Design and Run a Simulation

PROMOTIONS A company that produces bottles of water runs a promotion in which 1 out of every 8 bottle caps wins the customer a free bottle of water.

Example 2

EXPAND

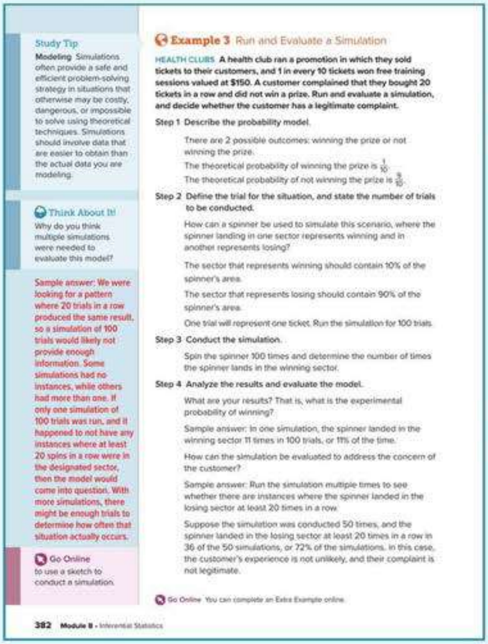
Students tap to see the steps to design and run a simulation.

DRAG & DROP

Students define a trial for a situation by selecting the appropriate tools for the situation.

WEB SKETCHPAD

Students use a sketch to run a simulation.



Study Tip

Modeling Simulations often provide a safe and efficient problem-solving strategy in situations that otherwise may be costly, dangerous, or impossible to solve using theoretical techniques. Simulations should involve data that are easier to obtain than the actual data you are modeling.

Think About It!

Why do you think multiple simulations were needed to evaluate this model?

Sample answer: We were looking for a pattern where 20 trials in a row produced the same result, so a simulation of 100 trials would likely not provide enough information. Some simulations had no instances, while others had more than one. If only one simulation of 100 trials was run, and it happened to not have any instances where at least 20 spins in a row were in the designated sector, then the model would come into question. With more simulations, there might be enough trials to determine how often that situation actually occurs.

Go Online to use a sketch to conduct a simulation.

Example 3 Run and Evaluate a Simulation

HEALTH CLUBS **A health club ran a promotion in which they sold tickets to their customers, and 1 in every 10 tickets won free training sessions valued at $150. A customer complained that they bought 20 tickets in a row and did not win a prize. Run and evaluate a simulation, and decide whether the customer has a legitimate complaint.**

Step 1 Describe the probability model.

There are 2 possible outcomes: winning the prize or not winning the prize.

The theoretical probability of winning the prize is $\frac{1}{10}$.

The theoretical probability of not winning the prize is $\frac{9}{10}$.

Step 2 Define the trial for the situation, and state the number of trials to be conducted.

How can a spinner be used to simulate this scenario, where the spinner landing in one sector represents winning and in another represents losing?

The sector that represents winning should contain 10% of the spinner's area.

The sector that represents losing should contain 90% of the spinner's area.

One trial will represent one ticket. Run the simulation for 100 trials.

Step 3 Conduct the simulation.

Spin the spinner 100 times and determine the number of times the spinner lands in the winning sector.

Step 4 Analyze the results and evaluate the model.

What are your results? That is, what is the experimental probability of winning?

Sample answer: In one simulation, the spinner landed in the winning sector 11 times in 100 trials, or 11% of the time.

How can the simulation be evaluated to address the concern of the customer?

Sample answer: Run the simulation multiple times to see whether there are instances where the spinner landed in the losing sector at least 20 times in a row.

Suppose the simulation was conducted 50 times, and the spinner landed in the losing sector at least 20 times in a row in 36 of the 50 simulations, or 72% of the simulations. In this case, the customer's experience is not unlikely, and their complaint is not legitimate.

Go Online You can complete an Extra Example online.

382 Module 8 • Inferential Statistics

Interactive Presentation

Example 3

WEB SKETCHPAD

Students use a sketch to run a simulation.

TYPE

Students report the results of the simulations.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Run and Evaluate a Simulation

Teaching the Mathematical Practices

8 Attend to Details Mathematically proficient students continually ask themselves, "Does this make sense?" Point out that in Example 3, students will use a simulation to evaluate the reasonableness of the customer's complaint.

Questions for Mathematical Discourse

AL Is it likely that a customer will win buying a single ticket? Explain. No; sample answer: because the theoretical probability is only $\frac{1}{10}$, according to the health club, it is unlikely that a customer would win with 1 ticket.

OL What other tools could have been used for this simulation? Sample answer: a random number generator with integers 1–10; the number cards and ace from one suit of a deck of cards

BL Why does the size of the set matter when using the random number generator simulation? Sample answer: The random number simulation more closely approximates the situation when the set is larger because a ticket being removed from the set has less effect on the probability of the next trial.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–5
2	exercises that use a variety of skills from this lesson	9–17
2	exercises that extend concepts learned in this lesson to new contexts	18, 19
3	exercises that emphasize higher-order and critical-thinking skills	20–24

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

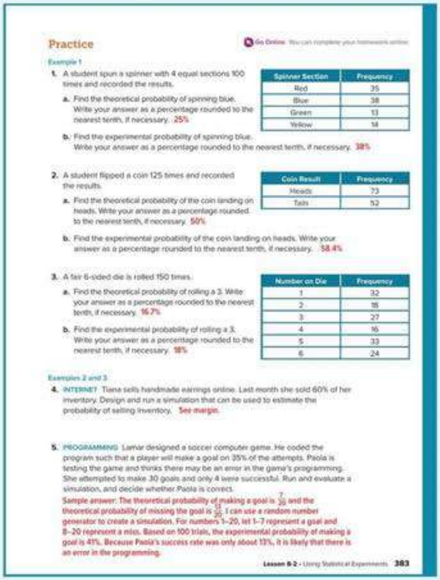
- Practice, Exercises 1–19 odd, 20–24
- Extension: Game Shows

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice, Exercises 1–23 odd
- Remediation, Review Resources: Sample Spaces
- Personal Tutors
- Extra Examples 1–3
- ALEKS Probability of Simple Events

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice, Exercises 1–5 odd
- Remediation, Review Resources: Sample Spaces
- *Quick Review Math Handbook*: Designing a Study
- ALEKS Probability of Simple Events



Practice

Go Online You can complete your homework online.

Example 1

1. A student spun a spinner with 4 equal sections 100 times and recorded the results.

Spinner Section	Frequency
Red	35
Blue	38
Green	13
Yellow	14

a. Find the theoretical probability of spinning blue. Write your answer as a percentage rounded to the nearest tenth, if necessary. 25%

b. Find the experimental probability of spinning blue. Write your answer as a percentage rounded to the nearest tenth, if necessary. 38%

2. A student flipped a coin 125 times and recorded the results.

Coin Result	Frequency
Heads	73
Tails	52

a. Find the theoretical probability of the coin landing on heads. Write your answer as a percentage rounded to the nearest tenth, if necessary. 50%

b. Find the experimental probability of the coin landing on heads. Write your answer as a percentage rounded to the nearest tenth, if necessary. 58.4%

3. A fair 6-sided die is rolled 150 times.

Number on Die	Frequency
1	32
2	18
3	27
4	16
5	33
6	24

a. Find the theoretical probability of rolling a 3. Write your answer as a percentage rounded to the nearest tenth, if necessary. 16.7%

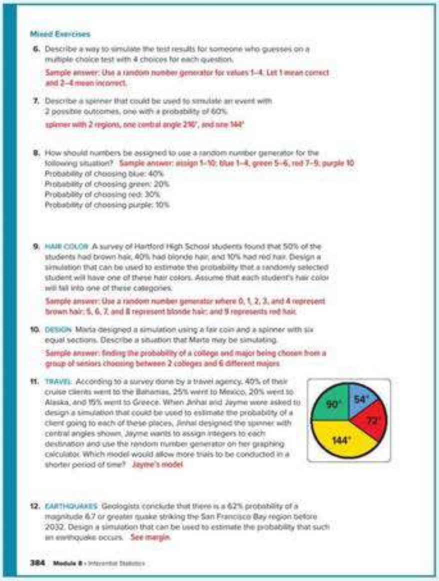
b. Find the experimental probability of rolling a 3. Write your answer as a percentage rounded to the nearest tenth, if necessary. 18%

Examples 2 and 3

4. INTERNET Tiana sells handmade earrings online. Last month she sold 60% of her inventory. Design and run a simulation that can be used to estimate the probability of selling inventory. See margin.

5. PROGRAMMING Lamar designed a soccer computer game. He coded the program such that a player will make a goal on 35% of the attempts. Paola is testing the game and thinks there may be an error in the game's programming. She attempted to make 30 goals and only 4 were successful. Run and evaluate a simulation, and decide whether Paola is correct.
Sample answer: The theoretical probability of making a goal is $\frac{7}{20}$ and the theoretical probability of missing the goal is $\frac{13}{20}$. I can use a random number generator to create a simulation. For numbers 1–20, let 1–7 represent a goal and 8–20 represent a miss. Based on 100 trials, the experimental probability of making a goal is 41%. Because Paola's success rate was only about 13%, it is likely that there is an error in the programming.

Lesson 8-2 • Using Statistical Experiments 383



Mixed Exercises

6. Describe a way to simulate the test results for someone who guesses on a multiple choice test with 4 choices for each question.
Sample answer: Use a random number generator for values 1–4. Let 1 mean correct and 2–4 mean incorrect.

7. Describe a spinner that could be used to simulate an event with 2 possible outcomes, one with a probability of 60%.
spinner with 2 regions, one central angle 216°, and one 144°

8. How should numbers be assigned to use a random number generator for the following situation? Sample answer: assign 1–10: blue 1–4, green 5–6, red 7–9, purple 10
Probability of choosing blue: 40%
Probability of choosing green: 20%
Probability of choosing red: 30%
Probability of choosing purple: 10%

9. HAIR COLOR A survey of Hartford High School students found that 50% of the students had brown hair, 40% had blonde hair, and 10% had red hair. Design a simulation that can be used to estimate the probability that a randomly selected student will have one of these hair colors. Assume that each student's hair color will fall into one of these categories.
Sample answer: Use a random number generator where 0, 1, 2, 3, and 4 represent brown hair; 5, 6, 7, and 8 represent blonde hair; and 9 represents red hair.

10. DESIGN Marta designed a simulation using a fair coin and a spinner with six equal sections. Describe a situation that Marta may be simulating.
Sample answer: finding the probability of a college and major being chosen from a group of seniors choosing between 2 colleges and 6 different majors

11. TRAVEL According to a survey done by a travel agency, 40% of their cruise clients went to the Bahamas, 25% went to Mexico, 20% went to Alaska, and 15% went to Greece. When Jinhai and Jayme were asked to design a simulation that could be used to estimate the probability of a client going to each of these places, Jinhai designed the spinner with central angles shown. Jayme wants to assign integers to each destination and use the random number generator on her graphing calculator. Which model would allow more trials to be conducted in a shorter period of time? Jayme's model

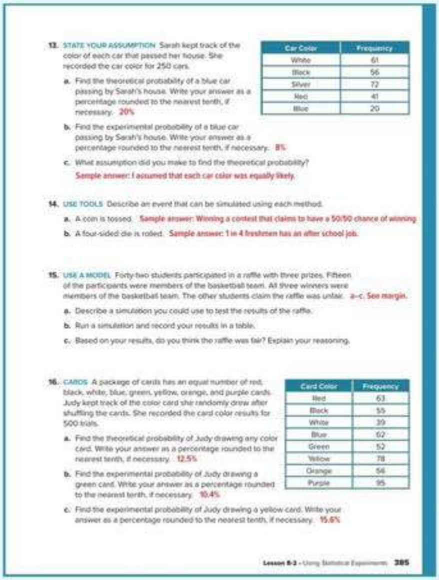
12. EARTHQUAKES Geologists conclude that there is a 62% probability of a magnitude 6.7 or greater quake striking the San Francisco Bay region before 2032. Design a simulation that can be used to estimate the probability that such an earthquake occurs. See margin.

384 Module 8 • Inferential Statistics

3 REFLECT AND PRACTICE

S.IC.2, S.IC.5

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION



13. STATE YOUR ASSUMPTION Sarah kept track of the color of each car that passed her house. She recorded the car color for 250 cars.

Car Color	Frequency
White	61
Black	56
Silver	72
Red	41
Blue	20

a. Find the theoretical probability of a blue car passing by Sarah's house. Write your answer as a percentage rounded to the nearest tenth, if necessary. 20%

b. Find the experimental probability of a blue car passing by Sarah's house. Write your answer as a percentage rounded to the nearest tenth, if necessary. 8%

c. What assumption did you make to find the theoretical probability? Sample answer: I assumed that each car color was equally likely.

14. USE TOOLS Describe an event that can be simulated using each method.

a. A coin is tossed. Sample answer: Winning a contest that claims to have a 50/50 chance of winning

b. A four-sided die is rolled. Sample answer: 1 in 4 freshmen has an after school job.

15. USE A MODEL Forty-two students participated in a raffle with three prizes. Fifteen of the participants were members of the basketball team. All three winners were members of the basketball team. The other students claim the raffle was unfair. a–c. See margin.

a. Describe a simulation you could use to test the results of the raffle.

b. Run a simulation and record your results in a table.

c. Based on your results, do you think the raffle was fair? Explain your reasoning.

16. CARDS A package of cards has an equal number of red, black, white, blue, green, yellow, orange, and purple cards. Judy kept track of the color card she randomly drew after shuffling the cards. She recorded the card color results for 500 trials.

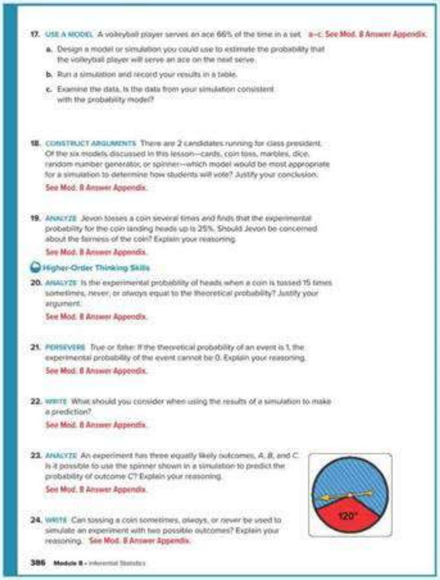
Card Color	Frequency
Red	63
Black	55
White	39
Blue	62
Green	52
Yellow	78
Orange	56
Purple	95

a. Find the theoretical probability of Judy drawing any color card. Write your answer as a percentage rounded to the nearest tenth, if necessary. 12.5%

b. Find the experimental probability of Judy drawing a green card. Write your answer as a percentage rounded to the nearest tenth, if necessary. 10.4%

c. Find the experimental probability of Judy drawing a yellow card. Write your answer as a percentage rounded to the nearest tenth, if necessary. 15.6%

Lesson 8-2 • Using Statistical Experiments 385



17. USE A MODEL A volleyball player serves an ace 66% of the time in a set. a–c. See Mod. 8 Answer Appendix.

a. Design a model or simulation you could use to estimate the probability that the volleyball player will serve an ace on the next serve.

b. Run a simulation and record your results in a table.

c. Examine the data. Is the data from your simulation consistent with the probability model?

18. CONSTRUCT ARGUMENTS There are 2 candidates running for class president. Of the six models discussed in this lesson—cards, coin toss, marbles, dice, random number generator, or spinner—which model would be most appropriate for a simulation to determine how students will vote? Justify your conclusion. See Mod. 8 Answer Appendix.

19. ANALYZE Jevon tosses a coin several times and finds that the experimental probability for the coin landing heads up is 25%. Should Jevon be concerned about the fairness of the coin? Explain your reasoning. See Mod. 8 Answer Appendix.

Higher-Order Thinking Skills

20. ANALYZE Is the experimental probability of heads when a coin is tossed 15 times sometimes, never, or always equal to the theoretical probability? Justify your argument. See Mod. 8 Answer Appendix.

21. PERSEVERE True or false: If the theoretical probability of an event is 1, the experimental probability of the event cannot be 0. Explain your reasoning. See Mod. 8 Answer Appendix.

22. WRITE What should you consider when using the results of a simulation to make a prediction? See Mod. 8 Answer Appendix.

23. ANALYZE An experiment has three equally likely outcomes, A, B, and C. Is it possible to use the spinner shown in a simulation to predict the probability of outcome C? Explain your reasoning. See Mod. 8 Answer Appendix.

24. WRITE Can tossing a coin sometimes, always, or never be used to simulate an experiment with two possible outcomes? Explain your reasoning. See Mod. 8 Answer Appendix.

386 Module 8 • Inferential Statistics

Answers

4. Sample answer: Use a spinner that is divided into 2 sectors, one at 60% area, or 216°, the other at 40%, or 144°. Perform 50 trials and record answers in a frequency table. Based on the simulation, probability of selling inventory is 0.64, and the probability of not selling inventory is 0.36.

Sample data:

Outcomes	Frequency
Sold	32
Not sold	18
Total	50

12. Sample answer: Use a random number generator to generate integers 1–50. 1–31 will represent an earthquake occurring, and 32–50 will represent an earthquake not occurring. Use 50 trials.

15a. Sample answer: Use a random number generator to generate a set of three numbers from 1 to 42. Numbers 1–15 will represent players on the basketball team and numbers 16–42 will represent the other students. Discard any trial that repeats any of the same number. Run the trial fifty times.

15b. Sample data is shown:

35, 13, 36	15, 32, 9	6, 14, 12	16, 26, 27	4, 34, 20	16, 32, 24	12, 9, 18	10, 17, 26	
15, 13, 23	18, 15, 41	13, 9, 20	33, 38, 30	34, 40, 35	37, 18, 36	22, 16, 36	23, 5, 28	
29, 7, 2	21, 22, 1	33, 31, 22	30, 39, 18	31, 7, 17	35, 40, 36	35, 9, 29	32, 20, 20	
21, 3, 23	16, 36, 5	30, 36, 11	28, 35, 10	5, 42, 21	11, 18, 40	26, 10, 2	28, 35, 41	
13, 11, 23	12, 7, 24	5, 3, 7	15, 39, 2	30, 21, 24	32, 25, 7	18, 9, 34	29, 20, 2	
18, 26, 6	32, 6, 39	17, 3, 21	31, 18, 27	24, 29, 31	25, 37, 21	32, 23, 13	21, 15, 28	
3, 34, 15	12, 2, 20							

15c. Sample answer: Only 2 of the 50 trials, or 4%, resulted in only team members winning the raffle. According to this simulation it is highly unlikely that all the winners are members of the basketball team. There is not enough data to determine if the raffle is unfair. If it happens again for another raffle, then the fairness should be examined more closely.

Lesson 8-3

Analyzing Population Data

S.IC.4

LESSON GOAL

Students use statistics to summarize, represent, and compare sets of data.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Describing Distributions
- Find a Standard Deviation
- Calculate Statistics

Comparing Distributions
- Compare Distributions

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Measures of Center	●	●		●
Extension: The Harmonic Mean		●	●	●

Language Development Handbook

Assign page 43 of the *Language Development Handbook* to help your students build mathematical language related to using statistics to summarize, represent, and compare sets of data.

ELL You can use the tips and suggestions on page T43 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Statistics and Probability

Standards for Mathematical Content:

S.IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

4 Model with mathematics.

Coherence

Vertical Alignment

Previous

Students collected, analyzed, and used data to compare theoretical and experimental probabilities. **S.IC.2, S.IC.5**

Now

Students use statistics to summarize, represent, and compare sets of data. **S.IC.4**

Next

Students will use statistics and normal distributions to analyze data. **S.ID.4, S.IC.6**

Rigor

The Three Pillars of Rigor

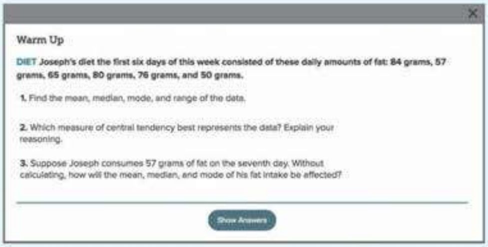
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students build fluency by calculating statistics and using those statistics to compare distributions. They apply their understanding by solving real-world problems.

Mathematical Background

A distribution of data shows the frequency of each possible data value. The shape of a distribution can be determined by looking at its histogram or box-and-whisker plot.

Interactive Presentation

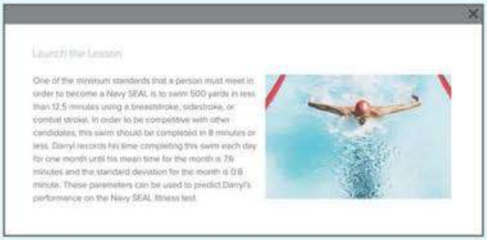


Warm Up

DIET Joseph's diet the first six days of this week consisted of these daily amounts of fat: 84 grams, 57 grams, 65 grams, 80 grams, 76 grams, and 50 grams.

1. Find the mean, median, mode, and range of the data.
2. Which measure of central tendency best represents the data? Explain your reasoning.
3. Suppose Joseph consumes 57 grams of fat on the seventh day. Without calculating, how will the mean, median, and mode of his fat intake be affected?

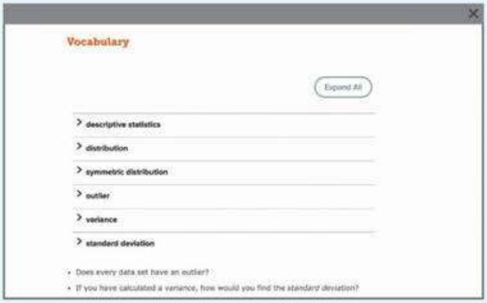
Warm Up



Launch the Lesson

One of the minimum standards that a person must meet in order to become a Navy SEAL is to swim 500 yards in less than 12.5 minutes using a breaststroke, sidestroke, or combat stroke. In order to be competitive with other candidates, this swim should be completed in 8 minutes or less. Darryl records his time completing this swim each day for one month until his mean time for the month is 7.6 minutes and the standard deviation for the month is 0.8 minute. These parameters can be used to predict Darryl's performance on the Navy SEAL fitness test.

Launch the Lesson



Vocabulary

- descriptive statistics
- distribution
- symmetric distribution
- outlier
- variance
- standard deviation

- Does every data set have an outlier?
- If you have calculated a variance, how would you find the standard deviation?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skills for this lesson:

- finding the mean, median, mode, and range
- analyzing measures of center in context

Answers:

1. $68\frac{2}{3}$70.5; no mode; 34 g
2. Answers will vary.
3. The mean and median will decrease, and there will be a mode, 57.

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will learn how descriptive statistics can be used to predict a person's performance.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the question below with the class.

Learn Describing Distributions

Objective

Students describe distributions by finding their mean and standard deviation.

MP Teaching the Mathematical Practices

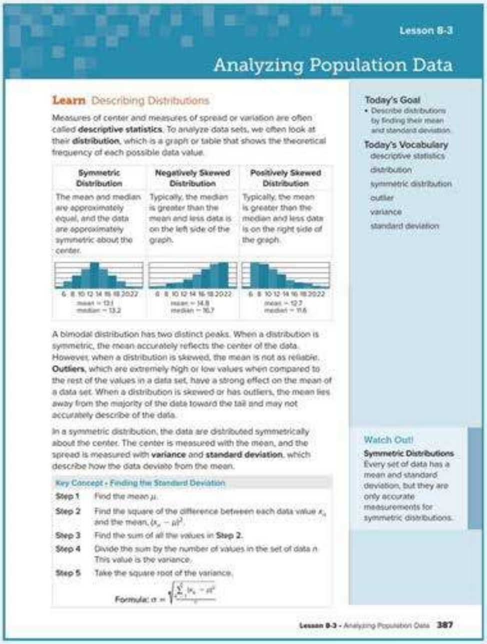
3 Analyze Cases Guide students to examine the cases of data distributions. Encourage students to familiarize themselves with the important features of each type of distribution.

About the Key Concept

The formula for calculating the standard deviation can seem overwhelming to students. Students will become more familiar with it as they apply it more frequently. Compare the concept of standard deviation to concepts of spread that students have encountered earlier, such as range, interquartile range, or mean absolute deviation.

Go Online

- F ind additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 8-3

Analyzing Population Data

Learn Describing Distributions

Measures of center and measures of spread or variation are often called **descriptive statistics**. To analyze data sets, we often look at their **distribution**, which is a graph or table that shows the theoretical frequency of each possible data value.

Symmetric Distribution	Negatively Skewed Distribution	Positively Skewed Distribution
The mean and median are approximately equal, and the data are approximately symmetric about the center.	Typically, the median is greater than the mean and less data is on the left side of the graph.	Typically, the mean is greater than the median and less data is on the right side of the graph.
mean ≈ 13.1 median ≈ 13.2	mean ≈ 14.8 median ≈ 16.7	mean ≈ 12.7 median ≈ 11.6

A bimodal distribution has two distinct peaks. When a distribution is symmetric, the mean accurately reflects the center of the data. However, when a distribution is skewed, the mean is not as reliable. **Outliers**, which are extremely high or low values when compared to the rest of the values in a data set, have a strong effect on the mean of a data set. When a distribution is skewed or has outliers, the mean lies away from the majority of the data toward the tail and may not accurately describe of the data.

In a symmetric distribution, the data are distributed symmetrically about the center. The center is measured with the mean, and the spread is measured with **variance** and **standard deviation**, which describe how the data deviate from the mean.

Key Concept • Finding the Standard Deviation

Step 1 Find the mean μ.

Step 2 Find the square of the difference between each data value x_n and the mean, $(x_n - \mu)^2$.

Step 3 Find the sum of all the values in **Step 2**.

Step 4 Divide the sum by the number of values in the set of data n. This value is the variance.

Step 5 Take the square root of the variance.

Formula: $\sigma = \sqrt{\frac{\sum_{k=1}^{n}(x_k - \mu)^2}{n}}$

Today's Goal
- Describe distributions by finding their mean and standard deviation.

Today's Vocabulary
descriptive statistics
distribution
symmetric distribution
outlier
variance
standard deviation

Watch Out!

Symmetric Distributions Every set of data has a mean and standard deviation, but they are only accurate measurements for symmetric distributions.

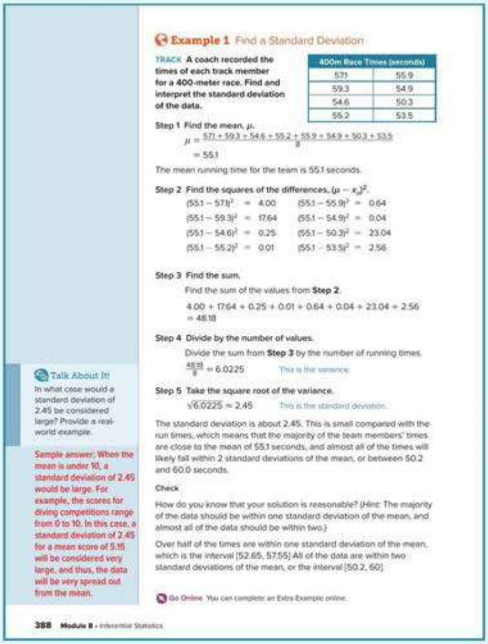
Lesson 8-3 • Analyzing Population Data **387**

Interactive Presentation

Learn

SWIPE

Students move through slides to see different kinds of distributions.



Example 1 Find a Standard Deviation

TRACK **A coach recorded the times of each track member for a 400-meter race. Find and interpret the standard deviation of the data.**

400m Race Times (seconds)	
57.1	55.9
59.3	54.9
54.6	50.3
55.2	53.5

Step 1 Find the mean, μ.

$$\mu = \frac{57.1 + 59.3 + 54.6 + 55.2 + 55.9 + 54.9 + 50.3 + 53.5}{8}$$
$$= 55.1$$

The mean running time for the team is 55.1 seconds.

Step 2 Find the squares of the differences, $(\mu - x_n)^2$.

$$(55.1 - 57.1)^2 = 4.00 \qquad (55.1 - 55.9)^2 = 0.64$$
$$(55.1 - 59.3)^2 = 17.64 \qquad (55.1 - 54.9)^2 = 0.04$$
$$(55.1 - 54.6)^2 = 0.25 \qquad (55.1 - 50.3)^2 = 23.04$$
$$(55.1 - 55.2)^2 = 0.01 \qquad (55.1 - 53.5)^2 = 2.56$$

Step 3 Find the sum.

Find the sum of the values from **Step 2**.

$$4.00 + 17.64 + 0.25 + 0.01 + 0.64 + 0.04 + 23.04 + 2.56$$
$$= 48.18$$

Step 4 Divide by the number of values.

Divide the sum from **Step 3** by the number of running times.

$\frac{48.18}{8} = 6.0225$ This is the variance.

Step 5 Take the square root of the variance.

$\sqrt{6.0225} \approx 2.45$ This is the standard deviation.

The standard deviation is about 2.45. This is small compared with the run times, which means that the majority of the team members' times are close to the mean of 55.1 seconds, and almost all of the times will likely fall within 2 standard deviations of the mean, or between 50.2 and 60.0 seconds.

Check

How do you know that your solution is reasonable? (Hint: The majority of the data should be within one standard deviation of the mean, and almost all of the data should be within two.)

Over half of the times are within one standard deviation of the mean, which is the interval [52.65, 57.55]. All of the data are within two standard deviations of the mean, or the interval [50.2, 60].

Talk About It!

In what case would a standard deviation of 2.45 be considered large? Provide a real-world example.

Sample answer: When the mean is under 10, a standard deviation of 2.45 would be large. For example, the scores for diving competitions range from 0 to 10. In this case, a standard deviation of 2.45 for a mean score of 5.15 will be considered very large, and thus, the data will be very spread out from the mean.

Go Online You can complete an Extra Example online.

388 **Module 8** • Inferential Statistics

Interactive Presentation

TRACK **A track athlete recorded her running times while training for a 400-meter race. Find and interpret the standard deviation of the data.**

400m Practice Times (seconds)	
57.1	55.9
59.3	54.9

Example 1

TYPE

Students complete a table to organize their calculations.

Example 1 Find a Standard Deviation

Teaching the Mathematical Practices

2 Attend to Quantities Point out that it is important to note the meaning of the standard deviation used in this problem. Encourage students to explain what the standard deviation means in the context of the problem.

Questions for Mathematical Discourse

AL How does the standard deviation indicate the spread of the data values? The standard deviation tells you a range around the mean in which most of the data occur. A larger standard deviation indicates points deviate more from the mean than a data set with a smaller standard deviation.

OL What does a small standard deviation mean in this situation? The majority of the team members have times that are very close to the mean.

BL Why is the average difference from the mean not used as a measure of spread? The result would be zero.

Example 2 Calculate Statistics

Teaching the Mathematical Practices

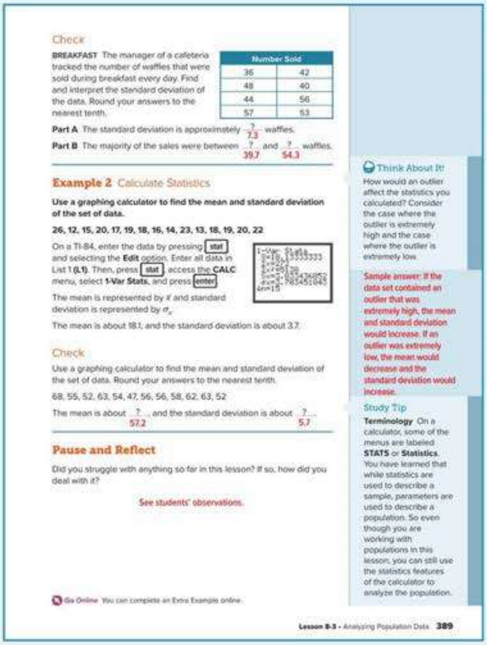
5 Use Mathematical Tools To solve the problem in Example 2, students must use a graphing calculator to calculate the statistics of the set of data. Work with students to explore and deepen their understanding of calculating statistics.

Questions for Mathematical Discourse

AL How do you use the standard deviation to find the interval in which most of the values occur? Most of the data is in the interval found by adding or subtracting the standard deviation from the mean, or $[\mu - \sigma, \mu + \sigma]$.

OL How can you check that most of the data is within one standard deviation of the mean? Sample answer: count how many points are within the interval $[\mu - \sigma, \mu + \sigma]$ and divide the number by the total number of the data points in the set. If the result is greater than 0.5, most of the data is within one standard deviation.

BL What would happen to the mean and standard deviation if the 26 was removed from the data set? The mean and standard deviation would both decrease.



Check

BREAKFAST The manager of a cafeteria tracked the number of waffles that were sold during breakfast every day. Find and interpret the standard deviation of the data. Round your answers to the nearest tenth.

Number Sold	
36	42
48	40
44	56
57	53

Part A The standard deviation is approximately ? 7.3 waffles.

Part B The majority of the sales were between ? 39.7 and ? 54.3 waffles.

Example 2 Calculate Statistics

Use a graphing calculator to find the mean and standard deviation of the set of data.

26, 12, 15, 20, 17, 19, 18, 16, 14, 23, 13, 18, 19, 20, 22

On a TI-84, enter the data by pressing stat and selecting the **Edit** option. Enter all data in List 1 **(L1)**. Then, press stat, access the **CALC** menu, select **1-Var Stats**, and press enter.

The mean is represented by $\bar{x}$ and standard deviation is represented by σ_x.

The mean is about 18.1, and the standard deviation is about 3.7.

Check

Use a graphing calculator to find the mean and standard deviation of the set of data. Round your answers to the nearest tenth.

68, 55, 52, 63, 54, 47, 56, 56, 58, 62, 63, 52

The mean is about ? 57.2 and the standard deviation is about ? 5.7

Pause and Reflect

Did you struggle with anything so far in this lesson? If so, how did you deal with it?

See students' observations.

Go Online You can complete an Extra Example online.

Think About It!

How would an outlier affect the statistics you calculated? Consider the case where the outlier is extremely high and the case where the outlier is extremely low.

Sample answer: If the data set contained an outlier that was extremely high, the mean and standard deviation would increase. If an outlier was extremely low, the mean would decrease and the standard deviation would increase.

Study Tip

Terminology On a calculator, some of the menus are labeled **STATS** or **Statistics**. You have learned that while statistics are used to describe a sample, parameters are used to describe a population. So even though you are working with populations in this lesson, you can still use the statistics features of the calculator to analyze the population.

Lesson 8-3 • Analyzing Population Data 389

Interactive Presentation

Calculate Statistics

Use a graphing calculator to find the mean and standard deviation of the set of data.

26, 12, 15, 20, 17, 19, 18, 16, 14, 23, 13, 18, 19, 20, 22

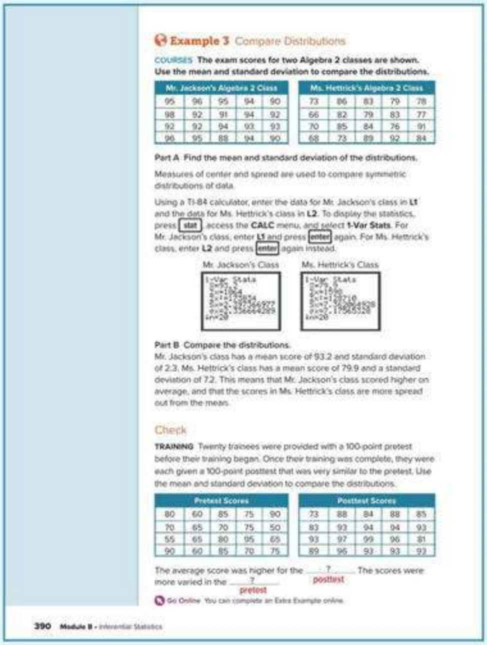
Example 2

SWIPE

Students move through the slides to see how to use a graphing calculator to find statistics.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Example 3 Compare Distributions

COURSES **The exam scores for two Algebra 2 classes are shown. Use the mean and standard deviation to compare the distributions.**

Mr. Jackson's Algebra 2 Class				
95	96	95	94	90
98	92	91	94	92
92	92	94	93	93
96	95	88	94	90

Ms. Hettrick's Algebra 2 Class				
73	86	83	79	78
66	82	79	83	77
70	85	84	76	91
68	73	89	92	84

Part A **Find the mean and standard deviation of the distributions.**

Measures of center and spread are used to compare symmetric distributions of data.

Using a TI-84 calculator, enter the data for Mr. Jackson's class in **L1** and the data for Ms. Hettrick's class in **L2**. To display the statistics, press stat, access the **CALC** menu, and select **1-Var Stats**. For Mr. Jackson's class, enter **L1** and press enter again. For Ms. Hettrick's class, enter **L2** and press enter again instead.

Mr. Jackson's Class

Ms. Hettrick's Class

Part B **Compare the distributions.**

Mr. Jackson's class has a mean score of 93.2 and standard deviation of 2.3. Ms. Hettrick's class has a mean score of 79.9 and a standard deviation of 7.2. This means that Mr. Jackson's class scored higher on average, and that the scores in Ms. Hettrick's class are more spread out from the mean.

Check

TRAINING Twenty trainees were provided with a 100-point pretest before their training began. Once their training was complete, they were each given a 100-point posttest that was very similar to the pretest. Use the mean and standard deviation to compare the distributions.

Pretest Scores				
80	60	85	75	90
70	65	70	75	50
55	65	80	95	65
90	60	85	70	75

Posttest Scores				
73	88	84	88	85
83	93	94	94	93
93	97	99	96	81
89	96	93	93	93

The average score was higher for the ___?___. The scores were more varied in the ___?___.
posttest
pretest

Go Online You can complete an Extra Example online.

390 Module 8 • Inferential Statistics

Interactive Presentation

COURSES The exam scores for two Algebra 2 classes are shown. Use the mean and standard deviation to compare the distributions.

Example 3

SWIPE

Students move through the slides to see how to use a graphing calculator to find statistics.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 3 Compare Distributions

Teaching the Mathematical Practices

4 Interpret Mathematical Results In Example 3, point out that to solve the problem, students should interpret their mathematical results in the context of the problem.

Questions for Mathematical Discourse

AL Why does Ms. Hettrick's class have a greater standard deviation than Mr. Jackson's class? The exam scores have a greater spread in Ms. Hettrick's class than in Mr. Jackson's class.

OL What could explain the differences in the two distributions? Sample answer: The students in Ms. Hettrick's class had a wider range of math ability than the students in Mr. Jackson's class.

BL On another exam, Mr. Jackson's class had an average test score of 80 out of 100 points, with a standard deviation of 1.2. What might this indicate about the exam? Sample answer: Because most of the students scored between 78 points and 82 points, the exam may have questions that none of the students were able to answer correctly.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–8
2	exercises that use a variety of skills from this lesson	9–13
3	exercises that emphasize higher-order and critical-thinking skills	14–17

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **BL**
THEN assign:

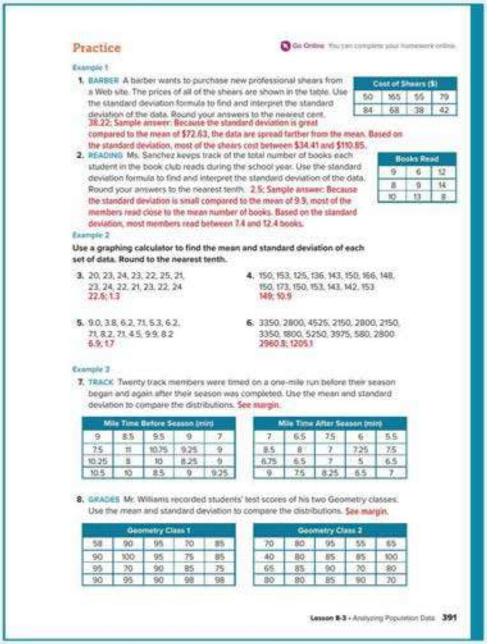
- Practice, Exercises 1–13 odd, 14–17
- Extension: The Harmonic Mean

IF students score 66%–89% on the Checks, **OL**
THEN assign:

- Practice, Exercises 1–17 odd
- Remediation, Review Resources: Measures of Variation
- Personal Tutors
- Extra Examples 1–3
- ALEKS Measures of Variation

IF students score 65% or less on the Checks, **AL**
THEN assign:

- Practice, Exercises 1–7 odd
- Remediation, Review Resources: Measures of Variation
- ALEKS Measures of Variation



Practice

Go Online You can complete your homework online.

Example 1

1. BARBER A barber wants to purchase new professional shears from a Web site. The prices of all of the shears are shown in the table. Use the standard deviation formula to find and interpret the standard deviation of the data. Round your answers to the nearest cent. 38.22; Sample answer: Because the standard deviation is great compared to the mean of $72.63, the data are spread farther from the mean. Based on the standard deviation, most of the shears cost between $34.41 and $110.85.

Cost of Shears ($)			
50	165	55	79
84	68	38	42

2. READING Ms. Sanchez keeps track of the total number of books each student in the book club reads during the school year. Use the standard deviation formula to find and interpret the standard deviation of the data. Round your answers to the nearest tenth. 2.5; Sample answer: Because the standard deviation is small compared to the mean of 9.9, most of the members read close to the mean number of books. Based on the standard deviation, most members read between 7.4 and 12.4 books.

Books Read		
9	6	12
8	9	14
10	13	8

Example 2

Use a graphing calculator to find the mean and standard deviation of each set of data. Round to the nearest tenth.

3. 20, 23, 24, 23, 22, 25, 21, 23, 24, 22, 21, 23, 22, 24
22.6; 1.3

4. 150, 153, 125, 136, 143, 150, 166, 148, 150, 173, 150, 153, 143, 142, 153
149; 10.9

5. 9.0, 3.8, 6.2, 7.1, 5.3, 6.2, 7.1, 8.2, 7.1, 4.5, 9.9, 8.2
6.9; 1.7

6. 3350, 2800, 4525, 2150, 2800, 2150, 3350, 1800, 5250, 3975, 580, 2800
2960.8; 1205.1

Example 3

7. TRACK Twenty track members were timed on a one-mile run before their season began and again after their season was completed. Use the mean and standard deviation to compare the distributions. See margin.

Mile Time Before Season (min)				
9	8.5	9.5	9	7
7.5	11	10.75	9.25	9
10.25	8	10	8.25	9
10.5	10	8.5	9	9.25

Mile Time After Season (min)				
7	6.5	7.5	6	5.5
8.5	8	7	7.25	7.5
6.75	6.5	7	5	6.5
9	7.5	8.25	6.5	7

8. GRADES Mr. Williams recorded students' test scores of his two Geometry classes. Use the mean and standard deviation to compare the distributions. See margin.

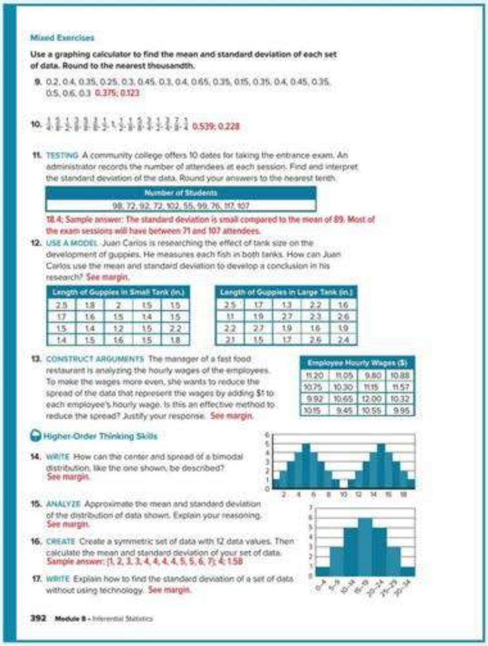
Geometry Class 1				
58	90	95	70	85
90	100	95	75	85
95	70	90	85	75
90	95	90	98	98

Geometry Class 2				
70	80	95	55	65
40	80	85	85	100
65	85	90	70	80
80	80	85	90	70

Lesson 8-3 • Analyzing Population Data 391

Answers

7. Before the season, the mean time was 9.2 minutes and the standard deviation was 1.0 minute. After the season, the mean time was 7.0 minutes and the standard deviation was 0.9 minutes. This means that before the season, the mile times were higher on average, but that the mile times were generally spread the same before and after the season.
8. Geometry Class 1 had a mean score of 86.5 and a standard deviation of 11.0. Geometry Class 2 had a mean score of 77.5 and a standard deviation of 13.7. This means that the students in Geometry Class 1 scored higher on average, and their scores were less spread out.



Mixed Exercises

Use a graphing calculator to find the mean and standard deviation of each set of data. Round to the nearest thousandth.

9. 0.2, 0.4, 0.35, 0.25, 0.3, 0.45, 0.3, 0.4, 0.65, 0.35, 0.15, 0.35, 0.4, 0.45, 0.35, 0.5, 0.6, 0.3 0.375; 0.123

10. 0.539; 0.228

11. TESTING A community college offers 10 dates for taking the entrance exam. An administrator records the number of attendees at each session. Find and interpret the standard deviation of the data. Round your answers to the nearest tenth.

Number of Students
98, 72, 92, 72, 102, 55, 99, 76, 117, 107

18.4; Sample answer: The standard deviation is small compared to the mean of 89. Most of the exam sessions will have between 71 and 107 attendees.

12. USE A MODEL Juan Carlos is researching the effect of tank size on the development of guppies. He measures each fish in both tanks. How can Juan Carlos use the mean and standard deviation to develop a conclusion in his research? See margin.

Length of Guppies in Small Tank (in.)				
2.5	1.8	2	1.5	1.5
1.7	1.6	1.5	1.4	1.5
1.5	1.4	1.2	1.5	2.2
1.4	1.5	1.6	1.5	1.8

Length of Guppies in Large Tank (in.)				
2.5	1.7	1.3	2.2	1.6
1.1	1.9	2.7	2.3	2.6
2.2	2.7	1.9	1.6	1.9
2.1	1.5	1.7	2.6	2.4

13. CONSTRUCT ARGUMENTS The manager of a fast food restaurant is analyzing the hourly wages of the employees. To make the wages more even, she wants to reduce the spread of the data that represent the wages by adding $1 to each employee's hourly wage. Is this an effective method to reduce the spread? Justify your response. See margin.

Employee Hourly Wages ($)			
11.20	11.05	9.80	10.88
10.75	10.30	11.15	11.57
9.92	10.65	12.00	10.32
10.15	9.45	10.55	9.95

Higher-Order Thinking Skills

14. WRITE How can the center and spread of a bimodal distribution, like the one shown, be described? See margin.

15. ANALYZE Approximate the mean and standard deviation of the distribution of data shown. Explain your reasoning. See margin.

16. CREATE Create a symmetric set of data with 12 data values. Then calculate the mean and standard deviation of your set of data. Sample answer: {1, 2, 3, 3, 4, 4, 4, 4, 5, 5, 6, 7}; 4; 1.58

17. WRITE Explain how to find the standard deviation of a set of data without using technology. See margin.

392 Module 8 • Inferential Statistics

Answers

12. The guppies in the small tank have a mean length of 1.63 inches with a standard deviation of 0.3 inches. The guppies in the large tank have a mean length of 2.03 inches with a standard deviation of 0.47 inches. From the mean and standard deviation, Juan Carlos can conclude that the guppies in the larger tank grew longer. Because the standard deviation for the smaller tank was less, he can also conclude that the guppies in the small tank are slightly closer in length to each other.

13. No; sample answer: adding $1 to each employee's hourly wage has no effect on the standard deviation of the data. The standard deviation both before and after the wage change is 0.67.

14. Sample answer: Since the distribution has two clusters that are each symmetric, summarize the center and spread of each cluster individually, using its mean and standard deviation.

15. Sample answer: Assume that each data value falls in the center of each bar of the histogram. Using technology, the mean is 17, and the standard deviation is about 7.4.

17. Find the mean of the set of data. Then, find the square of the difference between each data value and the mean. Next, find the sum of all of the squared differences. Then, divide the sum by the number of values in the set of data, which is the variance. Finally, take the square root of the quotient, or variance.

Lesson 8-4

Normal Distributions

S.ID.4, S.IC.6

LESSON GOAL

Students use statistics and normal distributions to analyze data.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Probability Distributions

Develop:

Probability Distributions

- Classify Random Variables
- Analyze a Probability Distribution
- Misleading Distributions

Normal Distributions

- Approximate Data by Using a Normal Distribution
- Use the Empirical Rule to Analyze Data

The Standard Normal Distribution

- Use *z*-values to Locate Position
- Find Area Under the Standard Normal Curve by Using a Table
- Find Area Under the Standard Normal Curve by Using a Calculator

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Measures of Spread	●	●		●
Extension: Working Backward with Normal Distributions		●	●	●

Language Development Handbook

Assign page 44 of the *Language Development Handbook* to help your students build mathematical language related to using sample data to make inferences about populations.

Reveal
INTEGRATED III

ELL You can use the tips and suggestions on page T44 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Statistics and Probability

Standards for Mathematical Content:

S.ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.

S.IC.6 Evaluate reports based on data.

Standards for Mathematical Practice:

5 Use appropriate tools strategically.

6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students used statistics to summarize, represent, and compare sets of data.
S.IC.4

Now
Students use statistics and normal distributions to analyze data.
S.IC.6, S.ID.4

Next
Students will use sample data to make inferences about populations.
S.IC.4, S.IC.6

Rigor

The Three Pillars of Rigor

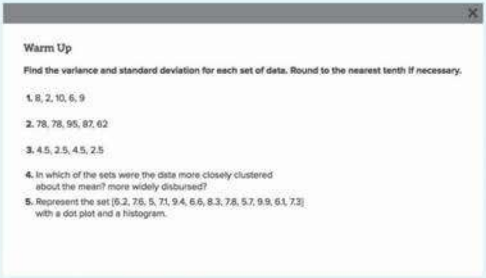
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand on their understanding of distributions by extending to normal distributions and build fluency by estimating population percentages. They apply their understanding of normal distributions by solving real-world problems.

Mathematical Background

The graphs of all normally distributed variables have essentially the same shape. With appropriate labeling of the mean and the points that are one standard deviation from the mean, the same normal curve can represent any normal distribution.

Interactive Presentation



Warm Up

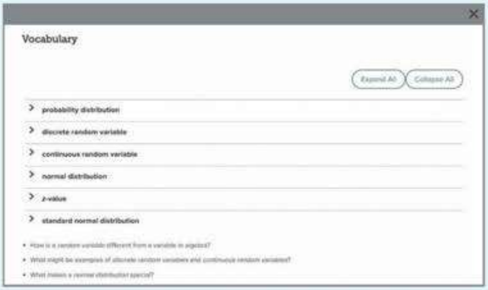
Find the variance and standard deviation for each set of data. Round to the nearest tenth if necessary.

1. 8, 2, 10, 6, 9
2. 78, 78, 95, 87, 62
3. 4.5, 2.5, 4.5, 2.5
4. In which of the sets were the data more closely clustered about the mean? more widely disbursed?
5. Represent the set {6.2, 7.6, 5, 7.1, 9.4, 6.6, 8.3, 7.8, 5.7, 9.9, 6.1, 7.3} with a dot plot and a histogram.

Warm Up

Launch the Lesson

Launch the Lesson



Vocabulary

- probability distribution
- discrete random variable
- continuous random variable
- normal distribution
- z-value
- standard normal distribution

Today's Vocabulary

Warm Up

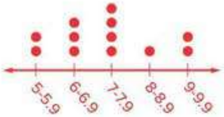
Prerequisite Skills

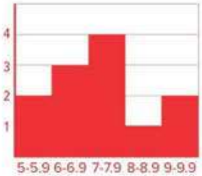
The Warm Up exercises address the following prerequisite skill for this lesson:

- finding the variance and standard deviation

Answers:

1. 8; 2.8
2. 121.2; 11.0
3. 1; 1
4. Exercise 3; Exercise 2
5. (See below.)





Launch the Lesson

Teaching the Mathematical Practices

4 Analyze Mathematical Relationships In the Launch the Lesson, students will learn about how the gestation time for animal species can be approximated by a normal distribution. Encourage students to analyze the data in the table about the mean times and standard deviations for some species. Ask students to draw conclusions about the gestation time for 95% of the young for each species.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Probability Distributions

Objective

Students examine the mean of a probability distribution as the sample size increases.

MP Teaching the Mathematical Practices

5 Decide When to Use Tools Mathematically proficient students can make sound decisions about when to use mathematical tools such as a concrete model or sketch. Help them see why using a sketch will help them solve problems and what the limitations are of using a concrete model to complete the Explore.

7 Look for a Pattern Help students to see the pattern in the probability distribution of the data as the sample size increases.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will be provided with an Inquiry Question to answer at the end of the activity. They will use a simulation and complete guiding exercises to explore the relationship between expected value and sample size. Then students will answer the Inquiry Question.

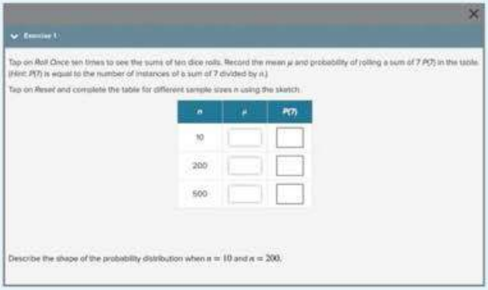
(continued on the next page)

Interactive Presentation

Probability Distributions

INQUIRY What is the relationship between the expected value of a discrete random variable and the mean of the distribution of that variable as the sample size increases?

Explore



Exercise 1

Tap on *Roll Once* ten times to see the sums of ten dice rolls. Record the mean μ and probability of rolling a sum of 7 $P(7)$ in the table. (Hint: $P(7)$ is equal to the number of instances of a sum of 7 divided by n.)

Tap on *Reset* and complete the table for different sample sizes n using the sketch.

n	μ	$P(7)$
10	☐	☐
200	☐	☐
500	☐	☐

Describe the shape of the probability distribution when $n = 10$ and $n = 200$.

Explore

WEB SKETCHPAD

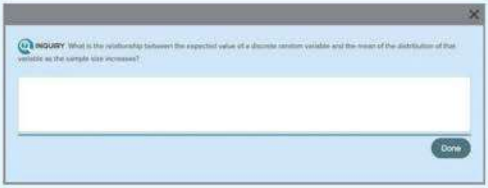
Students use a sketch to simulate rolling two dice.

TYPE

a|

Students record the results of their simulation in a table and describe distributions.

Interactive Presentation



INQUIRY What is the relationship between the expected value of a discrete random variable and the mean of the distribution of that variable as the sample size increases?

Explore

Explore Probability Distributions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Suppose a similar simulation is conducted that records the sum of two integers x and y, where $1 \leq x, y \leq 10$. What is the expected mean? 11 Describe the distribution as the number of trials is increased. Sample answer: The distribution is discrete and is bell-shaped. The highest bar will be at 11.
- What if the random number generator is used to find the sum of two real numbers a and b, where $1 \leq a, b \leq 10$. What is the expected mean? 11 Describe the distribution as the number of trials is increased. Sample answer: The distribution is continuous and is bell-shaped. The curve of the distribution will be at the highest at 11.

Inquiry

What is the relationship between the expected value of a discrete random variable and the mean of the distribution of that variable as the sample size increases? Sample answer: The mean of the distribution is equal to the expected value.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Probability Distributions

Objective

Students classify variables and analyze probability distributions to determine expected outcomes.

Teaching the Mathematical Practices

3 Construct Arguments Students will use stated assumptions and definitions to construct an argument about the probability of each value of a random variable.

Essential Question Follow-Up

Students have explored probability.

Ask:

How can probability be used in decision making? Sample answer: You can use probability to predict the most likely outcomes, and then make a decision based on those findings.

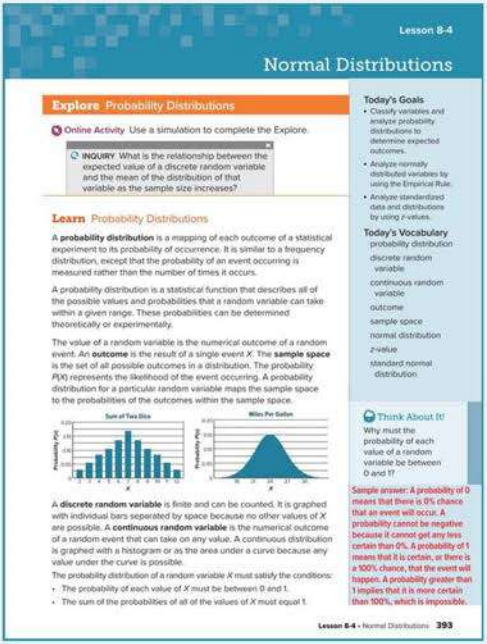
DIFFERENTIATE

Language Development Activity AL BL ELL

Social Learners It is important to be aware that some students may have cultural or familial prohibitions against cards, dice, or gambling of any kind. Explain that, historically, the laws of probability were developed in the context of gambling but are now applied in many other contexts, including medicine and meteorology.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 8-4

Normal Distributions

Explore Probability Distributions

Online Activity Use a simulation to complete the Explore.

INQUIRY What is the relationship between the expected value of a discrete random variable and the mean of the distribution of that variable as the sample size increases?

Learn Probability Distributions

A **probability distribution** is a mapping of each outcome of a statistical experiment to its probability of occurrence. It is similar to a frequency distribution, except that the probability of an event occurring is measured rather than the number of times it occurs.

A probability distribution is a statistical function that describes all of the possible values and probabilities that a random variable can take within a given range. These probabilities can be determined theoretically or experimentally.

The value of a random variable is the numerical outcome of a random event. An **outcome** is the result of a single event X. The **sample space** is the set of all possible outcomes in a distribution. The probability $P(X)$ represents the likelihood of the event occurring. A probability distribution for a particular random variable maps the sample space to the probabilities of the outcomes within the sample space.

A **discrete random variable** is finite and can be counted. It is graphed with individual bars seperated by space because no other values of X are possible. A **continuous random variable** is the numerical outcome of a random event that can take on any value. A continuous distribution is graphed with a histogram or as the area under a curve because any value under the curve is possible.

The probability distribution of a random variable X must satisfy the conditions:

- The probability of each value of X must be between 0 and 1.
- The sum of the probabilities of all of the values of X must equal 1.

Today's Goals

- Classify variables and analyze probability distributions to determine expected outcomes.
- Analyze normally distributed variables by using the Empirical Rule.
- Analyze standardized data and distributions by using z-values.

Today's Vocabulary

probability distribution
discrete random variable
continuous random variable
outcome
sample space
normal distribution
z-value
standard normal distribution

Think About It!

Why must the probability of each value of a random variable be between 0 and 1?

Sample answer: A probability of 0 means that there is 0% chance that an event will occur. A probability cannot be negative because it cannot get any less certain than 0%. A probability of 1 means that it is certain, or there is a 100% chance, that the event will happen. A probability greater than 1 implies that it is more certain than 100%, which is impossible.

Lesson 8-4 • Normal Distributions 393

Interactive Presentation

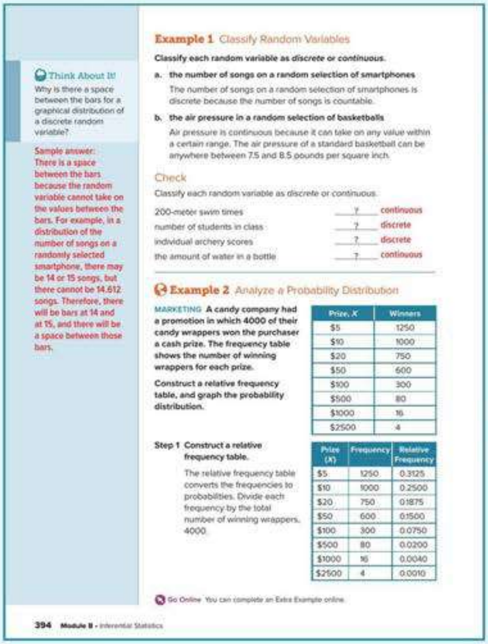
Learn

TAP

Students tap on buttons to learn about probability distributions.

TYPE

Students answer a question about the possible values for probabilities of random variables.



Think About It!

Why is there a space between the bars for a graphical distribution of a discrete random variable?

Sample answer: There is a space between the bars because the random variable cannot take on the values between the bars. For example, in a distribution of the number of songs on a randomly selected smartphone, there may be 14 or 15 songs, but there cannot be 14.612 songs. Therefore, there will be bars at 14 and at 15, and there will be a space between those bars.

Example 1 Classify Random Variables

Classify each random variable as discrete or continuous.

a. the number of songs on a random selection of smartphones

The number of songs on a random selection of smartphones is discrete because the number of songs is countable.

b. the air pressure in a random selection of basketballs

Air pressure is continuous because it can take on any value within a certain range. The air pressure of a standard basketball can be anywhere between 7.5 and 8.5 pounds per square inch.

Check

Classify each random variable as discrete or continuous.

200-meter swim times	___?___ continuous
number of students in class	___?___ discrete
individual archery scores	___?___ discrete
the amount of water in a bottle	___?___ continuous

Example 2 Analyze a Probability Distribution

MARKETING **A candy company had a promotion in which 4000 of their candy wrappers won the purchaser a cash prize. The frequency table shows the number of winning wrappers for each prize.**

Construct a relative frequency table, and graph the probability distribution.

Prize, X	Winners
$5	1250
$10	1000
$20	750
$50	600
$100	300
$500	80
$1000	16
$2500	4

Step 1 **Construct a relative frequency table.**

The relative frequency table converts the frequencies to probabilities. Divide each frequency by the total number of winning wrappers, 4000.

Prize (X)	Frequency	Relative Frequency
$5	1250	0.3125
$10	1000	0.2500
$20	750	0.1875
$50	600	0.1500
$100	300	0.0750
$500	80	0.0200
$1000	16	0.0040
$2500	4	0.0010

Go Online You can complete an Extra Example online.

394 Module 8 • Inferential Statistics

Interactive Presentation

Example 2

EXPAND

Students can tap to see how to construct the relative frequency table and probability distribution.

Example 1 Classify Random Variables

Teaching the Mathematical Practices

2 Attend to Quantities Point out that it is important to note the meaning of the quantities used in Example 1. Students must consider the meaning of the quantities to determine whether they are discrete or continuous.

Questions for Mathematical Discourse

AL Does a continuous variable have to have an infinite range? Explain. No; sample answer: A variable can be continuous over an interval.

OL Does a decimal value indicate a continuous variable? Explain. No; sample answer: A variable does not need to be limited to integers to be discrete. For example, a variable could be scaled, such as 2.3 million people.

BL In previous examples, continuous functions were used to model discrete variables. Can discrete models be applied to continuous variables? Yes; sample answer: You can define a discrete set using ranges of the variable, and then graph with a histogram.

Example 2 Analyze a Probability Distribution

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the table and graph used in Example 2.

Questions for Mathematical Discourse

AL What should the sum of the relative frequencies of a probability distribution be? Explain. 1; Sample answer: The sum of all prizes is 4000, and the relative frequencies are calculated by dividing each frequency by 4000. Dividing the sum of 4000 by 4000 equals 1.

OL Does the sum of the relative frequencies, 1, indicate that buying a candy bar guarantees winning a prize? Explain. No; sample answer: The probabilities in the example are for the types of prizes given a winning wrapper.

BL If the company produces 400,000 candy bars during the promotion, how do you find the probability of one candy bar purchase resulting in a $2500 prize? Multiply the probability of getting a winning candy bar, $\frac{1}{100}$, by the probability for the $2500 prize $\frac{1}{1000}$. The probability of winning the $2500 prize is $\frac{1}{100,000}$.

Common Error

Remind students that the relatively frequency is out of the 4000 wrappers that show the word "Winner!" when they are opened, not out of all wrappers on candy from the candy company. Encourage students to think about whether the relative frequency reflects the probability of winning a prize when a candy is purchased, depending on how many total candy bars are manufactured.

Example 3 Misleading Distributions

Teaching the Mathematical Practices

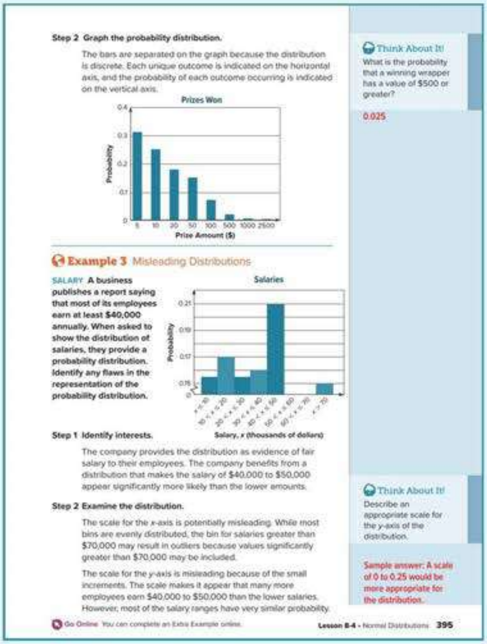
6 Use Quantities Use the Think About It! question to guide students to clarifying their use of quantities in Example 3. Ensure that they specify the units of measure used in the problem and label axes appropriately. Encourage students to explain why labeling axes without the appropriate scale and label can be misleading.

Questions for Mathematical Discourse

AL What is the difference between a median and a mean? A median is the value that splits the data in half, half of the data is greater than the median, half is less than the median. A mean is the sum of the data divided by the number of data.

OL What is implied by saying "most of its employees," and does the data support the claim? Sample answer: The claim implies that the median salary is at least $40,000, which is not accurate.

BL What would be a more accurate claim for the company to make? Sample answer: The company could report the median salary, which should fall between $30,000 and $40,000.



Step 2 Graph the probability distribution.

The bars are separated on the graph because the distribution is discrete. Each unique outcome is indicated on the horizontal axis, and the probability of each outcome occurring is indicated on the vertical axis.

Think About It!
What is the probability that a winning wrapper has a value of $500 or greater?

0.025

Example 3 Misleading Distributions

SALARY A business publishes a report saying that most of its employees earn at least $40,000 annually. When asked to show the distribution of salaries, they provide a probability distribution. Identify any flaws in the representation of the probability distribution.

Step 1 Identify interests.

The company provides the distribution as evidence of fair salary to their employees. The company benefits from a distribution that makes the salary of $40,000 to $50,000 appear significantly more likely than the lower amounts.

Step 2 Examine the distribution.

The scale for the x-axis is potentially misleading. While most bins are evenly distributed, the bin for salaries greater than $70,000 may result in outliers because values significantly greater than $70,000 may be included.

The scale for the y-axis is misleading because of the small increments. The scale makes it appear that many more employees earn $40,000 to $50,000 than the lower salaries. However, most of the salary ranges have very similar probability.

Think About It!
Describe an appropriate scale for the y-axis of the distribution.

Sample answer: A scale of 0 to 0.25 would be more appropriate for the distribution.

Go Online You can complete an Extra Example online.

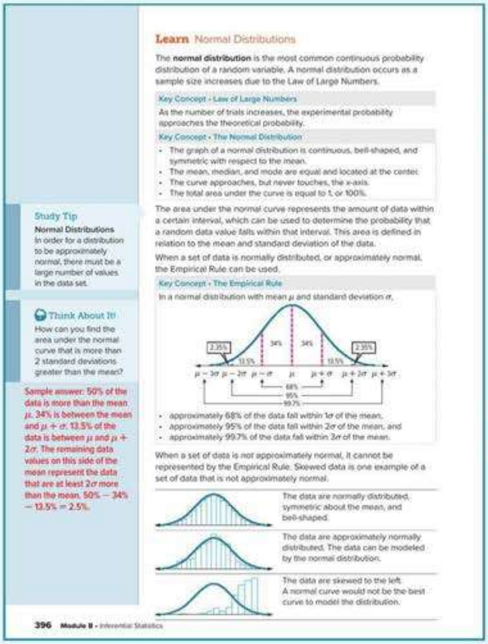
Lesson 8-4 • Normal Distributions 395

Interactive Presentation

Check

CHECK

Students complete the Check online to determine whether they are ready to move on.



Study Tip

Normal Distributions In order for a distribution to be approximately normal, there must be a large number of values in the data set.

Think About It!

How can you find the area under the normal curve that is more than 2 standard deviations greater than the mean?

Sample answer: 50% of the data is more than the mean μ. 34% is between the mean and $\mu + \sigma$. 13.5% of the data is between μ and $\mu + 2\sigma$. The remaining data values on this side of the mean represent the data that are at least 2σ more than the mean. 50% − 34% − 13.5% = 2.5%.

Learn Normal Distributions

The **normal distribution** is the most common continuous probability distribution of a random variable. A normal distribution occurs as a sample size increases due to the Law of Large Numbers.

Key Concept • Law of Large Numbers

As the number of trials increases, the experimental probability approaches the theoretical probability.

Key Concept • The Normal Distribution

- The graph of a normal distribution is continuous, bell-shaped, and symmetric with respect to the mean.
- The mean, median, and mode are equal and located at the center.
- The curve approaches, but never touches, the x-axis.
- The total area under the curve is equal to 1, or 100%.

The area under the normal curve represents the amount of data within a certain interval, which can be used to determine the probability that a random data value falls within that interval. This area is defined in relation to the mean and standard deviation of the data.

When a set of data is normally distributed, or approximately normal, the Empirical Rule can be used.

Key Concept • The Empirical Rule

In a normal distribution with mean μ and standard deviation σ,

- approximately 68% of the data fall within 1σ of the mean,
- approximately 95% of the data fall within 2σ of the mean, and
- approximately 99.7% of the data fall within 3σ of the mean.

When a set of data is not approximately normal, it cannot be represented by the Empirical Rule. Skewed data is one example of a set of data that is not approximately normal.

The data are normally distributed, symmetric about the mean, and bell-shaped.

The data are approximately normally distributed. The data can be modeled by the normal distribution.

The data are skewed to the left. A normal curve would not be the best curve to model the distribution.

396 Module 8 • Inferential Statistics

Interactive Presentation

Learn

SWIPE

Students move through slides to see how the Law of Large Numbers affects a histogram.

TYPE

Students answer a question about the area under the curve for a specific interval.

Learn Normal Distributions

Objective

Students analyze normally distributed variables by using the Empirical Rule.

Teaching the Mathematical Practices

7 Use Structure Guide students to see how the structure of a normal distribution and normal curve can be used to estimate the area under the curve.

DIFFERENTIATE

Language Development Activity ELL

Intermediate Before students read the text, have them take a close look at the distributions. Have them use the graph in the Key Concept as they work in pairs to form questions about the normal distribution. After students finish reading, have partners discuss how their ideas changed or stayed the same.

Advanced Have student pairs take turns reading the text to one another. Remind them that they are reading for an audience and that they need to maintain the audience's interest with their voices. Move around the room, correcting pronunciation as necessary. Ask students what information they learn from the title and how the title affects their approach to the text. What do they know about distributions, and what do they expect to learn? Have them share with the class.

Advanced High Have students write a paragraph explaining and evaluating each distribution used in relation to the content of the lesson.

Example 4 Approximate Data by Using a Normal Distribution

Teaching the Mathematical Practices

4 Apply Mathematics In Example 4, students apply what they have learned about the normal distribution to analyzing a real-world problem.

Questions for Mathematical Discourse

AL What does it mean to say that the data is skewed? More of the data are on one side of the range than the other.

OL If a data set has a mean that is close to the minimum or maximum value of the variable, what does this indicate? The distribution is likely skewed.

BL How would removing the outliers affect the mean and standard deviation? Both the mean and standard deviation would decrease.

Example 5 Use the Empirical Rule to Analyze Data

Teaching the Mathematical Practices

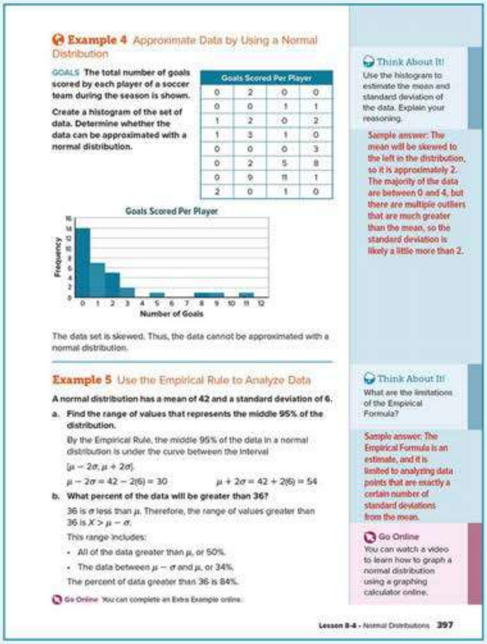
7 Use Structure Help students to use the structure of the Empirical Rule to analyze a normal distribution.

Questions for Mathematical Discourse

AL Why is σ multiplied by 2 in part **a**? We want the range of values that represent the middle 95% of the distribution. 68% of the distribution is between $\mu - \sigma$ and $\mu + \sigma$, while 95% is between $\mu - 2\sigma$ and $\mu + 2\sigma$.

OL What range of values represents 100% of the data? $(-\infty, \infty)$

BL Could a skewed set of data have the same standard deviation and mean as the normal distribution in the example? yes What is a way you could show that skewed data is not well represented by the normal distribution? Sample answer: find the percent of the data that is within a standard deviation greater than and less than the mean, and compare to the expected percent for that interval for a normal distribution.



Example 4 Approximate Data by Using a Normal Distribution

GOALS **The total number of goals scored by each player of a soccer team during the season is shown.**

Create a histogram of the set of data. Determine whether the data can be approximated with a normal distribution.

Goals Scored Per Player			
0	2	0	0
0	0	1	1
1	2	0	2
1	3	1	0
0	0	0	3
0	2	5	8
0	9	11	1
2	0	1	0

The data set is skewed. Thus, the data cannot be approximated with a normal distribution.

Think About It! Use the histogram to estimate the mean and standard deviation of the data. Explain your reasoning.

Sample answer: The mean will be skewed to the left in the distribution, so it is approximately 2. The majority of the data are between 0 and 4, but there are multiple outliers that are much greater than the mean, so the standard deviation is likely a little more than 2.

Example 5 Use the Empirical Rule to Analyze Data

A normal distribution has a mean of 42 and a standard deviation of 6.

a. **Find the range of values that represents the middle 95% of the distribution.**

By the Empirical Rule, the middle 95% of the data in a normal distribution is under the curve between the interval $[\mu - 2\sigma, \mu + 2\sigma]$.

$\mu - 2\sigma = 42 - 2(6) = 30$ $\quad$ $\mu + 2\sigma = 42 + 2(6) = 54$

b. **What percent of the data will be greater than 36?**

36 is σ less than μ. Therefore, the range of values greater than 36 is $X > \mu - \sigma$.

This range includes:

- All of the data greater than μ, or 50%.
- The data between $\mu - \sigma$ and μ, or 34%.

The percent of data greater than 36 is 84%.

Go Online You can complete an Extra Example online.

Think About It! What are the limitations of the Empirical Formula?

Sample answer: The Empirical Formula is an estimate, and it is limited to analyzing data points that are exactly a certain number of standard deviations from the mean.

Go Online You can watch a video to learn how to graph a normal distribution using a graphing calculator online.

Lesson 8-4 • Normal Distributions 397

Interactive Presentation

A normal distribution has a mean of 42 and a standard deviation of 6.

a. Find the range of values that represent the middle 95% of the distribution.

Example 5

SWIPE

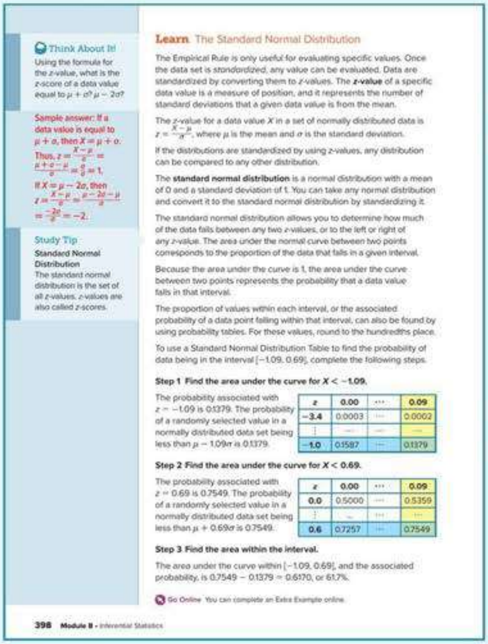
Students move through the slides to see how the Empirical Rule is applied.

WATCH

Students can watch videos to see how to graph a normal distribution using a graphing calculator.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Think About It!

Using the formula for the z-value, what is the z-score of a data value equal to $\mu + \sigma$? $\mu - 2\sigma$?

Sample answer: If a data value is equal to $\mu + \sigma$, then $X = \mu + \sigma$. Thus, $z = \frac{X - \mu}{\sigma} = \frac{\mu + \sigma - \mu}{\sigma} = \frac{\sigma}{\sigma} = 1$. If $X = \mu - 2\sigma$, then $z = \frac{X - \mu}{\sigma} = \frac{\mu - 2\sigma - \mu}{\sigma} = \frac{-2\sigma}{\sigma} = -2$.

Study Tip

Standard Normal Distribution The standard normal distribution is the set of all z-values. z-values are also called z-scores.

Learn The Standard Normal Distribution

The Empirical Rule is only useful for evaluating specific values. Once the data set is standardized, any value can be evaluated. Data are standardized by converting them to z-values. The **z-value** of a specific data value is a measure of position, and it represents the number of standard deviations that a given data value is from the mean.

The z-value for a data value X in a set of normally distributed data is $z = \frac{X - \mu}{\sigma}$, where μ is the mean and σ is the standard deviation.

If the distributions are standardized by using z-values, any distribution can be compared to any other distribution.

The **standard normal distribution** is a normal distribution with a mean of 0 and a standard deviation of 1. You can take any normal distribution and convert it to the standard normal distribution by standardizing it.

The standard normal distribution allows you to determine how much of the data falls between any two z-values, or to the left or right of any z-value. The area under the normal curve between two points corresponds to the proportion of the data that falls in a given interval.

Because the area under the curve is 1, the area under the curve between two points represents the probability that a data value falls in that interval.

The proportion of values within each interval, or the associated probability of a data point falling within that interval, can also be found by using probability tables. For these values, round to the hundredths place.

To use a Standard Normal Distribution Table to find the probability of data being in the interval [−1.09, 0.69], complete the following steps.

Step 1 Find the area under the curve for $X < -1.09$.

The probability associated with $z = -1.09$ is 0.1379. The probability of a randomly selected value in a normally distributed data set being less than $\mu - 1.09\sigma$ is 0.1379.

z	0.00	⋯	0.09
−3.4	0.0003	⋯	0.0002
⋮	⋯	⋯	⋯
−1.0	0.1587	⋯	0.1379

Step 2 Find the area under the curve for $X < 0.69$.

The probability associated with $z = 0.69$ is 0.7549. The probability of a randomly selected value in a normally distributed data set being less than $\mu + 0.69\sigma$ is 0.7549.

z	0.00	⋯	0.09
0.0	0.5000	⋯	0.5359
⋮	⋯	⋯	⋯
0.6	0.7257	⋯	0.7549

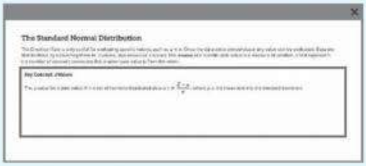
Step 3 Find the area within the interval.

The area under the curve within [−1.09, 0.69], and the associated probability, is 0.7549 − 0.1379 = 0.6170, or 61.7%.

Go Online You can complete an Extra Example online.

398 Module 8 • Inferential Statistics

Interactive Presentation



The Standard Normal Distribution

Key Concept: z-Values

Learn

TYPE

Students will answer a question to show the relationship between the z-value and standard deviation.

1 CONCEPTUAL UNDERSTANDING 2 FLUENCY 3 APPLICATION

Learn The Standard Normal Distribution

Objective

Students analyze standardized data and distributions by using z-scores.

Teaching the Mathematical Practices

1 Explain Correspondences Guide students as they use the z-score tables to interpret the meaning of the values in relation to the data.

Example 6 Use *z*-Values to Locate Position

MP Teaching the Mathematical Practices

6 Use Precision In this Example, students will calculate accurately and efficiently using the formula for calculating *z*-values.

Questions for Mathematical Discourse

AL What is the *z*-value of the mean? Explain. 0; Sample answer: A *z*-value is the number of standard deviations a data value is from the mean.

OL What does it mean when a *z*-value is negative? The data value is below the mean.

BL What is the *z*-value of *X* when the distribution is a standard normal distribution? *X*

Example 7 Find Area Under the Standard Normal Curve by Using a Table

MP Teaching the Mathematical Practices

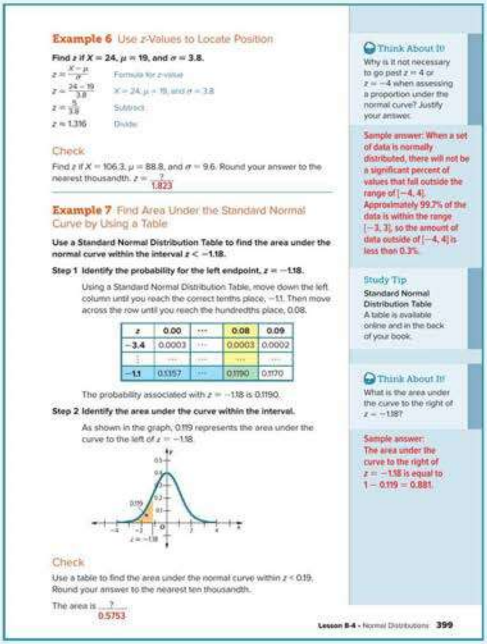
1 Explain Correspondences Encourage students to explain the relationships between the *z*-values in the table and the area under the curve.

Questions for Mathematical Discourse

AL How would you use the *z*-table to find the probability for $z = -1.50$? Move down the first column to -1.5, and then move over to the value that is in the 0.00 column.

OL Without looking at the *z*-table, what do you expect the table to show for the 0.0 row and 0.00 column? Explain. 0.5000; Sample answer: The *z*-score for the mean is 0. Because 50% of the distribution is to the left of the mean, the value is 0.5000.

BL How do you find the area under the standard normal curve to the right of any *z*-value? Subtract the table value from 1.



Example 6 Use *z*-Values to Locate Position

Find z if $X = 24$, $\mu = 19$, and $\sigma = 3.8$.

$z = \frac{X - \mu}{\sigma}$ Formula for *z*-value

$z = \frac{24 - 19}{3.8}$ $X = 24$, $\mu = 19$, and $\sigma = 3.8$

$z = \frac{5}{3.8}$ Subtract

$z \approx 1.316$ Divide

Check

Find z if $X = 106.3$, $\mu = 88.8$, and $\sigma = 9.6$. Round your answer to the nearest thousandth. $z = \underline{\ ?\ }$ 1.823

Example 7 Find Area Under the Standard Normal Curve by Using a Table

Use a Standard Normal Distribution Table to find the area under the normal curve within the interval $z < -1.18$.

Step 1 Identify the probability for the left endpoint, $z = -1.18$.

Using a Standard Normal Distribution Table, move down the left column until you reach the correct tenths place, −1.1. Then move across the row until you reach the hundredths place, 0.08.

z	0.00	⋯	0.08	0.09
−3.4	0.0003	⋯	0.0003	0.0002
⋮	⋯	⋯	⋯	⋯
−1.1	0.1357	⋯	0.1190	0.1170

The probability associated with $z = -1.18$ is 0.1190.

Step 2 Identify the area under the curve within the interval.

As shown in the graph, 0.119 represents the area under the curve to the left of $z = -1.18$.

Check

Use a table to find the area under the normal curve within $z < 0.19$. Round your answer to the nearest ten thousandth.

The area is $\underline{\ ?\ }$ 0.5753

Think About It!

Why is it not necessary to go past $z = 4$ or $z = -4$ when assessing a proportion under the normal curve? Justify your answer.

Sample answer: When a set of data is normally distributed, there will not be a significant percent of values that fall outside the range of [−4, 4]. Approximately 99.7% of the data is within the range [−3, 3], so the amount of data outside of [−4, 4] is less than 0.3%.

Study Tip

Standard Normal Distribution Table A table is available online and in the back of your book.

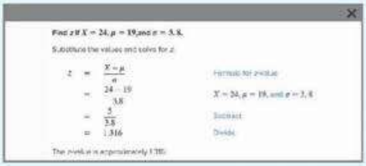
Think About It!

What is the area under the curve to the right of $z = -1.18$?

Sample answer: The area under the curve to the right of $z = -1.18$ is equal to $1 - 0.119 = 0.881$.

Lesson 8-4 • Normal Distributions 399

Interactive Presentation



Find z if $X = 24$, $\mu = 19$, and $\sigma = 3.8$.

Substitute the values and solve for z.

$z = \frac{X - \mu}{\sigma}$ Formula for *z*-value

$= \frac{24 - 19}{3.8}$ $X = 24$, $\mu = 19$, and $\sigma = 3.8$

$= \frac{5}{3.8}$ Subtract

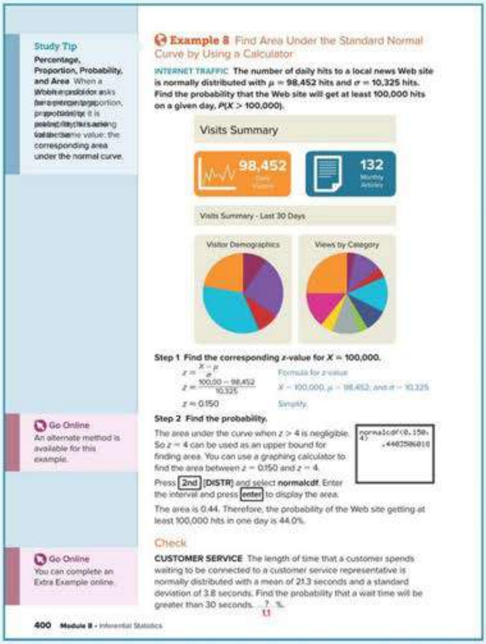
$= 1.316$ Divide

The *z*-value is approximately 1.316.

Example 6

TYPE

Students use the formula for *z*-values to find a missing quantity.



Study Tip

Percentage, Proportion, Probability, and Area When a [illegible] asks for a [illegible]portion, pr[illegible] it is [illegible]ng [illegible]me value: the corresponding area under the normal curve.

Example 8 Find Area Under the Standard Normal Curve by Using a Calculator

INTERNET TRAFFIC **The number of daily hits to a local news Web site is normally distributed with $\mu = 98{,}452$ hits and $\sigma = 10{,}325$ hits. Find the probability that the Web site will get at least 100,000 hits on a given day, $P(X > 100{,}000)$.**

Step 1 Find the corresponding z-value for $X = 100{,}000$.

$z = \frac{X - \mu}{\sigma}$	Formula for z-value
$z = \frac{100{,}00 - 98{,}452}{10{,}325}$	$X = 100{,}000$, $\mu = 98{,}452$, and $\sigma = 10{,}325$
$z = 0.150$	Simplify.

Step 2 Find the probability.

The area under the curve when $z > 4$ is negligible. So $z = 4$ can be used as an upper bound for finding area. You can use a graphing calculator to find the area between $z = 0.150$ and $z = 4$.

```
normalcdf(0.150,
4)
          .4403586018
```

Press 2nd [DISTR] and select **normalcdf**. Enter the interval and press enter to display the area.

The area is 0.44. Therefore, the probability of the Web site getting at least 100,000 hits in one day is 44.0%.

Go Online An alternate method is available for this example.

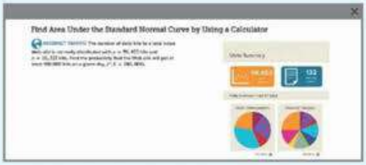
Check

CUSTOMER SERVICE The length of time that a customer spends waiting to be connected to a customer service representative is normally distributed with a mean of 21.3 seconds and a standard deviation of 3.8 seconds. Find the probability that a wait time will be greater than 30 seconds. ? %.

1.1

Go Online You can complete an Extra Example online.

400 Module 8 • Inferential Statistics



Find Area Under the Standard Normal Curve by Using a Calculator

Example 8

SWIPE

Students move through the slides to see how to find the area under a standard normal curve by using a calculator.

EXPAND

Students can tap to see an alternate method.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 8 Find Area Under the Standard Normal Curve by Using a Calculator

Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to solve the problem in Example 8, students will need to use a graphing calculator to determine the area under the standard normal curve.

Questions for Mathematical Discourse

AL Why do you need to find the corresponding z-value for X? Sample answer: the given distribution is a normal distribution, but not a standard normal distribution. The probabilities for a standard normal distribution can be found with the table or a calculator, and the probability for X will be the same as the probability for the associated z-value.

OL What are alternative ways to determine the probability of the Web site getting less than 100,000 hits on a given day? Sample answer: use the calculator to find the area from $z = -4$ to $z = 0.150$; subtract 0.44 from 1; use the table to find the value for $z = 0.150$

BL Why is $z = 4$ used as the other endpoint for finding the area? Sample answer: though the distribution continues to approach zero as z approaches infinity, for the purpose of estimation, the curve is effectively zero when $z = 4$.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–14
2	exercises that use a variety of skills from this lesson	15–18
3	exercises that emphasize higher-order and critical-thinking skills	19–21

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **BL**
THEN assign:

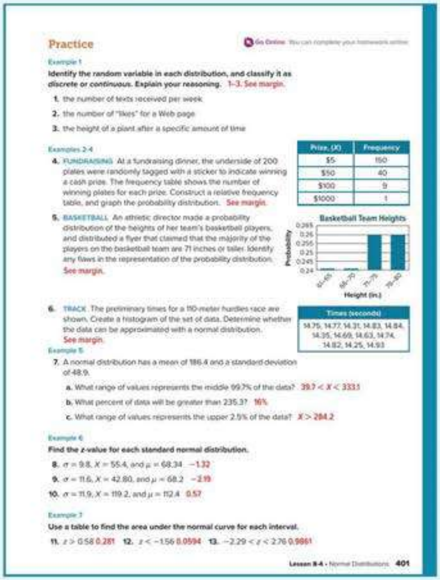
- Practice, Exercises 1–17 odd, 19–21
- Extension: Working Backward with Normal Distributions
- ALEKS Standard Deviation and the Normal Distribution

IF students score 66%–89% on the Checks, **OL**
THEN assign:

- Practice, Exercises 1–21 odd
- Remediation, Review Resources: Measures of Spread
- Personal Tutors
- Extra Examples 1–8
- ALEKS Measures of Center and Spread

IF students score 65% or less on the Checks, **AL**
THEN assign:

- Practice, Exercises 1–13 odd
- Remediation, Review Resources: Measures of Spread
- *Quick Review Math Handbook:* The Normal Distribution
- ALEKS Measures of Center and Spread



Practice

Go Online You can complete your homework online.

Example 1

Identify the random variable in each distribution, and classify it as discrete or continuous. Explain your reasoning. 1–3. See margin.

1. the number of texts received per week
2. the number of "likes" for a Web page
3. the height of a plant after a specific amount of time

Examples 2-4

4. FUNDRAISING At a fundraising dinner, the underside of 200 plates were randomly tagged with a sticker to indicate winning a cash prize. The frequency table shows the number of winning plates for each prize. Construct a relative frequency table, and graph the probability distribution. See margin.

Prize, (X)	Frequency
$5	150
$50	40
$100	9
$1000	1

5. BASKETBALL An athletic director made a probability distribution of the heights of her team's basketball players, and distributed a flyer that claimed that the majority of the players on the basketball team are 71 inches or taller. Identify any flaws in the representation of the probability distribution. See margin.

6. TRACK The preliminary times for a 110-meter hurdles race are shown. Create a histogram of the set of data. Determine whether the data can be approximated with a normal distribution. See margin.

Times (seconds)
14.75, 14.77, 14.31, 14.83, 14.84, 14.35, 14.69, 14.63, 14.74, 14.82, 14.25, 14.93

Example 5

7. A normal distribution has a mean of 186.4 and a standard deviation of 48.9.
   a. What range of values represents the middle 99.7% of the data? $39.7 < X < 333.1$
   b. What percent of data will be greater than 235.3? 16%
   c. What range of values represents the upper 2.5% of the data? $X > 284.2$

Example 6

Find the z-value for each standard normal distribution.

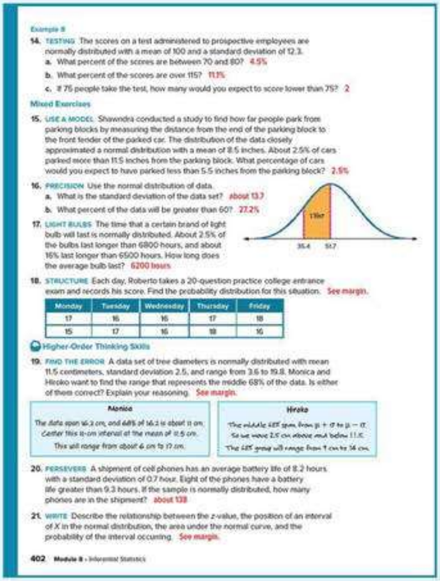
8. $\sigma = 9.8$, $X = 55.4$, and $\mu = 68.34$ -1.32
9. $\sigma = 11.6$, $X = 42.80$, and $\mu = 68.2$ -2.19
10. $\sigma = 11.9$, $X = 119.2$, and $\mu = 112.4$ 0.57

Example 7

Use a table to find the area under the normal curve for each interval.

11. $z > 0.58$ 0.281 12. $z < -1.56$ 0.0594 13. $-2.29 < z < 2.76$ 0.9861

Lesson 8-4 • Normal Distributions 401



Example 8

14. TESTING The scores on a test administered to prospective employees are normally distributed with a mean of 100 and a standard deviation of 12.3.
    a. What percent of the scores are between 70 and 80? 4.5%
    b. What percent of the scores are over 115? 11.1%
    c. If 75 people take the test, how many would you expect to score lower than 75? 2

Mixed Exercises

15. USE A MODEL Shawndra conducted a study to find how far people park from parking blocks by measuring the distance from the end of the parking block to the front fender of the parked car. The distribution of the data closely approximated a normal distribution with a mean of 8.5 inches. About 2.5% of cars parked more than 11.5 inches from the parking block. What percentage of cars would you expect to have parked less than 5.5 inches from the parking block? 2.5%

16. PRECISION Use the normal distribution of data.
    a. What is the standard deviation of the data set? about 13.7
    b. What percent of the data will be greater than 60? 27.2%

17. LIGHT BULBS The time that a certain brand of light bulb will last is normally distributed. About 2.5% of the bulbs last longer than 6800 hours, and about 16% last longer than 6500 hours. How long does the average bulb last? 6200 hours

18. STRUCTURE Each day, Roberto takes a 20-question practice college entrance exam and records his score. Find the probability distribution for this situation. See margin.

Monday	Tuesday	Wednesday	Thursday	Friday
17	16	16	17	18
15	17	16	18	16

Higher-Order Thinking Skills

19. FIND THE ERROR A data set of tree diameters is normally distributed with mean 11.5 centimeters, standard deviation 2.5, and range from 3.6 to 19.8. Monica and Hiroko want to find the range that represents the middle 68% of the data. Is either of them correct? Explain your reasoning. See margin.

Monica

The data span 16.2 cm, and 68% of 16.2 is about 11 cm.
Center this 11-cm interval at the mean of 11.5 cm.
This will range from about 6 cm to 17 cm.

Hiroko

The middle 68% span from $\mu + \sigma$ to $\mu - \sigma$.
So we move 2.5 cm above and below 11.5.
The 68% group will range from 9 cm to 14 cm.

