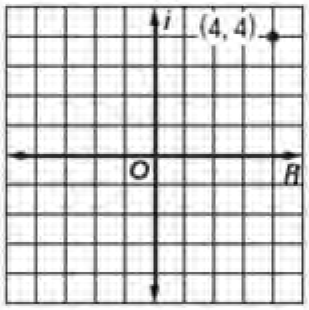
9-5 Complex Numbers and De Moivre's Theorem

Graph each number in the complex plane and find its absolute value.

1. $z = 4 + 4\boldsymbol{i}$

SOLUTION:

For $z = 4 + 4\boldsymbol{i}$, $(a, b) = (4, 4)$. Graph the point (4, 4) in the complex plane.

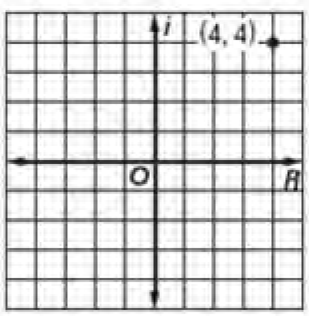


Use the absolute value of a complex number formula.

$$\begin{aligned}|z| &= \sqrt{a^2 + b^2}\\ &= \sqrt{4^2 + 4^2}\\ &= \sqrt{32}\\ &= 4\sqrt{2} \text{ or about } 5.66\end{aligned}$$

ANSWER:

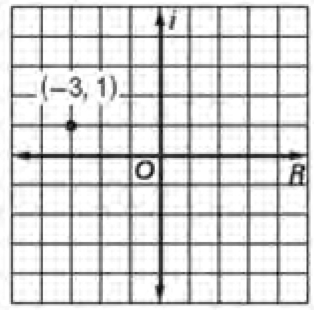
$4\sqrt{2} \approx 5.66$



2. $z = -3 + \boldsymbol{i}$

SOLUTION:

For $z = -3 + \boldsymbol{i}$, $(a, b) = (-3, 1)$. Graph the point (–3, 1) in the complex plane.

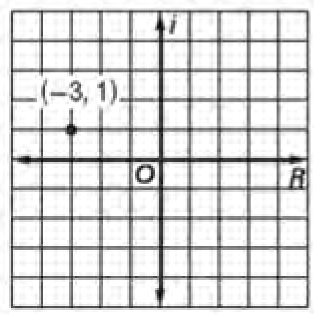


Use the absolute value of a complex number formula.

$$\begin{aligned}|z| &= \sqrt{a^2 + b^2}\\ &= \sqrt{(-3)^2 + 1^2}\\ &= \sqrt{10} \text{ or about } 3.16\end{aligned}$$

ANSWER:

$\sqrt{10} \approx 3.16$



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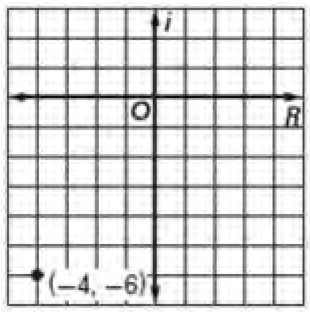
3. $z = -4 - 6\boldsymbol{i}$

SOLUTION:

For $z = -4 - 6\boldsymbol{i}$, $(a, b) = (-4, -6)$. Graph the point $(-4, -6)$ in the complex plane.

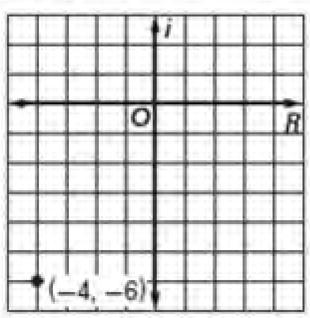


Use the absolute value of a complex number formula.

$$\begin{aligned}|z| &= \sqrt{a^2 + b^2}\\ &= \sqrt{(-4)^2 + (-6)^2}\\ &= \sqrt{52}\\ &= 2\sqrt{13} \text{ or about } 7.21\end{aligned}$$

ANSWER:

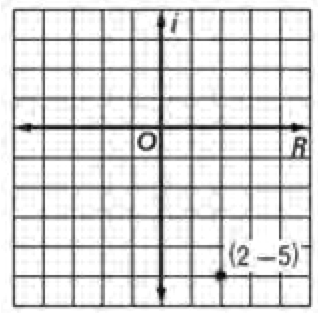
$2\sqrt{13} \approx 7.21$



4. $z = 2 - 5\boldsymbol{i}$

SOLUTION:

For $z = 2 - 5\boldsymbol{i}$, $(a, b) = (2, -5)$. Graph the point $(2, -5)$ in the complex plane.

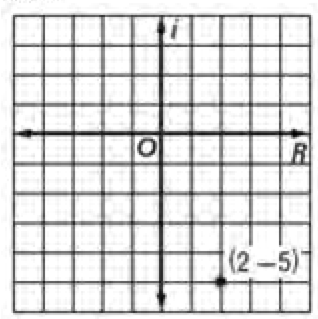


Use the absolute value of a complex number formula.

$$\begin{aligned}|z| &= \sqrt{a^2 + b^2}\\ &= \sqrt{2^2 + (-5)^2}\\ &= \sqrt{29} \text{ or about } 5.39\end{aligned}$$

ANSWER:

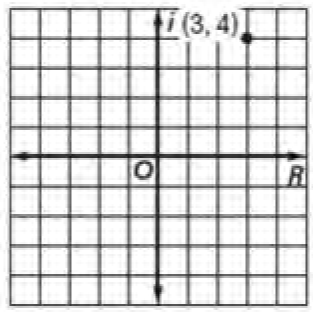
$\sqrt{29} \approx 5.39$



5. $z = 3 + 4\boldsymbol{i}$

SOLUTION:

For $z = 3 + 4\boldsymbol{i}$, $(a, b) = (3, 4)$. Graph the point (3, 4) in the complex plane.

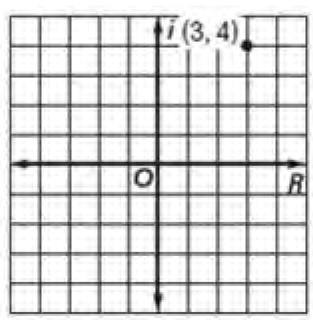


Use the absolute value of a complex number formula.

$$\begin{aligned}|z| &= \sqrt{a^2 + b^2}\\ &= \sqrt{3^2 + 4^2}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

ANSWER:

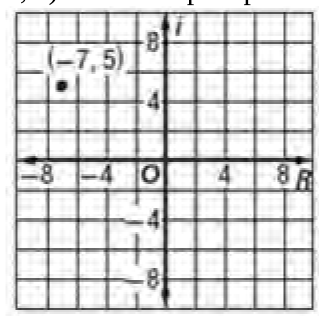
5



6. $z = -7 + 5\boldsymbol{i}$

SOLUTION:

For $z = -7 + 5\boldsymbol{i}$, $(a, b) = (-7, 5)$. Graph the point (–7, 5) in the complex plane.

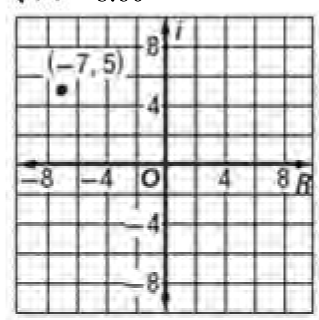


Use the absolute value of a complex number formula.

$$\begin{aligned}|z| &= \sqrt{a^2 + b^2}\\ &= \sqrt{(-7)^2 + 5^2}\\ &= \sqrt{74} \text{ or about } 8.60\end{aligned}$$

ANSWER:

$\sqrt{74} \approx 8.60$

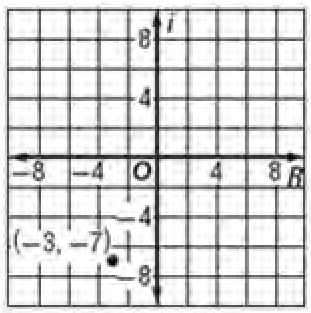


9-5 Complex Numbers and De Moivre's Theorem

7. $z = -3 - 7\boldsymbol{i}$

SOLUTION:

For $z = -3 - 7\boldsymbol{i}$, $(a, b) = (-3, -7)$. Graph the point (–3,   7) in the complex plane.

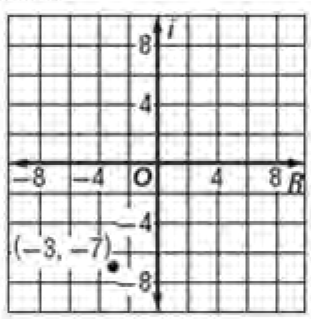


Use the absolute value of a complex number formula.

$$\begin{aligned} |z| &= \sqrt{a^2 + b^2} \\ &= \sqrt{(-3)^2 + (-7)^2} \\ &= \sqrt{58} \text{ or about } 7.62 \end{aligned}$$

ANSWER:

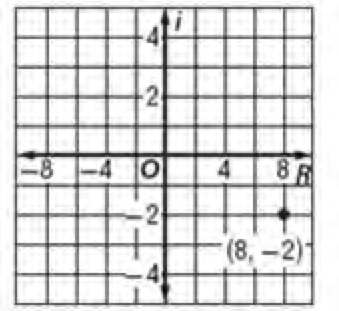
$\sqrt{58} \approx 7.62$



8. $z = 8 - 2\boldsymbol{i}$

SOLUTION:

For $z = 8 - 2\boldsymbol{i}$, $(a, b) = (8, -2)$. Graph the point (8, –2) in the complex plane.

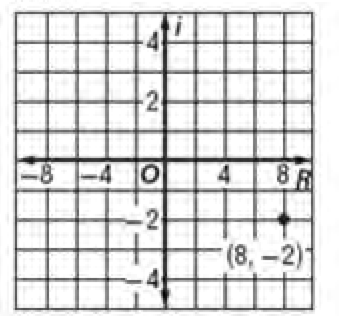


Use the absolute value of a complex number formula.

$$\begin{aligned} |z| &= \sqrt{a^2 + b^2} \\ &= \sqrt{8^2 + (-2)^2} \\ &= \sqrt{68} \\ &= 2\sqrt{17} \text{ or about } 8.25 \end{aligned}$$

ANSWER:

$2\sqrt{17} \approx 8.25$



9. **VECTORS** The force on an object is given by $z = 10 + 15\boldsymbol{i}$, where the components are measured in newtons (N).

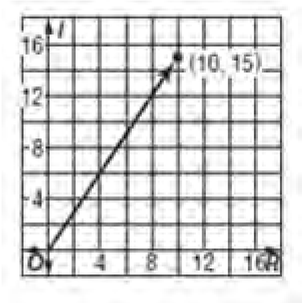
a. Represent z as a vector in the complex plane.

b. Find the magnitude and direction angle of the vector.

SOLUTION:

a. For $z = 10 + 15\boldsymbol{i}$, $(a, b) = (10, 15)$. Graph the point (10, 15) in the complex plane. Then draw a vector with an initial point at the origin and a terminal point at (10, 15).

9-5 Complex Numbers and De Moivre's Theorem



b. Use the absolute value of a complex number formula.

$$|z| = \sqrt{a^2 + b^2}$$
$$= \sqrt{10^2 + 15^2}$$
$$= \sqrt{325}$$
$$= 5\sqrt{13} \text{ or about } 18.03$$

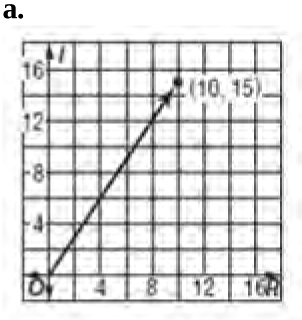
Find the measure of the angle θ that the vector makes with the positive real axis.

$$\theta = \tan^{-1}\frac{b}{a}$$
$$= \tan^{-1}\frac{15}{10}$$
$$\approx 56.31°$$

The magnitude of the force is about 18.03 newtons at an angle of about 56.31°.

ANSWER:

a.



b. 18.03 N; about 56.31°

Express each complex number in polar form.

10. 4 + 4***i***

SOLUTION:

4 + 4***i***

Find the modulus *r* and argument θ.

$$r = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\frac{b}{a}$$
$$= \sqrt{4^2 + 4^2} \qquad = \tan^{-1}\frac{4}{4}$$
$$= \sqrt{32} \qquad = \tan^{-1} 1 \text{ or } \frac{\pi}{4}$$
$$= 4\sqrt{2}$$

The polar form of 4+ 4***i*** is $4\sqrt{2}\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)$.

ANSWER:

$$4\sqrt{2}\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)$$

11. –2 + ***i***

SOLUTION:

–2 + ***i***

Find the modulus *r* and argument θ.

$$r = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\frac{b}{a} + \pi$$
$$= \sqrt{(-2)^2 + 1^2} \qquad = \tan^{-1}\frac{1}{-2} + \pi$$
$$= \sqrt{5} \qquad \approx 2.68$$

The polar form of −2 + ***i*** is $\sqrt{5}$ (cos 2.68 + ***i*** sin 2.68).

ANSWER:

$\sqrt{5}$ (cos 2.68 + ***i*** sin 2.68)

9-5 Complex Numbers and De Moivre's Theorem

12. $4 - \sqrt{2}\,\boldsymbol{i}$

SOLUTION:

$4 - \sqrt{2}\,\boldsymbol{i}$

Find the modulus r and argument θ.

$$\begin{aligned} r &= \sqrt{a^2+b^2} & \theta &= \tan^{-1}\frac{b}{a} \\ &= \sqrt{4^2+\left(-\sqrt{2}\right)^2} & &= \tan^{-1}\frac{-\sqrt{2}}{4} \\ &= \sqrt{18} & &\approx -0.34 \\ &= 3\sqrt{2} \end{aligned}$$

The polar form of $4 - \sqrt{2}\,\boldsymbol{i}$ is $3\sqrt{2}$ (cos −0.34 + $\boldsymbol{i}$ sin −0.34).

ANSWER:

$3\sqrt{2}$ (cos −0.34 + $\boldsymbol{i}$ sin −0.34)

13. $2 - 2\boldsymbol{i}$

SOLUTION:

$2 - 2\boldsymbol{i}$

Find the modulus r and argument θ.

$$\begin{aligned} r &= \sqrt{a^2+b^2} & \theta &= \tan^{-1}\frac{b}{a} \\ &= \sqrt{2^2+(-2)^2} & &= \tan^{-1}\frac{-2}{2} \\ &= \sqrt{8} & &= \tan^{-1}-1 \text{ or } \frac{7\pi}{4} \\ &= 2\sqrt{2} \end{aligned}$$

The polar form of $2 - 2\boldsymbol{i}$ is

$$2\sqrt{2}\left(\cos\frac{7\pi}{4} + \boldsymbol{i}\sin\frac{7\pi}{4}\right).$$

ANSWER:

$$2\sqrt{2}\left(\cos\frac{7\pi}{4} + \boldsymbol{i}\sin\frac{7\pi}{4}\right)$$

14. $4 + 5\boldsymbol{i}$

SOLUTION:

$4 + 5\boldsymbol{i}$

Find the modulus r and argument θ.

$$\begin{aligned} r &= \sqrt{a^2+b^2} & \theta &= \tan^{-1}\frac{b}{a} \\ &= \sqrt{4^2+5^2} & &= \tan^{-1}\frac{5}{4} \\ &= \sqrt{41} & &\approx 0.90 \end{aligned}$$

The polar form of 4+ 5$\boldsymbol{i}$ is $\sqrt{41}$ (cos 0.90 + $\boldsymbol{i}$ sin 0.90).

ANSWER:

$\sqrt{41}$ (cos 0.90 + $\boldsymbol{i}$ sin 0.90)

15. $-2 + 4\boldsymbol{i}$

SOLUTION:

$2 + 4\boldsymbol{i}$

Find the modulus r and argument θ.

$$\begin{aligned} r &= \sqrt{a^2+b^2} & \theta &= \tan^{-1}\frac{b}{a}+\pi \\ &= \sqrt{(-2)^2+4^2} & &= \tan^{-1}\frac{4}{-2}+\pi \\ &= \sqrt{20} & &\approx 2.03 \\ &= 2\sqrt{5} \end{aligned}$$

The polar form of $-2 + 4\boldsymbol{i}$ is $2\sqrt{5}$ (cos 2.03 + $\boldsymbol{i}$ sin 2.03).

ANSWER:

$2\sqrt{5}$ (cos 2.03 + $\boldsymbol{i}$ sin 2.03)

9-5 Complex Numbers and De Moivre's Theorem

16. $-1 - \sqrt{3}\,\boldsymbol{i}$

SOLUTION:

$-1 - \sqrt{3}\,\boldsymbol{i}$

Find the modulus r and argument θ.

$$\begin{aligned} r &= \sqrt{a^2 + b^2} & \theta &= \tan^{-1}\frac{b}{a} + \pi \\ &= \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} & &= \tan^{-1}\frac{-\sqrt{3}}{-1} + \pi \\ &= \sqrt{4} & &= \frac{\pi}{3} + \pi \text{ or } \frac{4\pi}{3} \\ &= 2 \end{aligned}$$

The polar form of $-1 - \sqrt{3}\,\boldsymbol{i}$ is

$2\left(\cos\frac{4\pi}{3} + \boldsymbol{i}\sin\frac{4\pi}{3}\right)$.

ANSWER:

$2\left(\cos\frac{4\pi}{3} + \boldsymbol{i}\sin\frac{4\pi}{3}\right)$

17. $3 + 3\boldsymbol{i}$

SOLUTION:

$3 + 3\boldsymbol{i}$

Find the modulus r and argument θ.

$$\begin{aligned} r &= \sqrt{a^2 + b^2} & \theta &= \tan^{-1}\frac{b}{a} \\ &= \sqrt{3^2 + 3^2} & &= \tan^{-1}\frac{3}{3} \\ &= \sqrt{18} & &= \tan^{-1} 1 \text{ or } \frac{\pi}{4} \\ &= 3\sqrt{2} \end{aligned}$$

The polar form of 3+ 3$\boldsymbol{i}$ is $3\sqrt{2}\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)$.

ANSWER:

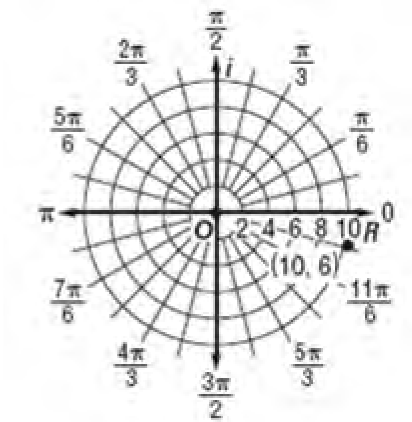
$3\sqrt{2}\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)$

Graph each complex number on a polar grid. Then express it in rectangular form.

10(cos 6 + $\boldsymbol{i}$ sin 6)

SOLUTION:

The value of r is 10, and the value of θ is 6. Plot the polar coordinates (10, 6).



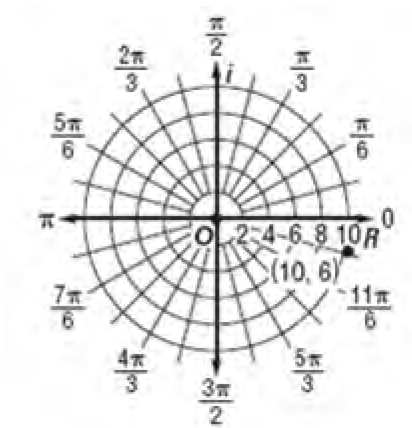
To express the number in rectangular form, evaluate the trigonometric values and simplify.

$$\begin{aligned} 10(\cos 6 + i\sin 6) &= 10\left[0.960 + (-0.279)\boldsymbol{i}\right] \\ &= 9.60 - 2.79\boldsymbol{i} \end{aligned}$$

The rectangular form of $10(\cos 6 + i\sin 6)$ is $9.60 - 2.79\boldsymbol{i}$.

ANSWER:

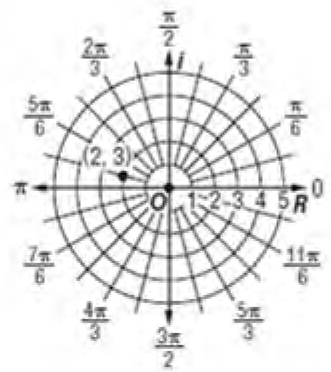
9.60 – 2.79$\boldsymbol{i}$



19. 2(cos 3 + ***i*** sin 3)

SOLUTION:

The value of *r* is 2, and the value of θ is 3. Plot the polar coordinates (2, 3). Notice that 3 radians is slightly greater than $\frac{11\pi}{12}$ but less than π.



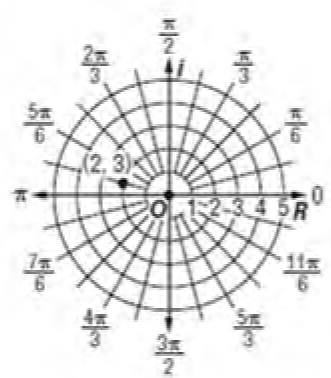
To express the number in rectangular form, evaluate the trigonometric values and simplify.

$$\begin{aligned} 2(\cos 3 + \boldsymbol{i} \sin 3) &= 2(-0.99 + 0.14\boldsymbol{i}) \\ &= -1.98 + 0.28\boldsymbol{i} \end{aligned}$$

The rectangular form of 2(cos 3 + ***i*** sin 3) is −1.98 + 0.28***i***.

ANSWER:

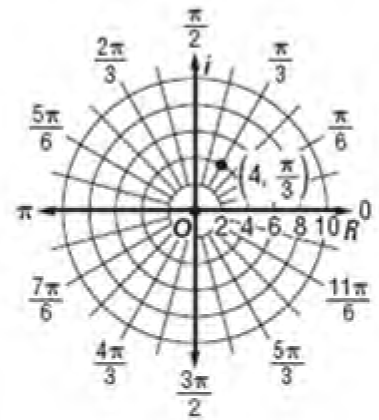
−1.98 + 0.28***i***



20. $4\left(\cos \frac{\pi}{3} + \boldsymbol{i} \sin \frac{\pi}{3}\right)$

SOLUTION:

The value of *r* is 4, and the value of θ is $\frac{\pi}{3}$. Plot the polar coordinates $\left(4, \frac{\pi}{3}\right)$.



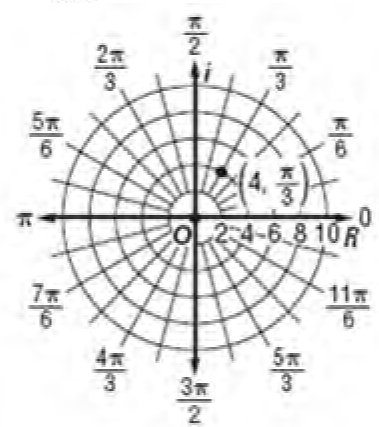
To express the number in rectangular form, evaluate the trigonometric values and simplify.

$$\begin{aligned} 4\left(\cos \frac{\pi}{3} + \boldsymbol{i} \sin \frac{\pi}{3}\right) &= 4\left[\frac{1}{2} + \frac{\sqrt{3}}{2}\boldsymbol{i}\right] \\ &= 2 + 2\sqrt{3}\boldsymbol{i} \end{aligned}$$

The rectangular form of $4\left(\cos \frac{\pi}{3} + \boldsymbol{i} \sin \frac{\pi}{3}\right)$ is $2 + 2\sqrt{3}\boldsymbol{i}$.

ANSWER:

$2 + 2\sqrt{3}\,\boldsymbol{i}$

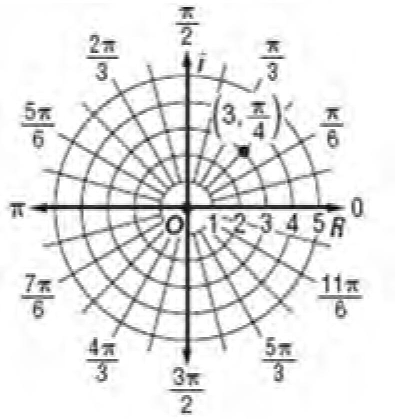


9-5 Complex Numbers and De Moivre's Theorem

21. $3\left(\cos\dfrac{\pi}{4} + \boldsymbol{i}\sin\dfrac{\pi}{4}\right)$

SOLUTION:

The value of r is 3, and the value of θ is $\dfrac{\pi}{4}$. Plot the polar coordinates $\left(3, \dfrac{\pi}{4}\right)$.



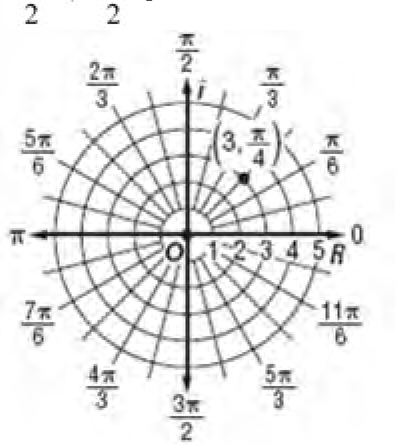
To express the number in rectangular form, evaluate the trigonometric values and simplify.

$$\begin{aligned}3\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right) &= 3\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\boldsymbol{i}\right)\\ &= \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}\boldsymbol{i}\end{aligned}$$

The rectangular form of $3\left(\cos\dfrac{\pi}{4} + \boldsymbol{i}\sin\dfrac{\pi}{4}\right)$ is $\dfrac{3\sqrt{2}}{2} + \dfrac{3\sqrt{2}}{2}\boldsymbol{i}$.

ANSWER:

$\dfrac{3\sqrt{2}}{2} + \dfrac{3\sqrt{2}}{2}\boldsymbol{i}$

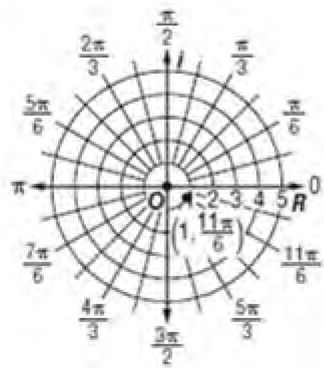


22. $\left(\cos\dfrac{11\pi}{6} + \boldsymbol{i}\sin\dfrac{11\pi}{6}\right)$

SOLUTION:

The value of r is 1, and the value of θ is $\dfrac{11\pi}{6}$.

Plot the polar coordinates $\left(1, \dfrac{11\pi}{6}\right)$.



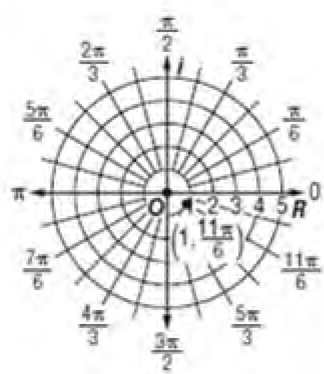
To express the number in rectangular form, evaluate the trigonometric values and simplify.

$$\begin{aligned}\left(\cos\frac{11\pi}{6} + \boldsymbol{i}\sin\frac{11\pi}{6}\right) &= \frac{\sqrt{3}}{2} + \boldsymbol{i}\left(-\frac{1}{2}\right)\\ &= \frac{\sqrt{3}}{2} - \frac{1}{2}\boldsymbol{i}\end{aligned}$$

The rectangular form of $\left(\cos\dfrac{11\pi}{6} + \boldsymbol{i}\sin\dfrac{11\pi}{6}\right)$ is $\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}\boldsymbol{i}$.

ANSWER:

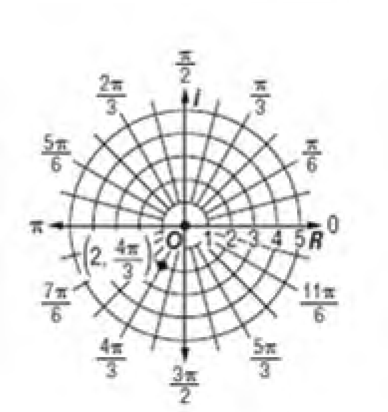
$\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}\boldsymbol{i}$



23. $2\left(\cos\frac{4\pi}{3} + \boldsymbol{i}\sin\frac{4\pi}{3}\right)$

SOLUTION:

The value of r is 2, and the value of θ is $\frac{4\pi}{3}$. Plot the polar coordinates $\left(2, \frac{4\pi}{3}\right)$.



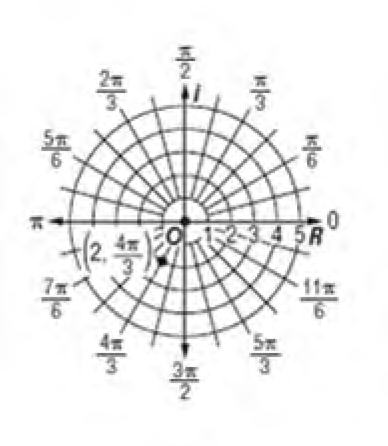
To express the number in rectangular form, evaluate the trigonometric values and simplify.

$$\begin{aligned} 2\left(\cos\frac{4\pi}{3} + \boldsymbol{i}\sin\frac{4\pi}{3}\right) &= 2\left[-\frac{1}{2} + \boldsymbol{i}\left(-\frac{\sqrt{3}}{2}\right)\right] \\ &= -1 - \sqrt{3}\boldsymbol{i} \end{aligned}$$

The rectangular form of $2\left(\cos\frac{4\pi}{3} + \boldsymbol{i}\sin\frac{4\pi}{3}\right)$ is $-1 - \sqrt{3}\,\boldsymbol{i}$.

ANSWER:

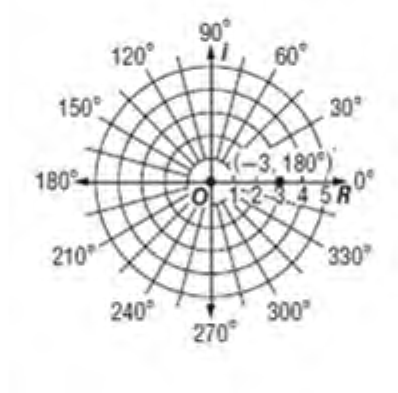
$-1 - \sqrt{3}\,\boldsymbol{i}$



24. $-3(\cos 180° + \boldsymbol{i}\sin 180°)$

SOLUTION:

The value of r is -3, and the value of θ is 180°. Plot the polar coordinates $(-3, 180°)$.