20. PERSEVERE A shipment of cell phones has an average battery life of 8.2 hours with a standard deviation of 0.7 hour. Eight of the phones have a battery life greater than 9.3 hours. If the sample is normally distributed, how many phones are in the shipment? about 138

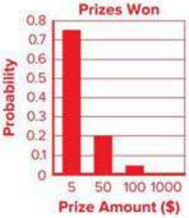
21. WRITE Describe the relationship between the z-value, the position of an interval of X in the normal distribution, the area under the normal curve, and the probability of the interval occurring. See margin.

402 Module 8 • Inferential Statistics

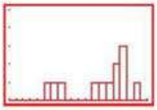
Answers

1. texts; discrete; The number of texts can only be represented as a whole number.
2. likes; discrete; The number of "likes" can only be represented as a whole number.
3. height of plant; continuous; The height of a plant may be any positive value.
4.

Prize (X)	Frequency	Relative Frequency
$5	150	0.75
$50	40	0.2
$100	9	0.045
$1000	1	0.005



5. The scale of the y-axis misleadingly shows the differences in probabilities.
6.



[14, 15] scl: 0.05 by [0, 5] scl: 1

The data set is skewed. Thus, the data cannot be approximated with the normal distribution.

18.

Score (x)	15	16	17	18	19	20
$P(X = x)$	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{0}{10}$	$\frac{0}{10}$

19. Hiroko; Sample answer: Monica’s solution would work with a uniform distribution.

21. Sample answer: The z-value represents the position of a value X in a normal distribution. The area under a curve to the left of a z-value represents the probability that a value from the distribution will be less than the given value X.

Lesson 8-5

Estimating Population Parameters

S.IC.4, S.IC.6

LESSON GOAL

Students use sample data to make inferences about populations.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Estimating the Population Mean
- Find the Maximum Error of the Estimate
- Estimate a Population Mean

Estimating the Population Proportion
- Estimate a Population Proportion
- Misleading Population Estimates

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Measures of Spread	●	●		●
Extension: Hypothesis Testing		●	●	●

Language Development Handbook

Assign page 45 of the *Language Development Handbook* to help your students build mathematical language related to using sample data to make inferences about populations.

ELL You can use the tips and suggestions on page T45 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Statistics and Probability

Standards for Mathematical Content:

S.IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.

S.IC.6 Evaluate reports based on data.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

3 Construct viable arguments and critique the reasoning of others.

Coherence

Vertical Alignment

Previous
Students used statistics and normal distributions to analyze data.
S.ID.4, S.IC.6

Now
Students use sample data to make inferences about populations.
S.IC.4, S.IC.6

Next
Students will perform calculations and graph with angles and trigonometric expressions, equations, and functions.
F.TF.3, F.TF.5

Rigor

The Three Pillars of Rigor

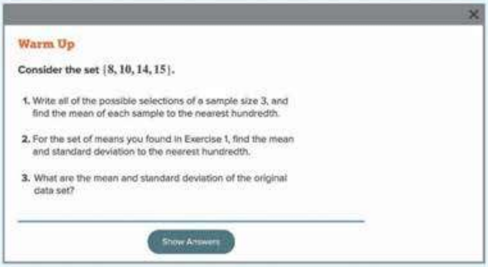
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of population mean and proportion. They build fluency and apply their understanding by solving real-world problems.

Mathematical Background

A population parameter is a statistical measure that, for a given population, is fixed and is used as the value of a variable in a general distribution to make it descriptive of that population. The mean and variance of a population are population parameters. The confidence level describes the likelihood that a particular sampling method will produce a confidence interval that includes the true population parameter. The confidence level describes the uncertainty of a sampling method.

Interactive Presentation



Warm Up

Consider the set $\{8, 10, 14, 15\}$.

1. Write all of the possible selections of a sample size 3, and find the mean of each sample to the nearest hundredth.
2. For the set of means you found in Exercise 1, find the mean and standard deviation to the nearest hundredth.
3. What are the mean and standard deviation of the original data set?

Warm Up

Estimating Population Parameters

The Positive Experience Index score is the mean of measures of the positive responses to questions asked of adults in 140 countries. Higher scores mean that positive emotions are more pervasive in a country. These scores strongly relate to people's perceptions about their living standards, personal freedoms, and the presence of social networks. Paraguay had the highest Positive Experience Index of 84, with a margin of error of 3.5 at 95% confidence. The population estimate, margin of error, and confidence level are determined from sample data.

Launch the Lesson

Vocabulary

- inferential statistics
- sampling error
- standard error of the mean
- confidence interval
- maximum error of the estimate
- population proportion

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- finding the sample mean and standard deviation

Answers:

1. {8, 10, 14}, 10.67; {8, 10, 15}, 11; {8, 14, 15}, 12.33; {10, 14, 15}, 13
2. mean 11.75; standard deviation 0.95
3. mean 11.75; standard deviation 2.86

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will learn how sample data are used to estimate the Positive Experience Index of a population.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Learn Estimating the Population Mean

Objective

Students use sample data to infer a population mean by using confidence intervals.

MP Teaching the Mathematical Practices

2 Attend to Quantities Point out that it is important to know the meaning of quantities used in statistical formulas. Throughout this lesson, encourage students to attend to the meaning of variables and mathematical results.

3 Reason Inductively Throughout this lesson, students will use inductive reasoning to make plausible arguments and inferences about a population from a sample.

DIFFERENTIATE

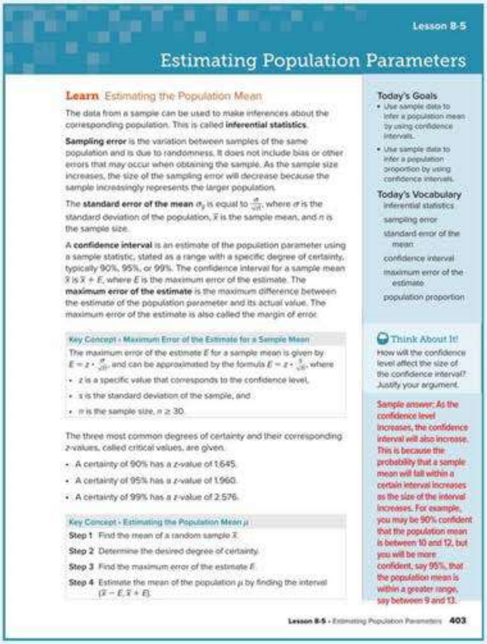
Language Development Activity AL BL ELL

IF students are having difficulty keeping track of the various key concepts and equations, many of which are similar to each other in name and structure,

THEN have the students create a list of the key terms, like standard deviation and maximum error of the estimate. Identify the variables that represent these terms, as well as equations used to calculate them.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 8-5

Estimating Population Parameters

Learn Estimating the Population Mean

The data from a sample can be used to make inferences about the corresponding population. This is called **inferential statistics**.

Sampling error is the variation between samples of the same population and is due to randomness. It does not include bias or other errors that may occur when obtaining the sample. As the sample size increases, the size of the sampling error will decrease because the sample increasingly represents the larger population.

The **standard error of the mean** $\sigma_{\bar{x}}$ is equal to $\frac{\sigma}{\sqrt{n}}$, where σ is the standard deviation of the population, $\bar{x}$ is the sample mean, and n is the sample size.

A **confidence interval** is an estimate of the population parameter using a sample statistic, stated as a range with a specific degree of certainty, typically 90%, 95%, or 99%. The confidence interval for a sample mean $\bar{x}$ is $\bar{x} \pm E$, where E is the maximum error of the estimate. The **maximum error of the estimate** is the maximum difference between the estimate of the population parameter and its actual value. The maximum error of the estimate is also called the margin of error.

Key Concept • Maximum Error of the Estimate for a Sample Mean

The maximum error of the estimate E for a sample mean is given by $E = z \cdot \frac{\sigma}{\sqrt{n}}$, and can be approximated by the formula $E = z \cdot \frac{s}{\sqrt{n}}$, where

- z is a specific value that corresponds to the confidence level,
- s is the standard deviation of the sample, and
- n is the sample size, $n \geq 30$.

The three most common degrees of certainty and their corresponding z-values, called critical values, are given.

- A certainty of 90% has a z-value of 1.645.
- A certainty of 95% has a z-value of 1.960.
- A certainty of 99% has a z-value of 2.576.

Key Concept • Estimating the Population Mean μ

Step 1 Find the mean of a random sample $\bar{x}$.

Step 2 Determine the desired degree of certainty.

Step 3 Find the maximum error of the estimate E.

Step 4 Estimate the mean of the population μ by finding the interval $(\bar{x} - E, \bar{x} + E)$.

Today's Goals

- Use sample data to infer a population mean by using confidence intervals.
- Use sample data to infer a population proportion by using confidence intervals.

Today's Vocabulary

inferential statistics
sampling error
standard error of the mean
confidence interval
maximum error of the estimate
population proportion

Think About It!

How will the confidence level affect the size of the confidence interval? Justify your argument.

Sample answer: As the confidence level increases, the confidence interval will also increase. This is because the probability that a sample mean will fall within a certain interval increases as the size of the interval increases. For example, you may be 90% confident that the population mean is between 10 and 12, but you will be more confident, say 95%, that the population mean is within a greater range, say between 9 and 13.

Lesson 8-5 • Estimating Population Parameters 403

Interactive Presentation

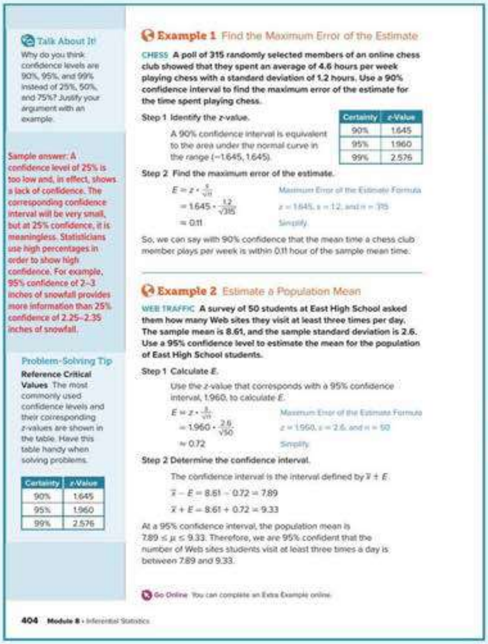
Learn

SWIPE

Students can swipe to see how the sample size n relates to the distribution of all possible sample means n.

TYPE

Students answer a question about the relationship between confidence level and confidence interval.



Talk About It!

Why do you think confidence levels are 90%, 95%, and 99% instead of 25%, 50%, and 75%? Justify your argument with an example.

Sample answer: A confidence level of 25% is too low and, in effect, shows a lack of confidence. The corresponding confidence interval will be very small, but at 25% confidence, it is meaningless. Statisticians use high percentages in order to show high confidence. For example, 95% confidence of 2–3 inches of snowfall provides more information than 25% confidence of 2.25–2.35 inches of snowfall.

Problem-Solving Tip

Reference Critical Values The most commonly used confidence levels and their corresponding z-values are shown in the table. Have this table handy when solving problems.

Certainty	z-Value
90%	1.645
95%	1.960
99%	2.576

Example 1 Find the Maximum Error of the Estimate

CHESS **A poll of 315 randomly selected members of an online chess club showed that they spent an average of 4.6 hours per week playing chess with a standard deviation of 1.2 hours. Use a 90% confidence interval to find the maximum error of the estimate for the time spent playing chess.**

Step 1 Identify the z-value.

A 90% confidence interval is equivalent to the area under the normal curve in the range (−1.645, 1.645).

Certainty	z-Value
90%	1.645
95%	1.960
99%	2.576

Step 2 Find the maximum error of the estimate.

$E = z \cdot \frac{s}{\sqrt{n}}$ Maximum Error of the Estimate Formula

$= 1.645 \cdot \frac{1.2}{\sqrt{315}}$ $z = 1.645$, $s = 1.2$, and $n = 315$

$= 0.11$ Simplify.

So, we can say with 90% confidence that the mean time a chess club member plays per week is within 0.11 hour of the sample mean time.

Example 2 Estimate a Population Mean

WEB TRAFFIC **A survey of 50 students at East High School asked them how many Web sites they visit at least three times per day. The sample mean is 8.61, and the sample standard deviation is 2.6. Use a 95% confidence level to estimate the mean for the population of East High School students.**

Step 1 Calculate E.

Use the z-value that corresponds with a 95% confidence interval, 1.960, to calculate E.

$E = z \cdot \frac{s}{\sqrt{n}}$ Maximum Error of the Estimate Formula

$= 1.960 \cdot \frac{2.6}{\sqrt{50}}$ $z = 1.960$, $s = 2.6$, and $n = 50$

≈ 0.72 Simplify.

Step 2 Determine the confidence interval.

The confidence interval is the interval defined by $\bar{x} \pm E$.

$\bar{x} - E = 8.61 - 0.72 = 7.89$

$\bar{x} + E = 8.61 + 0.72 = 9.33$

At a 95% confidence interval, the population mean is $7.89 \leq \mu \leq 9.33$. Therefore, we are 95% confident that the number of Web sites students visit at least three times a day is between 7.89 and 9.33.

Go Online You can complete an Extra Example online.

404 Module 8 • Inferential Statistics

Interactive Presentation

Example 1

TYPE

Students determine why 90%, 95%, and 99% are usually chosen for confidence intervals.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 1 Find the Maximum Error of the Estimate

Teaching the Mathematical Practices

3 Construct Arguments In the Talk About It! feature, students will use stated assumptions and definitions to construct an argument about confidence intervals.

Questions for Mathematical Discourse

AL What is the difference between σ and s? σ is the standard deviation for the entire population, and s is the standard deviation for the sample.

OL What happens to the maximum error of the estimate as the sample size increases? The maximum error of the estimate decreases as the sample size increases.

BL Without calculating it, estimate the maximum error of the estimate. Explain your reasoning. 0.165; Sample answer: Because the z-value for 99% certainty is approximately 1.5 times greater than the z-value for 90% certainty, I multiplied the maximum error of the estimate by 1.5.

Example 2 Estimate a Population Mean

Teaching the Mathematical Practices

4 Interpret Mathematical Results In Example 2, point out that to solve the problem, students should interpret their mathematical results in the context of the problem.

Questions for Mathematical Discourse

AL What is the relationship between $\bar{x}$ and μ? Sample answer: $\bar{x}$ is the sample mean, and μ is the population mean. You can use the sample mean to estimate the population mean to varying levels of confidence using the sample standard deviation, the sample size, and the z-value of the desired confidence interval.

OL Why would you expect the z-value for 95% to be close to 2? Sample answer: The empirical rule tells you that about 95% of the data is within 2 standard deviations of the mean.

BL Suppose you took multiple random samples of 50 students and looked at the distribution of the sample means. What do you think the distribution of the sample means would look like? The sample means would be normally distributed.

Learn Estimating the Population Proportion

Objective

Students use sample data to infer a population proportion by using confidence intervals.

Teaching the Mathematical Practices

2 Make Sense of Quantities Mathematically proficient students need to be able to make sense of quantities and their relationships in the context of a problem. Notice the relationship between the variables when calculating sample proportions and the maximum error of the estimate for a population proportion.

What Students Are Learning

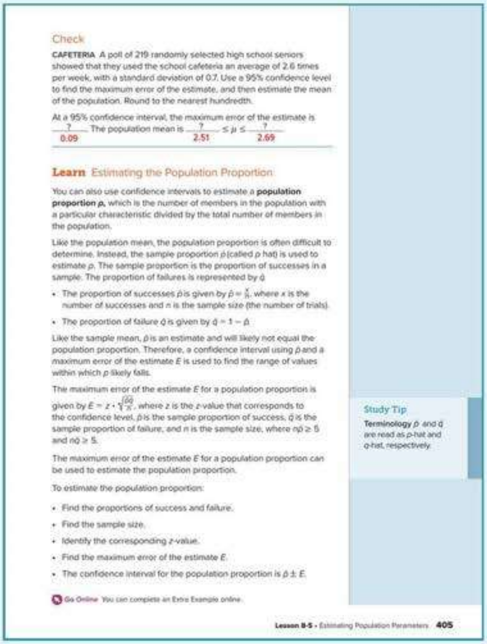
Remind students that the sample proportion is a means of estimating the population proportion. Encourage students to consider what it would take to determine the population proportion directly and to justify why using the sample proportion as an estimate is often more realistic. Remind students to consider bias and sampling errors when calculating a sample proportion.

Essential Question Follow-Up

Students have explored probability distributions.

Ask:

Can statistics lie? Sample answer: Statistics are valid only when the related biases of the sample populations are understood and taken into account. When used to make claims that involve influences which were not part of the original sample, statistics can be misleading. Statistics can "lie" when they are manipulated and used to influence the intended audience's beliefs and behaviors.



Check

CAFETERIA A poll of 219 randomly selected high school seniors showed that they used the school cafeteria an average of 2.6 times per week, with a standard deviation of 0.7. Use a 95% confidence level to find the maximum error of the estimate, and then estimate the mean of the population. Round to the nearest hundredth.

At a 95% confidence interval, the maximum error of the estimate is $\underset{0.09}{\underline{\ ?\ }}$. The population mean is $\underset{2.51}{\underline{\ ?\ }} \leq \mu \leq \underset{2.69}{\underline{\ ?\ }}$

Learn Estimating the Population Proportion

You can also use confidence intervals to estimate a **population proportion** p, which is the number of members in the population with a particular characteristic divided by the total number of members in the population.

Like the population mean, the population proportion is often difficult to determine. Instead, the sample proportion $\hat{p}$ (called p hat) is used to estimate p. The sample proportion is the proportion of successes in a sample. The proportion of failures is represented by $\hat{q}$.

- The proportion of successes $\hat{p}$ is given by $\hat{p} = \frac{x}{n}$, where x is the number of successes and n is the sample size (the number of trials).
- The proportion of failure $\hat{q}$ is given by $\hat{q} = 1 - \hat{p}$.

Like the sample mean, $\hat{p}$ is an estimate and will likely not equal the population proportion. Therefore, a confidence interval using $\hat{p}$ and a maximum error of the estimate E is used to find the range of values within which p likely falls.

The maximum error of the estimate E for a population proportion is given by $E = z \cdot \sqrt{\frac{\hat{p}\hat{q}}{n}}$, where z is the z-value that corresponds to the confidence level, $\hat{p}$ is the sample proportion of success, $\hat{q}$ is the sample proportion of failure, and n is the sample size, where $n\hat{p} \geq 5$ and $n\hat{q} \geq 5$.

The maximum error of the estimate E for a population proportion can be used to estimate the population proportion.

To estimate the population proportion:

- Find the proportions of success and failure.
- Find the sample size.
- Identify the corresponding z-value.
- Find the maximum error of the estimate E.
- The confidence interval for the population proportion is $\hat{p} \pm E$.

Go Online You can complete an Extra Example online.

Study Tip

Terminology $\hat{p}$ and $\hat{q}$ are read as p-hat and q-hat, respectively.

Lesson 8-5 • Estimating Population Parameters 405

Interactive Presentation

Learn

TAP

Students tap to see a Study Tip about terminology.

Apply Example 3 Estimate a Population Proportion

LUNCH PERIOD **A survey of 150 students at West High School asked them if they agree with the administration's plan to split the lunch period into three different periods, with 72 of the respondents saying that they agree. Use a 90% confidence level to estimate the population proportion for all 1334 West High School students.**

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to find $\hat{p}$ and $\hat{q}$ and then use those values to estimate the population proportion with 90% confidence.

Is the sample size large enough to use a confidence interval? What formulas do I need to use?

I can test the sample size. I can use the previous example within the lesson to check that I am using the correct formulas.

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I will find $\hat{p}$ and $\hat{q}$ and check that the sample size is appropriate to use a confidence interval. Next, I will find the maximum error of the estimate. Then I will use the confidence interval to find the population proportion.

3 What is your solution?

Use your strategy to solve the problem.

What are the values of $\hat{p}$ and $\hat{q}$? $\hat{p} = 0.48$; $\hat{q} = 0.52$

How can you ensure that proportions have large enough sample sizes to use a confidence interval?

Sample answer: I can verify that $n\hat{p} \geq 5$ and $n\hat{q} \geq 5$. Because the sample size was 150 students, $n\hat{p} = 150 \cdot 0.48$, or 72 and $n\hat{q} = 150 \cdot 0.52$, or 78. So the sample size is large enough.

What is the confidence interval? $CI = [0.413, 0.547]$

With 90% confidence, what proportion of students agree with the administration's plan? between 41.3% and 54.7%

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: The proportion of success of the sample falls within the range I found using the confidence interval. Because the administration used a large enough sample size, it is reasonable to assume that proportion of success within the population sample will be representative of the population.

Go Online You can complete an Extra Example online.

406 Module 8 • Inferential Statistics

Interactive Presentation

LUNCH PERIOD A survey of 150 students at West High School asked them if they agree with the administration's plan to split the lunch period into three different periods, with 72 of the respondents saying that they agree. Use a 90% confidence level to estimate the population proportion for all 1334 West High School students.

Apply Example 3

TYPE

a| Students answer questions about the steps to solving the problem.

Apply Example 3 Estimate a Population Proportion

Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them, 4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- What is the z-value that corresponds to a 90% confidence level?
- In order to find the confidence interval, what other value must be determined and what formula can be used to determine that value?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.

Example 4 Misleading Population Estimates

Teaching the Mathematical Practices

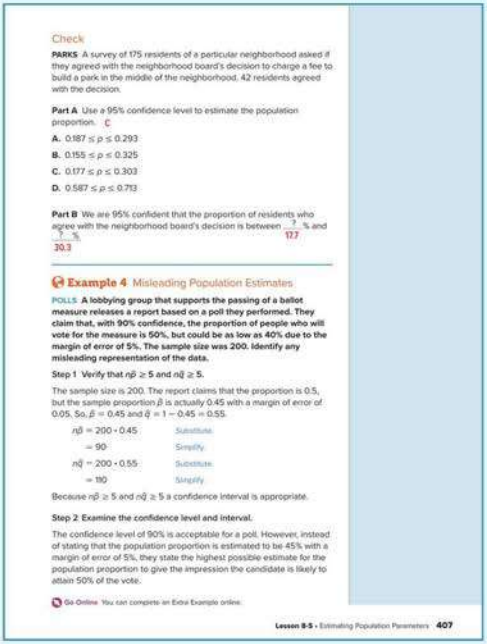
3 Find the Error Example 4 requires students to read the claims of others, decide whether they are misleading or flawed, and explain the flaw.

Questions for Mathematical Discourse

AL What does the margin of error represent in this example? the maximum error of the estimate

OL How do you know the sample proportion is 0.45? The low end of the estimate is 40% and the margin of error is 5%.

BL If the lobbying group used the same tactic with a 95% confidence interval, what impression could they give? Sample answer: They could use the larger interval to claim the proportion will be above 50%.



Check

PARKS A survey of 175 residents of a particular neighborhood asked if they agreed with the neighborhood board's decision to charge a fee to build a park in the middle of the neighborhood. 42 residents agreed with the decision.

Part A Use a 95% confidence level to estimate the population proportion. C

A. $0.187 \le p \le 0.293$

B. $0.155 \le p \le 0.325$

C. $0.177 \le p \le 0.303$

D. $0.587 \le p \le 0.713$

Part B We are 95% confident that the proportion of residents who agree with the neighborhood board's decision is between $\underset{17.7}{\underline{\ ?\ }}$ % and $\underset{30.3}{\underline{\ ?\ }}$ %.

Example 4 Misleading Population Estimates

POLLS **A lobbying group that supports the passing of a ballot measure releases a report based on a poll they performed. They claim that, with 90% confidence, the proportion of people who will vote for the measure is 50%, but could be as low as 40% due to the margin of error of 5%. The sample size was 200. Identify any misleading representation of the data.**

Step 1 Verify that $n\hat{p} \ge 5$ and $n\hat{q} \ge 5$.

The sample size is 200. The report claims that the proportion is 0.5, but the sample proportion $\hat{p}$ is actually 0.45 with a margin of error of 0.05. So, $\hat{p} = 0.45$ and $\hat{q} = 1 - 0.45 = 0.55$.

$$
\begin{aligned}
n\hat{p} &= 200 \cdot 0.45 && \text{Substitute.}\\
&= 90 && \text{Simplify.}\\
n\hat{q} &= 200 \cdot 0.55 && \text{Substitute.}\\
&= 110 && \text{Simplify.}
\end{aligned}
$$

Because $n\hat{p} \ge 5$ and $n\hat{q} \ge 5$ a confidence interval is appropriate.

Step 2 Examine the confidence level and interval.

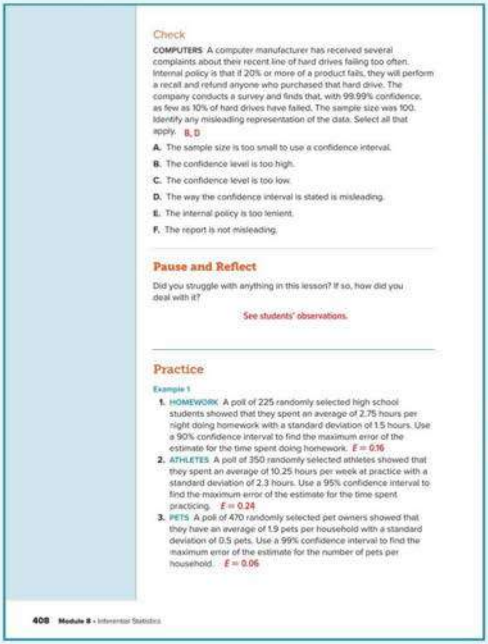
The confidence level of 90% is acceptable for a poll. However, instead of stating that the population proportion is estimated to be 45% with a margin of error of 5%, they state the highest possible estimate for the population proportion to give the impression the candidate is likely to attain 50% of the vote.

Go Online You can complete an Extra Example online.

Lesson 8-5 • Estimating Population Parameters **407**

Interactive Presentation

Example 4



Check

COMPUTERS A computer manufacturer has received several complaints about their recent line of hard drives failing too often. Internal policy is that if 20% or more of a product fails, they will perform a recall and refund anyone who purchased that hard drive. The company conducts a survey and finds that, with 99.99% confidence, as few as 10% of hard drives have failed. The sample size was 100. Identify any misleading representation of the data. Select all that apply. B, D

A. The sample size is too small to use a confidence interval.

B. The confidence level is too high.

C. The confidence level is too low.

D. The way the confidence interval is stated is misleading.

E. The internal policy is too lenient.

F. The report is not misleading.

Pause and Reflect

Did you struggle with anything in this lesson? If so, how did you deal with it?

See students' observations.

Practice

Example 1

1. HOMEWORK A poll of 225 randomly selected high school students showed that they spent an average of 2.75 hours per night doing homework with a standard deviation of 1.5 hours. Use a 90% confidence interval to find the maximum error of the estimate for the time spent doing homework. $E = 0.16$
2. ATHLETES A poll of 350 randomly selected athletes showed that they spent an average of 10.25 hours per week at practice with a standard deviation of 2.3 hours. Use a 95% confidence interval to find the maximum error of the estimate for the time spent practicing. $E = 0.24$
3. PETS A poll of 470 randomly selected pet owners showed that they have an average of 1.9 pets per household with a standard deviation of 0.5 pets. Use a 99% confidence interval to find the maximum error of the estimate for the number of pets per household. $E = 0.06$

408 Module 8 • Inferential Statistics

Interactive Presentation

Question 2

Check

CHECK

Students complete the Check online to determine whether they are ready to move on.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

3 REFLECT AND PRACTICE

S.IC.4, S.IC.6

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–11
2	exercises that use a variety of skills from this lesson	12–14
3	exercises that emphasize higher-order and critical-thinking skills	15–17

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **BL**
THEN assign:

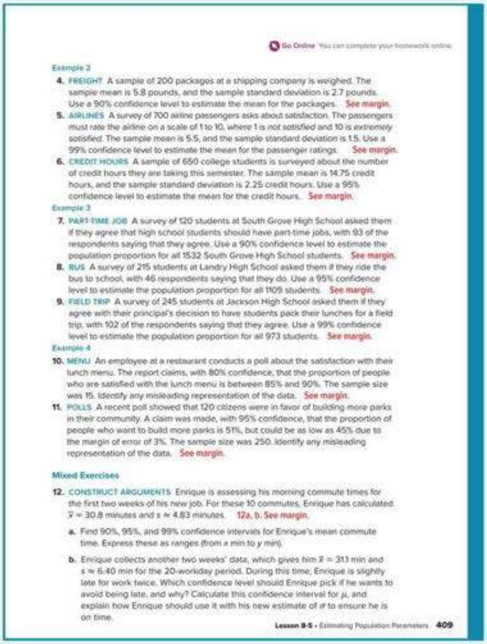
- Practice, Exercises 1–13 odd, 15–17
- Extension: Hypothesis Testing

IF students score 66%–89% on the Checks, **OL**
THEN assign:

- Practice, Exercises 1–17 odd
- Remediation, Review Resources: Measures of Spread
- Personal Tutors
- Extra Examples 1–4
- ALEKS Measures of Center and Spread

IF students score 65% or less on the Checks, **AL**
THEN assign:

- Practice, Exercises 1–11 odd
- Remediation, Review Resources: Measures of Spread
- *Quick Review Math Handbook*: Simulations and Margin of Error
- ALEKS Measures of Center and Spread



Go Online You can complete your homework online.

Example 2

4. FREIGHT A sample of 200 packages at a shipping company is weighed. The sample mean is 5.8 pounds, and the sample standard deviation is 2.7 pounds. Use a 90% confidence level to estimate the mean for the packages. See margin.

5. AIRLINES A survey of 700 airline passengers asks about satisfaction. The passengers must rate the airline on a scale of 1 to 10, where 1 is *not satisfied* and 10 is *extremely satisfied*. The sample mean is 5.5, and the sample standard deviation is 1.5. Use a 99% confidence level to estimate the mean for the passenger ratings. See margin.

6. CREDIT HOURS A sample of 650 college students is surveyed about the number of credit hours they are taking this semester. The sample mean is 14.75 credit hours, and the sample standard deviation is 2.25 credit hours. Use a 95% confidence level to estimate the mean for the credit hours. See margin.

Example 3

7. PART-TIME JOB A survey of 120 students at South Grove High School asked them if they agree that high school students should have part-time jobs, with 93 of the respondents saying that they agree. Use a 90% confidence level to estimate the population proportion for all 1532 South Grove High School students. See margin.

8. BUS A survey of 215 students at Landry High School asked them if they ride the bus to school, with 46 respondents saying that they do. Use a 95% confidence level to estimate the population proportion for all 1109 students. See margin.

9. FIELD TRIP A survey of 245 students at Jackson High School asked them if they agree with their principal's decision to have students pack their lunches for a field trip, with 102 of the respondents saying that they agree. Use a 99% confidence level to estimate the population proportion for all 973 students. See margin.

Example 4

10. MENU An employee at a restaurant conducts a poll about the satisfaction with their lunch menu. The report claims, with 80% confidence, that the proportion of people who are satisfied with the lunch menu is between 85% and 90%. The sample size was 15. Identify any misleading representation of the data. See margin.

11. POLLS A recent poll showed that 120 citizens were in favor of building more parks in their community. A claim was made, with 95% confidence, that the proportion of people who want to build more parks is 51%, but could be as low as 45% due to the margin of error of 3%. The sample size was 250. Identify any misleading representation of the data. See margin.

Mixed Exercises

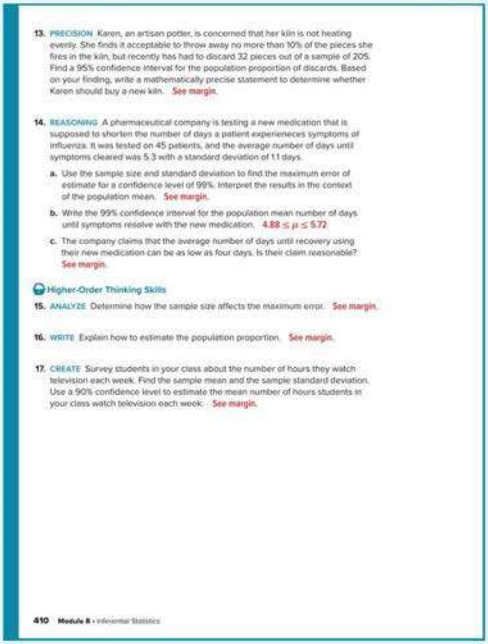
12. CONSTRUCT ARGUMENTS Enrique is assessing his morning commute times for the first two weeks of his new job. For these 10 commutes, Enrique has calculated $\bar{x} \approx 30.8$ minutes and $s \approx 4.83$ minutes. 12a, b. See margin.
    a. Find 90%, 95%, and 99% confidence intervals for Enrique's mean commute time. Express these as ranges (from x min to y min).
    b. Enrique collects another two weeks' data, which gives him $\bar{x} = 31.1$ min and $s \approx 6.40$ min for the 20-workday period. During this time, Enrique is slightly late for work twice. Which confidence level should Enrique pick if he wants to avoid being late, and why? Calculate this confidence interval for μ, and explain how Enrique should use it with his new estimate of σ to ensure he is on time.

Lesson 8-5 • Estimating Population Parameters 409

3 REFLECT AND PRACTICE

S.IC.4, S.IC.6

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**



13. PRECISION Karen, an artisan potter, is concerned that her kiln is not heating evenly. She finds it acceptable to throw away no more than 10% of the pieces she fires in the kiln, but recently has had to discard 32 pieces out of a sample of 205. Find a 95% confidence interval for the population proportion of discards. Based on your finding, write a mathematically precise statement to determine whether Karen should buy a new kiln. See margin.

14. REASONING A pharmaceutical company is testing a new medication that is supposed to shorten the number of days a patient experieneces symptoms of influenza. It was tested on 45 patients, and the average number of days until symptoms cleared was 5.3 with a standard deviation of 1.1 days.
    a. Use the sample size and standard deviation to find the maximum error of estimate for a confidence level of 99%. Interpret the results in the context of the population mean. See margin.
    b. Write the 99% confidence interval for the population mean number of days until symptoms resolve with the new medication. $4.88 \leq \mu \leq 5.72$
    c. The company claims that the average number of days until recovery using their new medication can be as low as four days. Is their claim reasonable? See margin.

Higher-Order Thinking Skills

15. ANALYZE Determine how the sample size affects the maximum error. See margin.

16. WRITE Explain how to estimate the population proportion. See margin.

17. CREATE Survey students in your class about the number of hours they watch television each week. Find the sample mean and the sample standard deviation. Use a 90% confidence level to estimate the mean number of hours students in your class watch television each week. See margin.

Answers

4. At a 90% confidence interval, the population mean is $5.49 \leq \mu \leq 6.11$. Therefore, we are 90% confident that the weight of packages is between 5.49 pounds and 6.11 pounds.
5. At a 99% confidence interval, the population mean is $5.35 \leq \mu \leq 5.65$. Therefore, we are 99% confident that the rating of the airline is between 5.35 and 5.65.
6. At a 95% confidence interval, the population mean is $14.58 \leq \mu \leq 14.92$. Therefore, we are 95% confident that the number of credit hours taken by a college student is between 14.58 and 14.92.
7. At a 90% confidence interval, the population proportion is $0.712 \leq p \leq 0.838$. Therefore, we are 90% confident that the proportion of students who agree that high school students should have a part-time job is between 71.2% and 83.8%.
8. At a 95% confidence interval, the population proportion is $0.159 \leq p \leq 0.269$. Therefore, we are 95% confident that the proportion of students who ride the bus to school is between 15.9% and 26.9%.
9. At a 99% confidence interval, the population proportion is $0.335 \leq p \leq 0.497$. Therefore, we are 99% confident that the proportion of students who agree with the principal's plan is between 33.5% and 49.7%.
10. The sample size is too small to use a confidence interval, and the confidence level is too low.
11. Instead of stating that the population proportion is estimated at $\frac{120}{250} = 0.48 = 48\%$ with a margin of error of 3%, the claim uses the highest possible estimate for the population proportion to give the impression that the proportion of people who want parks to be built is greater than 50%.

12a. 90%: $E = 2.5$, so the confidence interval is from 28.3 min to 33.3 min
95%: $E = 3.0$, so the confidence interval is from 27.8 min to 33.8 min
99%: $E = 3.9$, so the confidence interval is from 26.9 min to 34.7 min

12b. Sample answer: Enrique should use a 99% confidence level to provide information he can really count on. $E = 2.58 \cdot \frac{6.40}{\sqrt{20}} \approx 3.7$, so $CI = 31.1 \pm 3.7$; Enrique can be 99% confident his mean journey time is no longer than 34.8 min, and adding 15 min safety margin based on doubling the S.D. of 6.4 min he obtained, 50 min should ensure he is on time.

13. $\hat{p} = \frac{32}{205} \approx 0.156$ and $ME = 1.96\sqrt{\frac{0.156(1 - 0.156)}{205}} \approx 0.050$, so $CI = 15.6\% \pm 5.0\%$; Sample answer: With 95% confidence, the mean proportion of discards for the population of all pieces fired in her kiln is between 10.6% and 20.6%, so Karen should probably buy a new kiln on the basis that the discard rate is most likely higher than 10%.

14a. $E = 0.42$; Sample answer: You can be 99% confident that the mean number of days of recovery for all patients taking this drug lies within 0.42 days of 5.3.

14c. Because 4 lies outside the 99% confidence interval, it is not reasonable. It is possible because 1% of cases will fall outside of the interval, but it is not reasonable.

15. As the sample size increases, the maximum error decreases.
16. To estimate the population proportion, first find the proportions of success and failure. Then find the sample size. Next, identify the corresponding z-value. Then, find the maximum error of the estimate E. The confidence interval for the population proportion is $\hat{p} \pm E$.
17. Sample answer: sample mean: 18.8 hours; sample standard deviation: 8.8; the mean number of hours with a 90% confidence level is $15.4 \leq \mu \leq 22.2$.

Hours of Television Watched Each Week

12	8	19	27	20	18
0	33	25	14	9	19
23	6	20	26	30	29

Module 8 • Inferential Statistics

Review

Rate Yourself

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have them respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- How can you use information to make decisions?
- How can probability be used in decision making?
- Can statistics lie?

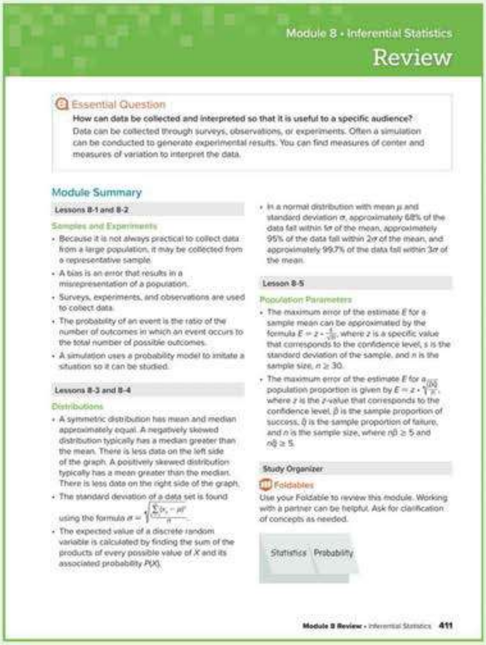
Then have them write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

ELL A completed Foldable for this module should include the key concepts related to statistics and probability.

LS **LearnSmart** Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Inferences and Conclusions from Data.**

- Data Measurements
- Data Collection Methods
- Inferences and Conclusions



Module 8 • Inferential Statistics

Review

Essential Question

How can data be collected and interpreted so that it is useful to a specific audience?

Data can be collected through surveys, observations, or experiments. Often a simulation can be conducted to generate experimental results. You can find measures of center and measures of variation to interpret the data.

Module Summary

Lessons 8-1 and 8-2

Samples and Experiments

- Because it is not always practical to collect data from a large population, it may be collected from a representative sample.
- A bias is an error that results in a misrepresentation of a population.
- Surveys, experiments, and observations are used to collect data.
- The probability of an event is the ratio of the number of outcomes in which an event occurs to the total number of possible outcomes.
- A simulation uses a probability model to imitate a situation so it can be studied.

Lessons 8-3 and 8-4

Distributions

- A symmetric distribution has mean and median approximately equal. A negatively skewed distribution typically has a median greater than the mean. There is less data on the left side of the graph. A positively skewed distribution typically has a mean greater than the median. There is less data on the right side of the graph.
- The standard deviation of a data set is found using the formula $\sigma = \sqrt{\frac{\sum (x_i - \mu)^2}{n}}$.
- The expected value of a discrete random variable is calculated by finding the sum of the products of every possible value of X and its associated probability $P(X)$.
- In a normal distribution with mean μ and standard deviation σ, approximately 68% of the data fall within 1σ of the mean, approximately 95% of the data fall within 2σ of the mean, and approximately 99.7% of the data fall within 3σ of the mean.

Lesson 8-5

Population Parameters

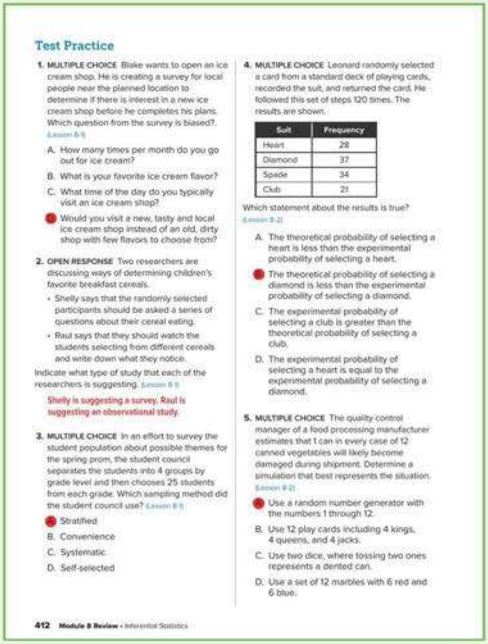
- The maximum error of the estimate E for a sample mean can be approximated by the formula $E = z \cdot \frac{s}{\sqrt{n}}$, where z is a specific value that corresponds to the confidence level, s is the standard deviation of the sample, and n is the sample size, $n \geq 30$.
- The maximum error of the estimate E for a population proportion is given by $E = z \cdot \sqrt{\frac{\hat{p}\hat{q}}{n}}$, where z is the z-value that corresponds to the confidence level, $\hat{p}$ is the sample proportion of success, $\hat{q}$ is the sample proportion of failure, and n is the sample size, where $n\hat{p} \geq 5$ and $n\hat{q} \geq 5$.

Study Organizer

Foldables

Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.

Module 8 Review • Inferential Statistics 411



Test Practice

1. **MULTIPLE CHOICE** Blake wants to open an ice cream shop. He is creating a survey for local people near the planned location to determine if there is interest in a new ice cream shop before he completes his plans. Which question from the survey is biased?. (Lesson 8-1)

   A. How many times per month do you go out for ice cream?

   B. What is your favorite ice cream flavor?

   C. What time of the day do you typically visit an ice cream shop?

   [✓ D.] Would you visit a new, tasty and local ice cream shop instead of an old, dirty shop with few flavors to choose from?

2. **OPEN RESPONSE** Two researchers are discussing ways of determining children's favorite breakfast cereals.

   - Shelly says that the randomly selected participants should be asked a series of questions about their cereal eating.
   - Raul says that they should watch the students selecting from different cereals and write down what they notice.

Indicate what type of study that each of the researchers is suggesting. (Lesson 8-1)

Shelly is suggesting a survey. Raul is suggesting an observational study.

3. **MULTIPLE CHOICE** In an effort to survey the student population about possible themes for the spring prom, the student council separates the students into 4 groups by grade level and then chooses 25 students from each grade. Which sampling method did the student council use? (Lesson 8-1)

   [✓ A.] Stratified

   B. Convenience

   C. Systematic

   D. Self-selected

4. **MULTIPLE CHOICE** Leonard randomly selected a card from a standard deck of playing cards, recorded the suit, and returned the card. He followed this set of steps 120 times. The results are shown.

Suit	Frequency
Heart	28
Diamond	37
Spade	34
Club	21

Which statement about the results is true? (Lesson 8-2)

   A. The theoretical probability of selecting a heart is less than the experimental probability of selecting a heart.

   [✓ B.] The theoretical probability of selecting a diamond is less than the experimental probability of selecting a diamond.

   C. The experimental probability of selecting a club is greater than the theoretical probability of selecting a club.

   D. The experimental probability of selecting a heart is equal to the experimental probability of selecting a diamond.

5. **MULTIPLE CHOICE** The quality control manager of a food processing manufacturer estimates that 1 can in every case of 12 canned vegetables will likely become damaged during shipment. Determine a simulation that best represents the situation. (Lesson 8-2)

   [✓ A.] Use a random number generator with the numbers 1 through 12.

   B. Use 12 play cards including 4 kings, 4 queens, and 4 jacks.

   C. Use two dice, where tossing two ones represents a dented can.

   D. Use a set of 12 marbles with 6 red and 6 blue.

412 Module 8 Review • Inferential Statistics

Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Vocabulary Activity
Module Review

Assessment Resources

Vocabulary Test
AL Module Test Form B
OL Module Test Form A
BL Module Test Form C
Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

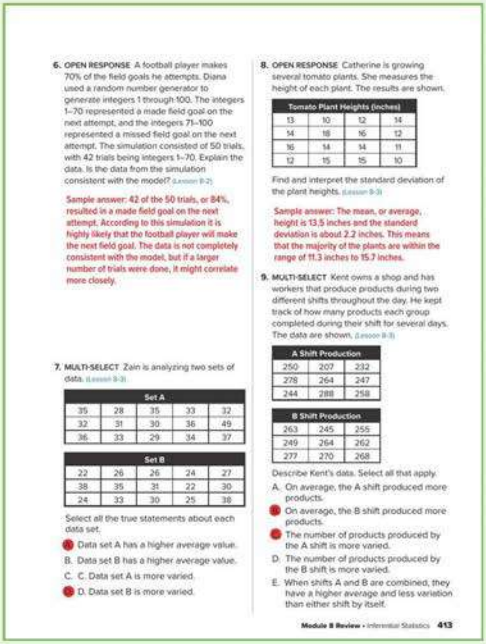
Test Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1–15 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Multiple Choice	Students select one correct answer.	1, 3–5, 12, 14
Multi-Select	Multiple answers may be correct. Students must select all correct answers.	9
Table Item	Students complete a table by entering the correct values.	2, 7
Open Response	Students construct their own response.	6, 8, 10, 11, 13

To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
S.IC.1	8-1	1
S.IC.2	8-2	4, 5
S.IC.3	8-1	2, 3
S.IC.4	8-3, 8-5	7, 8, 9, 12, 13
S.IC.5	8-2	6
S.IC.6	8-5	14
S.ID.4	8-4	10, 11



6. **OPEN RESPONSE** A football player makes 70% of the field goals he attempts. Diana used a random number generator to generate integers 1 through 100. The integers 1–70 represented a made field goal on the next attempt, and the integers 71–100 represented a missed field goal on the next attempt. The simulation consisted of 50 trials, with 42 trials being integers 1–70. Explain the data. Is the data from the simulation consistent with the model? (Lesson 8-2)

Sample answer: 42 of the 50 trials, or 84%, resulted in a made field goal on the next attempt. According to this simulation it is highly likely that the football player will make the next field goal. The data is not completely consistent with the model, but if a larger number of trials were done, it might correlate more closely.

7. **MULTI-SELECT** Zain is analyzing two sets of data. (Lesson 8-3)

Set A				
35	28	35	33	32
32	31	30	36	49
36	33	29	34	37

Set B				
22	26	26	24	27
38	35	31	22	30
24	33	30	25	38

Select all the true statements about each data set.

● Data set A has a higher average value.
B. Data set B has a higher average value.
C. C. Data set A is more varied.
● D. Data set B is more varied.

8. **OPEN RESPONSE** Catherine is growing several tomato plants. She measures the height of each plant. The results are shown.

Tomato Plant Heights (inches)			
13	10	12	14
14	18	16	12
16	14	14	11
12	15	15	10

Find and interpret the standard deviation of the plant heights. (Lesson 8-3)

Sample answer: The mean, or average, height is 13.5 inches and the standard deviation is about 2.2 inches. This means that the majority of the plants are within the range of 11.3 inches to 15.7 inches.

9. **MULTI-SELECT** Kent owns a shop and has workers that produce products during two different shifts throughout the day. He kept track of how many products each group completed during their shift for several days. The data are shown. (Lesson 8-3)

A Shift Production		
250	207	232
278	264	247
244	288	258

B Shift Production		
263	245	255
249	264	262
277	270	268

Describe Kent's data. Select all that apply.

A. On average, the A shift produced more products.
● On average, the B shift produced more products.
● The number of products produced by the A shift is more varied.
D. The number of products produced by the B shift is more varied.
E. When shifts A and B are combined, they have a higher average and less variation than either shift by itself.

Module 8 Review • Inferential Statistics 413

Lesson 8-2

17a. The theoretical probably that she serves an ace on her next serve is 66%, and the theoretical probability that she doesn't is 34%. Use a random number generator to generate integers 1 through 100. The integers 1–66 will represent an ace, and the integers 67–100 will represent any other outcome of her serve. The simulation will consist of 30 trials.

17b. Sample data is shown:

14	16	8	94	9	87
41	41	68	42	66	11
74	38	86	41	26	92
28	36	4	12	25	68
24	4	76	3	61	35

17c. Sample answer: 22 of the 30 trials, or 73.3%, resulted in an ace on the next serve. According to this simulation, it is highly likely that her next serve will be an ace. The data is not completely consistent with the model, but if a larger trial were done, it might correlate more closely.

18. Coin toss; I recognized that there are only two choices of candidates, so the pattern of a coin toss is closest to voting.

19. Sample answer: Jevon's concern about the fairness of the coin should be dependent on the number of times he tossed it. If he completed only 4, or even 20, trials, then the sample size is not large enough to warrant concern. However, if he completed 100 or more trials, then he should be concerned since the experimental probability should be closer to the theoretical probability of $\frac{1}{2}$, or 50%.

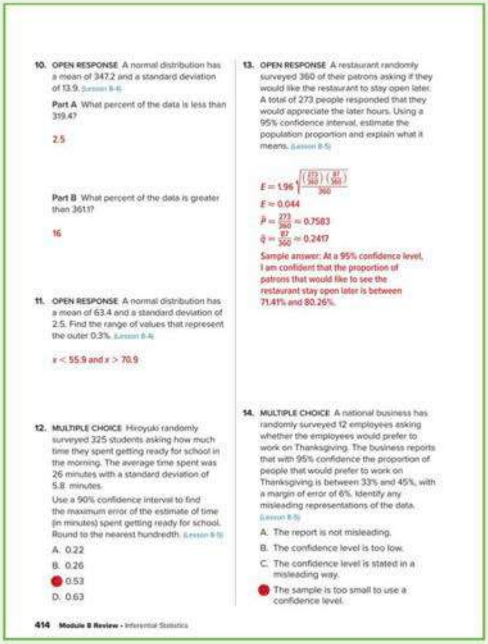
20. Never; sample answer: The theoretical probability of getting heads from a coin is one-half, or 7.5 times out of 15 tosses. However, when you toss a coin 15 times, getting 7.5 heads is not possible.

21. True; sample answer: The event will always occur when the theoretical probability is 1. Therefore, the experimental probability can never be 0. For example, the theoretical probability of rolling a die and getting 1, 2, 3, 4, 5, or 6 is 1 since it will always show one of these six numbers. So, when a die is rolled, the result will be one of these six numbers, making the experimental probability always greater than 0.

22. Sample answer: You should consider the design of the simulation, how many trials were used, and whether the theoretical and experimental probabilities are reasonably close.

23. Yes; sample answer: If the spinner were going to be divided equally into three outcomes, each sector would measure 120. Because you only want to know the probability of outcome *C*, you can record spins that end in the red area as a success, or the occurrence of outcome *C*, and spins that end in the blue area as a failure, or an outcome of *A* or *B*.

24. Sometimes; sample answer: Flipping a coin can be used to simulate an experiment with two possible outcomes when both outcomes are equally likely. If the probabilities of the occurrence of the two outcomes are different, flipping a coin is not an appropriate simulation.



10. **OPEN RESPONSE** A normal distribution has a mean of 347.2 and a standard deviation of 13.9. (Lesson 8-4)

Part A What percent of the data is less than 319.4?

2.5

Part B What percent of the data is greater than 361.1?

16

11. **OPEN RESPONSE** A normal distribution has a mean of 63.4 and a standard deviation of 2.5. Find the range of values that represent the outer 0.3%. (Lesson 8-4)

$x < 55.9$ and $x > 70.9$

12. **MULTIPLE CHOICE** Hiroyuki randomly surveyed 325 students asking how much time they spent getting ready for school in the morning. The average time spent was 26 minutes with a standard deviation of 5.8 minutes.

Use a 90% confidence interval to find the maximum error of the estimate of time (in minutes) spent getting ready for school. Round to the nearest hundredth. (Lesson 8-5)

A. 0.22

B. 0.26

● 0.53

D. 0.63

13. **OPEN RESPONSE** A restaurant randomly surveyed 360 of their patrons asking if they would like the restaurant to stay open later. A total of 273 people responded that they would appreciate the later hours. Using a 95% confidence interval, estimate the population proportion and explain what it means. (Lesson 8-5)

$E = 1.96\sqrt{\frac{\left(\frac{273}{360}\right)\left(\frac{87}{360}\right)}{360}}$

$E \approx 0.044$

$\hat{p} = \frac{273}{360} \approx 0.7583$

$\hat{q} = \frac{87}{360} \approx 0.2417$

Sample answer: At a 95% confidence level, I am confident that the proportion of patrons that would like to see the restaurant stay open later is between 71.41% and 80.26%.

14. **MULTIPLE CHOICE** A national business has randomly surveyed 12 employees asking whether the employees would prefer to work on Thanksgiving. The business reports that with 95% confidence the proportion of people that would prefer to work on Thanksgiving is between 33% and 45%, with a margin of error of 6%. Identify any misleading representations of the data. (Lesson 8-5)

A. The report is not misleading.

B. The confidence level is too low.

C. The confidence level is stated in a misleading way.

● The sample is too small to use a confidence level.

414 **Module 8 Review** • Inferential Statistics

Module 9

Trigonometric Functions

Module Goals

- Students draw angles in standard position and convert between degree and radian measures of angles.
- Students find the values of trigonometric functions by using general angles and reference angles.
- Students find trigonometric values by using the unit circle and the properties of periodic functions.
- Students graph and analyze sine and cosine functions.
- Students graph and analyze tangent and reciprocal trigonometric functions.

Focus

Domain: Functions

Standards for Mathematical Content:

F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Also addresses F.IF.4, F.TF.1, F.TF.3, F.TF.5, F.TF.7, and F.BF.3.

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this module.

Coherence

Vertical Alignment

Previous

Students solved problems involving right triangles using the Pythagorean Theorem and trigonometric ratios.

8.G.7, 8.G.8, G.SRT.8 (Course 2)

Now

Students evaluate and graph trigonometric functions.

F.TF.2, F.TF.5

Rigor

The Three Pillars of Rigor

To help students meet standards, they need to illustrate their ability to use the three pillars of rigor. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. Once they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Independent Practice.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

EXPLORE → **LEARN** → **EXAMPLE & PRACTICE**

Suggested Pacing

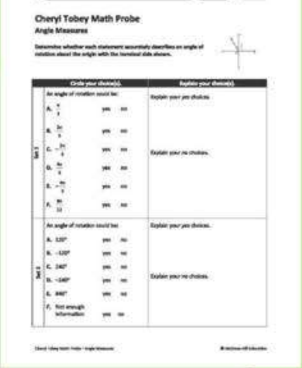
Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
9-1 Angles and Angle Measure	F.TF.1	2	1
9-2 Trigonometric Functions of General Angles	F.TF.3	2	1
9-3 Circular and Periodic Functions	F.TF.2, F.TF.5	1	0.5
9-4 Graphing Sine and Cosine Functions	F.IF.4; F.IF.7e	2	1
9-5 Graphing Other Trigonometric Functions	F.IF.4; F.IF.7e	1	0.5
9-6 Translations of Trigonometric Graphs	F.IF.7e; F.BF.3	1	0.5
9-7 Inverse Trigonometric Functions	F.TF.7	1	0.5
Module Review		1	0.5
Module Assessment		1	0.5
	Total Days	**13**	**6.5**

CHERYL TOBEY MATH PROBES

Formative Assessment Math Probe

Angle Measures

Module Resource



Correct Answers:
Set 1. No: A, C, D; Yes: B, E, F
Set 2. No: B, C, F; Yes: A, D, E

Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will determine whether statements accurately describe angles of rotation.

Targeted Concepts Understand the structure of the unit circle and angles of rotation using degrees and radians.

Targeted Misconceptions

- Students may confuse clockwise rotation with positive angle measures and counterclockwise rotation with negative angle measures.
- Students may confuse angles of rotation with reference angles.
- Students may have difficulty with radian measures that have different denominators than the one used in a reference angle.
- Students may not know how to use degree measures when a radian measure is given.
- Students may not understand that angle of rotation can be greater than 360° (or less than −360°).

Use the Probe after Lesson 9-1.

Collect and Assess Student Answers

If the student selects these responses...		**Then** the student likely...
1B. no **1C.** yes **1D.** yes **1E.** no	**2A.** no **2B.** yes **2C.** yes **2D.** no	believes that a clockwise rotation is positive and a counterclockwise rotation is negative.
1A. yes		confuses the reference angle with the angle of rotation/angle measure.
1F. no		is having difficulty with a denominator different from what is given in the reference angle.
2E. no		does not think that the angle of rotation can be greater than 360°.
2F. yes		does not understand how to change a radian measure into a degree measure.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

- ALEKS Angles
- Lesson 9-1, all Learns, all Examples

Revisit the Probe at the end of the module to be sure that your students no longer carry these misconceptions.

IGNITE!

The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

What are the key features of the graph of a trigonometric function and how do they represent real-world situations? Sample answer: Trigonometric functions have amplitudes, periods, and frequencies. This allows real-world cyclical situations to be modeled. The amplitude identifies maximum or minimum values. The period often tells the time to complete one cycle. The frequency tells the number of cycles the function completes in a given interval.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

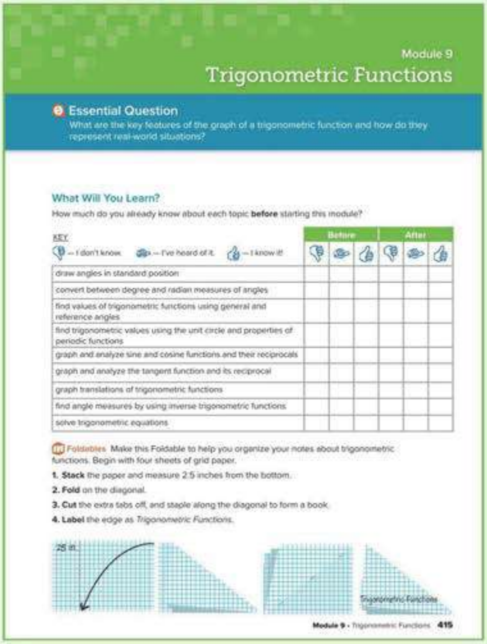
Focus Students write notes about new terms and concepts as they are presented in each lesson of this module.

Teach Have students construct their Foldable as illustrated. Have students write an explanation of each term or concept on the appropriate section of their Foldable while working through each lesson. Encourage students to record examples of each term or concept on the back of each flap.

When to Use It Encourage students to add to their Foldable as they work through the module, and to use it to review for the module test.

Launch the Module

For this module, the Launch the Module video uses trigonometric functions to model the rides at an amusement park. Students learn the basic components of trigonometric functions in a real-world situation.



Module 9

Trigonometric Functions

Essential Question

What are the key features of the graph of a trigonometric function and how do they represent real-world situations?

What Will You Learn?

How much do you already know about each topic **before** starting this module?

KEY

👎 — I don't know. 🤚 — I've heard of it. 👍 — I know it!

	Before			After		
	👎	🤚	👍	👎	🤚	👍
draw angles in standard position						
convert between degree and radian measures of angles						
find values of trigonometric functions using general and reference angles						
find trigonometric values using the unit circle and properties of periodic functions						
graph and analyze sine and cosine functions and their reciprocals						
graph and analyze the tangent function and its reciprocal						
graph translations of trigonometric functions						
find angle measures by using inverse trigonometric functions						
solve trigonometric equations						

Foldables Make this Foldable to help you organize your notes about trigonometric functions. Begin with four sheets of grid paper.

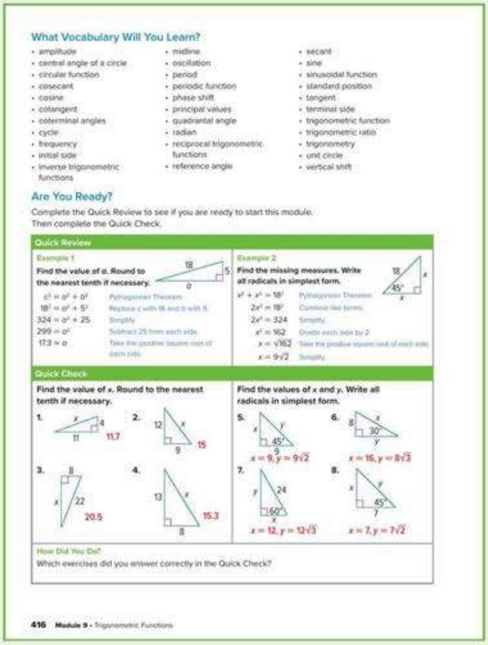
1. **Stack** the paper and measure 2.5 inches from the bottom.
2. **Fold** on the diagonal.
3. **Cut** the extra tabs off, and staple along the diagonal to form a book.
4. **Label** the edge as *Trigonometric Functions*.

2.5 in.

Trigonometric Functions

Module 9 • Trigonometric Functions 415

Interactive Presentation



What Vocabulary Will You Learn?

- amplitude
- central angle of a circle
- circular function
- cosecant
- cosine
- cotangent
- coterminal angles
- cycle
- frequency
- initial side
- inverse trigonometric functions
- midline
- oscillation
- period
- periodic function
- phase shift
- principal values
- quadrantal angle
- radian
- reciprocal trigonometric functions
- reference angle
- secant
- sine
- sinusoidal function
- standard position
- tangent
- terminal side
- trigonometric function
- trigonometric ratio
- trigonometry
- unit circle
- vertical shift

Are You Ready?

Complete the Quick Review to see if you are ready to start this module. Then complete the Quick Check.

Quick Review

Example 1

Find the value of *a*. Round to the nearest tenth if necessary.

$c^2 = a^2 + b^2$ Pythagorean Theorem
$18^2 = a^2 + 5^2$ Replace c with 18 and b with 5.
$324 = a^2 + 25$ Simplify.
$299 = a^2$ Subtract 25 from each side.
$17.3 \approx a$ Take the positive square root of each side.

Example 2

Find the missing measures. Write all radicals in simplest form.

$x^2 + x^2 = 18^2$ Pythagorean Theorem
$2x^2 = 18^2$ Combine like terms.
$2x^2 = 324$ Simplify.
$x^2 = 162$ Divide each side by 2.
$x = \sqrt{162}$ Take the positive square root of each side.
$x = 9\sqrt{2}$ Simplify.

Quick Check

Find the value of *x*. Round to the nearest tenth if necessary.

1. 11.7
2. 15
3. 20.5
4. 15.3

Find the values of *x* and *y*. Write all radicals in simplest form.

5. $x = 9, y = 9\sqrt{2}$
6. $x = 16, y = 8\sqrt{3}$
7. $x = 12, y = 12\sqrt{3}$
8. $x = 7, y = 7\sqrt{2}$

How Did You Do?

Which exercises did you answer correctly in the Quick Check?

416 **Module 9** • Trigonometric Functions

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define The phase shift of a graph is the horizontal translation of the graph of a trigonometric function; $y = \sin(x - h)$

Example $g(x) = \sin(x - \pi)$

Ask What is the value of h? π

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- converting measurements
- solving trigonometric ratios
- using circle terms
- finding function values
- finding vertical asymptotes
- predicting transformations
- finding inverses of functions

ALEKS

ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You can use the ALEKS pie report to see which students know the topics in the **Trigonometry** module—who is ready to learn these topics and who isn't quite ready to learn them yet—in order to adjust your instruction as appropriate.

Mindset Matters

Collaborative Risk Taking

Some students may be averse to taking risks during math class, like sharing an idea, strategy, or solution. They may worry about their grades or scores on tests, or some might feel less confident solving math problems, especially in front of their peers.

How Can I Apply It?

Assign the **Practice** problems of each lesson and encourage students to take risks as they solve problems, try new paths, and discuss their strategies with their partner or group.

Lesson 9-1

Angles and Angle Measures

F.TF.1

LESSON GOAL

Students draw angles in standard position and convert between degree and radian measures of angles.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Angles in Standard Position
- Draw an Angle in Standard Position
- Draw an Angle with More Than One Rotation
- Identify Coterminal Angles

Explore: Arc Length

Degrees and Radians
- Convert Degrees to Radians
- Convert Radians to Degrees
- Find Arc Length

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

Formative Assessment Math Probe

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Convert Customary Measurements	●	●		●
Extension: Planetary Orbits		● ●	●	
ELL Support				

Language Development Handbook

Assign page 46 of the *Language Development Handbook* to help your students build mathematical language related to converting between degree and radian measures of angles.

ELL You can use the tips and suggestions on page T46 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Functions

Standards for Mathematical Content:

F.TF1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

5 Use appropriate tools strategically.

Coherence

Vertical Alignment

Previous
Students know the circumference of a circle with radius 1 unit as 2π.
7.G.4

Now
Students draw angles in standard position and convert between degree and radian measures of angles.
F.TF.1

Next
Students will find values of trigonometric functions by using general angles and reference angles.
F.TF.3

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of radian measure. They build fluency by converting between degree and radian measures of angles, and they apply their understanding by solving real-world problems.

Mathematical Background

An angle on a coordinate plane is in standard position if one ray of the angle (initial side) is placed on the positive *x*-axis and the other ray (terminal side) rotates about the origin. The terminal side rotates counterclockwise to show an angle with a positive measure and rotates clockwise to show an angle with a negative measure.

Interactive Presentation

Warm Up

Convert each measure to the indicated unit by using the given conversion factors. Round to the nearest tenth.

1 mile ≈ 1.6 kilometers 1 ounce ≈ 28.35 grams
1 kilogram ≈ 2.2 pounds

1. 42 miles to kilometers
2. 42 kilometers to miles
3. 12 ounces to grams
4. 12 grams to ounces
5. 18.5 kilograms to pounds
6. 18.5 pounds to kilograms

Warm Up

Launch the Lesson

The geographic coordinate system (GCS) uses measures of latitude and longitude to determine locations on Earth. Given in degrees, these measures can be used to approximate distances between locations around the globe.

Launch the Lesson

Vocabulary

- central angle of a circle
- coterminal angles
- initial side
- radian
- standard position
- terminal side

- How can you select *radian* mode or *degree* mode on your calculator? Which is the default mode?
- Under which conditions can two angles be *coterminal angles*?
- What is the difference between degrees and radians?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- converting measurements

Answers:

1. 67.2 km
2. 26.3 mi
3. 340.2 g
4. 0.4 oz
5. 40.7 lb
6. 8.4 kg

Launch the Lesson

Teaching the Mathematical Practices

4 Make Assumptions Have students explain an assumption or approximation that must be made when calculating distance using the geographic coordinate system.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Arc Length

Objective

Students use a sketch to explore radians and the relationship between central angles and arc length.

Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. Throughout the Explore, students justify their conclusions about the relationships between a central angle, radius, and arc length.

5 Use Mathematical Tools Encourage students to use the sketch to explore the relationship between central angles and the lengths of intercepted arcs.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? Y ou may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of Activity

Students will be presented with an Inquiry Question to answer at the end of the activity. They will use a sketch to explore the relationship between central angles and the lengths of the generated arcs. Students will work through three different exercises. Then students will answer the Inquiry Question.

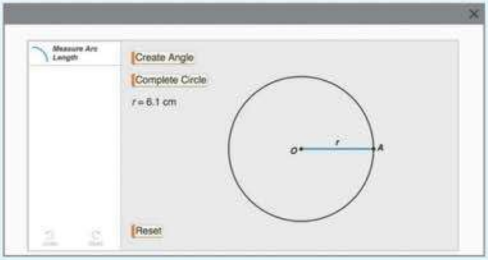
(continued on the next page)

Interactive Presentation

Arc Length

INQUIRY How can you use a central angle to determine the length of an arc?

Explore



Explore

WEB SKETCHPAD

Students use the sketch to explore the relationship between the measure of a central angle and the length of the arc created.

TYPE

Students move through the exercises and answer questions pertaining to the generated angles and arcs.

Interactive Presentation

INQUIRY How can you use a central angle to determine the length of an arc?

Explore

TYPE

Students will respond to the Inquiry Question and can view a sample answer.

Explore Arc Length (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What is the length of $\overset{\frown}{AC}$ in centimeters? 12.2 cm What is the length in terms of r? $2r$
- What is the length of the arc, in terms of r, generated by two radii that form a diameter of the circle? πr

Inquiry

How can you use a central angle to determine the length of an arc? Sample answer: Multiply the measure of the central angle in radians by the radius of the circle to find the length of the arc that the angle intercepts.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Angles in Standard Position

Objective

Students draw angles in standard position and identify coterminal angles.

MP Teaching the Mathematical Practices

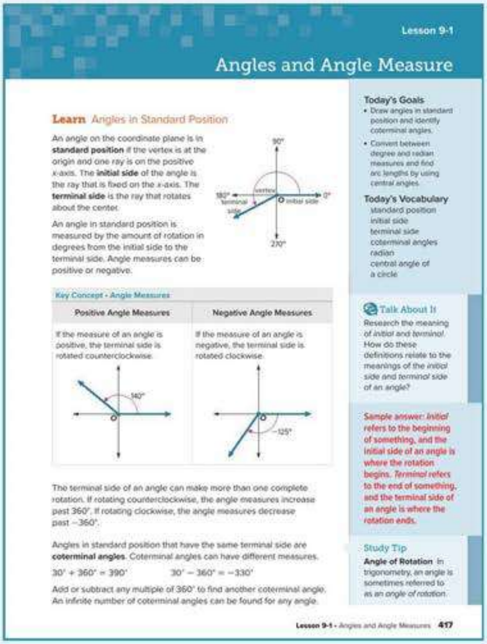
6 Communicate Precisely Encourage students to routinely write and explain their reasoning. Students should use clear definitions and mathematical language when they discuss their answer to the question in the Talk About It! feature.

Common Misconception

A common misconception some students may have is that a negative angle cannot have the same measure as a positive angle. A student may not realize that −75° shares the same measure as a 75° angle, but with a different rotation. Have students graph several pairs of positive and negative angles with the same measure to help solidify the concept.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 9-1

Angles and Angle Measure

Learn Angles in Standard Position

An angle on the coordinate plane is in **standard position** if the vertex is at the origin and one ray is on the positive x-axis. The **initial side** of the angle is the ray that is fixed on the x-axis. The **terminal side** is the ray that rotates about the center.

An angle in standard position is measured by the amount of rotation in degrees from the initial side to the terminal side. Angle measures can be positive or negative.

Key Concept • Angle Measures

Positive Angle Measures	Negative Angle Measures
If the measure of an angle is positive, the terminal side is rotated counterclockwise.	If the measure of an angle is negative, the terminal side is rotated clockwise.

The terminal side of an angle can make more than one complete rotation. If rotating counterclockwise, the angle measures increase past 360°. If rotating clockwise, the angle measures decrease past −360°.

Angles in standard position that have the same terminal side are **coterminal angles**. Coterminal angles can have different measures.

30° + 360° = 390°　　30° − 360° = −330°

Add or subtract any multiple of 360° to find another coterminal angle. An infinite number of coterminal angles can be found for any angle.

Today's Goals
- Draw angles in standard position and identify coterminal angles.
- Convert between degree and radian measures and find arc lengths by using central angles.

Today's Vocabulary
standard position
initial side
terminal side
coterminal angles
radian
central angle of a circle

Talk About It!

Research the meaning of *initial* and *terminal*. How do these definitions relate to the meanings of the *initial side* and *terminal side* of an angle?

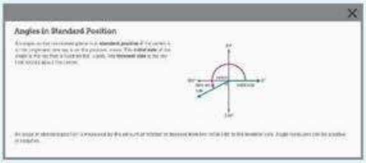
Sample answer: *Initial* refers to the beginning of something, and the initial side of an angle is where the rotation begins. *Terminal* refers to the end of something, and the terminal side of an angle is where the rotation ends.

Study Tip

Angle of Rotation In trigonometry, an angle is sometimes referred to as an *angle of rotation*.

Lesson 9-1 • Angles and Angle Measures **417**

Interactive Presentation



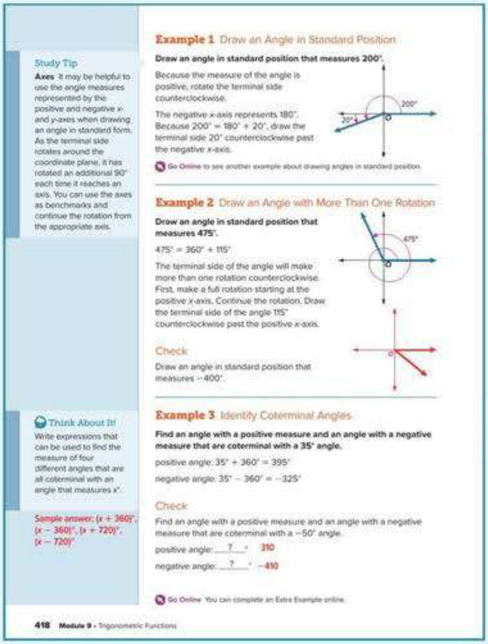
Angles in Standard Position

Learn

TYPE

a

Students research the meaning of *initial* and *terminal* as they are related to angles.



Study Tip

Axes It may be helpful to use the angle measures represented by the positive and negative *x*- and *y*-axes when drawing an angle in standard form. As the terminal side rotates around the coordinate plane, it has rotated an additional 90° each time it reaches an axis. You can use the axes as benchmarks and continue the rotation from the appropriate axis.

Example 1 Draw an Angle in Standard Position

Draw an angle in standard position that measures 200°.

Because the measure of the angle is positive, rotate the terminal side counterclockwise.

The negative *x*-axis represents 180°. Because 200° = 180° + 20°, draw the terminal side 20° counterclockwise past the negative *x*-axis.

Go Online to see another example about drawing angles in standard position.

Example 2 Draw an Angle with More Than One Rotation

Draw an angle in standard position that measures 475°.

475° = 360° + 115°

The terminal side of the angle will make more than one rotation counterclockwise. First, make a full rotation starting at the positive *x*-axis. Continue the rotation. Draw the terminal side of the angle 115° counterclockwise past the positive *x*-axis.

Check

Draw an angle in standard position that measures −400°.

Think About It!

Write expressions that can be used to find the measure of four different angles that are all coterminal with an angle that measures *x*°.

Sample answer: (*x* + 360)°, (*x* − 360)°, (*x* + 720)°, (*x* − 720)°

Example 3 Identify Coterminal Angles

Find an angle with a positive measure and an angle with a negative measure that are coterminal with a 35° angle.

positive angle: 35° + 360° = 395°

negative angle: 35° − 360° = −325°

Check

Find an angle with a positive measure and an angle with a negative measure that are coterminal with a −50° angle.

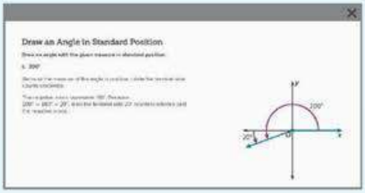
positive angle: __?__° 310

negative angle: __?__° −410

Go Online You can complete an Extra Example online.

418 Module 9 • Trigonometric Functions

Interactive Presentation



Draw an Angle in Standard Position

Example 1

SELECT

Students select the correct rotation in order to draw the given angle.

WEB SKETCHPAD

Students use a sketch to draw the given angle.

Example 1 Draw an Angle in Standard Position

MP Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to draw the angle in Example 1 students will need to use a protractor or the sketch online.

Questions for Mathematical Discourse

AL Rotating a steering wheel or a screw are often described using left and right. Why are clockwise and counterclockwise used to describe the rotation of the terminal side of the angle? Using left, right, up, or down does not make sense because the direction changes as you rotate. When people use left and right to describe rotation, it is a convention based on the direction from the top, or positive *y*-axis.

OL How would drawing a 70° angle compare to drawing a −70° angle? The amount of rotation from the positive *x*-axis is the same for both angles, but the positive angle rotates counterclockwise and the negative angle rotates clockwise.

BL For the 200° angle, how many degrees is the terminal side from the negative *y*-axis? Explain two ways to find the answer. 70°; You can take 90° − 20° for the quadrant, or 270° − 200° if you measure from the initial side.

Example 2 Draw an Angle with More Than One Rotation

MP Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated things as single objects or as being composed of several objects. In Example 2, guide students to see the angle as a 360° rotation plus an additional rotation.

Questions for Mathematical Discourse

AL How do you know when an angle involves more than one full rotation? A full rotation is 360°, so an angle with measure greater than 360° or less than −360° will be more than one full rotation.

OL Why is 475° broken into 360° + 115°? One full rotation is 360°, so the remaining rotation past the positive *x*-axis is 115°.

BL Do angles measuring −475° and 475° share the same terminal side when drawn? Explain. No; sample answer: Both angles involve one full rotation plus 115° degrees, but they are in opposite directions. The terminal side of the positive angle is in the second quadrant, while the terminal side of the negative angle is in the third quadrant.

Example 3 Identify Coterminal Angles

Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in the way coterminal angles are calculated to answer the question in the Think About It! feature.

Questions for Mathematical Discourse

AL When the angle 395° is drawn, how does it compare to the angle 35°? Sample answer: The terminal sides are in the same position. 395 degrees is one full rotation plus 35 degrees.

OL How would you find an angle with a negative measure that is coterminal with a 405° angle? Subtract 360° more than once.

BL How would you test whether two angles are coterminal? Check whether the difference between the angles is a multiple of 360°.

Learn Degrees and Radians

Objective

Students convert between degree measures and radian measures and find arc lengths by using central angles.

Teaching the Mathematical Practices

6 Use Precision In the Think About It! feature, students will accurately calculate the degree measure of 1 radian and express the answer with an appropriate degree of precision.

Common Misconception

A common misconception some students may have is that the ratio $\frac{\pi}{180^\circ}$ is always used to convert angles, regardless of their unit of measure. Remind students that the ratio must be set up to eliminate the given unit of measure.

Example 4 Convert Degrees to Radians

Teaching the Mathematical Practices

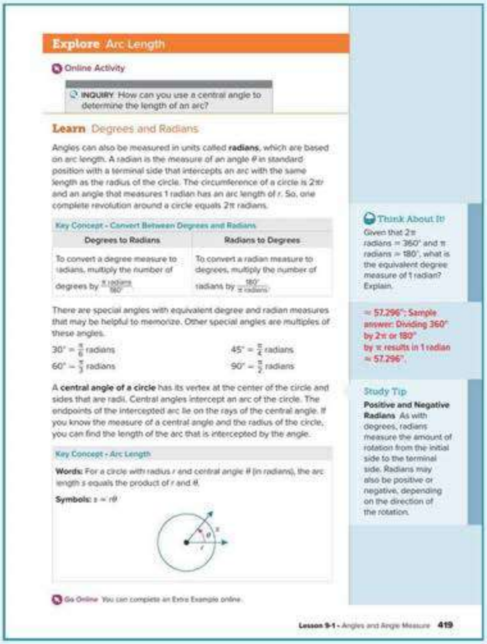
2 Make Sense of Quantities Mathematically proficient students need to be able to make sense of quantities and their relationships. Encourage students to note the relationship between degree and radian measures.

Questions for Mathematical Discourse

AL Why must 180° be in the denominator of the conversion factor? so that the degree units can divide out

OL Why is 180° used in the conversion factor when a circle has 360°? A circle has 2π radians, and $\frac{2\pi \text{ radians}}{360^\circ} = \frac{\pi \text{ radians}}{180^\circ}$.

BL Why are radians expressed in terms of π rather than being expressed as a numerical value? π is an irrational number, so the exact way to express it is by leaving it as π.



Explore Arc Length

Online Activity

INQUIRY How can you use a central angle to determine the length of an arc?

Learn Degrees and Radians

Angles can also be measured in units called **radians**, which are based on arc length. A radian is the measure of an angle θ in standard position with a terminal side that intercepts an arc with the same length as the radius of the circle. The circumference of a circle is $2\pi r$ and an angle that measures 1 radian has an arc length of r. So, one complete revolution around a circle equals 2π radians.

Key Concept • Convert Between Degrees and Radians

Degrees to Radians	Radians to Degrees
To convert a degree measure to radians, multiply the number of degrees by $\frac{\pi \text{ radians}}{180^\circ}$.	To convert a radian measure to degrees, multiply the number of radians by $\frac{180^\circ}{\pi \text{ radians}}$.

There are special angles with equivalent degree and radian measures that may be helpful to memorize. Other special angles are multiples of these angles.

$30^\circ = \frac{\pi}{6}$ radians

$45^\circ = \frac{\pi}{4}$ radians

$60^\circ = \frac{\pi}{3}$ radians

$90^\circ = \frac{\pi}{2}$ radians

A **central angle of a circle** has its vertex at the center of the circle and sides that are radii. Central angles intercept an arc of the circle. The endpoints of the intercepted arc lie on the rays of the central angle. If you know the measure of a central angle and the radius of the circle, you can find the length of the arc that is intercepted by the angle.

Key Concept • Arc Length

Words: For a circle with radius r and central angle θ (in radians), the arc length s equals the product of r and θ.

Symbols: $s = r\theta$

Go Online You can complete an Extra Example online.

Think About It!

Given that 2π radians = 360° and π radians = 180°, what is the equivalent degree measure of 1 radian? Explain.

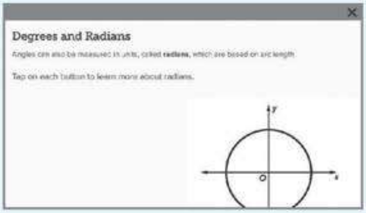
≈ 57.296°; Sample answer: Dividing 360° by 2π or 180° by π results in 1 radian ≈ 57.296°.

Study Tip

Positive and Negative Radians As with degrees, radians measure the amount of rotation from the initial side to the terminal side. Radians may also be positive or negative, depending on the direction of the rotation.

Lesson 9-1 • Angles and Angle Measures 419

Interactive Presentation



Degrees and Radians

Angles can also be measured in units, called **radians**, which are based on arc length.

Tap on each button to learn more about radians.

Learn

TAP

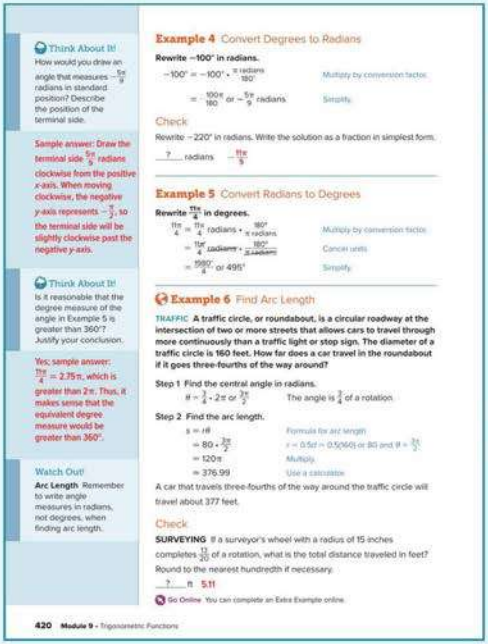
Students tap each button to learn more about radians.

TYPE

Students determine the degree equivalent of 1 radian.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Think About It!
How would you draw an angle that measures $-\frac{5\pi}{9}$ radians in standard position? Describe the position of the terminal side.

Sample answer: Draw the terminal side $\frac{5\pi}{9}$ radians clockwise from the positive x-axis. When moving clockwise, the negative y-axis represents $-\frac{\pi}{2}$, so the terminal side will be slightly clockwise past the negative y-axis.

Think About It!
Is it reasonable that the degree measure of the angle in Example 5 is greater than 360°? Justify your conclusion.

Yes; sample answer: $\frac{11\pi}{4} = 2.75\pi$, which is greater than 2π. Thus, it makes sense that the equivalent degree measure would be greater than 360°.

Watch Out!
Arc Length Remember to write angle measures in radians, not degrees, when finding arc length.

Example 4 Convert Degrees to Radians

Rewrite −100° in radians.

$-100° = -100° \cdot \frac{\pi \text{ radians}}{180°}$ Multiply by conversion factor.

$= -\frac{100\pi}{180}$ or $-\frac{5\pi}{9}$ radians Simplify.

Check

Rewrite −220° in radians. Write the solution as a fraction in simplest form.

___?___ radians $-\frac{11\pi}{9}$

Example 5 Convert Radians to Degrees

Rewrite $\frac{11\pi}{4}$ in degrees.

$\frac{11\pi}{4} = \frac{11\pi}{4}$ radians $\cdot \frac{180°}{\pi \text{ radians}}$ Multiply by conversion factor.

$= \frac{11\pi}{4} \cancel{\text{radians}} \cdot \frac{180°}{\cancel{\pi \text{ radians}}}$ Cancel units.

$= \frac{1980°}{4}$ or 495° Simplify.

Example 6 Find Arc Length

TRAFFIC **A traffic circle, or roundabout, is a circular roadway at the intersection of two or more streets that allows cars to travel through more continuously than a traffic light or stop sign. The diameter of a traffic circle is 160 feet. How far does a car travel in the roundabout if it goes three-fourths of the way around?**

Step 1 Find the central angle in radians.

$\theta = \frac{3}{4} \cdot 2\pi$ or $\frac{3\pi}{2}$ The angle is $\frac{3}{4}$ of a rotation.

Step 2 Find the arc length.

$s = r\theta$ Formula for arc length

$= 80 \cdot \frac{3\pi}{2}$ $r = 0.5d = 0.5(160)$ or 80 and $\theta = \frac{3\pi}{2}$

$= 120\pi$ Multiply.

≈ 376.99 Use a calculator.

A car that travels three-fourths of the way around the traffic circle will travel about 377 feet.

Check

SURVEYING If a surveyor's wheel with a radius of 15 inches completes $\frac{13}{20}$ of a rotation, what is the total distance traveled in feet? Round to the nearest hundredth if necessary.

___?___ ft 5.11

Go Online You can complete an Extra Example online.

420 Module 9 • Trigonometric Functions

Interactive Presentation

Convert Degrees to Radians

Example 4

TYPE

Students explain how to draw an angle given in radians.

Example 5 Convert Radians to Degrees

Teaching the Mathematical Practices

8 Attend to Details Mathematically proficient students continually ask themselves, "Does this make sense?" In the Think About It! feature, students will evaluate the reasonableness of their answer.

Questions for Mathematical Discourse

AL What is different about the conversion factor in this example compared to the previous example of converting degrees to radians? The conversion factor is flipped because you are converting radians to degrees, so the radian units must cancel.

OL Why are you able to multiply by a conversion factor without changing the measure of the angle? Multiplying by a conversion factor is effectively multiplying by 1; $180° = \pi$ radians, so $\frac{180°}{\pi \text{ radians}} = 1$

BL Explain how to use the equivalent degree and radian measures for special angles to rewrite $\frac{7\pi}{6}$ in degrees. Sample answer: $\frac{\pi}{6} = 30°$, $\frac{7\pi}{6} = 7(30°) = 210°$

Example 6 Find Arc Length

Teaching the Mathematical Practices

4 Interpret Mathematical Results In Example 6, point out that to solve the problem, students should interpret their mathematical results in the context of the problem.

Questions for Mathematical Discourse

AL Why is $\theta = \frac{3\pi}{2}$ for rotating $\frac{3}{4}$ of the circle? The angle of rotation for a full circle is 2π, so $\frac{3}{4} \bullet 2\pi = \frac{3\pi}{2}$.

OL How do the equations for arc length and circumference relate? The arc length for the full circle of 2π radians is $2\pi r$, which is equal to the circumference.

BL If r is in feet and θ is in radians, why is s in feet and not feet-radians? A radian is a dimensionless quantity defined as the ratio of the arc length and the radius.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

3 REFLECT AND PRACTICE

F.TF.1

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–53
2	exercises that use a variety of skills from this lesson	54–62
2	exercises that extend concepts learned in this lesson to new contexts	63–77
3	exercises that emphasize higher-order and critical-thinking skills	78–83

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: BL

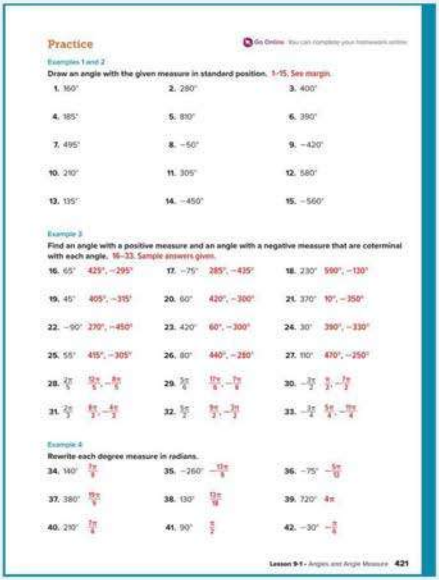
- Practice Exercises 1–77 odd, 78–83
- Extension: Degrees, Minutes, and Seconds
- **ALEKS** Angles and Their Measure

IF students score 66%–89% on the Checks, **THEN** assign: OL

- Practice Exercises 1–83 odd
- Remediation, Review Resources: Convert Customary Measurement Units
- Personal Tutors
- Extra Examples 1–6
- **ALEKS** U.S. Customary Units of Measurement

IF students score 65% or less on the Checks, **THEN** assign: AL

- Practice Exercises 1–53 odd
- Remediation, Review Resources: Convert Customary Measurement Units
- *Quick Review Math Handbook:* Angles and Angle Measure
- **ALEKS** U.S. Customary Units of Measurement



Practice

Go Online You can complete your homework online.

Examples 1 and 2

Draw an angle with the given measure in standard position. 1–15. See margin.

1. 160° 2. 280° 3. 400°
4. 185° 5. 810° 6. 390°
7. 495° 8. −50° 9. −420°
10. 210° 11. 305° 12. 580°
13. 135° 14. −450° 15. −560°

Example 3

Find an angle with a positive measure and an angle with a negative measure that are coterminal with each angle. 16–33. Sample answers given.

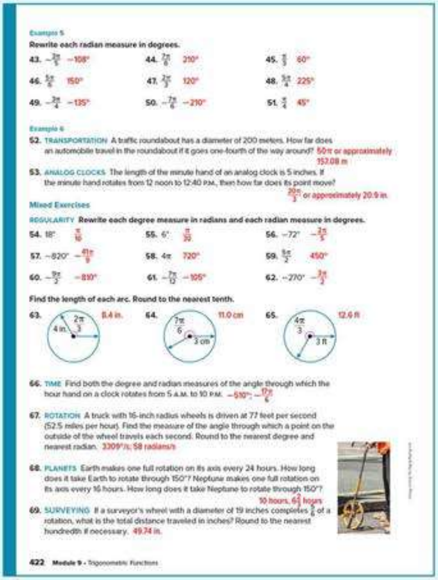
16. 65° 425°, −295° 17. −75° 285°, −435° 18. 230° 590°, −130°
19. 45° 405°, −315° 20. 60° 420°, −300° 21. 370° 10°, −350°
22. −90° 270°, −450° 23. 420° 60°, −300° 24. 30° 390°, −330°
25. 55° 415°, −305° 26. 80° 440°, −280° 27. 110° 470°, −250°
28. $\frac{2\pi}{5}$ $\frac{12\pi}{5}, -\frac{8\pi}{5}$ 29. $\frac{5\pi}{6}$ $\frac{17\pi}{6}, -\frac{7\pi}{6}$ 30. $-\frac{3\pi}{2}$ $\frac{\pi}{2}, -\frac{7\pi}{2}$
31. $\frac{2\pi}{3}$ $\frac{8\pi}{3}, -\frac{4\pi}{3}$ 32. $\frac{5\pi}{2}$ $\frac{9\pi}{2}, -\frac{3\pi}{2}$ 33. $-\frac{3\pi}{4}$ $\frac{5\pi}{4}, -\frac{11\pi}{4}$

Example 4

Rewrite each degree measure in radians.

34. 140° $\frac{7\pi}{9}$ 35. −260° $-\frac{13\pi}{9}$ 36. −75° $-\frac{5\pi}{12}$
37. 380° $\frac{19\pi}{9}$ 38. 130° $\frac{13\pi}{18}$ 39. 720° 4π
40. 210° $\frac{7\pi}{6}$ 41. 90° $\frac{\pi}{2}$ 42. −30° $-\frac{\pi}{6}$

Lesson 9-1 • Angles and Angle Measure 421



Example 5

Rewrite each radian measure in degrees.

43. $-\frac{3\pi}{5}$ −108° 44. $\frac{7\pi}{6}$ 210° 45. $\frac{\pi}{3}$ 60°
46. $\frac{5\pi}{6}$ 150° 47. $\frac{2\pi}{3}$ 120° 48. $\frac{5\pi}{4}$ 225°
49. $-\frac{3\pi}{4}$ −135° 50. $-\frac{7\pi}{6}$ −210° 51. $\frac{\pi}{4}$ 45°

Example 6

52. TRANSPORTATION A traffic roundabout has a diameter of 200 meters. How far does an automobile travel in the roundabout if it goes one-fourth of the way around? 50π or approximately 157.08 m

53. ANALOG CLOCKS The length of the minute hand of an analog clock is 5 inches. If the minute hand rotates from 12 noon to 12:40 P.M., then how far does its point move? $\frac{20\pi}{3}$ or approximately 20.9 in.

Mixed Exercises

REGULARITY Rewrite each degree measure in radians and each radian measure in degrees.

54. 18° $\frac{\pi}{10}$ 55. 6° $\frac{\pi}{30}$ 56. −72° $-\frac{2\pi}{5}$
57. −820° $-\frac{41\pi}{9}$ 58. 4π 720° 59. $\frac{5\pi}{2}$ 450°
60. $-\frac{9\pi}{2}$ −810° 61. $-\frac{7\pi}{12}$ −105° 62. −270° $-\frac{3\pi}{2}$

Find the length of each arc. Round to the nearest tenth.

63. 8.4 in. 64. 11.0 cm 65. 12.6 ft

66. TIME Find both the degree and radian measures of the angle through which the hour hand on a clock rotates from 5 A.M. to 10 P.M. −510°; $-\frac{17\pi}{6}$

67. ROTATION A truck with 16-inch radius wheels is driven at 77 feet per second (52.5 miles per hour). Find the measure of the angle through which a point on the outside of the wheel travels each second. Round to the nearest degree and nearest radian. 3309°/s; 58 radians/s

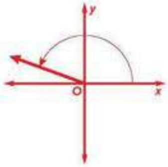
68. PLANETS Earth makes one full rotation on its axis every 24 hours. How long does it take Earth to rotate through 150°? Neptune makes one full rotation on its axis every 16 hours. How long does it take Neptune to rotate through 150°? 10 hours, $6\frac{2}{3}$ hours

69. SURVEYING If a surveyor's wheel with a diameter of 19 inches completes $\frac{5}{6}$ of a rotation, what is the total distance traveled in inches? Round to the nearest hundredth if necessary. 49.74 in.

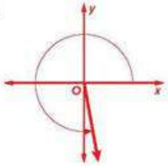
422 Module 9 • Trigonometric Functions

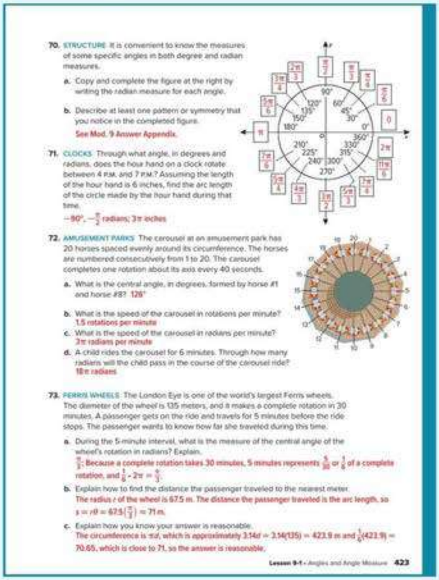
Answers

1.



2.





70. STRUCTURE It is convenient to know the measures of some specific angles in both degree and radian measures.

a. Copy and complete the figure at the right by writing the radian measure for each angle.

b. Describe at least one pattern or symmetry that you notice in the completed figure.

See Mod. 9 Answer Appendix.

71. CLOCKS Through what angle, in degrees and radians, does the hour hand on a clock rotate between 4 P.M. and 7 P.M.? Assuming the length of the hour hand is 6 inches, find the arc length of the circle made by the hour hand during that time.

−90°, $-\frac{\pi}{2}$ radians; 3π inches

72. AMUSEMENT PARKS The carousel at an amusement park has 20 horses spaced evenly around its circumference. The horses are numbered consecutively from 1 to 20. The carousel completes one rotation about its axis every 40 seconds.

a. What is the central angle, in degrees, formed by horse #1 and horse #8? 126°

b. What is the speed of the carousel in rotations per minute? 1.5 rotations per minute

c. What is the speed of the carousel in radians per minute? 3π radians per minute

d. A child rides the carousel for 6 minutes. Through how many radians will the child pass in the course of the carousel ride? 18π radians

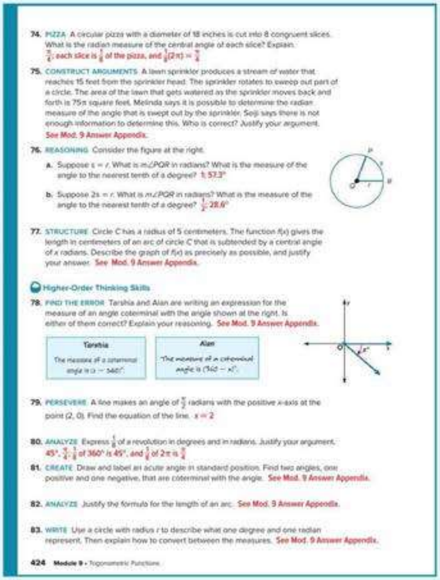
73. FERRIS WHEELS The London Eye is one of the world's largest Ferris wheels. The diameter of the wheel is 135 meters, and it makes a complete rotation in 30 minutes. A passenger gets on the ride and travels for 5 minutes before the ride stops. The passenger wants to know how far she traveled during this time.

a. During the 5-minute interval, what is the measure of the central angle of the wheel's rotation in radians? Explain.
$\frac{\pi}{3}$; Because a complete rotation takes 30 minutes, 5 minutes represents $\frac{5}{30}$ or $\frac{1}{6}$ of a complete rotation, and $\frac{1}{6} \cdot 2\pi = \frac{\pi}{3}$.

b. Explain how to find the distance the passenger traveled to the nearest meter.
The radius r of the wheel is 67.5 m. The distance the passenger traveled is the arc length, so $s = r\theta = 67.5\left(\frac{\pi}{3}\right) \approx 71$ m.

c. Explain how you know your answer is reasonable.
The circumference is πd, which is approximately $3.14d = 3.14(135) = 423.9$ m and $\frac{1}{6}(423.9) = 70.65$, which is close to 71, so the answer is reasonable.

Lesson 9-1 • Angles and Angle Measure 423



74. PIZZA A circular pizza with a diameter of 18 inches is cut into 8 congruent slices. What is the radian measure of the central angle of each slice? Explain.
$\frac{\pi}{4}$; each slice is $\frac{1}{8}$ of the pizza, and $\frac{1}{8}(2\pi) = \frac{\pi}{4}$

75. CONSTRUCT ARGUMENTS A lawn sprinkler produces a stream of water that reaches 15 feet from the sprinkler head. The sprinkler rotates to sweep out part of a circle. The area of the lawn that gets watered as the sprinkler moves back and forth is 75π square feet. Melinda says it is possible to determine the radian measure of the angle that is swept out by the sprinkler. Seiji says there is not enough information to determine this. Who is correct? Justify your argument.
See Mod. 9 Answer Appendix.

76. REASONING Consider the figure at the right.

a. Suppose $s = r$. What is $m\angle PQR$ in radians? What is the measure of the angle to the nearest tenth of a degree? 1; 57.3°

b. Suppose $2s = r$. What is $m\angle PQR$ in radians? What is the measure of the angle to the nearest tenth of a degree? $\frac{1}{2}$; 28.6°

77. STRUCTURE Circle C has a radius of 5 centimeters. The function $f(x)$ gives the length in centimeters of an arc of circle C that is subtended by a central angle of x radians. Describe the graph of $f(x)$ as precisely as possible, and justify your answer. See Mod. 9 Answer Appendix.

Higher-Order Thinking Skills

78. FIND THE ERROR Tarshia and Alan are writing an expression for the measure of an angle coterminal with the angle shown at the right. Is either of them correct? Explain your reasoning. See Mod. 9 Answer Appendix.

Tarshia
The measure of a coterminal angle is $(x - 360)°$.

Alan
The measure of a coterminal angle is $(360 - x)°$.

79. PERSEVERE A line makes an angle of $\frac{\pi}{2}$ radians with the positive x-axis at the point (2, 0). Find the equation of the line. $x = 2$

80. ANALYZE Express $\frac{1}{8}$ of a revolution in degrees and in radians. Justify your argument.
45°, $\frac{\pi}{4}$; $\frac{1}{8}$ of 360° is 45°, and $\frac{1}{8}$ of 2π is $\frac{\pi}{4}$

81. CREATE Draw and label an acute angle in standard position. Find two angles, one positive and one negative, that are coterminal with the angle. See Mod. 9 Answer Appendix.

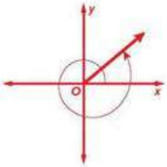
82. ANALYZE Justify the formula for the length of an arc. See Mod. 9 Answer Appendix.

83. WRITE Use a circle with radius r to describe what one degree and one radian represent. Then explain how to convert between the measures. See Mod. 9 Answer Appendix.

424 Module 9 • Trigonometric Functions

Answers

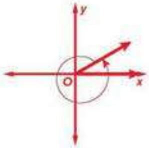
3.



4.

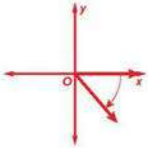
5.

6.

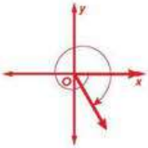


7.

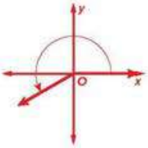
8.



9.



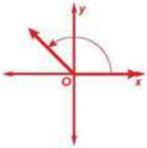
10.



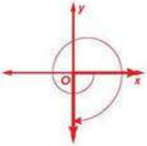
11.

12.

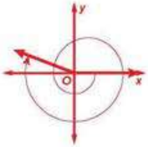
13.



14.



15.



Lesson 9-2

Trigonometric Functions of General Angles

F.TF.3

LESSON GOAL

Students find the values of trigonometric functions by using general angles and reference angles.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Trigonometric Functions in Right Triangles
- Evaluate Trigonometric Functions
- Find Trigonometric Ratios

Trigonometric Functions of General Angles
- Evaluate Trigonometric Functions Given a Point
- Evaluate Trigonometric Functions of Quadrantal Angles

Trigonometric Functions with Reference Angles
- Find Reference Angles
- Use a Reference Angle to Find a Trigonometric Value
- Use Trigonometric Functions

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Trigonometry	●	●		●
Extension: The Angle of Repose		●	●	●
ELL Support				

Language Development Handbook

Assign page 47 of the *Language Development Handbook* to help your students build mathematical language related to finding the values of trigonometric functions by using general angles and reference angles.

ELL You can use the tips and suggestions on page T47 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Functions

Standards for Mathematical Content:

F.TF.3 Use special triangles to determine geometrically the values of sine, cosine, tangent for $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

6 Attend to precision.

Coherence

Vertical Alignment

Previous

Students drew angles in standard position and converted between degree and radian measures of angles.

F.TF.1

Now

Students find the values of trigonometric functions by using general angles and reference angles.

F.TF.3

Next

Students will find trigonometric values by using the unit circle and the properties of periodic functions.

F.TF.2, F.TF.5

Rigor

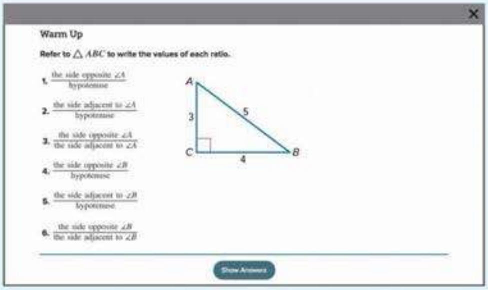
The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION
Conceptual Bridge In this lesson, students develop an understanding of periodicity. They build fluency and apply their understanding by solving problems related to periodic phenomena.		

Mathematical Background

When a point $P(x, y)$ on the terminal side of the angle θ is known, the value of the six trigonometric functions can be found. Draw a segment from the point perpendicular to the x-axis, forming a right triangle with a leg measuring x units, a leg measuring y units, and a hypotenuse measuring r units, where $r = \sqrt{x^2 + y^2}$. Thus, $\sin \theta = \frac{y}{r}$ and $\cos \theta = \frac{x}{r}$.

Interactive Presentation

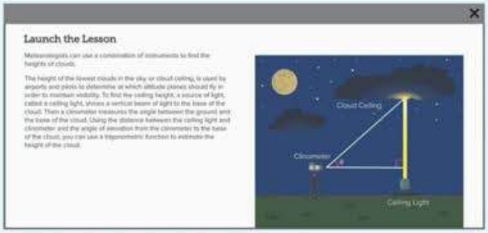


Warm Up

Refer to △ *ABC* to write the values of each ratio.

1. $\frac{\text{the side opposite } \angle A}{\text{hypotenuse}}$
2. $\frac{\text{the side adjacent to } \angle A}{\text{hypotenuse}}$
3. $\frac{\text{the side opposite } \angle A}{\text{the side adjacent to } \angle A}$
4. $\frac{\text{the side opposite } \angle B}{\text{hypotenuse}}$
5. $\frac{\text{the side adjacent to } \angle B}{\text{hypotenuse}}$
6. $\frac{\text{the side opposite } \angle B}{\text{the side adjacent to } \angle B}$

Warm Up

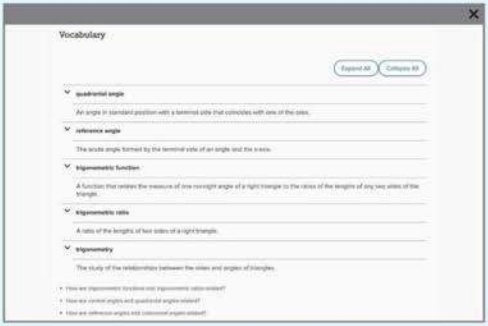


Launch the Lesson

Meteorologists can use a combination of instruments to find the heights of clouds.

The height of the lowest clouds in the sky, or cloud ceiling, is used by airports and pilots to determine at which altitude planes should fly in order to maintain visibility. To find the ceiling height, a source of light, called a ceiling light, shines a vertical beam of light to the base of the cloud. Then a clinometer measures the angle between the ground and the base of the cloud. Using the distance between the ceiling light and clinometer and the angle of elevation from the clinometer to the base of the cloud, you can use a trigonometric function to estimate the height of the cloud.

Launch the Lesson



Vocabulary

quadrantal angle

An angle in standard position with a terminal side that coincides with one of the axes.

reference angle

The acute angle formed by the terminal side of an angle and the x-axis.

trigonometric function

A function that relates the measure of one nonright angle of a right triangle to the ratios of the lengths of any two sides of the triangle.

trigonometric ratio

A ratio of the lengths of two sides of a right triangle.

trigonometry

The study of the relationships between the sides and angles of triangles.

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- writing ratios of side measures in a right triangle

Answers:

1. $\frac{4}{5}$
2. $\frac{3}{5}$
3. $\frac{4}{3}$
4. $\frac{3}{5}$
5. $\frac{4}{5}$
6. $\frac{3}{4}$

Launch the Lesson

Teaching the Mathematical Practices

4 Use Tool The diagram maps the relationships between the important quantities in the situation. Encourage students to use the diagram and what they know about trigonometry to determine which trigonometric function could be used to find the height of a cloud.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Learn Trigonometric Functions in Right Triangles

Objective

Students find values of trigonometric functions for acute angles by comparing side lengths of right triangles.

MP Teaching the Mathematical Practices

2 Different Properties Mathematically proficient students look for different ways to solve problems. Encourage students to consider how to write trigonometric functions by using ratios of the sides of a triangle and in terms of sine and/or cosine and to choose a method that works best for them.

Common Misconception

Students might think that opposite and adjacent sides are always the base and height of a right triangle, neglecting to consider the location of the reference angle. Reinforce to students that opposite and adjacent sides can be identified based only on the position of theta.

Example 1 Evaluate Trigonometric Functions

MP Teaching the Mathematical Practices

6 Use Precision Students will calculate the values of trigonometric functions accurately and efficiently.

Questions for Mathematical Discourse

AL Why are the values of the trigonometric functions left as fractions rather than decimals? The fractions are exact values. Decimal values would have to be rounded.

OL Did you need the figure to determine the cosecant, cosine, and cotangent, given the values for the sine, secant, and tangent? No, you can determine the missing trigonometric functions from their reciprocals.

BL Will the secant or cosecant ever be less than 1? Explain. No. The secant and the cosecant have the measure of the hypotenuse as the numerator, which will always be greater than the measure of each leg of the triangle.

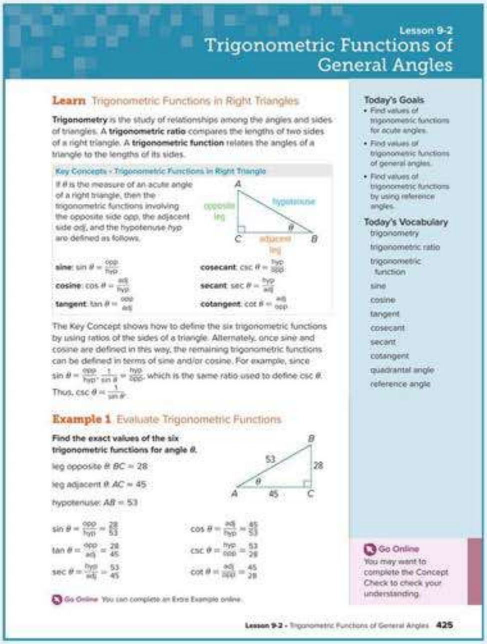
Common Error

Students often confuse the ratios for the secant and cosecant functions. Encourage students to memorize the ratios of all six functions to avoid errors.

DIFFERENTIATE

Language Development Activity AL ELL

IF students are struggling to remember the trigonometric functions, **THEN** have them memorize SOHCAHTOA (say it like so-cah-toe-ah). This helps place the trig function with the sides needed for its ratio. SOH means sine-opposite-hypotenuse. CAH means cosine-adjacent-hypotenuse. TOA means tangent-opposite-adjacent.



Lesson 9-2

Trigonometric Functions of General Angles

Learn Trigonometric Functions in Right Triangles

Trigonometry is the study of relationships among the angles and sides of triangles. A **trigonometric ratio** compares the lengths of two sides of a right triangle. A **trigonometric function** relates the angles of a triangle to the lengths of its sides.

Key Concepts • Trigonometric Functions in Right Triangle

If θ is the measure of an acute angle of a right triangle, then the trigonometric functions involving the opposite side opp, the adjacent side adj, and the hypotenuse hyp are defined as follows.

sine: $\sin \theta = \frac{opp}{hyp}$

cosine: $\cos \theta = \frac{adj}{hyp}$

tangent: $\tan \theta = \frac{opp}{adj}$

cosecant: $\csc \theta = \frac{hyp}{opp}$

secant: $\sec \theta = \frac{hyp}{adj}$

cotangent: $\cot \theta = \frac{adj}{opp}$

The Key Concept shows how to define the six trigonometric functions by using ratios of the sides of a triangle. Alternately, once sine and cosine are defined in this way, the remaining trigonometric functions can be defined in terms of sine and/or cosine. For example, since $\sin \theta = \frac{opp}{hyp}$, $\frac{1}{\sin \theta} = \frac{hyp}{opp}$, which is the same ratio used to define $\csc \theta$. Thus, $\csc \theta = \frac{1}{\sin \theta}$.

Example 1 Evaluate Trigonometric Functions

Find the exact values of the six trigonometric functions for angle θ.

leg opposite θ: $BC = 28$

leg adjacent θ: $AC = 45$

hypotenuse: $AB = 53$

$\sin \theta = \frac{opp}{hyp} = \frac{28}{53}$

$\cos \theta = \frac{adj}{hyp} = \frac{45}{53}$

$\tan \theta = \frac{opp}{adj} = \frac{28}{45}$

$\csc \theta = \frac{hyp}{opp} = \frac{53}{28}$

$\sec \theta = \frac{hyp}{adj} = \frac{53}{45}$

$\cot \theta = \frac{adj}{opp} = \frac{45}{28}$

Go Online You can complete an Extra Example online.

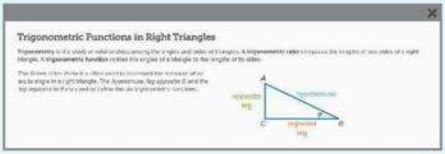
Today's Goals
- Find values of trigonometric functions for acute angles.
- Find values of trigonometric functions of general angles.
- Find values of trigonometric functions by using reference angles.

Today's Vocabulary
trigonometry
trigonometric ratio
trigonometric function
sine
cosine
tangent
cosecant
secant
cotangent
quadrantal angle
reference angle

Go Online You may want to complete the Concept Check to check your understanding.

Lesson 9-2 • Trigonometric Functions of General Angles 425

Interactive Presentation



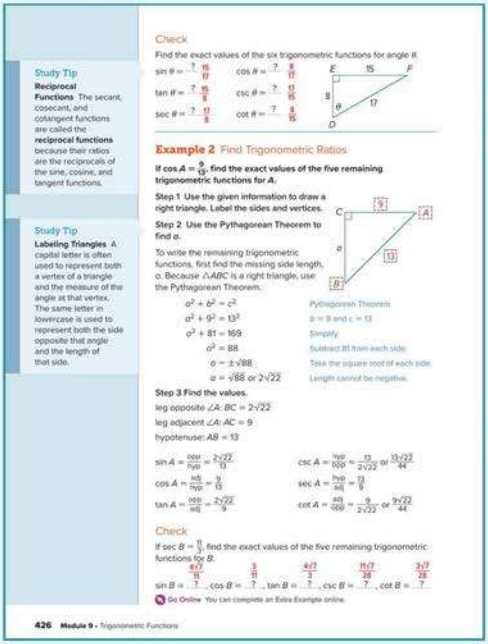
Trigonometric Functions in Right Triangles

Learn

TYPE

a|

Students complete a table to examine the definitions of trigonometric functions.



Study Tip

Reciprocal Functions The secant, cosecant, and cotangent functions are called the **reciprocal functions** because their ratios are the reciprocals of the sine, cosine, and tangent functions.

Study Tip

Labeling Triangles A capital letter is often used to represent both a vertex of a triangle and the measure of the angle at that vertex. The same letter in lowercase is used to represent both the side opposite that angle and the length of that side.

Check

Find the exact values of the six trigonometric functions for angle θ.

$\sin\theta = \underline{\ ?\ }$ $\frac{15}{17}$ $\cos\theta = \underline{\ ?\ }$ $\frac{8}{17}$

$\tan\theta = \underline{\ ?\ }$ $\frac{15}{8}$ $\csc\theta = \underline{\ ?\ }$ $\frac{17}{15}$

$\sec\theta = \underline{\ ?\ }$ $\frac{17}{8}$ $\cot\theta = \underline{\ ?\ }$ $\frac{8}{15}$

Example 2 Find Trigonometric Ratios

If $\cos A = \frac{9}{13}$, find the exact values of the five remaining trigonometric functions for A.

Step 1 Use the given information to draw a right triangle. Label the sides and vertices.

Step 2 Use the Pythagorean Theorem to find a.

To write the remaining trigonometric functions, first find the missing side length, a. Because $\triangle ABC$ is a right triangle, use the Pythagorean Theorem.

$$\begin{aligned} a^2 + b^2 &= c^2 && \text{Pythagorean Theorem}\\ a^2 + 9^2 &= 13^2 && b = 9 \text{ and } c = 13\\ a^2 + 81 &= 169 && \text{Simplify.}\\ a^2 &= 88 && \text{Subtract 81 from each side.}\\ a &= \pm\sqrt{88} && \text{Take the square root of each side.}\\ a &= \sqrt{88} \text{ or } 2\sqrt{22} && \text{Length cannot be negative.} \end{aligned}$$

Step 3 Find the values.

leg opposite $\angle A$: $BC = 2\sqrt{22}$

leg adjacent $\angle A$: $AC = 9$

hypotenuse: $AB = 13$

$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{2\sqrt{22}}{13}$ $\csc A = \frac{\text{hyp}}{\text{opp}} = \frac{13}{2\sqrt{22}}$ or $\frac{13\sqrt{22}}{44}$

$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{9}{13}$ $\sec A = \frac{\text{hyp}}{\text{adj}} = \frac{13}{9}$

$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{2\sqrt{22}}{9}$ $\cot A = \frac{\text{adj}}{\text{opp}} = \frac{9}{2\sqrt{22}}$ or $\frac{9\sqrt{22}}{44}$

Check

If $\sec B = \frac{11}{3}$, find the exact values of the five remaining trigonometric functions for B.

$\sin B = \underline{\ ?\ }$ $\frac{4\sqrt{7}}{11}$, $\cos B = \underline{\ ?\ }$ $\frac{3}{11}$, $\tan B = \underline{\ ?\ }$ $\frac{4\sqrt{7}}{3}$, $\csc B = \underline{\ ?\ }$ $\frac{11\sqrt{7}}{28}$, $\cot B = \underline{\ ?\ }$ $\frac{3\sqrt{7}}{28}$

Go Online You can complete an Extra Example online.

426 **Module 9** • Trigonometric Functions

Interactive Presentation

Find Trigonometric Ratios

If $\cos A = \frac{9}{13}$, find the exact values of the five remaining trigonometric functions for A.

Use the given information to draw a right triangle.

Use the Pythagorean Theorem to find a.

Example 2

DRAG & DROP

Students drag and drop the names and measures of the angles and sides to complete the right triangle.

TAP

Students move through the steps to find all trigonometric functions for a given angle. They also examine an Alternate Method.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Find T rigonometric Ratios

MP Teaching the Mathematical Practices

1 Explain Correspondences Guide students to use the information in Example 2 to draw a diagram to represent the problem.

Questions for Mathematical Discourse

AL How do you know where to place the 9 and the 13 on the triangle? cos *A* is the leg adjacent to *A* over the hypotenuse.

OL What would change about tan *A* if you were initially given $\sin A = \frac{9}{13}$? tan *A* would be the reciprocal of the original tan *A*.

BL If each side of the triangle was scaled by a common factor, how would the trigonometric ratios be affected? They would remain the same.

Common Error

Students may add the squares of two given sides without considering what the sides of the right triangle represent. Reinforce that two measures are not automatically the legs of the right triangle; one may be the hypotenuse.

Essential Question Follow-Up

Students have begun learning about trigonometric functions.

Ask:

Why are trigonometric functions in right triangles useful in the real world? Sample answer: Many situations are modeled by right triangles, and missing information can be solved for using trigonometric functions. For example, if we were standing 10 feet from a flagpole and knew the angle from our feet to the top of the flagpole, we could determine its height.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Learn Trigonometric Functions of General Angles

Objective

Students find values of trigonometric functions of general angles by using a point on the terminal side of the angle.

MP Teaching the Mathematical Practices

3 Construct Arguments In the Think About It! features, students will use stated assumptions, definitions, and previously established results to construct arguments about the value of *r*.

Common Misconception

A common misconception some students may have is that the trigonometric functions must be positive since they describe the ratio of sides in a right triangle. Remind students that the application of the trigonometric functions extends beyond a right triangle to the coordinate plane, which contains positive and negative values.

Example 3 Evaluate Trigonometric Functions Given a Point

MP Teaching the Mathematical Practices

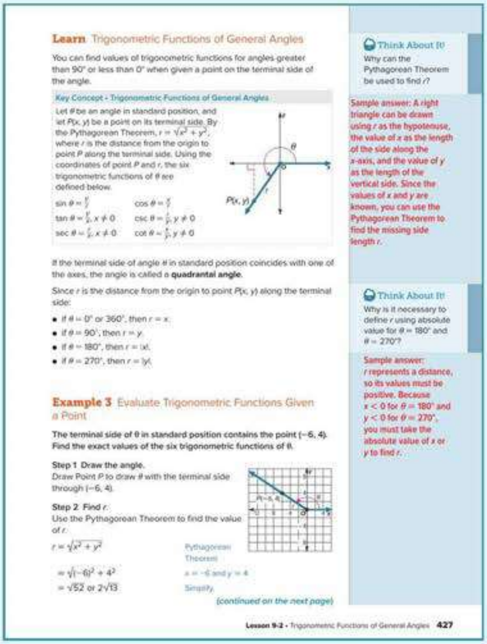
6 Use Precision Students must calculate accurately and precisely to determine the trigonometric ratios for an angle that has a terminal side passing through a given point.

Questions for Mathematical Discourse

AL How could you draw a triangle on the graph to illustrate finding *r*? Starting at the origin, draw a line left 6 units, then draw a line up 4 units. *r* is the hypotenuse from (−6, 4) back to the origin.

OL What would change about the trigonometric ratios if you were given a different point on the same line? The values would not change.

BL What would change about tan θ if the point was (6, −4)? It would remain the same. What does tan $(\theta + \pi)$ equal? tan θ



Learn Trigonometric Functions of General Angles

You can find values of trigonometric functions for angles greater than 90° or less than 0° when given a point on the terminal side of the angle.

Key Concept • Trigonometric Functions of General Angles

Let θ be an angle in standard position, and let $P(x, y)$ be a point on its terminal side. By the Pythagorean Theorem, $r = \sqrt{x^2 + y^2}$, where r is the distance from the origin to point P along the terminal side. Using the coordinates of point P and r, the six trigonometric functions of θ are defined below.

$\sin \theta = \frac{y}{r}$　　$\cos \theta = \frac{x}{r}$

$\tan \theta = \frac{y}{x}, x \neq 0$　　$\csc \theta = \frac{r}{y}, y \neq 0$

$\sec \theta = \frac{r}{x}, x \neq 0$　　$\cot \theta = \frac{x}{y}, y \neq 0$

If the terminal side of angle θ in standard position coincides with one of the axes, the angle is called a **quadrantal angle**.

Since r is the distance from the origin to point $P(x, y)$ along the terminal side:

- If $\theta = 0°$ or $360°$, then $r = x$.
- If $\theta = 90°$, then $r = y$.
- If $\theta = 180°$, then $r = |x|$.
- If $\theta = 270°$, then $r = |y|$.

Example 3 Evaluate Trigonometric Functions Given a Point

The terminal side of θ in standard position contains the point $(-6, 4)$. Find the exact values of the six trigonometric functions of θ.

Step 1 Draw the angle.
Draw Point P to draw θ with the terminal side through $(-6, 4)$.

Step 2 Find r.
Use the Pythagorean Theorem to find the value of r.

$r = \sqrt{x^2 + y^2}$　Pythagorean Theorem

$= \sqrt{(-6)^2 + 4^2}$　$x = -6$ and $y = 4$

$= \sqrt{52}$ or $2\sqrt{13}$　Simplify.

(continued on the next page)

Think About It!
Why can the Pythagorean Theorem be used to find r?

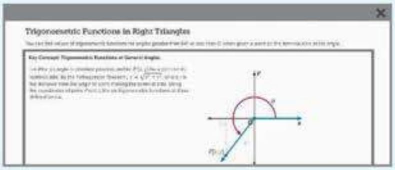
Sample answer: A right triangle can be drawn using r as the hypotenuse, the value of x as the length of the side along the x-axis, and the value of y as the length of the vertical side. Since the values of x and y are known, you can use the Pythagorean Theorem to find the missing side length r.

Think About It!
Why is it necessary to define r using absolute value for $\theta = 180°$ and $\theta = 270°$?

Sample answer: r represents a distance, so its values must be positive. Because $x < 0$ for $\theta = 180°$ and $y < 0$ for $\theta = 270°$, you must take the absolute value of x or y to find r.

Lesson 9-2 • Trigonometric Functions of General Angles 427

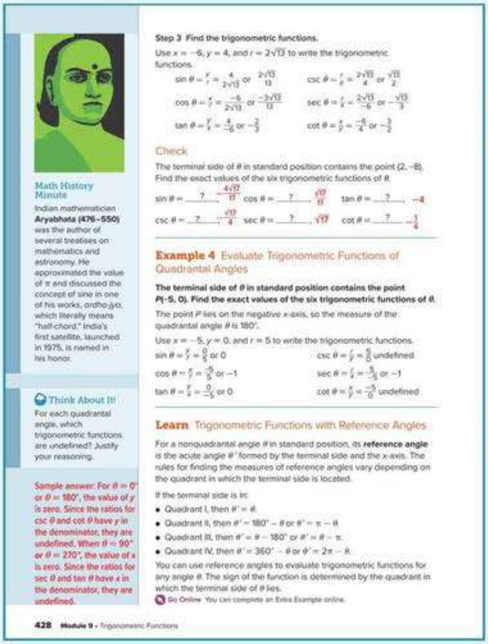
Interactive Presentation



Learn

TYPE

Students explain why the Pythagorean Theorem can be used to find *r*.



Math History Minute

Indian mathematician **Aryabhata (476–550)** was the author of several treatises on mathematics and astronomy. He approximated the value of π and discussed the concept of sine in one of his works, *ardha-jya*, which literally means "half-chord." India's first satellite, launched in 1975, is named in his honor.

Think About It!

For each quadrantal angle, which trigonometric functions are undefined? Justify your reasoning.

Sample answer: For $\theta = 0°$ or $\theta = 180°$, the value of y is zero. Since the ratios for csc θ and cot θ have y in the denominator, they are undefined. When $\theta = 90°$ or $\theta = 270°$, the value of x is zero. Since the ratios for sec θ and tan θ have x in the denominator, they are undefined.

Step 3 Find the trigonometric functions.

Use $x = -6$, $y = 4$, and $r = 2\sqrt{13}$ to write the trigonometric functions.

$\sin\theta = \frac{y}{r} = \frac{4}{2\sqrt{13}}$ or $\frac{2\sqrt{13}}{13}$ $\qquad$ $\csc\theta = \frac{r}{y} = \frac{2\sqrt{13}}{4}$ or $\frac{\sqrt{13}}{2}$

$\cos\theta = \frac{x}{r} = \frac{-6}{2\sqrt{13}}$ or $\frac{-3\sqrt{13}}{13}$ $\qquad$ $\sec\theta = \frac{r}{x} = \frac{2\sqrt{13}}{-6}$ or $-\frac{\sqrt{13}}{3}$

$\tan\theta = \frac{y}{x} = \frac{4}{-6}$ or $-\frac{2}{3}$ $\qquad$ $\cot\theta = \frac{x}{y} = \frac{-6}{4}$ or $-\frac{3}{2}$

Check

The terminal side of θ in standard position contains the point (2, −8). Find the exact values of the six trigonometric functions of θ.

$\sin\theta =$ __?__ $-\frac{4\sqrt{17}}{17}$ $\quad$ $\cos\theta =$ __?__ $\frac{\sqrt{17}}{17}$ $\quad$ $\tan\theta =$ __?__ -4

$\csc\theta =$ __?__ $-\frac{\sqrt{17}}{4}$ $\quad$ $\sec\theta =$ __?__ $\sqrt{17}$ $\quad$ $\cot\theta =$ __?__ $-\frac{1}{4}$

Example 4 Evaluate Trigonometric Functions of Quadrantal Angles

The terminal side of θ in standard position contains the point $P(-5, 0)$. Find the exact values of the six trigonometric functions of θ.

The point P lies on the negative x-axis, so the measure of the quadrantal angle θ is 180°.

Use $x = -5$, $y = 0$, and $r = 5$ to write the trigonometric functions.

$\sin\theta = \frac{y}{r} = \frac{0}{5}$ or 0 $\qquad$ $\csc\theta = \frac{r}{y} = \frac{5}{0}$ undefined

$\cos\theta = \frac{x}{r} = \frac{-5}{5}$ or -1 $\qquad$ $\sec\theta = \frac{r}{x} = \frac{5}{-5}$ or -1

$\tan\theta = \frac{y}{x} = \frac{0}{-5}$ or 0 $\qquad$ $\cot\theta = \frac{x}{y} = \frac{-5}{0}$ undefined

Learn Trigonometric Functions with Reference Angles

For a nonquadrantal angle θ in standard position, its **reference angle** is the acute angle θ' formed by the terminal side and the x-axis. The rules for finding the measures of reference angles vary depending on the quadrant in which the terminal side is located.

If the terminal side is in:

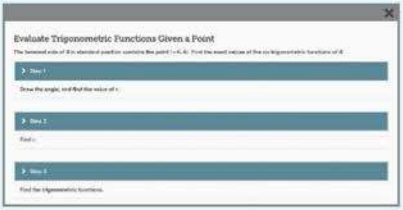
- Quadrant I, then $\theta' = \theta$.
- Quadrant II, then $\theta' = 180° - \theta$ or $\theta' = \pi - \theta$.
- Quadrant III, then $\theta' = \theta - 180°$ or $\theta' = \theta - \pi$.
- Quadrant IV, then $\theta' = 360° - \theta$ or $\theta' = 2\pi - \theta$.

You can use reference angles to evaluate trigonometric functions for any angle θ. The sign of the function is determined by the quadrant in which the terminal side of θ lies.

Go Online You can complete an Extra Example online.

428 Module 9 • Trigonometric Functions

Interactive Presentation



Evaluate Trigonometric Functions Given a Point

Step 1

Draw the angle, and find the value of r.

Step 2

Find r.

Step 3

Find the trigonometric functions.

Example 3

TAP

Students move through the steps to find all trigonometric functions for a given point.

TYPE

Students identify the values of x, y, and r to find the 6 trigonometric functions.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 4 Evaluate Trigonometric Functions of Quadrantal Angles

Teaching the Mathematical Practices

2 Make Sense of Quantities Mathematically proficient students need to be able to make sense of quantities and their relationships. In Example 4, notice the relationships between the x- and y-coordinates of point P, the value of r, and the trigonometric functions.

Questions for Mathematical Discourse

AL Why are csc θ and cot θ undefined for $P(-5, 0)$? Division by zero is undefined.

OL How would the ratios change if the point was (5, 0)? cos θ and sec θ would be 1, the others would remain the same.

BL Will the values of the six trigonometric functions be the same for any point on the negative x-axis? Explain. Yes; sample answer: The coordinates of any point on the negative x-axis are $(a, 0)$ where $a < 0$. At this point, $r = |a|$ or $-a$. This will result in the same six ratios as the point $(-5, 0)$.

Learn Trigonometric Functions with Reference Angles

Objective

Students find values of trigonometric functions by using reference angles.

Teaching the Mathematical Practices

3 Analyze Cases Students will learn how the determine the reference angle when the terminal side of the angle is in Quadrant I, II, III, or IV. Encourage students to familiarize themselves with all of the cases.

Common Misconception

A common misconception some students may have is that the reference angle is just a coterminal angle. Remind students that reference angles are acute angles and coterminal angles can be greater than 90˚.

Example 5 Find Reference Angles

Teaching the Mathematical Practices

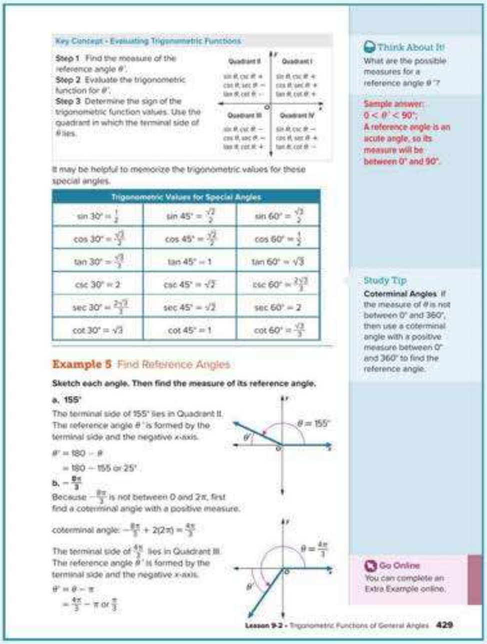
1 Explain Correspondences Encourage students to explain the relationships between the given angle, the terminal side of the angle, and the reference angle.

Questions for Mathematical Discourse

AL How do you know the terminal side of the 155° angle is in Quadrant II? 155° is between 90° and 180°.

OL For angles in the third quadrant, why do we subtract 180˚ or π from the measure to determine the reference angle? Sample answer: The vertical distance from the *x*-axis is the measure of the reference angle, and the measure of an angle in the third quadrant is greater than 180 degrees.

BL Why do you need to find a coterminal angle with a positive measure? You need a positive angle to use the given formulas for reference angles. If you used a negative angle measure, you could still find the reference angle if you correctly determined which values to subtract.



Key Concept • Evaluating Trigonometric Functions

Step 1 Find the measure of the reference angle θ'.

Step 2 Evaluate the trigonometric function for θ'.

Step 3 Determine the sign of the trigonometric function values. Use the quadrant in which the terminal side of θ lies.

It may be helpful to memorize the trigonometric values for these special angles.

Trigonometric Values for Special Angles

$\sin 30^\circ = \frac{1}{2}$	$\sin 45^\circ = \frac{\sqrt{2}}{2}$	$\sin 60^\circ = \frac{\sqrt{3}}{2}$
$\cos 30^\circ = \frac{\sqrt{3}}{2}$	$\cos 45^\circ = \frac{\sqrt{2}}{2}$	$\cos 60^\circ = \frac{1}{2}$
$\tan 30^\circ = \frac{\sqrt{3}}{3}$	$\tan 45^\circ = 1$	$\tan 60^\circ = \sqrt{3}$
$\csc 30^\circ = 2$	$\csc 45^\circ = \sqrt{2}$	$\csc 60^\circ = \frac{2\sqrt{3}}{3}$
$\sec 30^\circ = \frac{2\sqrt{3}}{3}$	$\sec 45^\circ = \sqrt{2}$	$\sec 60^\circ = 2$
$\cot 30^\circ = \sqrt{3}$	$\cot 45^\circ = 1$	$\cot 60^\circ = \frac{\sqrt{3}}{3}$

Example 5 Find Reference Angles

Sketch each angle. Then find the measure of its reference angle.

a. 155°

The terminal side of 155° lies in Quadrant II. The reference angle θ' is formed by the terminal side and the negative *x*-axis.

$$\theta' = 180 - \theta$$
$$= 180 - 155 \text{ or } 25^\circ$$

b. $-\frac{8\pi}{3}$

Because $-\frac{8\pi}{3}$ is not between 0 and 2π, first find a coterminal angle with a positive measure.

coterminal angle: $-\frac{8\pi}{3} + 2(2\pi) = \frac{4\pi}{3}$

The terminal side of $\frac{4\pi}{3}$ lies in Quadrant III. The reference angle θ' is formed by the terminal side and the negative *x*-axis.

$$\theta' = \theta - \pi$$
$$= \frac{4\pi}{3} - \pi \text{ or } \frac{\pi}{3}$$

Think About It!

What are the possible measures for a reference angle θ'?

Sample answer: $0 < \theta' < 90^\circ$; A reference angle is an acute angle, so its measure will be between 0° and 90°.

Study Tip

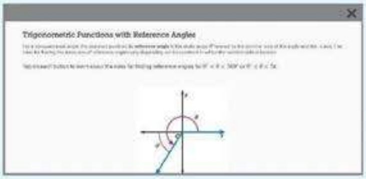
Coterminal Angles If the measure of θ is not between 0° and 360°, then use a coterminal angle with a positive measure between 0° and 360° to find the reference angle.

Go Online

You can complete an Extra Example online.

Lesson 9-2 • Trigonometric Functions of General Angles 429

Interactive Presentation



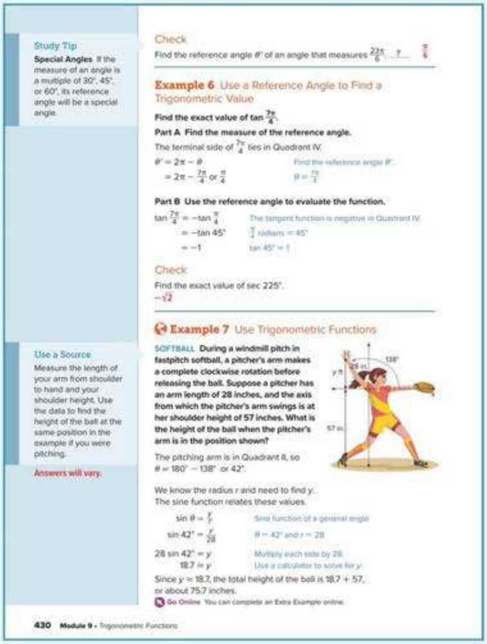
Learn

TAP

Students tap each button to learn more about the rules for finding reference angles.

TYPE

Students give possible measures for a reference angle.



Study Tip

Special Angles If the measure of an angle is a multiple of 30°, 45°, or 60°, its reference angle will be a special angle.

Check

Find the reference angle θ' of an angle that measures $\frac{23\pi}{6}$. ___ $\frac{\pi}{6}$

Example 6 Use a Reference Angle to Find a Trigonometric Value

Find the exact value of tan $\frac{7\pi}{4}$.

Part A Find the measure of the reference angle.

The terminal side of $\frac{7\pi}{4}$ lies in Quadrant IV.

$\theta' = 2\pi - \theta$ Find the reference angle θ'.

$= 2\pi - \frac{7\pi}{4}$ or $\frac{\pi}{4}$ $\theta = \frac{7\pi}{4}$

Part B Use the reference angle to evaluate the function.

$\tan\frac{7\pi}{4} = -\tan\frac{\pi}{4}$ The tangent function is negative in Quadrant IV.

$= -\tan 45°$ $\frac{\pi}{4}$ radians $= 45°$

$= -1$ $\tan 45° = 1$

Check

Find the exact value of sec 225°.

$-\sqrt{2}$

Example 7 Use Trigonometric Functions

Use a Source

Measure the length of your arm from shoulder to hand and your shoulder height. Use the data to find the height of the ball at the same position in the example if you were pitching.

Answers will vary.

SOFTBALL **During a windmill pitch in fastpitch softball, a pitcher's arm makes a complete clockwise rotation before releasing the ball. Suppose a pitcher has an arm length of 28 inches, and the axis from which the pitcher's arm swings is at her shoulder height of 57 inches. What is the height of the ball when the pitcher's arm is in the position shown?**

The pitching arm is in Quadrant II, so $\theta = 180° - 138°$ or 42°.

We know the radius r and need to find y. The sine function relates these values.

$\sin\theta = \frac{y}{r}$ Sine function of a general angle

$\sin 42° = \frac{y}{28}$ $\theta = 42°$ and $r = 28$

$28 \sin 42° = y$ Multiply each side by 28.

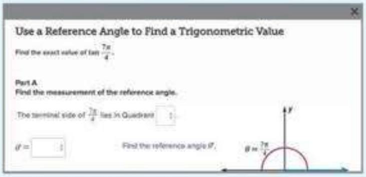
$18.7 \approx y$ Use a calculator to solve for y.

Since $y \approx 18.7$, the total height of the ball is 18.7 + 57, or about 75.7 inches.

Go Online You can complete an Extra Example online.

430 Module 9 • Trigonometric Functions

Interactive Presentation



Use a Reference Angle to Find a Trigonometric Value

Find the exact value of tan $\frac{7\pi}{4}$.

Part A

Find the measurement of the reference angle.

The terminal side of $\frac{7\pi}{4}$ lies in Quadrant ☐

θ' = ☐ Find the reference angle θ'.

Example 6

SELECT

Students select the quadrant in which the terminal side lies and the sign of the trigonometric function.

TYPE

Students enter the correct measures of the reference angle and then analyze a common error.

Example 6 Use a Reference Angle to Find a Trigonometric Value

Teaching the Mathematical Practices

3 Construct Arguments Students must use stated definitions and previously established results to complete the given example.

Questions for Mathematical Discourse

AL Why is the tangent function negative in quadrant IV? $\tan\theta = \frac{x}{y}$; x is positive and y is negative in quadrant IV.

OL Without referring to the reference table of special angles, how do you know that tan 45° = 1? A right triangle with an angle of 45° would be a 45°-45°-90° triangle. Thus, the lengths of the legs would be equal, making the ratio of the legs equal to 1.

BL If *n* is a positive integer, what is a rule for the value of $\tan\frac{n\pi}{4}$? Sample answer: Divide *n* by 4. If the remainder is 1, the tangent is 1; if the remainder is 2, the tangent is undefined; if the remainder is 3, the tangent is −1; if there is no remainder, the tangent is 0.

Example 7 Use T rigonometric Functions

Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to solve the problem in Example 7, students will need to analyze the mathematical relationships in the problem to draw a conclusion.

Questions for Mathematical Discourse

AL What part of the triangle does the pitcher's arm represent? the hypotenuse

OL What other trigonometric function could you use to solve this? You could use the cosecant because it is the reciprocal of the sine.

BL Would you get the same result if you used 138°? Y es, sin 42° = sin 138°. Using the reference angle makes it easier to use the triangle to visualize and determine which trigonometric function will yield the unknown variable.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–47
2	exercises that use a variety of skills from this lesson	48–62
2	exercises that extend concepts learned in this lesson to new contexts	63–68
3	exercises that emphasize higher-order and critical-thinking skills	69–76

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

BL

IF students score 90% or more on the Checks,
THEN assign:

- Practice Exercises 1–67 odd, 69–76
- Extension: The Angle of Repose
- ALEKS® Trigonometric Functions of Angles

OL

IF students score 66%–89% on the Checks,
THEN assign:

- Practice Exercises 1–75 odd
- Remediation, Review Resources: Trigonometry
- Personal Tutors
- Extra Examples 1–6
- ALEKS® Right Triangle Trigonometry

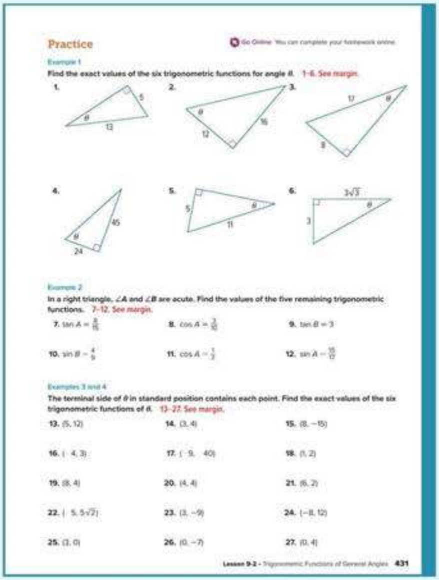
AL

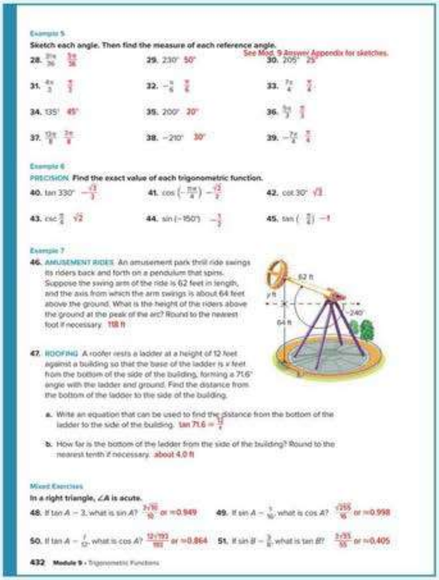
IF students score 65% or less on the Checks,
THEN assign:

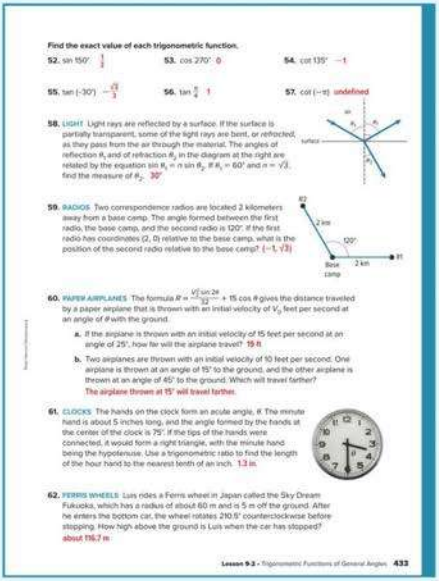
- Practice, Exercises 1–47 odd
- Remediation, Review Resources: Trigonometry
- ALEKS® Right Triangle Trigonometry

Answers

1. $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$, $\tan\theta = \frac{5}{12}$, $\csc\theta = \frac{13}{5}$, $\sec\theta = \frac{13}{12}$, $\cot\theta = \frac{12}{5}$
2. $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$, $\csc\theta = \frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = \frac{3}{4}$
3. $\sin\theta = \frac{8}{17}$, $\cos\theta = \frac{15}{17}$, $\tan\theta = \frac{8}{15}$, $\csc\theta = \frac{17}{8}$, $\sec\theta = \frac{17}{15}$, $\cot\theta = \frac{15}{8}$
4. $\sin\theta = \frac{15}{17}$, $\cos\theta = \frac{8}{17}$, $\tan\theta = \frac{15}{8}$, $\csc\theta = \frac{17}{15}$, $\sec\theta = \frac{17}{8}$, $\cot\theta = \frac{8}{15}$
5. $\sin\theta = \frac{5}{11}$, $\cos\theta = \frac{4\sqrt{6}}{11}$, $\tan\theta = \frac{5\sqrt{6}}{24}$, $\csc\theta = \frac{11}{5}$, $\sec\theta = \frac{11\sqrt{6}}{24}$, $\cot\theta = \frac{4\sqrt{6}}{5}$
6. $\sin\theta = \frac{1}{2}$, $\cos\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = \frac{\sqrt{3}}{3}$, $\csc\theta = 2$, $\sec\theta = \frac{2\sqrt{3}}{3}$, $\cot\theta = \sqrt{3}$
7. $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\csc A = \frac{17}{8}$, $\sec A = \frac{17}{15}$, $\cot A = \frac{15}{8}$
8. $\sin A = \frac{\sqrt{91}}{10}$, $\tan A = \frac{\sqrt{91}}{3}$, $\csc A = \frac{10\sqrt{91}}{91}$, $\sec A = \frac{10}{3}$, $\cot A = \frac{3\sqrt{91}}{91}$
9. $\sin B = \frac{3\sqrt{10}}{10}$, $\cos B = \frac{\sqrt{10}}{10}$, $\csc B = \frac{\sqrt{10}}{3}$, $\sec B = \sqrt{10}$, $\cot B = \frac{1}{3}$
10. $\cos B = \frac{\sqrt{65}}{9}$, $\tan B = \frac{4\sqrt{65}}{65}$, $\csc B = \frac{9}{4}$, $\sec B = \frac{9\sqrt{65}}{65}$, $\cot B = \frac{\sqrt{65}}{4}$







Find the exact value of each trigonometric function.

52. sin 150° $\frac{1}{2}$

53. cos 270° 0

54. cot 135° −1

55. tan (−30°) $-\frac{\sqrt{3}}{3}$

56. tan $\frac{\pi}{4}$ 1

57. cot (−π) undefined

58. LIGHT Light rays are reflected by a surface. If the surface is partially transparent, some of the light rays are bent, or refracted, as they pass from the air through the material. The angles of reflection θ_1 and of refraction θ_2 in the diagram at the right are related by the equation $\sin\theta_1 = n\sin\theta_2$. If $\theta_1 = 60°$ and $n = \sqrt{3}$, find the measure of θ_2. 30°

59. RADIOS Two correspondence radios are located 2 kilometers away from a base camp. The angle formed between the first radio, the base camp, and the second radio is 120°. If the first radio has coordinates (2, 0) relative to the base camp, what is the position of the second radio relative to the base camp? $(-1, \sqrt{3})$

60. PAPER AIRPLANES The formula $R = \frac{V_0^2 \sin 2\theta}{32} + 15\cos\theta$ gives the distance traveled by a paper airplane that is thrown with an initial velocity of V_0 feet per second at an angle of θ with the ground.

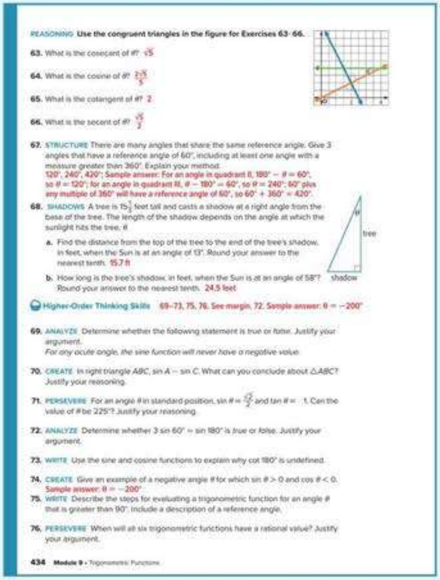
a. If the airplane is thrown with an initial velocity of 15 feet per second at an angle of 25°, how far will the airplane travel? 19 ft

b. Two airplanes are thrown with an initial velocity of 10 feet per second. One airplane is thrown at an angle of 15° to the ground, and the other airplane is thrown at an angle of 45° to the ground. Which will travel farther?
The airplane thrown at 15° will travel farther.

61. CLOCKS The hands on the clock form an acute angle, θ. The minute hand is about 5 inches long, and the angle formed by the hands at the center of the clock is 75°. If the tips of the hands were connected, it would form a right triangle, with the minute hand being the hypotenuse. Use a trigonometric ratio to find the length of the hour hand to the nearest tenth of an inch. 1.3 in.

62. FERRIS WHEELS Luis rides a Ferris wheel in Japan called the Sky Dream Fukuoka, which has a radius of about 60 m and is 5 m off the ground. After he enters the bottom car, the wheel rotates 210.5° counterclockwise before stopping. How high above the ground is Luis when the car has stopped? about 116.7 m

Lesson 9-3 • Trigonometric Functions of General Angles 433



REASONING **Use the congruent triangles in the figure for Exercises 63–66.**

63. What is the cosecant of θ? $\sqrt{5}$

64. What is the cosine of θ? $\frac{2\sqrt{5}}{5}$

65. What is the cotangent of θ? 2

66. What is the secant of θ? $\frac{\sqrt{5}}{2}$

67. STRUCTURE There are many angles that share the same reference angle. Give 3 angles that have a reference angle of 60°, including at least one angle with a measure greater than 360°. Explain your method.
120°, 240°, 420°; Sample answer: For an angle in quadrant II, 180° − θ = 60°, so θ = 120°; for an angle in quadrant III, θ − 180° = 60°, so θ = 240°; 60° plus any multiple of 360° will have a reference angle of 60°, so 60° + 360° = 420°.

68. SHADOWS A tree is $15\frac{1}{3}$ feet tall and casts a shadow at a right angle from the base of the tree. The length of the shadow depends on the angle at which the sunlight hits the tree, θ.

a. Find the distance from the top of the tree to the end of the tree's shadow, in feet, when the Sun is at an angle of 13°. Round your answer to the nearest tenth. 15.7 ft

b. How long is the tree's shadow, in feet, when the Sun is at an angle of 58°? Round your answer to the nearest tenth. 24.5 feet

Higher-Order Thinking Skills 69–73, 75, 76. See margin. 72. Sample answer: θ = −200°

69. ANALYZE Determine whether the following statement is true or false. Justify your argument.
For any acute angle, the sine function will never have a negative value.

70. CREATE In right triangle ABC, sin A = sin C. What can you conclude about △ABC? Justify your reasoning.

71. PERSEVERE For an angle θ in standard position, sin θ = $\frac{\sqrt{2}}{2}$ and tan θ = −1. Can the value of θ be 225°? Justify your reasoning.

72. ANALYZE Determine whether 3 sin 60° = sin 180° is *true* or *false*. Justify your argument.

73. WRITE Use the sine and cosine functions to explain why cot 180° is undefined.

74. CREATE Give an example of a negative angle θ for which sin θ > 0 and cos θ < 0.
Sample answer: θ = −200°

75. WRITE Describe the steps for evaluating a trigonometric function for an angle θ that is greater than 90°. Include a description of a reference angle.

76. PERSEVERE When will all six trigonometric functions have a rational value? Justify your argument.

434 Module 9 • Trigonometric Functions

Answers

11. $\sin A = \frac{\sqrt{3}}{2}$, $\tan A = \sqrt{3}$, $\csc A = \frac{2\sqrt{3}}{3}$, $\sec A = 2$, $\cot A = \frac{\sqrt{3}}{3}$

12. $\cos A = \frac{8}{17}$, $\tan A = \frac{15}{8}$, $\csc A = \frac{17}{15}$, $\sec A = \frac{17}{8}$, $\cot A = \frac{8}{15}$

13. $\sin\theta = \frac{12}{13}$, $\cos\theta = \frac{5}{13}$, $\tan\theta = \frac{12}{5}$, $\csc\theta = \frac{13}{12}$, $\sec\theta = \frac{13}{5}$, $\cot\theta = \frac{5}{12}$

14. $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$, $\csc\theta = \frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = \frac{3}{4}$

15. $\sin\theta = -\frac{15}{17}$, $\cos\theta = \frac{8}{17}$, $\tan\theta = -\frac{15}{8}$, $\csc\theta = -\frac{17}{15}$, $\sec\theta = \frac{17}{8}$,
$\cot\theta = -\frac{8}{15}$

16. $\sin\theta = \frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$, $\csc\theta = \frac{5}{3}$, $\sec\theta = -\frac{5}{4}$, $\cot\theta = -\frac{4}{3}$

17. $\sin\theta = -\frac{40}{41}$, $\cos\theta = -\frac{9}{41}$, $\tan\theta = \frac{40}{9}$, $\csc\theta = -\frac{41}{40}$, $\sec\theta = -\frac{41}{9}$,
$\cot\theta = \frac{9}{40}$

18. $\sin\theta = \frac{2\sqrt{5}}{5}$, $\cos\theta = \frac{\sqrt{5}}{5}$, $\tan\theta = 2$, $\csc\theta = \frac{\sqrt{5}}{2}$, $\sec\theta = \sqrt{5}$, $\cot\theta = \frac{1}{2}$

19. $\sin\theta = \frac{\sqrt{5}}{5}$, $\cos\theta = \frac{2\sqrt{5}}{5}$, $\tan\theta = \frac{1}{2}$, $\csc\theta = \sqrt{5}$, $\sec\theta = \frac{\sqrt{5}}{2}$, $\cot\theta = 2$

20. $\sin\theta = \frac{\sqrt{2}}{2}$, $\cos\theta = \frac{\sqrt{2}}{2}$, $\tan\theta = 1$, $\csc\theta = \sqrt{2}$, $\sec\theta = \sqrt{2}$, $\cot\theta = 1$

21. $\sin\theta = \frac{\sqrt{10}}{10}$, $\cos\theta = \frac{3\sqrt{10}}{10}$, $\tan\theta = \frac{1}{3}$, $\csc\theta = \sqrt{10}$, $\sec\theta = \frac{\sqrt{10}}{3}$, $\cot\theta = 3$

22. $\sin\theta = \frac{\sqrt{6}}{3}$, $\cos\theta = -\frac{\sqrt{3}}{3}$, $\tan\theta = -\sqrt{2}$, $\csc\theta = \frac{\sqrt{6}}{2}$, $\sec\theta = -\sqrt{3}$,
$\cot\theta = -\frac{\sqrt{2}}{2}$

23. $\sin\theta = \frac{-3\sqrt{10}}{10}$, $\cos\theta = \frac{\sqrt{10}}{10}$, $\tan\theta = -3$, $\csc\theta = -\frac{\sqrt{10}}{3}$, $\sec\theta = \sqrt{10}$,
$\cot\theta = -\frac{1}{3}$

24. $\sin\theta = \frac{-3\sqrt{13}}{13}$, $\cos\theta = -\frac{2\sqrt{13}}{13}$, $\tan\theta = -\frac{3}{2}$, $\csc\theta = \frac{\sqrt{13}}{3}$, $\sec\theta = -\frac{\sqrt{13}}{2}$
,
$\cot\theta = -\frac{2}{3}$

25. $\sin\theta = 0$, $\cos\theta = 1$, $\tan\theta = 0$, $\csc\theta =$ undefined, $\sec\theta = 1$,
$\cot\theta =$ undefined

26. $\sin\theta = -1$, $\cos\theta = 0$, $\tan\theta =$ undefined, $\csc\theta = -1$,
$\sec\theta =$ undefined, $\cot\theta = 0$

27. $\sin\theta = 1$, $\cos\theta = 0$, $\tan\theta =$ undefined, $\csc\theta = 1$, $\sec\theta =$ undefined,
$\cot\theta = 0$

69. True; $\sin\theta = \frac{opp}{hyp}$, and the values of the opposite side and the hypotenuse of an acute triangle are positive, so the value of the sine function is positive.

70. $\sin A = \sin C$, so $\frac{\text{side opp } A}{\text{hyp}} = \frac{\text{side opp } C}{\text{hyp}}$. Since the hypotenuse is the same, the length of the sides opposite angles *A* and *C* must be equal. Since the two sides have the same measure, the triangle is isosceles.

71. No; for $\sin\theta = \frac{\sqrt{2}}{2}$ and $\tan\theta = -1$, the reference angle is 45°. However, for sin θ to be positive and tan θ to be negative, the reference angle must be in the second quadrant. So, the value of θ must be 135° or an angle coterminal with 135°.

72. False; $3\sin 60° = 3 \cdot \frac{\sqrt{3}}{2}$ or $\frac{3\sqrt{3}}{2}$, and $\sin 180° = 0$.

73. We know that $\cot\theta = \frac{x}{y}$, $\sin\theta = \frac{y}{r}$, and $\cos\theta = \frac{x}{r}$. Since $\sin 180° = 0$, it must be true that $\cot 180° = \frac{x}{0}$, which is undefined.

75. First, sketch the angle and determine in which quadrant it is located. Then use the appropriate rule for finding its reference angle θ′. A reference angle is the acute angle formed by the terminal side of θ and the *x*-axis. Next, find the value of the trigonometric function for θ′. Finally, use the quadrant location to determine the sign of the trigonometric function value of θ.

76. When the side lengths of the right triangle form a Pythagorean triple, they are all integers. Then the trigonometric functions will all be ratios of integers, which are rational.

Lesson 9-3

Circular and Periodic Functions

F.TF.2, F.TF.5

LESSON GOAL

Students find trigonometric values by using the unit circle and the properties of periodic functions.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Trigonometric Functions of Special Angles

Develop:

Circular Functions
- Find Sine and Cosine Given a Point on the Unit Circle
- Find Trigonometric Values of Special Angles

Periodic Functions
- Identify the Period of a Function
- Graph Periodic Functions
- Evaluate Trigonometric Expressions

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Measuring Angles and Arcs	●	●		●
Extension: Polar Coordinates		●	●	●
ELL Support				

Language Development Handbook

Assign page 48 of the *Language Development Handbook* to help your students build mathematical language related to finding the values of trigonometric functions by using the unit circle and the properties of periodic functions.

ELL You can use the tips and suggestions on page T48 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Functions

Standards for Mathematical Content:

F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

4 Model with mathematics.

6 Attend to precision.

Coherence

Vertical Alignment

Previous
Students found the values of trigonometric functions by using general angles and reference angles.
F.TF.3

Now
Students find trigonometric values by using the unit circle and the properties of periodic functions.
F.TF.2, F.TF.5

Next
Students will graph and analyze sine and cosine functions.
F.IF.4, F.IF.7e

Rigor

The Three Pillars of Rigor

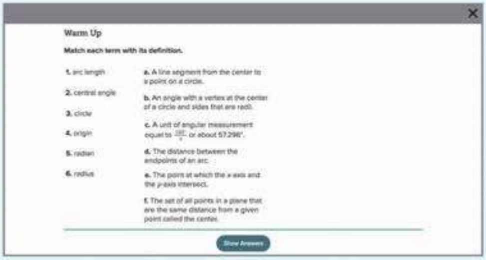
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students develop an understanding of the unit circle and circular functions. They build fluency and apply their understanding by solving real-world problems that involve modeling periodic phenomena.

Mathematical Background

A unit circle – a circle centered at the origin with a radius of 1 unit – can be used to generalize the sine and cosine functions. If the terminal side of an angle θ in standard position intersects the unit circle at a point P with coordinates (x, y), then $\cos \theta = x$ and $\sin \theta = y$.

Interactive Presentation



Warm Up

Match each term with its definition.

1. arc length
2. central angle
3. circle
4. origin
5. radian
6. radius

a. A line segment from the center to a point on a circle.
b. An angle with a vertex at the center of a circle and sides that are radii.
c. A unit of angular measurement equal to $\frac{180°}{\pi}$ or about 57.296°.
d. The distance between the endpoints of an arc.
e. The point at which the x-axis and the y-axis intersect.
f. The set of all points in a plane that are the same distance from a given point called the center.

Warm Up

Launch the Lesson

Most living things, including animals, plants, and even tiny microbes, follow circadian rhythms. Circadian rhythms are physical, mental, and behavioral changes that follow a roughly 24-hour cycle. Health experts have found that the circadian rhythm of your body temperature is related to your sleep cycle and metabolic rate. Graphs that represent these cycles are periodic functions. A **periodic function** has y-values that repeat at regular intervals.

Launch the Lesson

Vocabulary

circular function
A function that describes a point on a circle as the function of an angle defined in radians.

cycle
One complete pattern of a periodic function.

period
The horizontal length of a one cycle.

periodic function
A function with y-values that repeat at regular intervals.

unit circle
A circle with a radius of 1 unit centered at the origin on the coordinate plane.

- How many radians represent a full revolution?
- How can you use symmetry and reflections to help you remember the values on the unit circle?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- defining terms related to circles

Answers:

1. d
2. b
3. f
4. e
5. c
6. a

Launch the Lesson

MP Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Have students read through the infographic. Encourage students to analyze the mathematical relationship between time and body temperature. Ask them to identify the pattern in the graph.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Trigonometric Functions of Special Angles

Objective

Students use a sketch to explore radians and the relationship between central angles and arc length.

Teaching the Mathematical Practices

3 Construct Arguments Throughout the Explore, students will use stated assumptions, definitions, and previously established results to construct arguments about the trigonometric values of special angles.

7 Use Structure Help students to explore how the structure of special right triangles and the unit circle can be used find the trigonometric values of special angles.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will be presented with an Inquiry Question to answer at the end of the activity. They will use a sketch to explore the relationships among trigonometric values of special angles. Students will work through five different exercises. Then students will answer the Inquiry Question.

(continued on the next page)

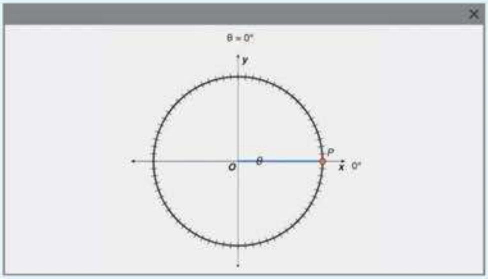
Interactive Presentation

Trigonometric Functions of Special Angles

INQUIRY How can you use special right triangles and the unit circle to find the exact trigonometric values of special angles?

In trigonometry, angles with a reference angle of 30°, 45°, or 60°, are called special angles because the exact trigonometric values for these angles can be found. These trigonometric values can be found by using the relationships between side lengths of special right triangles and a unit circle. A unit circle is a circle with a radius of 1 unit that is centered at the origin on the coordinate plane.

Explore



Explore

WEB SKETCHPAD

Students use the sketch to find the trigonometric values of special angles.

TYPE

Students move through the exercises and answer questions pertaining to the special angles and trigonometric values.

Interactive Presentation

Explore

TYPE

Students respond to the Inquiry Question and can view a sample answer.

Explore Trigonometric Functions of Special Angles (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Using reference angles, what angles will have the same trigonometric ratios as 30˚, ignoring the sign of the ratios? 150°, 210°, and 300°
- Using reference angles, what angles will have the same trigonometric ratios as 45˚, ignoring the sign of the ratios? 135°, 225°, and 315°

Inquiry

How can you use special right triangles and the unit circle to find the exact trigonometric values of special angles? Sample answer: When drawn in the unit circle, each special angle forms a special right triangle with a hypotenuse of length 1 unit. The side lengths of these triangles can then be found by applying the relationships between the sides of special right triangles. Once the side lengths are known, the trigonometric ratios can be used to find the trigonometric values for each special angle.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Circular Functions

Objective

Students find values of trigonometric functions given a point on a unit circle or the measure of a special angle.

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between an angle, the point at which the terminal side intersects the unit circle, and the values of sine and cosine.

What Students Are Learning

A unit circle is a circle with a radius of 1 unit centered at the origin on the coordinate plane. On a unit circle, the radian measure of a central angle $\theta = s$, so the radian measure of an angle is the length of the arc on the unit circle subtended by the angle. Any point $P(x, y)$ on a unit circle represents $(\cos\theta, \sin\theta)$.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

DIFFERENTIATE

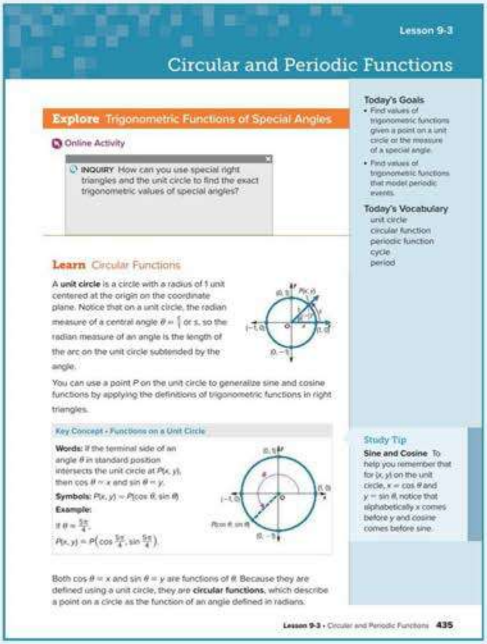
Reteaching Activity AL

IF students are struggling to remember the general coordinates $(\cos\theta, \sin\theta)$ of a unit circle,

THEN remind students that the coordinates are alphabetical, just like (x, y) are written alphabetically.

Language Development Activity ELL

Intermediate Instruct a small group of students to write a paragraph describing what is happening in the figure illustrating the unit circle. Their paragraphs should describe each part of the diagram in their own words. Ask for volunteers to read their paragraphs. Have students ask for clarification as needed.



Lesson 9-3

Circular and Periodic Functions

Explore Trigonometric Functions of Special Angles

Online Activity

INQUIRY How can you use special right triangles and the unit circle to find the exact trigonometric values of special angles?

Learn Circular Functions

A **unit circle** is a circle with a radius of 1 unit centered at the origin on the coordinate plane. Notice that on a unit circle, the radian measure of a central angle $\theta = \frac{s}{1}$ or s, so the radian measure of an angle is the length of the arc on the unit circle subtended by the angle.

You can use a point P on the unit circle to generalize sine and cosine functions by applying the definitions of trigonometric functions in right triangles.

Key Concept • Functions on a Unit Circle

Words: If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos\theta = x$ and $\sin\theta = y$.

Symbols: $P(x, y) = P(\cos\theta, \sin\theta)$

Example:

If $\theta = \frac{5\pi}{4}$,

$P(x, y) = P\left(\cos\frac{5\pi}{4}, \sin\frac{5\pi}{4}\right)$

Both $\cos\theta = x$ and $\sin\theta = y$ are functions of θ. Because they are defined using a unit circle, they are **circular functions**, which describe a point on a circle as the function of an angle defined in radians.

Today's Goals
- Find values of trigonometric functions given a point on a unit circle or the measure of a special angle.
- Find values of trigonometric functions that model periodic events.

Today's Vocabulary
unit circle
circular function
periodic function
cycle
period

Study Tip

Sine and Cosine To help you remember that for (x, y) on the unit circle, $x = \cos\theta$ and $y = \sin\theta$, notice that alphabetically x comes before y and cosine comes before sine.

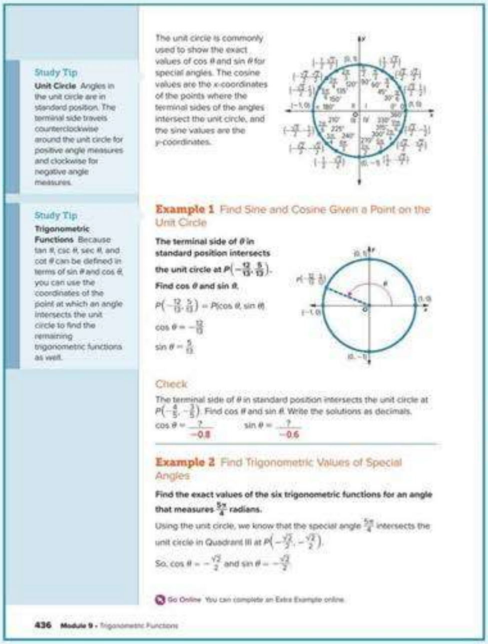
Lesson 9-3 • Circular and Periodic Functions 435

Interactive Presentation

Learn

TAP

Students tap each button to learn more about the relationship between a point P on the unit circle and the sine and cosine functions.



Study Tip

Unit Circle Angles in the unit circle are in standard position. The terminal side travels counterclockwise around the unit circle for positive angle measures and clockwise for negative angle measures.

Study Tip

Trigonometric Functions Because tan θ, csc θ, sec θ, and cot θ can be defined in terms of sin θ and cos θ, you can use the coordinates of the point at which an angle intersects the unit circle to find the remaining trigonometric functions as well.

The unit circle is commonly used to show the exact values of cos θ and sin θ for special angles. The cosine values are the x-coordinates of the points where the terminal sides of the angles intersect the unit circle, and the sine values are the y-coordinates.

Example 1 Find Sine and Cosine Given a Point on the Unit Circle

The terminal side of θ in standard position intersects the unit circle at $P\left(-\frac{12}{13}, \frac{5}{13}\right)$. Find cos θ and sin θ.

$P\left(-\frac{12}{13}, \frac{5}{13}\right) = P(\cos\theta, \sin\theta)$

$\cos\theta = -\frac{12}{13}$

$\sin\theta = \frac{5}{13}$

Check

The terminal side of θ in standard position intersects the unit circle at $P\left(-\frac{4}{5}, -\frac{3}{5}\right)$. Find cos θ and sin θ. Write the solutions as decimals.

$\cos\theta = \underline{\ ?\ }$ −0.8 $\sin\theta = \underline{\ ?\ }$ −0.6

Example 2 Find Trigonometric Values of Special Angles

Find the exact values of the six trigonometric functions for an angle that measures $\frac{5\pi}{4}$ radians.

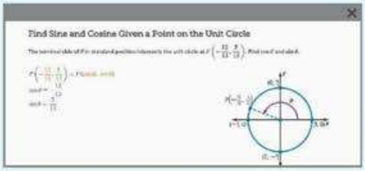
Using the unit circle, we know that the special angle $\frac{5\pi}{4}$ intersects the unit circle in Quadrant III at $P\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

So, $\cos\theta = -\frac{\sqrt{2}}{2}$ and $\sin\theta = -\frac{\sqrt{2}}{2}$

Go Online You can complete an Extra Example online.

436 Module 9 • Trigonometric Functions

Interactive Presentation



Find Sine and Cosine Given a Point on the Unit Circle

Example 1

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 1 Find Sine and Cosine Given a Point on the Unit Circle

Teaching the Mathematical Practices

6 Use Definitions In Example 1, students will use definitions of trigonometric functions on a unit circle to identify the values of sine and cosine for the angle that passes through the given point.

Questions for Mathematical Discourse

AL Why is the x-coordinate of the point equal to $\cos\theta$? Cosine is the adjacent, x, over hypotenuse. The hypotenuse is 1 because the radius of the unit circle is 1.

OL For a given value of θ, would the value of $\cos\theta$ and $\sin\theta$ change if the circle was not a unit circle? No, the values of the trigonometric functions would remain the same; the point on the circle associated with θ would change.

BL What must be true about a point for it to be on the unit circle? $x^2 + y^2 = 1$

Example 2 Find Trigonometric Values of Special Angles

Teaching the Mathematical Practices

7 Use Structure Students will use the structure of the unit circle to identify the values of sine and cosine for a special angle. Then they will calculate the other trigonometric values by using the structure of the functions.

Questions for Mathematical Discourse

AL Why are you able to find all the trigonometric values once you identify $\cos\theta$ and $\sin\theta$? All the other trigonometric ratios can be written in terms of $\cos\theta$ or $\sin\theta$.

OL How can you verify that $\left(-\sqrt{\frac{2}{2}}, -\sqrt{\frac{2}{2}}\right)$ is the correct point for $\frac{5\pi}{4}$ radians? $\frac{5\pi}{4}$ is halfway between π and $\frac{3\pi}{2}$, so the distance from the negative x-axis and the distance from the negative y-axis are the same. $x^2 + y^2 = 1$, substituting $y = x$ yields $x^2 + x^2 = 1$, which means $|x| = \sqrt{\frac{2}{2}}$, and the negatives make sense for the quadrant.

BL A student claims that to evaluate $\tan\theta$ for the special angles, only the numerators of the coordinates are necessary. Determine if the student's claim is correct. Yes; sample answer: When calculating $\frac{\sin\theta}{\cos\theta}$, the denominators of the fractions are always the same and will thus divide out; therefore, only the numerators are necessary.

Common Error

Students may struggle to identify where a special angle is located on the unit circle. Encourage students to identify the four "families" of angles present to help them locate angles more quickly. There are the quadrantal angles and then the angles that measure 30° or $\frac{\pi}{6}$ radians, 45° or $\frac{\pi}{4}$ radians, and 60° or $\frac{\pi}{3}$ radians.

Learn Periodic Functions

Objective

Students find values of trigonometric functions that model periodic events.

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the values of sine and cosine on the unit circle and the graphs of the sine and cosine functions.

Important to Know

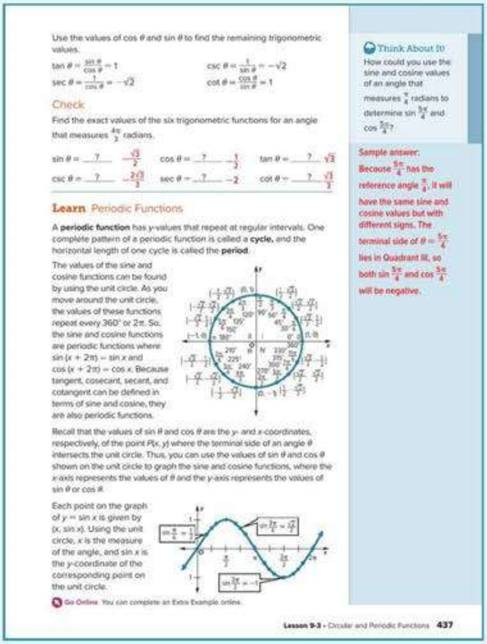
A periodic function has *y*-values that repeat at regular intervals. One complete pattern of a periodic function is called a cycle, and the horizontal length of one cycle is called the period.

Essential Question Follow-Up

Students have begun learning about the graphs of sine and cosine.

Ask:

Why are the graphs of sine and cosine useful in a real-world setting? Sample answer: Day-to-day events such as sunrise and sunset or ocean tides follow a periodic model. Sine and cosine allow us to use a mathematical model to predict values.



Use the values of cos θ and sin θ to find the remaining trigonometric values.

$\tan\theta = \frac{\sin\theta}{\cos\theta} = 1$ $\quad$ $\csc\theta = \frac{1}{\sin\theta} = -\sqrt{2}$

$\sec\theta = \frac{1}{\cos\theta} = -\sqrt{2}$ $\quad$ $\cot\theta = \frac{\cos\theta}{\sin\theta} = 1$

Check

Find the exact values of the six trigonometric functions for an angle that measures $\frac{4\pi}{3}$ radians.

$\sin\theta =$? $-\frac{\sqrt{3}}{2}$ $\quad$ $\cos\theta =$? $-\frac{1}{2}$ $\quad$ $\tan\theta =$? $\sqrt{3}$

$\csc\theta =$? $-\frac{2\sqrt{3}}{3}$ $\quad$ $\sec\theta =$? -2 $\quad$ $\cot\theta =$? $\frac{\sqrt{3}}{3}$

Learn Periodic Functions

A **periodic function** has *y*-values that repeat at regular intervals. One complete pattern of a periodic function is called a **cycle**, and the horizontal length of one cycle is called the **period**.

The values of the sine and cosine functions can be found by using the unit circle. As you move around the unit circle, the values of these functions repeat every 360° or 2π. So, the sine and cosine functions are periodic functions where $\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x$. Because tangent, cosecant, secant, and cotangent can be defined in terms of sine and cosine, they are also periodic functions.

Recall that the values of sin θ and cos θ are the *y*- and *x*-coordinates, respectively, of the point $P(x, y)$ where the terminal side of an angle θ intersects the unit circle. Thus, you can use the values of sin θ and cos θ shown on the unit circle to graph the sine and cosine functions, where the *x*-axis represents the values of θ and the *y*-axis represents the values of sin θ or cos θ.

Each point on the graph of $y = \sin x$ is given by $(x, \sin x)$. Using the unit circle, *x* is the measure of the angle, and sin *x* is the *y*-coordinate of the corresponding point on the unit circle.

Go Online You can complete an Extra Example online.

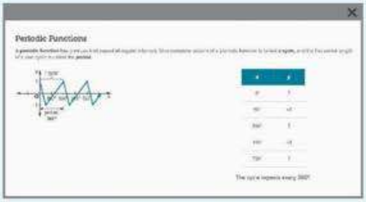
Lesson 9-3 • Circular and Periodic Functions 437

Think About It!

How could you use the sine and cosine values of an angle that measures $\frac{\pi}{4}$ radians to determine $\sin\frac{5\pi}{4}$ and $\cos\frac{5\pi}{4}$?

Sample answer: Because $\frac{5\pi}{4}$ has the reference angle $\frac{\pi}{4}$, it will have the same sine and cosine values but with different signs. The terminal side of $\theta = \frac{5\pi}{4}$ lies in Quadrant III, so both $\sin\frac{5\pi}{4}$ and $\cos\frac{5\pi}{4}$ will be negative.

Interactive Presentation

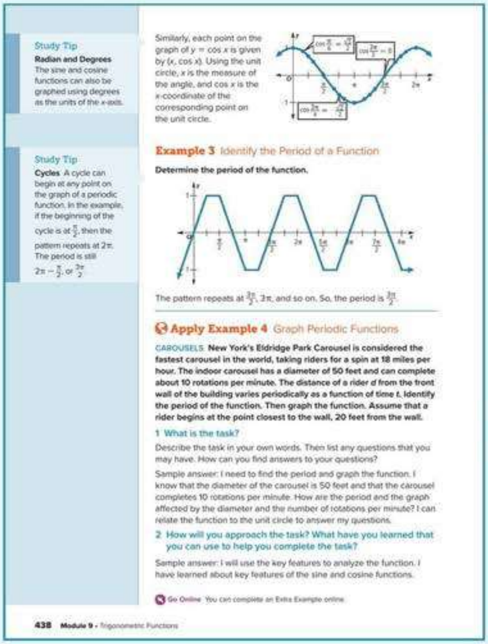


Periodic Functions

Learn

TAP

Students tap to see the relationship between the unit circle and the graphs of $y = \sin x$ and $y = \cos x$.



Study Tip

Radian and Degrees The sine and cosine functions can also be graphed using degrees as the units of the x-axis.

Study Tip

Cycles A cycle can begin at any point on the graph of a periodic function. In the example, if the beginning of the cycle is at $\frac{\pi}{2}$, then the pattern repeats at 2π. The period is still $2\pi - \frac{\pi}{2}$, or $\frac{3\pi}{2}$.

Similarly, each point on the graph of $y = \cos x$ is given by $(x, \cos x)$. Using the unit circle, x is the measure of the angle, and $\cos x$ is the x-coordinate of the corresponding point on the unit circle.

Example 3 Identify the Period of a Function

Determine the period of the function.

The pattern repeats at $\frac{3\pi}{2}$, 3π, and so on. So, the period is $\frac{3\pi}{2}$.

Apply Example 4 Graph Periodic Functions

CAROUSELS **New York's Eldridge Park Carousel is considered the fastest carousel in the world, taking riders for a spin at 18 miles per hour. The indoor carousel has a diameter of 50 feet and can complete about 10 rotations per minute. The distance of a rider *d* from the front wall of the building varies periodically as a function of time *t*. Identify the period of the function. Then graph the function. Assume that a rider begins at the point closest to the wall, 20 feet from the wall.**

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to find the period and graph the function. I know that the diameter of the carousel is 50 feet and that the carousel completes 10 rotations per minute. How are the period and the graph affected by the diameter and the number of rotations per minute? I can relate the function to the unit circle to answer my questions.

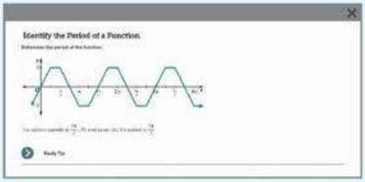
2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I will use the key features to analyze the function. I have learned about key features of the sine and cosine functions.

Go Online You can complete an Extra Example online.

438 Module 9 • Trigonometric Functions

Interactive Presentation



Identify the Period of a Function

Determine the period of the function.

Example 3

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Example 3 Identify the Period of a Function

MP Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in the graph to identify the period of the function.

Questions for Mathematical Discourse

AL What does the period of a function represent? length of one cycle

OL If the graph of the function were stretched vertically, would the period of the function change? Explain. No; sample answer: The period represents the horizontal length of a cycle, and stretching the graph vertically will not change the horizontal length.

BL If the graph was shifted right by $\frac{\pi}{4}$, how could you find the period? Sample answer: Y ou could use the *x*-intercepts: $\frac{7\pi}{4} - \frac{\pi}{4} = \frac{3\pi}{2}$.

Apply Example 4 Graph Periodic Functions

Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them, 4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

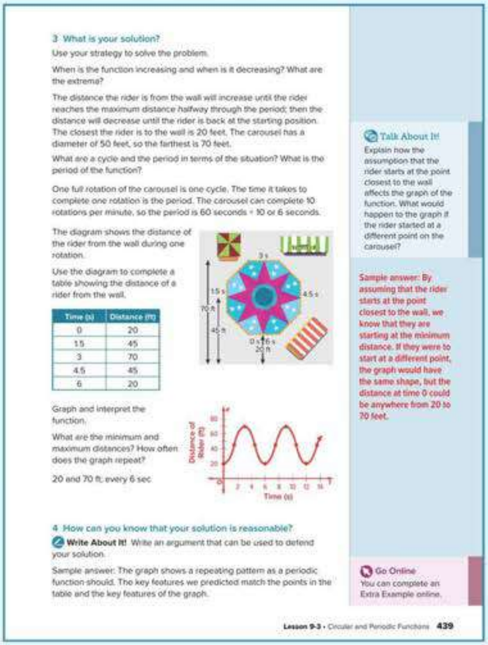
Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- If the diameter of the carousel increased, would the number of rotations per minute necessarily change? If yes, would the number increase or decrease?
- What are the dependent and independent variables?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.



3 What is your solution?

Use your strategy to solve the problem.

When is the function increasing and when is it decreasing? What are the extrema?

The distance the rider is from the wall will increase until the rider reaches the maximum distance halfway through the period; then the distance will decrease until the rider is back at the starting position. The closest the rider is to the wall is 20 feet. The carousel has a diameter of 50 feet, so the farthest is 70 feet.

What are a cycle and the period in terms of the situation? What is the period of the function?

One full rotation of the carousel is one cycle. The time it takes to complete one rotation is the period. The carousel can complete 10 rotations per minute, so the period is 60 seconds ÷ 10 or 6 seconds.

The diagram shows the distance of the rider from the wall during one rotation.

Use the diagram to complete a table showing the distance of a rider from the wall.

Time (s)	Distance (ft)
0	20
1.5	45
3	70
4.5	45
6	20

Graph and interpret the function.

What are the minimum and maximum distances? How often does the graph repeat?

20 and 70 ft; every 6 sec

Talk About It! Explain how the assumption that the rider starts at the point closest to the wall affects the graph of the function. What would happen to the graph if the rider started at a different point on the carousel?

Sample answer: By assuming that the rider starts at the point closest to the wall, we know that they are starting at the minimum distance. If they were to start at a different point, the graph would have the same shape, but the distance at time 0 could be anywhere from 20 to 70 feet.

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: The graph shows a repeating pattern as a periodic function should. The key features we predicted match the points in the table and the key features of the graph.

Go Online You can complete an Extra Example online.

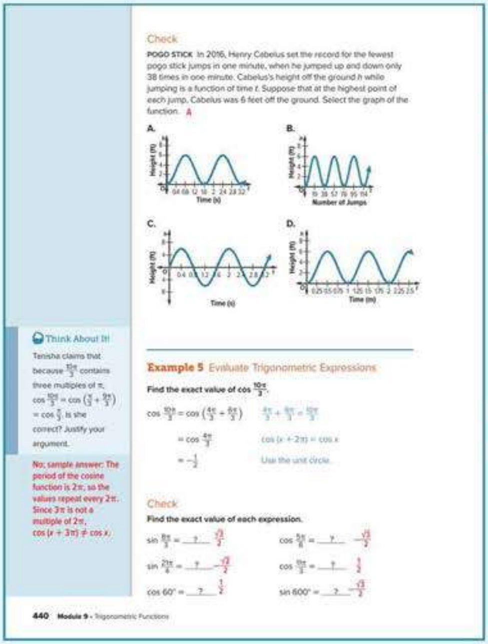
Lesson 9-3 • Circular and Periodic Functions 439

Interactive Presentation

Apply Example 4

TYPE

Students explain how the given assumption may affect the graph.



Check

POGO STICK In 2016, Henry Cabelus set the record for the fewest pogo stick jumps in one minute, when he jumped up and down only 38 times in one minute. Cabelus's height off the ground h while jumping is a function of time t. Suppose that at the highest point of each jump, Cabelus was 6 feet off the ground. Select the graph of the function. A

Think About It!

Tenisha claims that because $\frac{10\pi}{3}$ contains three multiples of π, $\cos \frac{10\pi}{3} = \cos\left(\frac{\pi}{3} + \frac{9\pi}{3}\right) = \cos \frac{\pi}{3}$. Is she correct? Justify your argument.

No; sample answer: The period of the cosine function is 2π, so the values repeat every 2π. Since 3π is not a multiple of 2π, $\cos(x + 3\pi) \neq \cos x$.

Example 5 Evaluate Trigonometric Expressions

Find the exact value of $\cos \frac{10\pi}{3}$.

$\cos \frac{10\pi}{3} = \cos\left(\frac{4\pi}{3} + \frac{6\pi}{3}\right)$ $\frac{4\pi}{3} + \frac{6\pi}{3} = \frac{10\pi}{3}$

$= \cos \frac{4\pi}{3}$ $\cos(x + 2\pi) = \cos x$

$= -\frac{1}{2}$ Use the unit circle.

Check

Find the exact value of each expression.

$\sin \frac{8\pi}{3} = \underline{\ ?\ }$ $\frac{\sqrt{3}}{2}$

$\cos \frac{5\pi}{6} = \underline{\ ?\ }$ $-\frac{\sqrt{3}}{2}$

$\sin \frac{21\pi}{4} = \underline{\ ?\ }$ $-\frac{\sqrt{2}}{2}$

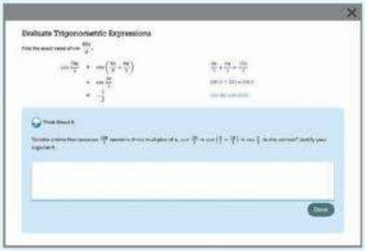
$\cos \frac{19\pi}{3} = \underline{\ ?\ }$ $\frac{1}{2}$

$\cos 60^\circ = \underline{\ ?\ }$ $\frac{1}{2}$

$\sin 600^\circ = \underline{\ ?\ }$ $-\frac{\sqrt{3}}{2}$

440 **Module 9** • Trigonometric Functions

Interactive Presentation



Evaluate Trigonometric Expressions

Example 5

TYPE

Students evaluate a claim and explain their reasoning.

Example 5 Evaluate T rigonometric Expressions

Teaching the Mathematical Practices

3 Find the Error The Think About It! feature requires students read Tanisha's argument, decide whether it makes sense, and explain the flaw in her reasoning.

Questions for Mathematical Discourse

AL Why is $\frac{10\pi}{3}$ split into $\frac{4\pi}{3} + \frac{6\pi}{3}$? This allows you to use an angle on the unit circle because $\frac{6\pi}{3} = 2\pi$, and $\cos(x + 2\pi) = \cos x$, so $\cos\left(\frac{4\pi}{3} + 2\pi\right) = \cos\left(\frac{4\pi}{3}\right)$. $\frac{4\pi}{3}$ is a special angle on the unit circle for which you know the value of the cosine function.

OL How do you use the unit circle to find trigonometric values for angle measures greater than 2π? Subtract 2π or an integer multiple of 2π from the angle to get a coterminal angle with measure less than 2π.

BL Given n is an integer, what is $\cos(4 + 6n)\frac{\pi}{3}$? $-\frac{1}{2}$

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–26
2	exercises that use a variety of skills from this lesson	27–36
2	exercises that extend concepts learned in this lesson to new contexts	37–40
3	exercises that emphasize higher-order and critical-thinking skills	41–45

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **BL**
THEN assign:

- Practice Exercises 1–39 odd, 41–45
- Extension: Polar Coordinates
- ALEKS Trigonometric Functions of Angles; The Unit Circle

IF students score 66%–89% on the Checks, **OL**
THEN assign:

- Practice Exercises 1–45 odd
- Remediation, Review Resources: Measuring Angles and Arcs
- Personal Tutors
- Extra Examples 1–5
- ALEKS Chords and Arcs

IF students score 65% or less on the Checks, **AL**
THEN assign:

- Practice Exercises 1–25 odd
- Remediation, Review Resources: Measuring Angles and Arcs
- *Quick Review Math Handbook*: Circular and Periodic Functions
- ALEKS Chords and Arcs

Answers

7. $\sin\theta = \frac{1}{2}, \cos\theta = \frac{\sqrt{3}}{2}, \tan\theta = \frac{\sqrt{3}}{3}, \csc\theta = 2, \sec\theta = \frac{2\sqrt{3}}{3}, \cot\theta = \sqrt{3}$

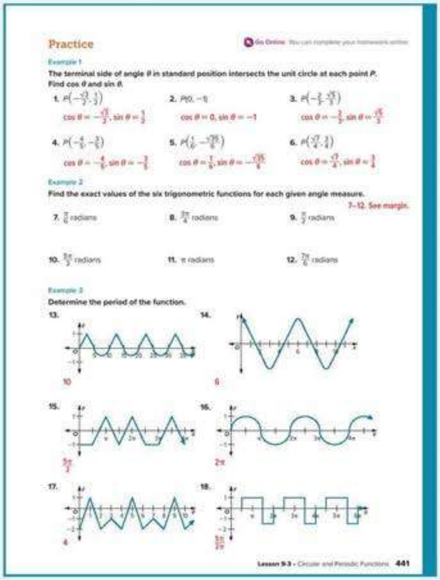
8. $\sin\theta = \frac{\sqrt{2}}{2}; \cos\theta = -\frac{\sqrt{2}}{2}, \tan\theta = -1, \csc\theta = \sqrt{2}, \sec\theta = -\sqrt{2}, \cot\theta = -1$

9. $\sin\theta = 1; \cos\theta = 0, \tan\theta =$ undefined, $\csc\theta = 1, \sec\theta =$ undefined, $\cot\theta = 0$

10. $\sin\theta = -\frac{\sqrt{3}}{2}; \cos\theta = \frac{1}{2}, \tan\theta = -\sqrt{3}, \csc\theta = -\frac{2\sqrt{3}}{3} \sec\theta = 2, \cot\theta = -\frac{\sqrt{3}}{3}$

11. $\sin\theta = 0; \cos\theta = -1, \tan\theta = 0, \csc\theta =$ undefined, $\sec\theta = -1, \cot\theta =$ undefined

12. $\sin\theta = -\frac{1}{2}; \cos\theta = -\frac{\sqrt{3}}{2}, \tan\theta = \frac{\sqrt{3}}{3}, \csc\theta = -2, \sec\theta = \frac{-2\sqrt{3}}{3}, \cot\theta = \sqrt{3}$



Practice

Go Online You can complete your homework online.