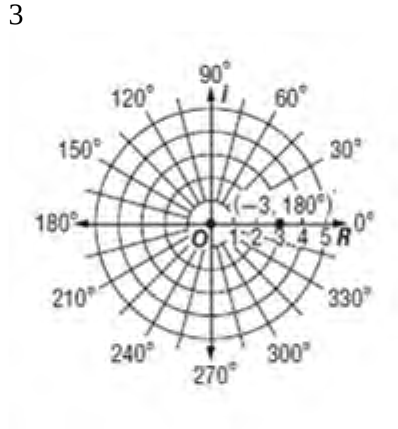
To express the number in rectangular form, evaluate the trigonometric values and simplify.

$$\begin{aligned} -3(\cos 180° + \boldsymbol{i}\sin 180°) &= -3[-1 + \boldsymbol{i}(0)] \\ &= 3 \end{aligned}$$

The rectangular form of $-3(\cos 180° + \boldsymbol{i}\sin 180°)$ is 3.

ANSWER:

3

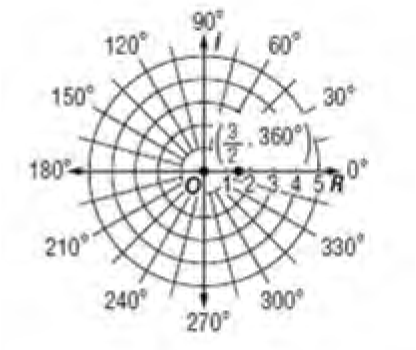


25. $\frac{3}{2}$(cos 360º + ***i*** sin 360º)

SOLUTION:

The value of r is $\frac{3}{2}$, and the value of θ is 360°.

Plot the polar coordinates $\left(\frac{3}{2}, 360^\circ\right)$



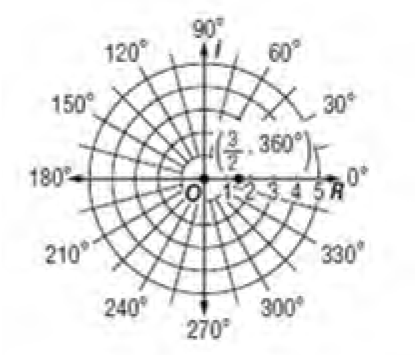
To express the number in rectangular form, evaluate the trigonometric values and simplify.

$$\frac{3}{2}(\cos 360^\circ + i \sin 360^\circ) = \frac{3}{2}[1 + i(0)]$$
$$= \frac{3}{2}$$

The rectangular form of $\frac{3}{2}$(cos 360º + ***i*** sin 360º) is $\frac{3}{2}$.

ANSWER:

$\frac{3}{2}$



Find each product or quotient and express it in rectangular form.

26. $6\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \cdot 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

SOLUTION:

Use the Product Formula to find the product in polar form.

$$6\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \cdot 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$= 6(4)\left[\cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right)\right]$$
$$= 24\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Now find the rectangular form of the product.

$$24\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 24\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$
$$= -12\sqrt{2} + 12\sqrt{2}i$$

The polar form is $24\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$. The rectangular form is $-12\sqrt{2} + 12\sqrt{2}i$.

ANSWER:

$24\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right); -12\sqrt{2} + 12\sqrt{2}i$

27. 5(cos 135° + ***i*** sin 135°) · 2 (cos 45° + ***i*** sin 45°)

SOLUTION:

Use the Product Formula to find the product in polar form.

$$5(\cos 135^\circ + i\sin 135^\circ) \cdot 2(\cos 45^\circ + i\sin 45^\circ)$$
$$= 5(2)[\cos(135^\circ + 45^\circ) + i\sin(135^\circ + 45^\circ)]$$
$$= 10(\cos 180^\circ + i\sin 180^\circ)$$

Now find the rectangular form of the product.

$$10(\cos 180^\circ + i\sin 180^\circ) = 10(-1 + 0)$$
$$= -10$$

The polar form is 10(cos 180° + ***i*** sin 180°). The rectangular form is −10.

ANSWER:

10(cos 180° + ***i*** sin 180°); −10

9-5 Complex Numbers and De Moivre's Theorem

28. $3\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\div\frac{1}{2}(\cos\pi+i\sin\pi)$

SOLUTION:

Use the Quotient Formula to find the quotient in polar form.

$$3\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\div\frac{1}{2}(\cos\pi+i\sin\pi)$$
$$=\frac{3}{\frac{1}{2}}\left[\cos\left(\frac{3\pi}{4}-\pi\right)+i\sin\left(\frac{3\pi}{4}-\pi\right)\right]$$
$$=6\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$$

Now find the rectangular form.

$$6\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]=6\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)$$
$$=3\sqrt{2}-3\sqrt{2}i$$

The polar form is $6\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$. The rectangular form is $3\sqrt{2}-3\sqrt{2}i$.

ANSWER:

$6\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]; 3\sqrt{2}-3\sqrt{2}i$

29. $2(\cos 90^\circ + i\sin 90^\circ) \cdot 2(\cos 270^\circ + i\sin 270^\circ)$

SOLUTION:

Use the Product Formula to find the product in polar form.

$$2(\cos 90^\circ + i\sin 90^\circ)\cdot 2(\cos 270^\circ + i\sin 270^\circ)$$
$$=2(2)\left[\cos(90^\circ+270^\circ)+i\sin(90^\circ+270^\circ)\right]$$
$$=4(\cos 360^\circ + i\sin 360^\circ)$$

Now find the rectangular form of the product.

$$4(\cos 360^\circ + i\sin 360^\circ)=4(1+0)$$
$$=4$$

The polar form is $4(\cos 360^\circ + i\sin 360^\circ)$. The rectangular form is 4.

ANSWER:

$4(\cos 360^\circ + i\sin 360^\circ)$; 4

30. $3\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\div 4\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

SOLUTION:

Use the Quotient Formula to find the quotient in polar form.

$$3\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\div 4\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$
$$=\frac{3}{4}\left[\cos\left(\frac{\pi}{6}-\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{6}-\frac{2\pi}{3}\right)\right]$$
$$=\frac{3}{4}\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$$

Now find the rectangular form.

$$\frac{3}{4}\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]=\frac{3}{4}(0-i)$$
$$=-\frac{3}{4}i$$

The polar form of the quotient is $\frac{3}{4}\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$. The rectangular form of the quotient is $-\frac{3}{4}i$.

ANSWER:

$\frac{3}{4}\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]; -\frac{3}{4}i$

9-5 Complex Numbers and De Moivre's Theorem

31. $4\left(\cos\frac{9\pi}{4}+\boldsymbol{i}\sin\frac{9\pi}{4}\right)\div 2\left(\cos\frac{3\pi}{2}+\boldsymbol{i}\sin\frac{3\pi}{2}\right)$

SOLUTION:

Use the Quotient Formula to find the quotient in polar form.

$$4\left(\cos\frac{9\pi}{4}+\boldsymbol{i}\sin\frac{9\pi}{4}\right)\div 2\left(\cos\frac{3\pi}{2}+\boldsymbol{i}\sin\frac{3\pi}{2}\right)$$
$$=\frac{4}{2}\left[\cos\left(\frac{9\pi}{4}-\frac{3\pi}{2}\right)+\boldsymbol{i}\sin\left(\frac{9\pi}{4}-\frac{3\pi}{2}\right)\right]$$
$$=2\left[\cos\frac{3\pi}{4}+\boldsymbol{i}\sin\frac{3\pi}{4}\right]$$

Now find the rectangular form.

$$2\left[\cos\frac{3\pi}{4}+\boldsymbol{i}\sin\frac{3\pi}{4}\right]=2\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\boldsymbol{i}\right)$$
$$=-\sqrt{2}+\sqrt{2}\boldsymbol{i}$$

The polar form of the quotient is $2\left[\cos\frac{3\pi}{4}+\boldsymbol{i}\sin\frac{3\pi}{4}\right]$. The rectangular form of the quotient is $-\sqrt{2}+\sqrt{2}\boldsymbol{i}$.

ANSWER:

$2\left(\cos\frac{3\pi}{4}+\boldsymbol{i}\sin\frac{3\pi}{4}\right); -\sqrt{2}+\sqrt{2}\boldsymbol{i}$

32. $\frac{1}{2}$(cos 60º + $\boldsymbol{i}$ sin 60º) · 6(cos 150º + $\boldsymbol{i}$ sin 150º)

SOLUTION:

Use the Product Formula to find the product in polar form.

$$\frac{1}{2}(\cos 60^\circ+\boldsymbol{i}\sin 60^\circ)\cdot 6(\cos 150^\circ+\boldsymbol{i}\sin 150^\circ)$$
$$=\frac{1}{2}(6)\left[\cos(60^\circ+150^\circ)+\boldsymbol{i}\sin(60^\circ+150^\circ)\right]$$
$$=3(\cos 210^\circ+\boldsymbol{i}\sin 210^\circ)$$

Now find the rectangular form of the product.

$$3(\cos 210^\circ+\boldsymbol{i}\sin 210^\circ)=3\left(-\frac{\sqrt{3}}{2}+\left(-\frac{1}{2}\right)\boldsymbol{i}\right)$$
$$=-\frac{3\sqrt{3}}{2}-\frac{3}{2}\boldsymbol{i}$$

The polar form is $3(\cos 210^\circ+\boldsymbol{i}\sin 210^\circ)$. The rectangular form is $-\frac{3\sqrt{3}}{2}-\frac{3}{2}\boldsymbol{i}$.

ANSWER:

3(cos 210° + $\boldsymbol{i}$ sin 210°); $-\frac{3\sqrt{3}}{2}-\frac{3}{2}\boldsymbol{i}$

33. $6\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\div 2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

SOLUTION:

Use the Quotient Formula to find the quotient in polar form.

$$6\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\div 2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$
$$=\frac{6}{2}\left[\cos\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)\right]$$
$$=3\left(\cos\frac{2\pi}{4}+i\sin\frac{2\pi}{4}\right)$$
$$=3\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

Now find the rectangular form.

$$3\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=3(0+i)$$
$$=3i$$

The polar form of the quotient is $3\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$.

The rectangular form of the quotient is $3i$.

ANSWER:

$3\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right); 3i$

34. $5(\cos 180^\circ + i\sin 180^\circ)\cdot 2(\cos 135^\circ + i\sin 135^\circ)$

SOLUTION:

Use the Product Formula to find the product in polar form.

$$5(\cos 180^\circ + i\sin 180^\circ)\cdot 2(\cos 135^\circ + i\sin 135^\circ)$$
$$=5(2)\left[\cos(180^\circ+135^\circ)+i\sin(180^\circ+135^\circ)\right]$$
$$=10(\cos 315^\circ + i\sin 315^\circ)$$

Now find the rectangular form of the product.

$$10(\cos 315^\circ + i\sin 315^\circ)=10\left(\frac{\sqrt{2}}{2}+\left(-\frac{\sqrt{2}}{2}\right)i\right)$$
$$=5\sqrt{2}-5\sqrt{2}i$$

The polar form is $10(\cos 315^\circ + i\sin 315^\circ)$. The rectangular form is $5\sqrt{2}-5\sqrt{2}i$.

ANSWER:

$10(\cos 315^\circ + i\sin 315^\circ)$; $5\sqrt{2} - 5\sqrt{2}i$

35. $\frac{1}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\div 3\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

SOLUTION:

Use the Quotient Formula to find the quotient in polar form.

$$\frac{1}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\div 3\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$
$$=\frac{\frac{1}{2}}{3}\left[\cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\right]$$
$$=\frac{1}{6}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

Now find the rectangular form of the product.

$$\frac{1}{6}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=\frac{1}{6}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$
$$=\frac{\sqrt{3}}{12}+\frac{1}{12}i$$

The polar form of the quotient is $\frac{1}{6}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$. The rectangular form of the quotient is $\frac{\sqrt{3}}{12}+\frac{1}{12}i$.

ANSWER:

$\frac{1}{6}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right); \frac{\sqrt{3}}{12}+\frac{1}{12}i$

9-5 Complex Numbers and De Moivre's Theorem

Find each power and express it in rectangular form.

36. $(2 + 2\sqrt{3}\,\boldsymbol{i})^6$

SOLUTION:

First, write $2 + 2\sqrt{3}\,\boldsymbol{i}$ in polar form.

$$\begin{aligned} r &= \sqrt{a^2+b^2} \\ &= \sqrt{2^2+(2\sqrt{3})^2} \\ &= \sqrt{4+12} \\ &= 4 \end{aligned} \qquad \begin{aligned} \theta &= \tan^{-1}\frac{b}{a} \\ &= \tan^{-1}\frac{2\sqrt{3}}{2} \\ &= \tan^{-1}\sqrt{3} \\ &= \frac{\pi}{3} \end{aligned}$$

The polar form of $2 + 2\sqrt{3}\,\boldsymbol{i}$ is $4\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)$.

Now use De Moivre's Theorem to find the sixth power.

$$\begin{aligned} \left(2+2\sqrt{3}i\right)^6 &= \left[4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]^6 \\ &= 4^6\left[\cos 6\left(\frac{\pi}{3}\right)+i\sin 6\left(\frac{\pi}{3}\right)\right] \\ &= 4096(\cos 2\pi + i\sin 2\pi) \\ &= 4096(1+0i) \\ &= 4096 \end{aligned}$$

Therefore, $\left(2+2\sqrt{3}i\right)^6 = 4096$.

ANSWER:

4096

37. $(12\boldsymbol{i} - 5)^3$

SOLUTION:

First, write $12\boldsymbol{i} - 5$ in polar form.

$$\begin{aligned} r &= \sqrt{a^2+b^2} \\ &= \sqrt{(-5)^2+12^2} \\ &= \sqrt{25+144} \\ &= 13 \end{aligned} \qquad \begin{aligned} \theta &= \tan^{-1}\frac{b}{a}+\pi \\ &= \tan^{-1}\frac{12}{-5}+\pi \\ &\approx 1.97 \end{aligned}$$

The polar form of $12\boldsymbol{i} - 5$ is $13(\cos 1.97 + \boldsymbol{i}\sin 1.97)$. Now use De Moivre's Theorem to find the third power.

$$\begin{aligned} (12i-5)^3 &= \left[13\left(\cos\left(\tan^{-1}\frac{12}{-5}+\pi\right)+i\sin\left(\tan^{-1}\frac{12}{-5}+\pi\right)\right)\right]^3 \\ &= 13^3\left[\cos 3\left(\tan^{-1}\frac{12}{-5}+\pi\right)+i\sin 3\left(\tan^{-1}\frac{12}{-5}+\pi\right)\right] \\ &= 2197\left[\cos 3\left(\tan^{-1}\frac{12}{-5}+\pi\right)+i\sin 3\left(\tan^{-1}\frac{12}{-5}+\pi\right)\right] \\ &= 2035-828i \end{aligned}$$

Therefore, $(12\boldsymbol{i}-5)^3 = 2035 - 828\boldsymbol{i}$.