Example 1

The terminal side of angle θ in standard position intersects the unit circle at each point P. Find cos θ and sin θ.

2. P(0, −1) cos θ = 0, sin θ = −1

Example 2

Find the exact values of the six trigonometric functions for each given angle measure. 7–12. See margin.

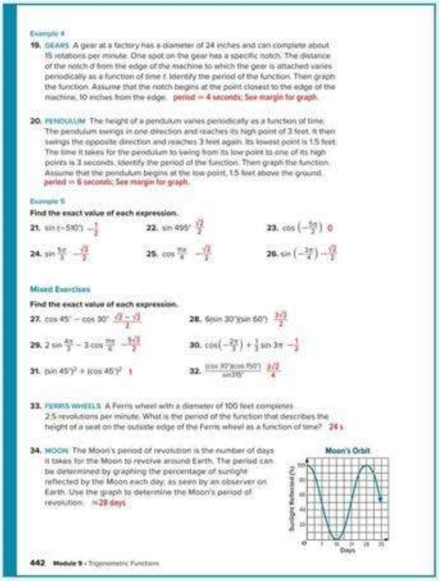
7. $\frac{\pi}{6}$ radians
8. $\frac{3\pi}{4}$ radians
9. $\frac{\pi}{2}$ radians
10. $\frac{5\pi}{3}$ radians
11. π radians
12. $\frac{7\pi}{6}$ radians

Example 3

Determine the period of the function.

13. 10
14. 6
15.
16. 2π
17. 4
18.

Lesson 9-3 • Circular and Periodic Functions 441



Example 4

19. GEARS A gear at a factory has a diameter of 24 inches and can complete about 15 rotations per minute. One spot on the gear has a specific notch. The distance of the notch d from the edge of the machine to which the gear is attached varies periodically as a function of time t. Identify the period of the function. Then graph the function. Assume that the notch begins at the point closest to the edge of the machine, 10 inches from the edge. period = 4 seconds; See margin for graph.

20. PENDULUM The height of a pendulum varies periodically as a function of time. The pendulum swings in one direction and reaches its high point of 3 feet. It then swings the opposite direction and reaches 3 feet again. Its lowest point is 1.5 feet. The time it takes for the pendulum to swing from its low point to one of its high points is 3 seconds. Identify the period of the function. Then graph the function. Assume that the pendulum begins at the low point, 1.5 feet above the ground. period = 6 seconds; See margin for graph.

Example 5

Find the exact value of each expression.

21. sin (−510°) $-\frac{1}{2}$
22. sin 495° $\frac{\sqrt{2}}{2}$
23.
24.
25.
26.

Mixed Exercises

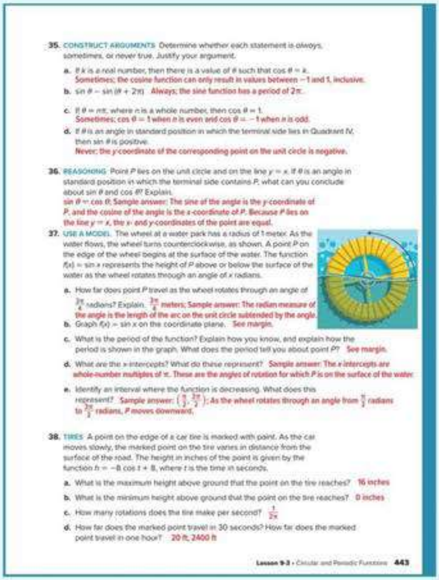
Find the exact value of each expression.

27. cos 45° − cos 30° $\frac{\sqrt{2}-\sqrt{3}}{2}$
28. 6(sin 30°)(sin 60°) $\frac{3\sqrt{3}}{2}$
29.
30.
31. $(\sin 45°)^2 + (\cos 45°)^2$ 1
32.

33. FERRIS WHEELS A Ferris wheel with a diameter of 100 feet completes 2.5 revolutions per minute. What is the period of the function that describes the height of a seat on the outside edge of the Ferris wheel as a function of time? 24 s

34. MOON The Moon's period of revolution is the number of days it takes for the Moon to revolve around Earth. The period can be determined by graphing the percentage of sunlight reflected by the Moon each day, as seen by an observer on Earth. Use the graph to determine the Moon's period of revolution. ≈28 days

442 Module 9 • Trigonometric Functions



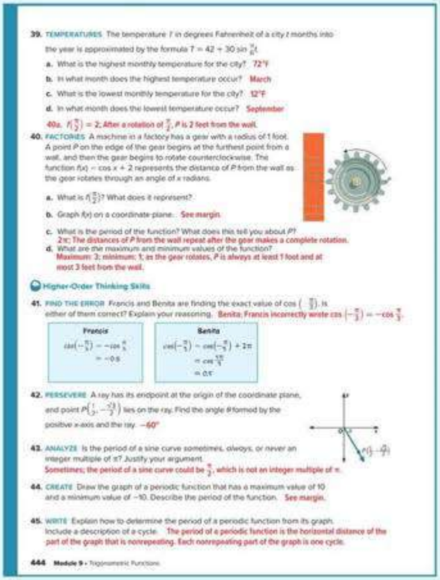
35. CONSTRUCT ARGUMENTS Determine whether each statement is always, sometimes, or never true. Justify your argument.
   a. If k is a real number, then there is a value of θ such that $\cos\theta = k$. Sometimes; the cosine function can only result in values between −1 and 1, inclusive.
   b. $\sin\theta = \sin(\theta + 2\pi)$ Always; the sine function has a period of 2π.
   c. If $\theta = n\pi$, where n is a whole number, then $\cos\theta = 1$. Sometimes; $\cos\theta = 1$ when n is even and $\cos\theta = -1$ when n is odd.
   d. If θ is an angle in standard position in which the terminal side lies in Quadrant IV, then $\sin\theta$ is positive. Never; the y-coordinate of the corresponding point on the unit circle is negative.

36. REASONING Point P lies on the unit circle and on the line $y = x$. If θ is an angle in standard position in which the terminal side contains P, what can you conclude about $\sin\theta$ and $\cos\theta$? Explain. $\sin\theta = \cos\theta$; Sample answer: The sine of the angle is the y-coordinate of P, and the cosine of the angle is the x-coordinate of P. Because P lies on the line $y = x$, the x- and y-coordinates of the point are equal.

37. USE A MODEL The wheel at a water park has a radius of 1 meter. As the water flows, the wheel turns counterclockwise, as shown. A point P on the edge of the wheel begins at the surface of the water. The function $f(x) = \sin x$ represents the height of P above or below the surface of the water as the wheel rotates through an angle of x radians.
   a. How far does point P travel as the wheel rotates through an angle of $\frac{3\pi}{4}$ radians? Explain. $\frac{3\pi}{4}$ meters; Sample answer: The radian measure of the angle is the length of the arc on the unit circle subtended by the angle.
   b. Graph $f(x) = \sin x$ on the coordinate plane. See margin.
   c. What is the period of the function? Explain how you know, and explain how the period is shown in the graph. What does the period tell you about point P? See margin.
   d. What are the x-intercepts? What do these represent? Sample answer: The x-intercepts are whole-number multiples of π. These are the angles of rotation for which P is on the surface of the water.
   e. Identify an interval where the function is decreasing. What does this represent? Sample answer: $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$; As the wheel rotates through an angle from $\frac{\pi}{2}$ radians to $\frac{3\pi}{2}$ radians, P moves downward.

38. TIRES A point on the edge of a car tire is marked with paint. As the car moves slowly, the marked point on the tire varies in distance from the surface of the road. The height in inches of the point is given by the function $h = -8\cos t + 8$, where t is the time in seconds.
   a. What is the maximum height above ground that the point on the tire reaches? 16 inches
   b. What is the minimum height above ground that the point on the tire reaches? 0 inches
   c. How many rotations does the tire make per second? $\frac{1}{2\pi}$
   d. How far does the marked point travel in 30 seconds? How far does the marked point travel in one hour? 20 ft, 2400 ft

Lesson 9-3 • Circular and Periodic Functions 443



39. TEMPERATURES The temperature T in degrees Fahrenheit of a city t months into the year is approximated by the formula $T = 42 + 30\sin\frac{\pi}{6}t$.
   a. What is the highest monthly temperature for the city? 72°F
   b. In what month does the highest temperature occur? March
   c. What is the lowest monthly temperature for the city? 12°F
   d. In what month does the lowest temperature occur? September

40a. $f\left(\frac{\pi}{2}\right) = 2$; After a rotation of $\frac{\pi}{2}$, P is 2 feet from the wall.

40. FACTORIES A machine in a factory has a gear with a radius of 1 foot. A point P on the edge of the gear begins at the furthest point from a wall, and then the gear begins to rotate counterclockwise. The function $f(x) = \cos x + 2$ represents the distance of P from the wall as the gear rotates through an angle of x radians.
   a. What is $f\left(\frac{\pi}{2}\right)$? What does it represent?
   b. Graph $f(x)$ on a coordinate plane. See margin.
   c. What is the period of the function? What does this tell you about P? 2π; The distances of P from the wall repeat after the gear makes a complete rotation.
   d. What are the maximum and minimum values of the function? Maximum: 3; minimum: 1; as the gear rotates, P is always at least 1 foot and at most 3 feet from the wall.

Higher-Order Thinking Skills

41. FIND THE ERROR Francis and Benita are finding the exact value of $\cos\left(-\frac{\pi}{3}\right)$. Is either of them correct? Explain your reasoning. Benita; Francis incorrectly wrote $\cos\left(-\frac{\pi}{3}\right) = -\cos\frac{\pi}{3}$.

Francis	Benita
$\cos\left(-\frac{\pi}{3}\right) = -\cos\frac{\pi}{3}$ $= -0.5$	$\cos\left(-\frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3}\right) + 2\pi$ $= \cos\frac{5\pi}{3}$ $= 0.5$

42. PERSEVERE A ray has its endpoint at the origin of the coordinate plane, and point $P\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ lies on the ray. Find the angle θ formed by the positive x-axis and the ray. −60°

43. ANALYZE Is the period of a sine curve sometimes, always, or never an integer multiple of π? Justify your argument. Sometimes; the period of a sine curve could be $\frac{\pi}{2}$, which is not an integer multiple of π.

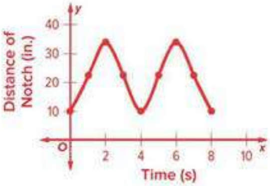
44. CREATE Draw the graph of a periodic function that has a maximum value of 10 and a minimum value of −10. Describe the period of the function. See margin.

45. WRITE Explain how to determine the period of a periodic function from its graph. Include a description of a cycle. The period of a periodic function is the horizontal distance of the part of the graph that is nonrepeating. Each nonrepeating part of the graph is one cycle.

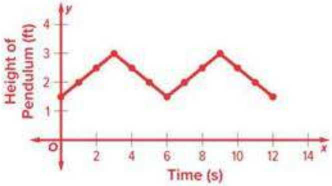
444 Module 9 • Trigonometric Functions

Answers

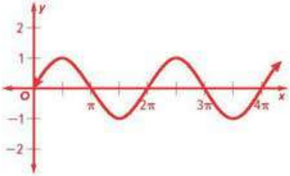
19.



20.

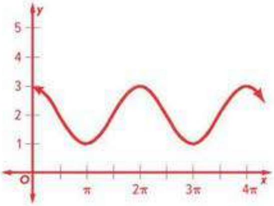


37b.

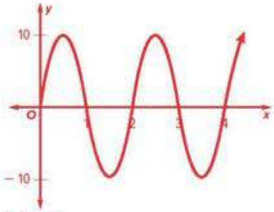


37c. The period is 2π because the values of the function repeat every 2π radians. This is shown in the graph when the shape of the curve from 0 to 2π is repeated from 2π to 4π. The heights of point *P* above or below the surface of the water repeat once as the wheel makes a complete rotation (2π radians).

40b.



44. Sample answer:



Period: 2

Lesson 9-4

Graphing Sine and Cosine Functions

F.IF.4, F.IF.7e

LESSON GOAL

Students graph and analyze sine and cosine functions.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Graphing Sine and Cosine Functions
- Identify the Amplitude and Period from a Graph
- Identify the Amplitude and Period from an Equation
- Graph a Sine Function
- Graph a Cosine Function

Modeling with Sine and Cosine Functions
- Characteristics of the Sine and Cosine Functions
- Model Periodic Situations

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Trigonometric Functions of General Angles	●	●		●
Extension: Blueprints		●	●	●
ELL Support				

Language Development Handbook

Assign page 49 of the *Language Development Handbook* to help your students build mathematical language related to graphing and analyzing sine and cosine functions.

ELL You can use the tips and suggestions on page T49 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.
2 Reason abstractly and quantitatively.
7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students found trigonometric values by using the unit circle and the properties of periodic functions.
F.TF.2, F.TF.5

Now
Students graph and analyze sine and cosine functions.
F.IF.4, F.IF.7e

Next
Students will graph and analyze tangent and reciprocal trigonometric functions.
F.IF.4, F.IF.7e

Rigor

The Three Pillars of Rigor

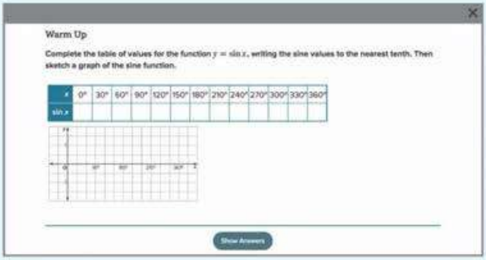
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand on their understanding of trigonometric ratios to include graphs of trigonometric functions. They build fluency by graphing the functions, and they apply their understanding by solving real-world problems related to trigonometric functions.

Mathematical Background

The graphs of the sine and cosine functions have amplitude. The graphs of the other trigonometric functions do not have amplitude because there is no maximum or minimum value. The period is the distance along the horizontal axis required for the graph to complete one cycle. The period is easy to determine from the graph.

Interactive Presentation



Warm Up

Complete the table of values for the function $y = \sin x$, writing the sine values to the nearest tenth. Then sketch a graph of the sine function.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin x$													

Warm Up

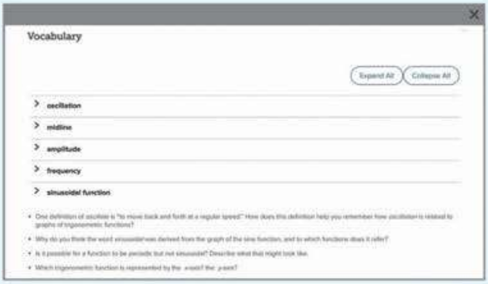
Launch the Lesson

The power of a radio signal is similar to the volume of a sound. The radio signal varies in terms of the frequency and wavelength of the radio waves much like the variation in the pitch of sound waves. This audio clip was recorded by Voyager when it passed Jupiter.

While somewhat difficult to hear, the whistling tone, called a whistler, provided confirmation that there was lightning in Jupiter's atmosphere. Whistlers are just one part of the electromagnetic spectrum of a lightning strike.

Sound waves and radio signals can both be modeled by trigonometric functions.

Launch the Lesson



Vocabulary

- oscillation
- midline
- amplitude
- frequency
- sinusoidal function

- One definition of oscillate is "to move back and forth at a regular speed." How does this definition help you remember how oscillation is related to graphs of trigonometric functions?
- Why do you think the word sinusoidal was derived from the graph of the sine function, and to which functions does it refer?
- Is it possible for a function to be periodic but not sinusoidal? Describe what that might look like.
- Which trigonometric function is represented by the x-axis? the y-axis?

Today's Vocabulary

Warm Up

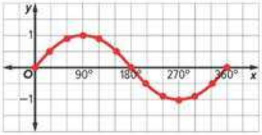
Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- graphing a sine function

Answers:

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sin *x*	**0**	**0.5**	**0.9**	**1**	**0.9**	**0.5**	**0**	**−0.5**	**−0.9**	**−1**	**−0.9**	**−0.5**	**0**



Launch the Lesson

Teaching the Mathematical Practices

2 Reason abstractly and quantitatively Encourage students to make sense of the quantities of frequency and wavelength and their relationship to the graphs of the sine and cosine functions.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Learn Graphing Sine and Cosine Functions

Objective

Students graph and analyze sine and cosine functions and identify the periods, midlines, and amplitudes.

MP Teaching the Mathematical Practices

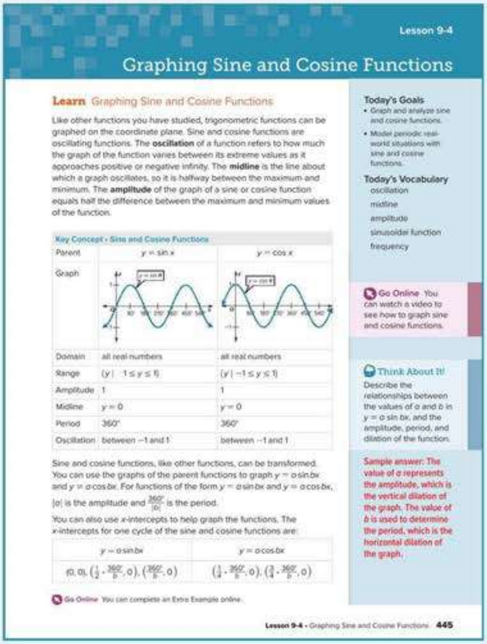
1 Explain Correspondences The Think About It! feature asks students to explain the relationships between the parameters of a sine function and the key features of the graph of the function.

About the Key Concept

Trigonometric functions can be graphed on the coordinate plane. The graphs of the sine and cosine functions oscillate about a midline and have amplitude. They can also be transformed. The graphs of the parent functions help graph $y = a \sin bx$ and $y = a \cos bx$. For functions of the form $y = a \sin bx$ and $y = a \cos bx$, $|a|$ is the amplitude and $\frac{360°}{|b|}$ is the period.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 9-4

Graphing Sine and Cosine Functions

Learn Graphing Sine and Cosine Functions

Like other functions you have studied, trigonometric functions can be graphed on the coordinate plane. Sine and cosine functions are oscillating functions. The **oscillation** of a function refers to how much the graph of the function varies between its extreme values as it approaches positive or negative infinity. The **midline** is the line about which a graph oscillates, so it is halfway between the maximum and minimum. The **amplitude** of the graph of a sine or cosine function equals half the difference between the maximum and minimum values of the function.

Key Concept • Sine and Cosine Functions

Parent	$y = \sin x$	$y = \cos x$
Graph		
Domain	all real numbers	all real numbers
Range	$\{y \mid -1 \le y \le 1\}$	$\{y \mid -1 \le y \le 1\}$
Amplitude	1	1
Midline	$y = 0$	$y = 0$
Period	360°	360°
Oscillation	between −1 and 1	between −1 and 1

Sine and cosine functions, like other functions, can be transformed. You can use the graphs of the parent functions to graph $y = a \sin bx$ and $y = a \cos bx$. For functions of the form $y = a \sin bx$ and $y = a \cos bx$, $|a|$ is the amplitude and $\frac{360°}{|b|}$ is the period.

You can also use x-intercepts to help graph the functions. The x-intercepts for one cycle of the sine and cosine functions are:

$y = a \sin bx$	$y = a \cos bx$
$(0, 0), \left(\frac{1}{2} \cdot \frac{360°}{b}, 0\right), \left(\frac{360°}{b}, 0\right)$	$\left(\frac{1}{4} \cdot \frac{360°}{b}, 0\right), \left(\frac{3}{4} \cdot \frac{360°}{b}, 0\right)$

Go Online You can complete an Extra Example online.

Today's Goals
- Graph and analyze sine and cosine functions.
- Model periodic real-world situations with sine and cosine functions.

Today's Vocabulary
oscillation
midline
amplitude
sinusoidal function
frequency

Go Online You can watch a video to see how to graph sine and cosine functions.

Think About It!
Describe the relationships between the values of a and b in $y = a \sin bx$, and the amplitude, period, and dilation of the function.

Sample answer: The value of a represents the amplitude, which is the vertical dilation of the graph. The value of b is used to determine the period, which is the horizontal dilation of the graph.

Lesson 9-4 • Graphing Sine and Cosine Functions 445

Interactive Presentation

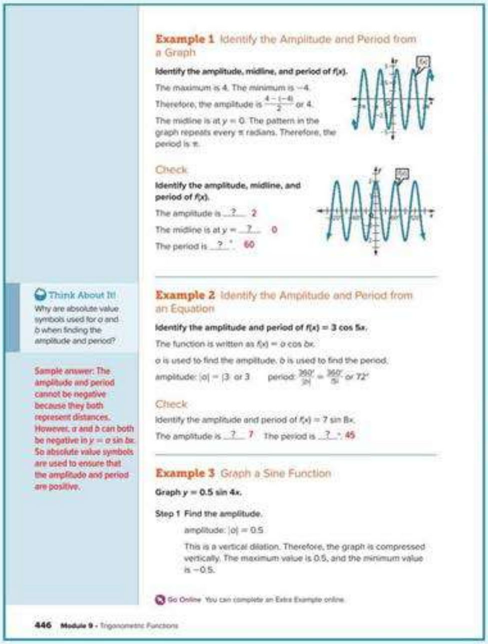
Graphing Sine and Cosine Functions

Learn

TYPE

a|

Students describe the relationship between the values of a and b in $y = a \sin bx$ and $y = a \cos bx$ and the amplitude, period, and dilation of the function.



Example 1 Identify the Amplitude and Period from a Graph

Identify the amplitude, midline, and period of $f(x)$.

The maximum is 4. The minimum is −4.

Therefore, the amplitude is $\frac{4-(-4)}{2}$ or 4.

The midline is at $y = 0$. The pattern in the graph repeats every π radians. Therefore, the period is π.

Check

Identify the amplitude, midline, and period of $f(x)$.

The amplitude is ? 2

The midline is at $y =$? 0

The period is ?° 60

Example 2 Identify the Amplitude and Period from an Equation

Identify the amplitude and period of $f(x) = 3 \cos 5x$.

The function is written as $f(x) = a \cos bx$.

a is used to find the amplitude. b is used to find the period.

amplitude: $|a| = |3|$ or 3 period: $\frac{360°}{|b|} = \frac{360°}{|5|}$ or 72°

Check

Identify the amplitude and period of $f(x) = 7 \sin 8x$.

The amplitude is ? 7 The period is ?° 45

Example 3 Graph a Sine Function

Graph $y = 0.5 \sin 4x$.

Step 1 **Find the amplitude.**

amplitude: $|a| = 0.5$

This is a vertical dilation. Therefore, the graph is compressed vertically. The maximum value is 0.5, and the minimum value is −0.5.

Go Online You can complete an Extra Example online.

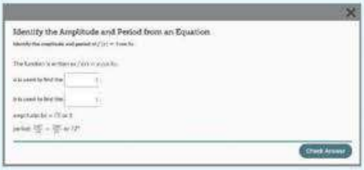
Think About It!

Why are absolute value symbols used for a and b when finding the amplitude and period?

Sample answer: The amplitude and period cannot be negative because they both represent distances. However, a and b can both be negative in $y = a \sin bx$. So absolute value symbols are used to ensure that the amplitude and period are positive.

446 Module 9 • Trigonometric Functions

Interactive Presentation



Identify the Amplitude and Period from an Equation

Example 2

SELECT

Students select the correct vocabulary word for each value.

TYPE

Students explain why absolute value symbols are necessary when finding amplitude and period.

Example 1 Identify the Amplitude and Period from a Graph

MP Teaching the Mathematical Practices

7 Look for a Pattern Help students to see the pattern in the graph to identify the period of the function.

Questions for Mathematical Discourse

AL What does the amplitude of the graph of a sine function represent? The amplitude is half of the distance between the maximum and minimum values of the function.

OL Assuming there is no horizontal translation, is the graphed function a transformation of the sine or cosine function? Explain. Sine; sample answer: The function intersects (0, 0).

BL What are the values of a and b for this transformation of the parent function? $a = 4$, $b = 2$

Example 2 Identify the Amplitude and Period from an Equation

MP Teaching the Mathematical Practices

2 Attend to Quantities The Think About It! feature asks students to consider the meaning of the quantities when finding amplitude and period.

Questions for Mathematical Discourse

AL How do you determine the minimum and maximum values of the function? The minimum is $-|a|$, which is −3, and the maximum is $|a|$, which is 3.

OL How would you find the period in radians? Use the period of the parent function in radians instead of degrees: $360° = 2\pi$ radians, so the period is $\frac{2\pi}{|b|}$.

BL As b increases, what happens to the period? Explain. It decreases because b is the denominator of the period.

Example 3 Graph a Sine Function

MP Teaching the Mathematical Practices

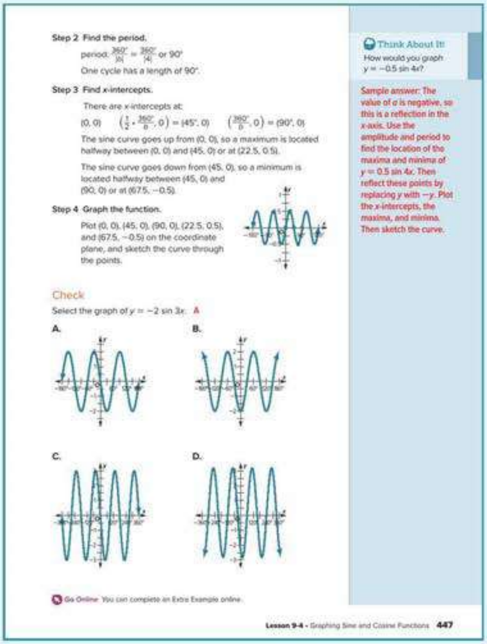
7 Use Structure Students should use the key features of the sine function to graph the function.

Questions for Mathematical Discourse

AL How do the period and amplitude of this function compare to those of the parent function? Both the period and the amplitude are smaller compared to the parent function.

OL How many periods of the function would occur in the interval [0°, 360°]? 4

BL What is the average rate of change for a 90° interval of the function? Explain. It is zero for any 90° interval of the function because the period is 90°.



Step 2 Find the period.

period: $\frac{360°}{|b|} = \frac{360°}{|4|}$ or 90°

One cycle has a length of 90°.

Step 3 Find *x*-intercepts.

There are *x*-intercepts at:

$(0, 0)$ $\left(\frac{1}{2} \cdot \frac{360°}{b}, 0\right) = (45°, 0)$ $\left(\frac{360°}{b}, 0\right) = (90°, 0)$

The sine curve goes up from (0, 0), so a maximum is located halfway between (0, 0) and (45, 0) or at (22.5, 0.5).

The sine curve goes down from (45, 0), so a minimum is located halfway between (45, 0) and (90, 0) or at (67.5, −0.5).

Step 4 Graph the function.

Plot (0, 0), (45, 0), (90, 0), (22.5, 0.5), and (67.5, −0.5) on the coordinate plane, and sketch the curve through the points.

Check

Select the graph of $y = -2 \sin 3x$. A

A.

B.

C.

D.

Go Online You can complete an Extra Example online.

Think About It!

How would you graph $y = -0.5 \sin 4x$?

Sample answer: The value of *a* is negative, so this is a reflection in the *x*-axis. Use the amplitude and period to find the location of the maxima and minima of $y = 0.5 \sin 4x$. Then reflect these points by replacing *y* with −*y*. Plot the *x*-intercepts, the maxima, and minima. Then sketch the curve.

Lesson 9-4 • Graphing Sine and Cosine Functions 447

Interactive Presentation

Graph a Sine Function

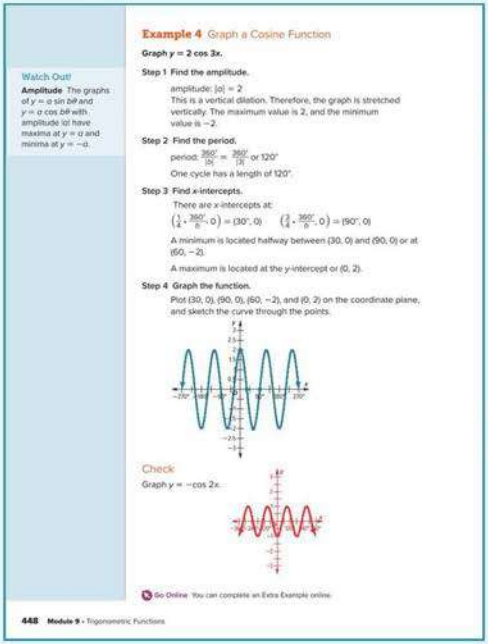
Example 3

TAP

Students move through the steps to see how to graph a sine function.

TYPE

Students explain how to graph a sine function.



Watch Out!

Amplitude The graphs of $y = a \sin b\theta$ and $y = a \cos b\theta$ with amplitude $|a|$ have maxima at $y = a$ and minima at $y = -a$.

Example 4 Graph a Cosine Function

Graph $y = 2 \cos 3x$.

Step 1 Find the amplitude.

amplitude: $|a| = 2$

This is a vertical dilation. Therefore, the graph is stretched vertically. The maximum value is 2, and the minimum value is −2.

Step 2 Find the period.

period: $\frac{360°}{|b|} = \frac{360°}{|3|}$ or 120°

One cycle has a length of 120°.

Step 3 Find x-intercepts.

There are x-intercepts at:

$\left(\frac{1}{4} \cdot \frac{360°}{6}, 0\right) = (30°, 0)$ $\left(\frac{3}{4} \cdot \frac{360°}{6}, 0\right) = (90°, 0)$

A minimum is located halfway between (30, 0) and (90, 0) or at (60, −2).

A maximum is located at the y-intercept or (0, 2).

Step 4 Graph the function.

Plot (30, 0), (90, 0), (60, −2), and (0, 2) on the coordinate plane, and sketch the curve through the points.

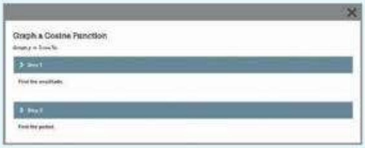
Check

Graph $y = -\cos 2x$.

Go Online You can complete an Extra Example online.

448 Module 9 • Trigonometric Functions

Interactive Presentation



Graph a Cosine Function

Example 4

EXPAND

Students move through the steps to learn how to graph a cosine function.

TYPE

Students explain how to graph a cosine function.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 4 Graph a Cosine Function

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the function, key features, and graph of a cosine function.

Questions for Mathematical Discourse

AL What are the similarities and differences in graphing a cosine function compared to a sine function? The functions have the same periodic behavior and transform similarly. The y-intercept for the parent sine function occurs at $y = 0$, while the y-intercept for the parent cosine function occurs at the maximum y-value.

OL How do you find the x-values where the minima occur? The minima occur at odd integer multiples of half the period: $60°(2n + 1)$, or $(120°n + 60°)$.

BL What happens if you shift the function to the right 30°? Sample answer: It looks like $y = 2 \sin 3x$.

Learn Modeling with Sine and Cosine Functions

Objective

Students model periodic real-world situations with sine and cosine functions.

Teaching the Mathematical Practices

6 Use Quantities Use the Think About It! question to guide students to clarifying their use of quantities when modeling periodic phenomena. Ensure that they specify the units of measure used in a problem and label axes appropriately.

About the Key Concept

A sinusoidal function is a function that can be produced by translating, reflecting, or dilating the sine function. The frequency is the number of cycles in a given unit of time. The frequency of the graph of a function is the reciprocal of the period of the function. So, if the period is $\frac{1}{100}$ of a second, then the frequency is 100 cycles per second.

Example 5 Characteristics of the Sine and Cosine Functions

Teaching the Mathematical Practices

4 Interpret Mathematical Results In Example 5, students must interpret the characteristics of the function in the context of the problem.

Questions for Mathematical Discourse

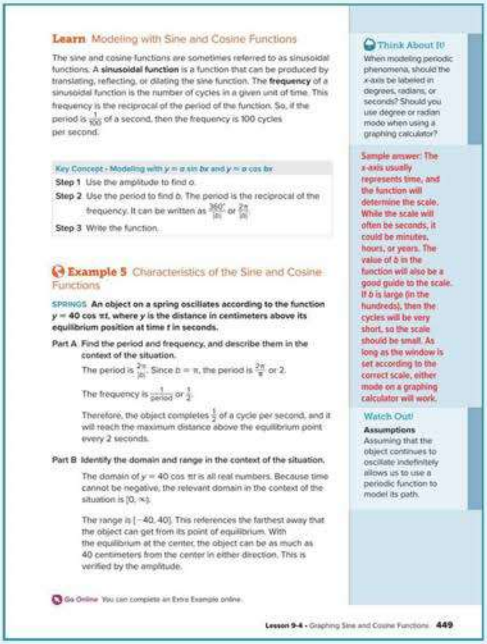
AL Where is the object located at $t = 0$? $y = 40$

OL At what times in the first period does the object reach the equilibrium point? Explain. The equilibrium occurs at $y = 0$, so it occurs at the t-intercepts. The t-intercepts will be $\frac{1}{4}$ and $\frac{3}{4}$ of the period: $\frac{1}{2}$ sec and $\frac{3}{2}$ sec.

BL What must be true of b for the frequency to be a rational number? b must have π as a factor.

Common Error

Many students think frequency and period represent the same thing, but remind students that frequency is the number of cycles in one unit of time.



Learn Modeling with Sine and Cosine Functions

The sine and cosine functions are sometimes referred to as sinusoidal functions. A **sinusoidal function** is a function that can be produced by translating, reflecting, or dilating the sine function. The **frequency** of a sinusoidal function is the number of cycles in a given unit of time. This frequency is the reciprocal of the period of the function. So, if the period is $\frac{1}{100}$ of a second, then the frequency is 100 cycles per second.

Key Concept • Modeling with $y = a \sin bx$ and $y = a \cos bx$

Step 1 Use the amplitude to find a.

Step 2 Use the period to find b. The period is the reciprocal of the frequency. It can be written as $\frac{360°}{|b|}$ or $\frac{2\pi}{|b|}$.

Step 3 Write the function.

Example 5 Characteristics of the Sine and Cosine Functions

SPRINGS **An object on a spring oscillates according to the function $y = 40 \cos \pi t$, where y is the distance in centimeters above its equilibrium position at time t in seconds.**

Part A Find the period and frequency, and describe them in the context of the situation.

The period is $\frac{2\pi}{|b|}$. Since $b = \pi$, the period is $\frac{2\pi}{\pi}$ or 2.

The frequency is $\frac{1}{\text{period}}$ or $\frac{1}{2}$.

Therefore, the object completes $\frac{1}{2}$ of a cycle per second, and it will reach the maximum distance above the equilibrium point every 2 seconds.

Part B Identify the domain and range in the context of the situation.

The domain of $y = 40 \cos \pi t$ is all real numbers. Because time cannot be negative, the relevant domain in the context of the situation is $[0, \infty)$.

The range is $[-40, 40]$. This references the farthest away that the object can get from its point of equilibrium. With the equilibrium at the center, the object can be as much as 40 centimeters from the center in either direction. This is verified by the amplitude.

Go Online You can complete an Extra Example online.

Think About It!

When modeling periodic phenomena, should the x-axis be labeled in degrees, radians, or seconds? Should you use degree or radian mode when using a graphing calculator?

Sample answer: The x-axis usually represents time, and the function will determine the scale. While the scale will often be seconds, it could be minutes, hours, or years. The value of b in the function will also be a good guide to the scale. If b is large (in the hundreds), then the cycles will be very short, so the scale should be small. As long as the window is set according to the correct scale, either mode on a graphing calculator will work.

Watch Out!

Assumptions Assuming that the object continues to oscillate indefinitely allows us to use a periodic function to model its path.

Lesson 9-4 • Graphing Sine and Cosine Functions 449

Interactive Presentation

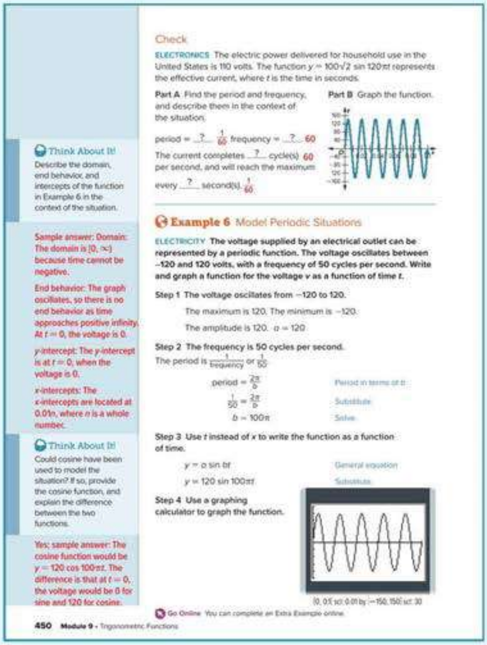
Modeling with Sine and Cosine Functions

Learn

TYPE

a|

Students discuss identifying the independent variable and setting up the coordinate plane when modeling real-world situations.



Check

ELECTRONICS The electric power delivered for household use in the United States is 110 volts. The function $y = 100\sqrt{2} \sin 120\pi t$ represents the effective current, where t is the time in seconds.

Part A Find the period and frequency, and describe them in the context of the situation.

period = ? $\frac{1}{60}$ frequency = ? 60

The current completes ? cycle(s) 60 per second, and will reach the maximum every ? second(s). $\frac{1}{60}$

Part B Graph the function.

Think About It!

Describe the domain, end behavior, and intercepts of the function in Example 6 in the context of the situation.

Sample answer: Domain: The domain is [0, ∞) because time cannot be negative.

End behavior: The graph oscillates, so there is no end behavior as time approaches positive infinity. At $t = 0$, the voltage is 0.

y-intercept: The y-intercept is at $t = 0$, when the voltage is 0.

x-intercepts: The x-intercepts are located at $0.01n$, where n is a whole number.

Example 6 Model Periodic Situations

ELECTRICITY **The voltage supplied by an electrical outlet can be represented by a periodic function. The voltage oscillates between −120 and 120 volts, with a frequency of 50 cycles per second. Write and graph a function for the voltage v as a function of time t.**

Step 1 The voltage oscillates from −120 to 120.

The maximum is 120. The minimum is −120.

The amplitude is 120. $a = 120$

Step 2 The frequency is 50 cycles per second.

The period is $\frac{1}{\text{frequency}}$ or $\frac{1}{50}$.

$\text{period} = \frac{2\pi}{b}$ Period in terms of b

$\frac{1}{50} = \frac{2\pi}{b}$ Substitute.

$b = 100\pi$ Solve.

Step 3 Use t instead of x to write the function as a function of time.

$y = a \sin bt$ General equation

$y = 120 \sin 100\pi t$ Substitute.

Step 4 Use a graphing calculator to graph the function.

[0, 0.1] scl: 0.01 by [−150, 150] scl: 30

Think About It!

Could cosine have been used to model the situation? If so, provide the cosine function, and explain the difference between the two functions.

Yes; sample answer: The cosine function would be $y = 120 \cos 100\pi t$. The difference is that at $t = 0$, the voltage would be 0 for sine and 120 for cosine.

Go Online You can complete an Extra Example online.

450 Module 9 • Trigonometric Functions

Interactive Presentation

Model Periodic Situations

Example 6

TAP

Students move through the steps to write and graph a function for the voltage as a function of time.

TYPE

Students describe the domain, end behavior, and intercepts of the function in the context of the situation.

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING | **2 FLUENCY** | **3 APPLICATION**

Example 6 Model Periodic Situations

Teaching the Mathematical Practices

2 Create Representations Guide students to write a function that models the situation in Example 6.

Questions for Mathematical Discourse

AL If you did not use a graphing calculator, what would you need to do to graph a cycle of this function? Use the period to find the t-intercepts and the t-values for the maximum and minimum.

OL What interval of t would you use to graph the first 15 cycles of the function? Explain. 5 cycles occur for every 10th of a second, so the interval would be [0, 0.3].

BL What happens to the value of b as the frequency increases? b increases.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–20
2	exercises that use a variety of skills from this lesson	21–30
2	exercises that extend concepts learned in this lesson to new contexts	31–34
3	exercises that emphasize higher-order and critical-thinking skills	35–39

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **BL**
THEN assign:

- Practice Exercises 1–33 odd, 35–39
- Extension: Blueprints
- ALEKS Graphs of Sine and Cosine Functions

IF students score 66%–89% on the Checks, **OL**
THEN assign:

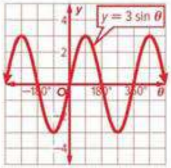
- Practice Exercises 1–39 odd
- Remediation, Review Resources: Trigonometric Functions of General Angles
- Personal Tutors
- Extra Examples 1–6
- ALEKS Trigonometric Functions of Angles

IF students score 65% or less on the Checks, **AL**
THEN assign:

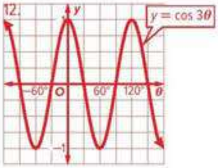
- Practice Exercises 1–19 odd
- Remediation, Review Resources: Trigonometric Functions of General Angles
- *Quick Review Math Handbook:* Graphing Trigonometric Functions
- ALEKS Trigonometric Functions of Angles

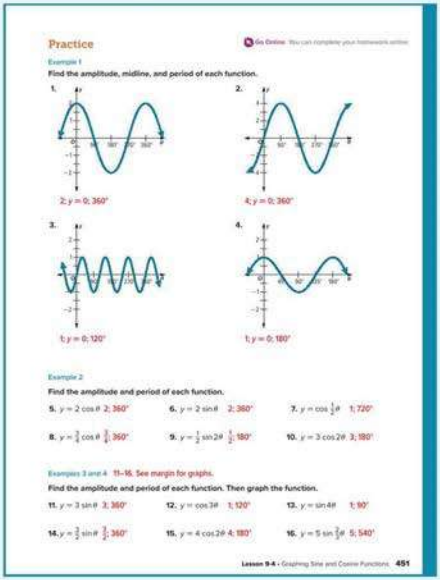
Answers

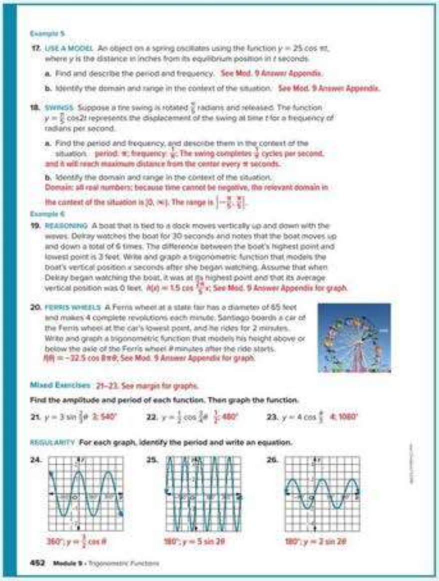
11.

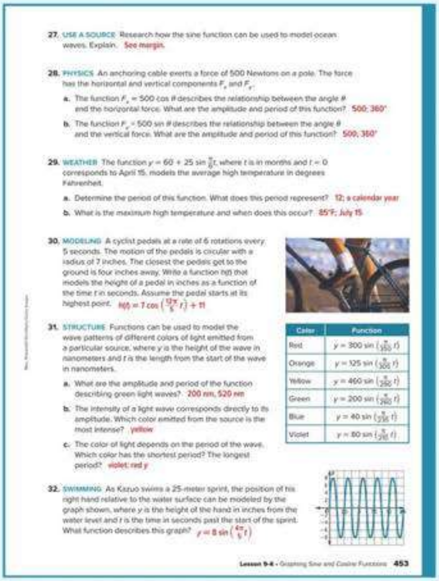


12.









27. USE A SOURCE Research how the sine function can be used to model ocean waves. Explain. See margin.

28. PHYSICS An anchoring cable exerts a force of 500 Newtons on a pole. The force has the horizontal and vertical components F_x and F_y.

a. The function $F_x = 500 \cos \theta$ describes the relationship between the angle θ and the horizontal force. What are the amplitude and period of this function? 500; 360°

b. The function $F_y = 500 \sin \theta$ describes the relationship between the angle θ and the vertical force. What are the amplitude and period of this function? 500; 360°

29. WEATHER The function $y = 60 + 25 \sin \frac{\pi}{6}t$, where t is in months and $t = 0$ corresponds to April 15, models the average high temperature in degrees Fahrenheit.

a. Determine the period of this function. What does this period represent? 12; a calendar year

b. What is the maximum high temperature and when does this occur? 85°F; July 15

30. MODELING A cyclist pedals at a rate of 6 rotations every 5 seconds. The motion of the pedals is circular with a radius of 7 inches. The closest the pedals get to the ground is four inches away. Write a function $h(t)$ that models the height of a pedal in inches as a function of the time t in seconds. Assume the pedal starts at its highest point. $h(t) = 7 \cos\left(\frac{12\pi}{5}t\right) + 11$

31. STRUCTURE Functions can be used to model the wave patterns of different colors of light emitted from a particular source, where y is the height of the wave in nanometers and t is the length from the start of the wave in nanometers.

Color	Function
Red	$y = 300 \sin\left(\frac{\pi}{350}t\right)$
Orange	$y = 125 \sin\left(\frac{\pi}{305}t\right)$
Yellow	$y = 460 \sin\left(\frac{\pi}{290}t\right)$
Green	$y = 200 \sin\left(\frac{\pi}{260}t\right)$
Blue	$y = 40 \sin\left(\frac{\pi}{235}t\right)$
Violet	$y = 80 \sin\left(\frac{\pi}{210}t\right)$

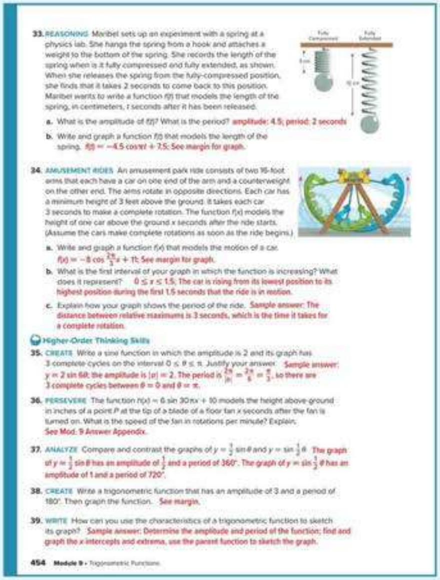
a. What are the amplitude and period of the function describing green light waves? 200 nm, 520 nm

b. The intensity of a light wave corresponds directly to its amplitude. Which color emitted from the source is the most intense? yellow

c. The color of light depends on the period of the wave. Which color has the shortest period? The longest period? violet; red

32. SWIMMING As Kazuo swims a 25-meter sprint, the position of his right hand relative to the water surface can be modeled by the graph shown, where y is the height of the hand in inches from the water level and t is the time in seconds past the start of the sprint. What function describes this graph? $y = 8 \sin\left(\frac{4\pi}{5}t\right)$

Lesson 9-4 • Graphing Sine and Cosine Functions 453



33. REASONING Maribel sets up an experiment with a spring at a physics lab. She hangs the spring from a hook and attaches a weight to the bottom of the spring. She records the length of the spring when is it fully compressed and fully extended, as shown. When she releases the spring from the fully-compressed position, she finds that it takes 2 seconds to come back to this position. Maribel wants to write a function $f(t)$ that models the length of the spring, in centimeters, t seconds after it has been released.

a. What is the amplitude of $f(t)$? What is the period? amplitude: 4.5; period: 2 seconds

b. Write and graph a function $f(t)$ that models the length of the spring. $f(t) = -4.5 \cos \pi t + 7.5$; See margin for graph.

34. AMUSEMENT RIDES An amusement park ride consists of two 16-foot arms that each have a car on one end of the arm and a counterweight on the other end. The arms rotate in opposite directions. Each car has a minimum height of 3 feet above the ground. It takes each car 3 seconds to make a complete rotation. The function $f(x)$ models the height of one car above the ground x seconds after the ride starts. (Assume the cars make complete rotations as soon as the ride begins.)

a. Write and graph a function $f(x)$ that models the motion of a car. $f(x) = -8 \cos \frac{2\pi}{3}x + 11$; See margin for graph.

b. What is the first interval of your graph in which the function is increasing? What does it represent? $0 \le x \le 1.5$; The car is rising from its lowest position to its highest position during the first 1.5 seconds that the ride is in motion.

c. Explain how your graph shows the period of the ride. Sample answer: The distance between relative maximums is 3 seconds, which is the time it takes for a complete rotation.

Higher-Order Thinking Skills

35. CREATE Write a sine function in which the amplitude is 2 and its graph has 3 complete cycles on the interval $0 \le \theta \le \pi$. Justify your answer. Sample answer: $y = 2 \sin 6\theta$; the amplitude is $|a| = 2$. The period is $\frac{2\pi}{|b|} = \frac{2\pi}{6} = \frac{\pi}{3}$, so there are 3 complete cycles between $\theta = 0$ and $\theta = \pi$.

36. PERSEVERE The function $h(x) = 6 \sin 30\pi x + 10$ models the height above ground in inches of a point P at the tip of a blade of a floor fan x seconds after the fan is turned on. What is the speed of the fan in rotations per minute? Explain. See Mod. 9 Answer Appendix.

37. ANALYZE Compare and contrast the graphs of $y = \frac{1}{2} \sin \theta$ and $y = \sin \frac{1}{2}\theta$. The graph of $y = \frac{1}{2} \sin \theta$ has an amplitude of $\frac{1}{2}$ and a period of 360°. The graph of $y = \sin \frac{1}{2}\theta$ has an amplitude of 1 and a period of 720°.

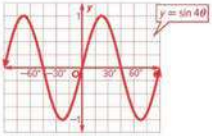
38. CREATE Write a trigonometric function that has an amplitude of 3 and a period of 180°. Then graph the function. See margin.

39. WRITE How can you use the characteristics of a trigonometric function to sketch its graph? Sample answer: Determine the amplitude and period of the function; find and graph the x-intercepts and extrema, use the parent function to sketch the graph.

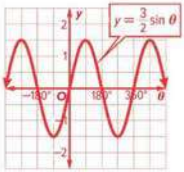
454 Module 9 • Trigonometric Functions

Answers

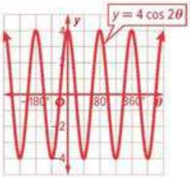
13.



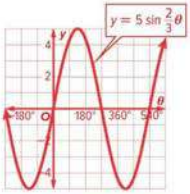
14.



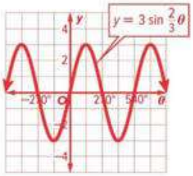
15.



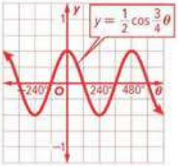
16.



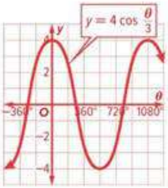
21.



22.

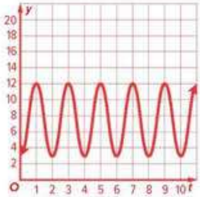


23.

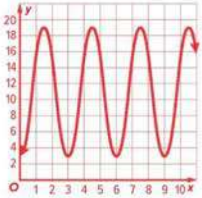


27. Sample answer: The sine function has the same shape as an ocean wave. Additionally, the sine function and ocean waves share similar behaviors, as they both repeat themselves over a consistent interval of time and distance.

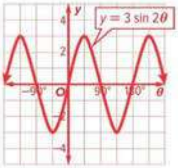
33b.



34a.



38. Sample answer: $y = 3 \sin 2\theta$



Lesson 9-5

Graphing Other Trigonometric Functions

LESSON GOAL

Students graph and analyze tangent and reciprocal trigonometric functions.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Graphing Tangent Functions

- Graph a Tangent Function with a Dilation
- Graph a Tangent Function with a Dilation and a Reflection

Graphing Reciprocal Trigonometric Functions

- Graph a Cosecant Function
- Graph a Cotangent Function
- Apply a Reciprocal Trigonometric Function

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Graphing Reciprocal Functions	●	●		●
Extension: Hyperbolic Functions		●	●	●
ELL Support				

Language Development Handbook

Assign page 50 of the *Language Development Handbook* to help your students build mathematical language related to graphing and analyzing tangent and reciprocal trigonometric functions.

ELL You can use the tips and suggestions on page T50 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous
Students graphed and analyzed sine and cosine functions.
F.IF.4, F.IF.7e

Now
Students graph and analyze tangent and other trigonometric functions.
F.IF.4, F.IF.7e

Next
Students will graph translations of trigonometric functions.
F.IF.7e, F.BF.3

Rigor

The Three Pillars of Rigor

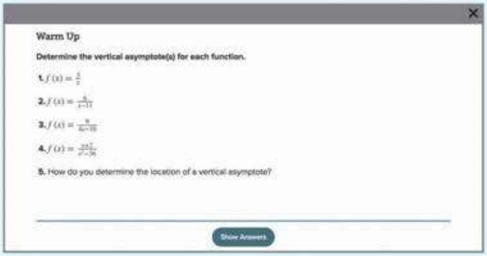
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand their understanding of the sine and cosine functions to include graphs of other trigonometric functions. They build fluency by graphing the functions, and they apply their understanding by solving real-world problems related to trigonometric functions.

Mathematical Background

The cosecant is the reciprocal of the sine. When the sine is zero, the cosecant will be undefined, so there will be a vertical asymptote. The cotangent is the reciprocal of the tangent. When the tangent is zero, the cotangent will have a vertical asymptote; when the tangent has a vertical asymptote, the cotangent will have a zero.

Interactive Presentation

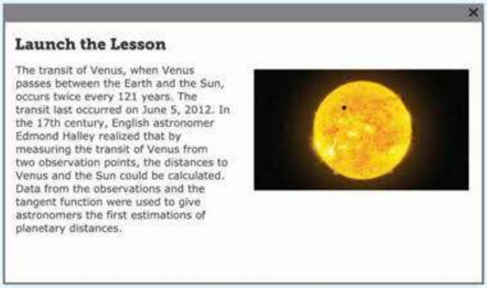


Warm Up

Determine the vertical asymptote(s) for each function.

1. $f(x) =$ [illegible]
2. $f(x) =$ [illegible]
3. $f(x) =$ [illegible]
4. $f(x) =$ [illegible]
5. How do you determine the location of a vertical asymptote?

Warm Up



Launch the Lesson

The transit of Venus, when Venus passes between the Earth and the Sun, occurs twice every 121 years. The transit last occurred on June 5, 2012. In the 17th century, English astronomer Edmond Halley realized that by measuring the transit of Venus from two observation points, the distances to Venus and the Sun could be calculated. Data from the observations and the tangent function were used to give astronomers the first estimations of planetary distances.

Launch the Lesson

Vocabulary

reciprocal trigonometric functions

Trigonometric functions that are reciprocals of each other.

- One definition of reciprocal is "the quantity obtained by dividing the number 1 by a given quantity." How does this definition help you remember what reciprocal trigonometric functions are?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- finding vertical asymptotes of a rational function

Answers:

1. $x = 0$

2. $x = 11$

3. $x = 2.5$

4. $x = -6, 6$

5. Find where the denominator equals 0 and numerator is nonzero.

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students to learn how tangent functions were used to estimate planetary distances.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then, discuss the question below with the class.

Learn Graphing Tangent Functions

Objective

Students graph and analyze tangent functions and identify the periods and midlines.

Teaching the Mathematical Practices

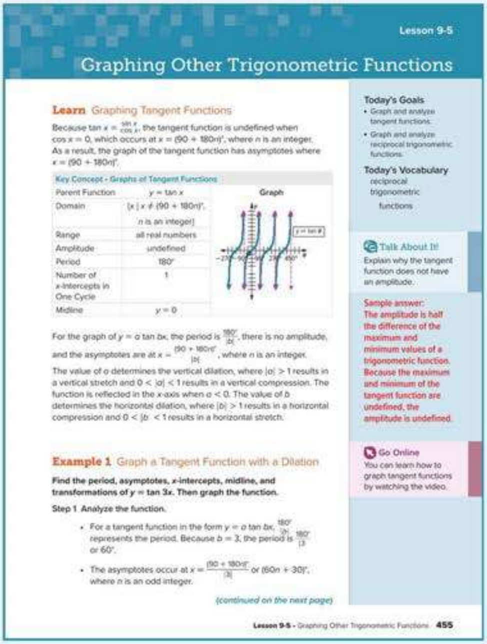
6 Communicate Precisely Encourage students to routinely write and explain their reasoning. The Talk About It! feature requires students to use clear definitions and mathematical language to explain why the tangent function does not have an amplitude.

About the Key Concept

Because $\tan x = \frac{\sin x}{\cos x}$, the tangent function is undefined when $\cos x = 0$, which occurs at $x = (90 + 180n)°$, where n is an integer. As a result, the graph of the tangent function has asymptotes at $x = (90 + 180n)°$. For the graph of $y = a \tan bx$, the period is $\frac{180°}{|b|}$, and the asymptotes are odd multiples of $\frac{180°}{2|b|}$. The value of a determines the vertical dilation; the function is reflected in the x-axis when $a < 0$. The value of b determines the horizontal dilation, where $|b| > 1$ results in a horizontal compression and $0 < |b| < 1$ results in a horizontal stretch.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 9-5

Graphing Other Trigonometric Functions

Learn Graphing Tangent Functions

Because $\tan x = \frac{\sin x}{\cos x}$, the tangent function is undefined when $\cos x = 0$, which occurs at $x = (90 + 180n)°$, where n is an integer. As a result, the graph of the tangent function has asymptotes where $x = (90 + 180n)°$.

Key Concept • Graphs of Tangent Functions

		Graph
Parent Function	$y = \tan x$	
Domain	$\{x \mid x \neq (90 + 180n)°,$ n is an integer$\}$	
Range	all real numbers	
Amplitude	undefined	
Period	180°	
Number of x-intercepts in One Cycle	1	
Midline	$y = 0$	

For the graph of $y = a \tan bx$, the period is $\frac{180°}{|b|}$, there is no amplitude, and the asymptotes are at $x = \frac{(90 + 180n)°}{|b|}$, where n is an integer.

The value of a determines the vertical dilation, where $|a| > 1$ results in a vertical stretch and $0 < |a| < 1$ results in a vertical compression. The function is reflected in the x-axis when $a < 0$. The value of b determines the horizontal dilation, where $|b| > 1$ results in a horizontal compression and $0 < |b| < 1$ results in a horizontal stretch.

Example 1 Graph a Tangent Function with a Dilation

Find the period, asymptotes, x-intercepts, midline, and transformations of $y = \tan 3x$. Then graph the function.

Step 1 Analyze the function.

- For a tangent function in the form $y = a \tan bx$, $\frac{180°}{|b|}$ represents the period. Because $b = 3$, the period is $\frac{180°}{|3|}$ or 60°.
- The asymptotes occur at $x = \frac{(90 + 180n)°}{|3|}$ or $(60n + 30)°$, where n is an odd integer.

(continued on the next page)

Today's Goals
- Graph and analyze tangent functions.
- Graph and analyze reciprocal trigonometric functions.

Today's Vocabulary
reciprocal trigonometric functions

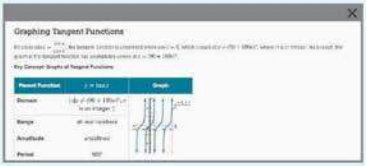
Talk About It!
Explain why the tangent function does not have an amplitude.

Sample answer: The amplitude is half the difference of the maximum and minimum values of a trigonometric function. Because the maximum and minimum of the tangent function are undefined, the amplitude is undefined.

Go Online
You can learn how to graph tangent functions by watching the video.

Lesson 9-5 • Graphing Other Trigonometric Functions 455

Interactive Presentation



Graphing Tangent Functions

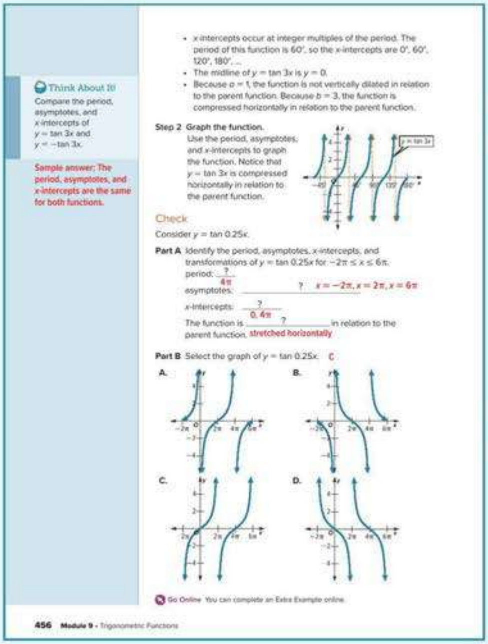
Learn

TYPE

Students explain why the tangent function does not have an amplitude.

WATCH

Students can watch a video to see how to graph tangent functions.



- x-intercepts occur at integer multiples of the period. The period of this function is 60°, so the x-intercepts are 0°, 60°, 120°, 180°, ...
- The midline of $y = \tan 3x$ is $y = 0$.
- Because $a = 1$, the function is not vertically dilated in relation to the parent function. Because $b = 3$, the function is compressed horizontally in relation to the parent function.

Step 2 **Graph the function.**
Use the period, asymptotes, and x-intercepts to graph the function. Notice that $y = \tan 3x$ is compressed horizontally in relation to the parent function.

Think About It!
Compare the period, asymptotes, and x-intercepts of $y = \tan 3x$ and $y = -\tan 3x$.

Sample answer: The period, asymptotes, and x-intercepts are the same for both functions.

Check

Consider $y = \tan 0.25x$.

Part A Identify the period, asymptotes, x-intercepts, and transformations of $y = \tan 0.25x$ for $-2\pi \le x \le 6\pi$.
period: ? 4π
asymptotes: ? $x = -2\pi, x = 2\pi, x = 6\pi$
x-intercepts: ? $0, 4\pi$
The function is ? in relation to the parent function. stretched horizontally

Part B Select the graph of $y = \tan 0.25x$. C

A. B. C. D.

Go Online You can complete an Extra Example online.

456 Module 9 • Trigonometric Functions

Interactive Presentation

Graph a Tangent Function with a Dilation

Analyze the function.

Graph the function.

Example 1

TAP

Students move through the steps to graph the dilated graph of the tangent function.

TYPE

Students compare the period, asymptotes, and x-intercepts for the graph and its reflection.

Example 1 Graph a Tangent Function with a Dilation

Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in where the asymptotes of the tangent function occur and use the asymptotes to help graph the function.

Questions for Mathematical Discourse

AL What are the similarities and differences of the tangent function compared to the sine and cosine functions? Sample answer: All three functions are periodic. The period of the parent sine and cosine functions is 360°, and the period for the parent tangent function is 180°. Sine and cosine functions are continuous, and the range is $[-1, 1]$. Tangent functions have discontinuities in the domain in the form of asymptotes, and the range is $(-\infty, \infty)$.

OL Why do x-intercepts occur at integer multiples of the period? $\tan x = 0$ when x is $180°n$, where n is an integer. Thus, for $3x = 180°n$, $x = \frac{180°n}{3} = 60°n$.

BL Why does the parent tangent function have a period of 180° rather than 360°? Sample answer: The tangent function is the ratio of the sine and cosine functions. Looking at the unit circle, the absolute values of the cosine and sine functions repeat in a 180° period, but the signs change. They are both positive from 0° to 90°, one is positive and one is negative from 90° to 180° and 270° to 360°, and both are negative from 180° to 270°. This makes the ratio the same for 0° to 180° and 180° to 360°, so the period of a parent tangent function is 180°.

Common Error

Many students struggle to find the asymptotes of the graph of the tangent function. Encourage students to write the formula each time and then substitute $n = 0, 1, 2$ to find the pattern for asymptotes.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students are having difficulty understanding how the graph of tan x is generated,

THEN have students create a table of values using the unit circle. They can use different angles from each quadrant and calculate the value of $\frac{\sin \theta}{\cos \theta}$. Then they can graph the angles with the function values to see the general shape and key features of tangent.

Example 2 Graph a Tangent Function with a Dilation and a Reflection

MP Teaching the Mathematical Practices

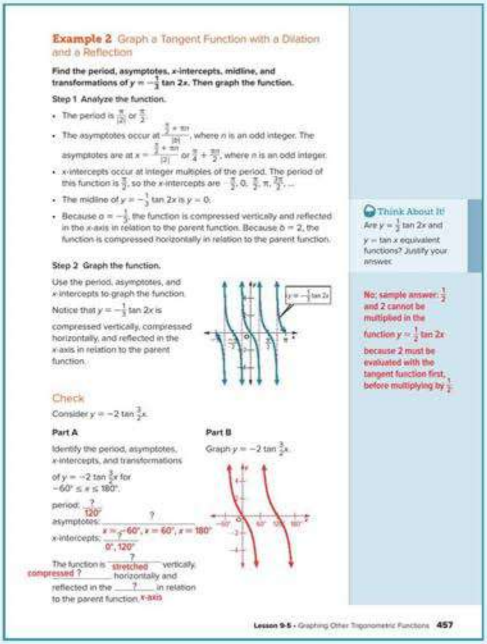
7 Use Structure Guide students to use the structure of the function to identify the transformations in and key features of the graph of the function.

Questions for Mathematical Discourse

AL How does the sign of a affect the graph of a tangent function compared to other function types? A negative value for a reflects the function in the x-axis, which is the same effect as for other types of functions.

OL Would the asymptotes change if a had been positive? Explain. No; sample answer: The asymptotes are affected by b, not a, so the value doesn't change the asymptotes.

BL What transformation would need to be performed to change the midline from $y = 0$ to $y = 4$? The graph would have to be shifted up 4 units by adding 4 as a constant term.



Example 2 Graph a Tangent Function with a Dilation and a Reflection

Find the period, asymptotes, x-intercepts, midline, and transformations of $y = -\frac{1}{3}\tan 2x$. Then graph the function.

Step 1 Analyze the function.

- The period is $\frac{\pi}{|2|}$ or $\frac{\pi}{2}$.
- The asymptotes occur at $\frac{\frac{\pi}{2} + \pi n}{|b|}$, where n is an odd integer. The asymptotes are at $x = \frac{\frac{\pi}{2} + \pi n}{|2|}$ or $\frac{\pi}{4} + \frac{\pi n}{2}$, where n is an odd integer.
- x-intercepts occur at integer multiples of the period. The period of this function is $\frac{\pi}{2}$, so the x-intercepts are $-\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$
- The midline of $y = -\frac{1}{3}\tan 2x$ is $y = 0$.
- Because $a = -\frac{1}{3}$, the function is compressed vertically and reflected in the x-axis in relation to the parent function. Because $b = 2$, the function is compressed horizontally in relation to the parent function.

Step 2 Graph the function.

Use the period, asymptotes, and x-intercepts to graph the function.

Notice that $y = -\frac{1}{3}\tan 2x$ is compressed vertically, compressed horizontally, and reflected in the x-axis in relation to the parent function.

Check

Consider $y = -2\tan\frac{3}{2}x$.

Part A

Identify the period, asymptotes, x-intercepts, and transformations of $y = -2\tan\frac{3}{2}x$ for $-60° \le x \le 180°$.

period: ? 120°

asymptotes: ? $x = -60°, x = 60°, x = 180°$

x-intercepts: ? 0°, 120°

The function is ? stretched vertically, ? compressed horizontally and reflected in the ? x-axis in relation to the parent function.

Part B

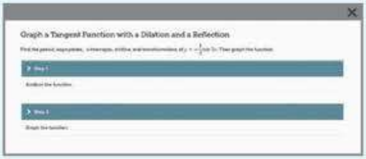
Graph $y = -2\tan\frac{3}{2}x$.

Think About It! Are $y = \frac{1}{2}\tan 2x$ and $y = \tan x$ equivalent functions? Justify your answer.

No; sample answer: $\frac{1}{2}$ and 2 cannot be multiplied in the function $y = \frac{1}{2}\tan 2x$ because 2 must be evaluated with the tangent function first, before multiplying by $\frac{1}{2}$.

Lesson 9-5 • Graphing Other Trigonometric Functions 457

Interactive Presentation



Graph a Tangent Function with a Dilation and a Reflection

Example 2

TAP

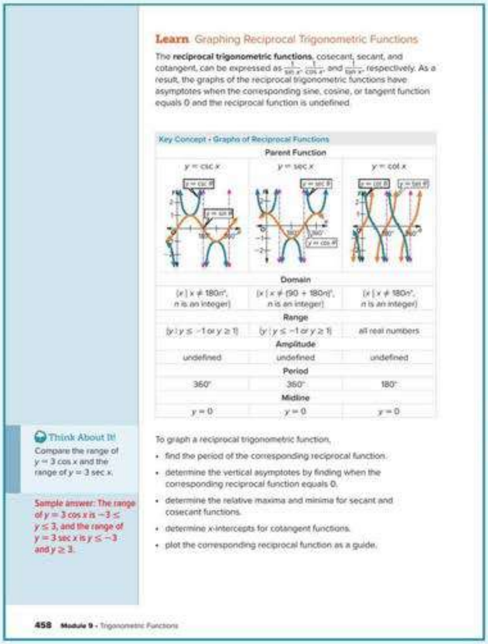
Students move through the steps to graph the dilated and reflected graph of the tangent function.

TYPE

Students determine whether two tangent functions are equivalent.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Learn Graphing Reciprocal Trigonometric Functions

The **reciprocal trigonometric functions**, cosecant, secant, and cotangent, can be expressed as $\frac{1}{\sin x}$, $\frac{1}{\cos x}$, and $\frac{1}{\tan x}$, respectively. As a result, the graphs of the reciprocal trigonometric functions have asymptotes when the corresponding sine, cosine, or tangent function equals 0 and the reciprocal function is undefined.

Key Concept • Graphs of Reciprocal Functions

Parent Function		
$y = \csc x$	$y = \sec x$	$y = \cot x$
Domain		
$\{x \mid x \neq 180n°$, n is an integer$\}$	$\{x \mid x \neq (90 + 180n)°$, n is an integer$\}$	$\{x \mid x \neq 180n°$, n is an integer$\}$
Range		
$\{y \mid y \leq -1$ or $y \geq 1\}$	$\{y \mid y \leq -1$ or $y \geq 1\}$	all real numbers
Amplitude		
undefined	undefined	undefined
Period		
360°	360°	180°
Midline		
$y = 0$	$y = 0$	$y = 0$

Think About It!
Compare the range of $y = 3 \cos x$ and the range of $y = 3 \sec x$.

Sample answer: The range of $y = 3 \cos x$ is $-3 \leq y \leq 3$, and the range of $y = 3 \sec x$ is $y \leq -3$ and $y \geq 3$.

To graph a reciprocal trigonometric function,

- find the period of the corresponding reciprocal function.
- determine the vertical asymptotes by finding when the corresponding reciprocal function equals 0.
- determine the relative maxima and minima for secant and cosecant functions.
- determine x-intercepts for cotangent functions.
- plot the corresponding reciprocal function as a guide.

458 Module 9 • Trigonometric Functions

Interactive Presentation

Graphing Reciprocal Trigonometric Functions

Learn

TYPE

Students compare the ranges of a cosine function and a secant function.

1 CONCEPTUAL UNDERSTANDING | 2 FLUENCY | 3 APPLICATION

Learn Graphing Reciprocal Trigonometric Functions

Objective

Students graph and analyze reciprocal trigonometric functions and identify the periods and midlines.

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the reciprocal trigonometric functions and their corresponding trigonometric functions. The Think About It! feature ask students to consider the correspondence between the range of a trigonometric function and the range of its reciprocal function.

About the Key Concept

The reciprocal trigonometric functions, cosecant, secant, and cotangent, can be expressed as $\frac{1}{\sin x}$, $\frac{1}{\cos x}$, and $\frac{1}{\tan x}$, respectively. As a result, the graphs of the reciprocal trigonometric functions have asymptotes when the corresponding sine, cosine, or tangent function equals 0 and the reciprocal function is undefined. To graph a reciprocal trigonometric function, find the period of the corresponding reciprocal function. Then determine the vertical asymptotes by finding when the corresponding reciprocal function equals 0, and find the relative maxima and minima for secant and cosecant functions. Next, determine x-intercepts for cotangent functions, and, finally, plot the corresponding reciprocal function as a guide.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students do not understand the asymptotes on the graphs of secant and cosecant,

THEN show students how the x-intercepts of the graphs of sine and cosine are related to the asymptotes of cosecant and secant, respectively. Seeing the asymptote generated from the sine or cosine function may help students visualize asymptotes in the future.

Example 3 Graph a Cosecant Function

Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. Example 3 requires student to justify their conclusions and explain their reasoning as they work to solve the problem.

Questions for Mathematical Discourse

AL For what y-values will the graphs of the cosecant and its reciprocal sine function intersect? They will intersect at the extrema; in this example, $y = 1$ and $y = -1$.

OL How would the asymptotes change if the value of b were 2 instead of 0.5? Sample answer: The asymptotes would be half of the x-intercepts of the graph of sine, so instead of 0, π, 2π, and so on, the asymptotes would be at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$.

BL Will the graph of a cosecant function ever have x-intercepts? Explain. Y es; sample answer: if the graph of cosecant is shifted up or down more than 1 unit, the graph will cross the x-axis.

Example 4 Graph a Cotangent Function

Teaching the Mathematical Practices

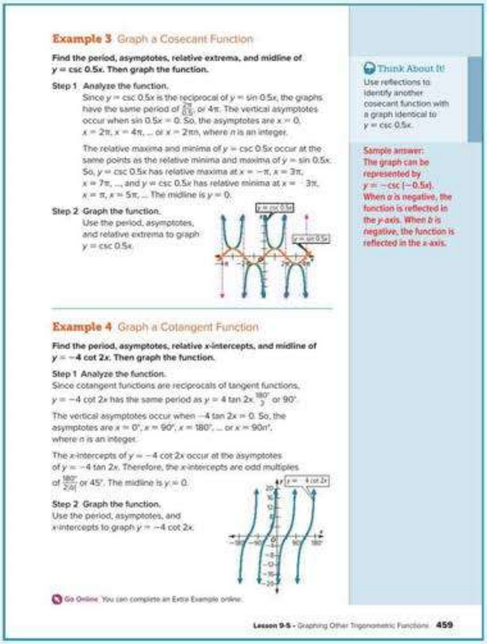
1 Explain Correspondences Encourage students to explain the relationships between the graphs and key features of the reciprocal function to the graph of its related trigonometric function.

Questions for Mathematical Discourse

AL What is the relationship between the asymptotes and x-intercepts of a cotangent function to the asymptotes and x-intercepts of its reciprocal tangent function? The asymptotes of one are the x-intercepts of the other.

OL Where will the graphs of $y = -4 \cot 2x$ and $y = -4 \tan 2x$ intersect? $22.5° + 45°n$ or $22.5°(1 + 2n)$, where n is an integer.

BL Does reflecting a cotangent function in the x-axis make it a tangent function? It can be written as a tangent function, but an equivalent tangent function would also have to have a phase shift in the x-axis equal to half the period. $-\cot x = \tan (x - 90°)$ or $\cot x = \tan (90° - x)$



Example 3 Graph a Cosecant Function

Find the period, asymptotes, relative extrema, and midline of $y = \csc 0.5x$. Then graph the function.

Step 1 **Analyze the function.**

Since $y = \csc 0.5x$ is the reciprocal of $y = \sin 0.5x$, the graphs have the same period of $\frac{2\pi}{0.5}$, or 4π. The vertical asymptotes occur when $\sin 0.5x = 0$. So, the asymptotes are $x = 0$, $x = 2\pi$, $x = 4\pi$, ... or $x = 2\pi n$, where n is an integer.

The relative maxima and minima of $y = \csc 0.5x$ occur at the same points as the relative minima and maxima of $y = \sin 0.5x$. So, $y = \csc 0.5x$ has relative maxima at $x = -\pi$, $x = 3\pi$, $x = 7\pi$, ..., and $y = \csc 0.5x$ has relative minima at $x = -3\pi$, $x = \pi$, $x = 5\pi$, ... The midline is $y = 0$.

Step 2 **Graph the function.**

Use the period, asymptotes, and relative extrema to graph $y = \csc 0.5x$.

Think About It!

Use reflections to identify another cosecant function with a graph identical to $y = \csc 0.5x$.

Sample answer: The graph can be represented by $y = -\csc (-0.5x)$. When a is negative, the function is reflected in the y-axis. When b is negative, the function is reflected in the x-axis.

Example 4 Graph a Cotangent Function

Find the period, asymptotes, relative x-intercepts, and midline of $y = -4 \cot 2x$. Then graph the function.

Step 1 **Analyze the function.**

Since cotangent functions are reciprocals of tangent functions, $y = -4 \cot 2x$ has the same period as $y = 4 \tan 2x$, $\frac{180°}{3}$ or 90°.

The vertical asymptotes occur when $-4 \tan 2x = 0$. So, the asymptotes are $x = 0°$, $x = 90°$, $x = 180°$, ... or $x = 90n°$, where n is an integer.

The x-intercepts of $y = -4 \cot 2x$ occur at the asymptotes of $y = -4 \tan 2x$. Therefore, the x-intercepts are odd multiples of $\frac{180°}{2|4|}$ or 45°. The midline is $y = 0$.

Step 2 **Graph the function.**

Use the period, asymptotes, and x-intercepts to graph $y = -4 \cot 2x$.

Go Online You can complete an Extra Example online.

Lesson 9-5 • Graphing Other Trigonometric Functions 459

Interactive Presentation

Graph a Cosecant Function

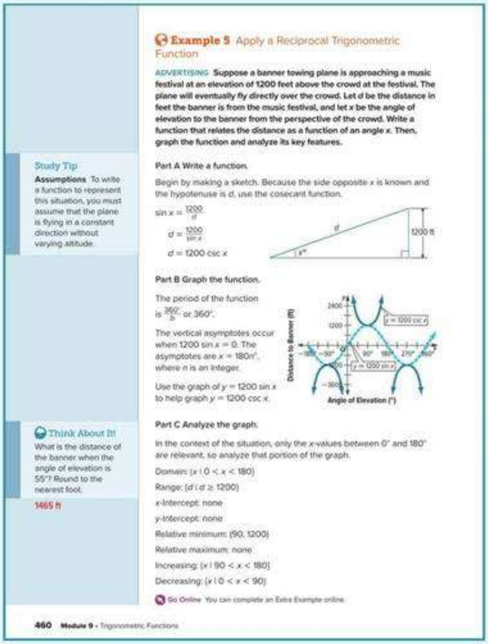
Example 3

TAP

Students move through the steps to graph a cosecant function.

TYPE

Students use reflections to write another cosecant function that has a graph identical to the given function.



Example 5 Apply a Reciprocal Trigonometric Function

ADVERTISING **Suppose a banner towing plane is approaching a music festival at an elevation of 1200 feet above the crowd at the festival. The plane will eventually fly directly over the crowd. Let *d* be the distance in feet the banner is from the music festival, and let *x* be the angle of elevation to the banner from the perspective of the crowd. Write a function that relates the distance as a function of an angle *x*. Then, graph the function and analyze its key features.**

Study Tip

Assumptions To write a function to represent this situation, you must assume that the plane is flying in a constant direction without varying altitude.

Part A Write a function.

Begin by making a sketch. Because the side opposite *x* is known and the hypotenuse is *d*, use the cosecant function.

$$\sin x = \frac{1200}{d}$$

$$d = \frac{1200}{\sin x}$$

$$d = 1200 \csc x$$

Part B Graph the function.

The period of the function is $\frac{360°}{b}$ or 360°.

The vertical asymptotes occur when $1200 \sin x = 0$. The asymptotes are $x = 180n°$, where *n* is an integer.

Use the graph of $y = 1200 \sin x$ to help graph $y = 1200 \csc x$.

Part C Analyze the graph.

In the context of the situation, only the *x*-values between 0° and 180° are relevant, so analyze that portion of the graph.

Domain: $\{x \mid 0 < x < 180\}$

Range: $\{d \mid d \geq 1200\}$

x-intercept: none

y-intercept: none

Relative minimum: (90, 1200)

Relative maximum: none

Increasing: $\{x \mid 90 < x < 180\}$

Decreasing: $\{x \mid 0 < x < 90\}$

Think About It!

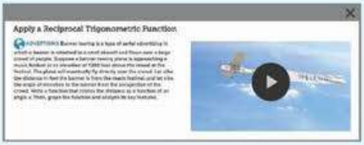
What is the distance of the banner when the angle of elevation is 55°? Round to the nearest foot.

1465 ft

Go Online You can complete an Extra Example online.

460 Module 9 • Trigonometric Functions

Interactive Presentation



Apply a Reciprocal Trigonometric Function

Example 5

TAP

Students move through the steps to apply a reciprocal trigonometric function in a real-world situation.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 5 Apply a Reciprocal Trigonometric Function

Teaching the Mathematical Practices

2 Create Representations Guide students to write a function that models the situation in Example 5. Then graph and analyze the function to solve the problem.

Questions for Mathematical Discourse

AL What happens to *d* and *x* as the plane approaches the festival? *d* decreases, *x* increases

OL Does it make sense that there is a relative minimum when $x = 90°$? Explain. Yes; sample answer: When $x = 90°$, the plane is directly overhead, which should be the closest it gets to the crowd.

BL How do asymptotes at 0° and 180° make sense in the context of the situation? If the plane maintains an elevation of 1200 feet, there is no distance it can travel away from the festival that would make $x = 0°$ or 180°.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–17
2	exercises that use a variety of skills from this lesson	18–23
2	exercises that extend concepts learned in this lesson to new contexts	24–25
3	exercises that emphasize higher-order and critical-thinking skills	26–29

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **BL**
THEN assign:

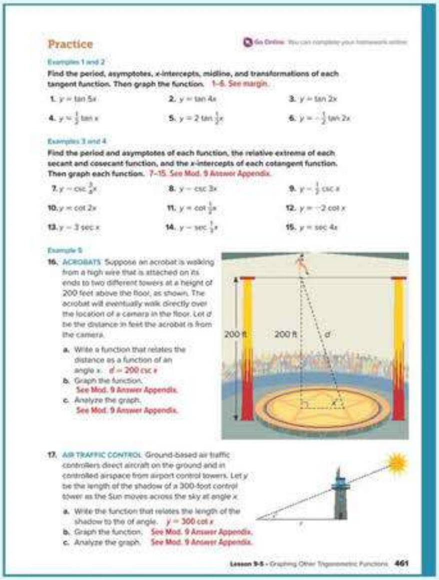
- Practice Exercises 1–25 odd, 26–29
- Extension: Hyperbolic Functions
- ALEKS Trigonometry

IF students score 66%–89% on the Checks, **OL**
THEN assign:

- Practice Exercises 1–29 odd
- Remediation, Review Resources: Graphing Reciprocal Functions
- Personal Tutors
- Extra Examples 1–5
- ALEKS Graphs of Rational Functions

IF students score 65% or less on the Checks, **AL**
THEN assign:

- Practice Exercises 1–17 odd
- Remediation, Review Resources: Graphing Reciprocal Functions
- *Quick Review Math Handbook:* Graphing Trigonometric Functions
- ALEKS Graphs of Rational Functions



Practice

Go Online You can complete your homework online.

Examples 1 and 2

Find the period, asymptotes, *x*-intercepts, midline, and transformations of each tangent function. Then graph the function. 1–6. See margin.

1. $y = \tan 5x$
2. $y = \tan 4x$
3. $y = \tan 2x$
4. $y = \frac{1}{2} \tan x$
5. $y = 2 \tan \frac{1}{2}x$
6. $y = -\frac{1}{2} \tan 2x$

Examples 3 and 4

Find the period and asymptotes of each function, the relative extrema of each secant and cosecant function, and the *x*-intercepts of each cotangent function. Then graph each function. 7–15. See Mod. 9 Answer Appendix.

7. $y = \csc \frac{3}{4}x$
8. $y = \csc 3x$
9. $y = \frac{1}{2} \csc x$
10. $y = \cot 2x$
11. $y = \cot \frac{1}{2}x$
12. $y = -2 \cot x$
13. $y = 3 \sec x$
14. $y = \sec \frac{1}{3}x$
15. $y = \sec 4x$

Example 5

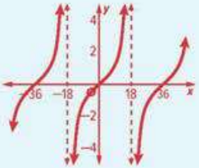
16. ACROBATS Suppose an acrobat is walking from a high wire that is attached on its ends to two different towers at a height of 200 feet above the floor, as shown. The acrobat will eventually walk directly over the location of a camera in the floor. Let *d* be the distance in feet the acrobat is from the camera.
    a. Write a function that relates the distance as a function of an angle *x*. $d = 200 \csc x$
    b. Graph the function. See Mod. 9 Answer Appendix.
    c. Analyze the graph. See Mod. 9 Answer Appendix.

17. AIR TRAFFIC CONTROL Ground-based air traffic controllers direct aircraft on the ground and in controlled airspace from airport control towers. Let *y* be the length of the shadow of a 300-foot control tower as the Sun moves across the sky at angle *x*.
    a. Write the function that relates the length of the shadow to the of angle. $y = 300 \cot x$
    b. Graph the function. See Mod. 9 Answer Appendix.
    c. Analyze the graph. See Mod. 9 Answer Appendix.

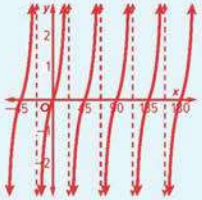
Lesson 9-5 • Graphing Other Trigonometric Functions 461

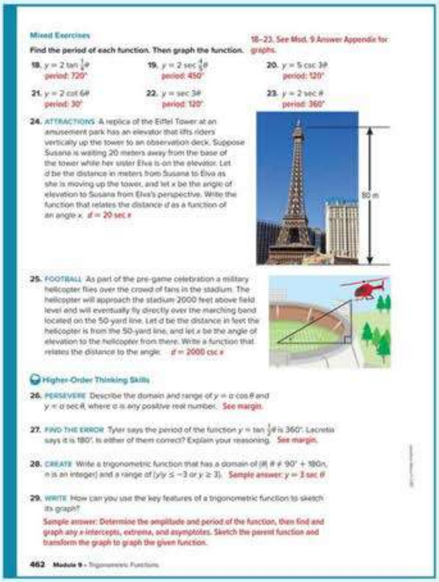
Answers

1. period = 36°; asymptotes: $(36n + 18)°$, where n is an odd integer; x-intercepts: 0°, 36°, 72°, 108°, ...; midline: $y = 0$; Because $a = 1$, the function is not vertically dilated in relation to the parent function. Because $b = 5$, the function is compressed horizontally in relation to the parent function.



2. period = 45°; asymptotes: $(45n + 22.5)°$, where n is an integer; x-intercepts: 0°, 45°, 90°, 135°, ...; midline: $y = 0$; Because $a = 1$, the function is not vertically dilated in relation to the parent function. Because $b = 4$, the function is compressed horizontally in relation to the parent function.





Mixed Exercises

Find the period of each function. Then graph the function. 18–23. See Mod. 9 Answer Appendix for graphs.

18. $y = 2 \tan \frac{1}{2}\theta$ period: 720°
19. $y = 2 \sec \frac{4}{5}\theta$ period: 450°
20. $y = 5 \csc 3\theta$ period: 120°
21. $y = 2 \cot 6\theta$ period: 30°
22. $y = \sec 3\theta$ period: 120°
23. $y = 2 \sec \theta$ period: 360°

24. ATTRACTIONS A replica of the Eiffel Tower at an amusement park has an elevator that lifts riders vertically up the tower to an observation deck. Suppose Susana is waiting 20 meters away from the base of the tower while her sister Elva is on the elevator. Let d be the distance in meters from Susana to Elva as she is moving up the tower, and let x be the angle of elevation to Susana from Elva's perspective. Write the function that relates the distance d as a function of an angle x. $d = 20 \sec x$

25. FOOTBALL As part of the pre-game celebration a military helicopter flies over the crowd of fans in the stadium. The helicopter will approach the stadium 2000 feet above field level and will eventually fly directly over the marching band located on the 50-yard line. Let d be the distance in feet the helicopter is from the 50-yard line, and let x be the angle of elevation to the helicopter from there. Write a function that relates the distance to the angle. $d = 2000 \csc x$

Higher-Order Thinking Skills

26. PERSEVERE Describe the domain and range of $y = a \cos \theta$ and $y = a \sec \theta$, where a is any positive real number. See margin.

27. FIND THE ERROR Tyler says the period of the function $y = \tan \frac{1}{2}\theta$ is 360°. Lacretia says it is 180°. Is either of them correct? Explain your reasoning. See margin.

28. CREATE Write a trigonometric function that has a domain of $\{\theta \mid \theta \neq 90° + 180n$, n is an integer$\}$ and a range of $\{y \mid y \leq -3$ or $y \geq 3\}$. Sample answer: $y = 3 \sec \theta$

29. WRITE How can you use the key features of a trigonometric function to sketch its graph?

Sample answer: Determine the amplitude and period of the function, then find and graph any x-intercepts, extrema, and asymptotes. Sketch the parent function and transform the graph to graph the given function.

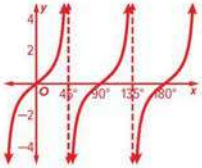
462 Module 9 • Trigonometric Functions

26. The domain of $y = a \cos \theta$ is the set of all real numbers.
The domain of $y = a \sec \theta$ is the set of all real numbers except for the values for which $\cos \theta = 0$.
The range of $y = a \cos \theta$ is $-a \leq y \leq a$.
The range of $y = a \sec \theta$ is $y \leq -a$ and $y \geq a$.

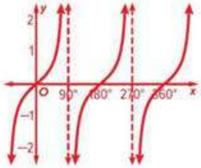
27. Tyler; The period of the tangent function is $\frac{180°}{|b|}$, and b is $\frac{1}{2}$. So the period is $\frac{180°}{\frac{1}{2}}$, or 360°.

Answers

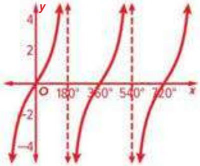
3. period = 90°; asymptotes: $(90n + 45)°$, where n is an odd integer; x-intercepts: 0°, 90°, 180°, 270°, ...; midline: $y = 0$; Because $a = 1$, the function is not vertically dilated in relation to the parent function. Because $b = 2$, the function is compressed horizontally in relation to the parent function.



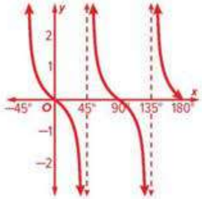
4. period = 180°; asymptotes: $(180n + 90)°$, where n is an odd integer; x-intercepts: 0°, 180°, 360°, 540°, ...; midline: $y = 0$; Because $a = \frac{1}{2}$, the function is compressed vertically in relation to the parent function. Because $b = 1$, the function is not dilated horizontally in relation to the parent function.



5. period = 360°; asymptotes: $(360n + 180)°$, where n is an odd integer; x-intercepts: 0°, 360°, 720°, 1080°, ...; midline: $y = 0$; Because $a = 2$, the function is stretched vertically in relation to the parent function. Because $b = \frac{1}{2}$, the function is stretched horizontally in relation to the parent function.



6. period = 90°; asymptotes: $(90n + 45)°$, where n is an integer; x-intercepts: 0°, 90°, 180°, 270°, ...; midline: $y = 0$; Because $a = \frac{1}{2}$, the function is compressed vertically and reflected in the x-axis. Because $b = 2$, the function is compressed horizontally in relation to the parent function.



Lesson 9-6

Translations of Trigonometric Graphs

F.IF.7e, F.BF.3

LESSON GOAL

Students graph translations of trigonometric functions.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Analyzing Graphs of Trigonometric Functions by Using Technology

Develop:

Horizontal Translations of Trigonometric Functions

- Graph a Phase Shift
- Graph a Transformation of a Trigonometric Function

Vertical Translations of Trigonometric Functions

- Graph a Vertical Shift
- Model Transformations of Trigonometric Functions
- Write a Trigonometric Function from a Graph

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Transformations of Functions	●	●		●
Extension: Simple Harmonic Motion		●	●	●
ELL Support				

Language Development Handbook

Assign page 51 of the *Language Development Handbook* to help your students build mathematical language related to graphing translations of trigonometric functions.

ELL You can use the tips and suggestions on page T51 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

5 Use appropriate tools strategically.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous
Students transformed linear, exponential, and quadratic functions.
F.BF.3 (Course 1, Course 2)

Now
Students graph translations of trigonometric functions.
F.IF.7e, F.BF.3

Next
Students will find the values of angle measures by using inverse trigonometric functions.
F.TF.7

Rigor

The Three Pillars of Rigor

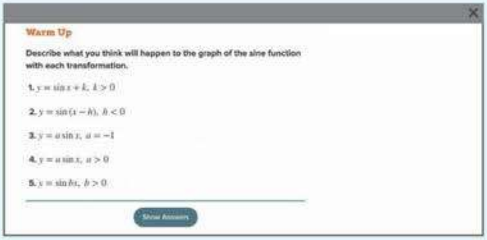
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand their understanding of the basic trigonometric functions to include translations of these graphs. They build fluency by using translations to graph the functions, and they apply their understanding by solving real-world problems related to trigonometric functions.

Mathematical Background

The graphs of the functions $y = a \sin b(\theta - h) + k$, $y = a \cos b(\theta - h) + k$, and $y = a \tan b(\theta - h) + k$ are affected by changing the a, b, h, and k. A horizontal translation, or phase shift, is affected by h, and a vertical shift of the horizontal midline is affected by k. The amplitude is determined by the value of $|a|$ and the period is determined by $|b|$.

Interactive Presentation



Warm Up

Describe what you think will happen to the graph of the sine function with each transformation.

1. $y = \sin x + k,\ k > 0$
2. $y = \sin(x - h),\ h < 0$
3. $y = a \sin x,\ a = -1$
4. $y = a \sin x,\ a > 0$
5. $y = \sin bx,\ b > 0$

Warm Up

Launch the Lesson

A water wheel is a machine used for converting the energy of flowing water into power. It consists of a wheel with a number of blades or buckets, which form the driving surface.

The kinetic energy it produces depends on how fast the water is moving when it enters the wheel. The change in the water's height from the time it enters the water wheel to the time it leaves the wheel determines the potential energy. The power is the speed that the energy is delivered, which is determined by the flow rate. The flow of the water in a water wheel can be modeled by the transformation of a trigonometric function.

Launch the Lesson

Vocabulary

phase shift

A horizontal translation of the graph of a trigonometric function.

vertical shift

A vertical translation of the graph of a trigonometric function.