ANSWER:

$2035 - 828\boldsymbol{i}$

38. $\left[4\left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right)\right]^4$

SOLUTION:

$4\left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right)$ is already in polar form. Use De Moivre's Theorem to find the fourth power.

$$\begin{aligned} \left[4\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\right]^4 &= 4^4\left[\left(\cos 4\left(\frac{\pi}{2}\right)+i\sin 4\left(\frac{\pi}{2}\right)\right)\right] \\ &= 256(\cos 2\pi + i\sin 2\pi) \\ &= 256(1+0) \\ &= 256 \end{aligned}$$

ANSWER:

256

39. $(\sqrt{3} - \boldsymbol{i})^3$

SOLUTION:

First, write $\sqrt{3} - \boldsymbol{i}$ in polar form.

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{(\sqrt{3})^2 + (-1)^2} \\ &= \sqrt{3+1} \\ &= 2 \end{aligned} \qquad \begin{aligned} \theta &= \tan^{-1}\frac{b}{a} \\ &= \tan^{-1}\frac{-1}{\sqrt{3}} \\ &= \frac{\pi}{6} \end{aligned}$$

The polar form of $\sqrt{3} - \boldsymbol{i}$ is $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$. Now use De Moivre's Theorem to find the third power.

$$\begin{aligned} (\sqrt{3} - i)^3 &= \left[2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)\right]^3 \\ &= 2^3\left[\cos 3\left(\frac{\pi}{6}\right) + i\sin 3\left(\frac{\pi}{6}\right)\right] \\ &= 8\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] \\ &= 8(0 + (-1)i) \\ &= -8i \end{aligned}$$

Therefore, $(\sqrt{3} - \boldsymbol{i})^3 = -8\boldsymbol{i}$.

ANSWER:

$-8\boldsymbol{i}$

40. $(3 - 5\boldsymbol{i})^4$

SOLUTION:

First, write $3 - 5\boldsymbol{i}$ in polar form.

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{3^2 + (-5)^2} \\ &= \sqrt{9 + 25} \\ &= \sqrt{34} \end{aligned} \qquad \begin{aligned} \theta &= \tan^{-1}\frac{b}{a} \\ &= \tan^{-1}\frac{-5}{3} \\ &\approx -1.03 \end{aligned}$$

The polar form of $3 - 5\boldsymbol{i}$ is $\sqrt{34}\left(\cos(-1.03) + \boldsymbol{i}\sin(-1.03)\right)$. Now use De Moivre's Theorem to find the fourth power.

$$\begin{aligned} (3-5i)^4 &= \left[\sqrt{34}\left(\cos\left(\tan^{-1}\frac{-5}{3}\right) + i\sin\left(\tan^{-1}\frac{-5}{3}\right)\right)\right]^4 \\ &= \left(\sqrt{34}\right)^4\left[\left(\cos 4\left(\tan^{-1}\frac{-5}{3}\right) + i\sin 4\left(\tan^{-1}\frac{-5}{3}\right)\right)\right] \\ &= 1156\left[\left(\cos 4\left(\tan^{-1}\frac{-5}{3}\right) + i\sin 4\left(\tan^{-1}\frac{-5}{3}\right)\right)\right] \\ &= -644 + 960i \end{aligned}$$

Therefore, $(3 - 5\boldsymbol{i})^4 = -644 + 960\boldsymbol{i}$.

ANSWER:

$-644 + 960\boldsymbol{i}$

41. $(2 + 4\boldsymbol{i})^4$

SOLUTION:

First, write $2 + 4\boldsymbol{i}$ in polar form.

$$\begin{aligned} r &= \sqrt{a^2+b^2} \\ &= \sqrt{2^2+4^2} \\ &= \sqrt{4+16} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned} \qquad \begin{aligned} \theta &= \tan^{-1}\frac{b}{a} \\ &= \tan^{-1}\frac{4}{2} \\ &= \tan^{-1}2 \\ &\approx 1.11 \end{aligned}$$

The polar form of $2 + 4\boldsymbol{i}$ is $2\sqrt{5}\left(\cos(1.11) + \boldsymbol{i}\sin(1.11)\right)$. Now use De Moivre's Theorem to find the fourth power.

$$\begin{aligned} (2+4\boldsymbol{i})^4 &= \left[2\sqrt{5}\left(\cos\left(\tan^{-1}2\right) + \boldsymbol{i}\sin\left(\tan^{-1}2\right)\right)\right]^4 \\ &= \left(2\sqrt{5}\right)^4\left[\left(\cos 4\left(\tan^{-1}2\right) + \boldsymbol{i}\sin 4\left(\tan^{-1}2\right)\right)\right] \\ &= 400\left[\left(\cos 4\left(\tan^{-1}2\right) + \boldsymbol{i}\sin 4\left(\tan^{-1}2\right)\right)\right] \\ &= -112 - 384\boldsymbol{i} \end{aligned}$$

Therefore, $(2+4\boldsymbol{i})^4 = -112 - 384\boldsymbol{i}$.

ANSWER:

$-112 - 384\boldsymbol{i}$

42. $(3 - 6\boldsymbol{i})^4$

SOLUTION:

First, write $3 - 6\boldsymbol{i}$ in polar form.

$$\begin{aligned} r &= \sqrt{a^2+b^2} \\ &= \sqrt{3^2+(-6)^2} \\ &= \sqrt{9+36} \\ &= \sqrt{45} \end{aligned} \qquad \begin{aligned} \theta &= \tan^{-1}\frac{b}{a} \\ &= \tan^{-1}\frac{-6}{3} \\ &= \tan^{-1}(-2) \\ &\approx -1.11 \end{aligned}$$

The polar form of $3 - 6\boldsymbol{i}$ is $\sqrt{45}\left(\cos(-1.11) + \boldsymbol{i}\sin(-1.11)\right)$. Now use De Moivre's Theorem to find the fourth power.

$$\begin{aligned} (3-6i)^4 &= \left[\sqrt{45}\left(\cos\left(\tan^{-1}(-2)\right) + i\sin\left(\tan^{-1}(-2)\right)\right)\right]^4 \\ &= \left(\sqrt{45}\right)^4\left[\left(\cos 4\left(\tan^{-1}(-2)\right) + i\sin 4\left(\tan^{-1}(-2)\right)\right)\right] \\ &= 2025\left[\left(\cos 4\left(\tan^{-1}(-2)\right) + i\sin 4\left(\tan^{-1}(-2)\right)\right)\right] \\ &= -567 + 1944i \end{aligned}$$

Therefore, $(3-6i)^4 = -567 + 1944i$.

ANSWER:

$-567 + 1944\boldsymbol{i}$

43. $(2 + 3\boldsymbol{i})^2$

SOLUTION:

First, write $2 + 3\boldsymbol{i}$ in polar form.

$$\begin{aligned} r &= \sqrt{a^2+b^2} \\ &= \sqrt{2^2+3^2} \\ &= \sqrt{4+9} \\ &= \sqrt{13} \end{aligned} \qquad \begin{aligned} \theta &= \tan^{-1}\frac{b}{a} \\ &= \tan^{-1}\frac{3}{2} \\ &\approx 0.98 \end{aligned}$$

The polar form of $2 + 3\boldsymbol{i}$ is $\sqrt{13}\left(\cos 0.98 + \boldsymbol{i}\sin 0.98\right)$. Now use De Moivre's Theorem to find the second power.

$$\begin{aligned} (2+3\boldsymbol{i})^2 &= \left[\sqrt{13}\left(\cos\left(\tan^{-1}\left(\frac{3}{2}\right)\right) + \boldsymbol{i}\sin\left(\tan^{-1}\left(\frac{3}{2}\right)\right)\right)\right]^2 \\ &= \left(\sqrt{13}\right)^2\left[\left(\cos 2\left(\tan^{-1}\left(\frac{3}{2}\right)\right) + \boldsymbol{i}\sin 2\left(\tan^{-1}\left(\frac{3}{2}\right)\right)\right)\right] \\ &= 13\left[\left(\cos 2\left(\tan^{-1}\left(\frac{3}{2}\right)\right) + \boldsymbol{i}\sin 2\left(\tan^{-1}\left(\frac{3}{2}\right)\right)\right)\right] \\ &= -5 + 12\boldsymbol{i} \end{aligned}$$

Therefore, $(2+3\boldsymbol{i})^2 = -5 + 12\boldsymbol{i}$.

ANSWER:

$-5 + 12\boldsymbol{i}$

44. $\left[3\left(\cos\frac{\pi}{6} + \boldsymbol{i}\sin\frac{\pi}{6}\right)\right]^3$

SOLUTION:

$3\left(\cos\frac{\pi}{6} + \boldsymbol{i}\sin\frac{\pi}{6}\right)$ is already in polar form. Use De Moiver's Theorem to find the third power.

$$\begin{aligned} \left[3\left(\cos\frac{\pi}{6} + \boldsymbol{i}\sin\frac{\pi}{6}\right)\right]^3 &= 3^3\left[\left(\cos 3\left(\frac{\pi}{6}\right) + \boldsymbol{i}\sin 3\left(\frac{\pi}{6}\right)\right)\right] \\ &= 27\left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right) \\ &= 27(0 + \boldsymbol{i}) \\ &= 27\boldsymbol{i} \end{aligned}$$

ANSWER:

$27\boldsymbol{i}$

45. $\left[2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^4$

SOLUTION:

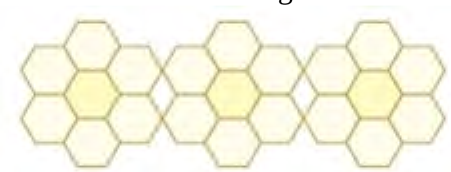
$2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ is already in polar form. Use De Moiver's Theorem to find the fourth power.

$$\begin{aligned}\left[2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^4&=2^4\left[\left(\cos 4\left(\frac{\pi}{4}\right)+i\sin 4\left(\frac{\pi}{4}\right)\right)\right]\\&=16(\cos\pi+i\sin\pi)\\&=16(-1+0)\\&=-16\end{aligned}$$

ANSWER:

−16

46. **DESIGN** Stella works for an advertising agency. She wants to incorporate a design comprised of regular hexagons as the artwork for one of her proposals. Stella can locate the vertices of one of the central regular hexagons by graphing the solutions to $x^6 - 1 = 0$ in the complex plane. Find the vertices of this hexagon.



SOLUTION:

The equation $x^6 - 1 = 0$ can be written as $x^6 = 1$. To find the location of the six vertices for one of the hexagons, find the sixth roots of 1.
First, write 1 in polar form.

$$\begin{aligned}r&=\sqrt{a^2+b^2}\\&=\sqrt{1^2+0^2}\\&=\sqrt{1+0}\\&=1\end{aligned}$$

$$\begin{aligned}\theta&=\tan^{-1}\frac{b}{a}\\&=\tan^{-1}\frac{0}{1}\\&=\tan^{-1}0\\&=0\end{aligned}$$

The polar form of 1 is $1 \cdot (\cos 0 + i\sin 0)$. Now write an expression for the sixth roots.

$$1\left(\cos\frac{0+2n\pi}{6}+i\sin\frac{0+2n\pi}{6}\right)=\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}$$

Let $n = 0$ to find the first root of 1.

$$\begin{aligned}&\cos\frac{(0)\pi}{3}+i\sin\frac{(0)\pi}{3}\\&=\cos 0+i\sin 0=1\end{aligned}$$

Notice that the modulus of each complex number is 1. The arguments are found by $\frac{n\pi}{3}$, resulting in θ increasing by $\frac{\pi}{3}$ for each successive root.
Therefore, we can calculate the remaining roots by adding $\frac{\pi}{3}$ to each previous θ.

$n = 1$

$$\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

$n = 2$

$$\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

$n = 3$

$$\cos\pi+i\sin\pi=-1$$

$n = 4$

$$\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$n = 5$

$$\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}=\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

The vertices are located at

$1, \frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -1, -\frac{1}{2}-\frac{\sqrt{3}}{2}i$, and $\frac{1}{2}-\frac{\sqrt{3}}{2}i$.

ANSWER:

$1, \frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -1, -\frac{1}{2}-\frac{\sqrt{3}}{2}i, \frac{1}{2}-\frac{\sqrt{3}}{2}i$

Find all of the distinct p th roots of the complex number.

47. sixth roots of i

SOLUTION:

First, write ***i*** in polar form.

$$r=\sqrt{a^2+b^2} \qquad \theta=\tan^{-1}\frac{b}{a}$$
$$=\sqrt{0^2+1^2} \qquad =\tan^{-1}\frac{1}{0}$$
$$=\sqrt{1} \qquad =\frac{\pi}{2}$$
$$=1$$

The polar form of ***i*** is $1(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})$. Now write an expression for the sixth roots.

$$(1)^{\frac{1}{6}}\left(\cos\frac{\frac{\pi}{2}+2n\pi}{6}+i\sin\frac{\frac{\pi}{2}+2n\pi}{6}\right)$$
$$=\cos\frac{\pi+4n\pi}{12}+i\sin\frac{\pi+4n\pi}{12}$$

Let n = 0, 1, 2, 3, 4 and 5 successively to find the sixth roots.

Let n = 0.

$$\cos\frac{\pi+4(0)\pi}{12}+i\sin\frac{\pi+4(0)\pi}{12}$$
$$=\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}$$
$$=0.97+0.26i$$

Let n = 1.

$$\cos\frac{\pi+4(1)\pi}{12}+i\sin\frac{\pi+4(1)\pi}{12}$$
$$=\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}$$
$$=0.26+0.97i$$

Let n = 2.

$$\cos\frac{\pi+4(2)\pi}{12}+i\sin\frac{\pi+4(2)\pi}{12}$$
$$=\cos\frac{9\pi}{12}+i\sin\frac{9\pi}{12}$$
$$=-0.71+0.71i$$

Let n = 3.

$$\cos\frac{\pi+4(3)\pi}{12}+i\sin\frac{\pi+4(3)\pi}{12}$$
$$=\cos\frac{13\pi}{12}+i\sin\frac{13\pi}{12}$$
$$=-0.97-0.26i$$

Let n = 4.

$$\cos\frac{\pi+4(4)\pi}{12}+i\sin\frac{\pi+4(4)\pi}{12}$$
$$=\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}$$
$$=-0.26-0.97i$$

Let n = 4

$$\cos\frac{\pi+4(5)\pi}{12}+i\sin\frac{\pi+4(5)\pi}{12}$$
$$=\cos\frac{21\pi}{12}+i\sin\frac{21\pi}{12}$$
$$=0.71-0.71i$$

The sixth roots of ***i*** are approximately 0.97 + 0.26***i***, 0.26 + 0.97***i***, –0.71 + 0.71***i***, –0.97 – 0.26 ***i***, – 0.26 – 0.97***i***, 0.71 – 0.71***i***.

ANSWER:

0.97 + 0.26***i***, 0.26 + 0.97***i***, –0.71 + 0.71***i***, –0.97 – 0.26***i***, – 0.26 – 0.97***i***, 0.71 – 0.71***i***

48. fifth roots of –***i***

SOLUTION:

First, write $-i$ in polar form.

$$r=\sqrt{a^2+b^2} \qquad \theta=\tan^{-1}\frac{b}{a}$$
$$=\sqrt{0^2+(-1)^2} \qquad =\tan^{-1}\frac{-1}{0}$$
$$=\sqrt{1} \qquad =-\frac{\pi}{2}$$
$$=1$$

The polar form of $-i$ is $1\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)$.

Now write an expression for the fifth roots.

$$(1)^{\frac{1}{5}}\left(\cos\frac{-\frac{\pi}{2}+2n\pi}{5}+i\sin\frac{-\frac{\pi}{2}+2n\pi}{5}\right)$$
$$=\cos\frac{-\pi+4n\pi}{10}+i\sin\frac{-\pi+4n\pi}{10}$$

Let n = 0, 1, 2, 3 and 4 successively to find the fifth roots.

Let n = 0.

$$\cos\frac{-\pi+4(0)\pi}{10}+i\sin\frac{-\pi+4(0)\pi}{10}$$
$$=\cos\left(-\frac{\pi}{10}\right)+i\sin\left(-\frac{\pi}{10}\right)$$
$$=0.95-0.31i$$

Let $n = 1$.

$$\cos\frac{-\pi+4(1)\pi}{10}+i\sin\frac{-\pi+4(1)\pi}{10}$$
$$=\cos\left(\frac{3\pi}{10}\right)+i\sin\left(\frac{3\pi}{10}\right)$$
$$=0.59+0.81i$$

Let $n = 2$.

$$\cos\frac{-\pi+4(2)\pi}{10}+i\sin\frac{-\pi+4(2)\pi}{10}$$
$$=\cos\left(\frac{7\pi}{10}\right)+i\sin\left(\frac{7\pi}{10}\right)$$
$$=-0.59+0.81i$$

Let $n = 3$.

$$\cos\frac{-\pi+4(3)\pi}{10}+i\sin\frac{-\pi+4(3)\pi}{10}$$
$$=\cos\left(\frac{11\pi}{10}\right)+i\sin\left(\frac{11\pi}{10}\right)$$
$$=-0.95-0.31i$$

Let $n = 4$.

$$\cos\frac{-\pi+4(4)\pi}{10}+i\sin\frac{-\pi+4(4)\pi}{10}$$
$$=\cos\left(\frac{15\pi}{10}\right)+i\sin\left(\frac{15\pi}{10}\right)$$
$$=-i$$

The sixth roots of $\boldsymbol{i}$ are approximately $0.59 + 0.81\boldsymbol{i}$, $-0.59 + 0.81\boldsymbol{i}$, $-0.95 - 0.31\boldsymbol{i}$, $-\boldsymbol{i}$, $0.95 - 0.31\boldsymbol{i}$.

ANSWER:

$0.59 + 0.81\boldsymbol{i}$, $-0.59 + 0.81\boldsymbol{i}$, $-0.95 - 0.31\boldsymbol{i}$, $-\boldsymbol{i}$, $0.95 - 0.31\boldsymbol{i}$

49. fourth roots of $4\sqrt{3} - 4\boldsymbol{i}$

SOLUTION:

First, write $4\sqrt{3} - 4\boldsymbol{i}$ in polar form.

$$r=\sqrt{a^2+b^2}=\sqrt{\left(4\sqrt{3}\right)^2+(-4)^2}=\sqrt{48+16}=8$$

$$\theta=\tan^{-1}\frac{b}{a}=\tan^{-1}\frac{-4}{4\sqrt{3}}=\tan^{-1}-\frac{\sqrt{3}}{3}=-\frac{\pi}{6}$$

The polar form of $4\sqrt{3} - 4\boldsymbol{i}$ is $8\left(\cos\left(-\frac{\pi}{6}\right)+\boldsymbol{i}\sin\left(-\frac{\pi}{6}\right)\right)$. Now write an expression for the fourth roots.

$$(8)^{\frac{1}{4}}\left(\cos\frac{-\frac{\pi}{6}+2n\pi}{4}+\boldsymbol{i}\sin\frac{-\frac{\pi}{6}+2n\pi}{4}\right)$$
$$=(8)^{\frac{1}{4}}\left(\cos\frac{-\pi+12n\pi}{24}+\boldsymbol{i}\sin\frac{-\pi+12n\pi}{24}\right)$$

Let $n = 0, 1, 2$ and 3 successively to find the fourth roots.

Let $n = 0$.

$$(8)^{\frac{1}{4}}\left(\cos\frac{-\pi+12(0)\pi}{24}+i\sin\frac{-\pi+12(0)\pi}{24}\right)$$
$$=(8)^{\frac{1}{4}}\left(\cos\left(-\frac{\pi}{24}\right)+i\sin\left(-\frac{\pi}{24}\right)\right)$$
$$=1.67-0.22i$$

Let $n = 1$.

$$(8)^{\frac{1}{4}}\left(\cos\frac{-\pi+12(1)\pi}{24}+i\sin\frac{-\pi+12(1)\pi}{24}\right)$$
$$=(8)^{\frac{1}{4}}\left(\cos\left(\frac{11\pi}{24}\right)+i\sin\left(\frac{11\pi}{24}\right)\right)$$
$$=0.22+1.67i$$

Let $n = 2$.

$$(8)^{\frac{1}{4}}\left(\cos\frac{-\pi+12(2)\pi}{24}+i\sin\frac{-\pi+12(2)\pi}{24}\right)$$
$$=(8)^{\frac{1}{4}}\left(\cos\left(\frac{23\pi}{24}\right)+i\sin\left(\frac{23\pi}{24}\right)\right)$$
$$=-1.67+0.22i$$

Let $n = 3$.

$$(8)^{\frac{1}{4}}\left(\cos\frac{-\pi+12(3)\pi}{24}+i\sin\frac{-\pi+12(3)\pi}{24}\right)$$
$$=(8)^{\frac{1}{4}}\left(\cos\left(\frac{35\pi}{24}\right)+i\sin\left(\frac{35\pi}{24}\right)\right)$$
$$=-0.22-1.67i$$

The fourth roots of $4\sqrt{3} - 4\boldsymbol{i}$ are approximately $0.22 + 1.67\boldsymbol{i}$, $-1.67 + 0.22\boldsymbol{i}$, $-0.22 - 1.67\boldsymbol{i}$, $1.67 - 0.22\boldsymbol{i}$.

ANSWER:

$0.22 + 1.67\boldsymbol{i}$, $-1.67 + 0.22\boldsymbol{i}$, $-0.22 - 1.67\boldsymbol{i}$, $1.67 - 0.22\boldsymbol{i}$

50. cube roots of $-117 + 44\boldsymbol{i}$

SOLUTION:

First, write $-117 + 44\boldsymbol{i}$ in polar form.

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{(-117)^2 + 44^2} \\ &= \sqrt{13689 + 1936} \\ &= \sqrt{15625} \\ &= 125 \end{aligned} \qquad \begin{aligned} \theta &= \tan^{-1}\frac{b}{a} + \pi \\ &= \tan^{-1}\frac{44}{-117} + \pi \\ &\approx 2.78 \end{aligned}$$

The polar form of $-117 + 44\boldsymbol{i}$ is $125(\cos 2.78 + \boldsymbol{i}\sin 2.78)$. Now write an expression for the cube roots.

$$\begin{aligned} &(125)^{\frac{1}{3}}\left(\cos\frac{2.78 + 2n\pi}{3} + \boldsymbol{i}\sin\frac{2.78 + 2n\pi}{3}\right) \\ &= 5\left(\cos\frac{2.78 + 2n\pi}{3} + \boldsymbol{i}\sin\frac{2.78 + 2n\pi}{3}\right) \end{aligned}$$

Let $n = 0$, 1 and 2 successively to find the cube roots.

Let $n = 0$.

$$\begin{aligned} &5\left(\cos\frac{2.78 + 2(0)\pi}{3} + i\sin\frac{2.78 + 2(0)\pi}{3}\right) \\ &= 5\left(\cos\frac{2.78}{3} + i\sin\frac{2.78}{3}\right) \\ &= 3 + 4i \end{aligned}$$

Let $n = 1$.

$$\begin{aligned} &5\left(\cos\frac{2.78 + 2(1)\pi}{3} + i\sin\frac{2.78 + 2(1)\pi}{3}\right) \\ &= 5\left(\cos\frac{2.78 + 2\pi}{3} + i\sin\frac{2.78 + 2\pi}{3}\right) \\ &= -4.96 + 0.60i \end{aligned}$$

Let $n = 2$.

$$\begin{aligned} &5\left(\cos\frac{2.78 + 2(2)\pi}{3} + i\sin\frac{2.78 + 2(2)\pi}{3}\right) \\ &= 5\left(\cos\frac{2.78 + 4\pi}{3} + i\sin\frac{2.78 + 4\pi}{3}\right) \\ &= 1.96 - 4.60i \end{aligned}$$

The cube roots of $-117 + 44\boldsymbol{i}$ are approximately $3 + 4\boldsymbol{i}$, $-4.96 + 0.60\boldsymbol{i}$, $1.96 - 4.60\boldsymbol{i}$.

ANSWER:

$3 + 4\boldsymbol{i}$, $-4.96 + 0.60\boldsymbol{i}$, $1.96 - 4.60\boldsymbol{i}$

51. fifth roots of $-1 + 11\sqrt{2}\,\boldsymbol{i}$

SOLUTION:

First, write $-1 + 11\sqrt{2}\,\boldsymbol{i}$ in polar form.

$$r = \sqrt{a^2+b^2} = \sqrt{(-1)^2+(11\sqrt{2})^2} = \sqrt{1+242} = \sqrt{243}$$

$$\theta = \tan^{-1}\frac{b}{a}+\pi = \tan^{-1}\frac{11\sqrt{2}}{-1}+\pi = \tan^{-1}-11\sqrt{2}+\pi \approx 1.63$$

The polar form of $-1 + 11\sqrt{2}\,\boldsymbol{i}$ is $\sqrt{243}$ (cos 1.63 + $\boldsymbol{i}$ sin 1.63). Now write an expression for the fifth roots.

$$\left(\sqrt{243}\right)^{\frac{1}{5}}\left(\cos\frac{1.63+2n\pi}{5}+i\sin\frac{1.63+2n\pi}{5}\right) = \sqrt{3}\left(\cos\frac{1.63+2n\pi}{5}+i\sin\frac{1.63+2n\pi}{5}\right)$$

Let n = 0, 1, 2, 3, and 4 successively to find the fifth roots.

Let n = 0.

$$\sqrt{3}\left(\cos\frac{1.63+2(0)\pi}{5}+i\sin\frac{1.63+2(0)\pi}{5}\right) = \sqrt{3}\left(\cos\frac{1.63}{5}+i\sin\frac{1.63}{5}\right) = 1.64+0.55i$$

Let n = 1.

$$\sqrt{3}\left(\cos\frac{1.63+2(1)\pi}{5}+i\sin\frac{1.63+2(1)\pi}{5}\right) = \sqrt{3}\left(\cos\frac{1.63+2\pi}{5}+i\sin\frac{1.63+2\pi}{5}\right) = -0.02+1.73i$$

Let n = 2.

$$\sqrt{3}\left(\cos\frac{1.63+2(2)\pi}{5}+i\sin\frac{1.63+2(2)\pi}{5}\right) = \sqrt{3}\left(\cos\frac{1.63+4\pi}{5}+i\sin\frac{1.63+4\pi}{5}\right) = -1.65+0.52i$$

Let n = 3.

$$\sqrt{3}\left(\cos\frac{1.63+2(3)\pi}{5}+i\sin\frac{1.63+2(3)\pi}{5}\right) = \sqrt{3}\left(\cos\frac{1.63+6\pi}{5}+i\sin\frac{1.63+6\pi}{5}\right) = -1.00-1.41i$$

Let n = 4.

$$\sqrt{3}\left(\cos\frac{1.63+2(4)\pi}{5}+i\sin\frac{1.63+2(4)\pi}{5}\right) = \sqrt{3}\left(\cos\frac{1.63+8\pi}{5}+i\sin\frac{1.63+8\pi}{5}\right) = 1.03-1.39i$$

The fifth roots of $-1 + 11\sqrt{2}\,\boldsymbol{i}$ are approximately 1.64 + 0.55$\boldsymbol{i}$, –0.02 + 1.73$\boldsymbol{i}$, −1.65 + 0.52$\boldsymbol{i}$, –1.00 – 1.41$\boldsymbol{i}$, and 1.03 – 1.39$\boldsymbol{i}$.

ANSWER:

1.64 + 0.55$\boldsymbol{i}$, –0.02 + 1.73$\boldsymbol{i}$, −1.65 + 0.52$\boldsymbol{i}$, –1.00 – 1.41$\boldsymbol{i}$, 1.03 – 1.39$\boldsymbol{i}$

52. square root of –3 – 4***i***

SOLUTION:

First, write –3 – 4***i*** in polar form.

$$r = \sqrt{a^2 + b^2} = \sqrt{(-3)^2 + (-4)^2} = \sqrt{9 + 16} = 5$$

$$\theta = \tan^{-1}\frac{b}{a} + \pi = \tan^{-1}\frac{-4}{-3} + \pi \approx 4.07$$

The polar form of –3 – 4***i*** is 5(cos 4.07 + ***i*** sin 4.07). Now write an expression for the square roots.

$$5^{\frac{1}{2}}\left(\cos\frac{4.07 + 2n\pi}{2} + i\sin\frac{4.07 + 2n\pi}{2}\right)$$
$$= \sqrt{5}\left(\cos\frac{4.07 + 2n\pi}{2} + i\sin\frac{4.07 + 2n\pi}{2}\right)$$

Let n = 0 and 1 successively to find the square roots.

Let n = 0.

$$\sqrt{5}\left(\cos\frac{4.07 + 2n\pi}{2} + i\sin\frac{4.07 + 2n\pi}{2}\right)$$
$$= \sqrt{5}\left(\cos\frac{4.07 + 2(0)\pi}{2} + i\sin\frac{4.07 + 2(0)\pi}{2}\right)$$
$$= -1 + 2i$$

Let n = 1.

$$\sqrt{5}\left(\cos\frac{4.07 + 2n\pi}{2} + i\sin\frac{4.07 + 2n\pi}{2}\right)$$
$$= \sqrt{5}\left(\cos\frac{4.07 + 2(1)\pi}{2} + i\sin\frac{4.07 + 2(1)\pi}{2}\right)$$
$$= 1 - 2i$$

The square roots of –3 – 4***i*** are approximately −1 + 2***i*** and 1 − 2***i***.

ANSWER:

−1 + 2***i***, 1 − 2***i***

53. find the square roots of unity

SOLUTION:

First, write 1 in polar form.

$$r = \sqrt{a^2 + b^2} = \sqrt{1^2 + 0^2} = \sqrt{1 + 0} = 1$$

$$\theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{0}{1} = \tan^{-1} 0 = 0$$

The polar form of 1 is 1 · (cos 0 + ***i*** sin 0). Now write an expression for the square roots.

$$1\left(\cos\frac{0 + 2n\pi}{2} + i\sin\frac{0 + 2n\pi}{2}\right) = \cos n\pi + i\sin n\pi$$

Let n = 0 to find the first root of 1.

$$\cos n\pi + i\sin n\pi$$
$$= \cos 0 + i\sin 0$$
$$= 1$$

Notice that the modulus of each complex number is 1. The arguments are found by $n\pi$, resulting in θ increasing by $n\pi$ for each successive root. Therefore, we can calculate the remaining root by adding $n\pi$ to the previous θ.

n = 1

$$\cos \pi + i\sin \pi = -1$$

The square roots of 1 are ±1.