- Since a phase shift represents a horizontal translation, how could you tell by looking at the equation that a function has a phase shift?
- Since a vertical shift represents a vertical translation, how could you tell by looking at the equation that a function has a vertical shift?

Today’s Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- describing transformations to a sine function

Answers:

1. Moves *k* units up.
2. Moves *h* units to the left.
3. Reflects the graph in the *x*-axis.
4. Vertically stretches the graph.
5. Horizontally compresses the graph.

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will learn how the flow of water in a water wheel can be modeled by the transformation of a trigonometric function.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today’s Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today’s Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Analyzing Trigonometric Functions by Using Technology

Objective

Students use a graphing calculator to explore transformations and key features of trigonometric functions.

Teaching the Mathematical Practices

5 Analyze Graphs Throughout the Explore, students will analyze the graphs they have generated using graphing calculators to explore transformations of trigonometric functions. Point out that to see the important features of the graphs, students may need to adjust the viewing window.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will be presented with an Inquiry Question to answer at the end of the activity. They will use a graphing calculator to explore transformations of trigonometric functions. Students will work through four different exercises. Then students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Analyzing Trigonometric Functions by Using Technology

Explore

Select a calculator.

Explore

TAP

Students select the calculator they will use for the Explore activity and then move through the slides.

TYPE

Students move through the exercises and answer questions pertaining to transformations of trigonometric functions.

Interactive Presentation

INQUIRY How does adding a constant to, subtracting a constant from, or multiplying a constant by a function affect the graph of a trigonometric function?

Explore

TYPE

Students will respond to the Inquiry Question and can view a sample answer.

Explore Analyzing Trigonometric Functions by Using Technology (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How will a vertical shift affect the midline of the graph of $y = \sin x + k$? Sample answer: The midline will shift from $y = 0$ to $y = k$, when k is the value shifted vertically.
- Which values of $y = a \sin b(x - h) + k$ affect the range of the function? a and k Which affect the domain? b and h

Inquiry

How does adding a constant to, subtracting a constant from, or multiplying a constant by a function affect the graph of a trigonometric function? Sample answer: Adding or subtracting a constant causes the graph to be shifted horizontally or vertically. Multiplying by a constant stretches or compresses the graph. Multiplying by a negative constant causes the graph to be reflected in the x-axis.

Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Horizontal Translations of Trigonometric Functions

Objective

Students graph horizontal translations of trigonometric functions.

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationship between the parameter *h* and the graph of a trigonometric function.

What Students Are Learning

A horizontal translation of the graph of a trigonometric function is called a phase shift. In the form $y = \sin(x - h)$, when $h > 0$, the parent graph is translated *h* units right. If $h < 0$, the parent function is translated $|h|$ units left.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students are having difficulty determining the phase shift, **THEN** have students set the argument of the sine function equal to zero and solve for *x*. This will give the phase shift.

Example 1 Graph a Phase Shift

MP Teaching the Mathematical Practices

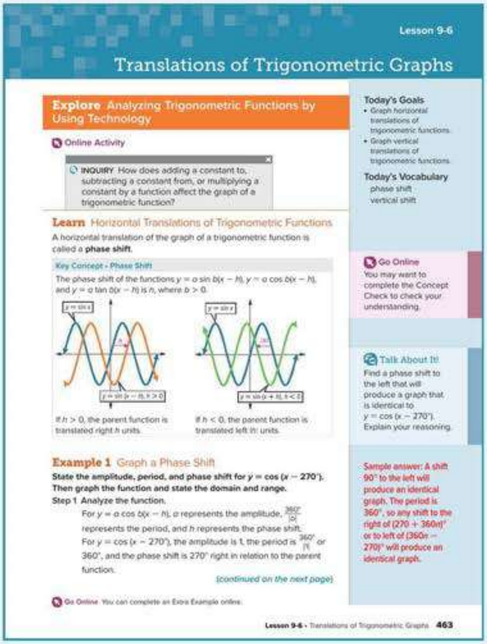
3 Justify Conclusions Students will explain the conclusions drawn when answering the question in the Talk About It! feature.

Questions for Mathematical Discourse

- **AL** How does a phase shift of a trigonometric function compare to horizontal translations of other function types? The form of the translation is the same for trigonometric functions as for other function types, $f(x - h)$.
- **OL** What other phase shifts of the parent cosine function would result in the same graph as $\cos(x - 270°)$? The function could be shifted by 270° plus any integer multiple of the period.
- **BL** How can the graph of $y = \sin x$ be shifted horizontally to be the same as the graph of $y = \cos x$? Sample answer: If the graph of $y = \sin x$ is shifted left 90°, it coincides with the graph of $y = \cos x$. So the graph of $y = \sin(x + 90°)$ is the same as the graph of $y = \cos x$.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 9-6

Translations of Trigonometric Graphs

Explore Analyzing Trigonometric Functions by Using Technology

Online Activity

INQUIRY How does adding a constant to, subtracting a constant from, or multiplying a constant by a function affect the graph of a trigonometric function?

Learn Horizontal Translations of Trigonometric Functions

A horizontal translation of the graph of a trigonometric function is called a **phase shift**.

Key Concept • Phase Shift

The phase shift of the functions $y = a \sin b(x - h)$, $y = a \cos b(x - h)$, and $y = a \tan b(x - h)$ is *h*, where $b > 0$.

If $h > 0$, the parent function is translated right *h* units.

If $h < 0$, the parent function is translated left $|h|$ units.

Example 1 Graph a Phase Shift

State the amplitude, period, and phase shift for $y = \cos(x - 270°)$. Then graph the function and state the domain and range.

Step 1 Analyze the function.

For $y = a \cos b(x - h)$, *a* represents the amplitude, $\frac{360°}{|b|}$ represents the period, and *h* represents the phase shift. For $y = \cos(x - 270°)$, the amplitude is 1, the period is $\frac{360°}{|1|}$ or 360°, and the phase shift is 270° right in relation to the parent function.

(continued on the next page)

Go Online You can complete an Extra Example online.

Today's Goals
- Graph horizontal translations of trigonometric functions.
- Graph vertical translations of trigonometric functions.

Today's Vocabulary
phase shift
vertical shift

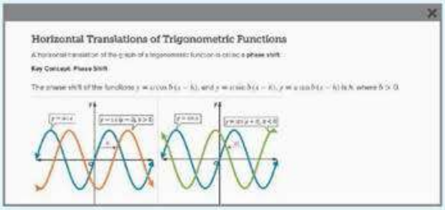
Go Online
You may want to complete the Concept Check to check your understanding.

Talk About It!
Find a phase shift to the left that will produce a graph that is identical to $y = \cos(x - 270°)$. Explain your reasoning.

Sample answer: A shift 90° to the left will produce an identical graph. The period is 360°, so any shift to the right of $(270 + 360n)°$ or to left of $(360n - 270)°$ will produce an identical graph.

Lesson 9-6 • Translations of Trigonometric Graphs 463

Interactive Presentation



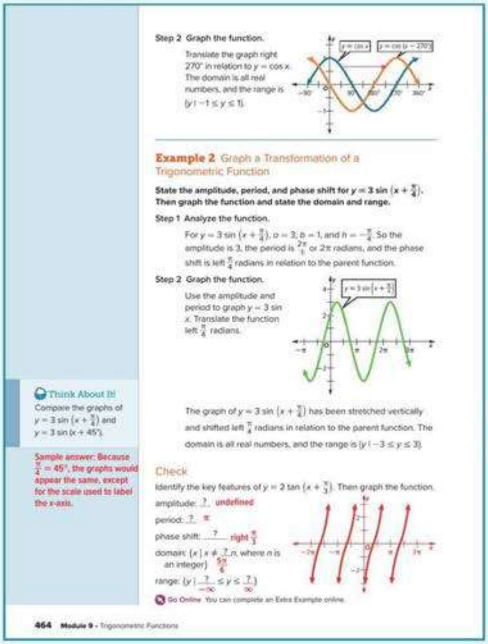
Horizontal Translations of Trigonometric Functions

Key Concept: Phase Shift

Learn

DRAG & DROP

Students drag functions to describe how their graphs have been transformed.



Step 2 **Graph the function.**

Translate the graph right 270° in relation to $y = \cos x$. The domain is all real numbers, and the range is $\{y \mid -1 \le y \le 1\}$.

Example 2 Graph a Transformation of a Trigonometric Function

State the amplitude, period, and phase shift for $y = 3 \sin \left(x + \frac{\pi}{4}\right)$. Then graph the function and state the domain and range.

Step 1 **Analyze the function.**

For $y = 3 \sin \left(x + \frac{\pi}{4}\right)$, $a = 3$, $b = 1$, and $h = -\frac{\pi}{4}$. So the amplitude is 3, the period is $\frac{2\pi}{1}$ or 2π radians, and the phase shift is left $\frac{\pi}{4}$ radians in relation to the parent function.

Step 2 **Graph the function.**

Use the amplitude and period to graph $y = 3 \sin x$. Translate the function left $\frac{\pi}{4}$ radians.

The graph of $y = 3 \sin \left(x + \frac{\pi}{4}\right)$ has been stretched vertically and shifted left $\frac{\pi}{4}$ radians in relation to the parent function. The domain is all real numbers, and the range is $\{y \mid -3 \le y \le 3\}$.

Think About It!

Compare the graphs of $y = 3 \sin \left(x + \frac{\pi}{4}\right)$ and $y = 3 \sin (x + 45°)$.

Sample answer: Because $\frac{\pi}{4} = 45°$, the graphs would appear the same, except for the scale used to label the x-axis.

Check

Identify the key features of $y = 2 \tan \left(x + \frac{\pi}{3}\right)$. Then graph the function.

amplitude: ? undefined

period: ? π

phase shift: ? right $\frac{\pi}{3}$

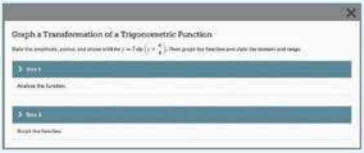
domain: $\{x \mid x \ne$? n, where n is an integer$\}$ $\frac{5\pi}{6}$

range: $\{y \mid$? $\le y \le$? $\}$ $-\infty$ ∞

Go Online You can complete an Extra Example online.

464 Module 9 • Trigonometric Functions

Interactive Presentation



Graph a Transformation of a Trigonometric Function

Step 1

Analyze the function.

Step 2

Graph the function.

Example 2

TAP

Students move through the steps to graph a transformation of the sine function.

TYPE

Students compare the graphs of two transformed sine functions.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Graph a Transformation of a Trigonometric Function

Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. Guide students to see what information they can gather about the function just by looking at it.

Questions for Mathematical Discourse

AL Why is the phase shift to the left? The form of a horizontal translation is $f(x - h)$, so $h = -\frac{\pi}{4}$. The negative indicates a shift to the left.

OL How can the period of 2π be verified on the graph of the transformed function? Sample answer: Find the distance between the x-values of two consecutive maxima.

BL When does a phase shift to the left by a value have the same result as a phase shift to the right by the same value? The effect is the same when the phase shift is an integer multiple of half of the period.

Common Error

Many students try to shift a graph before stretching or compressing. Although in this example the graph would still be the same, when multiple transformations are present, the graph must be stretched or compressed before being shifted. Remind students that the order of operations is always mandatory.

Learn Vertical Translations of Trigonometric Functions

Objective

Students graph vertical translations of trigonometric functions.

Teaching the Mathematical Practices

3 Construct Arguments In the Think About It! feature, students will use definitions and previously established results to construct an argument about the midline of a tangent function.

About the Key Concept

A vertical translation of the graph of a trigonometric function is called a vertical shift. In the form $y = \sin x + k$, when $k < 0$, the parent graph is translated k units down. If $k > 0$, the parent function is translated k units up. The vertical shift affects the midline, so if the graph is shifted k units up, the midline is at $y = k$.

DIFFERENTIATE

Reteaching Activity AL ELL

Identify the horizontal and vertical shift of $y = 6 - 3 \sin (2x + \pi)$.

Vertical shift: up 6

Horizontal shift: left $\frac{\pi}{2}$ units

Example 3 Graph a Vertical Shift

Teaching the Mathematical Practices

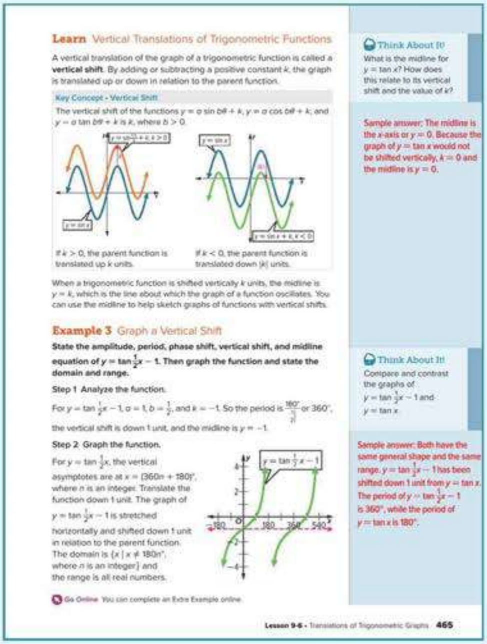
1 Explain Correspondences Encourage students to explain the relationships between the function, key features, and graph used in Example 3.

Questions for Mathematical Discourse

AL How does a vertical translation of a trigonometric function compare to vertical translations of other function types? Sample answer: The vertical translation is determined by the constant, k, for any $f(x) + k$. The translation can be applied by using key features. For a trigonometric function, the midline is $y = k$. For an exponential function, the asymptote is $y = k$. For a parabola, the y-value of the vertex is k.

OL How does a vertical shift affect the range of the graph of trigonometric functions? A vertical shift does not affect the range of tangent or cotangent functions, but it does affect the range of sine, cosine, cosecant, and secant functions by shifting the range up or down by $|k|$ units.

BL Where would the asymptotes for $y = \cot\left(\frac{1}{2x}\right) - 1$ be? They would occur at the x-values where $y = \tan\left(\frac{1}{2x}\right) - 1 = 1$.



Learn Vertical Translations of Trigonometric Functions

A vertical translation of the graph of a trigonometric function is called a **vertical shift**. By adding or subtracting a positive constant k, the graph is translated up or down in relation to the parent function.

Key Concept - Vertical Shift

The vertical shift of the functions $y = a \sin b\theta + k$, $y = a \cos b\theta + k$, and $y = a \tan b\theta + k$ is k, where $b > 0$.

If $k > 0$, the parent function is translated up k units.

If $k < 0$, the parent function is translated down $|k|$ units.

When a trigonometric function is shifted vertically k units, the midline is $y = k$, which is the line about which the graph of a function oscillates. You can use the midline to help sketch graphs of functions with vertical shifts.

Think About It!

What is the midline for $y = \tan x$? How does this relate to its vertical shift and the value of k?

Sample answer: The midline is the x-axis or $y = 0$. Because the graph of $y = \tan x$ would not be shifted vertically, $k = 0$ and the midline is $y = 0$.

Example 3 Graph a Vertical Shift

State the amplitude, period, phase shift, vertical shift, and midline equation of $y = \tan \frac{1}{2}x - 1$. Then graph the function and state the domain and range.

Step 1 Analyze the function.

For $y = \tan \frac{1}{2}x - 1$, $a = 1$, $b = \frac{1}{2}$, and $k = -1$. So the period is $\frac{180°}{\frac{1}{2}}$ or 360°, the vertical shift is down 1 unit, and the midline is $y = -1$.

Step 2 Graph the function.

For $y = \tan \frac{1}{2}x$, the vertical asymptotes are at $x = (360n + 180)°$, where n is an integer. Translate the function down 1 unit. The graph of $y = \tan \frac{1}{2}x - 1$ is stretched horizontally and shifted down 1 unit in relation to the parent function. The domain is $\{x \mid x \neq 180n°$, where n is an integer$\}$ and the range is all real numbers.

Go Online You can complete an Extra Example online.

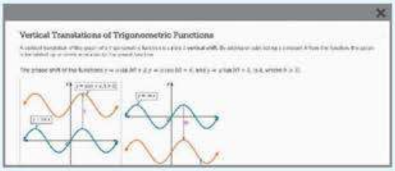
Think About It!

Compare and contrast the graphs of $y = \tan \frac{1}{2}x - 1$ and $y = \tan x$

Sample answer: Both have the same general shape and the same range. $y = \tan \frac{1}{2}x - 1$ has been shifted down 1 unit from $y = \tan x$. The period of $y = \tan \frac{1}{2}x - 1$ is 360°, while the period of $y = \tan x$ is 180°.

Lesson 9-6 • Translations of Trigonometric Graphs 465

Interactive Presentation



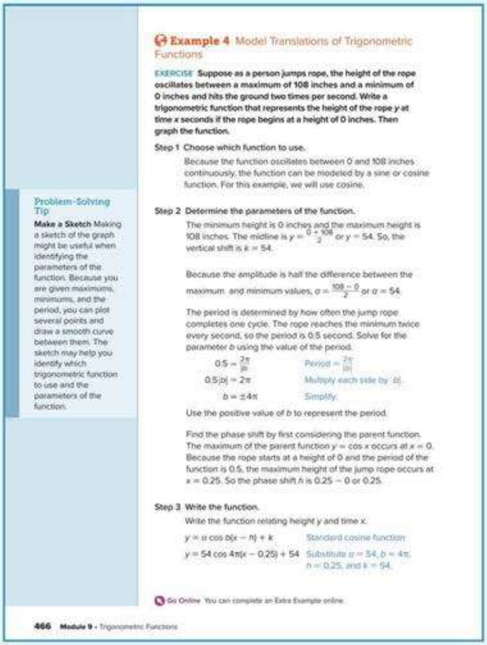
Vertical Translations of Trigonometric Functions

Learn

TYPE

a

Students identify the midline for the graph of the tangent function and then discuss how it is related to the vertical shift.



Example 4 Model Translations of Trigonometric Functions

EXERCISE **Suppose as a person jumps rope, the height of the rope oscillates between a maximum of 108 inches and a minimum of 0 inches and hits the ground two times per second. Write a trigonometric function that represents the height of the rope y at time x seconds if the rope begins at a height of 0 inches. Then graph the function.**

Step 1 **Choose which function to use.**

Because the function oscillates between 0 and 108 inches continuously, the function can be modeled by a sine or cosine function. For this example, we will use cosine.

Step 2 **Determine the parameters of the function.**

The minimum height is 0 inches and the maximum height is 108 inches. The midline is $y = \frac{0 + 108}{2}$ or $y = 54$. So, the vertical shift is $k = 54$.

Because the amplitude is half the difference between the maximum and minimum values, $a = \frac{108 - 0}{2}$ or $a = 54$.

The period is determined by how often the jump rope completes one cycle. The rope reaches the minimum twice every second, so the period is 0.5 second. Solve for the parameter b using the value of the period.

$0.5 = \frac{2\pi}{|b|}$ Period $= \frac{2\pi}{|b|}$

$0.5|b| = 2\pi$ Multiply each side by $|b|$.

$b = \pm 4\pi$ Simplify.

Use the positive value of b to represent the period.

Find the phase shift by first considering the parent function. The maximum of the parent function $y = \cos x$ occurs at $x = 0$. Because the rope starts at a height of 0 and the period of the function is 0.5, the maximum height of the jump rope occurs at $x = 0.25$. So the phase shift h is $0.25 - 0$ or 0.25.

Step 3 **Write the function.**

Write the function relating height y and time x.

$y = a \cos b(x - h) + k$ Standard cosine function

$y = 54 \cos 4\pi(x - 0.25) + 54$ Substitute $a = 54$, $b = 4\pi$, $h = 0.25$, and $k = 54$.

Go Online You can complete an Extra Example online.

Problem-Solving Tip

Make a Sketch Making a sketch of the graph might be useful when identifying the parameters of the function. Because you are given maximums, minimums, and the period, you can plot several points and draw a smooth curve between them. The sketch may help you identify which trigonometric function to use and the parameters of the function.

466 Module 9 • Trigonometric Functions

Interactive Presentation

Model Translations of Trigonometric Functions

Step 1

Choose which function to use.

Example 4

TAP

Students move through the steps of writing a trigonometric function that models a real-world situation.

Example 4 Model Translations of Trigonometric Functions

Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the verbal description, function, and graph used in this example.

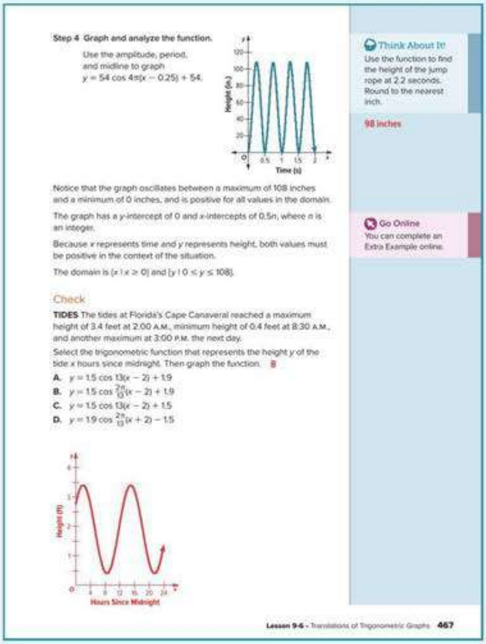
4 Analyze Relationships Mathematically Students will need to analyze the mathematical relationships in the problem to write and graph a function that models the situation in Example 4.

Questions for Mathematical Discourse

AL Why was cosine selected for this example? Either sine or cosine could be selected because they both have periodic behavior that matches the context. Both need a phase shift to model the situation, so it is an arbitrary choice.

OL Why are a and k the same value? This results in the minimum of the function being zero, which makes sense in the context.

BL What is the difference in the phase shift needed for using a cosine versus a sine function for the model? The cosine function needed a phase shift of half the period. The sine function would need to be shifted right by $\frac{1}{4}$ of the period.



Step 4 Graph and analyze the function.

Use the amplitude, period, and midline to graph $y = 54 \cos 4\pi(x - 0.25) + 54$.

Notice that the graph oscillates between a maximum of 108 inches and a minimum of 0 inches, and is positive for all values in the domain.

The graph has a *y*-intercept of 0 and *x*-intercepts of 0.5*n*, where *n* is an integer.

Because *x* represents time and *y* represents height, both values must be positive in the context of the situation.

The domain is $\{x \mid x \geq 0\}$ and $\{y \mid 0 \leq y \leq 108\}$.

Check

TIDES The tides at Florida's Cape Canaveral reached a maximum height of 3.4 feet at 2:00 A.M., minimum height of 0.4 feet at 8:30 A.M., and another maximum at 3:00 P.M. the next day.

Select the trigonometric function that represents the height *y* of the tide *x* hours since midnight. Then graph the function. B

A. $y = 1.5 \cos 13(x - 2) + 1.9$
B. $y = 1.5 \cos \frac{2\pi}{13}(x - 2) + 1.9$
C. $y = 1.5 \cos 13(x - 2) + 1.5$
D. $y = 1.9 \cos \frac{2\pi}{13}(x + 2) - 1.5$

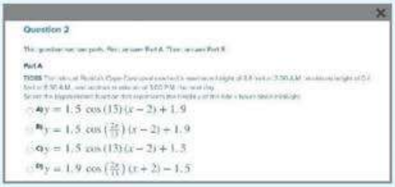
Think About It! Use the function to find the height of the jump rope at 2.2 seconds. Round to the nearest inch.

98 inches

Go Online You can complete an Extra Example online.

Lesson 9-6 • Translations of Trigonometric Graphs 467

Interactive Presentation



Question 2

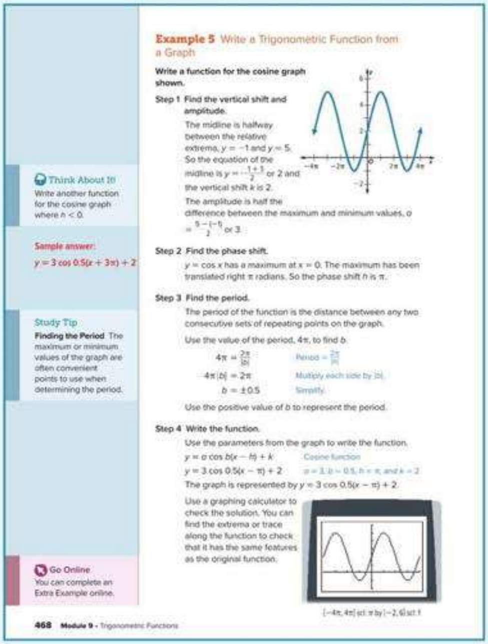
Part A

○ A) $y = 1.5 \cos(13)(x - 2) + 1.9$
○ B) $y = 1.5 \cos\left(\frac{2\pi}{13}\right)(x - 2) + 1.9$
○ C) $y = 1.5 \cos(13)(x - 2) + 1.5$
○ D) $y = 1.9 \cos\left(\frac{2\pi}{13}\right)(x + 2) - 1.5$

Learn

TYPE

Students use the function to evaluate at a specific time.



Example 5 Write a Trigonometric Function from a Graph

Write a function for the cosine graph shown.

Step 1 **Find the vertical shift and amplitude.**

The midline is halfway between the relative extrema, $y = -1$ and $y = 5$. So the equation of the midline is $y = \frac{-1+5}{2}$ or 2 and the vertical shift k is 2.

The amplitude is half the difference between the maximum and minimum values, $a = \frac{5-(-1)}{2}$ or 3.

Step 2 **Find the phase shift.**

$y = \cos x$ has a maximum at $x = 0$. The maximum has been translated right π radians. So the phase shift h is π.

Step 3 **Find the period.**

The period of the function is the distance between any two consecutive sets of repeating points on the graph.

Use the value of the period, 4π, to find b.

$4\pi = \frac{2\pi}{|b|}$ Period $= \frac{2\pi}{|b|}$

$4\pi|b| = 2\pi$ Multiply each side by $|b|$.

$b = \pm 0.5$ Simplify.

Use the positive value of b to represent the period.

Step 4 **Write the function.**

Use the parameters from the graph to write the function.

$y = a \cos b(x - h) + k$ Cosine function

$y = 3 \cos 0.5(x - \pi) + 2$ $a = 3, b = 0.5, h = \pi,$ and $k = 2$

The graph is represented by $y = 3 \cos 0.5(x - \pi) + 2$.

Use a graphing calculator to check the solution. You can find the extrema or trace along the function to check that it has the same features as the original function.

[−4π, 4π] scl: π by [−2, 6] scl: 1

Think About It!
Write another function for the cosine graph where $h < 0$.

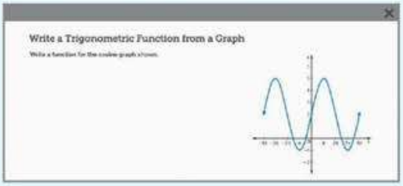
Sample answer:
$y = 3 \cos 0.5(x + 3\pi) + 2$

Study Tip
Finding the Period The maximum or minimum values of the graph are often convenient points to use when determining the period.

Go Online
You can complete an Extra Example online.

468 Module 9 • Trigonometric Functions

Interactive Presentation



Write a Trigonometric Function from a Graph

Write a function for the cosine graph shown.

Example 5

TAP

Students move through the steps to write the equation of a trigonometric function from a graph.

TYPE

Students write another function for the transformed graph.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 5 Write a Trigonometric Function from a Graph

Teaching the Mathematical Practices

5 Analyze Graphs To check their solutions, students should analyze the graph by using graphing calculators. Point out that to see the entire graph, students may need to adjust the viewing window.

Questions for Mathematical Discourse

AL How do you find the period of the graphed function? You can use the difference between two consecutive maxima or minima.

OL What would be the phase shift from the parent function if this was a sine function? There would not be a phase shift.

BL For any graph of an unspecified trigonometric function, how many function types could it represent? Two. Cosine and sine can each be represented as a phase shift of the other, likewise for secant and cosecant. Tangent and cotangent can be interchanged with a sign flip plus phase shift.

Common Error

Encourage students to find the midline of the graph before any other transformation because the midline is very helpful in identifying reflections, amplitude, and even phase shift.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–22
2	exercises that use a variety of skills from this lesson	23–39
2	exercises that extend concepts learned in this lesson to new contexts	40–44
3	exercises that emphasize higher-order and critical-thinking skills	45–52

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign: **BL**

- Practice Exercises 1–43 odd, 45–52
- Extension: Simple Harmonic Motion
- **ALEKS** Graphs of Sine and Cosine Functions

IF students score 66%–89% on the Checks, **THEN** assign: **OL**

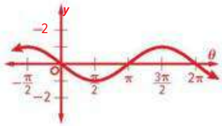
- Practice Exercises 1–51 odd
- Remediation, Review Resources: Transformations of Functions
- Personal Tutors
- Extra Examples 1–5
- **ALEKS** Transformations

IF students score 65% or less on the Checks, **THEN** assign: **AL**

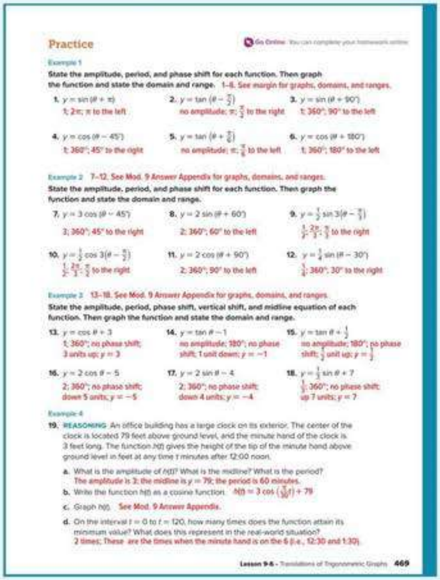
- Practice Exercises 1–21 odd
- Remediation, Review Resources: Transformations of Functions
- *Quick Review Math Handbook*: Translations of Trigonometric Graphs
- **ALEKS** Transformations

Answers

1.



domain: $-\infty < \theta < \infty$; range: $-1 \le f(\theta) \le 1$



Practice

Go Online You can complete your homework online.

Example 1

State the amplitude, period, and phase shift for each function. Then graph the function and state the domain and range. 1–6. See margin for graphs, domains, and ranges.

1. $y = \sin(\theta + \pi)$ 1; 2π; π to the left
2. $y = \tan\left(\theta - \frac{\pi}{2}\right)$ no amplitude; π; $\frac{\pi}{2}$ to the right
3. $y = \sin(\theta + 90°)$ 1; 360°; 90° to the left
4. $y = \cos(\theta - 45°)$ 1; 360°; 45° to the right
5. $y = \tan\left(\theta + \frac{\pi}{6}\right)$ no amplitude; π; $\frac{\pi}{6}$ to the left
6. $y = \cos(\theta + 180°)$ 1; 360°; 180° to the left

Example 2 7–12. See Mod. 9 Answer Appendix for graphs, domains, and ranges.

State the amplitude, period, and phase shift for each function. Then graph the function and state the domain and range.

7. $y = 3\cos(\theta - 45°)$ 3; 360°; 45° to the right
8. $y = 2\sin(\theta + 60°)$ 2; 360°; 60° to the left
9. $y = \frac{1}{2}\sin 3\left(\theta - \frac{\pi}{3}\right)$ $\frac{1}{2}$; $\frac{2\pi}{3}$; $\frac{\pi}{3}$ to the right
10. $y = \frac{1}{2}\cos 3\left(\theta - \frac{\pi}{2}\right)$ $\frac{1}{2}$; $\frac{2\pi}{3}$; $\frac{\pi}{2}$ to the right
11. $y = 2\cos(\theta + 90°)$ 2; 360°; 90° to the left
12. $y = \frac{1}{4}\sin(\theta - 30°)$ $\frac{1}{4}$; 360°; 30° to the right

Example 3 13–18. See Mod. 9 Answer Appendix for graphs, domains, and ranges.

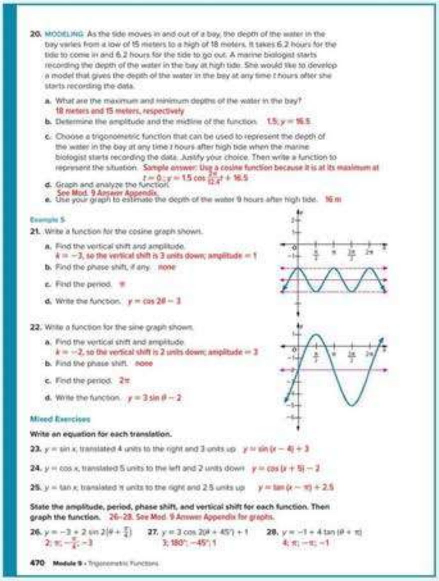
State the amplitude, period, phase shift, vertical shift, and midline equation of each function. Then graph the function and state the domain and range.

13. $y = \cos\theta + 3$ 1; 360°; no phase shift; 3 units up; $y = 3$
14. $y = \tan\theta - 1$ no amplitude; 180°; no phase shift; 1 unit down; $y = -1$
15. $y = \tan\theta + \frac{1}{2}$ no amplitude; 180°; no phase shift; $\frac{1}{2}$ unit up; $y = \frac{1}{2}$
16. $y = 2\cos\theta - 5$ 2; 360°; no phase shift; down 5 units; $y = -5$
17. $y = 2\sin\theta - 4$ 2; 360°; no phase shift; down 4 units; $y = -4$
18. $y = \frac{1}{3}\sin\theta + 7$ $\frac{1}{3}$; 360°; no phase shift; up 7 units; $y = 7$

Example 4

19. REASONING An office building has a large clock on its exterior. The center of the clock is located 79 feet above ground level, and the minute hand of the clock is 3 feet long. The function $h(t)$ gives the height of the tip of the minute hand above ground level in feet at any time t minutes after 12:00 noon.
    a. What is the amplitude of $h(t)$? What is the midline? What is the period? The amplitude is 3; the midline is $y = 79$; the period is 60 minutes.
    b. Write the function $h(t)$ as a cosine function. $h(t) = 3\cos\left(\frac{\pi}{30}t\right) + 79$
    c. Graph $h(t)$. See Mod. 9 Answer Appendix.
    d. On the interval $t = 0$ to $t = 120$, how many times does the function attain its minimum value? What does this represent in the real-world situation? 2 times; These are the times when the minute hand is on the 6 (i.e., 12:30 and 1:30).

Lesson 9-6 • Translations of Trigonometric Graphs 469



20. MODELING As the tide moves in and out of a bay, the depth of the water in the bay varies from a low of 15 meters to a high of 18 meters. It takes 6.2 hours for the tide to come in and 6.2 hours for the tide to go out. A marine biologist starts recording the depth of the water in the bay at high tide. She would like to develop a model that gives the depth of the water in the bay at any time t hours after she starts recording the data.
    a. What are the maximum and minimum depths of the water in the bay? 18 meters and 15 meters, respectively
    b. Determine the amplitude and the midline of the function. 1.5; $y = 16.5$
    c. Choose a trigonometric function that can be used to represent the depth of the water in the bay at any time t hours after high tide when the marine biologist starts recording the data. Justify your choice. Then write a function to represent the situation. Sample answer: Use a cosine function because it is at its maximum at $t = 0$; $y = 1.5\cos\frac{2\pi}{12.4}t + 16.5$
    d. Graph and analyze the function. See Mod. 9 Answer Appendix.
    e. Use your graph to estimate the depth of the water 9 hours after high tide. 16 m

Example 5

21. Write a function for the cosine graph shown.
    a. Find the vertical shift and amplitude. $k = -3$, so the vertical shift is 3 units down; amplitude = 1
    b. Find the phase shift, if any. none
    c. Find the period. π
    d. Write the function. $y = \cos 2\theta - 3$

22. Write a function for the sine graph shown.
    a. Find the vertical shift and amplitude. $k = -2$, so the vertical shift is 2 units down; amplitude = 3
    b. Find the phase shift. none
    c. Find the period. 2π
    d. Write the function. $y = 3\sin\theta - 2$

Mixed Exercises

Write an equation for each translation.

23. $y = \sin x$, translated 4 units to the right and 3 units up $y = \sin(x - 4) + 3$
24. $y = \cos x$, translated 5 units to the left and 2 units down $y = \cos(x + 5) - 2$
25. $y = \tan x$, translated π units to the right and 2.5 units up $y = \tan(x - \pi) + 2.5$

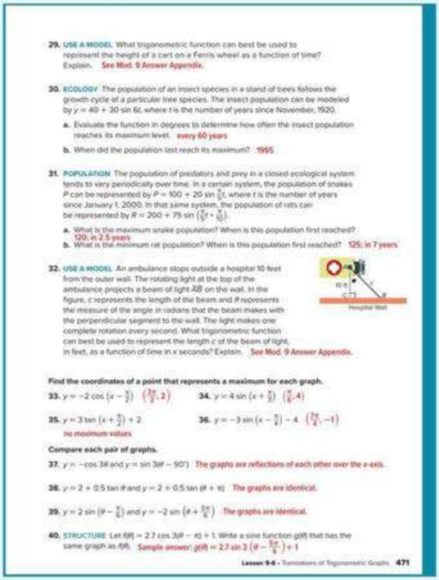
State the amplitude, period, phase shift, and vertical shift for each function. Then graph the function. 26–28. See Mod. 9 Answer Appendix for graphs.

26. $y = -3 + 2\sin 2\left(\theta + \frac{\pi}{4}\right)$ 2; π; $-\frac{\pi}{4}$; −3
27. $y = 3\cos 2(\theta + 45°) + 1$ 3; 180°; −45°; 1
28. $y = -1 + 4\tan(\theta + \pi)$ 4; π; −π; −1

470 Module 9 • Trigonometric Functions

3 REFLECT AND PRACTICE

F.IF.7e, F.BF.3



29. **USE A MODEL** What trigonometric function can best be used to represent the height of a cart on a Ferris wheel as a function of time? Explain. See Mod. 9 Answer Appendix.

30. **ECOLOGY** The population of an insect species in a stand of trees follows the growth cycle of a particular tree species. The insect population can be modeled by $y = 40 + 30 \sin 6t$, where t is the number of years since November, 1920.

a. Evaluate the function in degrees to determine how often the insect population reaches its maximum level. every 60 years

b. When did the population last reach its maximum? 1995

31. **POPULATION** The population of predators and prey in a closed ecological system tends to vary periodically over time. In a certain system, the population of snakes P can be represented by $P = 100 + 20 \sin \frac{\pi}{5}t$, where t is the number of years since January 1, 2000. In that same system, the population of rats can be represented by $R = 200 + 75 \sin\left(\frac{\pi}{5}t + \frac{\pi}{10}\right)$.

a. What is the maximum snake population? When is this population first reached? 120; in 2.5 years

b. What is the minimum rat population? When is this population first reached? 125; in 7 years

32. **USE A MODEL** An ambulance stops outside a hospital 10 feet from the outer wall. The rotating light at the top of the ambulance projects a beam of light $\overline{AB}$ on the wall. In the figure, c represents the length of the beam and θ represents the measure of the angle in radians that the beam makes with the perpendicular segment to the wall. The light makes one complete rotation every second. What trigonometric function can best be used to represent the length c of the beam of light, in feet, as a function of time in x seconds? Explain. See Mod. 9 Answer Appendix.

Find the coordinates of a point that represents a maximum for each graph.

33. $y = -2\cos\left(x - \frac{\pi}{2}\right)$ $\left(\frac{3\pi}{2}, 2\right)$

34. $y = 4\sin\left(x + \frac{\pi}{3}\right)$ $\left(\frac{\pi}{6}, 4\right)$

35. $y = 3\tan\left(x + \frac{\pi}{2}\right) + 2$ no maximum values

36. $y = -3\sin\left(x - \frac{\pi}{4}\right) - 4$ $\left(\frac{7\pi}{4}, -1\right)$

Compare each pair of graphs.

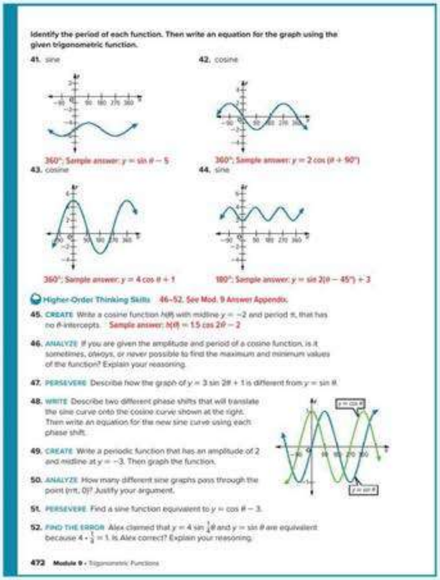
37. $y = -\cos 3\theta$ and $y = \sin(3\theta - 90°)$ The graphs are reflections of each other over the x-axis.

38. $y = 2 + 0.5\tan\theta$ and $y = 2 + 0.5\tan(\theta + \pi)$ The graphs are identical.

39. $y = 2\sin\left(\theta - \frac{\pi}{6}\right)$ and $y = -2\sin\left(\theta + \frac{5\pi}{6}\right)$ The graphs are identical.

40. **STRUCTURE** Let $f(\theta) = 2.7\cos 3(\theta - \pi) + 1$. Write a sine function $g(\theta)$ that has the same graph as $f(\theta)$. Sample answer: $g(\theta) = 2.7\sin 3\left(\theta - \frac{5\pi}{6}\right) + 1$

Lesson 9-6 • Translations of Trigonometric Graphs 471



Identify the period of each function. Then write an equation for the graph using the given trigonometric function.

41. sine

360°; Sample answer: $y = \sin\theta - 5$

42. cosine

360°; Sample answer: $y = 2\cos(\theta + 90°)$

43. cosine

360°; Sample answer: $y = 4\cos\theta + 1$

44. sine

180°; Sample answer: $y = \sin 2(\theta - 45°) + 3$

Higher-Order Thinking Skills 46–52. See Mod. 9 Answer Appendix.

45. **CREATE** Write a cosine function $h(\theta)$ with midline $y = -2$ and period π, that has no θ-intercepts. Sample answer: $h(\theta) = 1.5\cos 2\theta - 2$

46. **ANALYZE** If you are given the amplitude and period of a cosine function, is it sometimes, always, or never possible to find the maximum and minimum values of the function? Explain your reasoning.

47. **PERSEVERE** Describe how the graph of $y = 3\sin 2\theta + 1$ is different from $y = \sin\theta$.

48. **WRITE** Describe two different phase shifts that will translate the sine curve onto the cosine curve shown at the right. Then write an equation for the new sine curve using each phase shift.

49. **CREATE** Write a periodic function that has an amplitude of 2 and midline at $y = -3$. Then graph the function.

50. **ANALYZE** How many different sine graphs pass through the point $(n\pi, 0)$? Justify your argument.

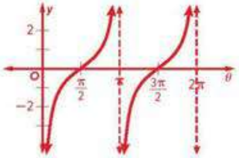
51. **PERSEVERE** Find a sine function equivalent to $y = \cos\theta - 3$.

52. **FIND THE ERROR** Alex claimed that $y = 4\sin\frac{1}{4}\theta$ and $y = \sin\theta$ are equivalent because $4 \cdot \frac{1}{4} = 1$. Is Alex correct? Explain your reasoning.

472 Module 9 • Trigonometric Functions

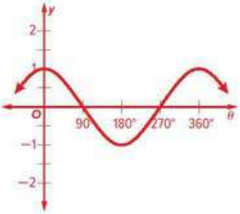
Answers

2.



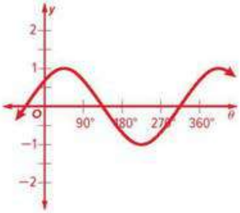
domain: $\pi n < \theta < \pi + \pi n$, where n is an integer; range: $-\infty < f(\theta) < \infty$

3.



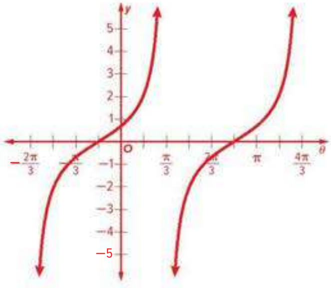
domain: $-\infty < \theta < \infty$; range: $-1 \le f(\theta) \le 1$

4.



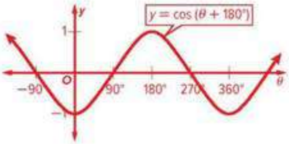
domain: $-\infty < \theta < \infty$; range: $-1 \le f(\theta) \le 1$

5.



domain: $\frac{\pi}{3}\pi n < \theta < +\frac{\pi}{3}\pi(n + 1)$, where n is an integer; range: $-\infty < f(\theta) < \infty$

6.



domain: $-\infty < \theta < \infty$; range: $-1 \le f(\theta) \le 1$

Lesson 9-7

Inverse Trigonometric Functions

F.TF.7

LESSON GOAL

Students find the values of angle measures by using inverse trigonometric functions.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Develop:

Inverse Trigonometric Functions

- Evaluate Inverse Trigonometric Functions
- Find a Trigonometric Value by Using a Calculator
- Find an Angle Measure by Using a Graphing Calculator
- Use Inverse Trigonometric Functions

You may want your students to complete the **Check** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources	AL	OL	BL	ELL
Remediation: Inverse Relations and Functions	●	●		●
Extension: Snell's Law		●	●	●

Language Development Handbook

Assign page 52 of the *Language Development Handbook* to help your students build mathematical language related to using inverse trigonometric functions to find the values of angle measures.

ELL You can use the tips and suggestions on page T52 of the handbook to support students who are building English proficiency.

Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Functions

Standards for Mathematical Content:

F.TF.7 Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.

Standards for Mathematical Practice:

4 Model with mathematics.

6 Attend to precision.

Coherence

Vertical Alignment

Previous

Students found the inverses of linear, quadratic, and exponential functions.

F.IF.5, F.IF.7e, F.BF.4a

Now

Students find the values of angle measures by using inverse trigonometric functions.

F.TF.7

Rigor

The Three Pillars of Rigor

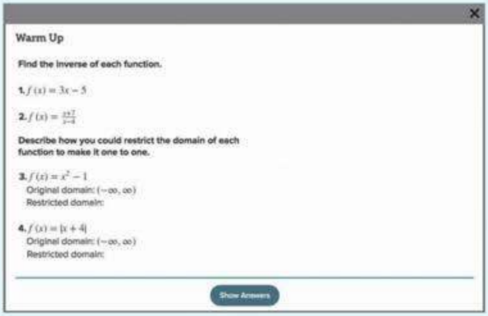
1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students expand on their understanding of the basic trigonometric functions to include inverse trigonometric functions. They build fluency by using inverses to solve trigonometric equations, and they apply their understanding by solving real-world problems related to trigonometric functions.

Mathematical Background

Because the trigonometric functions are periodic, many angles correspond to the same function value. Therefore, the inverse of any of the trigonometric functions is not a function. However, if the domain is restricted to an appropriate interval, the inverse is a function.

Interactive Presentation



Warm Up

Find the inverse of each function.

1. $f(x) = 3x - 5$

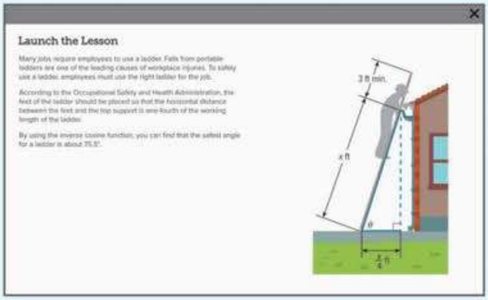
2. $f(x) = \frac{x+7}{x-4}$

Describe how you could restrict the domain of each function to make it one to one.

3. $f(x) = x^2 - 1$
   Original domain: $(-\infty, \infty)$
   Restricted domain:

4. $f(x) = |x + 4|$
   Original domain: $(-\infty, \infty)$
   Restricted domain:

Warm Up



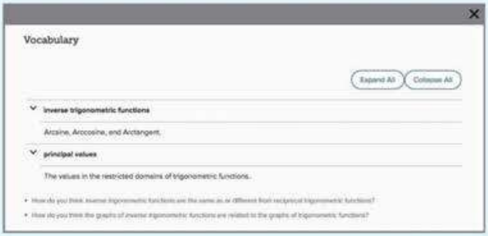
Launch the Lesson

Many jobs require employees to use a ladder. Falls from portable ladders are one of the leading causes of workplace injuries. To safely use a ladder, employees must use the right ladder for the job.

According to the Occupational Safety and Health Administration, the feet of the ladder should be placed so that the horizontal distance between the feet and the top support is one-fourth of the working length of the ladder.

By using the inverse cosine function, you can find that the safest angle for a ladder is about 75.5°.

Launch the Lesson



Vocabulary

inverse trigonometric functions

Arcsine, Arccosine, and Arctangent.

principal values

The values in the restricted domains of trigonometric functions.

- How do you think inverse trigonometric functions are the same as or different from reciprocal trigonometric functions?
- How do you think the graphs of inverse trigonometric functions are related to the graphs of trigonometric functions?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- finding inverse functions and restricted domains

Answers:

1. $f(x) = \frac{x+5}{3}$

2. $f(x) = \frac{4x+7}{x-1}$

3. $[0, \infty)$ or $(-\infty, 0]$

4. $[-4, \infty)$ or $(-\infty, -4]$

Launch the Lesson

Teaching the Mathematical Practices

4 Apply Mathematics Encourage students to model the relationship between the distance of the feet of the ladder from the wall and the angle created using an inverse trigonometric function.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?* and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Learn Inverse Trigonometric Functions

Objective

Students find values of angle measures by using inverse trigonometric functions.

Teaching the Mathematical Practices

4 Model with mathematics Students will model real-world situations by using inverse trigonometric functions.

6 Attend to precision Students will communicate mathematical reasoning precisely.

What Students Are Learning

When the value of a trigonometric function for an angle is known, the inverse function can be used to find the angle measure. The domain of inverse trigonometric functions must be restricted in order for the inverse to be a function. The values in the restricted domain are called principal values.

Essential Question Follow-Up

Students have begun learning about the inverse trigonometric functions.

Ask:

Why are inverse functions important? Sample answer: Inverse functions are important because they can be used to solve for unknown angle measures.

Example 1 Evaluate Inverse Trigonometric Functions

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to write their solution method using clear, logical steps.

Questions for Mathematical Discourse

AL How can you convert 60° to radians? Multiply by $\frac{\pi}{180°}$.

OL Why are we only solving for values of θ between $-90°$ and $90°$? Sample answer: Because $\text{Tan}^{-1}\sqrt{3}$ is capitalized, only the principal values should be included in the solution.

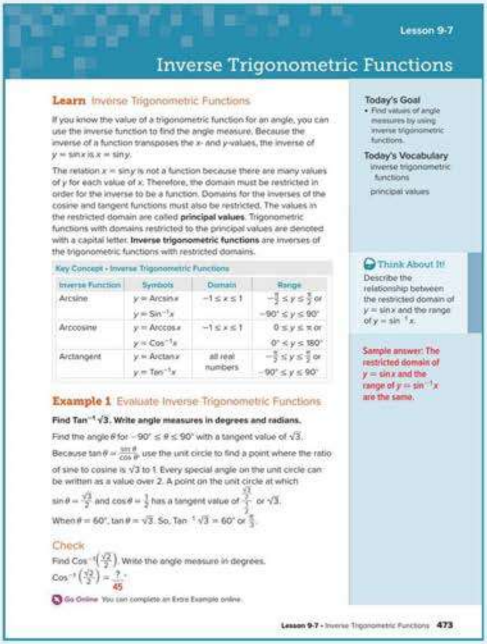
BL Find $\text{Tan}(-\sqrt{3})$. $-60°$

Common Error

Remind students that when they are finding evaluating inverse trigonometric functions using the unit circle, they only need to consider the quadrants of the principal values. For example, when evaluating Arccosine, only values in Quadrant I and Quadrant II are relevant.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Lesson 9-7

Inverse Trigonometric Functions

Learn Inverse Trigonometric Functions

If you know the value of a trigonometric function for an angle, you can use the inverse function to find the angle measure. Because the inverse of a function transposes the x- and y-values, the inverse of $y = \sin x$ is $x = \sin y$.

The relation $x = \sin y$ is not a function because there are many values of y for each value of x. Therefore, the domain must be restricted in order for the inverse to be a function. Domains for the inverses of the cosine and tangent functions must also be restricted. The values in the restricted domain are called **principal values**. Trigonometric functions with domains restricted to the principal values are denoted with a capital letter. **Inverse trigonometric functions** are inverses of the trigonometric functions with restricted domains.

Key Concept • Inverse Trigonometric Functions

Inverse Function	Symbols	Domain	Range
Arcsine	$y = \text{Arcsin}\,x$ $y = \text{Sin}^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ or $-90° \le y \le 90°$
Arccosine	$y = \text{Arccos}\,x$ $y = \text{Cos}^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$ or $0° \le y \le 180°$
Arctangent	$y = \text{Arctan}\,x$ $y = \text{Tan}^{-1}x$	all real numbers	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ or $-90° \le y \le 90°$

Example 1 Evaluate Inverse Trigonometric Functions

Find $\text{Tan}^{-1}\sqrt{3}$. Write angle measures in degrees and radians.

Find the angle θ for $-90° \le \theta \le 90°$ with a tangent value of $\sqrt{3}$.

Because $\tan\theta = \frac{\sin\theta}{\cos\theta}$, use the unit circle to find a point where the ratio of sine to cosine is $\sqrt{3}$ to 1. Every special angle on the unit circle can be written as a value over 2. A point on the unit circle at which $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2}$ has a tangent value of $\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$ or $\sqrt{3}$.

When $\theta = 60°$, $\tan\theta = \sqrt{3}$. So, $\text{Tan}^{-1}\sqrt{3} = 60°$ or $\frac{\pi}{3}$.

Check

Find $\text{Cos}^{-1}\left(\frac{\sqrt{2}}{2}\right)$. Write the angle measure in degrees.

$\text{Cos}^{-1}\left(\frac{\sqrt{2}}{2}\right) = \underline{\ ?\ }°$ 45

Go Online You can complete an Extra Example online.

Today's Goal
- Find values of angle measures by using inverse trigonometric functions.

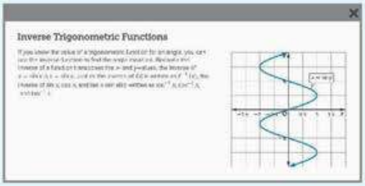
Today's Vocabulary
inverse trigonometric functions
principal values

Think About It!
Describe the relationship between the restricted domain of $y = \sin x$ and the range of $y = \sin^{-1}x$.

Sample answer: The restricted domain of $y = \sin x$ and the range of $y = \sin^{-1}x$ are the same.

Lesson 9-7 • Inverse Trigonometric Functions 473

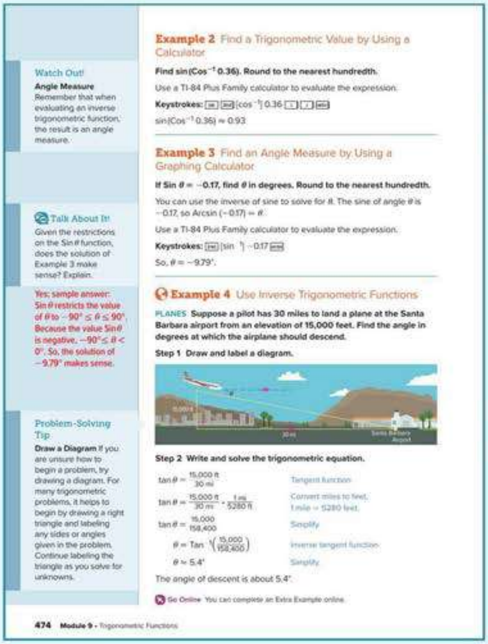
Interactive Presentation



Learn

TYPE

Students describe the relationship between the domain of a trigonometric function and the range of its inverse.



Watch Out!

Angle Measure Remember that when evaluating an inverse trigonometric function, the result is an angle measure.

Talk About It!

Given the restrictions on the Sin θ function, does the solution of Example 3 make sense? Explain.

Yes; sample answer: Sin θ restricts the value of θ to $-90° \le \theta \le 90°$. Because the value Sin θ is negative, $-90° \le \theta < 0°$. So, the solution of $-9.79°$ makes sense.

Problem-Solving Tip

Draw a Diagram If you are unsure how to begin a problem, try drawing a diagram. For many trigonometric problems, it helps to begin by drawing a right triangle and labeling any sides or angles given in the problem. Continue labeling the triangle as you solve for unknowns.

Example 2 Find a Trigonometric Value by Using a Calculator

Find $\sin(\text{Cos}^{-1} 0.36)$. Round to the nearest hundredth.

Use a TI-84 Plus Family calculator to evaluate the expression.

Keystrokes: sin 2nd [cos⁻¹] 0.36)) enter

$\sin(\text{Cos}^{-1} 0.36) \approx 0.93$

Example 3 Find an Angle Measure by Using a Graphing Calculator

If Sin $\theta = -0.17$, find θ in degrees. Round to the nearest hundredth.

You can use the inverse of sine to solve for θ. The sine of angle θ is -0.17, so Arcsin $(-0.17) = \theta$.

Use a TI-84 Plus Family calculator to evaluate the expression.

Keystrokes: 2nd [sin⁻¹] −0.17 enter

So, $\theta = -9.79°$.

Example 4 Use Inverse Trigonometric Functions

PLANES **Suppose a pilot has 30 miles to land a plane at the Santa Barbara airport from an elevation of 15,000 feet. Find the angle in degrees at which the airplane should descend.**

Step 1 Draw and label a diagram.

Step 2 Write and solve the trigonometric equation.

$\tan\theta = \frac{15{,}000 \text{ ft}}{30 \text{ mi}}$ Tangent function

$\tan\theta = \frac{15{,}000 \text{ ft}}{30 \text{ mi}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}}$ Convert miles to feet. 1 mile = 5280 feet.

$\tan\theta = \frac{15{,}000}{158{,}400}$ Simplify.

$\theta = \text{Tan}^{-1}\left(\frac{15{,}000}{158{,}400}\right)$ Inverse tangent function

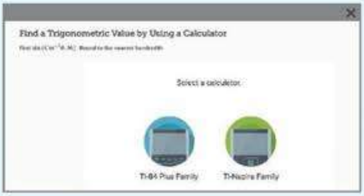
$\theta \approx 5.4°$ Simplify.

The angle of descent is about 5.4°.

Go Online You can complete an Extra Example online.

474 Module 9 • Trigonometric Functions

Interactive Presentation



Find a Trigonometric Value by Using a Calculator

Find $\sin(\text{Cos}^{-1} 0.36)$. Round to the nearest hundredth.

Example 2

TAP

Students select their preferred calculator and then move through the steps to evaluate an inverse trigonometric function.

TYPE

Students explain why the mode of the calculator was irrelevant in this example.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Find a Trigonometric Value by Using a Calculator

Questions for Mathematical Discourse

AL Using a calculator, what is the value of $\cos(\text{Sin}^{-1} 0.75)$, rounded to the nearest thousandth? 0.661

OL Why does $\sin(\text{Cos}^{-1} 1.5)$ result in an error on the graphing calculator? Sample answer: 1.5 is not in the domain of the inverse cosine function.

BL When will $\sin(\sin^{-1} x) = x$? when x is in the restricted domain of sine

Example 3 Find an Angle Measure by Using a Graphing Calculator

Questions for Mathematical Discourse

AL What does the inverse of a trigonometric function represent? the angle measure that corresponds to the given ratio

OL How can you check your answer? Sample answer: Evaluate $\sin(-9.79°)$ and check that it is approximately -0.17.

BL Would the value of $\text{Cos}^{-1}(-0.99)$ be positive or negative? Explain. Positive. Sample answer: The restricted domain of Cos θ is $0 \le \theta \le 180°$, so the value of Cos^{-1} will be positive.

Example 4 Use Inverse Trigonometric Functions

Questions for Mathematical Discourse

AL What is the ratio for $\tan\theta$ in terms of the sides of a right triangle in the context of this situation? altitude of the plane over its distance from the airport

OL Why must we convert miles to feet? Sample answer: The units in the ratio must be the same.

BL What would be the angle of descent for the same plane if the pilot has 45 miles to land from an elevation of 15,000 feet? 3.6°

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–32
2	exercises that use a variety of skills from this lesson	39–38
2	exercises that extend concepts learned in this lesson to new contexts	39
3	exercises that emphasize higher-order and critical-thinking skills	40–45

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **BL**
THEN assign:

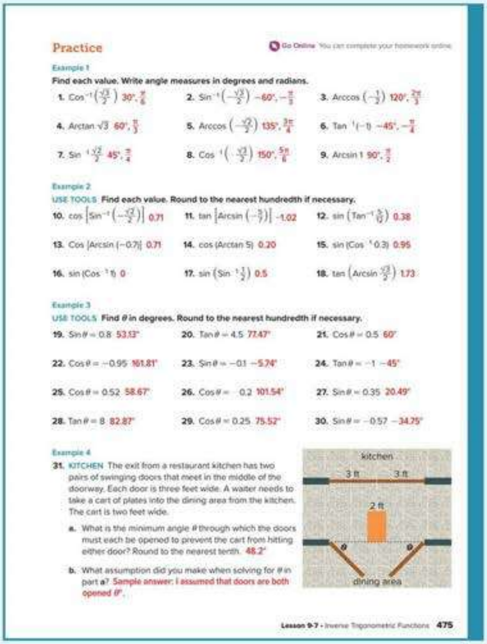
- Practice Exercises 1–39 odd, 40–45
- Extension: Snell's Law

IF students score 66%–89% on the Checks, **OL**
THEN assign:

- Practice Exercises 1–45 odd
- Remediation, Review Resources: Inverse Relations and Functions
- Personal Tutors
- Extra Examples 1–4
- **ALEKS** Function Operations, Inverse Functions

IF students score 65% or less on the Checks, **AL**
THEN assign:

- Practice Exercises 1–31 odd
- Remediation, Review Resources: Inverse Relations and Functions
- **ALEKS** Function Operations, Inverse Functions



Practice

Go Online You can complete your homework online.

Example 1

Find each value. Write angle measures in degrees and radians.

1. $\text{Cos}^{-1}\left(\frac{\sqrt{3}}{2}\right)$ 30°, $\frac{\pi}{6}$
2. $\text{Sin}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ −60°, $-\frac{\pi}{3}$
3. $\text{Arccos}\left(-\frac{1}{2}\right)$ 120°, $\frac{2\pi}{3}$
4. $\text{Arctan}\sqrt{3}$ 60°, $\frac{\pi}{3}$
5. $\text{Arccos}\left(-\frac{\sqrt{2}}{2}\right)$ 135°, $\frac{3\pi}{4}$
6. $\text{Tan}^{-1}(-1)$ −45°, $-\frac{\pi}{4}$
7. $\text{Sin}^{-1}\frac{\sqrt{2}}{2}$ 45°, $\frac{\pi}{4}$
8. $\text{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ 150°, $\frac{5\pi}{6}$
9. Arcsin 1 90°, $\frac{\pi}{2}$

Example 2

USE TOOLS **Find each value. Round to the nearest hundredth if necessary.**

10. $\cos\left[\text{Sin}^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right]$ 0.71
11. $\tan\left[\text{Arcsin}\left(-\frac{5}{7}\right)\right]$ −1.02
12. $\sin\left(\text{Tan}^{-1}\frac{5}{12}\right)$ 0.38
13. $\text{Cos}\,[\text{Arcsin}\,(-0.7)]$ 0.71
14. $\cos\,(\text{Arctan}\,5)$ 0.20
15. $\sin\,(\text{Cos}^{-1}\,0.3)$ 0.95
16. $\sin\,(\text{Cos}^{-1}\,1)$ 0
17. $\sin\left(\text{Sin}^{-1}\frac{1}{2}\right)$ 0.5
18. $\tan\left(\text{Arcsin}\frac{\sqrt{3}}{2}\right)$ 1.73

Example 3

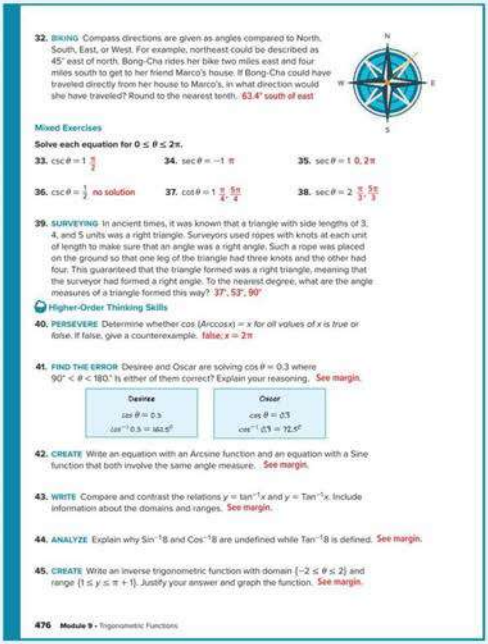
USE TOOLS **Find θ in degrees. Round to the nearest hundredth if necessary.**

19. $\text{Sin}\,\theta = 0.8$ 53.13°
20. $\text{Tan}\,\theta = 4.5$ 77.47°
21. $\text{Cos}\,\theta = 0.5$ 60°
22. $\text{Cos}\,\theta = -0.95$ 161.81°
23. $\text{Sin}\,\theta = -0.1$ −5.74°
24. $\text{Tan}\,\theta = -1$ −45°
25. $\text{Cos}\,\theta = 0.52$ 58.67°
26. $\text{Cos}\,\theta = -0.2$ 101.54°
27. $\text{Sin}\,\theta = 0.35$ 20.49°
28. $\text{Tan}\,\theta = 8$ 82.87°
29. $\text{Cos}\,\theta = 0.25$ 75.52°
30. $\text{Sin}\,\theta = -0.57$ −34.75°

Example 4

31. KITCHEN The exit from a restaurant kitchen has two pairs of swinging doors that meet in the middle of the doorway. Each door is three feet wide. A waiter needs to take a cart of plates into the dining area from the kitchen. The cart is two feet wide.
   a. What is the minimum angle θ through which the doors must each be opened to prevent the cart from hitting either door? Round to the nearest tenth. 48.2°
   b. What assumption did you make when solving for θ in part **a**? Sample answer: I assumed that doors are both opened θ°.

Lesson 9-7 • Inverse Trigonometric Functions 475



32. BIKING Compass directions are given as angles compared to North, South, East, or West. For example, northeast could be described as 45° east of north. Bong-Cha rides her bike two miles east and four miles south to get to her friend Marco's house. If Bong-Cha could have traveled directly from her house to Marco's, in what direction would she have traveled? Round to the nearest tenth. 63.4° south of east

Mixed Exercises

Solve each equation for $0 \le \theta \le 2\pi$.

33. $\csc\theta = 1$ $\frac{\pi}{2}$

34. $\sec\theta = -1$ π

35. $\sec\theta = 1$ $0, 2\pi$

36. $\csc\theta = \frac{1}{2}$ no solution

37. $\cot\theta = 1$ $\frac{\pi}{4}, \frac{5\pi}{4}$

38. $\sec\theta = 2$ $\frac{\pi}{3}, \frac{5\pi}{3}$

39. SURVEYING In ancient times, it was known that a triangle with side lengths of 3, 4, and 5 units was a right triangle. Surveyors used ropes with knots at each unit of length to make sure that an angle was a right angle. Such a rope was placed on the ground so that one leg of the triangle had three knots and the other had four. This guaranteed that the triangle formed was a right triangle, meaning that the surveyor had formed a right angle. To the nearest degree, what are the angle measures of a triangle formed this way? 37°, 53°, 90°

Higher-Order Thinking Skills

40. PERSEVERE Determine whether cos (Arccos$x$) = x for *all* values of x is *true* or *false*. If false, give a counterexample. false; $x = 2\pi$

41. FIND THE ERROR Desiree and Oscar are solving $\cos\theta = 0.3$ where $90° < \theta < 180°$. Is either of them correct? Explain your reasoning. See margin.

Desiree	Oscar
$\cos\theta = 0.3$ $\cos^{-1}0.3 = 162.5°$	$\cos\theta = 0.3$ $\cos^{-1}0.3 = 72.5°$

42. CREATE Write an equation with an Arcsine function and an equation with a Sine function that both involve the same angle measure. See margin.

43. WRITE Compare and contrast the relations $y = \tan^{-1}x$ and $y = \text{Tan}^{-1}x$. Include information about the domains and ranges. See margin.

44. ANALYZE Explain why $\text{Sin}^{-1}8$ and $\text{Cos}^{-1}8$ are undefined while $\text{Tan}^{-1}8$ is defined. See margin.

45. CREATE Write an inverse trigonometric function with domain $\{-2 \le \theta \le 2\}$ and range $\{1 \le y \le \pi + 1\}$. Justify your answer and graph the function. See margin.

476 Module 9 • Trigonometric Functions

Answers

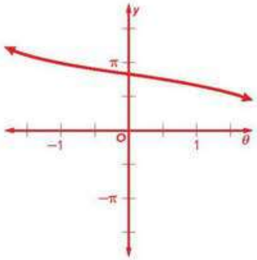
41. Sample answer: Neither; cosine is not positive in the second quadrant.

42. Sample answer: $\text{Arcsin}\ \frac{1}{2} = 30°$
$\frac{1}{2} = \text{Sin}\ 30°$

43. Sample answer: $y = \tan^{-1}x$ is a relation that has a domain of all real numbers and a range of all real numbers except odd multiples of $\frac{\pi}{2}$. The relation is not a function. $y = \text{Tan}^{-1}x$ is a function that has a domain of all real numbers and a range of $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

44. Sample answer: The range of $y = \text{Sin}\ x$ and $y = \text{Cos}\ x$ is $-1 \le x \le 1$. The range of $y = \text{Tan}^{-1}x$ is all real numbers.

45. Sample answer: $y = \text{Cos}^{-1}\frac{1}{2}\theta + 1$; this function is a horizontal stretch by a factor 2 of the parent function $y = \text{Cos}^{-1}\theta$, so the domain is $\{-2 \le \theta \le 2\}$. It is also a vertical translation 1 unit up, so the range is $\{1 \le y \le \pi + 1\}$.



Module 9 • Trigonometric Functions

Review

Rate Yourself!

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have them respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before they answer the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- Why are trigonometric functions in right triangles useful in the real world?
- Why are the graphs of sine and cosine useful in a real world setting?

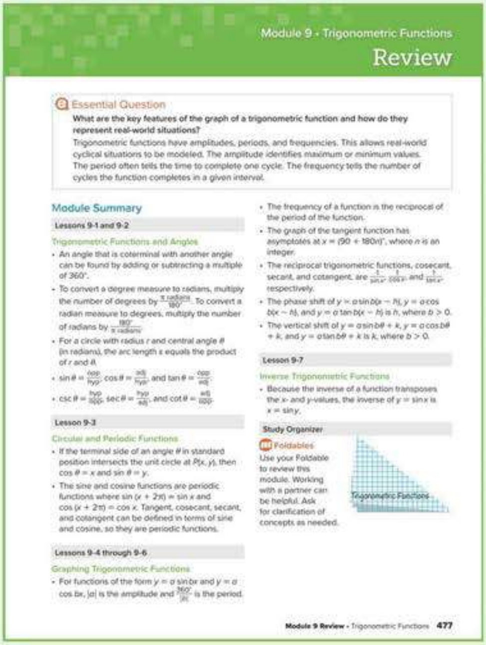
Then have them write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

ELL A completed Foldable for this module should include the key concepts related to trigonometric functions.

LS **LearnSmart** Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Trigonometric Functions** and **Modeling with Functions.**

- Trigonometric Functions and the Unit Circle
- Periodic Phenomena
- Creating Function Models



Module 9 • Trigonometric Functions

Review

Essential Question

What are the key features of the graph of a trigonometric function and how do they represent real-world situations?

Trigonometric functions have amplitudes, periods, and frequencies. This allows real-world cyclical situations to be modeled. The amplitude identifies maximum or minimum values. The period often tells the time to complete one cycle. The frequency tells the number of cycles the function completes in a given interval.

Module Summary

Lessons 9-1 and 9-2

Trigonometric Functions and Angles

- An angle that is coterminal with another angle can be found by adding or subtracting a multiple of 360°.
- To convert a degree measure to radians, multiply the number of degrees by $\frac{\pi \text{ radians}}{180°}$. To convert a radian measure to degrees, multiply the number of radians by $\frac{180°}{\pi \text{ radians}}$.
- For a circle with radius r and central angle θ (in radians), the arc length s equals the product of r and θ.
- $\sin \theta = \frac{\text{opp}}{\text{hyp}}$, $\cos \theta = \frac{\text{adj}}{\text{hyp}}$, and $\tan \theta = \frac{\text{opp}}{\text{adj}}$
- $\csc \theta = \frac{\text{hyp}}{\text{opp}}$, $\sec \theta = \frac{\text{hyp}}{\text{adj}}$, and $\cot \theta = \frac{\text{adj}}{\text{opp}}$

Lesson 9-3

Circular and Periodic Functions

- If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos \theta = x$ and $\sin \theta = y$.
- The sine and cosine functions are periodic functions where $\sin (x + 2\pi) = \sin x$ and $\cos (x + 2\pi) = \cos x$. Tangent, cosecant, secant, and cotangent can be defined in terms of sine and cosine, so they are periodic functions.

Lessons 9-4 through 9-6

Graphing Trigonometric Functions

- For functions of the form $y = a \sin bx$ and $y = a \cos bx$, $|a|$ is the amplitude and $\frac{360°}{|b|}$ is the period.
- The frequency of a function is the reciprocal of the period of the function.
- The graph of the tangent function has asymptotes at $x = (90 + 180n)°$, where n is an integer.
- The reciprocal trigonometric functions, cosecant, secant, and cotangent, are $\frac{1}{\sin x}$, $\frac{1}{\cos x}$, and $\frac{1}{\tan x}$, respectively.
- The phase shift of $y = a \sin b(x - h)$, $y = a \cos b(x - h)$, and $y = a \tan b(x - h)$ is h, where $b > 0$.
- The vertical shift of $y = a \sin b\theta + k$, $y = a \cos b\theta + k$, and $y = a \tan b\theta + k$ is k, where $b > 0$.

Lesson 9-7

Inverse Trigonometric Functions

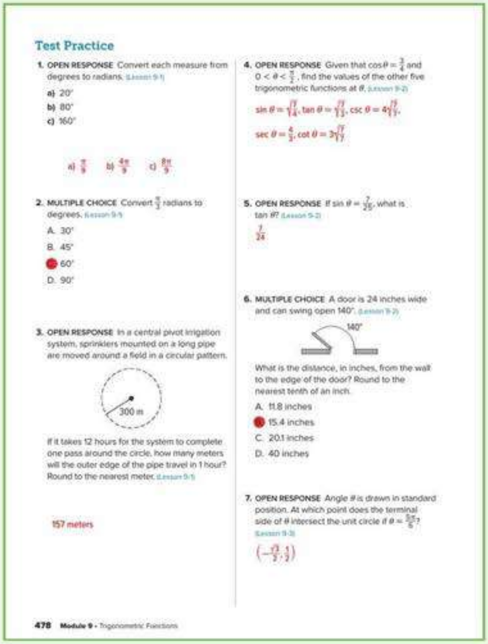
- Because the inverse of a function transposes the x- and y-values, the inverse of $y = \sin x$ is $x = \sin y$.

Study Organizer

Foldables

Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.

Module 9 Review • Trigonometric Functions 477



Test Practice

1. **OPEN RESPONSE** Convert each measure from degrees to radians. (Lesson 9-1)
   a) 20°
   b) 80°
   c) 160°

   a) $\frac{\pi}{9}$ b) $\frac{4\pi}{9}$ c) $\frac{8\pi}{9}$

2. **MULTIPLE CHOICE** Convert $\frac{\pi}{3}$ radians to degrees. (Lesson 9-1)
   A. 30°
   B. 45°
   [✓ C.] 60°
   D. 90°

3. **OPEN RESPONSE** In a central pivot irrigation system, sprinklers mounted on a long pipe are moved around a field in a circular pattern.

   If it takes 12 hours for the system to complete one pass around the circle, how many meters will the outer edge of the pipe travel in 1 hour? Round to the nearest meter. (Lesson 9-1)

   157 meters

4. **OPEN RESPONSE** Given that $\cos\theta = \frac{3}{4}$ and $0 < \theta < \frac{\pi}{2}$, find the values of the other five trigonometric functions at θ. (Lesson 9-2)

   $\sin\theta = \frac{\sqrt{7}}{4}$, $\tan\theta = \frac{\sqrt{7}}{3}$, $\csc\theta = \frac{4\sqrt{7}}{7}$,
   $\sec\theta = \frac{4}{3}$, $\cot\theta = \frac{3\sqrt{7}}{7}$

5. **OPEN RESPONSE** If $\sin\theta = \frac{7}{25}$, what is $\tan\theta$? (Lesson 9-2)

   $\frac{7}{24}$

6. **MULTIPLE CHOICE** A door is 24 inches wide and can swing open 140°. (Lesson 9-2)

   What is the distance, in inches, from the wall to the edge of the door? Round to the nearest tenth of an inch.
   A. 11.8 inches
   [✓ B.] 15.4 inches
   C. 20.1 inches
   D. 40 inches

7. **OPEN RESPONSE** Angle θ is drawn in standard position. At which point does the terminal side of θ intersect the unit circle if $\theta = \frac{5\pi}{6}$? (Lesson 9-3)

   $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

478 **Module 9** • Trigonometric Functions

Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Vocabulary Activity
Module Review

Assessment Resources

Vocabulary Test
AL Module Test Form B
OL Module Test Form A
BL Module Test Form C
Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

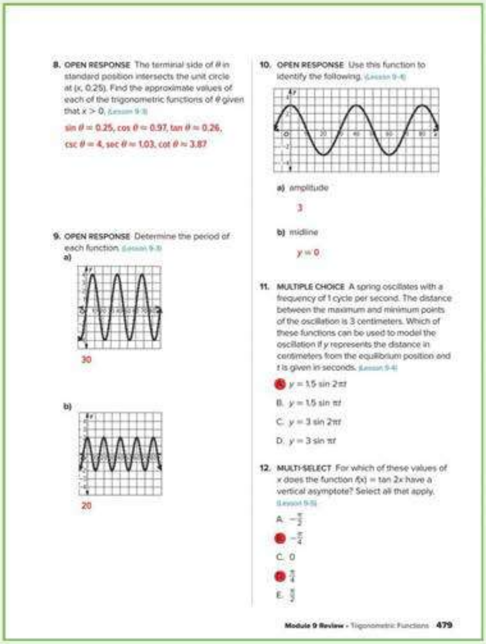
Test Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1-18 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Multiple Choice	Students select one correct answer.	2, 6, 11, 13, 17
Multi-Select	Multiple answers may be correct. Students must select all correct answers.	12, 14, 15
Table Item	Students complete a table by entering in the correct values.	4, 8
Open Response	Students construct their own response.	1, 3, 5, 7, 9, 10, 16, 18

To ensure that students understand the Standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
F.BF.3	9-6	14, 15, 16
F.TF.1	9-1	1, 2, 3
F.TF.2	9-3	7, 8
F.TF.3	9-2	4, 5, 6
F.TF.5	9-3, 9-4	9, 10, 11
F.TF.7	11-7	17, 18
F.IF.4	9-5, 9-6	12, 13
F.IF.7e	9-5	13, 14
A.CED.2	9-4	11



8. **OPEN RESPONSE** The terminal side of θ in standard position intersects the unit circle at $(x, 0.25)$. Find the approximate values of each of the trigonometric functions of θ given that $x > 0$. (Lesson 9-3)

$\sin \theta = 0.25$, $\cos \theta \approx 0.97$, $\tan \theta \approx 0.26$, $\csc \theta = 4$, $\sec \theta \approx 1.03$, $\cot \theta \approx 3.87$

9. **OPEN RESPONSE** Determine the period of each function. (Lesson 9-3)

a)

30

b)

20

10. **OPEN RESPONSE** Use this function to identify the following. (Lesson 9-4)

a) amplitude

3

b) midline

$y = 0$

11. **MULTIPLE CHOICE** A spring oscillates with a frequency of 1 cycle per second. The distance between the maximum and minimum points of the oscillation is 3 centimeters. Which of these functions can be used to model the oscillation if y represents the distance in centimeters from the equilibrium position and t is given in seconds. (Lesson 9-4)

[✓ A.] $y = 1.5 \sin 2\pi t$
B. $y = 1.5 \sin \pi t$
C. $y = 3 \sin 2\pi t$
D. $y = 3 \sin \pi t$

12. **MULTI-SELECT** For which of these values of x does the function $f(x) = \tan 2x$ have a vertical asymptote? Select all that apply. (Lesson 9-5)

A. $-\frac{\pi}{2}$
[✓ B.] $-\frac{\pi}{4}$
C. 0
[✓ D.] $\frac{\pi}{4}$
E. $\frac{\pi}{2}$

Lesson 9-1

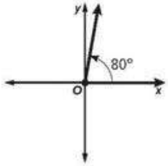
70b. Sample answer: As you move around the circle in 45°-increments, the radian measures increase by increments of $\frac{\pi}{4}$, resulting in the pattern $0, \frac{\pi}{4}, \frac{2\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{4}\frac{4\pi}{4}, = \pi, \frac{5\pi}{4}\frac{6\pi}{4} = \frac{3\pi}{2}, \frac{7\pi}{4}$, and $\frac{8\pi}{4} = 2\pi$.

75. Melinda is correct. The area of the complete circle is $\pi r^2 = \pi(15)^2 =$ 225π ft^2. Because the area that gets watered by the sprinkler is 75π ft^2; this area is $\frac{75\pi}{225\pi} = \frac{1}{3}$ of the circle. Therefore, the measure of the central angle is $\frac{1}{3}(2\pi) = \frac{2\pi}{3}$.

77. Because $s = r\theta$ and $r = 5$, the function may be written as $f(x) = 5x$. This means the graph is a straight line with a slope of 5 that passes through the origin.

78. Tarshia; a coterminal can be found by adding a multiple of 360° or by subtracting a multiple of 360°. Alan incorrectly subtracted the original angle measure from 360°.

81. Sample answer: 440° and −280°



82. $\frac{\text{measure of central angle}}{\text{measure of an entire circle}} = \frac{\text{length of arc}}{\text{circumference}}$

Use a proportion.

$\frac{\theta}{2\pi} = \frac{s}{2\pi r}$ Substitute.

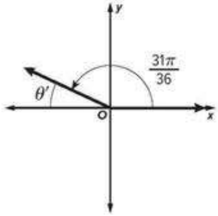
$2\pi r\theta = 2\pi s$ Multiply.

$r\theta = s$ Divide each side by 2π.

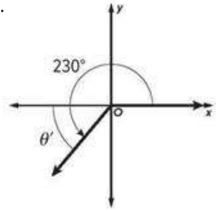
83. One degree represents an angle measure that equals $\frac{1}{360}$ rotation around a circle. One radian represents the measure of an angle in standard position that intercepts an arc of length r. To convert from degrees to radians, multiply the number of degrees by $\frac{\pi \text{ radians}}{180°}$. To convert from radians to degrees, multiply the number of radians by $\frac{180°}{\pi \text{ radians}}$.

Lesson 9-2

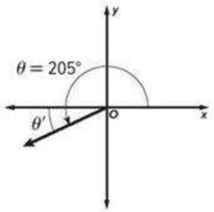
28.



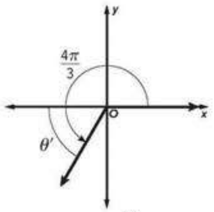
29.



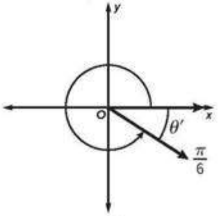
30.



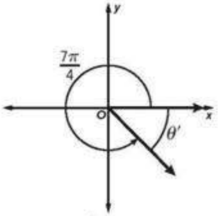
31.



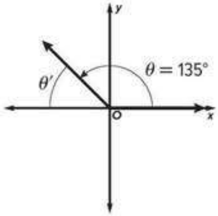
32.



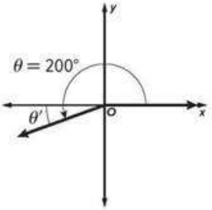
33.



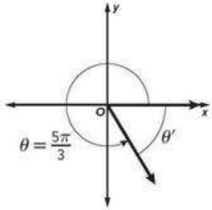
34.



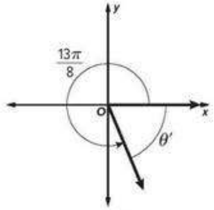
35.



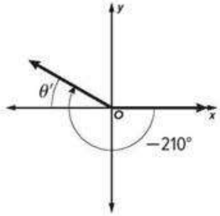
36.



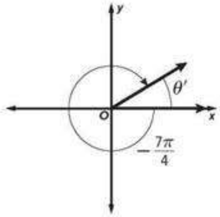
37.



38.



39.

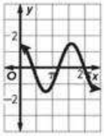


Lesson 9-4

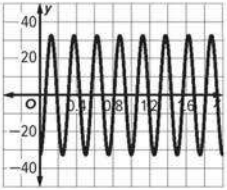
17a. period: 2; frequency: $\frac{1}{2}$; The object completes $\frac{1}{2}$ of a cycle per second, and it will reach maximum distance from the equilibrium point every 2 seconds.

17b. Domain: all real numbers; sample answer: Because time cannot be negative, the relevant domain in the context of the situation is $[0, \infty)$. The range is $[-25, 25]$.

19.



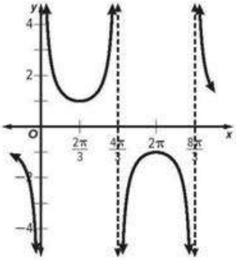
20.



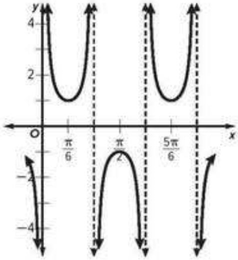
36. 900 rpm; the period is $\frac{2\pi}{|b|} = \frac{2\pi}{30\pi} = \frac{1}{15}$ second, or $\frac{1}{15} \cdot \frac{1}{60} = \frac{1}{900}$ minute; the frequency is the reciprocal of the period, so the blade makes 900 complete rotations per minute.

Lesson 9-5

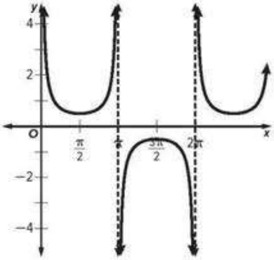
7. period $= \frac{8\pi}{3}$; asymptotes: $x = \frac{4\pi}{3} + \frac{4\pi}{3}n$, where n is an integer; relative minima at $x = \frac{2\pi}{3}$, $x = \frac{10\pi}{3}$, or $x = \frac{2\pi}{3} + \frac{8\pi}{3}n$, where n is an integer; relative maxima at $x = 2\pi$, $x = \frac{14\pi}{3}$, or $x = 2\pi + \frac{8\pi}{3}n$, where n is an integer.



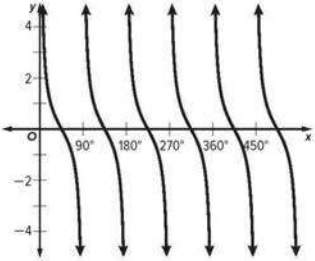
8. period $= \frac{2\pi}{3}$; asymptotes: $x = \frac{\pi}{3} + \frac{\pi}{3}n$, where n is an integer; relative minima at $x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$, or $x = \frac{\pi}{6} + \frac{2\pi}{3}n$, where n is an integer; relative maxima at $x = \frac{\pi}{2}$, $x = \frac{7\pi}{6}$, or $x = \frac{\pi}{2} + \frac{2\pi}{3}n$, where n is an integer.



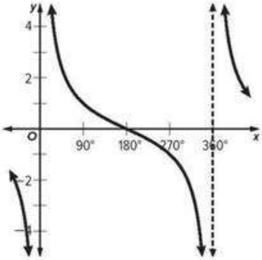
9. period $= 2\pi$; asymptotes: $x = 2\pi + 2\pi n$, where n is an integer; relative minima at $x = \frac{\pi}{2}$, $x = \frac{5\pi}{2}$, or $x = \frac{\pi}{2} + 2\pi n$, where n is an integer; relative maxima at $\frac{3\pi}{2}, \frac{7\pi}{2}$, or $x = \frac{3\pi}{2} + 2\pi n$, where n is an integer.



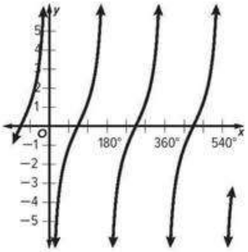
10. period $= 90°$; asymptotes: $x = 90n°$, where n is an integer; x-intercepts: 45°, 135°, 225° ...; (odd multiples of 45°).



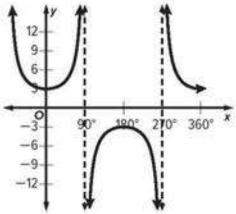
11. period $= 360°$; asymptotes: $x = 360n°$, where n is an integer; x-intercepts: 180°, 540°, ...; (odd multiples of 180°).



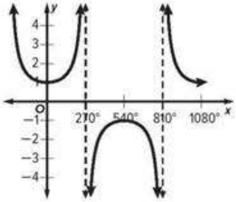
12. period $= 180°$; asymptotes: $x = 180n°$, where n is an integer; x-intercepts: 90°, 270°, ...; (odd multiples of 90°).



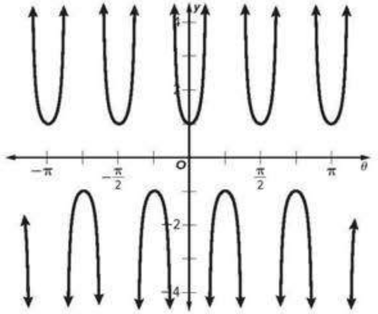
13. period $= 360°$; asymptotes: $x = (90 + 180n)°$, relative maxima at $x = (180 + 360n)°$, where n is an integer, relative minima at $x = 360n°$ where n is an integer



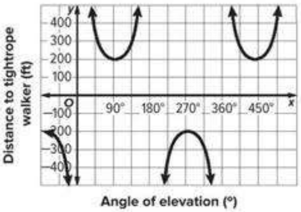
14. period $= 1080°$; asymptotes: $x = (270 + 540)°$, relative maxima at $x = (540 + 1080n)°$, where n is an integer, relative minima at $x = 1080n°$ where n is an integer



15. period $= \frac{\pi}{2}$; asymptotes: $x = \frac{\pi}{8} + \frac{\pi}{4}n$, relative maxima at $x = \frac{\pi}{4} + \frac{\pi}{2}n$ where n is an integer, relative minima at $x = \left(\frac{\pi}{2}\right)n$ where n is an integer



16b.

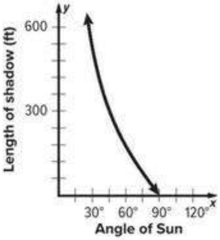


16c. Sample answer: In the context of the situation, only x-values between 0° and 180° are relevant, so focus on analyzing that portion of the graph.

domain: $\{x \mid 0 < x < 180\}$
range: $\{d \mid d \geq 200\}$
x-intercept: none
y-intercept: none

relative minimum: (90, 200)
relative maximum: none
increasing: $\{x \mid 90 < x < 180\}$
decreasing: $\{x \mid 0 < x < 90\}$

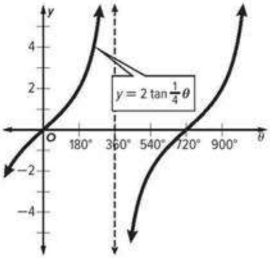
17b.



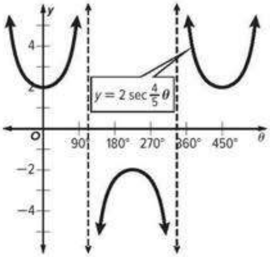
17c. Sample answer: In the context of the situation, only x-values between 0° and 90° are relevant, so focus on analyzing that portion of the graph.

domain: $\{x \mid 0 < x < 90\}$
range: $\{y \mid 0 \leq y \leq \infty\}$
x-intercept: 90
y-intercept: none
relative minimum: none
relative maximum: none
The period of $y = 300 \cot x$ is 180°.

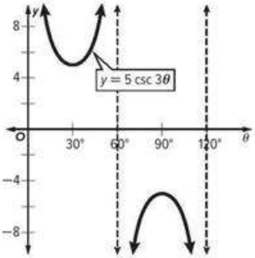
18.



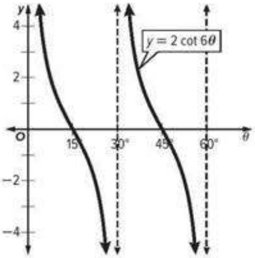
19.



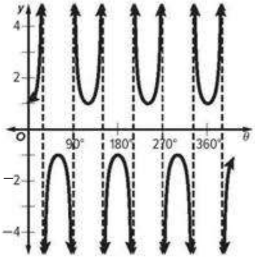
20.



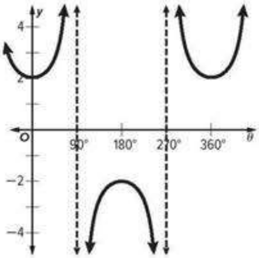
21.



22.

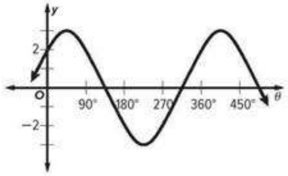


23.



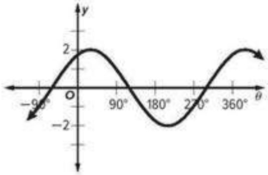
Lesson 9-6

7.



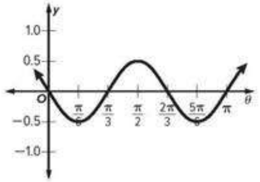
domain: $-\infty < \theta < \infty$; range: $-3 \leq f(\theta) \leq 3$

8.



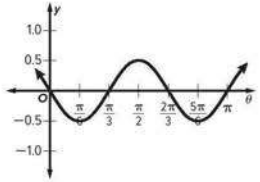
domain: $-\infty < \theta < \infty$; range: $-2 \leq f(\theta) \leq 2$

9.



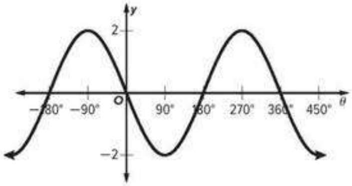
domain: $-\infty < \theta < \infty$; range: $-0.5 \leq f(\theta) \leq 0.5$

10.



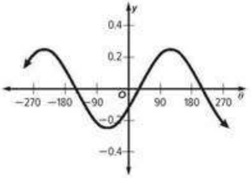
domain: $-\infty < \theta < \infty$; range: $-0.5 \leq f(\theta) \leq 0.5$

11.



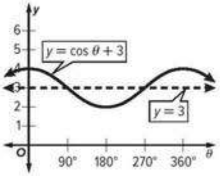
domain: $-\infty < \theta < \infty$; range: $-2 \leq f(\theta) \leq 2$

12.



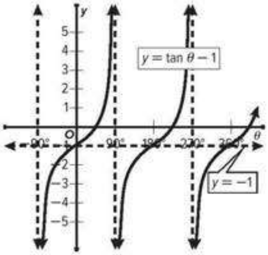
domain: $-\infty < \theta < \infty$; range: $-0.25 \leq f(\theta) \leq 0.25$

13.



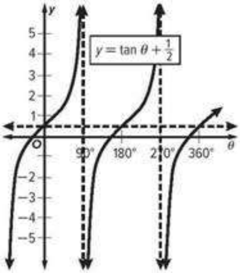
domain: $-\infty < \theta < \infty$; range: $2 \leq f(\theta) \leq 4$

14.



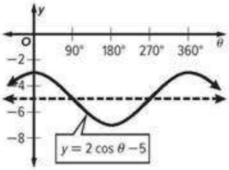
domain: 90 + 180*n* < θ < 270 + 180*n*, where *n* is an integer; range: $-\infty < f(\theta) < \infty$

15.



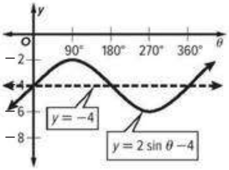
domain: 90 + 180*n* < θ < 270 + 180*n*, where *n* is an integer; range: $-\infty < f(\theta) < \infty$

16.



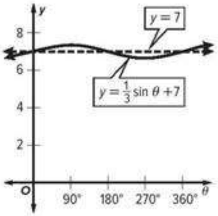
domain: $-\infty < \theta < \infty$; range: $-7 \leq f(\theta) \leq -3$

17.



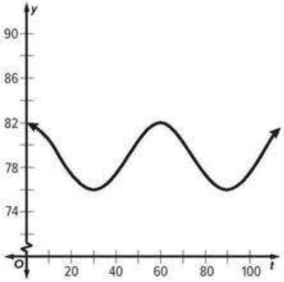
domain: $-\infty < \theta < \infty$; range: $-6 \leq f(\theta) \leq -2$

18.

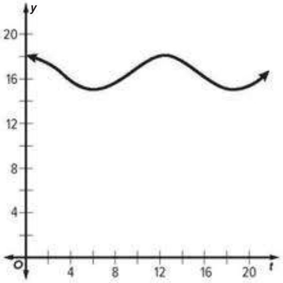


domain: $-\infty < \theta < \infty$; range: $\frac{20}{3} \leq f(\theta) \leq \frac{22}{3}$

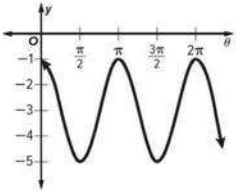
19c.



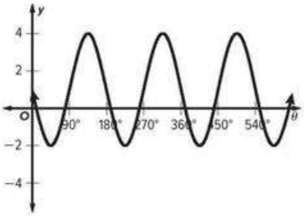
20d. Notice that the graph oscillates between a maximum of 18 meters and a minimum of 15 meters and is positive for all values in the domain. The graph has a y-intercept of 18 and no x-intercepts. Because t represents time and y represents depth, both values must be positive in the context of the situation. The domain is $\{t \mid t \geq 0\}$, and the range is $y \mid \{15 \leq y \leq 18\}$.



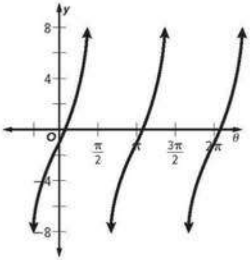
26.



27.



28.



29. Sample answer: sine; As a Ferris wheel turns, the rider's distance above the ground varies sinusoidally with time. Note: any phenomenon that can be modeled with a sine function can also be modeled with a cosine function using the appropriate horizontal shift and/or reflection about the horizontal axis.

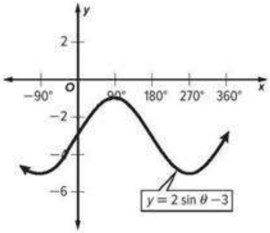
32. Sample answer: Secant; even though cosine and secant can both be used to write a trigonometric ratio involving θ, the function can best be modeled by a secant graph. Secant has no x-intercepts, which makes sense because the length of the beam of light is never 0. The asymptotes of the secant graph occur at values of x that represent times when the beam of light is parallel to the hospital wall.

46. Sometimes; if the function is shifted vertically, then you also need to know the value of the midline. The maximum value is the value of the midline plus the amplitude. The minimum value is the midline value minus the amplitude.

47. The graph of $y = 3 \sin 2\theta + 1$ has an amplitude of 3 rather than an amplitude of 1. It is shifted up 1 unit from the parent graph and is compressed so that it has a period of 180°.

48. Sample answer: a phase shift 90° left, $y = \sin(\theta + 90°)$; a phase shift 270° right, $y = \sin(\theta - 270°)$

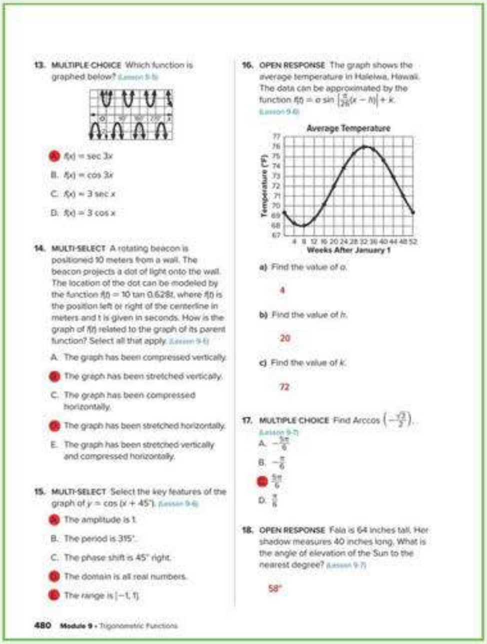
49. Sample answer:



50. Sample answer: Infinitely many; any change in amplitude will create a different graph that has the same θ-intercepts.

51. Sample answer: $y = \sin\left(\theta + \frac{\pi}{2}\right) - 3$ is equivalent to $y = \cos\theta - 3$.

52. Sample answer: The 4 cannot be distributed to eliminate the $\frac{1}{4}$ because the $\frac{1}{4}$ is part of the angle and needs to be evaluated by the sine function before the result can be multiplied by 4. This is similar to $4 \cdot f\left(\frac{1}{4}x\right) \neq f(x)$. The $\frac{1}{4}$ represents a horizontal dilation, which shrinks the period of the graph. The 4 represents a vertical dilation, which expands the amplitude of the graph.



13. MULTIPLE CHOICE Which function is graphed below? (Lesson 9-5)

[✓ A.] $f(x) = \sec 3x$

B. $f(x) = \cos 3x$

C. $f(x) = 3 \sec x$

D. $f(x) = 3 \cos x$

14. MULTI-SELECT A rotating beacon is positioned 10 meters from a wall. The beacon projects a dot of light onto the wall. The location of the dot can be modeled by the function $f(t) = 10 \tan 0.628t$, where $f(t)$ is the position left or right of the centerline in meters and t is given in seconds. How is the graph of $f(t)$ related to the graph of its parent function? Select all that apply. (Lesson 9-5)

A. The graph has been compressed vertically.

[✓ B.] The graph has been stretched vertically.

C. The graph has been compressed horizontally.

[✓ D.] The graph has been stretched horizontally.

E. The graph has been stretched vertically and compressed horizontally.

15. MULTI-SELECT Select the key features of the graph of $y = \cos(x + 45°)$. (Lesson 9-6)

[✓ A.] The amplitude is 1.

B. The period is 315°.

C. The phase shift is 45° right.

[✓ D.] The domain is all real numbers.

[✓ E.] The range is $[-1, 1]$.

16. OPEN RESPONSE The graph shows the average temperature in Haleiwa, Hawaii. The data can be approximated by the function $f(t) = a \sin\left[\frac{\pi}{26}(x - h)\right] + k$. (Lesson 9-6)

a) Find the value of a.

4

b) Find the value of h.

20

c) Find the value of k.

72

17. MULTIPLE CHOICE Find Arccos $\left(-\frac{\sqrt{3}}{2}\right)$. (Lesson 9-7)

A. $-\frac{5\pi}{6}$

B. $-\frac{\pi}{6}$

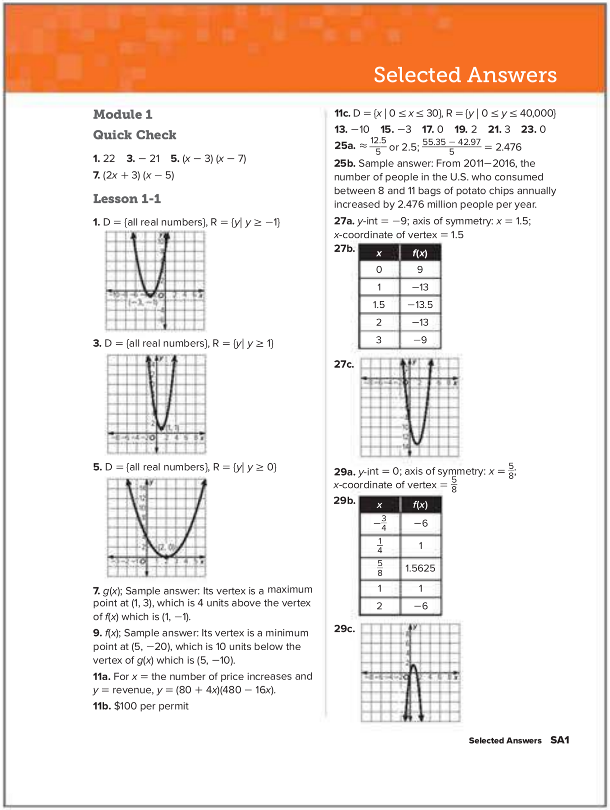
[✓ C.] $\frac{5\pi}{6}$

D. $\frac{\pi}{6}$

18. OPEN RESPONSE Fala is 64 inches tall. Her shadow measures 40 inches long. What is the angle of elevation of the Sun to the nearest degree? (Lesson 9-7)

58°

480 Module 9 • Trigonometric Functions



Selected Answers

Module 1

Quick Check

1. 22 **3.** − 21 **5.** $(x - 3)(x - 7)$

7. $(2x + 3)(x - 5)$

Lesson 1-1

1. D = {all real numbers}, R = $\{y | y \geq -1\}$

3. D = {all real numbers}, R = $\{y | y \geq 1\}$

5. D = {all real numbers}, R = $\{y | y \geq 0\}$

7. $g(x)$; Sample answer: Its vertex is a maximum point at (1, 3), which is 4 units above the vertex of $f(x)$ which is (1, −1).

9. $f(x)$; Sample answer: Its vertex is a minimum point at (5, −20), which is 10 units below the vertex of $g(x)$ which is (5, −10).

11a. For x = the number of price increases and y = revenue, $y = (80 + 4x)(480 - 16x)$.

11b. $100 per permit

11c. $D = \{x \mid 0 \leq x \leq 30\}$, $R = \{y \mid 0 \leq y \leq 40{,}000\}$

13. −10 **15.** −3 **17.** 0 **19.** 2 **21.** 3 **23.** 0

25a. $\approx \frac{12.5}{5}$ or 2.5; $\frac{55.35 - 42.97}{5} = 2.476$

25b. Sample answer: From 2011–2016, the number of people in the U.S. who consumed between 8 and 11 bags of potato chips annually increased by 2.476 million people per year.

27a. y-int = −9; axis of symmetry: $x = 1.5$; x-coordinate of vertex = 1.5

27b.

x	$f(x)$
0	9
1	−13
1.5	−13.5
2	−13
3	−9

27c.

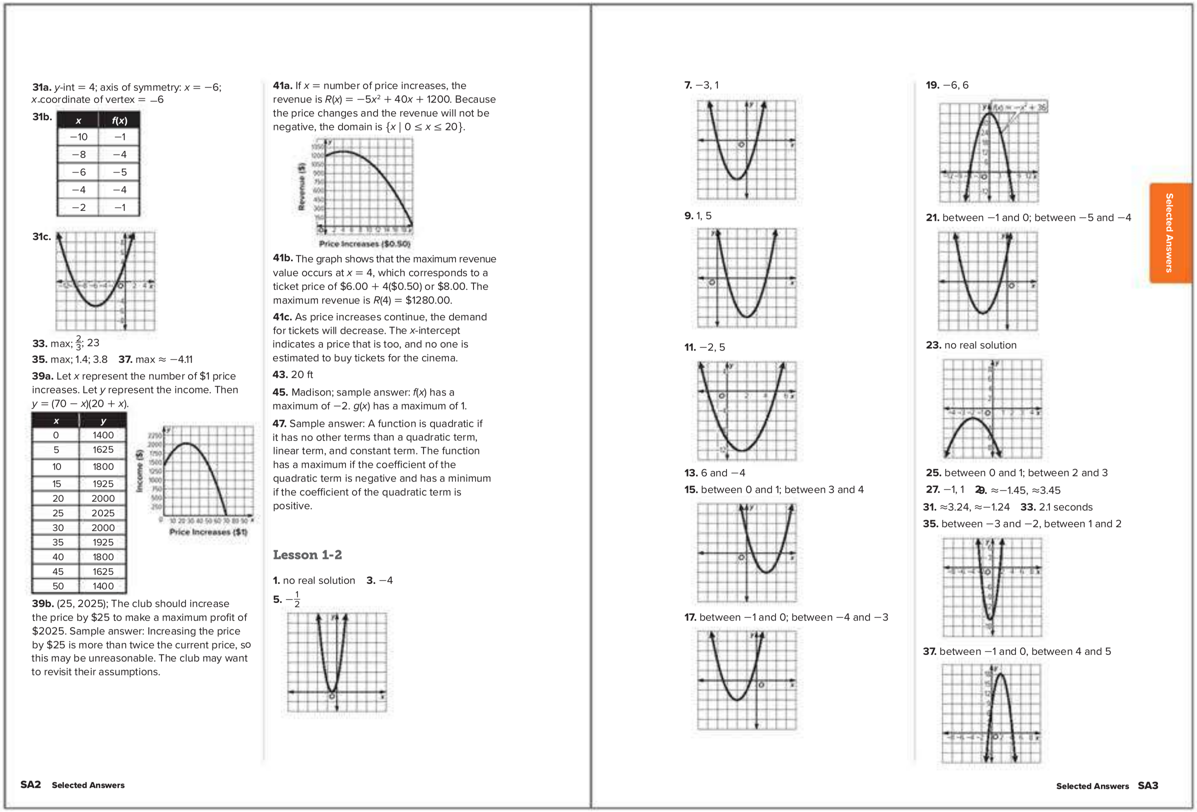
29a. y-int = 0; axis of symmetry: $x = \frac{5}{8}$; x-coordinate of vertex = $\frac{5}{8}$

29b.

x	$f(x)$
$-\frac{3}{4}$	−6
$\frac{1}{4}$	1
$\frac{5}{8}$	1.5625
1	1
2	−6

29c.

Selected Answers SA1



31a. *y*-int = 4; axis of symmetry: $x = -6$; *x*-coordinate of vertex = -6

31b.

x	$f(x)$
−10	−1
−8	−4
−6	−5
−4	−4
−2	−1

31c.

33. max; $\frac{2}{3}$; 23

35. max; 1.4; 3.8 **37.** max ≈ −4.11

39a. Let *x* represent the number of $1 price increases. Let *y* represent the income. Then $y = (70 - x)(20 + x)$.

x	y
0	1400
5	1625
10	1800
15	1925
20	2000
25	2025
30	2000
35	1925
40	1800
45	1625
50	1400

39b. (25, 2025); The club should increase the price by $25 to make a maximum profit of $2025. Sample answer: Increasing the price by $25 is more than twice the current price, so this may be unreasonable. The club may want to revisit their assumptions.

41a. If x = number of price increases, the revenue is $R(x) = -5x^2 + 40x + 1200$. Because the price changes and the revenue will not be negative, the domain is $\{x \mid 0 \leq x \leq 20\}$.

41b. The graph shows that the maximum revenue value occurs at $x = 4$, which corresponds to a ticket price of $6.00 + 4($0.50) or $8.00. The maximum revenue is $R(4) = \$1280.00$.

41c. As price increases continue, the demand for tickets will decrease. The *x*-intercept indicates a price that is too, and no one is estimated to buy tickets for the cinema.

43. 20 ft

45. Madison; sample answer: $f(x)$ has a maximum of −2. $g(x)$ has a maximum of 1.

47. Sample answer: A function is quadratic if it has no other terms than a quadratic term, linear term, and constant term. The function has a maximum if the coefficient of the quadratic term is negative and has a minimum if the coefficient of the quadratic term is positive.

Lesson 1-2

1. no real solution **3.** −4

5. $-\frac{1}{2}$

7. −3, 1

9. 1, 5

11. −2, 5

13. 6 and −4

15. between 0 and 1; between 3 and 4

17. between −1 and 0; between −4 and −3

19. −6, 6

21. between −1 and 0; between −5 and −4

23. no real solution

25. between 0 and 1; between 2 and 3

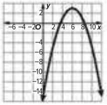
27. −1, 1 **29.** ≈−1.45, ≈3.45

31. ≈3.24, ≈−1.24 **33.** 2.1 seconds

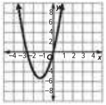
35. between −3 and −2, between 1 and 2

37. between −1 and 0, between 4 and 5

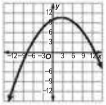
39. between 3 and 4, between 8 and 9



41. between −3 and −2; 0



43. between −8 and −7; between 12 and 13

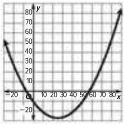


45. no real solution **47.** −2

49. between −3 and −2, between 4 and 5

51. about −5 and 17 **53.** 11 and −19

55. 64 m



57. 5 seconds

59. No; sample answer: Hakeem is right about the location of one of the roots, but his reason is not accurate. The roots are located where $f(x)$ changes signs.

61. 5; Sample answer: The intercepts are equidistant from the axis of symmetry.

63. Sample answer: Graph the function using the axis of symmetry. Determine where the graph intersects the x-axis. The x-coordinates of those points are solutions of the quadratic equation.

Lesson 1-3

1. $4i\sqrt{3}$ **3.** $6i\sqrt{2}$ **5.** $2i\sqrt{21}$ **7.** $-23\sqrt{2}$

9. $-5\sqrt{2}$ **11.** $-i$ **13.** $\pm 3i$ **15.** $\pm i$ **17.** $\pm 5i$

19. 3, 3 **21.** $-\frac{11}{2}, -3$ **23.** 4, −3 **25.** $10 - 4i$

27. $2 + i$ **29.** $10 - 5i$ **31.** $7 + i$ **33.** $-10i$

35. $\frac{3}{2} - \frac{1}{2}i$ **37.** $-\frac{5}{3} - 2i$ **39.** $\frac{7}{9} - \frac{4i\sqrt{2}}{9}$

41. $(3 + i)x^2 + (-2 + i)x - 8i + 7$

43. $10 + 10j$ volts **45.** $8 - 2j$ ohms

47. Zoe; $i^3 = -i$, not −1.

49. Always; Sample answer: the value of 5 can be represented by $5 + 0i$, and the value of $3i$ can be represented by $0 + 3i$.

51. Some quadratic equations have complex solutions and cannot be solved using only the real numbers.

Lesson 1-4

1. $0, \frac{1}{3}$ **3.** $0, -\frac{5}{4}$ **5.** −11, −3 **7.** 7 ft by 9 ft

9. $\frac{3}{2}, -1$ **11.** $\frac{2}{5}, -6$ **13.** $8, -\frac{5}{2}$

15. −8, 8 **17.** −17, 17 **19.** −13, 13 **21.** $\frac{7}{2}$

23. $\frac{3}{4}$ **25.** $-\frac{8}{5}$ **27.** $5i, -5i$

29. $25i, -25i$ **31.** $\frac{5}{6}i, -\frac{5}{6}i$ **33.** $-3, \frac{1}{2}$

35. $9i, -9i$ **37.** $\frac{1}{5}i, -\frac{1}{5}i$

39. 13 inches by 16 inches

41. 24, 26

43. Neither; both students made a mistake in Step 3.

45. Sample answer: 3 and 6 → $x^2 - 9x + 18 = 0$. −3 and −6 → $x^2 + 9x + 18 = 0$. The linear term changes sign.

47. Sample answer: Standard form is $ax^2 + bx + c$. Multiply a and c. Then find a pair of integers, g and h, that multiply to equal ac and add to equal b. Then write the quadratic expression, substituting the middle term, bx, with $gx + hx$. The expression is now $ax^2 + gx + hx + c$. Then factor the GCF from the first two terms and factor the GCF from the second two terms. So, the expression becomes $\text{GCF}(x - q) + \text{GCF}_2(x - q)$. Simplify to get $(\text{GCF} + \text{GCF}_2)(x - q)$ or $(x - p)(x - q)$.

Lesson 1-5

1. 2, 16 **3.** $0, \frac{4}{3}$ **5.** $\frac{15}{2}, -\frac{1}{2}$

7. $\frac{-1 \pm 3\sqrt{2}}{6}$ **9.** $\frac{-2 \pm 5\sqrt{3}}{5}$ **11.** $\frac{3 \pm 4\sqrt{6}}{5}$

13. $8 \pm 8i$ **15.** $-7 \pm 10i$

17. $-4 \pm 6i$ **19.** 25; $(x + 5)^2$

21. 144; $(x + 12)^2$ **23.** $\frac{81}{4}$; $\left(x - \frac{9}{2}\right)^2$

25. 4, 9 **27.** $2 \pm \sqrt{17}$ **29.** $\frac{1 \pm \sqrt{13}}{2}$

31. 16 in. by 16 in. by 16 in.

33. $1, \frac{1}{2}$ **35.** $\frac{1}{5}, -\frac{9}{5}$ **37.** $\frac{1}{3}, -1$

39. $\frac{3 \pm i\sqrt{31}}{4}$ **41.** $1 \pm i\sqrt{2}$ **43.** $1 \pm i\sqrt{3}$

45. $y = (x + 3)^2 - 8$; $x = -3$; $(h, k) = (-3, -8)$; minimum

47. $y = -(x + 4)^2 + 11$; $x = -4$; $(h, k) = (-4, 11)$; maximum

49. $y = 3(x + 1)^2 - 4$; $x = -1$; $(h, k) = (-1, -4)$; minimum

51a. $h(t) = -4.9(t - 0.427)^2 + 8.931$

51b. $t = 0.427$; points equidistant from the axis of symmetry represent the times when the diver will be at the same height during his dive. Vertex = (0.427, 8.391); Malik reaches a maximum height of about 8.391 meters approximately 0.427 second after he begins his dive.

53. −2.77, −0.56 **55.** −1.44, 0.24

57. $1.6 \pm 0.9i$ **59.** $-1.3 \pm 4.09i$

61. $h(t) = -4.9t^2 + 25.8$; 2.3 s **63.** 5%

65. $y = (x - 5)^2 + 3$; (5, 3)

67. $y = (x - 10)^2 + 4$; (10, 4)

69a. $n^2 + 90n$ **69b.** 10

71a. w = width, $V(w)$ = the volume; $V(w) = 6w^2 - 32w$

71b. 20.7 by 62.1 in.

73. Alsonso; Aika did not add 16 to each side; she added it only to the left side.

75a. 2; rational; 16 is a perfect square, so $x + 2$ and x are rational.

75b. 2; rational; 16 is a perfect square, so $x - 2$ and x are rational.

75c. 2; complex; if the opposite of a square is positive, the square is negative. The square root of a negative number is complex.

75d. 2; real; the square must equal 20. Since that is positive but not a perfect square, the solutions will be real but not rational.

75e. 1; rational; the expression must be equal to 0 and only −2 makes the expression equal to 0.

75f. 1; rational; the expressions $(x + 4)$ and $(x + 6)$ must either be equal or opposites. No value makes them equal, −5 makes them opposites. The only solution is −5.

77. Sample answer: Completing the square is rewriting one side of a quadratic equation in the form of a perfect square. Once in this form, the equation can be solved by using the Square Root Property.

Lesson 1-6

1. −3, −5 **3.** $\frac{3}{2}, \frac{1}{3}$ **5.** $-4 \pm \sqrt{11}$

7. 2.9 s

9. $\frac{1}{4}, -5$ **11.** $-2, \frac{1}{3}$ **13.** $\frac{-5 \pm \sqrt{57}}{16}$

15. $\frac{3 \pm \sqrt{34}}{4}$ **17.** $\pm 5i$ **19.** $\frac{1 \pm i}{4}$

21. $7 \pm 2i$ **23.** $\frac{3 \pm 2i\sqrt{6}}{3}$

25. 225; 2 rational roots

27. 289; 2 rational roots

29. 24; 2 irrational roots

31. 21; 2 irrational roots

33. −196; 2 complex roots

35. −7; 2 complex roots

37. $b^2 - 4ac > 0$;
2 real rational or irrational roots

39. 2 complex roots; $\frac{1 \pm 4i}{2}$

41. 2 rational roots; 0, $\frac{5}{7}$

43. 2 rational roots; 3, 8

45. 2 rational roots; 4, $\frac{4}{3}$

47. 2 irrational roots; $\frac{-5 \pm \sqrt{3}}{2}$

49. about 2.9 seconds

51. x = the length of a side of the base, $(x + 2)^2(3x + 4) = x^2(3x + 1 + 531)$; 5 in. by 5 in. by 16 in.

53. $7x^2 + 6x + 2 = 0$ is different from the other 3 equations because it has 2 complex roots, where the other 3 equations each have 2 rational roots.

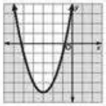
55a. Sample answer: Always; when a and c are opposite signs, then ac will always be negative and $-4ac$ will always be positive. Because b^2 will also always be positive, then $b^2 - 4ac$ represents the addition of two positive values, which will never be negative. Hence, the discriminant can never be negative and the solutions can never be imaginary.

55b. Sample answer: Sometimes; the roots will only be irrational if $b^2 - 4ac$ is not a perfect square.

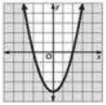
57. −0.75

Lesson 1-7

1.

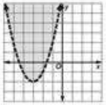


3.

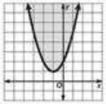


5.

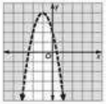
7.



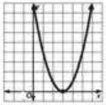
9.



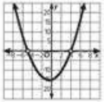
11.



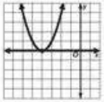
13. $\{x \mid x = 3\}$



15. $\{x \mid x < -5 \text{ or } x > 4\}$



17. {all real numbers}



19. 30 ft to 60 ft **21.** $\{x \mid x < -1.06 \text{ or } x > 7.06\}$

23. $\{x \mid x \leq -2.75 \text{ or } x \geq 1\}$

25. $\{x \mid x < 0.61 \text{ or } x > 2.72\}$

27. {all real numbers}

29. $\{x \mid x \leq -9.24 \text{ or } x \geq -0.76\}$

31. $\{x \mid -5 \leq x \leq -3\}$

33. $\left\{x \mid x \neq -\frac{1}{2}\right\}$ **35.** $\left\{x \mid -3 \leq x \leq -\frac{4}{9}\right\}$

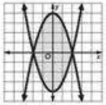
37. $y > x^2 - 4x - 6$ **39.** $y > -0.25x^2 - 4x + 2$

41. No; the graphs of the inequalities intersect the x-axis at the same points.

43a. Sample answer: $x^2 + 2x + 1 \geq 0$

43b. Sample answer: $x^2 - 4x + 6 < 0$

45.



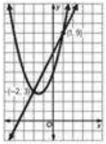
Lesson 1-8

1. (2, −1), (−1, −4) **3.** (−1, 2), (1.5, 4.5)

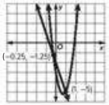
5. (2, −2), $\left(\frac{1}{2}, 1\right)$ **7.** (1, 2)

9. $\left(\frac{-1 + \sqrt{33}}{4}, \frac{-1 + \sqrt{33}}{4}\right), \left(\frac{-1 - \sqrt{33}}{4}, \frac{-1 - \sqrt{33}}{4}\right)$

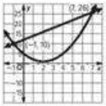
11. −2, 1



13. $-\frac{1}{4}$, 1



15. −1, 7

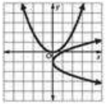


17a. $P = 2x^2 + 30x$ and $P = 50{,}000$

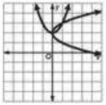
17b. $x \approx -165.79$, $x \approx \$150.79$

17c. Sample answer: Because the selling price cannot be negative, (150.79, 50,000) is the only viable solution in the context of the situation. This means that the business can earn a $50,000 profit when the selling price is about $150.79 per item.

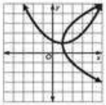
19. no solution



21. (0, 2) and (1, 3)



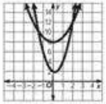
23. (1, 1) and (5, 5)



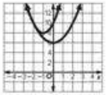
25. no solution **27.** (2, 2) and (2, −2)

29. (−1, −7) and (4, 23)

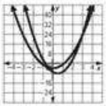
31. $-\frac{3}{2}$, 2



33. −2



35. $-\frac{1}{2}$, 3



37. −2 **39.** (−2, −6), (0, 8)

41. (−1, 2), (2, 5) **43.** (0.42, 0.352) and (1, 2)

45. (−1.4, 2) and (1.4, 2)

47. (−1.8, 3.2) and (1.1, 1.3)

49. (2, −2), (0, 0) **51.** no solution

53. no solution **55.** $(-\sqrt{31}, 7)$ and $(\sqrt{31}, 7)$

57. $-6\frac{1}{2}$, 1 **59.** −3, 1 **61.** $-\frac{1}{2}$, −1

63. −3 **65.** $-\frac{1}{2}$, 3

67. −6, −3 **69a.** 0.7 s

69b. about 20.9 m; The faster rocket lands after about 9.96 seconds. Solving $y = -4.9t^2 + 46.7t + c$ when $y = 0$ and $t = 9.96$, shows that $c \approx 20.9$.

71a. Yes; the maximum height of the ball is about 12.76 feet, which is higher than the net.

71b. Yes; the player's hands could strike the ball at a height of about 9 feet. Because that is above the height of the net, the ball may be blocked.

71c. Sample answer: I assumed that the student bumping the ball was far enough away from the net that she wouldn't hit it. I also assumed that the player attempting to block the ball is in the correct position for the path of the ball to intersect the path of her hands.

73. Sample answer: $y = x^2$ and $y = -x^2$

75. Danny is correct. Carol incorrectly solved for *y* in the second equation of the system before using the substitution method.

Module 1 Review

1. A **3.** 10 and 21 **5.** C **7.** $x = -3, x = 5$

9. D **11.** A, D **13.** C

15. between 0 and 5 feet

17. (2, 1), (7, 6)

Module 2

Quick Check

1. −5 + (−13) **3.** 5*mr* + (−7*mp*)

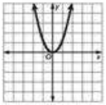
5. −4*a* − 20 **7.** $-m + \frac{5}{2}$

Lesson 2-1

1. As $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$ and $f(x) \to \infty$; D = $(-\infty, \infty)$, R = $[0, \infty)$

3. As $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$ and $f(x) \to -\infty$; D = $(-\infty, \infty)$, R = $(-\infty, \infty)$

5. D = $(-\infty, \infty)$, R = $[0, \infty)$



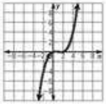
7. degree = 1, leading coefficient = 1

9. degree = 5, leading coefficient = −5

11. not in one variable because there are two variables, *u* and *t*

13a. 455

13b.



15. 1 **17.** 3 **19.** 3 **21a.** *f*(*x*)

21b. zeros: *f*(*x*): −1.5, 0.25, 3.25; *g*(*x*): −2.75, −0.5, 0.25

x−intercepts: *f*(*x*): −1.5, 0.25, 3.25; *g*(*x*): −2.75, −0.5, 0.25 *y*−intercept: *f*(*x*): 1; *g*(*x*): 1 end behavior: *f*(*x*): As $x \to -\infty$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$; *g*(*x*): As $x \to -\infty$, $g(x) \to \infty$, and as $x \to \infty$, $g(x) \to -\infty$.

23. As $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$. degree = 4; leading coefficient = −5

25. As $x \to -\infty$, $g(x) \to -\infty$ and as $x \to \infty$, $g(x) \to \infty$. degree = 5; leading coefficient = 5

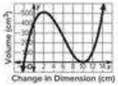
27. This is not a polynomial because there is a negative exponent.

29. As $x \to -\infty$, $h(x) \to \infty$ and as $x \to \infty$, $h(x) \to \infty$. degree = 2, leading coefficient = 3

31. As $x \to \infty$, $y \to -\infty$, and as $x \to -\infty$, $y \to \infty$; degree = 3; the leading coefficient is negative.

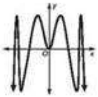
33. $f(1 - x) = a(1 - x)^3 - b(1 - x)^2 + (1 - x)$. So, $f(1 - x) = -ax^3 + (3a - b)x^2 + (-3a + 2b - 1)x + (a - b + 1)$. The function $f(1 - x)$ has the opposite leading coefficient, representing a reflection in the *y*-axis. So, it has the opposite end behavior.

35. Sample answer: The volume of the new box is modeled by the function $V(x) = (10 - x)^2(4 + 2x)$. The graph appears to have a relative maximum at $x = 2$ and $V(2) = 512$. So, the dimensions of the box with the greatest volume will be 8 centimeters by 8 centimeters by 8 centimeters.



37. *f*(*x*); *f*(*x*) has potential for 5 or more real zeros and a degree of 5 or more. *g*(*x*) has potential for 4 real zeros and a degree of 4.

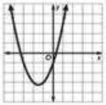
39. Sample answer:



Lesson 2-2

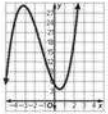
1. zeros between $x = -4$ and $x = -3$, and between $x = 0$ and $x = 1$

x	$f(x)$
−4	3
−3	−1
−2	−3
−1	−3
0	−1
1	3
2	9



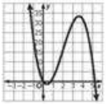
3a. zero between $x = -6$ and $x = -5$

x	$f(x)$
−6	37
−5	5
−4	25
−3	29
−2	23
−1	13
0	5
1	5
2	19
3	53



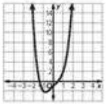
5. relative minimum between $x = 0$ and $x = 1$; relative maximum near $x = 4$

x	$f(x)$
−1	22
0	0
1	2
2	16
3	30
4	32
5	10
6	−48
7	−154



7. relative minimum near $x = -1$; no relative maximum

x	$f(x)$
−4	247
−3	74
−2	11
−1	−2
0	−1
1	2
2	19
3	86
4	263



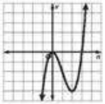
9. Domain and Range: The domain and range of the function are all real numbers. Because the function models years, the relevant domain is $\{n \mid n \geq 0\}$ and the relevant range is all real numbers.

Extrema: There is a relative minimum at $n = 2$ in the relevant domain.

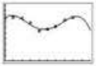
End Behavior: As $x \to \infty$, $v \to \infty$.

Intercepts: In the relevant domain, the v-intercept is at (0, 0). The n-intercept, or zero, is at about (3, 0).

Symmetry: In the relevant domain, the graph does not have symmetry.

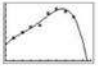


11. Curve of best fit: $y = -0.012x^4 + 0.217x^3 - 1.116x^2 + 1.486x + 7.552$; car sales in 2017: 9.991 million



[0, 10] scl: 1 by [0, 10] scl: 1

13. Curve of best fit: $y = -0.122x^4 + 1.565x^3 - 6.325x^2 + 20.321x + 33.125$, where x is the number of months since January; average volunteer hours for September: 44

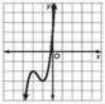


[0, 10] scl: 1 by [0, 120] scl: 12

15. average rate of change: 4500; From 2012 to 2017, the salesman's average rate of change in salary was an increase of $4500 per year.

17.

x	$f(x)$
−3	−7
−2	−2
−1	−3
0	2
1	25



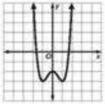
zero between $x = -1$ and $x = 0$;

rel. max. at $x = -2$, rel. min. at $x = -1$;

D = all real numbers; R = all real numbers

19.

x	$f(x)$
−3	61
−2	6
−1	−3
0	−2
1	−3
2	6
3	61



zeros between $x = -2$ and $x = -1$, and between $x = 1$ and $x = 2$;

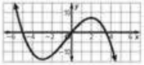
rel. max. at $x = 0$, min. at $x = -1$, and $x = 1$;

D = all real numbers; R = $\{f(x) \mid f(x) \geq -3\}$

21. rel. max: $x \approx -2.73$; rel. min: $x \approx 0.73$

23. max: $x \approx 1.34$; no rel. min

25. Sample answer: $f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$; the leading coefficient is negative.



27. The type of polynomial function that should be used to model the graph is a function with an even degree with a negative leading coefficient. This is based on the fact that the graph is reflected in the x-axis and as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$.

29. As the x-values approach large positive or negative numbers, the term with the largest degree becomes more and more dominant in determining the value of $f(x)$.

31. Sample answer:

33. Sample answer: No; $f(x) = x^2 + x$ is an even degree, but $f(1) \neq f(-1)$.

Lesson 2-3

1. yes; 2 **3.** no **5.** $2a^2 - a - 2$ **7.** $3g + 12$

9. $3x^2 + 4x - 5$ **11.** $-3x + 3$ **13.** $3np^2 - 3pz$

15. $-10c^2 + 5d^2$ **17.** $a^2 - 10a + 25$

19. $x^3 + x^2y - xy^2 - y^3$

21. $2x^3 + x^2y - 2xy^2 - y^3$ **23.** $r^2 - 4t^2$

25. $2x^5 - 7x^4 + 5x^3 - 4x^2 - x + 2$

27. $s^2 + 11s + 15$ **29a.** $4x^2 + 8x + 3$

29b. $8x^2 + 16x + 6$

31. $12a^2b + 8a^2b^2 - 15ab^2 + 4b^2$

33. $2n^4 - 3n^3p + 6n^4p^4$

35. $2n^5 - 14n^3 + 4n^2 - 28$

37. $64n^3 - 240n^2 + 300n - 125$

39. $7x - 2y$

41a. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$; 3rd degree

41b. $h(x)f(x) = (-x^2 - x)(3x^2 - 1) = -3x^4 + x^2 - 3x^3 + x$; 4th degree

41c. $[f(x)]^2 = (3x^2 - 1)^2 = 9x^4 - 6x^2 + 1$; 4th degree

43.

	Addition and Subtraction	Multiplication	Division
Integers	yes	yes	No; $3 \div 2 = 1.5$
Rational Numbers	yes	yes	yes
Polynomials	yes	yes	No; $\frac{3}{x^2} = 3x^{-2}$

45. $-0.022x + 1590$

47a. Area of sidewalk = area of larger circle − area of smaller circle, so $384\pi = \pi(r + 12)^2 - \pi r^2 = \pi(r^2 + 24r + 144) - \pi r^2 = 24\pi r + 144\pi$. $24\pi r = 240\pi$; $r = 10$; the radius of the smaller circle is 10 feet, and the radius of the larger circle is $r + 12 = 10 + 12$ or 22 feet.

47b.

47c. $48s + 576$ ft^2

49a. $m \cdot n$; each term of one polynomial multiplies each term of the other one time.

49b. 2; adding like terms may result in a sum of 0; but the first and last terms are unique.

51. $(3 - 2b)(3 + 2b)$

Lesson 2-4

1. $5y^2 + 2y + 1$ **3.** $2j - 3k$ **5.** $n + 2$

7. $\frac{2t + 1 + 9}{t + 6}$ **9.** $2g - 3$ **11.** $3v + 5 + \frac{10}{v - 4}$

13. $y^2 - 2y + 4 - \frac{2}{y + 2}$

15. $\frac{4}{3}p^2 + \frac{p}{9} + \frac{19}{27} + \frac{19}{27(3x - 1)}$

17. $m - 3 + \frac{6}{m + 4}$ **19.** $2x^2 - 3x + 1$

21. $x^2 - 2x + 4$ **23.** $2c^2d - \frac{3}{2}d$

25. $n^2 - n - 1$

27. $3z^4 - z^3 + 2z^2 - 4z + 9 - \frac{13}{z + 2}$

29. A is x^2; B is $6x$; C is 11. **31.** $\pi(x^2 - 8x + 16)$

33. Yes; Because $3x$ times the divisor is $9x^2 + 3x$, the divisor must be $3x + 1$. The second and third terms of the dividend must be $0x + 5$ because the first difference is $-3x + 5$.

35. $8x^3 - 8x^2 - 2x - 10$; I multiplied the divisor and the quotient and added the remainder: $(2x^2 + x + 1)(4x - 6) + (-4) = 8x^3 - 8x^2 - 2x - 10$.

37a. No; the degrees of $f(x)$ and $d(x)$ may be equal. For example, $\frac{3x^2 + 6}{x^2} = 3 + \frac{6}{x^2}$.

37b. Yes; if the degree of $r(x)$ is greater than or equal to the degree of $d(x)$, then the expression $\frac{r(x)}{d(x)}$ may be simplified by division. For example, if $\frac{r(x)}{d(x)} = \frac{8x + 1}{x}$, then $8x + 1$ may be divided by x to get $8 + \frac{1}{x}$.

37c. Yes; because $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$, the degree of $\frac{f(x)}{d(x)}$ must equal the degree of $q(x) + \frac{r(x)}{d(x)}$. The degree of $r(x)$ is less than the degree of $d(x)$, so the degree of $q(x) + \frac{r(x)}{d(x)}$ equals the degree of $q(x)$. This means the degree of $\frac{f(x)}{d(x)}$ equals the degree of $q(x)$, and the degree of $\frac{f(x)}{d(x)}$ is the degree of $f(x)$ minus the degree of $d(x)$. For example, in $\frac{2x^3 - 1}{x^2 + 3} = 2x + \frac{-6x - 1}{x^2 + 3}$, the degree of $q(x)$ is the degree of $f(x)$ minus the degree of $d(x)$.

39. The binomial is a factor of the polynomial.

41. Sample answer: $\frac{x^2 + 5x + 9}{x + 2}$

43a. $B^2 + 1$

43b. $B + \frac{1}{B^2 + B + 1}$

Lesson 2-5

1. $x^3 - 3x^2y + 3xy^2 - y^3$

3. $g^4 - 4g^3h + 6g^2h^2 - 4gh^3 + h^4$

5. $y^3 - 21y^2 + 147y - 343$

7. 38%

9. $243x^5 + 1620x^4y + 4320x^3y^2 + 5760x^2y^3 + 3840xy^4 + 1024y^5$

11. $4096h^4 - 6144h^3j + 3456h^2j^2 - 864hj^3 + 81j^4$

13. $x^5 + \frac{5}{2}x^4 + \frac{5}{2}x^3 + \frac{5}{4}x^2 + \frac{5}{16}x + \frac{1}{32}$

15. $32b^5 + 20b^4 + 5b^3 + \frac{5}{8}b^2 + \frac{5}{128}b + \frac{1}{1024}$

17. 23%

19. If C represents a correct circuit board and N represents an incorrect circuit board, then the number of ways for the robot to produce 5 of 7 circuit boards accurately is given by the coefficient of C^5N^2 in the expansion of $(C + N)^7$. Using the Binomial Theorem, there are 21 ways for the robot to produce a correct circuit board out of 128 possibilities. So, the probability that 5 of 7 are correct is $\frac{21}{128}$ or about 16% probability.

21. If c represents a correct answer and w represents a wrong answer, then the coefficients expansion of $(c + w)^{10}$ can be used to represent situation. Using Pascal's triangle, there are 45 ways to get 8 questions correct, 10 ways to get 9 questions correct, and 1 way to get all correct. So there are 56 ways to get 8 or more correct. By adding all of the values in this row of Pascal's triangle, I found that there are 1024 different ways he could answer the questions in the quiz. Matthew has a $\frac{56}{1024}$ or about a 5.5% chance of getting 8 or more correct.

23. There are 9 judges on the Supreme Court. The majority could be 5, 6, 7, 8, or 9 votes. So, there are ${}_9C_5 + {}_9C_6 + {}_9C_7 + {}_9C_8 + {}_9C_9 = 256$ combinations.

25. 1.21896; $(1.02)^{10} + 1.21899442$; the approximation differs from the value given by a calculator by 0.00003442.

27. Sample answer: While they have the same terms, the signs for $(x + y)^n$ will all be positive, while the signs for $(x - y)^n$ will alternate.

29. Sample answer: $\left(x + \frac{6}{5}y\right)^5$

Module 2 Review

1. B **3.** B **5.** $g(x) \to -\infty$ **7.** C

9. $x^6 + 7x^5 - 6x^3 + 8x^2 + 2x$

11. C **13.** $5x^2 + 7x - 3$ **15.** B

Module 3

Quick Check

1. $24x^3 + 8x^2 + 6x + 4$

3. $14x^3 - 40x^2 + 12x + 24$

5. $-4, 2$ **7.** $-\frac{4}{3}, \frac{1}{2}$

Lesson 3-1

1. -4.12 **3.** $-0.47, 0.54, 3.94$ **5.** -1.27

7a. $6x^3 + 110x^2 - 200x = 15{,}000$

7b. $y = 6x^3 + 110x^2 - 200x$, $y = 15{,}000$; $x = 10$ cm

7c. 25 cm by 20 cm by 30 cm

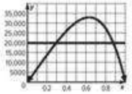
9a. $\pi x^3 + 3\pi x^2 = 628$

9b. $y = \pi x^3 + 3\pi x^2$, $y = 628$; $x = 5$

9c. radius = 5 in., height = 8 in.

11. $-3.63, -1.35, 1.35, 3.63$ **13.** no solution

15. \$0.29, 0.88; sample answer: graph $f(x) = 70{,}000(x - x^4)$ and $f(x) = 20{,}000$ and find the x-values of the points of intersections.



17. Sample answer: A function with an even degree reverses direction, so both ends extend in the same direction. That means that if the function opens up and the vertex is above the x-axis or if the function opens down and the vertex is below the x-axis, there are no real solutions. A function with an odd degree does not reverse direction, so the ends extend in opposite directions. Therefore, it must cross the x-axis at least once.

19. Sometimes; sample answer: The positive solution is often correct when a negative value doesn't make sense, such as for distance or time; however, sometimes a negative solution is reasonable, such as for temperature or position problems. Also, sometimes there are two positive solutions and one may be unreasonable.

21. $5x^2 = -2x - 11$; It is the only one that has no real solutions.

Lesson 3-2

1. $(2c - 3d)(4c^2 + 6cd + 9d^2)$

3. $a^2(a - b)(a^2 + ab + b^2)(a + b)(a^2 - ab + b^2)$

5. prime **7.** $(x + y)(x - y)(6f + g - 3h)$

9. $(a - b)(a^2 + ab + b^2)(x - 8)^2$ **11.** $0, 7, 2$

13. $0, -5, 8$

15a. Let x represent the length of a side of the leg and y represent the length of the notch; $x^3 - y^3$.

15b. 15 in.

17. $-15(x^2)^2 + 18(x^2) - 4$ **19.** not possible

21. $4(2x^5)^2 + 1(2x^5) + 6$ **23.** $\pm\sqrt{5}, \pm i\sqrt{2}$

25. $\pm\frac{2\sqrt{3}}{3}, \pm\frac{\sqrt{15}}{3}$ **27.** $\pm\frac{\sqrt{6}}{6}, \pm i\frac{\sqrt{3}}{2}$

29. $(x + 2)(x^2 - 2x + 4)(x - 2)(x^2 + 2x + 4)$

31. $x^2y^2(2x + 3y)(4x^2 + 6xy + 9y^2)$

33. $(y - 1)^3(y^2 + y + 1)^3$

35. $3, -3, \pm i\sqrt{10}$ **37.** $\pm\sqrt{11}, \pm 2i$

39. $-6, 3 \pm 3i\sqrt{3}$

41. $\pm\frac{1}{2}, \pm i\frac{\sqrt{6}}{2}$ **43.** $\pm\frac{3}{2}, \pm\frac{\sqrt{10}}{5}$

45. $\pm\frac{1}{2}, \pm\sqrt{2}$ **47.** $1, -2, \frac{-1 \pm i\sqrt{3}}{2}, 1 \pm i\sqrt{3}$

49. $-5, \frac{1}{2}, \frac{-1 \pm i\sqrt{3}}{2}, \frac{5 \pm 5i\sqrt{3}}{2}$ **51.** $\pm 3, \pm 1, \pm i$

53a. Sample answer: $\frac{1}{2} = 2\left(\frac{1}{4}\right)$

53b. $u = x^{\frac{1}{4}}$; $u^2 - 8u + 15 = 0$

53c. 81, 625

55. prime **57.** $(2a + 1)(k - 3)$ **59.** prime

61. $(d - 6)^2$ **63.** $(y + 9)^2$ **65.** $19x^2(x - 2)$

67. $(m^2 + 1)(m - 1)(m + 1)$ **69.** 2.4 in.

71. The solutions are $x = \pm\sqrt{m}$ and $\pm\sqrt{n}$. because both equations have the same coefficients, take the solutions for the quadratic equation and substitute x^2 for x.

73. $\frac{16}{3}, \frac{3}{4}$

75. Sample answer: $12x^6 + 6x^4 + 8x^2 + 4 = 12(x^2)^3 + 6(x^2)^2 + 8(x^2) + 4$

Lesson 3-3

1. $(x - y)^2 = x^2 - 2xy + y^2$ (Original equation)
$x^2 - 2xy + y^2 = x^2 - 2xy + y^2$ (Distributive Property)

3. $4(x - 7)^2 = 4x^2 - 56x + 196$ (Original equation)
$4(x^2 - 14x + 49) = 4x^2 - 56x + 196$ (Distributive Property)
$4x^2 - 56x + 196 = 4x^2 - 56x + 196$ (Distributive Property)

5. $a^2 - b^2 = (a + b)(a - b)$ (Original equation)
$= a^2 - ab + ab - b^2$ (Distributive Property)
$= a^2 - b^2$ (Simplify.)

7. $p^4 - q^4 = (p - q)(p + q)(p^2 - q^2)$ (Original equation)
$= (p^2 + pq - pq - q^2)(p^2 + q^2)$ (Distributive Property)
$= (p^2 - q^2)(p^2 + q^2)$ (Simplify.)
$= p^4 + p^2q^2 - p^2q^2 - q^4$ (Distributive Property)
$= p^4 - q^4$ (Simplify.)

9. $(3x + y)^2 = 9x^2 + 6xy + y^2$ (Original equation)
$9x^2 + 6xy + y^2 = 9x^2 + 6xy + y^2$ (Square the left side.)
$9x^2 + 6xy + y^2 = 9x^2 + 6xy + y^2$ (True)
Because the identity is true, this proves that Aponi is correct. Her process for finding the area of a square will always work.

11. identity **13.** not an identity **15.** not an identity

17. $g^6 + h^6$
$= (g^2 + h^2)(g^4 - g^2h^2 + h^4)$ (Original equation)
$= g^6 + g^4h^2 + g^2h^4 + g^4h^2 - g^2h^4 + h^6$ (Distributive Property)
$= g^6 + h^6$ (Simplify.)

19. $u^6 - w^6$
$= (u + w)(u - w)(u^2 + uw + w^2)(u^2 - uw + w^2)$ (Original equation)
$= (u^2 + w^2)(u^2 + uw + w^2)(u^2 - uw + w^2)$ (FOIL)
$= (u^4 + u^3w - uw^3 - w^4)(u^2 - uw + w^2)$ (Distributive Property)
$= u^6 + u^5w - u^3w^3 - u^2w^4 - u^5w - u^4w^2 + u^2w^4 + uw^5 + u^4w^2 + u^3w^3 - uw^5 - w^6$ (Distributive Property)
$= u^6 - w^6$ (Simplify.)

21. A polynomial identity is a polynomial equation that is satisfied for any values that are substituted for the variables. To prove that a polynomial equation is an identity you begin with the more complicated side of the equation and use algebra properties to transform that side of the equation until it is simplified to look like the other side.

23. $x^2 - y^2 = 3, 2xy = 4, x^2 + y^2 = 5; x = 2, y = 1$

25. When George multiplied b and a^2 in the second line he mistakenly made the term negative so it did not cancel when he simplified.

Lesson 3-4

1. -59; 11 **3.** 1707; 62 **5.** -98; 0 **7.** 13,190; 2

9. 21, 0 **11.** $-2, 1$ **13.** 3, 3 **15.** $-4, -7$

17. 15, -6 **19.** -22, 20

21a. \$20.4 billion **21b.** \$144.16 billion

23. $(x - 1)^2$ **25.** $x - 4, x + 1$

27. $x + 6, 2x + 7$ **29.** $x + 1, x^2 + 2x + 3$

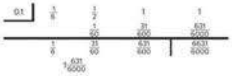
31. $x - 1, x + 2$ **33.** $x - 2, x + 2$

35. $x + 3, x - 6$ **37.** $x + 1, x - 4$

39. $2x + 1, x - 1$ **41.** $x - 1, x + 2$

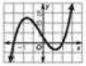
43. $x - 4, 3x - 2$ **45.** 11; 764

47.



49. 8 **51.** -3

53. By the Factor Theorem, $(x - 1)$ is a factor. Use synthetic division with $x = 1$. The remainder is $k - 3$. For $(x - 1)$ to be a factor, $k - 3 = 0$, so $k = 3$. The quotient is $10x^2 + 13x - 3$, which factors as $(5x - 1)(2x + 3)$. $P(x) = (x - 1)(5x - 1)(2x + 3)$. The cubic has a positive leading coefficient and zeros at -1.5, 0.2, and 1.

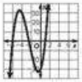


55a. (2003, 621), (2012, 1197), (2021, 1740), (2025, −15,255)

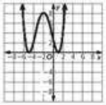
55b. The model still represents the situation after 25 years.; Sample answer: No, the average value is unlikely to fall so quickly.

57. -5; If $P(x)$ is symmetric to the y-axis, it must also contain the point $(-2, -5)$. According to the Remainder Theorem, the remainder when polynomial $P(x)$ is divided by $(x - r)$ is $P(r)$. Since $P(-2), = -5$, the remainder when $P(x)$ is divided by $(x + 2)$ is -5.

59. Use synthetic substitution with $a = -6$. The remainder must be zero, so $432 - 216k = 0$, or $k = 2$. The depressed polynomial is $2x^2 + 3x - 5$, which factors to $(2x + 5)(x - 1)$. Therefore, we can write $kx^3 + 15x^2 + 13x - 30 = (x + 6)(2x + 5)(x - 1)$.



61. $f(x) = 0.1(x - 1)^2(x + 5)^2$



63. If $x - a$ is a factor of $f(x)$, then $f(a)$ has a factor of $(a - a)$ or 0. Since a factor of $f(a)$ is 0, $f(a) = 0$. Now assume that $f(a) = 0$. If $f(a) = 0$, then the Remainder Theorem states that the remainder is 0 when $f(x)$ is divided by $x - a$. This means that $x - a$ is a factor of $f(x)$. This proves the Factor Theorem.

65. Sample answer: When $x = 1$, $f(1)$ is the sum of all of the coefficients and constants in $f(x)$, in this case, a, b, c, d, and e. The sum of a, b, c, d, and e is 0, so however the coefficients are arranged, $f(1)$ will always equal 0, and $f(x)$ will have a rational root.

67. Tyrone; Sample answer: By the Factor Theorem, $(x - r)$ is a factor when $P(r) = 0$. $P(r) = 0$ when $x = -1$ and $x = 2$.

Lesson 3-5

1. $-\frac{12}{5}$; 1 real

3. 0, 0, 0, $2i$, $-2i$; 3 real, 2 imaginary

5. $\frac{1 \pm \sqrt{2}}{2}$; 2 real

7. -2, $\frac{3}{2}$; 2 real

9. -1, $\frac{1 \pm i\sqrt{3}}{2}$; 1 real, 2 imaginary

11. $-\frac{8}{3}$, 1; 2 real

13. $-\frac{5}{2}$, $\frac{5}{2}$, $-\frac{5}{2}i$, $\frac{5}{2}i$; 2 real, 2 imaginary

15. -2, -2, 0, 2, 2; 5 real

17. 2 or 0; 1; 2 or 0

19. 3 or 1; 0; 2 or 0

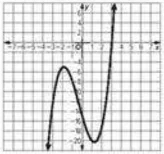
21. 1; 1; 2

23. 0 or 2; 0 or 2; 0, 2, or 4

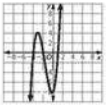
25. 0 or 2; 1; 2 or 4

27. 0 or 2; 0 or 2; 2, 4, or 6

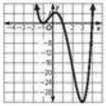
29. 3, $1 + \sqrt{2}$, $1 - \sqrt{2}$



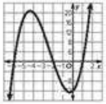
31. 1, -2, -3



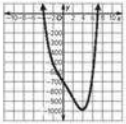
33. -1, -1, 1, 4



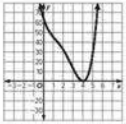
35. -6, -2, 1



37. -4, 7, $-5i$, $5i$



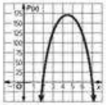
39. 4, 4, $-2i$, $2i$



41. $y = x^2 + x - 6$

43. $y = x^4 - 6x^3 + 7x^2 + 6x - 8$

45. Sample Answer:

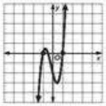


47. Sample answer: $f(x) = x^3 + 2x^2 - 23x - 60$

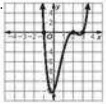
49. Sample answer: $f(x) = x^4 + 2x^3 + 6x^2 + 18x - 27$

51. Sample answer: $f(x) = x^4 - 3x^3 - 9x^2 + 77x + 150$

53.



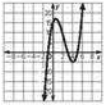
55.



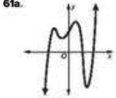
57a. $(x - 5)^2(7x + 9.5)\pi = 234\pi$; 7, $\frac{23 \pm 3\sqrt{65}}{28}$

57b. height 41.5 m, radius 2 m; $x = 7$ is the only reasonable solution in the context of the situation. The other possible values of x result in negative measures.

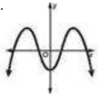
59. The graph is accurate as far as the location of the zeros, but does not consider any vertical dilation. It could be improved by finding more points between the roots.



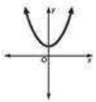
61a.



61b.



61c.



63. $r^4 + 1 = 0$; Sample answer: The equation has imaginary solutions and all of the others have real solutions.

65. Sample answer: To determine the number of positive real roots, determine how many time the signs change in the polynomial as you move from left to right. In this function there are 3 changes in sign. Therefore, there may be 3 or 1 positive real roots. To determine the number of negative real roots, I would first evaluate the polynomial for $-x$. All of the terms with an odd-degree variable would change signs. Then I would again count the number of sign changes as I move from left to right. There would be only one change. Therefore there may be 1 negative root.

Module 3 Review

1. A **3.** A, B, D

5. Set the expression for the volume of the first figure equal to the expression for the surface area of the second figure. Solve the equation $x^3 - 9x = 8x^2$. Solving the equation gives the solutions −1, 0, and 9. The value of x cannot be 0 because when substituting 0 for x in the original expressions, the volume of the first figure is $(0)3 - 9(0) = 0 - 0 = 0$ and surface area of the second figure is $8(0)^2 = 8(0) = 0$. Volume and surface area cannot be 0, so the value of x is −1 or 9.

7. $x = 6$

9. B **11.** C **13.** B

15. B, C

17. A

Module 4

Quick Check

1. between 0 and 1, and between 3 and 4

3. between −1 and 0, and between 1 and 2

5. $5x + 3$ **7.** $2x^2 + 5x - 6$

Lesson 4-1

1. $(f + g)(x) = -2x + 5$; $(f - g)(x) = 6x - 5$; $(f \cdot g)(x) = -8x^2 + 10x$; $\left(\frac{f}{g}\right)(x) = \frac{2x}{-4x+5}$, $x \neq \frac{5}{4}$

3. $(f + g)(x) = 3x - 9$; $(f - g)(x) = -x + 5$; $(f \cdot g)(x) = 2x^2 - 11x + 14$; $\left(\frac{f}{g}\right)(x) = \frac{x-2}{2x-7}$, $x \neq \frac{7}{2}$

5. $(f + g)(x) = x^2 + 3x + 1$; $(f - g)(x) = -3x^2 - 3x + 11$; $(f \cdot g)(x) = -2x^4 - 3x^3 + 17x^2 + 18x - 30$; $\left(\frac{f}{g}\right)(x) = \frac{-x^2+6}{2x^2+3x-5}$, $x \neq 1$ or $-\frac{5}{2}$

7a. $(a - b)(x)$; $(a - b)(x) = 85x + 1750$

7b.

9. $f \circ g = \{(-4, 4)\}$, D = {−4}, R = {4}; $g \circ f = \{(-8, 0), (0, -4), (2, -5), (-6, -1)\}$, D = {−6, 0, 2}, R = {−5, −4, −1, 0}

11. $f \circ g$ is undefined, D = Ø, R = Ø; $g \circ f$ is undefined, D = Ø, R = Ø

13. $[f \circ g](x) = 2x + 10$, D = {all real numbers}, R = {all even numbers}; $[g \circ f](x) = 2x + 5$, D = {all real numbers}, R = {all odd numbers}

15. $[f \circ g](x) = x^2 - 6x - 2$, D = {all real numbers}, R = $\{y \mid y \geq -11\}$; $[g \circ f](x) = x^2 + 6x - 8$, D = {all real numbers}, R = $\{y \mid y \geq -17\}$

17. $p(x) = 0.85x$; $t(x) = 1.065x$; $t[p(x)] = 1.065(0.85x)$; $1222.09

19. Sample answer: The order of the discounts does not matter. Either composition results in a final cost of $54.91.

21. 15 **23.** 1 **25.** 12 **27.** 189 **29.** 21

31. −3 **33.** 9 **35.** 440

37a. $V[r(t)] = \frac{\pi t^3}{6} + 2\pi t^2 + 8\pi t + \frac{32}{3}\pi$

37b. 2145 m^3

39a. $(f + g)(-1) = 4$

39b. $(h - g)(0) = 8$

39c. $(f \cdot h)(4) = 5$

39d. $\left(\frac{f}{g}\right)(3) = 4$

39e. $\left(\frac{g}{h}\right)(2) = 0$

39f. $\left(\frac{g}{f}\right)(1) =$ undefined

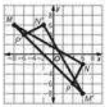
41. Sample answer: $f(x) = x - 9$, $g(x) = x + 5$

43a. D = {all real numbers}

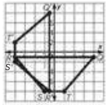
43b. D = $\{x \mid x \geq 0\}$

Lesson 4-2

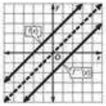
1. {(6, −8), (−2, 6), (−3, 7)}



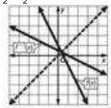
3. {(−1, 8), (−1, −8), (−8, −2), (−8, 2)}



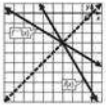
5. $f^{-1}(x) = x - 2$



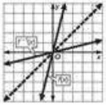
7. $f^{-1}(x) = -\frac{x}{2} + \frac{1}{2}$



9. $f^{-1}(x) = -\frac{3}{5}(x + 8)$



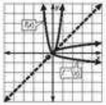
11. $f^{-1}(x) = \frac{1}{4}x$



13. $f^{-1}(x) = \pm\frac{x\sqrt{5}}{5}$

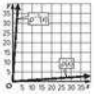
If the domain of $f(x)$ is restricted to $(-\infty, 0]$, then the inverse is $f^{-1}(x) = \frac{x\sqrt{5}}{5}$.

If the domain of $f(x)$ is restricted to $[0, \infty)$, then the inverse is $f^{-1}(x) = \frac{x\sqrt{5}}{5}$.



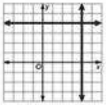
15a. $p^{-1}(x) = 14x$; Sample answer: The inverse converts stones to pounds, where x is the weight in stones.

15b. About 35 pounds

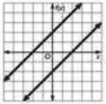


17. no **19.** yes **21.** no **23.** yes

25. $x = 4$

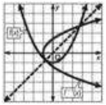


27. $f^{-1}(x) = x - 2$



29. $f^{-1}(x) = \pm\sqrt{2x + 2}$

If the domain of $f(x)$ is restricted to $(-\infty, 0]$, then the inverse is $f^{-1}(x) = \sqrt{2x + 2}$. If the domain of $f(x)$ is restricted to $[0, \infty)$, then the inverse is $f^{-1}(x) = -\sqrt{2x + 2}$.

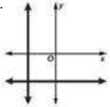


31. yes **33.** no **35.** yes

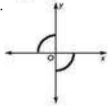
37. $\{x \mid x \geq 0\}$ or $\{x \mid x \leq 0\}$; $\{x \mid x \geq 5\}$

39. $\{x \mid x \geq -6\}$; $\{x \mid x \geq 0\}$

41.

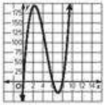


43.



45. $f^{-1} = 220 - \frac{x}{0.85}$; Sample answer: The inverse function gives the functional age of the clients based on their heart rate.

47a. $V(x) = x(12 - 2x)(16 - 2x)$; Sample answer: The value of x must be between 0 and 6 because the length of x must be positive, but the length of the side of the box $(12 - 2x)$ cannot be less than 0.



47b. No; sample answer: even with the restriction, the function does not pass the horizontal line test, so its inverse will not be a function.

49. Sample answer: Sometimes; $y = \pm\sqrt{x}$ is an example of a relation that is not a function, with an inverse being a function. A circle is an example of a relation that is not a function with an inverse not being a function.

51. Sample answer: $f(x) = x$ and $f^{-1}(x) = x$ or $f(x) = -x$ and $f^{-1}(x) = -x$

53. Sample answer: One of the functions carries out an operation on 5. Then the second function that is an inverse of the first function reverses the operation on 5. Thus, the result is 5.

Lesson 4-3

1. $\pm 11x^2y^8$ **3.** $\pm 7x^2$ **5.** $-9a^8b^{10}c^6$ **7.** $2|(x - 3)^3|$

9. $3|x - 4|$ **11.** $|a^3|$ **13.** $\sqrt[5]{8}$ **15.** $\sqrt{x^9}$

17. $5^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{2}{3}}$ **19.** $6^{\frac{1}{3}}t^{\frac{2}{3}}$ **21.** 3 **23.** $\frac{1}{64}$

25. 64 **27.** $x^{\frac{11}{15}}$ **29.** $\frac{b^{\frac{1}{4}}}{b}$ **31.** $3b^6c^4$

33. $3|(x + 4)|$ **35.** $(y^3 + 5)^6$

37. $14|c^3|d^2$ **39.** $-3a^5b^3$ **41.** a^3

43. $b^{\frac{1}{4}}$ **45.** $\frac{d^{\frac{1}{6}}}{d}$

47a. $5\sqrt{30}$ cm³

47b. $32\sqrt{3}$ cm³

47c. 288 cm³

49. $c^{\frac{1}{2}}$ **51.** $6^{\frac{1}{4}}h^{\frac{1}{2}}j^{\frac{1}{2}}$ **53.** $\frac{xy\sqrt[3]{z^2}}{z}$ **55.** 21 feet

57. There are four cases to consider. If $m > 0$ and b is even, then $\sqrt[b]{m^{3b}} = m^3$. If $m < 0$ and b is even, then $\sqrt[b]{m^{3b}} = -m^3$. If $m > 0$ and b is odd, then $\sqrt[b]{m^{3b}} = m^3$. If $m < 0$ and b is odd, then $\sqrt[b]{m^{3b}} = m^3$.

59. $x < 0$; If $x < 0$, then $\sqrt{x^2} = -x$.

61. Sample answer: It may be easier to simplify an expression when it has rational exponents because all the properties of exponents apply. We do not have as many properties dealing directly with radicals. However, we can convert all radicals to rational exponents, and then use the properties of exponents to simplify.

63. when x or $y = 0$ and the other variable is ≥ 0

65. $0 < x < 1, x < -1$ **67.** $2\sqrt[3]{2xy}$

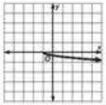
Lesson 4-4

1. $D = \{x \mid x \geq 9\}$; $R = \{y \mid y \geq 0\}$

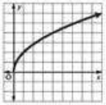
3. $D = \{x \mid x \geq 0\}$; $R = \{y \mid y \leq 0\}$

5. $D = \left\{x \mid x \geq \frac{4}{3}\right\}$; $R = \{y \mid y \geq 0\}$

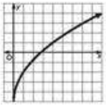
7. $D = \{x \mid x \geq -1\}$; $R = \{y \mid y \leq 0\}$; reflected in the x-axis, compressed vertically, and translated left 1 unit



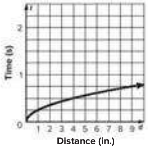
9. D = {x | $x \geq 0$}; R = {y | $y \geq 0$}
stretched vertically



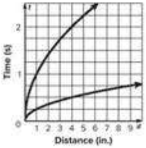
11. D = {x | $x \geq 0$}; R = {y | $y \geq -5$}; stretched vertically and translated down 5 units



13a. $t = \frac{1}{4}\sqrt{d}$

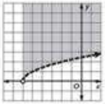


13b. D = {d | $d \geq 0$}; R = {t | $t \geq 0$}; increasing as $x \to \infty$; positive for $x > 0$; As $x \to \infty$, $y \to \infty$.



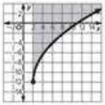
13c. compressed vertically

15.

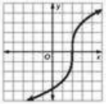


17.

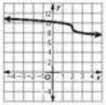
19.



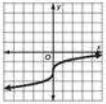
21. D = $(-\infty, \infty)$; R = $(-\infty, \infty)$



23. D = $(-\infty, \infty)$; R = $(-\infty, \infty)$



25a.

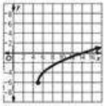


25b.

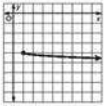
	$p(x) = \sqrt[3]{x} - 2$	$q(x)$
Domain	all real numbers	{x | $x \geq 0$}
Range	all real numbers	{y | $y \leq -3$}
Intercepts	x-int: 8; y-int: -2	x-int: 9; y-int: -3
Increasing/ Decreasing	increasing as $x \longrightarrow \infty$	increasing as $x \longrightarrow \infty$
Positive/ Negative	negative for $x < 8$; positive for $x > 8$	negative for $x < 9$; positive for $x > 9$
End Behavior	as $x \longrightarrow -\infty$, $p(x) \longrightarrow -\infty$ as $x \longrightarrow \infty$, $p(x) \longrightarrow \infty$	as $x \longrightarrow -\infty$, $q(x) \longrightarrow -2$ as $x \longrightarrow \infty$, $q(x) \longrightarrow \infty$

27. $f(x) = \sqrt[3]{x - 2} + 1$

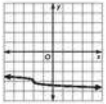
29. D = {x | $x \geq 5$}; R = {$f(x)$ | $f(x) \geq -6$} stretched vertically translated 5 units right and 6 units down



31. D = {x | $x \geq 1$}; R = {$f(x)$ | $f(x) \leq -4$} compressed vertically, translated 1 unit right and 4 units down, reflected across the x-axis



33. D = $(-\infty, \infty)$; R = $(-\infty, \infty)$



35.

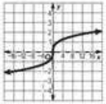
37. $f(x) = -\sqrt{x + 4} - 2$

39a. $f(x)$ has the greater maximum value because its maximum, $\frac{13}{2}$, is greater than 6, the maximum value of $g(x)$.

39b. The domain of $f(x)$ is $x \geq -3$, since any values less than 3 produce a negative value under the radical. The domain of $g(x)$ is $x \geq \frac{5}{2}$.

39c. The average rate of change over the interval is $-\frac{1}{7}$ for $f(x)$. It appears that the rate of change for $g(x)$ is the same.

41. D = {V | $-\infty < V < \infty$} or $(-\infty, \infty)$; R = {$r(V)$ | $-\infty < r(V) < \infty$} or $(-\infty, \infty)$; End behavior: The values of r increase as the values of V increase.



43. Sample answer: $y = -\sqrt{x + 4} + 6$

45. Sample answer: The domain is limited because square roots of negative numbers are imaginary. The range is limited due to the limitation of the domain.

47a. Sample answer: The original is $y = x^2 + 2$ and inverse is $y = \pm\sqrt{x - 2}$.

47b. Sample answer: The original is $y = \pm\sqrt{x + 4}$ and inverse is $y = (x - 4)^2$.

Lesson 4-5

1. $6a^4b^2\sqrt{2b}$ **3.** $2a^8b^4\sqrt{6cb}$ **5.** $2|ab|\sqrt[4]{4}$
7. $\frac{5}{6}|r|\sqrt{t}$ **9.** $\frac{4h \cdot \sqrt[5]{3h^3}}{3f}$ **11.** $8\sqrt{2}$ **13.** $\sqrt{5}$
15. $5\sqrt{7x} - \sqrt{14}$ **17.** $120y\sqrt{2z}$ **19.** $144ab^2\sqrt{2}$

21. $15xy\sqrt{14xy}$

23. $18 + 5\sqrt{6}\text{m}^2$

25. $56\sqrt{3} + 42\sqrt{6} - 36\sqrt{2} - 54$ **27.** 1260

29. $\frac{a^2\sqrt{5ab}}{b^7}$ **31.** $\frac{\sqrt[3]{150x^2y^2}}{5y}$

33. $6\sqrt{3} + 6\sqrt{2}$ **35.** $\frac{20 - 7\sqrt{3}}{11}$ **37.** $\frac{3\sqrt{7} + 3\sqrt{35}}{4}$

39. $2yz^4\sqrt[3]{2y}$

41. $\frac{(x + 1)(\sqrt{x} + 1)}{x - 1}$ or $\frac{x\sqrt{x} + \sqrt{x} + x + 1}{x - 1}$

43. $4\sqrt{3}$ **45.** $28\sqrt{7a}$ **47.** $10\sqrt[3]{28}$ in. **49.** $\frac{\sqrt{tu}}{u}$

51a. John: $\frac{0.8 - 4\sqrt{0.02}}{0.02}$ or $40 - 20\sqrt{2}$ min;
Jay: $\frac{0.8 - 4\sqrt{0.05}}{-0.01}$ or $40\sqrt{5} - 80$ min

51b. $2 - \sqrt{2} + \sqrt{5} + \frac{\sqrt{10}}{2}$

53a. $3x$; $(\sqrt{x} \cdot x) + (\sqrt{x} \cdot \sqrt{4x}) = (\sqrt{x^2} + \sqrt{4x^2}) = x + 2x = 3x$

53b. $6x$; $(\sqrt{x} \cdot \sqrt{x}) + (\sqrt{x} \cdot \sqrt{4x}) + (\sqrt{x} + \sqrt{9x}) = \left(\sqrt{x^2} + \sqrt{4x^2} + \sqrt{9x^2}\right) = x + 2x + 3x = 6x$

53c. $10x$; $(\sqrt{x} \cdot \sqrt{x}) + (\sqrt{x} \cdot \sqrt{4x}) + (\sqrt{x} \cdot \sqrt{9x}) + (\sqrt{x} \cdot \sqrt{16x}) = \left(\sqrt{x^2} + \sqrt{4x^2} + \sqrt{9x^2} + \sqrt{16x^2}\right) = x + 2x + 3x + 4x = 10x$

53d. $(\sqrt{x} \cdot \sqrt{x}) + (\sqrt{x} \cdot \sqrt{4x}) + \ldots + (\sqrt{x} + \sqrt{n^2x}) = x + 2x + \ldots + nx = (1 + 2 + \ldots + n)x = \left(\frac{n(n + 1)}{2}\right)x$

55a. $6\sqrt[3]{2} + 1$; $r = \sqrt[3]{\frac{3}{4\pi} \cdot 72\pi} = \sqrt[3]{54} = 3\sqrt[3]{2}$; $S = 3\sqrt[3]{2} + 2(2)$

55b. No; sample answer: a box with a volume of 384 cm^3 has a side that is $\sqrt[3]{384}$, or about 7.27 cm; $s = 6\sqrt[3]{2} + 4 \approx 11.55$ is the least possible value for a side of the gift box.

57. $a = 1, b = 256$; $a = 2, b = 16$; $a = 4, b = 4$; $a = 8, b = 2$

59. Sample answer: It is only necessary to use absolute values when it is possible that n could be odd or even and still be defined. It is when the radicand must be nonnegative in order for the root to be defined that the absolute values are not necessary.

Lesson 4-6

1. $\frac{1}{25}$ **3.** 8 **5.** 40 **7.** 11 **9.** 11 **11.** $-\frac{1}{2}$

13. no real solution; 16 is an extraneous solution.

15. 83 **17.** 61 **19.** 15,623 **21.** 43

23. 2.4 **25.** 2 **27.** 1.23 **29.** 36 cm

31. $\frac{1}{4}$ **33.** $\frac{81}{16}$

35. 2 **37.** 24.5

39. 3, 4 (extraneous solution)

41. 6, −5 (extraneous solution)

43. 12 units

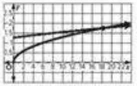
45a. $\sqrt{x^2 + \frac{289}{x^2}}$

45b. $\frac{17}{12} \leq x \leq 12$

47a. 20.2

47b. Silver; $5.713 \times 10^{-15} \approx (1.2 \times 10^{-15})(A)^{\frac{1}{3}}$; $4.761 \approx (A)^{\frac{1}{3}}$; $107.917 \approx A$

49. Graph functions $f(x) = \sqrt{\frac{x}{6}}$ and $g(x) = \sqrt{\frac{x + 20}{4\pi}}$, where x is surface area of the cube. The graphs intersect at $x \approx 18.27$. The surface areas of the cube and sphere are approximately 18.27 cm^2 and 38.27 cm^2.



51. $\sqrt{x + 2} - 7 = -10$; Sample answer: This equation does not have a real solution while the other three equations do.

53. never;

$$\frac{\sqrt{(x^2)^2}}{-x} = x$$
$$\frac{x^2}{-x} = x$$
$$x^2 = (x)(-x)$$
$$x^2 \neq -x^2$$

55. Never; sample answer: The radicand can be negative.

Module 4 Review

1. D

3. C

5. $(f \circ g)(x) = 12x + 8$

7. If the domain of $f(x)$ is restricted to $(-\infty, -2]$, then the inverse is $f^{-1}(x) = -2 - \sqrt{x + 1}$. If the domain of $f(x)$ is restricted to $[-2, \infty)$, then the inverse is $f^{-1}(x) = -2 + \sqrt{x + 1}$.

9. D

11. The parent function is $f(x) = \sqrt{x}$ and $g(x) = a\sqrt{x - h} + k$ where $a = -2$, $h = -5$, and $k = -3$. Because $a < 0$ and $|a| > 1$, the graph of $g(x)$ is the graph of the parent function reflected across the x-axis and stretched vertically by a factor of $|a|$, or 2. Also, because $h < 0$, the graph of $g(x)$ is the graph of the parent function translated left $|h|$ units, or 5 units. Finally, because $k < 0$, the graph of $g(x)$ is the graph of the parent function translated down $|k|$ units, or 3 units.

13. B **15.** $2\sqrt{3}$ **17.** B **19.** A

Module 5

Module 5 Opener

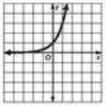
1. a^{12} **3.** $\frac{-3x^6}{2y^3z^5}$

5. $f^{-1}(x) = \frac{1}{2}x - \frac{5}{2}$

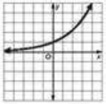
7. $f^{-1}(x) = -\frac{1}{4}x$

Lesson 5-1

1. Domain: all real numbers; Range: all positive real numbers; y-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$

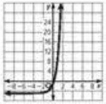


3. Domain: all real numbers; Range: all positive real numbers; y-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$

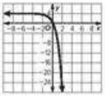


5.

7.



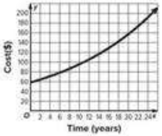
9.



11. $k = -3$; $g(x) = 2^x - 3$

13. $k = -1$; $g(x) = -\left(\frac{3}{2}\right)^x$

15. about \$201.73

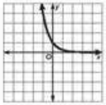


17. exponential growth

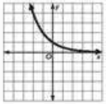
19. exponential decay

21. exponential decay

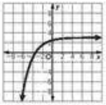
23. Domain: all real numbers; Range: all positive real numbers; y-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to 0$



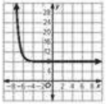
25. Domain: all real numbers; Range: all positive real numbers; y-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to 0$



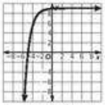
27.



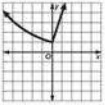
29.



31.

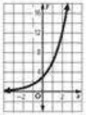


33a.

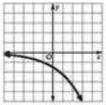


33b. Both $f(x)$ and $g(x)$ have a domain of all real numbers. $f(x)$ has a range of $y \geq 1$ and $g(x)$ has a range of $y > -1$, so the range of $f(x)$ is greater than the range of $g(x)$. $f(x)$ has no x-intercept and $g(x)$ has an x-intercept at (0, 0). $f(x)$ has a y-intercept at (0, 1) and $g(x)$ has a y-intercept at (0, 0), so the y-intercept of $f(x)$ is greater than the y-intercept of $g(x)$. $f(x)$ is increasing when $x > 0$ and $g(x)$ is increasing for all values of x. $f(x)$ is decreasing when $x < 0$ and $g(x)$ is never decreasing. $f(x)$ is positive for all values of x and $g(x)$ is positive when $x > 0$. $f(x)$ is never negative and $g(x)$ is negative when $x < 0$. $f(x)$ has a minimum at (0, 1) and $g(x)$ has a minimum slightly greater than −1, so the minimum of $f(x)$ is greater than the minimum of $g(x)$. Neither $f(x)$ nor $g(x)$ are symmetric. The end behavior of $f(x)$ is: As $x \to -\infty$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to \infty$. The end behavior of $g(x)$ is: As $x \to -\infty$, $g(x) \to -1$ and as $x \to \infty$, $g(x) \to \infty$.

35. D = {all real numbers}, R = $\{y \mid y > 0\}$; exponential growth



37. D = {all real numbers}, R = $\{y \mid y < 0\}$; exponential growth



39. $g(x) = -4^x$

41a. Sample answer: Choose $a = 2$, so $y = 2(1)^x$. All values for y are 2. Students should indicate that $y = a$ (regardless of their choice of a) is a constant function, not an exponential one, since the value of y never changes.

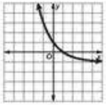
41b. Sample answer: Choose $a = 2$ and $b = -2$, so $y = 2(-2)^x$. The table of values for y are $-\frac{1}{4}$, $\frac{1}{2}$, −1, 2, −4, 8, −16. Students should indicate that this function is not exponential. Since values of y alternate signs, the function does not continuously increase or decrease.

43a. $V(t) = 28{,}000(0.85)^t$;

43b. Average annual value lost during Years 0–4 $= \frac{28{,}000(0.85)^0 - 28{,}000(0.85)^4}{5} =$ \$2676.77; Average annual value lost during Years 11–15: $\frac{28{,}000(0.85)^{11} - 28{,}000(0.85)^{15}}{5} =$ \$447.94

Selected Answers

45. Reflect $f(x)$ in the y-axis and translate the result one unit up to obtain the graph of $g(x)$. The y-intercept of $g(x)$ is (0, 2), which was translated up 1 unit from the y-intercept of $f(x)$, (0, 1). $g(x)$ is decreasing on its entire domain, $f(x)$ is increasing on its entire domain. Both $g(x)$ and $f(x)$ are positive on its entire domain. The asymptote of $g(x)$ is $y = 1$, which was translated up one unit from the asymptote of $f(x)$, $y = 0$. For $g(x)$ the end behavior is as $x \to -\infty$, $g(x) \to \infty$, as $x \to \infty$, $g(x) \to 1$. The end behavior of $f(x)$ is as $x \to -\infty$, $f(x) \to 0$, as $x \to \infty$, $f(x) \to \infty$.



47a. A is the amount in grams, t is time in years, $A(t) = 27.3(0.9)^t$

47b. rate of change for [0, 2] = −2.5935, rate of change for [3, 5] ≈ −1.3891; The amount of the compound decreases quickly at first and then more slowly as time passes.

49a. Always; sample answer: The domain of exponential functions is all real numbers, so (0, y) always exists.

49b. Sometimes; sample answer: The graph of an exponential function crosses the x-axis when $k < 0$.

49c. Sometimes; Sample answer: The function is not exponential if $b = 1$ or -1.

51. about 251 mg

53. Sample answer: The parent function, $g(x) = b^x$, is stretched if a is greater than 1 or compressed if a is less than 1. The parent function is translated up k units if k is positive and down $|k|$ units if k is negative. The parent function is translated h units to the right if h is positive and $|h|$ units to the left if h is negative.

Lesson 5-2

1. 4
3. 19
5. 2
7a. $y = 5000(1.05)^x$
7b. 33.0 years
9a. $y = 23(0.995)^x$
9b. 85 years
11. $1476.79
13. $5309.08
15. $x < 1$
17. $x \geq -2$
19. $x > -73$
21. 0
23. $a \leq -4$
25. $\frac{5}{3}$
27. $-\frac{2}{5}$
29. $c > \frac{3}{4}$
31. $\frac{11}{8}$
33. − 7
35. − 6
37. 1
39. 12
41. $t \geq -1$

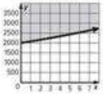
43a. Ingrid: $y = 2(2.924)^x$; Alberto: $y = 32(1.383)^x$

43b. Ingrid: 427; Alberto: 162

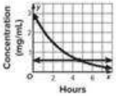
43c. Sample answer: No; a business cannot grow exponentially indefinitely.

45. $y = -1(2)^x$
47. $y = \frac{1}{2}(9)^x$
49. $y = -6\left(\frac{1}{4}\right)^x$
51. $y = (4)^x$

53.



55. No; A function to model the concentration is $f(t) = 3(0.5)^{0.5t}$. The intersection with $g(t) = 0.6$ occurs at $t \approx 4.6$, meaning the concentration drops below the effective level before 3 P.M.



57a. Sample answer: According to the US Census Bureau and the American Association of University Professors, the average salary for a professor in 2017 was $100,100. After 15 years, the professor's annual salary would be $100{,}100(1.02)^{15} = \$134{,}721.42$.

57b. Sample answer: It depends on how long the professor intends to work. For years 1–16, the average salary with a 2% raise earns more. After year 17, the lower starting salary with a 3% raise earns more. The professor should consider the cumulative salary over the time he or she intends to work to determine which is better.

59. Beth; sample answer: Liz added the exponents instead of multiplying them when taking the power of a power.

61. Reducing the term will be more beneficial. The multiplier is 1.3756 for the 4-year and 1.3828 for the 6.5%.

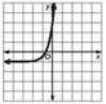
63. Sample answer: $4^x \leq 4^2$

65. Sample answer: Divide the final amount by the initial amount. If n is the number of time intervals that pass, take the nth root of the answer.

Lesson 5-3

1. e^{11}
3. e^2
5. $27e^{12x}$
7. $2e^3$
9. $-8e^6$

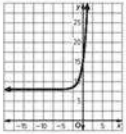
11a.



11b. The domain is all real numbers. The range is all real numbers greater than −1.

11c. ≈ 0.266
13a. $A = 5000e^{0.025t}$
13b. $5956.23
15a. $A = 12{,}750e^{0.055t}$
15b. $20,916.35
17. 0.17
19. 0.89
21. −1.29
23. $e^{-\frac{1}{4}}$
25. $-\frac{8}{125}e^{\frac{3}{4}x}$
27. $-24e^2$
29. −1.69
31. $f(x) = 900e^{0.012(5)}$; $955.65

33.



Domain: all real numbers; Range: all real numbers greater than 8; y-intercept: 15; no zeros; asymptote: $y = 8$; end behavior: as $x \to -\infty$, $f(x) \to 8$, as $x \to \infty$, $f(x) \to \infty$

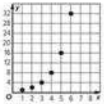
35. Final amount corresponds to final population, initial amount corresponds to initial population, interest rate corresponds to growth rate, and time of growth is the same for both equations.

37. Sample answer: Since e is irrational, evaluating any expression with e will approximate its value, which introduces error. To minimize the error, do as much work as possible with the exact value (e) before evaluating.

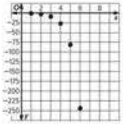
39. $g(x) = 2e^x + 9$ because it has a range of all real numbers greater than 9 and the other two functions have a range of all real numbers greater than 0.

Lesson 5-4

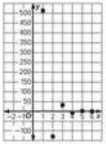
1. not geometric
3. geometric
5. not geometric
7. 1, 2, 4, 8, 16, 32



9. −1, −3, −9, −27, −81, −243



11. 512, −128, 32, −8, 2, −0.5



13. 32 **15.** $a_n = 3(3)^{n-1}$

17. $a_n = 2(-3)^{n-1}$ **19.** $a_n = 12(3)^{n-1}$

21. $a_n = 3(0.6)^{n-1}$ **23.** $a_n = -\left(\frac{1}{2}\right)^{n-1}$

25. ±8, 16, ±32 **27.** 1368

29a. The heights in feet after the first 6 seconds are 1000, 2010, 3030.1, 4060.401, 5101.00501, and 6152.01506.

29b. Yes; the height after 60 minutes is about 81,669.7 ft.

29c. 6.34×10^{29} **31.** $\frac{1}{2}$

33. −2 **35.** 255

37. 0 **39.** 4374

41. −5120 **43.** −0.2

45. geometric; $a_n = 25(7)^{n-1}$

47. geometric; $a^n = -15(-2)^{n-1}$

49. 31.9375 **51.** 9707.8189

53. 3, −6, 12, −24, 48 **55.** 1875

57. $a_n = 60\left(\frac{1}{2}\right)^{n-1}$ **59.** $\pm\frac{243}{16}, \frac{81}{4}, \pm 27$

61. −8, 4

63. Sample answer: The second option; receiving 20 annual payments represents a geometric series, and $S = 4{,}000{,}000\frac{1-(1.02)^{20}}{1-(1.02)}$ or $97,189,479.20. By choosing the annual payments, the winner will receive approximately $17 million more than by choosing the single payout.

65a. $347,514.00

65b. $206,000

67. Sample answer: $256 + 192 + 144 + 108 + 81 + \frac{243}{4}$

69. Sample answer:
Let a_n = the nth term of the sequence and r = the common ratio.

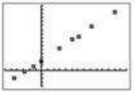
$a_2 = a_1 \cdot r$	Definition of the second term of a geometric sequence
$a_3 = a_2 \cdot r$	Definition of the third term of a geometric sequence
$a_3 = a_1 \cdot r \cdot r$	Substitution
$a_3 = a_1 \cdot r^2$	Associative Property of Multiplication
$a_3 = a_1 \cdot r^{3-1}$	$3 - 1 = 2$
$a_n = a_1 \cdot r^{n-1}$	$n = 3$

71. x^2y^2

73. Sample answer: A series is arithmetic if every pair of consecutive terms shares a common difference. A series is geometric if every pair of consecutive terms shares a common ratio. If the series displays both qualities, then it is both arithmetic and geometric. If the series displays neither quality, then it is neither geometric nor arithmetic.

Lesson 5-5

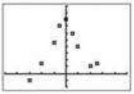
1.



[−6, 14] scl: 1 by [−4, 16] scl: 1

linear

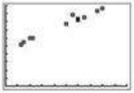
3.



[−5, 5] scl: 1 by [−2, 10] scl: 1

quadratic

5a.

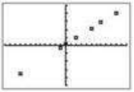


[0, 10] scl: 1 by [0, 100] scl: 10

linear; $y = 6.4x + 45.85$

5b. ≈2.2 hours

7.



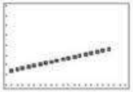
[−12, 12] scl: 1 by [−50, 50] scl: 10

linear

9. strong, positive

11. weak, positive

13.



[1999, 2020] scl: 1 by [40, 80] scl: 5

The scatter plot is linear, with a positive direction, and a strong correlation.

15. Enter the x-values in the table in L1 in a calculator. Enter the y-values in the table in L2 in a calculator. Perform linear, quadratic, and exponential regressions. Then compare the coefficients of determination, r^2. The function with a coefficient of determination closest to 1 will fit the data best.

17. quadratic; $SA = 3\pi r^2$

Module 5 Review

1. A

3. A, B, F

5. D

7. Sample answer: The x-coordinate of the intersection of the graphs will give the approximate solution of the equation.

9. C

11. 18.21

13. $a_n = 10\left(\frac{1}{2}\right)^{n-1}$

15. C

17. A

Module 6

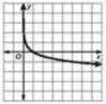
Module 6 Opener

1. \$2800 **3.** \$1674 **5.** $x = 8y + 20$
7. $x = \frac{y - 9}{-4}$

Lesson 6-1

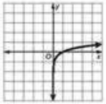
1. $15^2 = 225$ **3.** $5^{-2} = \frac{1}{25}$ **5.** $4^3 = 64$
7. $\log_2 128 = 7$ **9.** $\log_7 \frac{1}{49} = -2$
11. $\log_2 512 = 9$
13. 6 **15.** 4 **17.** $-\frac{5}{2}$ **19.** 3 **21.** $\frac{2}{3}$
23. $\log_{1.0065} \frac{y}{400} = x$

25.



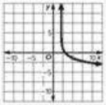
x-intercept: 1; no y-intercept; domain: $(0, \infty)$ range: $(0, \infty)$; end behavior: As $x \to 0^+$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to \infty$.

27.

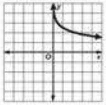


x-intercept: 1; no y-intercept; domain: $(0, \infty)$ range: $(0, \infty)$; end behavior: As $x \to 0^+$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$.

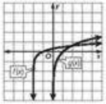
29.



31.



33a.



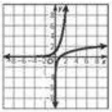
33b. $f(x)$: As $x \to -2+$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$.
$g(x)$: As $x \to 0+$, $g(x) \to -\infty$, and as $x \to \infty$, $g(x) \to \infty$.

35. $k = -1$; $g(x) = -(\log_5 x)$ **37.** $6^3 = 216$

39. $3^{-4} = \frac{1}{81}$ **41.** $\log_{\frac{1}{4}} \frac{1}{64} = 3$

43. 7.8 **45a.** Julio's; sample answer: I can use the graph to determine that $f(7) \approx 28$ and $g(7) \approx 1.7$. **45b.** Neither; sample answer: both $f(1)$ and $g(1)$ equal 0.

47.



$f(x)$: D: $(-\infty, \infty)$ R: $(0, \infty)$; y-int: (0, 1); asymptote: $y = 0$; $g(x)$: D: $(0, \infty)$ R: $(-\infty, \infty)$; x-int: (1, 0); asymptote: $x = 0$; The graphs have an inverse relationship with one another. They are symmetric about the line $y = x$.

49. $f(g(x)) = x$ and $g(f(x)) = x$, so $f(x) = \log_{10}(x)$ and $g(x) = 10^x$ are inverse of each other.

51. $\log_7 51$; sample answer: $\log_7 51$ equals a little more than 2. $\log_8 61$ equals a little less than 2. $\log_9 71$ equals a little less than 2. Therefore, $\log_7 51$ is the greatest.

53. No; sample answer: Elisa was closer. She should have $-y = 2$ or $y = -2$ instead of $y = 2$. Matthew used the definition of logarithms incorrectly.

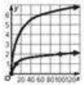
Lesson 6-2

1. 8 **3.** 3125 **5.** 9 **7.** 2 **9.** 4 **11.** $\frac{1}{3}$ or $\frac{1}{2}$
13. 2.4535 **15.** −0.2925 **17.** 1.5850
19. 4.6438 **21.** 6.9657 **23.** 4.755 **25.** 10^{11}
27. 8 **29.** 2 **31.** 2 **33.** −10 **35.** 2 of $-\frac{1}{3}$
37. 2.2921 **39.** −0.1788 **41.** 1.3652 **43.** 1.1
45. $\log F = \log k + \log m_1 + \log m_2 - 2 \log d$
47a. $E = 1.4 \log_{10} \frac{C_2}{C_1}$
47b. about 0.6679 kilocalories per mole
47c. about 0.4214 kilocalories per mole
49. Rewrite the equations as $3 \cdot \log_2 a + 2 \cdot \log_2 b = 19$ and $4 \cdot \log_2 a - 5 \cdot \log_2 b = 10$. Solve the system of equation to get $\log_2 a = 5$ and $\log_2 b = 2$. Then $a = 32$ and $b = 4$.
51. true **53.** true **55.** false **57.** true **59.** 4
61. $\frac{8}{5}$

Lesson 6-3

1. 1.2553 **3.** 2.0792 **5.** 1.6263 **7.** about 5.6 h
9. 0.5209 **11.** 1.3917 **13.** −0.3869
15. $\{x \mid x \geq 0.8549\}$ **17.** $\{x \mid x > 5\}$
19. $\{y \mid y > -2.6977\}$
21. $\frac{\log 22}{\log 4}$, 2.2297
23. $\frac{\log 50}{\log 2}$, 5.6439
25. $\frac{\log 2}{\log 3}$, 0.6309 **27.** \$16,000 **29.** ±2.3785
31. 0.9177 **33.** $\{p \mid p \leq 2.9437\}$
35. $\frac{2 \log 20}{2}$, 5.4537
37. $\frac{5 \log 4}{\log 8}$, 3.3333 **39.** $\frac{6 \log 3.6}{2}$, 11.0880
41. $\frac{\log 150}{2 \log 3}$, 2.2804 **43.** $\frac{\log 1600}{4 \log 5}$, 1.1460

45.