ANSWER:

±1

54. find the fourth roots of unity

SOLUTION:

First, write 1 in polar form.

$$r = \sqrt{a^2 + b^2} = \sqrt{1^2 + 0^2} = \sqrt{1 + 0} = 1 \qquad \theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{0}{1} = \tan^{-1} 0 = 0$$

The polar form of 1 is 1 · (cos 0 + ***i*** sin 0). Now write an expression for the fourth roots.

$$1\left(\cos\frac{0 + 2n\pi}{4} + \boldsymbol{i}\sin\frac{0 + 2n\pi}{4}\right) = \cos\frac{n\pi}{2} + \boldsymbol{i}\sin\frac{n\pi}{2}$$

Let n = 0 to find the first root of 1.

$$\cos\frac{(0)\pi}{2} + i\sin\frac{(0)\pi}{2} = \cos 0 + i\sin 0 = 1$$

Notice that the modulus of each complex number is 1. The arguments are found by $\frac{n\pi}{2}$, resulting in θ increasing by $\frac{\pi}{2}$ for each successive root.

Therefore, we can calculate the remaining roots by adding $\frac{\pi}{2}$ to each previous θ.

n = 1

$$\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2} = \boldsymbol{i}$$

n = 2

$$\cos\pi + \boldsymbol{i}\sin\pi = -1$$

n = 3

$$\cos\frac{3\pi}{2} + \boldsymbol{i}\sin\frac{3\pi}{2} = -\boldsymbol{i}$$

The fourth roots of 1 are ±1 and ±***i***.

ANSWER:

±1, ±***i***

55. **ELECTRICITY** The impedance in one part of a series circuit is 5(cos 0.9 +***j*** sin 0.9) ohms. In the second part of the circuit, it is 8(cos 0.4 +***j*** sin 0.4) ohms.

a. Convert each expression to rectangular form.
b. Add your answers from part **a** to find the total impedance in the circuit.
c. Convert the total impedance back to polar form.

SOLUTION:

a. Evaluate the trigonometric values and simplify.

$$5(\cos 0.9 + \boldsymbol{j}\sin 0.9) = 5\cos 0.9 + 5\boldsymbol{j}\sin 0.9 \approx 3.11 + 3.92\boldsymbol{j}\text{ ohms}$$

$$8(\cos 0.4 + \boldsymbol{j}\sin 0.4) = 8\cos 0.4 + 8\boldsymbol{j}\sin 0.4 \approx 7.37 + 3.12\boldsymbol{j}\text{ ohms}$$

b. Find the sum.

(3.11 + 3.92***j***) + (7.37 + 3.12***j***) = 10.48 + 7.04***j*** ohms

c. Find the modulus r and argument θ.

$$r = \sqrt{a^2 + b^2} = \sqrt{10.48^2 + 7.04^2} \approx 12.63 \qquad \theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{7.04}{10.48} \approx 0.59$$

The polar form of 10.48 + 7.04***j*** is 12.63(cos 0.59 + ***j*** sin 0.59) ohms.

ANSWER:

a. 3.11 + 3.92***j*** ohms, 7.37 + 3.12***j*** ohms
b. 10.48 + 7.04***j*** ohms
c. 12.63(cos 0.59 +***j*** sin 0.59) ohms

Find each product. Then repeat the process by multiplying the polar forms of each pair of complex numbers using the Product Formula.

56. $(1 - \boldsymbol{i})(4 + 4\boldsymbol{i})$

SOLUTION:

First, find each product.

$$\begin{aligned}(1-i)(4+4i) &= 4+4i-4i-4i^2\\ &= 4-4(-1)\\ &= 8\end{aligned}$$

Express each complex number in polar form.

For $1 - \boldsymbol{i}$, find the modulus r and argument θ.

$$\begin{aligned}r &= \sqrt{a^2+b^2}\\ &= \sqrt{1^2+(-1)^2}\\ &= \sqrt{2}\end{aligned} \qquad \begin{aligned}\theta &= \tan^{-1}\frac{b}{a}\\ &= \tan^{-1}\frac{-1}{1}\\ &= \tan^{-1}-1\\ &= -\frac{\pi}{4} \text{ or } \frac{7\pi}{4}\end{aligned}$$

The polar form of $1 - \boldsymbol{i}$ is $\sqrt{2}\left(\cos\frac{7\pi}{4} + \boldsymbol{i}\sin\frac{7\pi}{4}\right)$.

For $4 + 4\boldsymbol{i}$, find the modulus r and argument θ.

$$\begin{aligned}r &= \sqrt{a^2+b^2}\\ &= \sqrt{4^2+4^2}\\ &= 4\sqrt{2}\end{aligned} \qquad \begin{aligned}\theta &= \tan^{-1}\frac{b}{a}\\ &= \tan^{-1}\frac{4}{4}\\ &= \tan^{-1}1 \text{ or } \frac{\pi}{4}\end{aligned}$$

The polar form of $4 + 4\boldsymbol{i}$ is $4\sqrt{2}\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)$.

Use the Product Formula to find the product in polar form.

$$\begin{aligned}&\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)\cdot 4\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\\ &= \sqrt{2}(4\sqrt{2})\left[\cos\left(\frac{7\pi}{4}+\frac{\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}+\frac{\pi}{4}\right)\right]\\ &= 8(\cos 2\pi + i\sin 2\pi)\end{aligned}$$

Now find the rectangular form of the product.

$$\begin{aligned}8(\cos 2\pi + i\sin 2\pi) &= 8(1 + i\,0)\\ &= 8\end{aligned}$$

ANSWER:

$(1 - \boldsymbol{i})(4 + 4\boldsymbol{i}) = 8$;

$\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)\cdot 4\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = 8$

57. $(3 + \boldsymbol{i})(3 - \boldsymbol{i})$

SOLUTION:

First, find each product.

$$\begin{aligned}(3+i)(3-i) &= 9-3i+3i-i^2\\ &= 9-(-1)\\ &= 10\end{aligned}$$

Express each complex number in polar form.

For $3 + \boldsymbol{i}$, find the modulus r and argument θ.

$$\begin{aligned}r &= \sqrt{a^2+b^2}\\ &= \sqrt{3^2+1^2}\\ &= \sqrt{10}\end{aligned} \qquad \begin{aligned}\theta &= \tan^{-1}\frac{b}{a}\\ &= \tan^{-1}\frac{1}{3}\\ &\approx 0.3218\end{aligned}$$

The polar form of $3 + \boldsymbol{i}$ is $\sqrt{10}(\cos 0.3218 + \boldsymbol{i}\sin 0.3218)$.

For $3 - \boldsymbol{i}$, find the modulus r and argument θ.

$$\begin{aligned}r &= \sqrt{a^2+b^2}\\ &= \sqrt{3^2+(-1)^2}\\ &= \sqrt{10}\end{aligned} \qquad \begin{aligned}\theta &= \tan^{-1}\frac{b}{a}\\ &= \tan^{-1}\frac{-1}{3}\\ &\approx -0.3218\end{aligned}$$

The polar form of $3 - \boldsymbol{i}$ is $\sqrt{10}[\cos(-0.3218) + \boldsymbol{i}\sin(-0.3218)]$.

Use the Product Formula to find the product in polar form.

$$\begin{aligned}&\sqrt{10}(\cos 0.3218 + i\sin 0.3218)\cdot\sqrt{10}[\cos(-0.3218) + i\sin(-0.3218)]\\ &= \sqrt{10}(\sqrt{10})[\cos(0.3218-0.3218) + i\sin(0.3218-0.3218)]\\ &= 10(\cos 0 + i\sin 0)\end{aligned}$$

Now find the rectangular form of the product.

$$\begin{aligned}10(\cos 0 + i\sin 0) &= 10(1 + i\,0)\\ &= 10\end{aligned}$$

ANSWER:

$(3 + \boldsymbol{i})(3 - \boldsymbol{i}) = 10$; $\sqrt{10}$ (cos 0.3218 + $\boldsymbol{i}$ sin 0.3218) $\cdot$ $\sqrt{10}$ [cos (–0.3218) + $\boldsymbol{i}$ sin (–0.3218)] = 10

58. $(4 + \boldsymbol{i})(3 - \boldsymbol{i})$

SOLUTION:

First, find each product.

$$\begin{aligned}(3 - \boldsymbol{i})(4 + \boldsymbol{i}) &= 12 + 3\boldsymbol{i} - 4\boldsymbol{i} - \boldsymbol{i}^2 \\ &= 12 - \boldsymbol{i} - (-1) \\ &= 13 - \boldsymbol{i}\end{aligned}$$

Express each complex number in polar form.

For $3 - \boldsymbol{i}$, find the modulus r and argument θ.

$$\begin{aligned}r &= \sqrt{a^2 + b^2} & \theta &= \tan^{-1}\frac{b}{a} \\ &= \sqrt{3^2 + (-1)^2} & &= \tan^{-1}\frac{-1}{3} \\ &= \sqrt{10} & &\approx -0.3218\end{aligned}$$

The polar form of $3 - \boldsymbol{i}$ is

$\sqrt{10}\left[\cos(-0.3218) + \boldsymbol{i}\sin(-0.3218)\right]$.

For $4 + \boldsymbol{i}$, find the modulus r and argument θ.

$$\begin{aligned}r &= \sqrt{a^2 + b^2} & \theta &= \tan^{-1}\frac{b}{a} \\ &= \sqrt{4^2 + 1^2} & &= \tan^{-1}\frac{1}{4} \\ &= \sqrt{17} & &\approx 0.2450\end{aligned}$$

The polar form of $4 + \boldsymbol{i}$ is

$\sqrt{17}\left(\cos 0.2450 + \boldsymbol{i}\sin 0.2450\right)$.

Use the Product Formula to find the product in polar form.

$$\begin{aligned}&\sqrt{10}\left[\cos(-0.3218) + \boldsymbol{i}\sin(-0.3218)\right] \cdot \sqrt{17}\left(\cos 0.2450 + \boldsymbol{i}\sin 0.2450\right) \\ &= \sqrt{10}(\sqrt{17})\left[\cos(-0.3218 + 0.2450) + \boldsymbol{i}\sin(-0.3218 + 0.2450)\right] \\ &= \sqrt{170}\left[\cos(-0.0768) + \boldsymbol{i}\sin(-0.0768)\right]\end{aligned}$$

Now find the rectangular form of the product.

$$\begin{aligned}&\sqrt{170}\left[\cos(-0.0768) + \boldsymbol{i}\sin(-0.0768)\right] \\ &= \sqrt{170}(0.9971 - 0.0767\boldsymbol{i}) \\ &\approx 13 - \boldsymbol{i}\end{aligned}$$

ANSWER:

$(3 - \boldsymbol{i})(4 + \boldsymbol{i}) = 13 - \boldsymbol{i}$; $\sqrt{10}$ [cos (–0.3218) + $\boldsymbol{i}$ sin (–0.3218)] $\cdot$ $\sqrt{17}$ (cos 0.2450 + $\boldsymbol{i}$ sin 0.2450) ≈ 13 – $\boldsymbol{i}$

59. $(-6 + 5\boldsymbol{i})(2 - 3\boldsymbol{i})$

SOLUTION:

First, find each product.

$$\begin{aligned}(-6 + 5\boldsymbol{i})(2 - 3\boldsymbol{i}) &= -12 + 18\boldsymbol{i} + 10\boldsymbol{i} - 15\boldsymbol{i}^2 \\ &= -12 + 28\boldsymbol{i} - 15(-1) \\ &= 3 + 28\boldsymbol{i}\end{aligned}$$

Express each complex number in polar form.

For $-6 + 5\boldsymbol{i}$, find the modulus r and argument θ.

$$\begin{aligned}r &= \sqrt{a^2 + b^2} & \theta &= \tan^{-1}\frac{b}{a} + \pi \\ &= \sqrt{(-6)^2 + (5)^2} & &= \tan^{-1}\frac{5}{-6} + \pi \\ &= \sqrt{61} & &\approx 2.4469\end{aligned}$$

The polar form of $-6 + 5\boldsymbol{i}$ is

$\sqrt{61}(\cos 2.4469 + \boldsymbol{i}\sin 2.4469)$.

For $2 - 3\boldsymbol{i}$, find the modulus r and argument θ.

$$\begin{aligned}r &= \sqrt{a^2 + b^2} & \theta &= \tan^{-1}\frac{b}{a} \\ &= \sqrt{2^2 + (-3)^2} & &= \tan^{-1}\frac{-3}{2} \\ &= \sqrt{13} & &\approx -0.9828\end{aligned}$$

The polar form of $2 - 3\boldsymbol{i}$ is

$\sqrt{13}\left[\cos(-0.9828) + \boldsymbol{i}\sin(-0.9828)\right]$.

Use the Product Formula to find the product in polar form.

$$\begin{aligned}&\sqrt{61}(\cos 2.4469 + \boldsymbol{i}\sin 2.4469) \cdot \sqrt{13}\left[\cos(-0.9828) + \boldsymbol{i}\sin(-0.9828)\right] \\ &= \sqrt{61}(\sqrt{13})\left[\cos(2.4469 - 0.9828) + \boldsymbol{i}\sin(2.4469 - 0.9828)\right] \\ &= \sqrt{793}(\cos 1.4641 + \boldsymbol{i}\sin 1.4641)\end{aligned}$$

Now find the rectangular form of the product.

$$\begin{aligned}&\sqrt{793}(\cos 1.4641 + \boldsymbol{i}\sin 1.4641) \\ &= \sqrt{793}(0.1065 - 0.9943\boldsymbol{i}) \\ &\approx 3 + 28\boldsymbol{i}\end{aligned}$$

ANSWER:

$(-6 + 5\boldsymbol{i})(2 - 3\boldsymbol{i}) = 3 + 28\boldsymbol{i}$;

$\sqrt{61}$ (cos 2.4469 + $\boldsymbol{i}$ sin 2.4469) $\cdot$ $\sqrt{13}$ [cos (–0.9828) + $\boldsymbol{i}$ sin (–0.9828)] ≈ 3 + 28$\boldsymbol{i}$

60. $(\sqrt{2} + 2\boldsymbol{i})(1 + \boldsymbol{i})$

SOLUTION:

First, find each product.

$$\begin{aligned}(\sqrt{2}+2i)(1+i) &= \sqrt{2}+\sqrt{2}i+2i+2i^2\\ &= \sqrt{2}+\sqrt{2}i+2i+2(-1)\\ &= \sqrt{2}+\sqrt{2}i+2i-2\\ &= -0.586+3.414i\end{aligned}$$

Express each complex number in polar form.

For $\sqrt{2}$ + 2***i***, find the modulus *r* and argument θ.

$$r=\sqrt{a^2+b^2}=\sqrt{(\sqrt{2})^2+2^2}=\sqrt{6} \qquad \theta=\tan^{-1}\frac{b}{a}=\tan^{-1}\frac{2}{\sqrt{2}}\approx 0.9553$$

The polar form of $\sqrt{2}$ + 2***i*** is $\sqrt{6}(\cos 0.9553+i\sin 0.9553)$.