47a. $\log 4 = \log 2^2 = 2 \log 2 \approx 2 \cdot 0.3010 \approx 0.6020$, $\log 6 = \log (2 \cdot 3) = \log 2 + \log 3 \approx 0.3010 + 0.4771 \approx 0.7781$

47b. Sample answer:

$$\log 1.5 = \log 3 + \log 5 - \log 10$$
$$\approx 0.4771 + 0.6989 - 1$$
$$\approx 0.1760$$

49a. $3 \log_2 (2x) - \log_4 (3x) = 1$

$$3\left[\frac{\log(2x)}{\log(2)}\right] - \frac{\log(3x)}{\log(4)} = 1$$
$$\frac{3 \log(2x)}{\log(2)} - \frac{\log(3x)}{2 \log(2)} = 1$$
$$\frac{6 \log(2x) - \log(3x)}{2 \log(2)} = 1$$
$$6(\log (2) + \log (x)) - \log (3) - \log (x) = 2 \log (2)$$
$$5 \log (x) = \log (3) - 4 \log (2)$$

So, $x \approx 0.715$.

49b. $\log_6 (x^2) + \log_3 (x) = 3$

$$\frac{\log(x^2)}{\log(6)} + \frac{\log(x)}{\log(3)} = 3$$
$$\frac{\log(3) \log(x^2) + \log(6) \log(x)}{\log(3) \log(6)} = 3$$
$$2 \log (3) \log (x) + \log (6) \log (x) = 3 \log (3) \log (6)$$
$$\log (x) [2 \log (3) + \log (6)] = 3 \log (3) \log (6)$$

So, $x \approx 4.39$.

49c. $2 \log_2 (3x) = -8 \log_3 (x)$

$$2\left[\frac{\log(3x)}{\log(2)}\right] = -8\left[\frac{\log(x)}{\log(3)}\right]$$
$$2 \log (3) \log (3x) = -8 \log (2) \log (x)$$
$$2 \log (3)[\log (3) + \log (x)] = -8 \log (2) \log (x)$$
$$2 \log (x)[2 \log (3) + 8 \log (2)] = -2[\log (3)]^2$$

So, $x \approx 0.732$.

51. Rosamaria is correct. Sample answer: Sam forgot to bring the 3 down from the exponent when he applied the Property of Equality for Logarithms.

53. $\log_3 27 = 3$ and $\log_{27} 3 = \frac{1}{3}$;

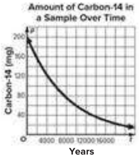
Conjecture: $\log_a b = \frac{1}{\log_b a}$

Proof:

$\log_a b \overset{?}{=} \frac{1}{\log_b a}$ Original statement
$\frac{\log_b b}{\log_b a} \overset{?}{=} \frac{1}{\log_b a}$ Change of Base Formula
$\frac{1}{\log_b a} = \frac{1}{\log_b a}$ Inverse Property of Exponents and Logarithms

Lesson 6-4

1. $\ln x = 15$ **3.** $-5x = \ln 0.2$ **5.** $x = \ln 3$
7. $\ln (x - 3) = 4$ **9.** $5 = \ln 10x$ **11.** $e^{2x} = 36$
13. $x = e^8$ **15.** $e^x = 0.0002$ **17.** $e^x = 15$
19. ln 3 **21.** $\ln 16x^2$ or $2 \ln 4x$ **23.** ln 5
25. $-8 \ln \frac{1}{3}$ **27.** $\ln \frac{x^8}{625}$ **29.** 1.0986
31. 1.7579 **33.** 2.7081 **35.** 18.0855
37. 36.7493 **39.** 2.4630
41a. \$813.28 **41b.** about 36.3 years
41c. \$873.72 **43.** $y = 804.30 - 52.90 \ln x$; about 624 grams **45.** $y = 2079.08 + 75.85 \ln x$; about 228,940 visitors **47.** $\frac{3}{5} \ln x$
49. 1.3900 **51.** about 24.75 years
53. $\ln(5 \cdot 5 \div 8) = \ln 5 + \ln 5 - \ln 8 = 1.1394$
55a. $P = P_0 e^{kt}$; $0.5P_0 = P_0 e^{5730k}$; $0.5 = e^{5730k}$; $\ln (0.5) = 5730k (\ln e)$; $k = \frac{\ln 0.5}{5730}$
55b. $P = 200e^{\left(\frac{\ln 0.5}{5730}\right)t}$



55c. Sample answer: at least about 7500 years. Solve inequality: $P \leq 200e^{\frac{\ln 0.5}{5730}t}$; $80 \leq 200e^{\frac{\ln 0.5}{5730}t}$; $0.4 \leq e^{\frac{\ln 0.5}{5730}t}$; $\ln (0.4) \leq \frac{\ln 0.5}{5730}t \cdot \ln (e)$; $t \geq \frac{5730 \cdot \ln 0.4}{\ln 0.5}$; ≈ 7575 years

57. Disagree; $e^{e^x} = x$ can be rewritten as $e^x(\ln e) = \ln x$, which can be written as $e^x = \ln x$. The functions $y = e^x$ and $y = \ln x$ are inverse functions which have no point of intersection so there are no values of x for which $e^{e^x} = x$.

59. Let $p = \ln a$ and $q = \ln b$. That means that $e^p = a$ and $e^q = b$.

$ab = e^p \times e^q$
$ab = e^{p + q}$
$\ln(ab) = (p + q)$
$\ln(ab) = \ln a + \ln b$

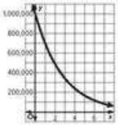
Lesson 6-5

1a. $y = 6.124e^{0.01238t}$ **1b.** 2016
3a. about 0.02166 or about 2.166%; $y = ae^{-0.02166t}$

3b. about 0.57 g **5.** about 12,618 years old
7a. $k \approx 0.071$ **7b.** about 60.8 min **7c.** about 30.85 min **9.** about 6.0 yr
11a. $y = 67{,}387e^{0.0198t}$ **11b.** about 9304
13. $P = 8e^{0.26t}$; about 4.4 years
15. Sample answer: Based on the data in the table, the decay constant is about -0.000124, which is approximately equal to the given constant. $f(t) = 1000e^{-0.000124t}$; Use the exponential model: $f(t) = ae^{kt}$; $999.628 = 1000^{k(3)}$; $0.999628 = e^{3k}$; $\ln 0.999628 = \ln e^{3k}$; $\ln 0.999628 = 3k$; $k = -0.000124$; Verify: $f(2) = 1000e^{-0.000124(2)} = 999.752$
17a.

t (min)	Surviving Cells After t Minutes	$(t, f(t))$
0	initial amount	(0, 1,000,000)
1	(0.70)(1,000,000) = 700,000 survive	(1, 700,000)
2	(0.70)(700,000) = 490,000	(2, 490,000)
3	(0.70)(490,000) = 343,000	(3, 343,000)

Sample answer: An exponential model best describes the points because the number of cells decreases by the same percentage every minute.

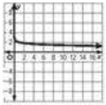


17b. In the model $f(t) = ae^{kt}$, $f(t)$ represents the number of cells remaining after t minutes, a represents the number of cells at the start of the experiment, t represents the number of minutes since the experiment began, and k is the growth or decay constant. $f(t) = 1{,}000{,}000e^{kt}$.

17c. $k \approx -0.356675$; Because k is negative, it is exponential decay. Substitute known values into the formula: $700{,}000 = 1{,}000{,}000e^{k(1)}$; $0.7 = e^k$; $\ln 0.7 = k$; $k \approx -0.356675$; The exponential decay model for this experiment is $f(t) = 1{,}000{,}000e^{-0.356675t}$.
17d. It will take approximately 19.367 minutes to have less than 1000 cells. Write the model and solve for t; $1000 = 1{,}000{,}000e^{-0.356675t}$; $0.001 = e^{-0.356675t}$; $\ln 0.001 = -0.356675t$; $t \approx 19.367$.
19. $t \approx 113.45$
21. Sample answer: Exponential functions can be used to model situations that incorporate a percentage of growth or decay for a specific number of times per year. Continuous exponential functions can be used to model situations that incorporate a percentage of growth or decay continuously.

Module 6 Review

1a. $\log_6 1296 = 4$ **1b.** $\log_{24} 27 = \frac{3}{5}$
1c. $\log_8 \frac{1}{64} = -2$
3.



5. Sample answer: If I divide 15,309 by 7, the result is 2,187, or 37. So, the expression that needs to be approximated can be written as $\log_3 7 \cdot 3^7$. Then, using the Product Property of Logarithms, the expression can be rewritten again as $\log_3 7 + \log_3 3^7$. I know that $\log_3 3^7 = 7$, so the approximate value of $\log_3 15{,}309$ is about 1.7712 + 7 or 8.7712.
7. D **9.** C, E **11.** B **13a.** $\ln 8x = 7$
13b. $\ln 17 = 2x$ **15.** B **17.** A **19.** 365 days

Module 7

Module 7 Opener

1. $x = \frac{15}{14}$ **3.** $M = \frac{56}{3}$ **5.** $-\frac{1}{8}$ **7.** $\frac{29}{30}$

Lesson 7-1

1. $\frac{x(x+6)}{x+4}$; $x = -4, 3$ **3.** $\frac{(x+3)(x-z)}{4}$; $x = -z, 3$

5. $\frac{x(x+2)}{6(x+5)}$; $x = -5, 0, 4$ **7.** $-\frac{x+2}{x+4}$ **9.** $-\frac{1}{x+6}$

11. $\frac{c}{4ab^2f^2}$ **13.** $\frac{32b}{3ac^3f^2}$ **15.** $\frac{5a^4c}{3b}$ **17.** $\frac{(4a+5)(a-4)}{3a+2}$

19. $\frac{t+12}{2(t+2)}$ **21.** $-\frac{6}{z}$ **23.** $\frac{b-a}{x+y}$ **25.** $\frac{c+2}{c(c-d)}$

27. $\frac{(a+1)(a-2)}{4(a-5)(a-1)}$ **29.** $\frac{x(x+2)(x-1)}{(x+3)(x-7)}$ **31.** $\frac{15y^3}{4a^2cxz}$

33. $\frac{x-4}{-4(x-3)}$ **35.** $\frac{(4x-1)^2(3x+1)(x+1)}{12(x+2)(x-4)(x^2-10x+6)}$

37. $\frac{48H+45c}{c+H}$ **39.** $\frac{r}{3}$

41a. The height can be found by dividing the volume by the product of the length and width of the box.

41b. $(x + 3)$ in.

41c. Sample answer: Substitute a value for x in each of the given expressions for the length, width, and volume, and the same value for x in the expression found for h, and then check that $V = \ell wh$.

CHECK: For $x = 5$,
length $= (5) + 10 = 15$ in.
width $= 2(5) = 10$ in.
volume $= 2(5)^3 + 26(5)^2 + 60(5) = 1200$ in^3
height $= (5) + 3 = 8$ in.
Verify $V = \ell wh$: $1200 = (15)(10)(8)$

43. By dividing the second week's speed by the first week's speed, you can determine how much faster she hopes to run on average. $\frac{60}{t-2} \div \frac{60}{t} = \frac{60}{t-2} \cdot \frac{t}{60} = \frac{t}{t-2}$. She hopes to run $\frac{t}{t-2}$ times faster during her second week than during her first week.

45a. C : B $= \frac{(x+8)}{(x+5)}$, B : A $= \frac{(x+5)}{x}$; For these two ratios to be equal, $\frac{C}{B} \cdot \frac{A}{B}$ must equal 1. To equal 1, their product must have the numerator equal to the denominator. So, $\frac{(x+8)}{(x+5)} \cdot \frac{x}{(x+5)} = \frac{x^2+8x}{(x+5)^2}$ and $x^2 + 8x = (x+5)^2$. Solving for x results in $x = -12.5$. Since x cannot be a negative value, there are no values of x that will make these ratios equal.

45b. Yes; C : B $= \frac{(x+6)}{(x+2)}$, B : A $= \frac{(x+2)}{x}$ and $\frac{(x+6)}{(x+2)} \cdot \frac{x}{(x+2)} = \frac{x^2+6x}{(x+2)^2}$. When $x^2 + 6x = (x+2)^2$, $x = 2$. The length of a side of square A would need to be 4 feet, and the fountain would need to be 2 feet in diameter.

47. Beverly; sample answer: Troy's mistake was multiplying by the reciprocal of the dividend instead of the divisor.

49. $\frac{x+1}{\sqrt{x+3}}$; The other three expressions are rational expressions. Since the denominator of $\frac{x+1}{\sqrt{x+3}}$ is not a polynomial, $\frac{x+1}{\sqrt{x+3}}$ is not a rational expression.

51. Sample answer: $\frac{x^2-1}{x^2+5x+4}$

53. Sample answer: $\frac{3a}{3a-15}$, $\frac{a^2}{a^2-5a}$, $\frac{a(a+1)}{(a-5)(a+1)}$

Lesson 7-2

1. $\frac{5x+3y}{xy}$ **3.** $\frac{2c+5}{3}$ **5.** $\frac{12z-2y}{5y^2z}$

7. $\frac{3w+7}{(w-3)(w+3)}$ **9.** $\frac{2k}{k-n}$ **11.** $\frac{n+2}{n-3}$ **13.** $\frac{1}{r(1-r)}$

15a. $\frac{100}{r}$, $\frac{100}{r+\frac{1}{2}}$, $\frac{100}{r-\frac{1}{2}}$, $\frac{100}{r-1}$

15b. $\frac{100(16r^3-12r^2-2r+1)}{4r^4-4r^3-r^2+r}$ **15c.** 10.97 m/s

17. $\frac{13x+21}{-3x+73}$ **19.** $\frac{-x^2+33x+16}{12x^2+11x-27}$

21. $\frac{15bd-6b-2d}{3bd(3b+d)}$ **23.** $\frac{2x^2+5x-2}{(x-5)(x+2)}$

25. $\frac{28by^2z-9bx}{105x^3y^4z}$

27. $\frac{20x^2y+120y+6x^2}{15x^3y}$ or $\frac{2(x^2y+60y+3x^2)}{15x^3y}$

29. $\frac{15b^3+100ab^2-216a}{240ab^3}$

31. $\frac{10y-4}{(y-7)(y+5)(y+4)}$ or $\frac{2(5y-2)}{(y-7)(y+5)(y+4)}$

33. $\frac{-10x-10}{(2x-1)(x+6)(x-3)}$ or $\frac{-10(x+1)}{(2x-1)(x+6)(x-3)}$

35. $\frac{2x^2+32x}{3(x-2)(x+3)(2x+5)}$ or $\frac{2x(x+16)}{3(x-2)(x+3)(2x+5)}$

37. Sample answer: Lord Brouncker is best known for the continued fraction approximation of π, $\pi = \frac{4}{1+\frac{1^2}{2+\frac{3^2}{2+\frac{5^2}{2+\frac{7^2}{\dots}}}}}$.

39. $\frac{15x^2-192x^2y^2-128y}{48x^2y^2}$ **41.** $\frac{19x-36}{12(2x+1)(x-3)(x+4)}$

43. $\frac{x^2+2x-29}{(x-1)(x-8)}$ **45.** $\frac{1}{y-x}$ **47.** $-\frac{4}{5}$

49. The average is the sum of the given numbers divided by the number of given numbers. So, the average of $\frac{1}{x}$, $\frac{1}{x-3}$, and $\frac{1}{2x}$ is $\frac{\frac{1}{x}+\frac{1}{x-3}+\frac{1}{2x}}{3} = \frac{\frac{1(2)(x-3)}{x(2)(x-3)}+\frac{1(2)(x)}{(x-3)(2)(x)}+\frac{1(x-3)}{2x(x-3)}}{3} = \frac{\frac{2(x-3)}{2x(x-3)}+\frac{2x}{2x(x-3)}+\frac{x-3}{2x(x-3)}}{3} = \frac{\frac{2(x-3)+2x+x-3}{2x(x-3)}}{3} = \frac{2(x-3)+2x+x-3}{6x(x-3)} = \frac{5x-9}{6x(x-3)}$ for $x \neq 0$ and $x \neq 3$.

51. $\frac{8x^3+2x^2+8x+10}{(x+1)(2x^2-1)}$

53. In 1 hour, Dell will plant $\frac{1}{3}$ of a flower bed, or $\frac{t}{3}$ flower beds in t hours. In 1 hour, Max will plant $\frac{1}{4}$ of a flower bed, or $\frac{t}{4}$ flower beds in t hours. Adding the two expressions, $\frac{t}{3} + \frac{t}{4} = \frac{7t}{12}$. So, $\frac{7t}{12}$ represents how many flowers beds will be planted in t hours.

55a. $\frac{x}{x+1}$, $x \neq 0$ **55b.** $\frac{x+1}{2x+1}$, $x \neq 0, -1$

55c. $\frac{2x+1}{3x+2}$, $x \neq 0, -1, -\frac{1}{2}$ **55d.** $\frac{3x+2}{5x+3}$, $x \neq 0, -1, -\frac{1}{2}, -\frac{2}{3}$ **55e.** $\frac{1}{2}; \frac{2}{3}; \frac{3}{5}; \frac{5}{8}$ **55f.** The values are ratios of two consecutive Fibonacci numbers. The next value should be $\frac{8}{13}$.

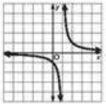
57. Sample answer: The set of rational expressions is closed under all of these operations because the sum, difference, product, and quotient of two rational expressions is a rational expression.

59. Sample answer: First, factor the denominators of all of the expressions. Find the LCD of the denominators. Convert each expression so they all have the LCD. Add or subtract the numerators. Then simplify. It is the same.

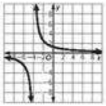
Lesson 7-3

1. $x \neq 0$ **3.** $x \neq 3$ **5.** $x \neq -\frac{3}{2}$

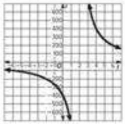
7. asymptotes: $x = 1$, $f(x) = 0$; D $= \{x \mid x \neq 1\}$; R $= \{f(x) \mid f(x) \neq 0\}$ The y-intercept is -1.



9. asymptotes: $x = -4$, $f(x) = 0$;
D $= \{x \mid x \neq -4\}$; R $= \{f(x) \mid f(x) \neq 0\}$
The y-intercept is 1.25.

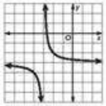


11a. $r = \frac{800}{t-2}$

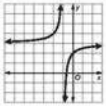


11b. Sample answer: D $= \{t \mid t > 2\}$; R $= \{r(t) \mid r(t) > 0\}$; intercepts: none; positive: when $t > 2$; negative: when $t < 2$; symmetry: symmetric about $(2, 0)$; end behavior: As $t \to -\infty$, $r \to 0$, and as $t \to \infty$, $r \to 0$. Because the plane's speed and travel time cannot be negative, only values in the domain $\{t \mid t > 2\}$ makes sense in the context of this situation.

13. D $= \{x \mid x \neq -3\}$; R $= \{f(x) \mid f(x) \neq -3\}$



15. D $= \{x \mid x \neq -1\}$; R $= \{f(x) \mid f(x) \neq 3\}$



17. $a = -2$, $h = 5$, $k = 0$; $g(x) = \frac{-2}{x-5}$

19. $a = 9, h = -3, k = 6; g(x) = \frac{9}{x+3} + 6$

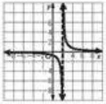
21. $a = -6, h = -4, k = -2; g(x) = \frac{-6}{x+4} - 2$

23. $\frac{5}{2}$ **25.** $-1, 3$

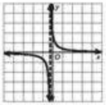
27. $-3, 5$

29. $\{x \mid x \neq -1\}$ and $\{f(x) \mid f(x) \neq 0\}$

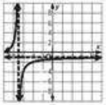
31. asymptotes: $x = 2$, $f(x) = 0$; $D = \{x \mid x \neq 2\}$; $R = \{f(x) \mid f(x) \neq 0\}$



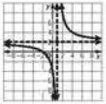
33. asymptotes: $x = \frac{1}{4}$, $f(x) = 0$; $D = \left\{x \mid x \neq -\frac{1}{4}\right\}$; $R = \{f(x) \mid f(x) \neq 0\}$



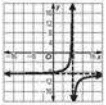
35. asymptotes: $x = -7$, $f(x) = -1$; $D = \{x \mid x \neq -7\}$; $R = \{f(x) \mid f(x) \neq -1\}$



37. asymptotes: $x = 1$, $f(x) = 2$; $D = \{x \mid x \neq 1\}$; $R = \{f(x) \mid f(x) \neq 2\}$



39. asymptotes: $x = 8$, $f(x) = -9$; $D = \{x \mid x \neq 8\}$; $R = \{f(x) \mid f(x) \neq -9\}$

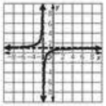


41. $f(x) = \frac{2}{x}$ **43.** $f(x) = \frac{2}{x+6} - 2$

45a. $C = \frac{125}{m} + 0.3$

45b. They cannot travel zero miles or a negative number of miles.

47. The graph of $g(x)$ is a reflection in the x-axis, a translation 2 units left, and a translation 1 unit up of the graph of $f(x)$.

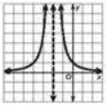


49. The graph of $k(x)$ is a reflection in the x-axis, a translation 2 units left, and a translation 2 units up of the graph of $f(x)$; so, $k(x) = -f(x + 2) + 2 = -\frac{1}{x+2} + 2$.

51. Sample answer: $f(x) = \frac{1}{x+4} + 6$

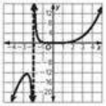
53. Sample answer: $g(x)$; all other choices have unknowns only in the denominator.

55. asymptotes: $x = -2$, $y = 0$

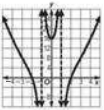


Lesson 7-4

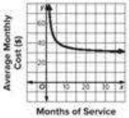
1.



3.



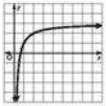
5a.



5b. x-intercept: -2; y-intercept: none; end behavior: As $x \to -\infty$, $f(x) \to 30$ and as $x \to -\infty$, $f(x) \to 30$.

5c. $37.50

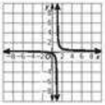
7a.



7b. x-intercept: $\frac{2}{3}$; y-intercept: none; end behavior: As $x \to -\infty$, $f(x) \to 3$ and as $x \to \infty$, $f(x) \to 3$.

7c. ≈ 2.7 cents

9a.

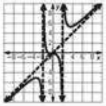


9b. $h(x)$, since $-\frac{1}{4} > -1$

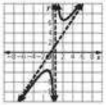
9c. Vertical asymptotes $h(x)$: $x = 1$ $j(x)$: $x = -3$
Horizontal asymptotes $h(x)$: $y = \frac{1}{4}$ $j(x)$: $y = \frac{1}{2}$

11. zero: $x = 4$; vertical asymptote: $x = -2$; oblique asymptote: $f(x) = x - 10$

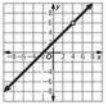
13. zero: $x = -1$; vertical asymptote: $x = -2$ and $x = 2$; oblique asymptote: $f(x) = x$



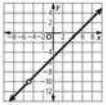
15. zero: none; vertical asymptote: $x = \frac{1}{2}$; oblique asymptote: $f(x) = \frac{3}{2}x + \frac{3}{4}$



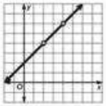
17. point discontinuity at $x = 4$



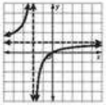
19. point discontinuity at $x = -5$



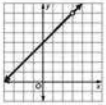
21. point discontinuity at $x = 2$ and $x = 4$



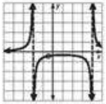
23.



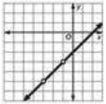
25.



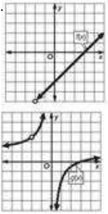
27.



29.



31a.

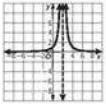


31b. $f(x)$

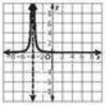
31c. The graph of $f(x)$ has a hole at $x = -2$, but no vertical asymptote. Its graph is a straight line with a hole in it at $(-2, -5)$. The graph of $g(x)$ also has a hole at $x = -2$, but has a vertical asymptote at $x = 0$. Its graph is not a straight line, but two curves having a hole in the graph at $\left(-2, \frac{5}{2}\right)$.

33. $x = 0$

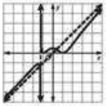
35.



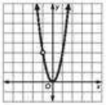
37.



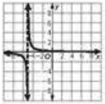
39.



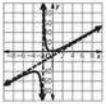
41.



43.



45.



47a. $y = \frac{2067 + 6767}{6767 + x}$

47b. vertical asymptote at $x = -6767$; horizontal asymptote at $y = 0$

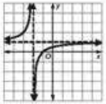
47c. As the number of at bats increases, the ratio grows closer and closer to zero, since the number of hits remains constant.

49. Sample answer: The domain is $\{x \mid x \neq -2$ and $x \neq 2\}$. The graph has zeros at $x = 1$ and $x = 4$ and a y-intercept at $(0, 1)$. The graph has vertical asymptotes at $x = -2$ and $x = 2$ and a horizontal asymptote at $f(x) = -1$ for $2 \leq x$ or $x \leq -2$. A function with these features is $f(x) = -\frac{(x-1)(x-4)}{(x+2)(x-2)} = -\frac{x^2 - 5x + 4}{x^2 - 4}$.

51. Sample answer:

53. Sample answer: If the degrees of the numerator and denominator are equal, there is a horizontal asymptote determined by the coefficients of the numerator and denominator. If the degree of the numerator is greater, there is no horizontal asymptote. If the degree of the numerator is exactly 1 greater than the degree of the denominator, there is an oblique asymptote. If the degree of the denominator is greater, the horizontal asymptote is the line $y = 0$.

55. Sample Answer:



57. The graph of $g(x)$ has a hole at $x = -3$.

59. Sample answer: By factoring the denominator of the expression in a rational function and determining the the values that cause each factor to equal zero, you can find the asymptotes or discontinuity of the function. If the denominator has a factor $x - c$ that does not appear in the numerator, then there is a vertical asymptote at $x = c$. If the numerator and denominator have a common factor of $x - c$, then there is point continuity at $x = c$.

Lesson 7-5

1. 1.5 **3.** −56 **5.** 55 **7.** −48 **9.** 72
11. 90 **13.** −22.5 **15.** 38

17. 4 **19.** −10 **21.** about 0.7 **23.** 0.000793 Ω **25.** 1.336 atm **27.** inverse; 4 **29.** inverse; 15 **31.** direct; 5280 **33.** inverse; −25 **35.** inverse; 8 **37.** direct; 2π **39.** inverse; −2 **41.** combined; 10 **43.** direct; 4 **45.** direct; −2 **47.** inverse; 7 **49.** joint; 20 **51.** 54 **53.** 7 **55.** $y = 30x$; 7 theaters **57.** inverse

59a. 11×14: $\ell = \frac{14}{11}w$; 12×16: $\ell = \frac{4}{3}w$; 16×20: $\ell = \frac{5}{4}w$; 18×24: $\ell = \frac{4}{3}w$; 20×24: $\ell = \frac{6}{5}w$; 24×30: $\ell = \frac{5}{4}w$; 24×36: $\ell = \frac{3}{2}w$; 30×40: $\ell = \frac{4}{3}w$; 36×48: $\ell = \frac{4}{3}w$; 48×60: $\ell = \frac{5}{4}w$; 48×72: $\ell = \frac{3}{2}w$ **59b.** 12×16, 18×24, 30×40, 36×48 all have the relationship $\ell = \frac{4}{3}w$. 16×20, 24×30, and 48×60 have the relationship $\ell = \frac{5}{4}w$. 24×36 and 48×72 have the relationship $\ell = \frac{3}{2}w$. **59c.** Sample answer: One canvas is an enlargement of the other.

61a. $\frac{m}{n}$ dollars **61b.** inversely **61c.** $280

63a. $d = 0.10hr$ **63b.** Joint variation; the amount deducted varies directly as the product of two quantities, the hourly wage and the number of hours worked. **63c.** Substitute $r = \$19.50$ and $h = 36$ in the formula for part a. The amount deducted was $70.20.

65. a and g are directly related.

67. Sample answer: The force of an object varies jointly as its mass and acceleration.

Lesson 7-6

1. −3 **3.** 9 **5.** 7 **7.** −12, 2 **9.** ø **11.** 6

13a. $f(x) = \frac{8(0.15) + x(0.75)}{8 + x}$ **13b.** 11.2 mL

15. 41 hours

17a. $\frac{14}{10 - C} - \frac{14}{10 + C} = \frac{1}{2}$ **17b.** 1.7 mph

19. $x < 0$ or $x > 1.75$ **21.** $x < -2$ or $2 < x < 14$

23. $x < -5$ or $4 < x < \frac{17}{3}$ **25a.** $\frac{1000}{N} \leq 15$

25b. 67 **27.** 56 in. **29.** no solution

31. $n < -3$ or $0 < n < 3$ **33.** $0 < x < \frac{3}{2}$

35. 9 **37.** $-5 < m < -2$

39a. $80 > \frac{60x + 17{,}000}{x - 50}$

39b. $x > 1050$; the company must produce and sell at least 1050 speakers in order to ensure that the revenue from each one is greater than the average cost of producing each one.

41. $x = 2$; this method works because the x-coordinate of the point of intersection will make both sides of the equation equal to the same value, and no extraneous roots are shown.

43. at least 40 students

45. Sample answer: 3 must be excluded as a solution, as it will make the denominator of $f(x)$ equal to 0. Only −4 is a solution.

47. Sample answer: I do not agree with the solution set. Multiplying both sides of the inequality by x does not result in an equivalent inequality if x is negative. The work is only valid for positive x-values. The student could have graphed the two equations $f(x) = \frac{1}{x}$ and $g(x) = 2$ and looked at x-values for which the graph of $f(x)$ is below the graph of $g(x)$.

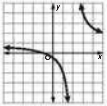
49. all real numbers except 5, −5, 0

51. Sample answer: Multiplying each side of a rational equation or inequality by the LCD can result in extraneous solutions. Therefore, you should check all solutions to make sure that they satisfy the original equation or inequality.

Module 7 Review

1. $\frac{2y^2}{x}$ **3.** A **5.** $\frac{2y^2 - 15x^2}{3x^2y}$ **7.** D

9.



11a. x-intercept: undefined; y-intercept: −1

11b. horizontal, $y = -2$; vertical, $x = -2$

13. D **15a.** 25 mph **15b.** 30 mph **17.** D

Module 8

Module 8 Opener

1. 89 customers, 88 customers, no mode

3. 7.7, 8, 10

5. $\frac{1}{4}$

7. $\frac{1}{4}$

Lesson 8-1

1. Sample: the T-shirts Berton selects; population: all of Berton's sports T-shirts; stratified: Berton divides the T-shirts by team before the sample is selected.

3. Sample: the customers who submit suggestions; population: all customers; self-selected: the customers voluntarily submit suggestions.

5. unbiased; Sample answer: the students are randomly selected. The fact that the sample is selected at band camp does not influence the response of the selected sample.

7. biased; Sample answer: The question is asking about two issues: whether the workout facility needs a new treadmill and whether the workout facility needs a new racquetball court.

9. unbiased; Sample answer: The question does not influence participants.

11. survey; sample: customers that take the online survey; population: all customers

13. experiment; sample: the 50 adults participating in the study; population: all adults

15. objective: to determine the number of juniors and seniors planning to attend prom; population: all juniors and seniors at the school; sample survey questions: What grade are you in? Do you plan on attending prom?

17. observational study; Sample answer: Teenagers are more willing than adults to try a new sports drink.

19. control group: students who do not receive gift cards; experimental group: students receiving gift cards; Sample answer: The students may not have any interest in the gift cards they are receiving. The principal should conduct a survey to find what gifts or activities might motivate the students.

21. convenience; survey

23. stratified; experiment

25. observational study

27. Sample answer: She should use an observational study to determine the average miles per gallon that identical cars are able to go on each type of fuel. She can then divide the current price of each fuel by the number of miles to find the average cost per mile. The group with the lower cost per mile would be the one that is more cost efficient. She might also look for the results of a survey by a motor club or magazine stating the average gas mileage for cars similar to her new car.

29. No; sample answer: the difference between the means of the experimental group and control group is only 0.1 ounce. Both groups had cats that gained significant weight and others that gained little. To compare the groups, I assumed that the cats in each group were similar in age, starting weight, and health. I also assumed that each cat was given the same amount of food.

31. Sample answer: This method of selecting a sample is valid. Each student has an equally likely chance of being selected for the sample. A weakness may be that this would not reflect that one grade may feel strongly about the dress code than another.

33. False; sample answer: A sample statistic is used to estimate a population parameter.

35. Sample answer:
objective: Determine the average amount of time that students spend studying at the library.
population: All students that study at the library.
sample: 25 randomly selected students studying at the library during a given week.

Study Time (minutes)				
38	16	45	41	63
18	20	17	8	15
41	28	55	19	15
30	11	20	79	24
78	24	26	32	19

mean: ≈31.3 min

Lesson 8-2

1a. 25%

1b. 38%

3a. 16.7%

3b. 18%

5. Sample answer: The theoretical probability of making a goal is $\frac{7}{20}$ and the theoretical probability of missing the goal is $\frac{13}{20}$. I can use a random number generator to create a simulation. For numbers 1–20, let 1–7 represent a goal and 8–20 represent a miss. Based on 100 trials, the experimental probability of making a goal is 41%. Because Paola's success rate was only about 13%, it is likely that there is an error in the programming.

7. spinner with 2 regions, one central angle 216°, and one 144°

9. Sample answer: Use a random number generator where 0, 1, 2, 3, and 4 represent brown hair, 5, 6, 7, and 8 represent blonde hair, and 9 represents red hair.

11. Jayme's model

13a. 20% **13b.** 8% **13c.** Sample answer: I assumed that each car color was equally likely.

15a. Sample answer: Use a random number generator to generate a set of three numbers from 1 to 42. Numbers 1–15 will represent players on the basketball team and numbers 16–42 will represent the other students. Discard any trial that repeats any of the same number. Run the trial fifty times.

15b.

35, 13, 36	15, 32, 9	6, 14, 12	16, 26, 27	4, 34, 20	16, 32, 24	12, 9, 18	10, 17, 26
15, 13, 23	18, 15, 41	13, 9, 20	33, 38, 30	34, 40, 35	37, 18, 36	22, 16, 36	23, 5, 28
29, 7, 2	21, 22, 1	33, 31, 22	30, 39, 18	31, 7, 17	35, 40, 36	35, 9, 29	32, 20, 20
21, 3, 23	16, 36, 5	30, 36, 11	28, 35, 10	5, 42, 21	11, 18, 40	26, 10, 2	28, 35, 41
13, 11, 23	12, 7, 24	5, 3, 7	15, 39, 2	30, 21, 24	32, 25, 7	18, 9, 34	29, 20, 2
18, 26, 6	32, 6, 39	17, 3, 21	31, 18, 27	24, 29, 31	25, 37, 21	32, 23, 13	21, 15, 28
3, 34, 15	12, 2, 20						

15c. Sample answer: Only 2 of the 50 trials, or 4%, resulted in only team members winning the raffle. According to this simulation it is highly unlikely that all the winners are members of the basketball team. There is not enough data to determine if the raffle is unfair. If it happens again for another raffle, then the fairness should be examined more closely.

17a. The theoretical probably that she serves an ace on her next serve is 66%, and the theoretical probability that she doesn't is 34%. Use a random number generator to generate integers 1 through 100. The integers 1—66 will represent an ace, and the integers 67—100 will represent any other outcome of her serve. The simulation will consist of 50 trials.

17b. Sample data is shown:

14	16	8	94	9	87
41	41	68	42	66	11
74	38	86	41	26	92
28	36	4	12	25	68
24	4	76	3	61	35

17c. Sample answer: 22 of the 30 trials, or 73.3%, resulted in an ace on the next serve. According to this simulation it is highly likely that her next serve will be an ace. The data is not completely consistent with the model, but if a larger trial were done it might correlate more closely.

19. Sample answer: Jevon's concern about the fairness of the coin should be dependent on the number of times he tossed it. If he completed only 4, or even 20, trials, then the sample size is not large enough to warrant concern. However, if he completed 100 or more trials, then he should be concerned since the experimental probability should be closer to the theoretical probability of $\frac{1}{2}$, or 50%.

21. True; sample answer: The event will always occur when the theoretical probability is 1. Therefore, the experimental probability can never be 0. For example, the theoretical probability of rolling a die and getting 1, 2, 3, 4, 5, or 6 is 1 since it will always show one of these six numbers. So, when a die is rolled, the result will be one of these six numbers, making the experimental probability always greater than 0.

23. Yes; sample answer: If the spinner were going to be divided equally into three outcomes, each sector would measure 120. Because you only want to know the probability of outcome *C*, you can record spins that end in the red area as a success, or the occurrence of outcome *C*, and spins that end in the blue area as a failure, or an outcome of *A* or *B*.

Lesson 8-3

1. 38.22; Sample answer: because the standard deviation is great compared to the mean of $78.17, the data are spread farther from the mean. Based on the standard deviation, most of the shears cost between $34.41 and $110.85.

3. 22.6; 1.3

5. 6.9; 1.7

7. Before the season the mean time was 9.2 minutes and the standard deviation was 1.0 minute. After the season the mean time was 7.0 minutes and the standard deviation was 0.9 minutes. This means that before the season the mile times were higher on average, but that the mile times were generally spread the same before and after the season.

9. 0.375; 0.123

11. 18.4; Sample answer: The standard deviation is small compared to the mean of 89. Most of the exam sessions will have between 71 and 107 attendees.

13. No; sample answer: adding $1 to each employee's hourly wage has no effect on the standard deviation of the data. The standard deviation both before and after the wage change is 0.67.

15. Sample answer: Assume that each data value falls in the center of each bar of the histogram. Using technology, the mean is 17 and the standard deviation is about 7.4.

17. Find the mean of the set of data. Then, find the square of the different between each data value and the mean. Next, find the sum of all of the differences. Then, divide the sum by the number of values in the set of data, which is the variance. Finally, take the square root of the quotient, or variance.

Lesson 8-4

1. texts; discrete; The number of texts can only be represented as a whole number.

3. height of plant; continuous; The height of a plant may be any positive value.

5. Sample answer: The scale of the *y*-axis misleadingly shows the differences in probabilities.

7a. $39.7 < X < 333.1$ **7b.** 16% **7c.** $X > 282.2$

9. −2.19 **11.** 0.281 **13.** 0.9861

15. 2.5% **17.** 6200 hours

19. Hiroko; Sample answer: Monica's solution would work with a uniform distribution.

21. Sample answer: The *z*-value represents the position of a value *X* in a normal distribution. The area under a curve to the left of a *z*-value represents the probability that a value from the distribution will be less than the given value *X*.

Lesson 8-5

1. $E = 0.16$

3. $E = 0.06$

5. At a 99% confidence interval, the population mean is $5.35 \leq \mu \leq 5.65$. Therefore, we are 99% confident that the rating of the airline is between 5.35 and 5.65.

7. At a 90% confidence interval, the population proportion is $0.737 \le p \le 0.813$. Therefore, we are 90% confident that the proportion of students who agree that high school students should have a part-time job is between 73.7% and 81.3%.

9. At a 99% confidence interval, the population proportion is $0.326 \le p \le 0.506$. Therefore, we are 99% confident that the proportion of students who agree with the principal's plan is between 32.6% and 50.6%.

11. Instead of stating that the population proportion is estimated at $\frac{120}{250} = 0.48 = 48\%$ with a margin of error of 3%, the claim uses the highest possible estimate for the population proportion to give the impression that the proportion of people who want parks to be built is greater than 50%.

13. $\hat{p} = \frac{32}{205} \approx 0.156$ and $ME = 1.96$ $\sqrt{\frac{0.156(1-0.156)}{205}} \approx 0.050$, so $CI = 15.6\% \pm 5.0\%$; Sample answer: With 95% confidence, the mean proportion of discards for the population of all pieces fired in her kiln is between 10.6% and 20.6%, so Karen should probably buy a new kiln on the basis that the discard rate is most likely higher than 10%.

15. As the sample size increase, the maximum error decreases.

17. Sample answer: sample mean: 18.8 hours; sample standard deviation: 8.8; the mean number of hours with a 90% confidence level is $15.4 \le \mu \le 22.2$.

Hours of Television Watched Each Week					
12	8	19	27	20	18
0	33	25	14	9	19
23	6	20	26	30	29

Module 8 Review

1. D **3.** A **5.** A

7. A, D **9.** B, C **11.** $x < 55.9$ and $x > 70.9$

13. $E = 1.96\sqrt{\frac{\left(\frac{273}{360}\right)\left(\frac{87}{360}\right)}{360}}$

$E \approx 0.0508$

$\hat{p} = \frac{273}{360} \approx 0.7583$

$\hat{q} = \frac{87}{360} \approx 0.2417$

Sample answer: At a 95% confidence level, I am confident that the proportion of patrons that would like to see the restaurant stay open later is between 70.75% and 80.91%.

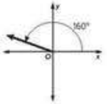
Module 9 Opener

1. 11.7 **3.** 20.5 **5.** $x = 9, y = 9\sqrt{2}$

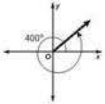
7. $x = 12, y = 12\sqrt{3}$

Lesson 9-1

1.



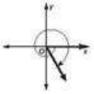
3.



5.

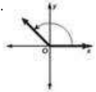
7.

9.

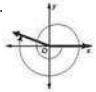


11.

13.



15.



17. 285°, −435° **19.** 405°, −315°

21. 10°, −350° **23.** 60°, −300°

25. 425°, −295° **27.** 470°, −250°

29. $\frac{17\pi}{6}, -\frac{7\pi}{6}$ **31.** $\frac{8\pi}{3}, -\frac{4\pi}{3}$ **33.** $\frac{5\pi}{4}, -\frac{11\pi}{4}$

35. $-\frac{13\pi}{9}$ **37.** $\frac{19\pi}{9}$ **39.** 4π **41.** $\frac{\pi}{2}$

43. −108° **45.** 60° **47.** 120° **49.** −135°

51. 45° **53.** $\frac{20\pi}{3}$ or approximately 20.9 in.

55. $\frac{\pi}{30}$ **57.** $-\frac{41\pi}{9}$ **59.** 450°

61. −105° **63.** 8.4 in. **65.** 12.6 ft

67. 3309°/s; 58 radians/s **69.** 49.74 in.

71. 90°, $\frac{\pi}{2}$ radians; 3π inches

73a. $\frac{\pi}{3}$; Because a complete rotation takes 30 minutes, 5 minutes represents $\frac{5}{30}$ or $\frac{1}{6}$ of a complete rotation, and $\frac{1}{6} \cdot 2\pi = \frac{\pi}{3}$.

73b. The radius r of the wheel is 67.5 m. The distance the passenger traveled is the arc length, so $s = r\theta = 67.5\left(\frac{\pi}{3}\right) \approx 71$ m.

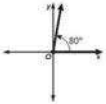
73c. The circumference is πd, which is approximately $3.14d = 3.14(135) = 423.9$ m and $\frac{1}{6}(423.9) = 70.65$, which is close to 71, so the answer is reasonable.

75. Melinda is correct. The area of the complete circle is $\pi r^2 = \pi(15)^2 = 225\pi$ ft². Because the area that gets watered by the sprinkler is 75π ft², this area is $\frac{75\pi}{225\pi} = \frac{1}{3}$ of the circle. Therefore, the measure of the central angle is $\frac{1}{3}(2\pi) = \frac{2\pi}{3}$.

77. Because $s = r\theta$ and $r = 5$, the function may be written as $f(x) = 5x$. This means the graph is a straight line with a slope of 5 that passes through the origin.

79. $x = 2$

81. 440° and −280°



83. One degree represents an angle measure that equals $\frac{1}{360}$ rotation around a circle. One radian represents the measure of an angle in standard position that intercepts an arc of length r. To convert from degrees to radians, multiply the number of degrees by $\frac{\pi \text{ radians}}{180°}$. To convert from radians to degrees, multiply the number of radians by $\frac{180°}{\pi \text{ radians}}$.

Lesson 9-2

1. $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$, $\tan\theta = \frac{5}{12}$, $\csc\theta = \frac{13}{5}$, $\sec\theta = \frac{13}{12}$, $\cot\theta = \frac{12}{5}$

3. $\sin\theta = \frac{8}{17}$, $\cos\theta = \frac{15}{17}$, $\tan\theta = \frac{8}{15}$, $\csc\theta = \frac{17}{8}$, $\sec\theta = \frac{17}{15}$, $\cot\theta = \frac{15}{8}$

5. $\sin\theta = \frac{5}{11}$, $\cos\theta = \frac{4\sqrt{6}}{11}$, $\tan\theta = \frac{5\sqrt{6}}{25}$, $\csc\theta = \frac{11}{5}$, $\sec\theta = \frac{11\sqrt{6}}{24}$, $\cot\theta = \frac{4\sqrt{6}}{5}$

7. $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\csc A = \frac{17}{8}$, $\sec A = \frac{17}{15}$, $\cot A = \frac{15}{8}$

9. $\sin B = \frac{3\sqrt{10}}{10}$, $\cos B = \frac{\sqrt{10}}{10}$, $\csc B = \frac{\sqrt{10}}{3}$, $\sec B = \sqrt{10}$, $\cot B = \frac{1}{3}$

11. $\sin A = \frac{\sqrt{3}}{2}$, $\tan A = \sqrt{3}$, $\csc A = \frac{2\sqrt{3}}{3}$, $\sec A = 2$, $\cot A = \frac{\sqrt{3}}{3}$

13. $\sin\theta = \frac{12}{13}$, $\cos\theta = \frac{5}{13}$, $\tan\theta = \frac{12}{5}$, $\csc\theta = \frac{13}{12}$, $\sec\theta = \frac{13}{5}$, $\cot\theta = \frac{5}{12}$

15. $\sin\theta = -\frac{15}{17}$, $\cos\theta = \frac{8}{17}$, $\tan\theta = -\frac{15}{8}$, $\csc\theta = -\frac{17}{15}$, $\sec\theta = \frac{17}{8}$, $\cot\theta = -\frac{8}{15}$

17. $\sin\theta = -\frac{40}{41}$, $\cos\theta = -\frac{9}{41}$, $\tan\theta = \frac{40}{9}$, $\csc\theta = -\frac{41}{40}$, $\sec\theta = -\frac{41}{9}$, $\cot\theta = \frac{9}{40}$

19. $\sin\theta = \frac{\sqrt{5}}{5}$, $\cos\theta = \frac{2\sqrt{5}}{5}$, $\tan\theta = \frac{1}{2}$, $\csc\theta = \sqrt{5}$, $\sec\theta = \frac{\sqrt{5}}{2}$, $\cot\theta = 2$

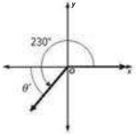
21. $\sin\theta = \frac{\sqrt{10}}{10}$, $\cos\theta = \frac{3\sqrt{10}}{10}$, $\tan\theta = \frac{1}{3}$, $\csc\theta = \sqrt{10}$, $\sec\theta = \frac{\sqrt{10}}{3}$, $\cot\theta = 3$

23. $\sin\theta = -\frac{3\sqrt{10}}{10}$, $\cos\theta = \frac{\sqrt{10}}{10}$, $\tan\theta = -3$, $\csc\theta = -\frac{\sqrt{10}}{3}$, $\sec\theta = \frac{\sqrt{10}}{1}$, $\cot\theta = -\frac{1}{3}$

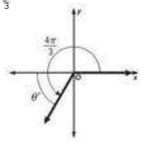
25. $\sin\theta = 0$, $\cos\theta = 1$, $\tan\theta = 0$, $\csc\theta =$ undefined, $\sec\theta = 1$, $\cot\theta =$ undefined

27. $\sin\theta = 1$, $\cos\theta = 0$, $\tan\theta =$ undefined, $\csc\theta = 1$, $\sec\theta =$ undefined, $\cot\theta = 0$

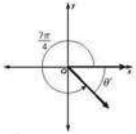
29. 50°



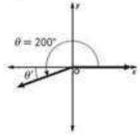
31. $\frac{\pi}{3}$



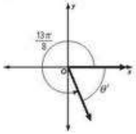
33. $\frac{\pi}{4}$



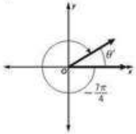
35. 20°



37. $\frac{3\pi}{8}$



39. $\frac{\pi}{4}$



41. $-\frac{\sqrt{2}}{2}$ **43.** $\sqrt{2}$ **45.** −1 **47a.** $\tan 71.6 = \frac{12}{x}$

47b. about 4.0 ft **49.** $\frac{\sqrt{255}}{16}$ or ≈ 0.998

51. $\frac{3\sqrt{55}}{55}$ or ≈ 0.405

53. 0 **55.** $-\frac{\sqrt{3}}{3}$ **57.** undefined **59.** $(-1, \sqrt{3})$

61. 75° **63.** $\sqrt{5}$ **65.** 2

67. Sample answer: 120°, 240°, 420°; For an angle in Quadrant II, $180° - \theta = 60°$, so $\theta = 120°$; for an angle in Quadrant III, $\theta - 180° = 60°$, so $\theta = 240°$; 60° plus any multiple of 360° will have a reference angle of 60°, so $60° + 360° = 420°$.

69. True; $\sin\theta = \frac{\text{opp}}{\text{hyp}}$ and the values of the opposite side and the hypotenuse of an acute triangle are positive, so the value of the sine function is positive.

71. No; for $\sin\theta = \frac{\sqrt{2}}{2}$ and $\tan\theta = -1$, the reference angle is 45°. However, for $\sin\theta$ to be positive and $\tan\theta$ to be negative, the reference angle must be in the second quadrant. So, the value of θ must be 135° or an angle coterminal with 135°.

73. We know that $\cot\theta = \frac{x}{y}$, $\sin\theta = \frac{y}{r}$, and $\cos\theta = \frac{x}{r}$. Because $\sin 180° = 0$, it must be true that $\cot 180° = \frac{x}{0}$, which is undefined.

75. First, sketch the angle and determine in which quadrant it is located. Then use the appropriate rule for finding its reference angle θ'. A reference angle is the acute angle formed by the terminal side of θ and the x-axis. Next, find the value of the trigonometric function for θ'. Finally, use the quadrant location to determine the sign of the trigonometric function value of θ.

Lesson 9-3

1. $\cos\theta = -\frac{\sqrt{3}}{2}$, $\sin\theta = \frac{1}{2}$

3. $\cos\theta = -\frac{2}{3}$, $\sin\theta = \frac{\sqrt{5}}{3}$

5. $\cos\theta = \frac{1}{6}$, $\sin\theta = -\frac{\sqrt{35}}{6}$

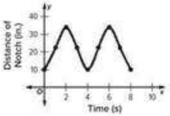
7. $\sin\theta = \frac{1}{2}$; $\cos\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = \frac{\sqrt{3}}{3}$, $\csc\theta = 2$, $\sec\theta = \frac{2\sqrt{3}}{3}$, $\cot\theta = \sqrt{3}$

9. $\sin\theta = 1$; $\cos\theta = 0$, $\tan\theta =$ undefined, $\csc\theta = 1$, $\sec\theta =$ undefined, $\cot\theta = 0$

11. $\sin\theta = 0$; $\cos\theta = -1$, $\tan\theta = 0$, $\csc\theta =$ undefined, $\sec\theta = -1$, $\cot\theta =$ undefined

13. 10 **15.** $\frac{5\pi}{2}$ **17.** 4

19. period = 4 seconds



21. $-\frac{1}{2}$ **23.** 0 **25.** $-\frac{\sqrt{2}}{2}$ **27.** $\frac{\sqrt{2}-\sqrt{3}}{2}$

29. $-\frac{5\sqrt{3}}{2}$ **31.** 1 **33.** 24 s

35a. Sometimes; the cosine function can only result in values between −1 and 1, inclusive.

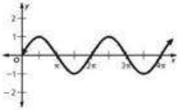
35b. Always; the sine function has a period of 2π.

35c. Sometimes; $\cos\theta = 1$ when n is even and $\cos\theta = -1$ when n is odd.

35d. Never; the y-coordinate of the corresponding point on the unit circle is negative.

37a. $\frac{3\pi}{4}$ meters; because the radian measure of the angle is the length of the arc on the unit circle subtended by the angle.

37b.



37c. The period is 2π because the values of the function repeat every 2π radians. This is shown in the graph when the shape of the curve from 0 to 2π is repeated from 2π to 4π. The heights of point P above or below the surface of the water repeat once as the wheel makes a complete rotation (2π radians).

37d. The x-intercepts are whole-number multiples of π. These are the angles of rotation for which P is on the surface of the water.

37e. Sample answer: $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$; as the wheel rotates through an angle from $\frac{\pi}{2}$ radians to $\frac{3\pi}{2}$ radians, P moves downward.

39a. 72°F **39b.** March **39c.** 12°F

39d. September

41. Benita; Francis incorrectly wrote $\cos\left(-\frac{\pi}{3}\right) = -\cos\frac{\pi}{3}$.

43. Sometimes; the period of a sine curve could be $\frac{\pi}{2}$, which is not a multiple of π.

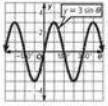
45. The period of a periodic function is the horizontal distance of the part of the graph that is nonrepeating. Each nonrepeating part of the graph is one cycle.

Lesson 9-4

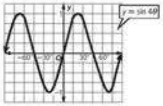
1. 2; $y = 0$; 360° **3.** 1; $y = 0$; 120°

5. 2; 360° **7.** 1; 720° **9.** $\frac{1}{2}$; 180°

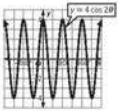
11. 3; 360°



13. 1; 90°



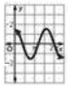
15. 4; 180°



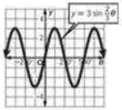
17a. period: 2; frequency: $\frac{1}{2}$ The object completes $\frac{1}{2}$ of a cycle per second, and it will reach maximum distance from the equilibrium point every 2 seconds.

17b. Domain: all real numbers; because time cannot be negative, the relevant domain in the context of the situation is $[0, \infty)$. The range is $[-25, 25]$.

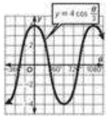
19. $h(x) = 1.5 \cos\frac{2\pi}{5}x$



21. 3; 540°



23. 4; 1080°



25. 180°; $y = 5 \sin 2\theta$

27. Sample answer: The sine function has the same shape as an ocean wave. Additionally, the sine function and ocean waves share similar behaviors, as they both repeat themselves over a consistent interval of time and distance.

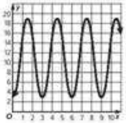
29a. 12; a calendar year

29b. 85°F; July 15 **31a.** 200 nm, 520 nm

31b. yellow **31c.** violet; red

33a. amplitude: 4.5; period: 2 seconds

33b. $f(t) = -4.5 \cos \pi t + 7.5$



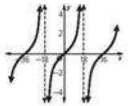
35. Sample answer: $y = 2 \sin 6\theta$; the amplitude is $|a| = 2$. The period is $\frac{2\pi}{|b|} = \frac{2\pi}{6} = \frac{\pi}{3}$, so there are 3 complete cycles between $\theta = 0$ and $\theta = \pi$.

37. The graph of $y = \frac{1}{2}\sin\theta$ has an amplitude of $\frac{1}{2}$ and a period of 360°. The graph of $y = \sin\frac{1}{2}\theta$ has an amplitude of 1 and a period of 720°.

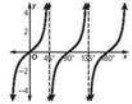
39. Sample answer: Determine the amplitude and period of the function; find and graph and x-intercepts, extrema, and asymptotes; use the parent function to sketch the graph.

Lesson 9-5

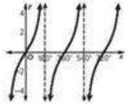
1. period = 36°; asymptotes: $(36n + 18)$°, where n is an odd integer; x-intercepts: 0°, 36°, 72°, 108°, ...; midline: $y = 0$; Because $a = 1$, the function is not vertically dilated in relation to the parent function. Because $b = 5$, the function is compressed horizontally in relation to the parent function.



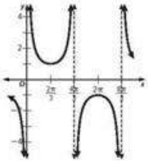
3. period = 90°; asymptotes: (90n + 45)°, where n is an odd integer; x-intercepts: 0°, 90°, 180°, 270°, ...; midline: $y = 0$; Because $a = 1$, the function is not vertically dilated in relation to the parent function. Because $b = 2$, the function is compressed horizontally in relation to the parent function.



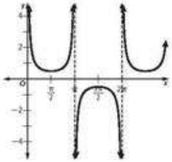
5. period = 360°; asymptotes: (360n + 180)°, where n is an odd integer; x-intercepts: 0°, 360°, 720°, 1080°, ...; midline: $y = 0$; Because a =2, the function is stretched vertically in relation to the parent function. Because $b = \frac{1}{2}$, the function is stretched horizontally in relation to the parent function.



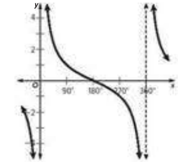
7. period = $\frac{8\pi}{3}$ asymptotes: $x = \frac{4\pi}{3} + \frac{4\pi}{3}n$, where n is an integer; relative minima at $x = \frac{2\pi}{3}$, $x = \frac{10\pi}{3}$, or $x = \frac{2\pi}{3} + \frac{8\pi}{3}n$, where n is an integer; relative maxima at $x = 2\pi$, $x = \frac{14\pi}{3}$, or $x = 2\pi + \frac{8\pi}{3}n$, where n is an integer.



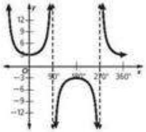
9. period = 2π; asymptotes: $x = 2\pi + 2\pi n$, where n is an integer; relative minima at $x = \frac{\pi}{2}$, $x = 3\pi$, or $x = \frac{\pi}{2} + 2\pi n$, where n is an integer; relative maxima at $\frac{3\pi}{2}$, $\frac{15\pi}{2}$, or $x = \frac{3\pi}{2} + 2\pi n$, where n is an integer.



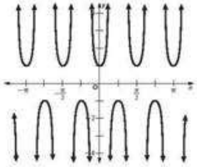
11. period = 360°; asymptotes: $x = 2\pi n$, where n is an integer; x-intercepts: 180°, 540°, ...; (odd multiples of 180°).



13. period = 360°; asymptotes: $x = 270 + 360n$, $x = 180 + 360n$, where n is an integer

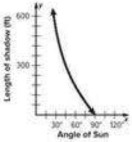


15. period = $\frac{\pi}{2}$; asymptotes: $x = \frac{3\pi}{8} + \frac{\pi}{2}n$, $x = \frac{\pi}{8} + \frac{\pi}{2}n$ where n is an integer



17a. $y = 300 \cot x$

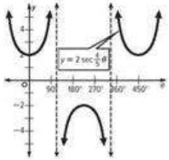
17b.



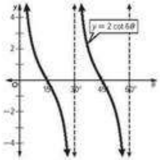
17c. Sample answer: In the context of the situation, only x-values between 0° and 90° are relevant, so focus on analyzing that portion of the graph.

domain: {x| 0 > x > 90}
range: {y| −∞ ≤ y ≤ ∞}
x-intercept: 90
y-intercept: none
relative minimum: none
relative maximum: none
The period of $y = 300 \cot x$ is 180°.

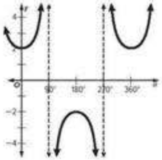
19. period 450°



21. period: 30°



23. period: 360°



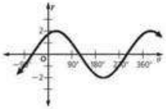
25. $d = 2000 \csc x$

27. Tyler; It appears that Lacretia gave the period of the function $y = \frac{1}{2} \tan \theta$, rather than $y = \tan \frac{1}{2}\theta$.

29. Sample answer: Determine the amplitude and period of the function, then find and graph any x-intercepts, extrema, and asymptotes. Sketch the parent function and transform the graph to graph the given function.

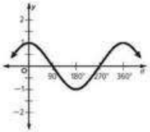
Lesson 9-6

1. 1; 2π; π to the left



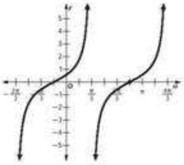
domain: $-\infty < \theta < \infty$; range: $-2 \leq f(\theta) \leq 2$

3. 1; 360°; 90° to the left



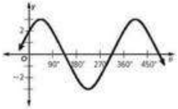
domain: $-\infty < \theta < \infty$; range: $-1 \leq f(\theta) \leq 1$

5. no amplitude; π; $\frac{\pi}{6}$ to the left



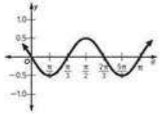
domain: $\pi n < \theta < \frac{\pi}{3} + \pi n$, where n is an integer; range: $-\infty < f(\theta) < \infty$

7. 3; 360°; 45° to the right



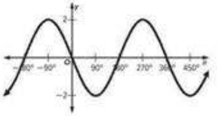
domain: $-\infty < \theta < \infty$; range: $-3 \leq f(\theta) \leq 3$

9. $\frac{1}{2}$; $\frac{2\pi}{3}$; $\frac{\pi}{3}$ to the right



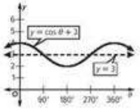
domain: $-\infty < \theta < \infty$; range: $-0.5 \leq f(\theta) \leq 0.5$

11. 2; 360°; 90° to the left



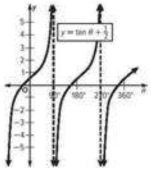
domain: $-\infty < \theta < \infty$; range: $-2 \leq f(\theta) \leq 2$

13. 1; 360°; $k = 3$; $y = 3$



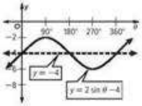
domain: $-\infty < \theta < \infty$; range: $2 \leq f(\theta) \leq 4$

15. no amplitude; 180° $k = \frac{1}{2}$; $y = \frac{1}{2}$



domain: $90 + 180n < \theta < 180 + 180n$, where n is an integer; range: $-\infty < f(\theta) < \infty$

17. 2; 360°; $k = -4$; $y = -4$

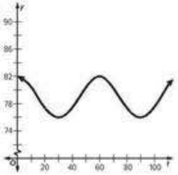


domain: $-\infty < \theta < \infty$; range: $-6 \leq f(\theta) \leq -2$

19a. The amplitude is 3; the midline is $y = 79$; the period is 60 minutes.

19b. $h(t) = 3 \cos\left(\frac{\pi}{30}t\right) + 79$

19c.

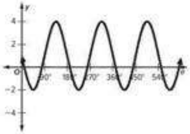


19d. 2 times; these are the times when the minute hand is on the 6 (i.e., 12:30 and 1:30).

21a. $k = -3$, so the vertical shift is 3 units down; amplitude = 1 **21b.** none **21c.** π

21d. $y = \cos 2\theta - 3$ **23.** $y = \sin(x - 4) + 3$

25. $y = \tan(x - \pi) + 2.5$ **27.** 3; 180°; −45°; 1



29. Sample answer: sine; Because as a Ferris wheel turns, the distance a rider is above the ground varies sinusoidally with time. Note: any phenomenon that can be modeled with a sine function can also be modeled with a cosine function using the appropriate horizontal shift and/or reflection about the horizontal axis.

31a. 120; in 2.5 years **31b.** 125; in 7 years

33. $\left(\frac{3\pi}{2}, 2\right)$ **35.** no maximum values

37. The graphs are reflections of each other over the x-axis.

39. The graphs are identical.

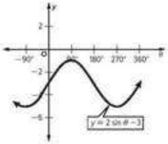
41. 360°; Sample answer: $y = \sin \theta - 5$

43. 360°; Sample answer: $y = 4 \cos \theta + 1$

45. Sample answer: $h(\theta) = 1.5 \cos 2\theta - 2$

47. The graph of $y = 3 \sin 2\theta + 1$ has an amplitude of 3 rather than an amplitude of 1. It is shifted up 1 unit from the parent graph and is compressed so that it has a period of 180°.

49. Sample answer: $y = 2 \sin \theta - 3$



51. Sample answer: $y = \sin\left(\theta + \frac{\pi}{2}\right) - 3$ is equivalent to $y = \cos(\theta)$. When the sine graph is shifted $\frac{\pi}{2}$ units to the left, it is equivalent to the cosine graph.

Lesson 9-7

1. 30°, $\frac{\pi}{6}$ **3.** 120°, $\frac{2\pi}{3}$ **5.** 135°, $\frac{3\pi}{4}$ **7.** 45°; $\frac{\pi}{4}$

9. 90°; $\frac{\pi}{2}$ **11.** −1.02 **13.** 0.71 **15.** 0.95

17. 0.5 **19.** 53.13° **21.** 60° **23.** −5.74°

25. 58.67° **27.** 20.49° **29.** 75.52°

31a. 48.2°

31b. Sample answer: I assumed that the angle at which the doors would be opened would be equal in measure.

33. $\frac{\pi}{2}$ **35.** 0, 2π **37.** $\frac{\pi}{4}$, $\frac{5\pi}{4}$

39. 37°, 53°, 90°

41. Sample answer: Neither; cosine is not positive in the second quadrant.

43. Sample answer: $y = \tan^{-1} x$ is a relation that has a domain of all real numbers and a range of all real numbers except odd multiples of $\frac{\pi}{2}$. The relation is not a function. $y = \text{Tan}^{-1} x$ is a function that has a domain of all real numbers and a range of $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

45. Sample answer: $y = \text{Cos}^{-1} \frac{1}{2}\theta + 1$; this function is a horizontal stretch by a factor 2 of the parent function $y = \text{Cos}^{-1} \theta$, so the domain is $\{-2 \leq \theta \leq 2\}$. It is also a vertical translation 1 unit up, so the range is $\{1 \leq y \leq \pi + 1\}$.

Module 9 Review

1a. $\frac{\pi}{9}$ **1b.** $\frac{4\pi}{9}$ **1c.** $\frac{8\pi}{9}$
3. 157 meters **5.** $\frac{7}{24}$ **7.** $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **9a.** 30
9b. 20 **11.** A **13.** A **15.** A, D, E **17.** C

Glossary

A

English	Español
30°-60°-90° triangle A right triangle with two acute angles that measure 30° and 60°.	triángulo 30°-60°-90° Un triángulo rectángulo con dos ángulos agudos que miden 30° y 60°.
45°-45°-90° triangle A right triangle with two acute angles that measure 45°.	triángulo 45°-45°-90° Un triángulo rectángulo con dos ángulos agudos que miden 45°.
absolute value The distance a number is from zero on the number line.	valor absoluto La distancia que un número es de cero en la línea numérica.
absolute value function A function written as $f(x) = \lvert x \rvert$, in which $f(x) \geq 0$ for all values of x.	función del valor absoluto Una función que se escribe $f(x) = \lvert x \rvert$, donde $f(x) \geq 0$, para todos los valores de x.
accuracy The nearness of a measurement to the true value of the measure.	exactitud La proximidad de una medida al valor verdadero de la medida.
additive identity Because the sum of any number a and 0 is equal to a, 0 is the additive identity.	identidad aditiva Debido a que la suma de cualquier número a y 0 es igual a, 0 es la identidad aditiva.
additive inverses Two numbers with a sum of 0.	inverso aditivos Dos números con una suma de 0.
adjacent angles Two angles that lie in the same plane and have a common vertex and a common side but have no common interior points.	ángulos adyacentes Dos ángulos que se encuentran en el mismo plano y tienen un vértice común y un lado común, pero no tienen puntos comunes en el interior.
adjacent arcs Arcs in a circle that have exactly one point in common.	arcos adyacentes Arcos en un circulo que tienen un solo punto en común.
algebraic expression A mathematical expression that contains at least one variable.	expresión algebraica Una expresión matemática que contiene al menos una variable.
algebraic notation Mathematical notation that describes a set by using algebraic expressions.	notación algebraica Notación matemática que describe un conjunto usando expresiones algebraicas.
alternate exterior angles When two lines are cut by a transversal, nonadjacent exterior angles that lie on opposite sides of the transversal.	ángulos alternos externos Cuando dos líneas son cortadas por un ángulo transversal, no adyacente exterior que se encuentran en lados opuestos de la transversal.
alternate interior angles When two lines are cut by a transversal, nonadjacent interior angles that lie on opposite sides of the transversal.	ángulos alternos internos Cuando dos líneas son cortadas por un ángulo transversal, no adyacente interior que se encuentran en lados opuestos de la transversal.
altitude of a parallelogram A perpendicular segment between any two parallel bases.	altitud de un paralelogramo Un segmento perpendicular entre dos bases paralelas.

Glossary

altitude of a prism or cylinder A segment perpendicular to the bases that joins the planes of the bases.	**altitud de un prisma o cilindro** Un segmento perpendicular a las bases que une los planos de las bases.
altitude of a pyramid or cone A segment perpendicular to the base that has the vertex as one endpoint and a point in the plane of the base as the other endpoint.	**altitud de una pirámide o cono** Un segmento perpendicular a la base que tiene el vértice como un punto final y un punto en el plano de la base como el otro punto final.
altitude of a triangle A segment from a vertex of the triangle to the line containing the opposite side and perpendicular to that side.	**altitud de triángulo** Un segmento de un vértice del triángulo a la línea que contiene el lado opuesto y perpendicular a ese lado.
ambiguous case When two different triangles could be created or described using the given information.	**caso ambiguo** Cuando dos triángulos diferentes pueden ser creados o descritos usando la información dada.
amplitude For functions of the form $y = a \sin b\theta$ or $y = a \cos b\theta$, the amplitude is $\|a\|$.	**amplitud** Para funciones de la forma $y = a \text{ sen } b\theta$ o $y = a \cos b\theta$, la amplitud es $\|a\|$.
analytic geometry The study of geometry that uses the coordinate system.	**geometría analítica** El estudio de la geometría que utiliza el sistema de coordenadas.
angle The intersection of two noncollinear rays at a common endpoint.	**ángulo** La intersección de dos rayos no colineales en un extremo común.
angle bisector A ray or segment that divides an angle into two congruent angles.	**bisectriz de un ángulo** Un rayo o segmento que divide un ángulo en dos ángulos congruentes.
angle of depression The angle formed by a horizontal line and an observer's line of sight to an object below the horizontal line.	**ángulo de depresión** El ángulo formado por una línea horizontal y la línea de visión de un observador a un objeto por debajo de la línea horizontal.
angle of elevation The angle formed by a horizontal line and an observer's line of sight to an object above the horizontal line.	**ángulo de elevación** El ángulo formado por una línea horizontal y la línea de visión de un observador a un objeto por encima de la línea horizontal.
angle of rotation The angle through which a figure rotates.	**ángulo de rotación** El ángulo a través del cual gira una figura.
apothem A perpendicular segment between the center of a regular polygon and a side of the polygon or the length of that line segment.	**apotema** Un segmento perpendicular entre el centro de un polígono regular y un lado del polígono o la longitud de ese segmento de línea.
approximate error The positive difference between an actual measurement and an approximate or estimated measurement.	**error aproximado** La diferencia positiva entre una medida real y una medida aproximada o estimada.
arc Part of a circle that is defined by two endpoints.	**arco** Parte de un círculo que se define por dos puntos finales.
arc length The distance between the endpoints of an arc measured along the arc in linear units.	**longitude de arco** La distancia entre los extremos de un arco medido a lo largo del arco en unidades lineales.
area The number of square units needed to cover a surface.	**área** El número de unidades cuadradas para cubrir una superficie.
arithmetic sequence A pattern in which each term after the first is found by adding a constant, the common difference *d*, to the previous term.	**secuencia aritmética** Un patrón en el cual cada término después del primero se encuentra añadiendo una constante, la diferencia común *d*, al término anterior.
asymptote A line that a graph approaches.	**asíntota** Una línea que se aproxima a un gráfico.
auxiliary line An extra line or segment drawn in a figure to help analyze geometric relationships.	**línea auxiliar** Una línea o segmento extra dibujado en una figura para ayudar a analizar las relaciones geométricas.
average rate of change The change in the value of the dependent variable divided by the change in the value of the independent variable.	**tasa media de cambio** El cambio en el valor de la variable dependiente dividido por el cambio en el valor de la variable independiente.
axiom A statement that is accepted as true without proof.	**axioma** Una declaración que se acepta como verdadera sin prueba.
axiomatic system A set of axioms from which theorems can be derived.	**sistema axiomático** Un conjunto de axiomas de los cuales se pueden derivar teoremas.
axis of symmetry A line about which a graph is symmetric.	**eje de simetría** Una línea sobre la cual un gráfica es simétrico.
axis symmetry If a figure can be mapped onto itself by a rotation between 0° and 360° in a line.	**eje simetría** Si una figura puede ser asignada sobre sí misma por una rotación entre 0° y 360° en una línea.

B

bar graph A graphical display that compares categories of data using bars of different heights.	**gráfico de barra** Una pantalla gráfica que compara las categorías de datos usando barras de diferentes alturas.
base In a power, the number being multiplied by itself.	**base** En un poder, el número se multiplica por sí mismo.
base angles of a trapezoid The two angles formed by the bases and legs of a trapezoid.	**ángulos de base de un trapecio** Los dos ángulos formados por las bases y patas de un trapecio.
base angles of an isosceles triangle The two angles formed by the base and the congruent sides of an isosceles triangle.	**ángulo de la base de un triángulo isosceles** Los dos ángulos formados por la base y los lados congruentes de un triángulo isosceles.
base edge The intersection of a lateral face and a base in a solid figure.	**arista de la base** La intersección de una cara lateral y una base en una figura sólida.
base of a parallelogram Any side of a parallelogram.	**base de un paralelogramo** Cualquier lado de un paralelogramo.
base of a pyramid or cone The face of the solid opposite the vertex of the solid.	**base de una pirámide o cono** La cara del sólido opuesta al vértice del sólido.

bases of a prism or cylinder The two parallel congruent faces of the solid.	**bases de un prisma o cilindro** Las dos caras congruentes paralelas de la figura sólida.
bases of a trapezoid The parallel sides in a trapezoid.	**bases de un trapecio** Los lados paralelos en un trapecio.
best-fit line The line that most closely approximates the data in a scatter plot.	**línea de ajuste óptimo** La línea que más se aproxima a los datos en un diagrama de dispersión.
betweenness of points Point C is between A and B if and only if A, B, and C are collinear and $AC + CB = AB$.	**intermediación de puntos** El punto C está entre A y B si y sólo si A, B, y C son colineales y $AC + CB = AB$.
bias An error that results in a misrepresentation of a population.	**sesgo** Un error que resulta en una tergiversación de una población.
biconditional statement The conjunction of a conditional and its converse.	**declaración bicondicional** La conjunción de un condicional y su inverso.
binomial The sum of two monomials.	**binomio** La suma de dos monomios.
bisect To separate a line segment into two congruent segments.	**bisecar** Separe un segmento de línea en dos segmentos congruentes.
bivariate data Data that consists of pairs of values.	**datos bivariate** Datos que constan de pares de valores.
boundary The edge of the graph of an inequality that separates the coordinate plane into regions.	**frontera** El borde de la gráfica de una desigualdad que separa el plano de coordenadas en regiones.
bounded When the graph of a system of constraints is a polygonal region.	**acotada** Cuando la gráfica de un sistema de restricciones es una región poligonal.
box plot A graphical representation of the five-number summary of a data set.	**diagram de caja** Una representación gráfica del resumen de cinco números de un conjunto de datos.

C

categorical data Data that can be organized into different categories.	**datos categóricos** Datos que pueden organizarse en diferentes categorías.
causation When a change in one variable produces a change in another variable.	**causalidad** Cuando un cambio en una variable produce un cambio en otra variable.
center of a circle The point from which all points on a circle are the same distance.	**centro de un círculo** El punto desde el cual todos los puntos de un círculo están a la misma distancia.
center of a regular polygon The center of the circle circumscribed about a regular polygon.	**centro de un polígono regular** El centro del círculo circunscrito alrededor de un polígono regular.
center of dilation The center point from which dilations are performed.	**centro de dilatación** Punto fijo en torno al cual se realizan las homotecias.

center of rotation The fixed point about which a figure rotates.	**centro de rotación** El punto fijo sobre el que gira una figura.
center of symmetry A point in which a figure can be rotated onto itself.	**centro de la simetría** Un punto en el que una figura se puede girar sobre sí misma.
central angle of a circle An angle with a vertex at the center of a circle and sides that are radii.	**ángulo central de un círculo** Un ángulo con un vértice en el centro de un círculo y los lados que son radios.
central angle of a regular polygon An angle with its vertex at the center of a regular polygon and sides that pass through consecutive vertices of the polygon.	**ángulo central de un polígono regular** Un ángulo con su vértice en el centro de un polígono regular y lados que pasan a través de vértices consecutivos del polígono.
centroid The point of concurrency of the medians of a triangle.	**baricentro** El punto de intersección de las medianas de un triángulo.
chord of a circle or sphere A segment with endpoints on the circle or sphere.	**cuerda de un círculo o esfera** Un segmento con extremos en el círculo o esfera.
circle The set of all points in a plane that are the same distance from a given point called the center.	**círculo** El conjunto de todos los puntos en un plano que están a la misma distancia de un punto dado llamado centro.
circular function A function that describes a point on a circle as the function of an angle defined in radians.	**función circular** Función que describe un punto en un círculo como la función de un ángulo definido en radianes.
circumcenter The point of concurrency of the perpendicular bisectors of the sides of a triangle.	**circuncentro** El punto de concurrencia de las bisectrices perpendiculares de los lados de un triángulo.
circumference The distance around a circle.	**circunferencia** La distancia alrededor de un círculo.
circumscribed angle An angle with sides that are tangent to a circle.	**ángulo circunscrito** Un ángulo con lados que son tangentes a un círculo.
circumscribed polygon A polygon with vertices outside the circle and sides that are tangent to the circle.	**polígono circunscrito** Un polígono con vértices fuera del círculo y lados que son tangentes al círculo.
closed If for any members in a set, the result of an operation is also in the set.	**cerrado** Si para cualquier número en el conjunto, el resultado de la operación es también en el conjunto.
closed half-plane The solution of a linear inequality that includes the boundary line.	**semi-plano cerrado** La solución de una desigualdad linear que incluye la línea de limite.
codomain The set of all the y-values that could possibly result from the evaluation of the function.	**codominar** El conjunto de todos los valores y que podrían resultar de la evaluación de la función.
coefficient The numerical factor of a term.	**coeficiente** El factor numérico de un término.
coefficient of determination An indicator of how well a function fits a set of data.	**coeficiente de determinación** Un indicador de lo bien que una función se ajusta a un conjunto de datos.

Glossary · Glosario

cofunction identities Identities that show the relationships between sine and cosine, tangent and cotangent, and secant and cosecant.	identidades de cofunción Identidades que muestran las relaciones entre seno y coseno, tangente y cotangente, y secante y cosecante.
collinear Lying on the same line.	colineal Acostado en la misma línea.
combination A selection of objects in which order is not important.	combinación Una selección de objetos en los que el orden no es importante.
combined variation When one quantity varies directly and/or inversely as two or more other quantities.	variación combinada Cuando una cantidad varía directamente y / o inversamente como dos o más cantidades.
common difference The difference between consecutive terms in an arithmetic sequence.	diferencia común La diferencia entre términos consecutivos de una secuencia aritmética.
common logarithms Logarithms of base 10.	logaritmos comunes Logaritmos de base 10.
common ratio The ratio of consecutive terms of a geometric sequence.	razón común El razón de términos consecutivos de una secuencia geométrica.
common tangent A line or segment that is tangent to two circles in the same plane.	tangente común Una línea o segmento que es tangente a dos círculos en el mismo plano.
complement of *A* All of the outcomes in the sample space that are not included as outcomes of event *A*.	complemento de *A* Todos los resultados en el espacio muestral que no se incluyen como resultados del evento *A*.
complementary angles Two angles with measures that have a sum of 90°.	ángulo complementarios Dos ángulos con medidas que tienen una suma de 90°.
completing the square A process used to make a quadratic expression into a perfect square trinomial.	completar el cuadrado Un proceso usado para hacer una expresión cuadrática en un trinomio cuadrado perfecto.
complex conjugates Two complex numbers of the form $a + bi$ and $a - bi$.	conjugados complejos Dos números complejos de la forma $a + bi$ y $a - bi$.
complex fraction A rational expression with a numerator and/or denominator that is also a rational expression.	fracción compleja Una expresión racional con un numerador y / o denominador que también es una expresión racional.
complex number Any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit.	número complejo Cualquier número que se puede escribir en la forma $a + bi$, donde a y b son números reales e i es la unidad imaginaria.
component form A vector written as $<x, y>$, which describes the vector in terms of its horizontal component x and vertical component y.	forma de componente Un vector escrito como $<x, y>$, que describe el vector en términos de su componente horizontal x y componente vertical y.
composite figure A figure that can be separated into regions that are basic figures, such as triangles, rectangles, trapezoids, and circles.	figura compuesta Una figura que se puede separar en regiones que son figuras básicas, tales como triángulos, rectángulos, trapezoides, y círculos.
composite solid A three-dimensional figure that is composed of simpler solids.	solido compuesta Una figura tridimensional que se compone de figuras más simples.
composition of functions An operation that uses the results of one function to evaluate a second function.	composición de funciones Operación que utiliza los resultados de una función para evaluar una segunda función.
composition of transformations When a transformation is applied to a figure and then another transformation is applied to its image.	composición de transformaciones Cuando una transformación se aplica a una figura y luego se aplica otra transformación a su imagen.
compound event Two or more simple events.	evento compuesto Dos o más eventos simples.
compound inequality Two or more inequalities that are connected by the words *and* or *or*.	desigualdad compuesta Dos o más desigualdades que están unidas por las palabras *y* u *o*.
compound interest Interest calculated on the principal and on the accumulated interest from previous periods.	interés compuesto Intereses calculados sobre el principal y sobre el interés acumulado de períodos anteriores.
compound statement Two or more statements joined by the word *and* or *or*.	enunciado compuesto Dos o más declaraciones unidas por la palabra *y* o *o*.
concave polygon A polygon with one or more interior angles with measures greater than 180°.	polígono cóncavo Un polígono con uno o más ángulos interiores con medidas superiores a 180°.
concentric circles Coplanar circles that have the same center.	círculos concéntricos Círculos coplanarios que tienen el mismo centro.
conclusion The statement that immediately follows the word *then* in a conditional.	conclusión La declaración que inmediatamente sigue la palabra *entonces* en un condicional.
concurrent lines Three or more lines that intersect at a common point.	líneas concurrentes Tres o más líneas que se intersecan en un punto común.
conditional probability The probability that an event will occur given that another event has already occurred.	probabilidad condicional La probabilidad de que un evento ocurra dado que otro evento ya ha ocurrido.
conditional relative frequency The ratio of the joint frequency to the marginal frequency.	frecuencia relativa condicional La relación entre la frecuencia de la articulación y la frecuencia marginal.
conditional statement A compound statement that consists of a premise, or hypothesis, and a conclusion, which is false only when its premise is true and its conclusion is false.	enunciado condicional Una declaración compuesta que consiste en una premisa, o hipótesis, y una conclusión, que es falsa solo cuando su premisa es verdadera y su conclusión es falsa.

G6 Glossary

Glossary G7

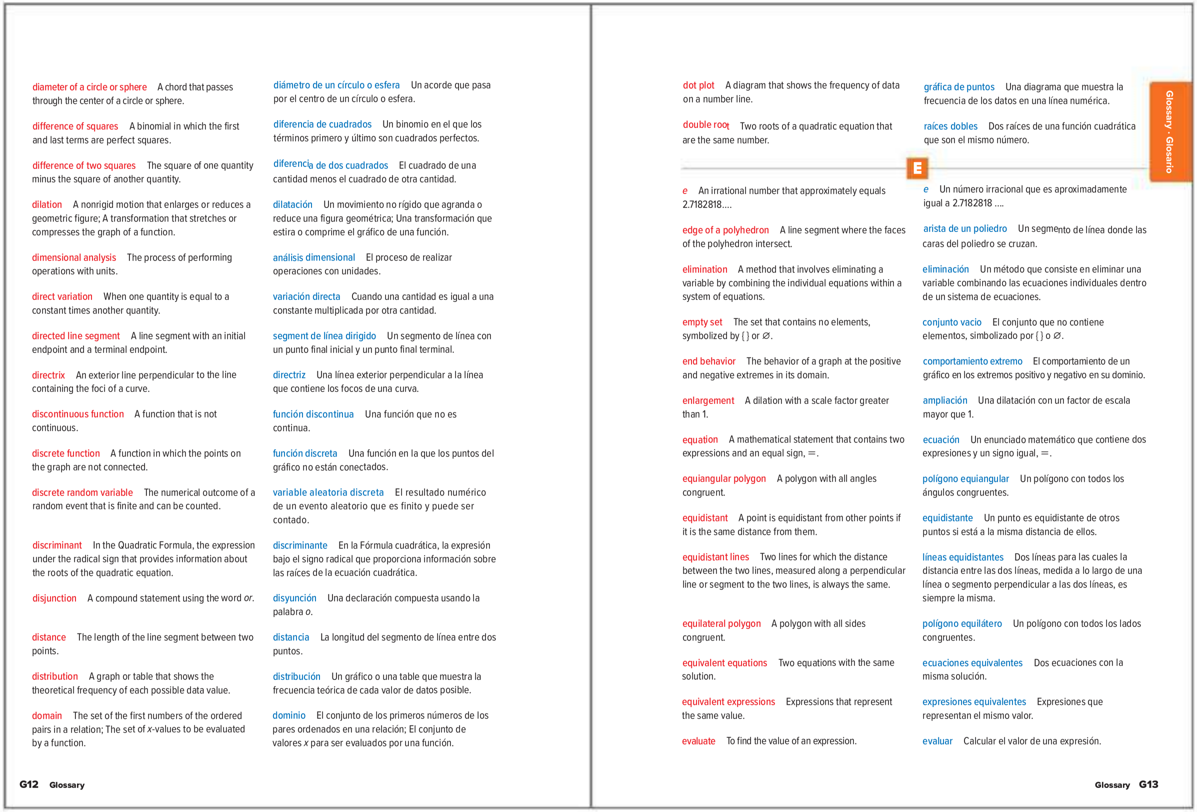
cone A solid figure with a circular base connected by a curved surface to a single vertex.	**cono** Una figura sólida con una base circular conectada por una superficie curvada a un solo vértice.
confidence interval An estimate of the population parameter stated as a range with a specific degree of certainty.	**intervalo de confianza** Una estimación del parámetro de población se indica como un rango con un grado específico de certeza.
congruent Having the same size and shape.	**congruente** Tener el mismo tamaño y forma.
congruent angles Two angles that have the same measure.	**ángulo congruentes** Dos ángulos que tienen la misma medida.
congruent arcs Arcs in the same or congruent circles that have the same measure.	**arcos congruentes** Arcos en los mismos círculos o congruentes que tienen la misma medida.
congruent polygons All of the parts of one polygon are congruent to the corresponding parts or matching parts of another polygon.	**poligonos congruentes** Todas las partes de un polígono son congruentes con las partes correspondientes o partes coincidentes de otro polígono.
congruent segments Line segments that are the same length.	**segmentos congruentes** Línea segmentos que son la misma longitud.
congruent solids Solid figures that have exactly the same shape, size, and a scale factor of 1:1.	**sólidos congruentes** Figuras sólidas que tienen exactamente la misma forma, tamaño y un factor de escala de 1:1.
conic sections Cross sections of a right circular cone.	**secciones cónicas** Secciones transversales de un cono circular derecho.
conjecture An educated guess based on known information and specific examples.	**conjetura** Una suposición educada basada en información conocida y ejemplos específicos.
conjugates Two expressions, each with two terms, in which the second terms are opposites.	**conjugados** Dos expresiones, cada una con dos términos, en la que los segundos términos son opuestos.
conjunction A compound statement using the word *and*.	**conjunción** Una declaración compuesta usando la palabra *y*.
consecutive interior angles When two lines are cut by a transversal, interior angles that lie on the same side of the transversal.	**ángulos internos consecutivos** Cuando dos líneas se cortan por un ángulo transversal, interior que se encuentran en el mismo lado de la transversal.
consistent A system of equations with at least one ordered pair that satisfies both equations.	**consistente** Una sistema de ecuaciones para el cual existe al menos un par ordenado que satisface ambas ecuaciones.
constant function A linear function of the form $y = b$; The function $f(x) = a$, where a is any number.	**función constante** Una función lineal de la forma $y = b$; La función $f(x) = a$, donde a es cualquier número.
constant of variation The constant in a variation function.	**constante de variación** La constante en una función de variación.

constant term A term that does not contain a variable.	**término constante** Un término que no contiene una variable.
constraint A condition that a solution must satisfy.	**restricción** Una condición que una solución debe satisfacer.
constructions Methods of creating figures without the use of measuring tools.	**construcciones** Métodos de creación de figuras sin el uso de herramientas de medición.
continuous function A function that can be graphed with a line or an unbroken curve.	**función continua** Una función que se puede representar gráficamente con una línea o una curva ininterrumpida.
continuous random variable The numerical outcome of a random event that can take on any value.	**variable aleatoria continua** El resultado numérico de un evento aleatorio que puede tomar cualquier valor.
contrapositive A statement formed by negating both the hypothesis and the conclusion of the converse of a conditional.	**antítesis** Una afirmación formada negando tanto la hipótesis como la conclusión del inverso del condicional.
convenience sample Members that are readily available or easy to reach are selected.	**muestra conveniente** Se seleccionan los miembros que están fácilmente disponibles o de fácil acceso.
converse A statement formed by exchanging the hypothesis and conclusion of a conditional statement.	**recíproco** Una declaración formada por el intercambio de la hipótesis y la conclusión de la declaración condicional.
convex polygon A polygon with all interior angles measuring less than 180°.	**polígono convexo** Un polígono con todos los ángulos interiores que miden menos de 180°.
coordinate proofs Proofs that use figures in the coordinate plane and algebra to prove geometric concepts.	**pruebas de coordenadas** Pruebas que utilizan figuras en el plano de coordenadas y álgebra para probar conceptos geométricos.
coplanar Lying in the same plane.	**coplanar** Acostado en el mismo plano.
corollary A theorem with a proof that follows as a direct result of another theorem.	**corolario** Un teorema con una prueba que sigue como un resultado directo de otro teorema.
correlation coefficient A measure that shows how well data are modeled by a regression function.	**coeficiente de correlación** Una medida que muestra cómo los datos son modelados por una función de regresión.
corresponding angles When two lines are cut by a transversal, angles that lie on the same side of a transversal and on the same side of the two lines.	**ángulos correspondientes** Cuando dos líneas se cortan transversalmente, los ángulos que se encuentran en el mismo lado de una transversal y en el mismo lado de las dos líneas.
corresponding parts Corresponding angles and corresponding sides of two polygons.	**partes correspondientes** Ángulos correspondientes y lados correspondientes.
cosecant The ratio of the length of a hypotenuse to the length of the leg opposite the angle.	**cosecante** Relación entre la longitud de la hipotenusa y la longitud de la pierna opuesta al ángulo.

cosine The ratio of the length of the leg adjacent to an angle to the length of the hypotenuse.	**coseno** Relación entre la longitud de la pierna adyacente a un ángulo y la longitud de la hipotenusa.
cotangent The ratio of the length of the leg adjacent to an angle to the length of the leg opposite the angle.	**cotangente** La relación entre la longitud de la pata adyacente a un ángulo y la longitud de la pata opuesta al ángulo.
coterminal angles Angles in standard position that have the same terminal side.	**ángulos coterminales** Ángulos en posición estándar que tienen el mismo lado terminal.
counterexample An example that contradicts the conjecture showing that the conjecture is not always true.	**contraejemplo** Un ejemplo que contradice la conjetura que muestra que la conjetura no siempre es cierta.
critical values The z-values corresponding to the most common degrees of certainty.	**valores críticos** Los valores z correspondientes a los grados de certeza más comunes.
cross section The intersection of a solid and a plane.	**sección transversal** Intersección de un sólido con un plano.
cube root One of three equal factors of a number.	**raíz cúbica** Uno de los tres factores iguales de un número.
cube root function A radical function that contains the cube root of a variable expression.	**función de la raíz del cubo** Función radical que contiene la raíz cúbica de una expresión variable.
curve fitting Finding a regression equation for a set of data that is approximated by a function.	**ajuste de curvas** Encontrar una ecuación de regresión para un conjunto de datos que es aproximado por una función.
cycle One complete pattern of a periodic function.	**ciclo** Un patron completo de una función periódica.
cylinder A solid figure with two congruent and parallel circular bases connected by a curved surface.	**cilindro** Una figura sólida con dos bases circulares congruentes y paralelas conectadas por una superficie curvada.

D

decay factor The base of an exponential expression, or $1 - r$.	**factor de decaimiento** La base de una expresión exponencial, o $1 - r$.
decomposition Separating a figure into two or more nonoverlapping parts.	**descomposición** Separar una figura en dos o más partes que no se solapan.
decreasing Where the graph of a function goes down when viewed from left to right.	**decreciente** Donde la gráfica de una función disminuye cuando se ve de izquierda a derecha.
deductive argument An argument that guarantees the truth of the conclusion provided that its premises are true.	**argumento deductivo** Un argumento que garantiza la verdad de la conclusión siempre que sus premisas sean verdaderas.
deductive reasoning The process of reaching a specific valid conclusion based on general facts, rules, definitions, or properties.	**razonamiento deductivo** El proceso de alcanzar una conclusión válida específica basada en hechos generales, reglas, definiciones, o propiedades.
define a variable To choose a variable to represent an unknown value.	**definir una variable** Para elegir una variable que represente un valor desconocido.
defined term A term that has a definition and can be explained.	**término definido** Un término que tiene una definición y se puede explicar.
definitions An explanation that assigns properties to a mathematical object.	**definiciones** Una explicación que asigna propiedades a un objeto matemático.
degree The value of the exponent in a power function; $\frac{1}{360}$ of the circular rotation about a point.	**grado** Valor del exponente en una función de potencia. $\frac{1}{360}$ de la rotación circular alrededor de un punto.
degree of a monomial The sum of the exponents of all its variables.	**grado de un monomio** La suma de los exponents de todas sus variables.
degree of a polynomial The greatest degree of any term in the polynomial.	**grado de un polinomio** El grado mayor de cualquier término del polinomio.
density A measure of the quantity of some physical property per unit of length, area, or volume.	**densidad** Una medida de la cantidad de alguna propiedad física por unidad de longitud, área o volumen.
dependent A consistent system of equations with an infinite number of solutions.	**dependiente** Una sistema consistente de ecuaciones con un número infinito de soluciones.
dependent events Two or more events in which the outcome of one event affects the outcome of the other events.	**eventos dependientes** Dos o más eventos en que el resultado de un evento afecta el resultado de los otros eventos.
dependent variable The variable in a relation, usually y, with values that depend on x.	**variable dependiente** La variable de una relación, generalmente y, con los valores que depende de x.
depressed polynomial A polynomial resulting from division with a degree one less than the original polynomial.	**polinomio reducido** Un polinomio resultante de la división con un grado uno menos que el polinomio original.
descriptive modeling A way to mathematically describe real-world situations and the factors that cause them.	**modelado descriptivo** Una forma de describir matemáticamente las situaciones del mundo real y los factores que las causan.
descriptive statistics The branch of statistics that focuses on collecting, summarizing, and displaying data.	**estadística descriptiva** Rama de la estadística cuyo enfoque es la recopilación, resumen y demostración de los datos.
diagonal A segment that connects any two nonconsecutive vertices within a polygon.	**diagonal** Un segmento que conecta cualquier dos vértices no consecutivos dentro de un polígono.

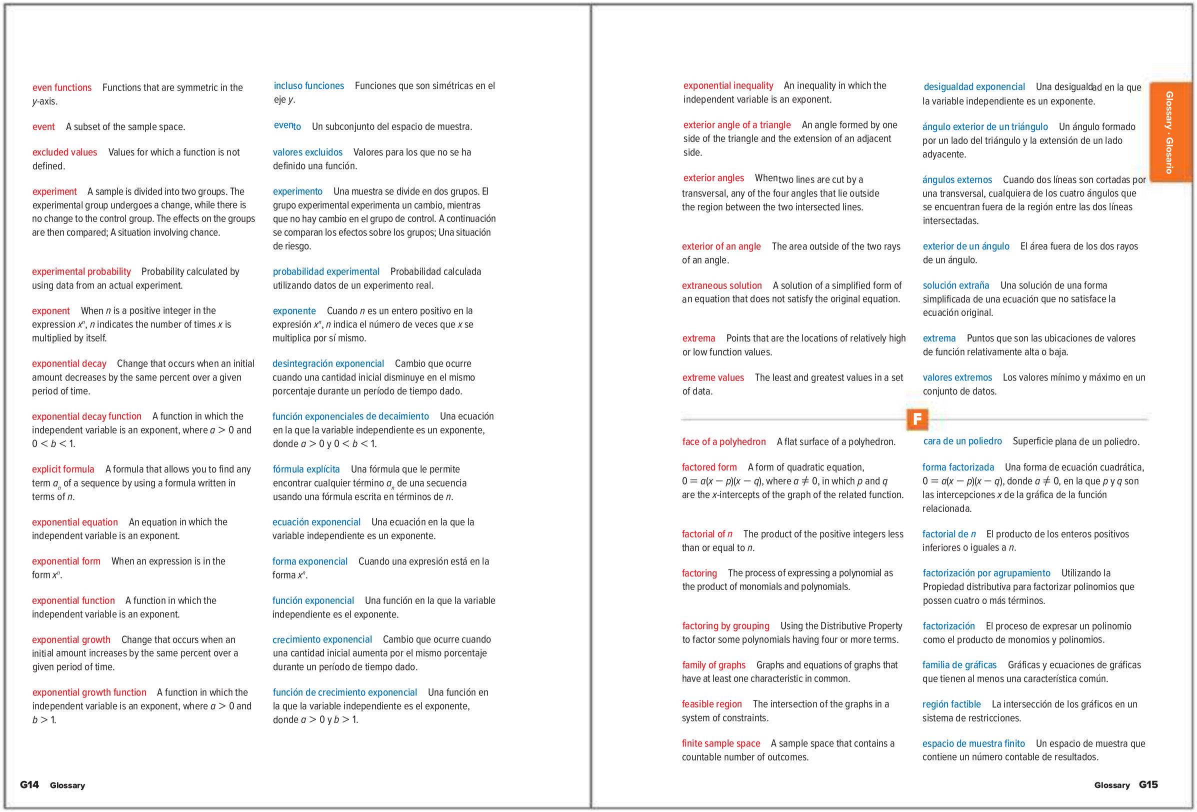


diameter of a circle or sphere A chord that passes through the center of a circle or sphere.	**diámetro de un círculo o esfera** Un acorde que pasa por el centro de un círculo o esfera.
difference of squares A binomial in which the first and last terms are perfect squares.	**diferencia de cuadrados** Un binomio en el que los términos primero y último son cuadrados perfectos.
difference of two squares The square of one quantity minus the square of another quantity.	**diferencia de dos cuadrados** El cuadrado de una cantidad menos el cuadrado de otra cantidad.
dilation A nonrigid motion that enlarges or reduces a geometric figure; A transformation that stretches or compresses the graph of a function.	**dilatación** Un movimiento no rígido que agranda o reduce una figura geométrica; Una transformación que estira o comprime el gráfico de una función.
dimensional analysis The process of performing operations with units.	**análisis dimensional** El proceso de realizar operaciones con unidades.
direct variation When one quantity is equal to a constant times another quantity.	**variación directa** Cuando una cantidad es igual a una constante multiplicada por otra cantidad.
directed line segment A line segment with an initial endpoint and a terminal endpoint.	**segment de línea dirigido** Un segmento de línea con un punto final inicial y un punto final terminal.
directrix An exterior line perpendicular to the line containing the foci of a curve.	**directriz** Una línea exterior perpendicular a la línea que contiene los focos de una curva.
discontinuous function A function that is not continuous.	**función discontinua** Una función que no es continua.
discrete function A function in which the points on the graph are not connected.	**función discreta** Una función en la que los puntos del gráfico no están conectados.
discrete random variable The numerical outcome of a random event that is finite and can be counted.	**variable aleatoria discreta** El resultado numérico de un evento aleatorio que es finito y puede ser contado.
discriminant In the Quadratic Formula, the expression under the radical sign that provides information about the roots of the quadratic equation.	**discriminante** En la Fórmula cuadrática, la expresión bajo el signo radical que proporciona información sobre las raíces de la ecuación cuadrática.
disjunction A compound statement using the word *or*.	**disyunción** Una declaración compuesta usando la palabra *o*.
distance The length of the line segment between two points.	**distancia** La longitud del segmento de línea entre dos puntos.
distribution A graph or table that shows the theoretical frequency of each possible data value.	**distribución** Un gráfico o una table que muestra la frecuencia teórica de cada valor de datos posible.
domain The set of the first numbers of the ordered pairs in a relation; The set of *x*-values to be evaluated by a function.	**dominio** El conjunto de los primeros números de los pares ordenados en una relación; El conjunto de valores *x* para ser evaluados por una función.

dot plot A diagram that shows the frequency of data on a number line.	**gráfica de puntos** Una diagrama que muestra la frecuencia de los datos en una línea numérica.
double root Two roots of a quadratic equation that are the same number.	**raíces dobles** Dos raíces de una función cuadrática que son el mismo número.

E

e An irrational number that approximately equals 2.7182818....	***e*** Un número irracional que es aproximadamente igual a 2.7182818 ....
edge of a polyhedron A line segment where the faces of the polyhedron intersect.	**arista de un poliedro** Un segmento de línea donde las caras del poliedro se cruzan.
elimination A method that involves eliminating a variable by combining the individual equations within a system of equations.	**eliminación** Un método que consiste en eliminar una variable combinando las ecuaciones individuales dentro de un sistema de ecuaciones.
empty set The set that contains no elements, symbolized by { } or ∅.	**conjunto vacío** El conjunto que no contiene elementos, simbolizado por { } o ∅.
end behavior The behavior of a graph at the positive and negative extremes in its domain.	**comportamiento extremo** El comportamiento de un gráfico en los extremos positivo y negativo en su dominio.
enlargement A dilation with a scale factor greater than 1.	**ampliación** Una dilatación con un factor de escala mayor que 1.
equation A mathematical statement that contains two expressions and an equal sign, =.	**ecuación** Un enunciado matemático que contiene dos expresiones y un signo igual, =.
equiangular polygon A polygon with all angles congruent.	**polígono equiangular** Un polígono con todos los ángulos congruentes.
equidistant A point is equidistant from other points if it is the same distance from them.	**equidistante** Un punto es equidistante de otros puntos si está a la misma distancia de ellos.
equidistant lines Two lines for which the distance between the two lines, measured along a perpendicular line or segment to the two lines, is always the same.	**líneas equidistantes** Dos líneas para las cuales la distancia entre las dos líneas, medida a lo largo de una línea o segmento perpendicular a las dos líneas, es siempre la misma.
equilateral polygon A polygon with all sides congruent.	**polígono equilátero** Un polígono con todos los lados congruentes.
equivalent equations Two equations with the same solution.	**ecuaciones equivalentes** Dos ecuaciones con la misma solución.
equivalent expressions Expressions that represent the same value.	**expresiones equivalentes** Expresiones que representan el mismo valor.
evaluate To find the value of an expression.	**evaluar** Calcular el valor de una expresión.



even functions Functions that are symmetric in the y-axis.	**incluso funciones** Funciones que son simétricas en el eje y.
event A subset of the sample space.	**evento** Un subconjunto del espacio de muestra.
excluded values Values for which a function is not defined.	**valores excluidos** Valores para los que no se ha definido una función.
experiment A sample is divided into two groups. The experimental group undergoes a change, while there is no change to the control group. The effects on the groups are then compared; A situation involving chance.	**experimento** Una muestra se divide en dos grupos. El grupo experimental experimenta un cambio, mientras que no hay cambio en el grupo de control. A continuación se comparan los efectos sobre los grupos; Una situación de riesgo.
experimental probability Probability calculated by using data from an actual experiment.	**probabilidad experimental** Probabilidad calculada utilizando datos de un experimento real.
exponent When n is a positive integer in the expression x^n, n indicates the number of times x is multiplied by itself.	**exponente** Cuando n es un entero positivo en la expresión x^n, n indica el número de veces que x se multiplica por sí mismo.
exponential decay Change that occurs when an initial amount decreases by the same percent over a given period of time.	**desintegración exponencial** Cambio que ocurre cuando una cantidad inicial disminuye en el mismo porcentaje durante un período de tiempo dado.
exponential decay function A function in which the independent variable is an exponent, where $a > 0$ and $0 < b < 1$.	**función exponenciales de decaimiento** Una ecuación en la que la variable independiente es un exponente, donde $a > 0$ y $0 < b < 1$.
explicit formula A formula that allows you to find any term a_n of a sequence by using a formula written in terms of n.	**fórmula explícita** Una fórmula que le permite encontrar cualquier término a_n de una secuencia usando una fórmula escrita en términos de n.
exponential equation An equation in which the independent variable is an exponent.	**ecuación exponencial** Una ecuación en la que la variable independiente es un exponente.
exponential form When an expression is in the form x^n.	**forma exponencial** Cuando una expresión está en la forma x^n.
exponential function A function in which the independent variable is an exponent.	**función exponencial** Una función en la que la variable independiente es el exponente.
exponential growth Change that occurs when an initial amount increases by the same percent over a given period of time.	**crecimiento exponencial** Cambio que ocurre cuando una cantidad inicial aumenta por el mismo porcentaje durante un período de tiempo dado.
exponential growth function A function in which the independent variable is an exponent, where $a > 0$ and $b > 1$.	**función de crecimiento exponencial** Una función en la que la variable independiente es el exponente, donde $a > 0$ y $b > 1$.

G14 Glossary

Glossary · Glosario

exponential inequality An inequality in which the independent variable is an exponent.	**desigualdad exponencial** Una desigualdad en la que la variable independiente es un exponente.
exterior angle of a triangle An angle formed by one side of the triangle and the extension of an adjacent side.	**ángulo exterior de un triángulo** Un ángulo formado por un lado del triángulo y la extensión de un lado adyacente.
exterior angles When two lines are cut by a transversal, any of the four angles that lie outside the region between the two intersected lines.	**ángulos externos** Cuando dos líneas son cortadas por una transversal, cualquiera de los cuatro ángulos que se encuentran fuera de la región entre las dos líneas intersectadas.
exterior of an angle The area outside of the two rays of an angle.	**exterior de un ángulo** El área fuera de los dos rayos de un ángulo.
extraneous solution A solution of a simplified form of an equation that does not satisfy the original equation.	**solución extraña** Una solución de una forma simplificada de una ecuación que no satisface la ecuación original.
extrema Points that are the locations of relatively high or low function values.	**extrema** Puntos que son las ubicaciones de valores de función relativamente alta o baja.
extreme values The least and greatest values in a set of data.	**valores extremos** Los valores mínimo y máximo en un conjunto de datos.

F

face of a polyhedron A flat surface of a polyhedron.	**cara de un poliedro** Superficie plana de un poliedro.
factored form A form of quadratic equation, $0 = a(x - p)(x - q)$, where $a \neq 0$, in which p and q are the x-intercepts of the graph of the related function.	**forma factorizada** Una forma de ecuación cuadrática, $0 = a(x - p)(x - q)$, donde $a \neq 0$, en la que p y q son las intercepciones x de la gráfica de la función relacionada.
factorial of n The product of the positive integers less than or equal to n.	**factorial de n** El producto de los enteros positivos inferiores o iguales a n.
factoring The process of expressing a polynomial as the product of monomials and polynomials.	**factorización por agrupamiento** Utilizando la Propiedad distributiva para factorizar polinomios que possen cuatro o más términos.
factoring by grouping Using the Distributive Property to factor some polynomials having four or more terms.	**factorización** El proceso de expresar un polinomio como el producto de monomios y polinomios.
family of graphs Graphs and equations of graphs that have at least one characteristic in common.	**familia de gráficas** Gráficas y ecuaciones de gráficas que tienen al menos una característica común.
feasible region The intersection of the graphs in a system of constraints.	**región factible** La intersección de los gráficos en un sistema de restricciones.
finite sample space A sample space that contains a countable number of outcomes.	**espacio de muestra finito** Un espacio de muestra que contiene un número contable de resultados.

Glossary G15

finite sequence A sequence that contains a limited number of terms.	**secuencia finita** Una secuencia que contiene un número limitado de términos.
five-number summary The minimum, quartiles, and maximum of a data set.	**resumen de cinco números** El mínimo, cuartiles y máximo de un conjunto de datos.
flow proof A proof that uses boxes and arrows to show the logical progression of an argument.	**demostración de flujo** Una prueba que usa cajas y flechas para mostrar la progresión lógica de un argumento.
focus A point inside a parabola having the property that the distances from any point on the parabola to them and to a fixed line have a constant ratio for any points on the parabola.	**foco** Un punto dentro de una parábola que tiene la propiedad de que las distancias desde cualquier punto de la parábola a ellos ya una línea fija tienen una relación constante para cualquier punto de la parábola.
formula An equation that expresses a relationship between certain quantities.	**fórmula** Una ecuación que expresa una relación entre ciertas cantidades.
fractional distance An intermediary point some fraction of the length of a line segment.	**distancia fraccionaria** Un punto intermediario de alguna fracción de la longitud de un segmento de línea.
frequency The number of cycles in a given unit of time.	**frecuencia** El número de ciclos en una unidad del tiempo dada.
function A relation in which each element of the domain is paired with exactly one element of the range.	**función** Una relación en que a cada elemento del dominio de corresponde un único elemento del rango.
function notation A way of writing an equation so that $y = f(x)$.	**notación functional** Una forma de escribir una ecuación para que $y = f(x)$.

G

geometric means The terms between two nonconsecutive terms of a geometric sequence; The nth root, where n is the number of elements in a set of numbers, of the product of the numbers.	**medios geométricos** Los términos entre dos términos no consecutivos de una secuencia geométrica; La enésima raíz, donde n es el número de elementos de un conjunto de números, del producto de los números.
geometric model A geometric figure that represents a real-life object.	**modelo geométrico** Una figura geométrica que representa un objeto de la vida real.
geometric probability Probability that involves a geometric measure such as length or area.	**probabilidad geométrica** Probabilidad que implica una medida geométrica como longitud o área.
geometric sequence A pattern of numbers that begins with a nonzero term and each term after is found by multiplying the previous term by a nonzero constant r.	**secuencia geométrica** Un patrón de números que comienza con un término distinto de cero y cada término después se encuentra multiplicando el término anterior por una constante no nula r.
geometric series The indicated sum of the terms in a geometric sequence.	**series geométricas** La suma indicada de los términos en una secuencia geométrica.

glide reflection The composition of a translation followed by a reflection in a line parallel to the translation vector.	**reflexión del deslizamiento** La composición de una traducción seguida de una reflexión en una línea paralela al vector de traslación.
greatest integer function A step function in which $f(x)$ is the greatest integer less than or equal to x.	**función entera más grande** Una función del paso en que $f(x)$ es el número más grande menos que o igual a x.
growth factor The base of an exponential expression, or $1 + r$.	**factor de crecimiento** La base de una expresión exponencial, o $1 + r$.

H

half-plane A region of the graph of an inequality on one side of a boundary.	**semi-plano** Una región de la gráfica de una desigualdad en un lado de un límite.
height of a parallelogram The length of an altitude of the parallelogram.	**altura de un paralelogramo** La longitud de la altitud del paralelogramo.
height of a solid The length of the altitude of a solid figure.	**altura de un sólido** La longitud de la altitud de una figura sólida.
height of a trapezoid The perpendicular distance between the bases of a trapezoid.	**altura de un trapecio** La distancia perpendicular entre las bases de un trapecio.
histogram A graphical display that uses bars to display numerical data that have been organized in equal intervals.	**histograma** Una exhibición gráfica que utiliza barras para exhibir los datos numéricos que se han organizado en intervalos iguales.
horizontal asymptote A horizontal line that a graph approaches.	**asíntota horizontal** Una línea horizontal que se aproxima a un gráfico.
hyperbola The graph of a reciprocal function.	**hipérbola** La gráfica de una función recíproca.
hypothesis The statement that immediately follows the word *if* in a conditional.	**hipótesis** La declaración que sigue inmediatamente a la palabra *si* en un condicional.

I

identity An equation that is true for every value of the variable.	**identidad** Una ecuación que es verdad para cada valor de la variable.
identity function The function $f(x) = x$.	**función identidad** La función $f(x) = x$.
if-then statement A compound statement of the form *if p, then q*, where p and q are statements.	**enunciado si-entonces** Enunciado compuesto de la forma *si p, entonces q*, donde p y q son enunciados.
image The new figure in a transformation.	**imagen** La nueva figura en una transformación.
imaginary unit i The principal square root of -1.	**unidad imaginaria i** La raíz cuadrada principal de -1.
incenter The point of concurrency of the angle bisectors of a triangle.	**incentro** El punto de intersección de las bisectrices interiores de un triángulo.

included angle The interior angle formed by two adjacent sides of a triangle.	**ángulo incluido** El ángulo interior formado por dos lados adyacentes de un triángulo.
included side The side of a triangle between two angles.	**lado incluido** El lado de un triángulo entre dos ángulos.
inconsistent A system of equations with no ordered pair that satisfies both equations.	**inconsistente** Una sistema de ecuaciones para el cual no existe par ordenado alguno que satisfaga ambas ecuaciones.
increasing Where the graph of a function goes up when viewed from left to right.	**creciente** Donde la gráfica de una función sube cuando se ve de izquierda a derecha.
independent A consistent system of equations with exactly one solution.	**independiente** Un sistema consistente de ecuaciones con exactamente una solución.
independent events Two or more events in which the outcome of one event does not affect the outcome of the other events.	**eventos independientes** Dos o más eventos en los que el resultado de un evento no afecta el resultado de los otros eventos.
independent variable The variable in a relation, usually x, with a value that is subject to choice.	**variable independiente** La variable de una relación, generalmente x, con el valor que sujeta a elección.
index In nth roots, the value that indicates to what root the value under the radicand is being taken.	**índice** En enésimas raíces, el valor que indica a qué raíz está el valor bajo la radicand.
indirect measurement Using similar figures and proportions to measure an object.	**medición indirecta** Usando figuras y proporciones similares para medir un objeto.
indirect proof One assumes that the statement to be proven is false and then uses logical reasoning to deduce that a statement contradicts a postulate, theorem, or one of the assumptions.	**demostración indirecta** Se supone que la afirmación a ser probada es falsa y luego utiliza el razonamiento lógico para deducir que una afirmación contradice un postulado, teorema o uno de los supuestos.
indirect reasoning Reasoning that eliminates all possible conclusions but one so that the one remaining conclusion must be true.	**razonamiento indirecto** Razonamiento que elimina todas las posibles conclusiones, pero una de manera que la conclusión que queda una debe ser verdad.
inductive reasoning The process of reaching a conclusion based on a pattern of examples.	**razonamiento inductive** El proceso de llegar a una conclusión basada en un patrón de ejemplos.
inequality A mathematical sentence that contains $<, >, \leq, \geq$, or $\neq$.	**desigualdad** Una oración matemática que contiene uno o más de $<, >, \leq, \geq$, o $\neq$.
inferential statistics When the data from a sample is used to make inferences about the corresponding population.	**estadísticas inferencial** Cuando los datos de una muestra se utilizan para hacer inferencias sobre la población correspondiente.
infinite sample space A sample space with outcomes that cannot be counted.	**espacio de muestra infinito** Un espacio de muestra con resultados que no pueden ser contados.
infinite sequence A sequence that continues without end.	**secuencia infinita** Una secuencia que continúa sin fin.

informal proof A paragraph that explains why the conjecture for a given situation is true.	**prueba informal** Un párrafo que explica por qué la conjetura para una situación dada es verdadera.
initial side The part of an angle that is fixed on the x-axis.	**lado inicial** La parte de un ángulo que se fija en el eje x.
inscribed angle An angle with its vertex on a circle and sides that contain chords of the circle.	**ángulo inscrito** Un ángulo con su vértice en un círculo y lados que contienen acordes del círculo.
inscribed polygon A polygon inside a circle in which all of the vertices of the polygon lie on the circle.	**polígono inscrito** Un polígono dentro de un círculo en el que todos los vértices del polígono se encuentran en el círculo.
intercept A point at which the graph of a function intersects an axis.	**interceptar** Un punto en el que la gráfica de una función corta un eje.
intercepted arc The part of a circle that lies between the two lines intersecting it.	**arco intersecado** La parte de un círculo que se encuentra entre las dos líneas que se cruzan.
interior angle of a triangle An angle at the vertex of a triangle.	**ángulo interior de un triángulo** Un ángulo en el vértice de un triángulo.
interior angles When two lines are cut by a transversal, any of the four angles that lie inside the region between the two intersected lines.	**ángulos interiores** Cuando dos líneas son cortadas por una transversal, cualquiera de los cuatro ángulos que se encuentran dentro de la región entre las dos líneas intersectadas.
interior of an angle The area between the two rays of an angle.	**interior de un ángulo** El área entre los dos rayos de un ángulo.
interquartile range The difference between the upper and lower quartiles of a data set.	**rango intercuartil** La diferencia entre el cuartil superior y el cuartil inferior de un conjunto de datos.
intersection A set of points common to two or more geometric figures; intersection The graph of a compound inequality containing *and*.	**intersección** Un conjunto de puntos communes a dos o más figuras geométricas; intersección La gráfica de una desigualdad compuesta que contiene la palabra *y*.
intersection of A and B The set of all outcomes in the sample space of event A that are also in the sample space of event B.	**intersección de A y B** El conjunto de todos los resultados en el espacio muestral del evento A que también se encuentran en el espacio muestral del evento B.
interval The distance between two numbers on the scale of a graph.	**intervalo** La distancia entre dos números en la escala de un gráfico.
interval notation Mathematical notation that describes a set by using endpoints with parentheses or brackets.	**notación de intervalo** Notación matemática que describe un conjunto utilizando puntos finales con paréntesis o soportes.
inverse A statement formed by negating both the hypothesis and conclusion of a conditional statement.	**inverso** Una declaración formada negando tanto la hipótesis como la conclusión de la declaración condicional.

inverse cosine The ratio of the length of the hypotenuse to the length of the leg adjacent to an angle.	**inverso del coseno** Relación de la longitud de la hipotenusa con la longitud de la pierna adyacente a un ángulo.
inverse functions Two functions, one of which contains points of the form (a, b) while the other contains points of the form (b, a).	**funciones inversas** Dos funciones, una de las cuales contiene puntos de la forma (a, b) mientras que la otra contiene puntos de la forma (b, a).
inverse relations Two relations, one of which contains points of the form (a, b) while the other contains points of the form (b, a).	**relaciones inversas** Dos relaciones, una de las cuales contiene puntos de la forma (a, b) mientras que la otra contiene puntos de la forma (b, a).
inverse sine The ratio of the length of the hypotenuse to the length of the leg opposite an angle.	**inverso del seno** Relación de la longitud de la hipotenusa con la longitud de la pierna opuesta a un ángulo.
inverse tangent The ratio of the length of the leg adjacent to an angle to the length of the leg opposite the angle.	**inverso del tangente** Relación de la longitud de la pierna adyacente a un ángulo con la longitud de la pierna opuesta a un ángulo.
inverse trigonometric functions Arcsine, Arccosine, and Arctangent.	**funciones trigonométricas inversas** Arcsine, Arccosine y Arctangent.
inverse variation When the product of two quantities is equal to a constant k.	**variación inversa** Cuando el producto de dos cantidades es igual a una constante k.
isosceles trapezoid A quadrilateral in which two sides are parallel and the legs are congruent.	**trapecio isósceles** Un cuadrilátero en el que dos lados son paralelos y las patas son congruentes.
isosceles triangle A triangle with at least two sides congruent.	**triángulo isósceles** Un triángulo con al menos dos lados congruentes.

J

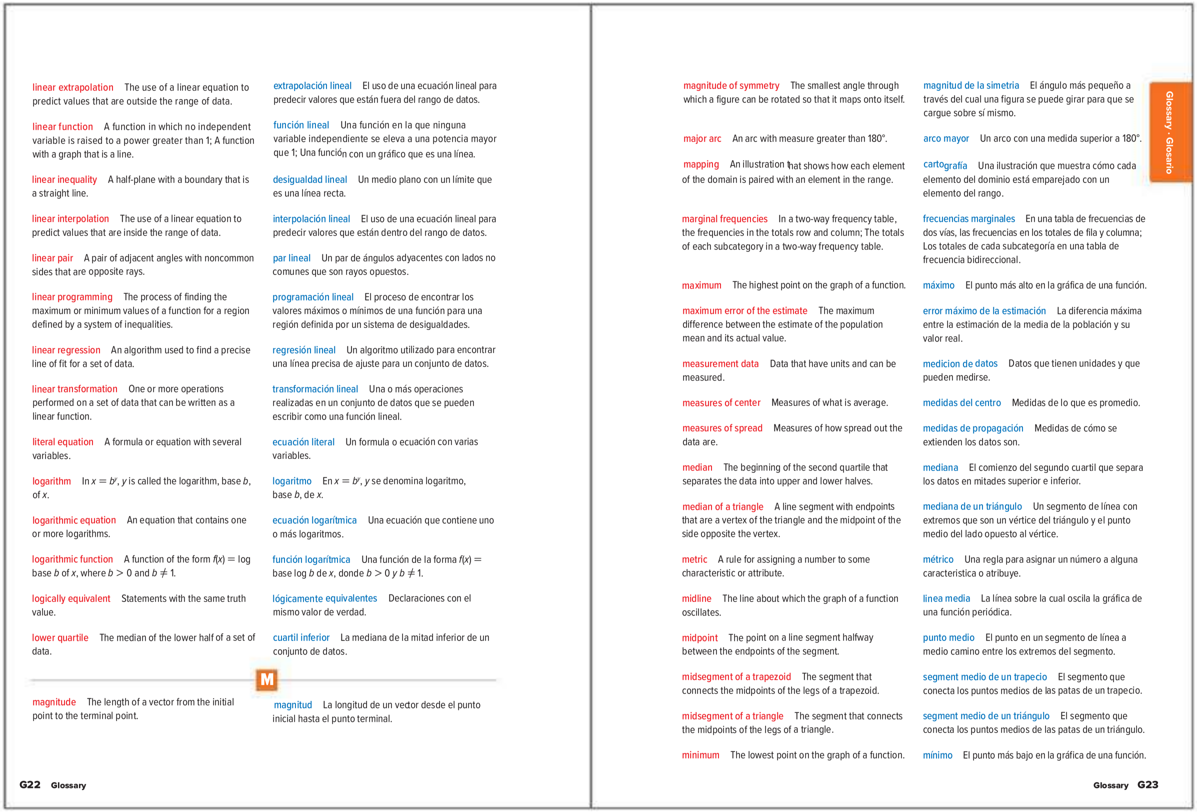
joint frequencies Entries in the body of a two-way frequency table. In a two-way frequency table, the frequencies in the interior of the table.	**frecuencias articulares** Entradas en el cuerpo de una tabla de frecuencias de dos vías. En una tabla de frecuencia bidireccional, las frecuencias en el interior de la tabla.
joint variation When one quantity varies directly as the product of two or more other quantities.	**variación conjunta** Cuando una cantidad varía directamente como el producto de dos o más cantidades.

K

kite A convex quadrilateral with exactly two distinct pairs of adjacent congruent sides.	**cometa** Un cuadrilátero convexo con exactamente dos pares distintos de lados congruentes adyacentes.

L

lateral area The sum of the areas of the lateral faces of the figure.	**área lateral** La suma de las áreas de las caras laterales de la figura.
lateral edges The intersection of two lateral faces.	**aristas laterales** La intersección de dos caras laterales.
lateral faces The faces that join the bases of a solid.	**caras laterales** Las caras que unen las bases de un sólido.
lateral surface of a cone The curved surface that joins the base of a cone to the vertex.	**superficie lateral de un cono** La superficie curvada que une la base de un cono con el vértice.
lateral surface of a cylinder The curved surface that joins the bases of a cylinder.	**superficie lateral de un cilindro** La superficie curvada que une las bases de un cilindro.
leading coefficient The coefficient of the first term when a polynomial is in standard form.	**coeficiente líder** El coeficiente del primer término cuando un polinomio está en forma estándar.
legs of a trapezoid The nonparallel sides in a trapezoid.	**patas de un trapecio** Los lados no paralelos en un trapezoide.
legs of an isosceles triangle The two congruent sides of an isosceles triangle.	**patas de un triángulo isósceles** Los dos lados congruentes de un triángulo isósceles.
like radical expressions Radicals in which both the index and the radicand are the same.	**expresiones radicales semejantes** Radicales en los que tanto el índice como el radicand son iguales.
like terms Terms with the same variables, with corresponding variables having the same exponent.	**términos semejantes** Términos con las mismas variables, con las variables correspondientes que tienen el mismo exponente.
line A line is made up of points, has no thickness or width, and extends indefinitely in both directions.	**línea** Una línea está formada por puntos, no tiene espesor ni anchura, y se extiende indefinidamente en ambas direcciones.
line of fit A line used to describe the trend of the data in a scatter plot.	**línea de ajuste** Una línea usada para describir la tendencia de los datos en un diagrama de dispersión.
line of reflection A line midway between a preimage and an image; The line in which a reflection flips the graph of a function.	**línea de reflexión** Una línea a medio camino entre una preimagen y una imagen; La línea en la que una reflexión voltea la gráfica de una función.
line of symmetry An imaginary line that separates a figure into two congruent parts.	**línea de simetría** Una línea imaginaria que separa una figura en dos partes congruentes.
line segment A measurable part of a line that consists of two points, called endpoints, and all of the points between them.	**segmento de línea** Una parte medible de una línea que consta de dos puntos, llamados extremos, y todos los puntos entre ellos.
line symmetry A graph has line symmetry if it can be reflected in a vertical line so that each half of the graph maps exactly to the other half.	**simetría de línea** Un gráfico tiene simetría de línea si puede reflejarse en una línea vertical, de modo que cada mitad del gráfico se asigna exactamente a la otra mitad.
linear equation An equation that can be written in the form $Ax + By = C$ with a graph that is a straight line.	**ecuación lineal** Una ecuación que puede escribirse de la forma $Ax + By = C$ con un gráfico que es una línea recta.



Glossary · Glosario

linear extrapolation The use of a linear equation to predict values that are outside the range of data.	**extrapolación lineal** El uso de una ecuación lineal para predecir valores que están fuera del rango de datos.
linear function A function in which no independent variable is raised to a power greater than 1; A function with a graph that is a line.	**función lineal** Una función en la que ninguna variable independiente se eleva a una potencia mayor que 1; Una función con un gráfico que es una línea.
linear inequality A half-plane with a boundary that is a straight line.	**desigualdad lineal** Un medio plano con un límite que es una línea recta.
linear interpolation The use of a linear equation to predict values that are inside the range of data.	**interpolación lineal** El uso de una ecuación lineal para predecir valores que están dentro del rango de datos.
linear pair A pair of adjacent angles with noncommon sides that are opposite rays.	**par lineal** Un par de ángulos adyacentes con lados no comunes que son rayos opuestos.
linear programming The process of finding the maximum or minimum values of a function for a region defined by a system of inequalities.	**programación lineal** El proceso de encontrar los valores máximos o mínimos de una función para una región definida por un sistema de desigualdades.
linear regression An algorithm used to find a precise line of fit for a set of data.	**regresión lineal** Un algoritmo utilizado para encontrar una línea precisa de ajuste para un conjunto de datos.
linear transformation One or more operations performed on a set of data that can be written as a linear function.	**transformación lineal** Una o más operaciones realizadas en un conjunto de datos que se pueden escribir como una función lineal.
literal equation A formula or equation with several variables.	**ecuación literal** Un formula o ecuación con varias variables.
logarithm In $x = b^y$, y is called the logarithm, base b, of x.	**logaritmo** En $x = b^y$, y se denomina logaritmo, base b, de x.
logarithmic equation An equation that contains one or more logarithms.	**ecuación logarítmica** Una ecuación que contiene uno o más logaritmos.
logarithmic function A function of the form $f(x) = \log$ base b of x, where $b > 0$ and $b \neq 1$.	**función logarítmica** Una función de la forma $f(x) =$ base $\log b$ de x, donde $b > 0$ y $b \neq 1$.
logically equivalent Statements with the same truth value.	**lógicamente equivalentes** Declaraciones con el mismo valor de verdad.
lower quartile The median of the lower half of a set of data.	**cuartil inferior** La mediana de la mitad inferior de un conjunto de datos.

M

magnitude The length of a vector from the initial point to the terminal point.	**magnitud** La longitud de un vector desde el punto inicial hasta el punto terminal.

G22 Glossary

magnitude of symmetry The smallest angle through which a figure can be rotated so that it maps onto itself.	**magnitud de la simetria** El ángulo más pequeño a través del cual una figura se puede girar para que se cargue sobre sí mismo.
major arc An arc with measure greater than 180°.	**arco mayor** Un arco con una medida superior a 180°.
mapping An illustration that shows how each element of the domain is paired with an element in the range.	**cartografía** Una ilustración que muestra cómo cada elemento del dominio está emparejado con un elemento del rango.
marginal frequencies In a two-way frequency table, the frequencies in the totals row and column; The totals of each subcategory in a two-way frequency table.	**frecuencias marginales** En una tabla de frecuencias de dos vías, las frecuencias en los totales de fila y columna; Los totales de cada subcategoría en una tabla de frecuencia bidireccional.
maximum The highest point on the graph of a function.	**máximo** El punto más alto en la gráfica de una función.
maximum error of the estimate The maximum difference between the estimate of the population mean and its actual value.	**error máximo de la estimación** La diferencia máxima entre la estimación de la media de la población y su valor real.
measurement data Data that have units and can be measured.	**medicion de datos** Datos que tienen unidades y que pueden medirse.
measures of center Measures of what is average.	**medidas del centro** Medidas de lo que es promedio.
measures of spread Measures of how spread out the data are.	**medidas de propagación** Medidas de cómo se extienden los datos son.
median The beginning of the second quartile that separates the data into upper and lower halves.	**mediana** El comienzo del segundo cuartil que separa los datos en mitades superior e inferior.
median of a triangle A line segment with endpoints that are a vertex of the triangle and the midpoint of the side opposite the vertex.	**mediana de un triángulo** Un segmento de línea con extremos que son un vértice del triángulo y el punto medio del lado opuesto al vértice.
metric A rule for assigning a number to some characteristic or attribute.	**métrico** Una regla para asignar un número a alguna caracteristica o atribuye.
midline The line about which the graph of a function oscillates.	**linea media** La línea sobre la cual oscila la gráfica de una función periódica.
midpoint The point on a line segment halfway between the endpoints of the segment.	**punto medio** El punto en un segmento de línea a medio camino entre los extremos del segmento.
midsegment of a trapezoid The segment that connects the midpoints of the legs of a trapezoid.	**segment medio de un trapecio** El segmento que conecta los puntos medios de las patas de un trapecio.
midsegment of a triangle The segment that connects the midpoints of the legs of a triangle.	**segment medio de un triángulo** El segmento que conecta los puntos medios de las patas de un triángulo.
minimum The lowest point on the graph of a function.	**mínimo** El punto más bajo en la gráfica de una función.

Glossary G23

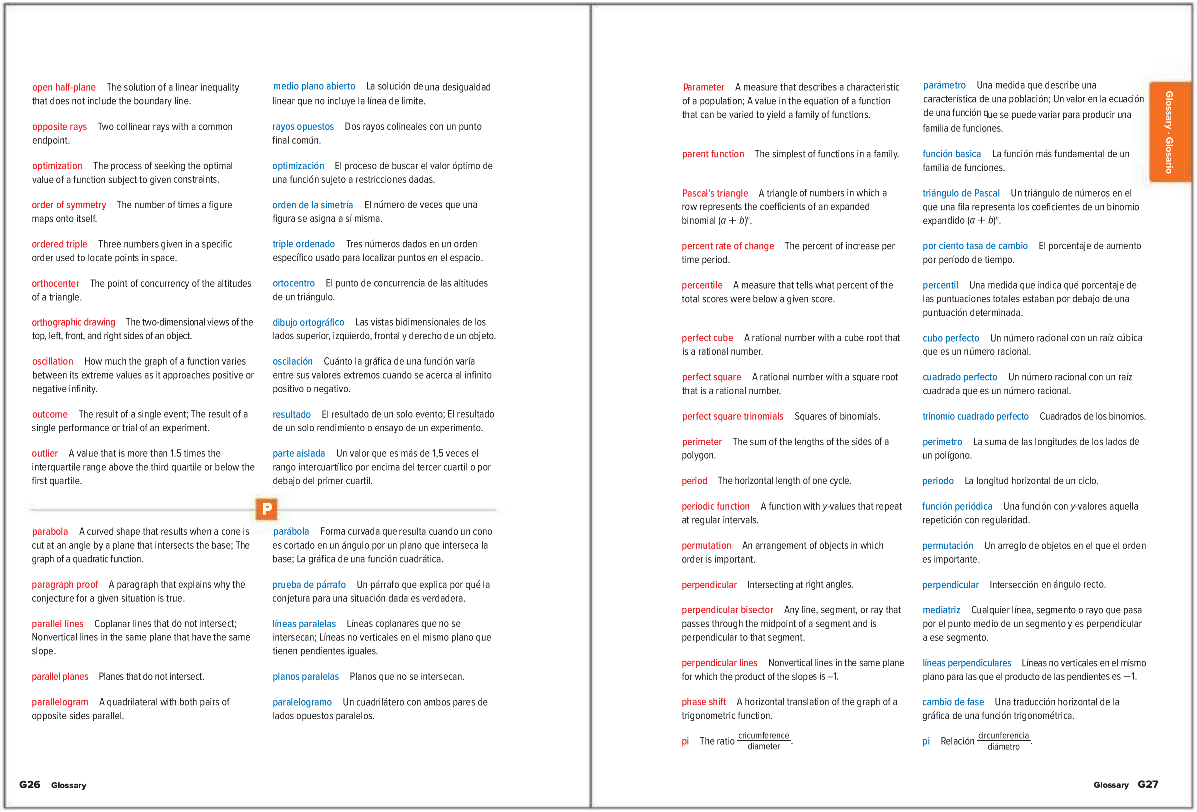
minor arc An arc with measure less than 180°.	arco menor Un arco con una medida inferior a 180°.
mixture problems Problems that involve creating a mixture of two or more kinds of things and then determining some quantity of the resulting mixture.	problemas de mezcla Problemas que implican crear una mezcla de dos o más tipos de cosas y luego determinar una cierta cantidad de la mezcla resultante.
monomial A number, a variable, or a product of a number and one or more variables.	monomio Un número, una variable, o un producto de un número y una o más variables.
monomial function A function of the form $f(x) = ax^n$, for which a is a nonzero real number and n is a positive integer.	función monomial Una función de la forma $f(x) = ax^n$, para la cual a es un número real no nulo y n es un entero positivo.
multi-step equation An equation that uses more than one operation to solve it.	ecuaciones de varios pasos Una ecuación que utiliza más de una operación para resolverla.
multiplicative identity Because the product of any number a and 1 is equal to a, 1 is the multiplicative identity.	identidad multiplicativa Dado que el producto de cualquier número a y 1 es igual a, 1 es la identidad multiplicativa.
multiplicative inverses Two numbers with a product of 1.	inversos multiplicativos Dos números con un producto es igual a 1.
multiplicity The number of times a number is a zero for a given polynomial.	multiplicidad El número de veces que un número es cero para un polinomio dado.
mutually exclusive Events that cannot occur at the same time.	mutuamente exclusivos Eventos que no pueden ocurrir al mismo tiempo.

N

natural base exponential function An exponential function with base e, written as $y = e^x$.	función exponencial de base natural Una función exponencial con base e, escrita como $y = e^x$.
natural logarithm The inverse of the natural base exponential function, most often abbreviated as ln x.	logaritmo natural La inversa de la función exponencial de base natural, más a menudo abreviada como ln x.
negation A statement that has the opposite meaning, as well as the opposite truth value, of an original statement.	negación Una declaración que tiene el significado opuesto, así como el valor de verdad opuesto, de una declaración original.
negative Where the graph of a function lies below the x-axis.	negativo Donde la gráfica de una función se encuentra debajo del eje x.
negative correlation Bivariate data in which y decreases as x increases.	correlación negativa Datos bivariate en el cual y disminuye a x aumenta.
negative exponent An exponent that is a negative number.	exponente negativo Un exponente que es un número negativo.
negatively skewed distribution A distribution that typically has a median greater than the mean and less data on the left side of the graph.	distribución negativamente sesgada Una distribución que típicamente tiene una mediana mayor que la media y menos datos en el lado izquierdo del gráfico.
net A two-dimensional figure that forms the surfaces of a three-dimensional object when folded.	red Una figura bidimensional que forma las superficies de un objeto tridimensional cuando se dobla.
no correlation Bivariate data in which x and y are not related.	sin correlación Datos bivariados en los que x e y no están relacionados.
nonlinear function A function in which a set of points cannot all lie on the same line	función no lineal Una función en la que un conjunto de puntos no puede estar en la misma línea
nonrigid motion A transformation that changes the dimensions of a given figure.	movimiento no rígida Una transformación que cambia las dimensiones de una figura dada.
normal distribution A continuous, symmetric, bell-shaped distribution of a random variable.	distribución normal Distribución con forma de campana, simétrica y continua de una variable aleatoria.
nth root If $a^n = b$ for a positive integer n, then a is the nth root of b.	raíz enésima Si $a^n = b$ para cualquier entero positive n, entonces a se llama una raíz enésima de b.
nth term of an arithmetic sequence The nth term of an arithmetic sequence with first term a_1 and common difference d is given by $a_n = a_1 + (n - 1)d$, where n is a positive integer.	enésimo término de una secuencia aritmética El enésimo término de una secuencia aritmética con el primer término a_1 y la diferencia común d viene dado por $a_n = a_1 + (n - 1)d$, donde n es un número entero positivo.
numerical expression A mathematical phrase involving only numbers and mathematical operations.	expresión numérica Una frase matemática que implica sólo números y operaciones matemáticas.

O

oblique asymptote An asymptote that is neither horizontal nor vertical.	asíntota oblicua Una asíntota que no es ni horizontal ni vertical.
observational study Members of a sample are measured or observed without being affected by the study.	estudio de observación Los miembros de una muestra son medidos o observados sin ser afectados por el estudio.
octant One of the eight divisions of three-dimensional space.	octante Una de las ocho divisiones del espacio tridimensional.
odd functions Functions that are symmetric in the origin.	funciones extrañas Funciones que son simétricas en el origen.
one-to-one function A function for which each element of the range is paired with exactly one element of the domain.	función biunívoca Función para la cual cada elemento del rango está emparejado con exactamente un elemento del dominio.
onto function A function for which the codomain is the same as the range.	sobre la función Función para la cual el codomain es el mismo que el rango.



open half-plane	The solution of a linear inequality that does not include the boundary line.	medio plano abierto	La solución de una desigualdad linear que no incluye la línea de límite.
opposite rays	Two collinear rays with a common endpoint.	rayos opuestos	Dos rayos colineales con un punto final común.
optimization	The process of seeking the optimal value of a function subject to given constraints.	optimización	El proceso de buscar el valor óptimo de una función sujeto a restricciones dadas.
order of symmetry	The number of times a figure maps onto itself.	orden de la simetría	El número de veces que una figura se asigna a sí misma.
ordered triple	Three numbers given in a specific order used to locate points in space.	triple ordenado	Tres números dados en un orden específico usado para localizar puntos en el espacio.
orthocenter	The point of concurrency of the altitudes of a triangle.	ortocentro	El punto de concurrencia de las altitudes de un triángulo.
orthographic drawing	The two-dimensional views of the top, left, front, and right sides of an object.	dibujo ortográfico	Las vistas bidimensionales de los lados superior, izquierdo, frontal y derecho de un objeto.
oscillation	How much the graph of a function varies between its extreme values as it approaches positive or negative infinity.	oscilación	Cuánto la gráfica de una función varía entre sus valores extremos cuando se acerca al infinito positivo o negativo.
outcome	The result of a single event; The result of a single performance or trial of an experiment.	resultado	El resultado de un solo evento; El resultado de un solo rendimiento o ensayo de un experimento.
outlier	A value that is more than 1.5 times the interquartile range above the third quartile or below the first quartile.	parte aislada	Un valor que es más de 1,5 veces el rango intercuartílico por encima del tercer cuartil o por debajo del primer cuartil.

P

parabola	A curved shape that results when a cone is cut at an angle by a plane that intersects the base; The graph of a quadratic function.	parábola	Forma curvada que resulta cuando un cono es cortado en un ángulo por un plano que interseca la base; La gráfica de una función cuadrática.
paragraph proof	A paragraph that explains why the conjecture for a given situation is true.	prueba de párrafo	Un párrafo que explica por qué la conjetura para una situación dada es verdadera.
parallel lines	Coplanar lines that do not intersect; Nonvertical lines in the same plane that have the same slope.	líneas paralelas	Líneas coplanares que no se intersecan; Líneas no verticales en el mismo plano que tienen pendientes iguales.
parallel planes	Planes that do not intersect.	planos paralelas	Planos que no se intersecan.
parallelogram	A quadrilateral with both pairs of opposite sides parallel.	paralelogramo	Un cuadrilátero con ambos pares de lados opuestos paralelos.

Parameter	A measure that describes a characteristic of a population; A value in the equation of a function that can be varied to yield a family of functions.	parámetro	Una medida que describe una característica de una población; Un valor en la ecuación de una función que se puede variar para producir una familia de funciones.
parent function	The simplest of functions in a family.	función basica	La función más fundamental de un familia de funciones.
Pascal's triangle	A triangle of numbers in which a row represents the coefficients of an expanded binomial $(a + b)^n$.	triángulo de Pascal	Un triángulo de números en el que una fila representa los coeficientes de un binomio expandido $(a + b)^n$.
percent rate of change	The percent of increase per time period.	por ciento tasa de cambio	El porcentaje de aumento por período de tiempo.
percentile	A measure that tells what percent of the total scores were below a given score.	percentil	Una medida que indica qué porcentaje de las puntuaciones totales estaban por debajo de una puntuación determinada.
perfect cube	A rational number with a cube root that is a rational number.	cubo perfecto	Un número racional con un raíz cúbica que es un número racional.
perfect square	A rational number with a square root that is a rational number.	cuadrado perfecto	Un número racional con un raíz cuadrada que es un número racional.
perfect square trinomials	Squares of binomials.	trinomio cuadrado perfecto	Cuadrados de los binomios.
perimeter	The sum of the lengths of the sides of a polygon.	perimetro	La suma de las longitudes de los lados de un polígono.
period	The horizontal length of one cycle.	periodo	La longitud horizontal de un ciclo.
periodic function	A function with y-values that repeat at regular intervals.	función periódica	Una función con y-valores aquella repetición con regularidad.
permutation	An arrangement of objects in which order is important.	permutación	Un arreglo de objetos en el que el orden es importante.
perpendicular	Intersecting at right angles.	perpendicular	Intersección en ángulo recto.
perpendicular bisector	Any line, segment, or ray that passes through the midpoint of a segment and is perpendicular to that segment.	mediatriz	Cualquier línea, segmento o rayo que pasa por el punto medio de un segmento y es perpendicular a ese segmento.
perpendicular lines	Nonvertical lines in the same plane for which the product of the slopes is –1.	líneas perpendiculares	Líneas no verticales en el mismo plano para las que el producto de las pendientes es –1.
phase shift	A horizontal translation of the graph of a trigonometric function.	cambio de fase	Una traducción horizontal de la gráfica de una función trigonométrica.
pi	The ratio $\frac{\text{cricumference}}{\text{diameter}}$.	pi	Relación $\frac{\text{circunferencia}}{\text{diámetro}}$.

piecewise-defined function A function defined by at least two subfunctions, each of which is defined differently depending on the interval of the domain.

piecewise-linear function A function defined by at least two linear subfunctions, each of which is defined differently depending on the interval of the domain.

plane A flat surface made up of points that has no depth and extends indefinitely in all directions.

plane symmetry When a plane intersects a three-dimensional figure so one half is the reflected image of the other half.

Platonic solid One of five regular polyhedra.

point A location with no size, only position.

point discontinuity An area that appears to be a hole in a graph.

point of concurrency The point of intersection of concurrent lines.

point of symmetry The point about which a figure is rotated.

point of tangency For a line that intersects a circle in one point, the point at which they intersect.

point symmetry A figure or graph has this when a figure is rotated 180° about a point and maps exactly onto the other part.

polygon A closed plane figure with at least three straight sides.

polyhedron A closed three-dimensional figure made up of flat polygonal regions.

polynomial A monomial or the sum of two or more monomials.

polynomial function A continuous function that can be described by a polynomial equation in one variable.

función definida por piezas Una función definida por al menos dos subfunciones, cada una de las cuales se define de manera diferente dependiendo del intervalo del dominio.

función lineal por piezas Una función definida por al menos dos subfunciones lineal, cada una de las cuales se define de manera diferente dependiendo del intervalo del dominio.

plano Una superficie plana compuesta de puntos que no tiene profundidad y se extiende indefinidamente en todas las direcciones.

simetría plana Cuando un plano cruza una figura tridimensional, una mitad es la imagen reflejada de la otra mitad.

sólido platónico Uno de cinco poliedros regulares.

punto Una ubicación sin tamaño, solo posición.

discontinuidad de punto Un área que parece ser un agujero en un gráfico.

punto de concurrencia El punto de intersección de líneas concurrentes.

punto de simetría El punto sobre el que se gira una figura.

punto de tangencia Para una línea que cruza un círculo en un punto, el punto en el que se cruzan.

simetría de punto Una figura o gráfica tiene esto cuando una figura se gira 180° alrededor de un punto y se mapea exactamente sobre la otra parte.

polígono Una figura plana cerrada con al menos tres lados rectos.

poliedros Una figura tridimensional cerrada formada por regiones poligonales planas.

polinomio Un monomio o la suma de dos o más monomios.

función polinómica Función continua que puede describirse mediante una ecuación polinómica en una variable.

polynomial identity A polynomial equation that is true for any values that are substituted for the variables.

population All of the members of a group of interest about which data will be collected.

population proportion The number of members in the population with a particular characteristic divided by the total number of members in the population.

positive Where the graph of a function lies above the *x*-axis.

positive correlation Bivariate data in which *y* increases as *x* increases.

positively skewed distribution A distribution that typically has a mean greater than the median.

postulate A statement that is accepted as true without proof.

power function A function of the form $f(x) = ax^n$, where a and n are nonzero real numbers.

precision The repeatability, or reproducibility, of a measurement.

preimage The original figure in a transformation.

prime polynomial A polynomial that cannot be written as a product of two polynomials with integer coefficients.

principal root The nonnegative root of a number.

principal square root The nonnegative square root of a number.

principal values The values in the restricted domains of trigonometric functions.

principle of superposition Two figures are congruent if and only if there is a rigid motion or series of rigid motions that maps one figure exactly onto the other.

prism A polyhedron with two parallel congruent bases connected by parallelogram faces.

identidad polinomial Una ecuación polinómica que es verdadera para cualquier valor que se sustituya por las variables.

población Todos los miembros de un grupo de interés sobre cuáles datos serán recopilados.

proporción de la población El número de miembros en la población con una característica particular dividida por el número total de miembros en la población.

positiva Donde la gráfica de una función se encuentra por encima del eje *x*.

correlación positiva Datos bivariate en el cual *y* aumenta a *x* disminuye.

distribución positivamente sesgada Una distribución que típicamente tiene una media mayor que la mediana.

postulado Una declaración que se acepta como verdadera sin prueba.

función de potencia Una ecuación polinomial que es verdadera para una función de la forma $f(x) = ax^n$, donde a y n son números reales no nulos.

precisión La repetibilidad, o reproducibilidad, de una medida.

preimagen La figura original en una transformación.

polinomio primo Un polinomio que no puede escribirse como producto de dos polinomios con coeficientes enteros.

raíz principal La raíz no negativa de un número.

raíz cuadrada principal La raíz cuadrada no negativa de un número.

valores principales Valores de los dominios restringidos de las funciones trigonométricas.

principio de superposición Dos figuras son congruentes si y sólo si hay un movimiento rígido o una serie de movimientos rígidos que traza una figura exactamente sobre la otra.

prisma Un poliedro con dos bases congruentes paralelas conectadas por caras de paralelogramo.

probability The number of outcomes in which a specified event occurs to the total number of trials.	**probabilidad** El número de resultados en los que se produce un evento especificado al número total de ensayos.
probability distribution A function that maps the sample space to the probabilities of the outcomes in the sample space for a particular random variable.	**distribución de probabilidad** Una función que mapea el espacio de muestra a las probabilidades de los resultados en el espacio de muestra para una variable aleatoria particular.
probability model A mathematical representation of a random event that consists of the sample space and the probability of each outcome.	**modelo de probabilidad** Una representación matemática de un evento aleatorio que consiste en el espacio muestral y la probabilidad de cada resultado.
projectile motion problems Problems that involve objects being thrown or dropped.	**problemas de movimiento del proyectil** Problemas que involucran objetos que se lanzan o caen.
proof A logical argument in which each statement is supported by a statement that is accepted as true.	**prueba** Un argumento lógico en el que cada sentencia está respaldada por una sentencia aceptada como verdadera.
proof by contradiction One assumes that the statement to be proven is false and then uses logical reasoning to deduce that a statement contradicts a postulate, theorem, or one of the assumptions.	**prueba por contradicción** Se supone que la afirmación a ser probada es falsa y luego utiliza el razonamiento lógico para deducir que una afirmación contradice un postulado, teorema o uno de los supuestos.
proportion A statement that two ratios are equivalent.	**proporción** Una declaración de que dos proporciones son equivalentes.
pure imaginary number A number of the form bi, where b is a real number and i is the imaginary unit.	**número imaginario puro** Un número de la forma bi, donde b es un número real e i es la unidad imaginaria.
pyramid A polyhedron with a polygonal base and three or more triangular faces that meet at a common vertex.	**pirámide** Poliedro con una base poligonal y tres o más caras triangulares que se encuentran en un vértice común.
Pythagorean identities Identities that express the Pythagorean Theorem in terms of the trigonometric functions.	**identidades pitagóricas** Identidades que expresan el Teorema de Pitágoras en términos de las funciones trigonométricas.
Pythagorean triple A set of three nonzero whole numbers that make the Pythagorean Theorem true.	**triplete Pitágorico** Un conjunto de tres números enteros distintos de cero que hacen que el Teorema de Pitágoras sea verdadero.

Q

quadrantal angle An angle in standard position with a terminal side that coincides with one of the axes.	**ángulo de cuadrante** Un ángulo en posición estándar con un lado terminal que coincide con uno de los ejes.
quadratic equation An equation that includes a quadratic expression.	**ecuación cuadrática** Una ecuación que incluye una expresión cuadrática.

quadratic expression An expression in one variable with a degree of 2.	**expresión cuadrática** Una expresión en una variable con un grado de 2.
quadratic form A form of polynomial equation, $au^2 + bu + c$, where u is an algebraic expression in x.	**forma cuadrática** Una forma de ecuación polinomial, $au^2 + bu + c$, donde u es una expresión algebraica en x.
quadratic function A function with an equation of the form $y = ax^2 + bx + c$, where $a \neq 0$.	**función cuadrática** Una función con una ecuación de la forma $y = ax^2 + bx + c$, donde $a \neq 0$.
quadratic inequality An inequality that includes a quadratic expression.	**desigualdad cuadrática** Una desigualdad que incluye una expresión cuadrática.
quadratic relations Equations of parabolas with horizontal axes of symmetry that are not functions.	**relaciones cuadráticas** Ecuaciones de parábolas con ejes horizontales de simetría que no son funciones.
quartic function A fourth-degree function.	**función cuartica** Una función de cuarto grado.
quartiles Measures of position that divide a data set arranged in ascending order into four groups, each containing about one fourth or 25% of the data.	**cuartiles** Medidas de posición que dividen un conjunto de datos dispuestos en orden ascendente en cuatro grupos, cada uno de los cuales contiene aproximadamente un cuarto o el 25% de los datos.
quintic function A fifth-degree function.	**función quíntica** Una función de quinto grado.

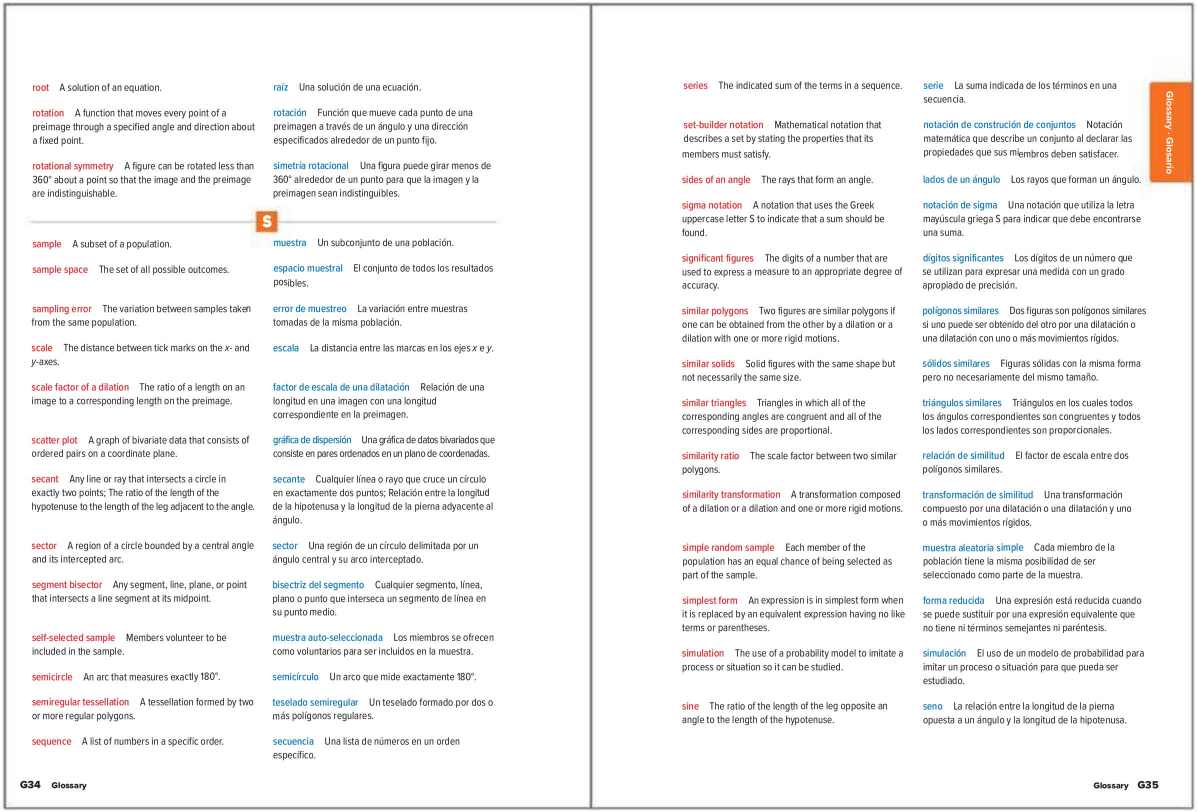
R

radian A unit of angular measurement equal to $\frac{180°}{\pi}$ or about 57.296°.	**radián** Una unidad de medida angular igual o $\frac{180°}{\pi}$ alrededor de 57.296°.
radical equation An equation with a variable in a radicand.	**ecuación radical** Una ecuación con una variable en un radicand.
radical expression An expression that contains a radical symbol, such as a square root.	**expresión radicales** Una expresión que contiene un símbolo radical, tal como una raíz cuadrada.
radical form When an expression contains a radical symbol.	**forma radical** Cuando una expresión contiene un símbolo radical.
radical function A function that contains radicals with variables in the radicand.	**función radical** Función que contiene radicales con variables en el radicand.
radicand The expression under a radical sign.	**radicando** La expresión debajo del signo radical.
radius of a circle or sphere A line segment from the center to a point on a circle or sphere.	**radio de un círculo o esfera** Un segmento de línea desde el centro hasta un punto en un círculo o esfera.
radius of a regular polygon The radius of the circle circumscribed about a regular polygon.	**radio de un polígono regular** El radio del círculo circunscrito alrededor de un polígono regular.

range The difference between the greatest and least values in a set of data; The set of second numbers of the ordered pairs in a relation; The set of y-values that actually result from the evaluation of the function.	**rango** La diferencia entre los valores de datos más grande or menos en un sistema de datos; El conjunto de los segundos números de los pares ordenados de una relación; El conjunto de valores y que realmente resultan de la evaluación de la función.
rate of change How a quantity is changing with respect to a change in another quantity.	**tasa de cambio** Cómo cambia una cantidad con respecto a un cambio en otra cantidad.
rational equation An equation that contains at least one rational expression.	**ecuación racional** Una ecuación que contiene al menos una expresión racional.
rational exponent An exponent that is expressed as a fraction.	**exponente racional** Un exponente que se expresa como una fracción.
rational expression A ratio of two polynomial expressions.	**expresión racional** Una relación de dos expresiones polinomiales.
rational function An equation of the form $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial expressions and $b(x) \neq 0$.	**función racional** Una ecuación de la forma $f(x) = \frac{a(x)}{b(x)}$, donde $a(x)$ y $b(x)$ son expresiones polinomiales y $b(x) \neq 0$.
rational inequality An inequality that contains at least one rational expression.	**desigualdad racional** Una desigualdad que contiene al menos una expresión racional.
rationalizing the denominator A method used to eliminate radicals from the denominator of a fraction or fractions from a radicand.	**racionalizando el denominador** Método utilizado para eliminar radicales del denominador de una fracción o fracciones de una radicand.
ray Part of a line that starts at a point and extends to infinity.	**rayo** Parte de una línea que comienza en un punto y se extiende hasta el infinito.
reciprocal function An equation of the form $f(x) = \frac{n}{b(x)}$, where n is a real number and $b(x)$ is a linear expression that cannot equal 0.	**función recíproca** Una ecuación de la forma $f(x) = \frac{n}{b(x)}$, donde n es un número real y $b(x)$ es una expresión lineal que no puede ser igual a 0.
reciprocal trigonometric functions Trigonometric functions that are reciprocals of each other.	**funciones trigonométricas recíprocas** Funciones trigonométricas que son reciprocales entre sí.
reciprocals Two numbers with a product of 1.	**recíprocos** Dos números con un producto de 1.
rectangle A parallelogram with four right angles.	**rectángulo** Un paralelogramo con cuatro ángulos rectos.
recursive formula A formula that gives the value of the first term in the sequence and then defines the next term by using the preceding term.	**formula recursiva** Una fórmula que da el valor del primer término en la secuencia y luego define el siguiente término usando el término anterior.
reduction A dilation with a scale factor between 0 and 1.	**reducción** Una dilatación con un factor de escala entre 0 y 1.
reference angle The acute angle formed by the terminal side of an angle and the x-axis.	**ángulo de referencia** El ángulo agudo formado por el lado terminal de un ángulo en posición estándar y el eje x.

reflection A function in which the preimage is reflected in the line of reflection; A transformation in which a figure, line, or curve is flipped across a line.	**reflexión** Función en la que la preimagen se refleja en la línea de reflexión; Una transformación en la que una figura, línea o curva se voltea a través de una línea.
regression function A function generated by an algorithm to find a line or curve that fits a set of data.	**función de regresión** Función generada por un algoritmo para encontrar una línea o curva que se ajuste a un conjunto de datos.
regular polygon A convex polygon that is both equilateral and equiangular.	**polígono regular** Un polígono convexo que es a la vez equilátero y equiangular.
regular polyhedron A polyhedron in which all of its faces are regular congruent polygons and all of the edges are congruent.	**poliedro regular** Un poliedro en el que todas sus caras son polígonos congruentes regulares y todos los bordes son congruentes.
regular pyramid A pyramid with a base that is a regular polygon.	**pirámide regular** Una pirámide con una base que es un polígono regular.
regular tessellation A tessellation formed by only one type of regular polygon.	**teselado regular** Un teselado formado por un solo tipo de polígono regular.
relation A set of ordered pairs.	**relación** Un conjunto de pares ordenados.
relative frequency In a two-way frequency table, the ratios of the number of observations in a category to the total number of observations; The ratio of the number of observations in a category to the total number of observations.	**frecuencia relativa** En una tabla de frecuencia bidireccional, las relaciones entre el número de observaciones en una categoría y el número total de observaciones; La relación entre el número de observaciones en una categoría y el número total de observaciones.
relative maximum A point on the graph of a function where no other nearby points have a greater y-coordinate.	**máximo relativo** Un punto en la gráfica de una función donde ningún otro punto cercano tiene una coordenada y mayor.
relative minimum A point on the graph of a function where no other nearby points have a lesser y-coordinate.	**mínimo relativo** Un punto en la gráfica de una función donde ningún otro punto cercano tiene una coordenada y menor.
remote interior angles Interior angles of a triangle that are not adjacent to an exterior angle.	**ángulos internos no adyacentes** Ángulos interiores de un triángulo que no están adyacentes a un ángulo exterior.
residual The difference between an observed y-value and its predicted y-value on a regression line.	**residual** La diferencia entre un valor de y observado y su valor de y predicho en una línea de regresión.
rhombus A parallelogram with all four sides congruent.	**rombo** Un paralelogramo con los cuatro lados congruentes.
rigid motion A transformation that preserves distance and angle measure.	**movimiento rígido** Una transformación que preserva la distancia y la medida del ángulo.

GLOSSARY



root A solution of an equation.	**raíz** Una solución de una ecuación.
rotation A function that moves every point of a preimage through a specified angle and direction about a fixed point.	**rotación** Función que mueve cada punto de una preimagen a través de un ángulo y una dirección especificados alrededor de un punto fijo.
rotational symmetry A figure can be rotated less than 360° about a point so that the image and the preimage are indistinguishable.	**simetría rotacional** Una figura puede girar menos de 360° alrededor de un punto para que la imagen y la preimagen sean indistinguibles.

S

sample A subset of a population.	**muestra** Un subconjunto de una población.
sample space The set of all possible outcomes.	**espacio muestral** El conjunto de todos los resultados posibles.
sampling error The variation between samples taken from the same population.	**error de muestreo** La variación entre muestras tomadas de la misma población.
scale The distance between tick marks on the *x*- and *y*-axes.	**escala** La distancia entre las marcas en los ejes *x* e *y*.
scale factor of a dilation The ratio of a length on an image to a corresponding length on the preimage.	**factor de escala de una dilatación** Relación de una longitud en una imagen con una longitud correspondiente en la preimagen.
scatter plot A graph of bivariate data that consists of ordered pairs on a coordinate plane.	**gráfica de dispersión** Una gráfica de datos bivariados que consiste en pares ordenados en un plano de coordenadas.
secant Any line or ray that intersects a circle in exactly two points; The ratio of the length of the hypotenuse to the length of the leg adjacent to the angle.	**secante** Cualquier línea o rayo que cruce un círculo en exactamente dos puntos; Relación entre la longitud de la hipotenusa y la longitud de la pierna adyacente al ángulo.
sector A region of a circle bounded by a central angle and its intercepted arc.	**sector** Una región de un círculo delimitada por un ángulo central y su arco interceptado.
segment bisector Any segment, line, plane, or point that intersects a line segment at its midpoint.	**bisectriz del segmento** Cualquier segmento, línea, plano o punto que interseca un segmento de línea en su punto medio.
self-selected sample Members volunteer to be included in the sample.	**muestra auto-seleccionada** Los miembros se ofrecen como voluntarios para ser incluidos en la muestra.
semicircle An arc that measures exactly 180°.	**semicírculo** Un arco que mide exactamente 180°.
semiregular tessellation A tessellation formed by two or more regular polygons.	**teselado semiregular** Un teselado formado por dos o más polígonos regulares.
sequence A list of numbers in a specific order.	**secuencia** Una lista de números en un orden específico.

series The indicated sum of the terms in a sequence.	**serie** La suma indicada de los términos en una secuencia.
set-builder notation Mathematical notation that describes a set by stating the properties that its members must satisfy.	**notación de construción de conjuntos** Notación matemática que describe un conjunto al declarar las propiedades que sus miembros deben satisfacer.
sides of an angle The rays that form an angle.	**lados de un ángulo** Los rayos que forman un ángulo.
sigma notation A notation that uses the Greek uppercase letter S to indicate that a sum should be found.	**notación de sigma** Una notación que utiliza la letra mayúscula griega S para indicar que debe encontrarse una suma.
significant figures The digits of a number that are used to express a measure to an appropriate degree of accuracy.	**dígitos significantes** Los dígitos de un número que se utilizan para expresar una medida con un grado apropiado de precisión.
similar polygons Two figures are similar polygons if one can be obtained from the other by a dilation or a dilation with one or more rigid motions.	**polígonos similares** Dos figuras son polígonos similares si uno puede ser obtenido del otro por una dilatación o una dilatación con uno o más movimientos rígidos.
similar solids Solid figures with the same shape but not necessarily the same size.	**sólidos similares** Figuras sólidas con la misma forma pero no necesariamente del mismo tamaño.
similar triangles Triangles in which all of the corresponding angles are congruent and all of the corresponding sides are proportional.	**triángulos similares** Triángulos en los cuales todos los ángulos correspondientes son congruentes y todos los lados correspondientes son proporcionales.
similarity ratio The scale factor between two similar polygons.	**relación de similitud** El factor de escala entre dos polígonos similares.
similarity transformation A transformation composed of a dilation or a dilation and one or more rigid motions.	**transformación de similitud** Una transformación compuesto por una dilatación o una dilatación y uno o más movimientos rígidos.
simple random sample Each member of the population has an equal chance of being selected as part of the sample.	**muestra aleatoria simple** Cada miembro de la población tiene la misma posibilidad de ser seleccionado como parte de la muestra.
simplest form An expression is in simplest form when it is replaced by an equivalent expression having no like terms or parentheses.	**forma reducida** Una expresión está reducida cuando se puede sustituir por una expresión equivalente que no tiene ni términos semejantes ni paréntesis.
simulation The use of a probability model to imitate a process or situation so it can be studied.	**simulación** El uso de un modelo de probabilidad para imitar un proceso o situación para que pueda ser estudiado.
sine The ratio of the length of the leg opposite an angle to the length of the hypotenuse.	**seno** La relación entre la longitud de la pierna opuesta a un ángulo y la longitud de la hipotenusa.

G34-G35 Glossary

sinusoidal function A function that can be produced by translating, reflecting, or dilating the sine function.	**función sinusoidal** Función que puede producirse traduciendo, reflejando o dilatando la función sinusoidal.
skew lines Noncoplanar lines that do not intersect.	**líneas alabeadas** Líneas no coplanares que no se cruzan.
slant height of a pyramid or right cone The length of a segment with one endpoint on the base edge of the figure and the other at the vertex.	**altura inclinada de una pirámide o cono derecho** La longitud de un segmento con un punto final en el borde base de la figura y el otro en el vértice.
slope The rate of change in the *y*-coordinates (rise) to the corresponding change in the *x*-coordinates (run) for points on a line.	**pendiente** La tasa de cambio en las coordenadas *y* (subida) al cambio correspondiente en las coordenadas *x* (carrera) para puntos en una línea.
slope criteria Outlines a method for proving the relationship between lines based on a comparison of the slopes of the lines.	**criterios de pendiente** Describe un método para probar la relación entre líneas basado en una comparación de las pendientes de las líneas.
solid of revolution A solid figure obtained by rotating a shape around an axis.	**sólido de revolución** Una figura sólida obtenida girando una forma alrededor de un eje.
solution A value that makes an equation true.	**solución** Un valor que hace que una ecuación sea verdadera.
solve an equation The process of finding all values of the variable that make the equation a true statement.	**resolver una ecuación** El proceso en que se hallan todos los valores de la variable que hacen verdadera la ecuación.
solving a triangle When you are given measurements to find the unknown angle and side measures of a triangle.	**resolver un triángulo** Cuando se le dan mediciones para encontrar el ángulo desconocido y las medidas laterales de un triángulo.
space A boundless three-dimensional set of all points.	**espacio** Un conjunto tridimensional ilimitado de todos los puntos.
sphere A set of all points in space equidistant from a given point called the center of the sphere.	**esfera** Un conjunto de todos los puntos del espacio equidistantes de un punto dado llamado centro de la esfera.
square A parallelogram with all four sides and all four angles congruent.	**cuadrado** Un paralelogramo con los cuatro lados y los cuatro ángulos congruentes.
square root One of two equal factors of a number.	**raíz cuadrada** Uno de dos factores iguales de un número.
square root function A radical function that contains the square root of a variable expression.	**función raíz cuadrada** Función radical que contiene la raíz cuadrada de una expresión variable.
square root inequality An inequality that contains the square root of a variable expression.	**square root inequality** Una desigualdad que contiene la raíz cuadrada de una expresión variable.
standard deviation A measure that shows how data deviate from the mean.	**desviación tipica** Una medida que muestra cómo los datos se desvían de la media.
standard error of the mean The standard deviation of the distribution of sample means taken from a population.	**error estandar de la media** La desviación estándar de la distribución de los medios de muestra se toma de una población.
standard form of a linear equation Any linear equation can be written in this form, $Ax + By = C$, where $A \geq 0$, A and B are not both 0, and A, B, and C are integers with a greatest common factor of 1.	**forma estándar de una ecuación lineal** Cualquier ecuación lineal se puede escribir de esta forma, $Ax + By = C$, donde $A \geq 0$, A y B no son ambos 0, y A, B y C son enteros con el mayor factor común de 1.
standard form of a polynomial A polynomial that is written with the terms in order from greatest degree to least degree.	**forma estándar de un polinomio** Un polinomio que se escribe con los términos en orden del grado más grande a menos grado.
standard form of a quadratic equation A quadratic equation can be written in the form $ax^2 + bx + c = 0$, where $a \neq 0$ and a, b, and c are integers.	**forma estándar de una ecuación cuadrática** Una ecuación cuadrática puede escribirse en la forma $ax^2 + bx + c = 0$, donde $a \neq 0$ y a, b, y c son enteros.
standard normal distribution A normal distribution with a mean of 0 and a standard deviation of 1.	**distribución normal estándar** Distribución normal con una media de 0 y una desviación estándar de 1.
standard position An angle positioned so that the vertex is at the origin and the initial side is on the positive *x*-axis.	**posición estándar** Un ángulo colocado de manera que el vértice está en el origen y el lado inicial está en el eje *x* positivo.
statement Any sentence that is either true or false, but not both.	**enunciado** Cualquier oración que sea verdadera o falsa, pero no ambas.
statistic A measure that describes a characteristic of a sample.	**estadística** Una medida que describe una característica de una muestra.
statistics An area of mathematics that deals with collecting, analyzing, and interpreting data.	**estadísticas** El proceso de recolección, análisis e interpretación de datos.
step function A type of piecewise-linear function with a graph that is a series of horizontal line segments.	**función escalonada** Un tipo de función lineal por piezas con un gráfico que es una serie de segmentos de línea horizontal.
straight angle An angle that measures 180°.	**ángulo recto** Un ángulo que mide 180°.
stratified sample The population is first divided into similar, nonoverlapping groups. Then members are randomly selected from each group.	**muestra estratificada** La población se divide primero en grupos similares, sin superposición. A continuación, los miembros se seleccionan aleatoriamente de cada grupo.
substitution A process of solving a system of equations in which one equation is solved for one variable in terms of the other.	**sustitución** Un proceso de resolución de un sistema de ecuaciones en el que una ecuación se resuelve para una variable en términos de la otra.
supplementary angles Two angles with measures that have a sum of 180°.	**ángulos suplementarios** Dos ángulos con medidas que tienen una suma de 180°.

G38-G39 Glossary

surface area	The sum of the areas of all faces and side surfaces of a three-dimensional figure.	área de superficie	La suma de las áreas de todas las caras y superficies laterales de una figura tridimensional.
survey	Data are collected from responses given by members of a group regarding their characteristics, behaviors, or opinions.	encuesta	Los datos se recogen de las respuestas dadas por los miembros de un grupo con respecto a sus características, comportamientos u opiniones.
symmetric distribution	A distribution in which the mean and median are approximately equal.	distribución simétrica	Un distribución en la que la media y la mediana son aproximadamente iguales.
symmetry	A figure has this if there exists a rigid motion—reflection, translation, rotation, or glide reflection—that maps the figure onto itself.	simetría	Una figura tiene esto si existe una reflexión-reflexión, una traducción, una rotación o una reflexión de deslizamiento rígida-que mapea la figura sobre sí misma.
synthetic division	An alternate method used to divide a polynomial by a binomial of degree 1.	división sintética	Un método alternativo utilizado para dividir un polinomio por un binomio de grado 1.
synthetic geometry	The study of geometric figures without the use of coordinates.	geometría sintética	El estudio de figuras geométricas sin el uso de coordenadas.
synthetic substitution	The process of using synthetic division to find a value of a polynomial function.	sustitución sintética	El proceso de utilizar la división sintética para encontrar un valor de una función polynomial.
system of equations	A set of two or more equations with the same variables.	sistema de ecuaciones	Un conjunto de dos o más ecuaciones con las mismas variables.
system of inequalities	A set of two or more inequalities with the same variables.	sistema de desigualdades	Un conjunto de dos o más desigualdades con las mismas variables.
systematic sample	Members are selected according to a specified interval from a random starting point.	muestra sistemática	Los miembros se seleccionan de acuerdo con un intervalo especificado desde un punto de partida aleatorio.

T

tangent	The ratio of the length of the leg opposite an angle to the length of the leg adjacent to the angle.	tangente	La relación entre la longitud de la pata opuesta a un ángulo y la longitud de la pata adyacente al ángulo.
tangent to a circle	A line or segment in the plane of a circle that intersects the circle in exactly one point and does not contain any points in the interior of the circle.	tangente a un círculo	Una línea o segmento en el plano de un círculo que interseca el círculo en exactamente un punto y no contiene ningún punto en el interior del círculo.
tangent to a sphere	A line that intersects the sphere in exactly one point.	tangente a una esfera	Una línea que interseca la esfera exactamente en un punto.
term	A number, a variable, or a product or quotient of numbers and variables.	término	Un número, una variable, o un producto o cociente de números y variables.

G38 Glossary

Glossary · Glosario

term of a sequence	A number in a sequence.	término de una sucesión	Un número en una secuencia.
terminal side	The part of an angle that rotates about the center.	lado terminal	La parte de un ángulo que gira alrededor de un centro.
tessellation	A repeating pattern of one or more figures that covers a plane with no overlapping or empty spaces.	teselado	Patrón repetitivo de una o más figuras que cubre un plano sin espacios superpuestos o vacíos.
theorem	A statement that can be proven true using undefined terms, definitions, and postulates.	teorema	Una afirmación o conjetura que se puede probar verdad utilizando términos, definiciones y postulados indefinidos.
theoretical probability	Probability based on what is expected to happen.	probabilidad teórica	Probabilidad basada en lo que se espera que suceda.
transformation	A function that takes points in the plane as inputs and gives other points as outputs. The movement of a graph on the coordinate plane.	transformación	Función que toma puntos en el plano como entradas y da otros puntos como salidas. El movimiento de un gráfico en el plano de coordenadas.
translation	A function in which all of the points of a figure move the same distance in the same direction; A transformation in which a figure is slid from one position to another without being turned.	traslación	Función en la que todos los puntos de una figura se mueven en la misma dirección; El movimiento de un gráfico en el plano de coordenadas.
translation vector	A directed line segment that describes both the magnitude and direction of the slide if the magnitude is the length of the vector from its initial point to its terminal point.	vector de traslación	Un segmento de línea dirigido que describe tanto la magnitud como la dirección de la diapositiva si la magnitud es la longitud del vector desde su punto inicial hasta su punto terminal.
transversal	A line that intersects two or more lines in a plane at different points.	transversal	Una línea que interseca dos o más líneas en un plano en diferentes puntos.
trapezoid	A quadrilateral with exactly one pair of parallel sides.	trapecio	Un cuadrilátero con exactamente un par de lados paralelos.
trend	A general pattern in the data.	tendencia	Un patrón general en los datos.
trigonometric equation	An equation that includes at least one trigonometric function.	ecuación trigonométrica	Una ecuación que incluye al menos una función trigonométrica.
trigonometric function	A function that relates the measure of one nonright angle of a right triangle to the ratios of the lengths of any two sides of the triangle.	función trigonométrica	Función que relaciona la medida de un ángulo no recto de un triángulo rectángulo con las relaciones de las longitudes de cualquiera de los dos lados del triángulo.
trigonometric identity	An equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.	identidad trigonométrica	Una ecuación que implica funciones trigonométricas que es verdadera para todos los valores para los cuales se define cada expresión en la ecuación.

Glossary G39

trigonometric ratio A ratio of the lengths of two sides of a right triangle.	**relación trigonométrica** Una relación de las longitudes de dos lados de un triángulo rectángulo.
trigonometry The study of the relationships between the sides and angles of triangles.	**trigonometría** El estudio de las relaciones entre los lados y los ángulos de los triángulos.
trinomial The sum of three monomials.	**trinomio** La suma de tres monomios.
truth value The truth or falsity of a statement.	**valor de verdad** La verdad o la falsedad de una declaración.
two-column proof A proof that contains statements and reasons organized in a two-column format.	**prueba de dos columnas** Una prueba que contiene declaraciones y razones organizadas en un formato de dos columnas.
two-way frequency table A table used to show frequencies of data classified according to two categories, with the rows indicating one category and the columns indicating the other.	**tabla de frecuencia bidireccional** Una tabla utilizada para mostrar las frecuencias de los datos clasificados de acuerdo con dos categorías, con las filas que indican una categoría y las columnas que indican la otra.
two-way relative frequency table A table used to show frequencies of data based on a percentage of the total number of observations.	**tabla de frecuencia relativa bidireccional** Una tabla usada para mostrar las frecuencias de datos basadas en un porcentaje del número total de observaciones.

U

unbounded When the graph of a system of constraints is open.	**no acotado** Cuando la gráfica de un sistema de restricciones está abierta.
undefined terms Words that are not formally explained by means of more basic words and concepts.	**términos indefinidos** Palabras que no se explican formalmente mediante palabras y conceptos más básicos.
uniform motion problems Problems that use the formula $d = rt$, where d is the distance, r is the rate, and t is the time.	**problemas de movimiento uniforme** Problemas que utilizan la fórmula $d = rt$, donde d es la distancia, r es la velocidad y t es el tiempo.
uniform tessellation A tessellation that contains the same arrangement of shapes and angles at each vertex.	**teselado uniforme** Un teselado que contiene la misma disposición de formas y ángulos en cada vértice.
union The graph of a compound inequality containing *or*.	**unión** La gráfica de una desigualdad compuesta que contiene la palabra *o*.
union of *A* and *B* The set of all outcomes in the sample space of event *A* combined with all outcomes in the sample space of event *B*.	**unión de *A* y *B*** El conjunto de todos los resultados en el espacio muestral del evento *A* combinado con todos los resultados en el espacio muestral del evento *B*.
unit circle A circle with a radius of 1 unit centered at the origin on the coordinate plane.	**círculo unitario** Un círculo con un radio de 1 unidad centrado en el origen en el plano de coordenadas.
univariate data Measurement data in one variable.	**datos univariate** Datos de medición en una variable.

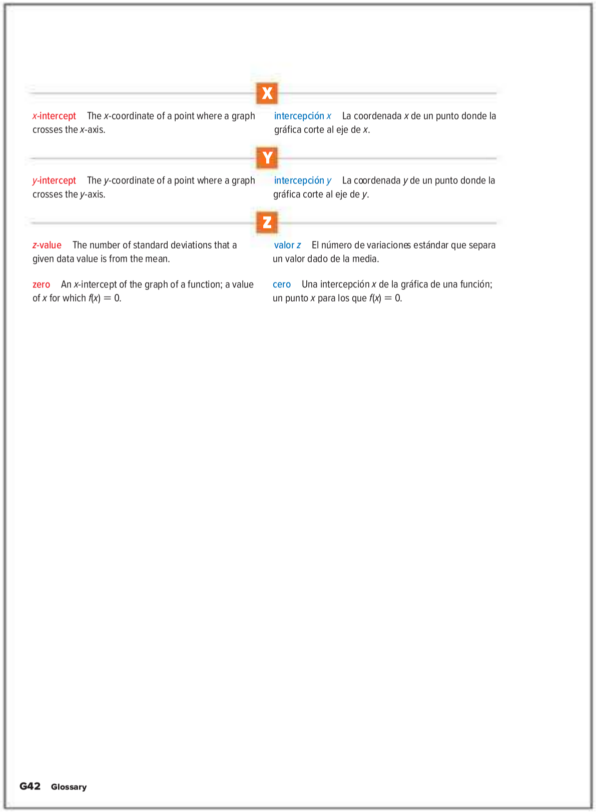
upper quartile The median of the upper half of a set of data.	**cuartil superior** La mediana de la mitad superior de un conjunto de datos.

V

valid argument An argument is valid if it is impossible for all of the premises, or supporting statements, of the argument to be true and its conclusion false.	**argumento válido** Un argumento es válido si es imposible que todas las premisas o argumentos de apoyo del argumento sean verdaderos y su conclusión sea falsa.
variable A letter used to represent an unspecified number or value; Any characteristic, number, or quantity that can be counted or measured.	**variable** Una letra utilizada para representar un número o valor no especificado; Cualquier característica, número, o cantidad que pueda ser contada o medida.
variable term A term that contains a variable.	**término variable** Un término que contiene una variable.
variance The square of the standard deviation.	**varianza** El cuadrado de la desviación estándar.
vertex Either the lowest point or the highest point of a function.	**vértice** El punto más bajo o el punto más alto en una función.
vertex angle of an isosceles triangle The angle between the sides that are the legs of an isosceles triangle.	**ángulo del vértice de un triángulo isósceles** El ángulo entre los lados que son las patas de un triángulo isósceles.
vertex form A quadratic function written in the form $f(x) = a(x - h)^2 + k$.	**forma de vértice** Una función cuadrática escribirse de la forma $f(x) = a(x - h)^2 + k$.
vertex of a polyhedron The intersection of three edges of a polyhedron.	**vértice de un polígono** La intersección de tres bordes de un poliedro.
vertex of an angle The common endpoint of the two rays that form an angle.	**vértice de un ángulo** El punto final común de los dos rayos que forman un ángulo.
vertical angles Two nonadjacent angles formed by two intersecting lines.	**ángulos verticales** Dos ángulos no adyacentes formados por dos líneas de intersección.
vertical asymptote A vertical line that a graph approaches.	**asíntota vertical** Una línea vertical que se aproxima a un gráfico.
vertical shift A vertical translation of the graph of a trigonometric function.	**cambio vertical** Una traducción vertical de la gráfica de una función trigonométrica.
volume The measure of the amount of space enclosed by a three-dimensional figure.	**volumen** La medida de la cantidad de espacio encerrada por una figura tridimensional.

W

work problems Problems that involve two people working at different rates who are trying to complete a single job.	**problemas de trabajo** Problemas que involucran a dos personas trabajando a diferentes ritmos que están tratando de completar un solo trabajo.



X

x-intercept The *x*-coordinate of a point where a graph crosses the *x*-axis.	**intercepción *x*** La coordenada *x* de un punto donde la gráfica corte al eje de *x*.

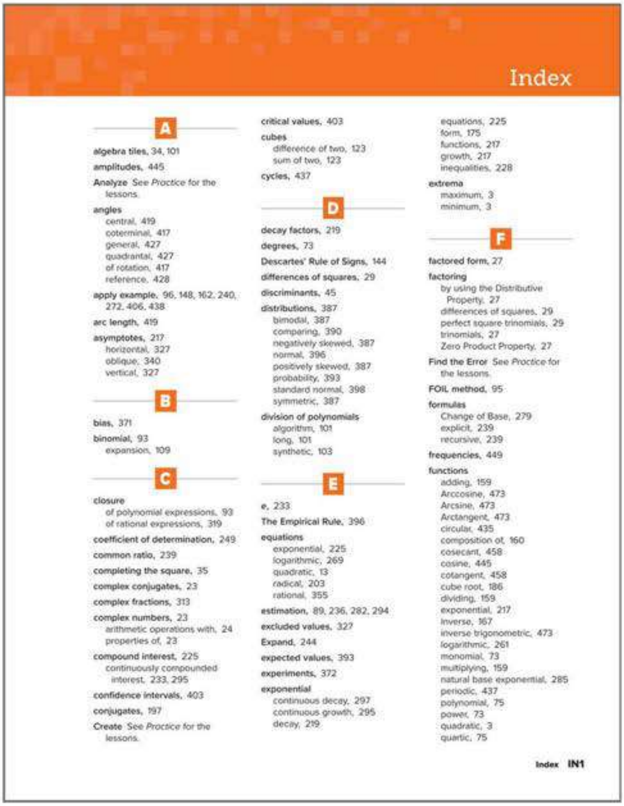
Y

y-intercept The *y*-coordinate of a point where a graph crosses the *y*-axis.	**intercepción *y*** La coordenada *y* de un punto donde la gráfica corte al eje de *y*.

Z

z-value The number of standard deviations that a given data value is from the mean.	**valor *z*** El número de variaciones estándar que separa un valor dado de la media.
zero An *x*-intercept of the graph of a function; a value of *x* for which $f(x) = 0$.	**cero** Una intercepción *x* de la gráfica de una función; un punto *x* para los que $f(x) = 0$.

Index



Index

A

algebra tiles, 34, 101

amplitudes, 445

Analyze See *Practice* for the lessons.

angles
- central, 419
- coterminal, 417
- general, 427
- quadrantal, 427
- of rotation, 417
- reference, 428

apply example, 96, 148, 162, 240, 272, 406, 438

arc length, 419

asymptotes, 217
- horizontal, 327
- oblique, 340
- vertical, 327

B

bias, 371

binomial, 93
- expansion, 109

C

closure
- of polynomial expressions, 93
- of rational expressions, 319

coefficient of determination, 249

common ratio, 239

completing the square, 35

complex conjugates, 23

complex fractions, 313

complex numbers, 23
- arithmetic operations with, 24
- properties of, 23

compound interest, 225
- continuously compounded interest, 233, 295

confidence intervals, 403

conjugates, 197

Create See *Practice* for the lessons.

critical values, 403

cubes
- difference of two, 123
- sum of two, 123

cycles, 437

D

decay factors, 219

degrees, 73

Descartes' Rule of Signs, 144

differences of squares, 29

discriminants, 45

distributions, 387
- bimodal, 387
- comparing, 390
- negatively skewed, 387
- normal, 396
- positively skewed, 387
- probability, 393
- standard normal, 398
- symmetric, 387

division of polynomials
- algorithm, 101
- long, 101
- synthetic, 103

E

e, 233

The Empirical Rule, 396

equations
- exponential, 225
- logarithmic, 269
- quadratic, 13
- radical, 203
- rational, 355

estimation, 89, 236, 282, 294

excluded values, 327

Expand, 244

expected values, 393

experiments, 372

exponential
- continuous decay, 297
- continuous growth, 295
- decay, 219
- equations, 225
- form, 175
- functions, 217
- growth, 217
- inequalities, 228

extrema
- maximum, 3
- minimum, 3

F

factored form, 27

factoring
- by using the Distributive Property, 27
- differences of squares, 29
- perfect square trinomials, 29
- trinomials, 27
- Zero Product Property, 27

Find the Error See *Practice* for the lessons.

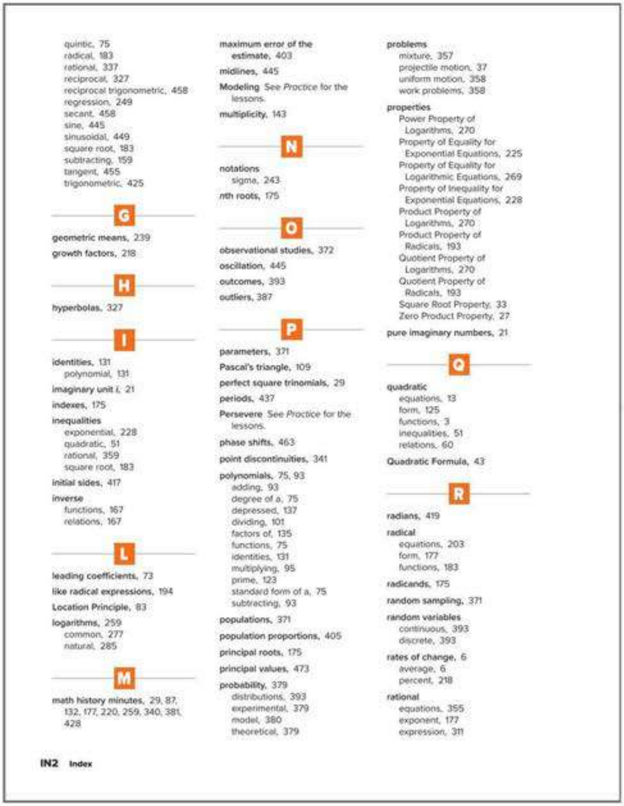
FOIL method, 95

formulas
- Change of Base, 279
- explicit, 239
- recursive, 239

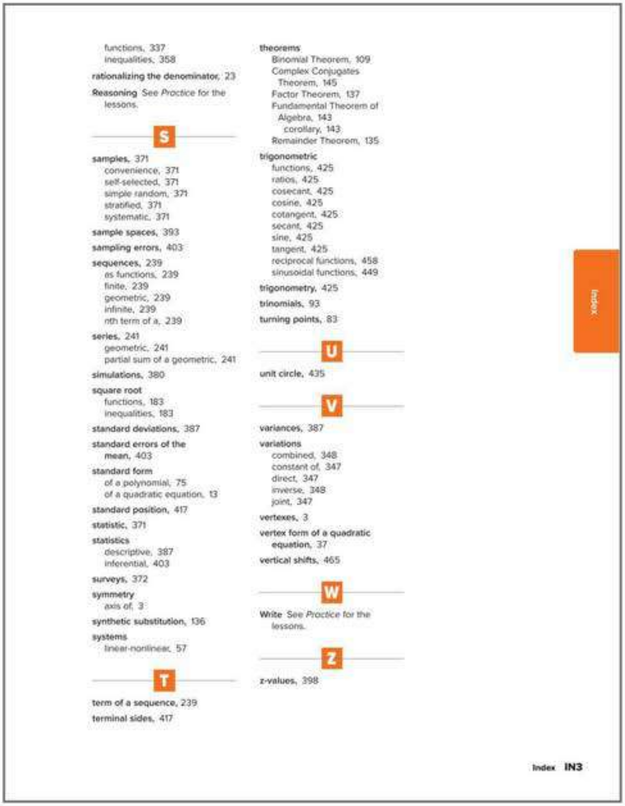
frequencies, 449

functions
- adding, 159
- Arccosine, 473
- Arcsine, 473
- Arctangent, 473
- circular, 435
- composition of, 160
- cosecant, 458
- cosine, 445
- cotangent, 458
- cube root, 186
- dividing, 159
- exponential, 217
- inverse, 167
- inverse trigonometric, 473
- logarithmic, 261
- monomial, 73
- multiplying, 159
- natural base exponential, 285
- periodic, 437
- polynomial, 75
- power, 73
- quadratic, 3
- quartic, 75

Index **IN1**



IN2 Index



Index IN3

Reveal MATH™

Integrated III • Volume 2

CONTENTS

mheducation.com/prek-12

978-0-07-703207-4
MHID 0-07-703207-1
99701
EAN
9 780077 032074

Mc Graw Hill