For 1 + ***i***, find the modulus *r* and argument θ.

$$r=\sqrt{a^2+b^2}=\sqrt{1^2+1^2}=\sqrt{2} \qquad \theta=\tan^{-1}\frac{b}{a}=\tan^{-1}\frac{1}{1}=\frac{\pi}{4}\text{ or }0.7854$$

The polar form of 1 + ***i*** is $\sqrt{2}(\cos 0.7854+i\sin 0.7854)$.

Use the Product Formula to find the product in polar form.

$$\begin{aligned}&\sqrt{6}(\cos 0.9553+i\sin 0.9553)\cdot\sqrt{2}(\cos 0.7854+i\sin 0.7854)\\ &=\sqrt{6}(\sqrt{2})\left[\cos(0.9553+0.7854)+i\sin(0.9553+0.7854)\right]\\ &=2\sqrt{3}(\cos 1.7407+i\sin 1.7407)\end{aligned}$$

Now find the rectangular form of the product.

$$\begin{aligned}&2\sqrt{3}(\cos 1.7407+i\sin 1.7407)\\ &=2\sqrt{3}(-0.1691+0.9856i)\\ &\approx -0.586+3.414i\end{aligned}$$

ANSWER:

($\sqrt{2}$ + 2***i***)(1 + ***i***) ≈ –0.586 + 3.414***i***; $\sqrt{6}$ (cos 0.9553 + ***i*** sin 0.9553) · $\sqrt{2}$ (cos 0.7854 + ***i*** sin 0.7854) ≈ –0.586 + 3.414***i***

61. (3 – 2***i***)(1 + $\sqrt{3}$ ***i***)

SOLUTION:

First, find each product.

$$\begin{aligned}(3-2i)(1+\sqrt{3}i) &= 3+3\sqrt{3}i-2i-2\sqrt{3}i^2\\ &= 3+3\sqrt{3}i-2i-2\sqrt{3}(-1)\\ &= 3+3\sqrt{3}i-2i+2\sqrt{3}\\ &= 6.464+3.196i\end{aligned}$$

Express each complex number in polar form.

For 3 – 2***i***, find the modulus *r* and argument θ.

$$r=\sqrt{a^2+b^2}=\sqrt{3^2+(-2)^2}=\sqrt{13} \qquad \theta=\tan^{-1}\frac{b}{a}=\tan^{-1}\frac{-2}{3}\approx -0.5880$$

The polar form of 3 – 2***i*** is $\sqrt{13}\left[\cos(-0.5880)+i\sin(-0.5880)\right]$.

For 1 + $\sqrt{3}$ ***i***, find the modulus *r* and argument θ.

$$r=\sqrt{a^2+b^2}=\sqrt{1^2+(\sqrt{3})^2}=2 \qquad \theta=\tan^{-1}\frac{b}{a}=\tan^{-1}\frac{\sqrt{3}}{1}=\frac{\pi}{3}\text{ or }1.0472$$

The polar form of 1 + $\sqrt{3}$ ***i*** is $2(\cos 1.0472+i\sin 1.0472)$.

Use the Product Formula to find the product in polar form.

$$\begin{aligned}&\sqrt{13}\left[\cos(-0.5880)+i\sin(-0.5880)\right]\cdot 2(\cos 1.0472+i\sin 1.0472)\\ &=\sqrt{13}(2)\left[\cos(-0.5880+1.0472)+i\sin(-0.5880+1.0472)\right]\\ &=2\sqrt{13}(\cos 0.4592+i\sin 0.4592)\end{aligned}$$

Now find the rectangular form of the product.

$$\begin{aligned}&2\sqrt{13}(\cos 0.4592+i\sin 0.4592)\\ &=2\sqrt{13}(0.8964+0.4432i)\\ &\approx 6.464+3.196i\end{aligned}$$

ANSWER:

(3 – 2***i***)(1 + $\sqrt{3}$ ***i***) ≈ 6.464 + 3.196***i***; $\sqrt{13}$ [cos (–0.5880) + ***i*** sin (–0.5880)] · 2(cos 1.0472 + ***i*** sin 1.0472) ≈ 6.464 + 3.196***i***

62. **FRACTALS** A *fractal* is a geometric figure that is made up of a pattern that is repeated indefinitely on successively smaller scales, as shown below.
Refer to the image on Page 578.
In this problem, you will generate a fractal through iterations of $f(z) = z^2$. Consider $z_0 = 0.8 + 0.5\boldsymbol{i}$.

a. Calculate $z_1, z_2, z_3, z_4, z_5, z_6$, and z_7 where $z_1 = f(z_0)$, $z_2 = f(z_1)$, and so on.

b. Graph each of the numbers on the complex plane.

c. Predict the location of z_{100}. Explain.

SOLUTION:

a. Calculate z_1.

$$\begin{aligned} z_1 &= f(z_0) \\ &= (0.8 + 0.5\boldsymbol{i})^2 \\ &= 0.64 + 0.8\boldsymbol{i} + 0.25\boldsymbol{i}^2 \\ &= 0.64 + 0.8\boldsymbol{i} + 0.25(-1) \\ &= 0.39 + 0.8\boldsymbol{i} \end{aligned}$$

Use the expression that you found for z_1 to find z_2.

$$\begin{aligned} z_2 &= f(z_1) \\ &= (0.39 + 0.8\boldsymbol{i})^2 \\ &= 0.1521 + 0.624\boldsymbol{i} + 0.64\boldsymbol{i}^2 \\ &= 0.1521 + 0.624\boldsymbol{i} + 0.64(-1) \\ &= -0.4879 + 0.624\boldsymbol{i} \\ &\approx -0.49 + 0.62\boldsymbol{i} \end{aligned}$$

Use the expression that you found for z_2 to find z_3.

$$\begin{aligned} z_3 &= f(z_2) \\ &= (-0.49 + 0.62\boldsymbol{i})^2 \\ &= 0.2401 - 0.6076\boldsymbol{i} + 0.3844\boldsymbol{i}^2 \\ &= 0.2401 - 0.6076\boldsymbol{i} + 0.3844(-1) \\ &= -0.1443 - 0.6076\boldsymbol{i} \\ &\approx -0.14 - 0.61\boldsymbol{i} \end{aligned}$$

Use the expression that you found for z_3 to find z_4.

$$\begin{aligned} z_4 &= f(z_3) \\ &= (-0.14 - 0.61\boldsymbol{i})^2 \\ &= 0.0196 + 0.1708\boldsymbol{i} + 0.3721\boldsymbol{i}^2 \\ &= 0.0196 + 0.1708\boldsymbol{i} + 0.3721(-1) \\ &= -0.3525 + 0.1708\boldsymbol{i} \\ &\approx -0.35 + 0.17\boldsymbol{i} \end{aligned}$$

Use the expression that you found for z_4 to find z_5.

$$\begin{aligned} z_5 &= f(z_4) \\ &= (-0.35 + 0.17\boldsymbol{i})^2 \\ &= 0.1225 - 0.119\boldsymbol{i} + 0.0289\boldsymbol{i}^2 \\ &= 0.1225 - 0.119\boldsymbol{i} + 0.0289(-1) \\ &= 0.0936 - 0.119\boldsymbol{i} \\ &\approx 0.09 - 0.12\boldsymbol{i} \end{aligned}$$

Use the expression that you found for z_5 to find z_6.

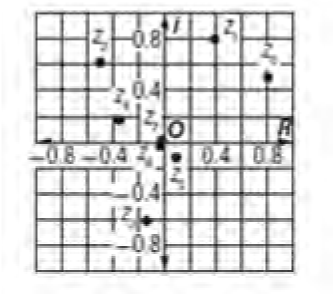
$$\begin{aligned} z_6 &= f(z_5) \\ &= (0.09 - 0.12\boldsymbol{i})^2 \\ &= 0.0081 - 0.0216\boldsymbol{i} + 0.0144\boldsymbol{i}^2 \\ &= 0.0081 - 0.0216\boldsymbol{i} + 0.0144(-1) \\ &= -0.0063 - 0.0216\boldsymbol{i} \\ &\approx -0.006 - 0.022\boldsymbol{i} \end{aligned}$$

Use the expression that you found for z_6 to find z_7.

$$\begin{aligned} z_7 &= f(z_6) \\ &= (-0.006 - 0.022\boldsymbol{i})^2 \\ &= 0.000036 + 0.000264\boldsymbol{i} + 0.000484\boldsymbol{i}^2 \\ &= 0.000036 + 0.000264\boldsymbol{i} + 0.000484(-1) \\ &= -0.000448 + 0.000264\boldsymbol{i} \\ &\approx -0.0004 + 0.0003\boldsymbol{i} \end{aligned}$$

b. For $z_1 = 0.39 + 0.8\boldsymbol{i}$, $(a, b) = (0.39, 0.8)$.
For $z_2 \approx -0.49 + 0.62\boldsymbol{i}$, $(a, b) = (-0.49, 0.62)$.
For $z_3 \approx -0.14 - 0.61\boldsymbol{i}$, $(a, b) = (-0.14, -0.61)$.
For $z_4 \approx -0.35 + 0.17\boldsymbol{i}$, $(a, b) = (-0.35, 0.17)$.
For $z_5 \approx 0.09 - 0.12\boldsymbol{i}$, $(a, b) = (0.09, -0.12)$.
For $z_6 \approx -0.006 - 0.022\boldsymbol{i}$, $(a, b) = (-0.006, -0.022)$.
For $z_7 \approx -0.0004 + 0.0003\boldsymbol{i}$, $(a, b) = (-0.0004, 0.0003)$.

Graph the points in the complex plane.

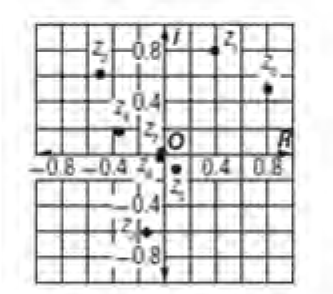


c. As more iterates are calculated and graphed, the iterates approach the origin. Sample answer: z_{100} will be located very close to the origin. With each iteration of $f(z) = z^2$, the iterates approach the origin.

ANSWER:

a. $z_1 = 0.39 + 0.8\boldsymbol{i}$; $z_2 \approx -0.49 + 0.62\boldsymbol{i}$; $z_3 \approx -0.14 - 0.61\boldsymbol{i}$; $z_4 \approx -0.35 + 0.17\boldsymbol{i}$, $z_5 \approx 0.09 - 0.12\boldsymbol{i}$, $z_6 \approx -0.006 - 0.022\boldsymbol{i}$, $z_7 \approx -0.0004 + 0.0003\boldsymbol{i}$

b.



c. Sample answer: z_{100} will be located very close to the origin. With each iteration of $f(z) = z^2$, the iterates approach the origin.

63. **TRANSFORMATIONS** There are certain operations with complex numbers that correspond to geometric transformations in the complex plane. Describe the transformation applied to point z to obtain point w in the complex plane for each of the following operations.

a. $w = z + (3 - 4\boldsymbol{i})$

b. w is the complex conjugate of z.

c. $w = \boldsymbol{i} \cdot z$

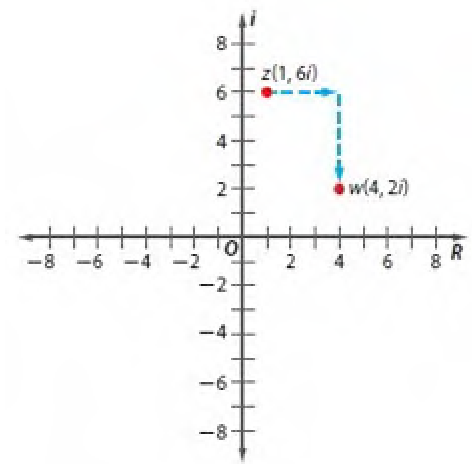
d. $w = 0.25z$

SOLUTION:

a. Let $z = a + b\boldsymbol{i}$. For z, $z = (a, b)$. For the transformation t represented by $3 - 4\boldsymbol{i}$, $t = (3, -4)$.

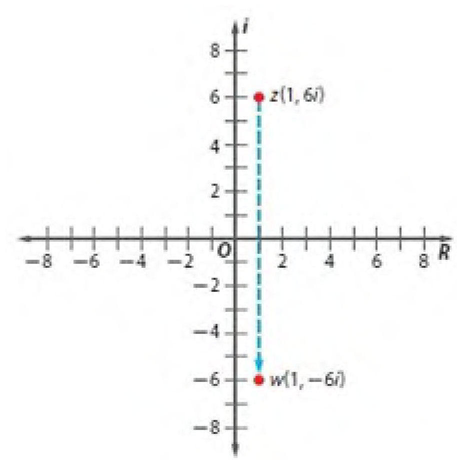
So, $w = (a, b) + (3, -4)$ or $(a + 3, b - 4)$. Thus, the transformation applied to point z to obtain point w is a translation 3 units to the right and 4 units down.

Let $z = (1, 6\boldsymbol{i})$.
$w = (4, 2\boldsymbol{i})$



b. Let $z = a + b\boldsymbol{i}$. For z, $z = (a, b)$. Let w be the complex conjugate of z. So, $w = a - b\boldsymbol{i}$. Thus, $w = (a, -b)$. Notice that the real component stayed the same, but the imaginary component changed signs. Therefore, the transformation applied to point z to obtain point w is a reflection in the real axis.

Let $z = (1, 6\boldsymbol{i})$.
$w = (1, -6\boldsymbol{i})$

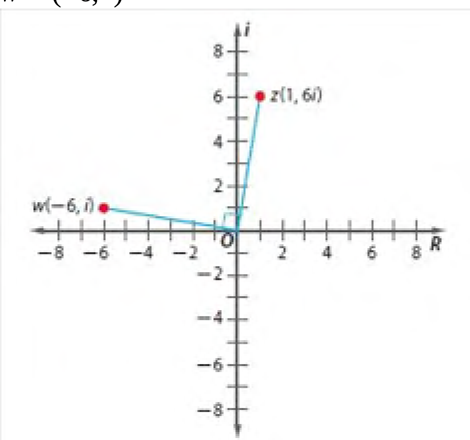


c. Let $z = a + b\boldsymbol{i}$. For z, $z = (a, b)$. Use substitution to find w.

$$\begin{aligned} w &= \boldsymbol{i} \cdot z \\ &= \boldsymbol{i} \cdot (a + b\boldsymbol{i}) \\ &= a\boldsymbol{i} + b\boldsymbol{i}^2 \\ &= a\boldsymbol{i} + b(-1) \\ &= -b + a\boldsymbol{i} \end{aligned}$$

So, $w = -b + a\boldsymbol{i}$. Thus, $w = (-b, a)$. Therefore, the transformation applied to point z to obtain point w is a rotation of 90° counterclockwise about the origin.

Let $z = (1, 6\boldsymbol{i})$.
$w = (-6, \boldsymbol{i})$

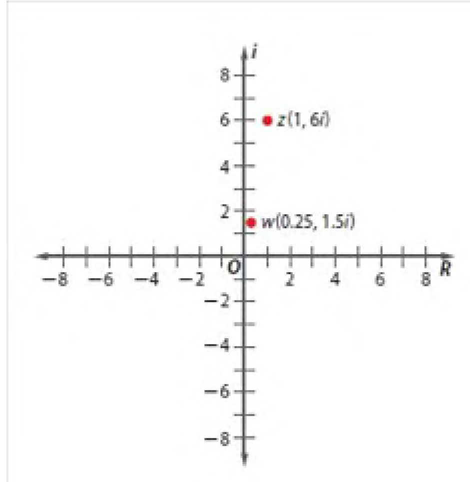


d. Let $z = a + b\boldsymbol{i}$. For z, $z = (a, b)$. Use substitution to find w.

$$\begin{aligned} w &= 0.25z \\ &= 0.25(a + b\boldsymbol{i}) \\ &= 0.25a + 0.25b\boldsymbol{i} \end{aligned}$$

So, $w = 0.25a + 0.25b\boldsymbol{i}$. Thus, $w = (0.25a, 0.25b)$. Therefore, the transformation applied to point z to obtain point w is a dilation by a factor of 0.25.

Let $z = (1, 6\boldsymbol{i})$.
$w = (0.25, 1.5\boldsymbol{i})$



ANSWER:

a. a translation 3 units to the right and 4 units down
b. a reflection in the real axis
c. a rotation of 90º counterclockwise about the origin
d. a dilation by a factor of 0.25

Find z and the p th roots of z given each of the following.

64. $p = 3$, one cube root is $\frac{5}{2} - \frac{5\sqrt{3}}{2}\boldsymbol{i}$

SOLUTION:

Cube $\frac{5}{2} - \frac{5\sqrt{3}}{2}\boldsymbol{i}$ to find z.

$$\begin{aligned} &\left(\frac{5}{2} - \frac{5\sqrt{3}}{2}i\right)^3 \\ &= \left(\frac{5}{2}\right)^3 + 3\left(\frac{5}{2}\right)^2\left(-\frac{5\sqrt{3}}{2}i\right) + 3\left(\frac{5}{2}\right)\left(-\frac{5\sqrt{3}}{2}i\right)^2 + \left(-\frac{5\sqrt{3}}{2}i\right)^3 \\ &= \frac{125}{8} - \frac{375\sqrt{3}}{8}i + \frac{1125}{8}i^2 - \frac{125\sqrt{27}}{2}i^3 \\ &= \frac{125}{8} - \frac{375\sqrt{3}}{8}i + \frac{1125}{8}(-1) - \frac{125\sqrt{27}}{8}i^2 \cdot i \\ &= \frac{125}{8} - \frac{375\sqrt{3}}{8}i - \frac{1125}{8} - \frac{375\sqrt{3}}{8}(-1) \cdot i \\ &= \frac{125}{8} - \frac{375\sqrt{3}}{8}i - \frac{1125}{8} + \frac{375\sqrt{3}}{8}i \\ &= -125 \end{aligned}$$

$z = -125$. To find the cube roots of -125, write -125 in polar form.

$$r = \sqrt{a^2 + b^2} = \sqrt{(-125)^2 + 0^2} = 125$$

$$\theta = \tan^{-1}\frac{b}{a} + \pi = \tan^{-1}\frac{0}{-125} + \pi = \tan^{-1} 0 + \pi = \pi$$

The polar form of −125 is 125(cos π + $\boldsymbol{i}$ sin π). Now write an expression for the cube roots.

$$125^{\frac{1}{3}}\left(\cos\frac{\pi + 2n\pi}{3} + \boldsymbol{i}\sin\frac{\pi + 2n\pi}{3}\right)$$
$$= 5\left(\cos\frac{\pi}{3} + \frac{2n\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3} + \frac{2n\pi}{3}\right)$$

Let n = 0, 1, and 2 successively to find the cube roots.

Let n = 0.

$$5\left(\cos\frac{\pi}{3} + \frac{2(0)\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3} + \frac{2(0)\pi}{3}\right) = 5\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right) = 5\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\boldsymbol{i}\right) = \frac{5}{2} + \frac{5\sqrt{3}}{2}\boldsymbol{i}$$

Let n = 1.

$$5\left(\cos\frac{\pi}{3} + \frac{2(1)\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3} + \frac{2(1)\pi}{3}\right) = 5(\cos\pi + \boldsymbol{i}\sin\pi) = 5(-1 + 0\boldsymbol{i}) = -5$$

Let n = 2.

$$5\left(\cos\frac{\pi}{3} + \frac{2(2)\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3} + \frac{2(2)\pi}{3}\right) = 5\left(\cos\frac{5\pi}{3} + \boldsymbol{i}\sin\frac{5\pi}{3}\right) = 5\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\boldsymbol{i}\right) = \frac{5}{2} - \frac{5\sqrt{3}}{2}\boldsymbol{i}$$

z = −125 and the cube roots of −125 are $\frac{5}{2} + \frac{5\sqrt{3}}{2}\boldsymbol{i}$, −5, and $\frac{5}{2} - \frac{5\sqrt{3}}{2}\boldsymbol{i}$.

ANSWER:

$z = -125$; $\frac{5}{2} + \frac{5\sqrt{3}}{2}\boldsymbol{i}$, −5, $\frac{5}{2} - \frac{5\sqrt{3}}{2}\boldsymbol{i}$

65. p = 4, one fourth root is –1 – $\boldsymbol{i}$

SOLUTION:

First, write −1 − $\boldsymbol{i}$ in polar form.

$$r = \sqrt{a^2 + b^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\frac{b}{a} + \pi = \tan^{-1}\frac{-1}{-1} + \pi = \tan^{-1} 1 + \pi = \frac{5\pi}{4}$$

The polar form of −1 − $\boldsymbol{i}$ is $\sqrt{2}\left(\cos\frac{5\pi}{4} + \boldsymbol{i}\sin\frac{5\pi}{4}\right)$.

Use De Moivre’s Theorem to find z.

$$\left[\sqrt{2}\left(\cos\frac{5\pi}{4} + \boldsymbol{i}\sin\frac{5\pi}{4}\right)\right]^4 = (\sqrt{2})^4\left[\cos 4\left(\frac{5\pi}{4}\right) + \boldsymbol{i}\sin 4\left(\frac{5\pi}{4}\right)\right] = 4(\cos 5\pi + \boldsymbol{i}\sin 5\pi) = 4(-1 + 0\boldsymbol{i}) = -4$$

To find the fourth roots of −4, write −4 in polar form.

$$r = \sqrt{a^2 + b^2} = \sqrt{(-4)^2 + 0^2} = 4$$

$$\theta = \tan^{-1}\frac{b}{a} + \pi = \tan^{-1}\frac{0}{-4} + \pi = \tan^{-1} 0 + \pi = \pi$$

The polar form of −4 is 4(cos π + $\boldsymbol{i}$ sin π). Now write an expression for the cube roots.

$$4^{\frac{1}{4}}\left(\cos\frac{\pi+2n\pi}{4}+i\sin\frac{\pi+2n\pi}{4}\right)$$
$$=\sqrt{2}\left(\cos\frac{\pi}{4}+\frac{n\pi}{2}+i\sin\frac{\pi}{4}+\frac{n\pi}{2}\right)$$

Let n = 0, 1, 2, and 3 successively to find the cube roots.

Let n = 0.

$$\sqrt{2}\left(\cos\frac{\pi}{4}+\frac{(0)\pi}{2}+i\sin\frac{\pi}{4}+\frac{(0)\pi}{2}\right)$$
$$=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$
$$=\sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$$
$$=1+i$$

Let n = 1.

$$\sqrt{2}\left(\cos\frac{\pi}{4}+\frac{(1)\pi}{2}+i\sin\frac{\pi}{4}+\frac{(1)\pi}{2}\right)$$
$$=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$
$$=\sqrt{2}\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$$
$$=-1+i$$

Let n = 2.

$$\sqrt{2}\left(\cos\frac{\pi}{4}+\frac{(2)\pi}{2}+i\sin\frac{\pi}{4}+\frac{(2)\pi}{2}\right)$$
$$=\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$$
$$=\sqrt{2}\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)$$
$$=-1-i$$

Let n = 3.

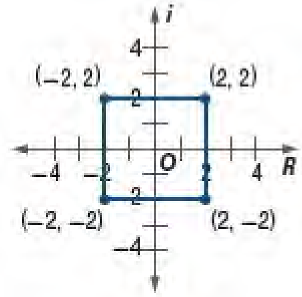
$$\sqrt{2}\left(\cos\frac{\pi}{4}+\frac{(3)\pi}{2}+i\sin\frac{\pi}{4}+\frac{(3)\pi}{2}\right)$$
$$=\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$$
$$=\sqrt{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)$$
$$=1-i$$

$z = -4$ and the fourth roots of -4 are $1 + \boldsymbol{i}$, $-1 + \boldsymbol{i}$, $-1 - \boldsymbol{i}$, and $1 - \boldsymbol{i}$.

ANSWER:

-4; $1 + \boldsymbol{i}$, $-1 + \boldsymbol{i}$, $-1 - \boldsymbol{i}$, $1 - \boldsymbol{i}$

66. **GRAPHICS** By representing each vertex by a complex number in polar form, a programmer dilates and then rotates the square below 45° counterclockwise so that the new vertices lie at the midpoints of the sides of the original square.



a. By what complex number should the programmer multiply each number to produce this transformation?
b. What happens if the numbers representing the original vertices are multiplied by the square of your answer to part **a**?

SOLUTION:

a. When the complex number representing the vertex (2, 2) is multiplied by the complex number z, the product will be located at the point (2, 0) because (2, 0) is the midpoint of the side of the original square located 45° counterclockwise of (2, 2).
Write the vertex (2, 2) in polar form.

$$r=\sqrt{a^2+b^2} \qquad \theta=\tan^{-1}\frac{b}{a}$$
$$=\sqrt{2^2+2^2} \qquad =\tan^{-1}\frac{2}{2}$$
$$=2\sqrt{2} \qquad =\frac{\pi}{4}$$

The polar form of (2, 2) is $2\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$.

Write (2, 0) in polar form.

$$r = \sqrt{a^2 + b^2} = \sqrt{2^2 + 0^2} = 2$$

$$\theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{0}{2} = \frac{\pi}{2}$$

The polar form of (2, 0) is $2\left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right)$. Use substitution to solve for z.

$$\begin{aligned}2\sqrt{2}\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right) \cdot z &= 2\left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right)\\ z &= 2\left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right) \div 2\sqrt{2}\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)\\ z &= \left(\frac{2}{2\sqrt{2}}\right)\left[\cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + \boldsymbol{i}\sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right]\\ z &= \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)\end{aligned}$$

To express z in rectangular form, evaluate the trigonometric values and simplify.

$$\begin{aligned}z &= \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)\\ &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\boldsymbol{i}\right)\\ &= \frac{1}{2} + \frac{1}{2}\boldsymbol{i}\end{aligned}$$

The programmer should multiply by $\frac{1}{2} + \frac{1}{2}\boldsymbol{i}$.

b. Square the answer found in part **a**.

$$\begin{aligned}\left(\frac{1}{2} + \frac{1}{2}\boldsymbol{i}\right)^2 &= \frac{1}{4} + \frac{1}{2}\boldsymbol{i} + \frac{1}{4}\boldsymbol{i}^2\\ &= \frac{1}{4} + \frac{1}{2}\boldsymbol{i} + \frac{1}{4}(-1)\\ &= \frac{1}{2}\boldsymbol{i}\end{aligned}$$

The vertex (2, 2) can be written as 2 + 2***i***. Multiply this vertex by $\frac{1}{2}\boldsymbol{i}$.

$$\begin{aligned}\frac{1}{2}\boldsymbol{i}(2 + 2\boldsymbol{i}) &= 1\boldsymbol{i} + \boldsymbol{i}^2\\ &= \boldsymbol{i} + (-1)\\ &= -1 + \boldsymbol{i}\end{aligned}$$

For −1 + ***i***, $(a, b) = (-1, 1)$.

The vertex (−2, 2) can be written as −2 + 2***i***. Multiply this vertex by $\frac{1}{2}\boldsymbol{i}$.

$$\begin{aligned}\frac{1}{2}\boldsymbol{i}(-2 + 2\boldsymbol{i}) &= -1\boldsymbol{i} + \boldsymbol{i}^2\\ &= -\boldsymbol{i} + (-1)\\ &= -1 - \boldsymbol{i}\end{aligned}$$

For −1 − ***i***, $(a, b) = (-1, -1)$.

The vertex (−2, −2) can be written as −2 − 2***i***. Multiply this vertex by $\frac{1}{2}\boldsymbol{i}$.

$$\begin{aligned}\frac{1}{2}\boldsymbol{i}(-2 - 2\boldsymbol{i}) &= -1\boldsymbol{i} - \boldsymbol{i}^2\\ &= -\boldsymbol{i} - (-1)\\ &= 1 - \boldsymbol{i}\end{aligned}$$

For 1 − ***i***, $(a, b) = (1, -1)$.

The vertex (2, −2) can be written as 2 − 2***i***. Multiply this vertex by $\frac{1}{2}\boldsymbol{i}$.

$$\begin{aligned}\frac{1}{2}\boldsymbol{i}(2 - 2\boldsymbol{i}) &= 1\boldsymbol{i} - \boldsymbol{i}^2\\ &= \boldsymbol{i} - (-1)\\ &= 1 + \boldsymbol{i}\end{aligned}$$

For 1 + ***i***, $(a, b) = (1, 1)$.

The vertices of the square are being rotated 90° counterclockwise and are dilated by a factor of $\frac{1}{2}$.

ANSWER:

a. $\frac{1}{2} + \frac{1}{2}\boldsymbol{i}$

b. The square is rotated 90° counterclockwise and dilated by a factor of $\frac{1}{2}$.

Use the Distinct Roots Formula to find all of the solutions of each equation. Express the solutions in rectangular form.

67. $x^3 = \boldsymbol{i}$

SOLUTION:

Solve for x.

$$x^3 = \boldsymbol{i}$$
$$x = \sqrt[3]{\boldsymbol{i}}$$

Find the cube roots of $\boldsymbol{i}$. First, write $\boldsymbol{i}$ in polar form.

$$r = \sqrt{a^2 + b^2} = \sqrt{0^2 + 1^2} = 1 \qquad \theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{1}{0} = \frac{\pi}{2}$$

The polar form of $\boldsymbol{i}$ is $\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}$. Now write an expression for the cube roots.

$$1\left(\cos\frac{\frac{\pi}{2} + 2n\pi}{3} + \boldsymbol{i}\sin\frac{\frac{\pi}{2} + 2n\pi}{3}\right)$$
$$= \cos\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right) + \boldsymbol{i}\sin\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right)$$

Let n = 0, 1, and 2 successively to find the cube roots.

Let n = 0.

$$\cos\left(\frac{\pi}{6} + \frac{2(0)\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2(0)\pi}{3}\right)$$
$$= \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Let n = 1.

$$\cos\left(\frac{\pi}{6} + \frac{2(1)\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2(1)\pi}{3}\right)$$
$$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$$
$$= -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

Let n = 2.

$$\cos\left(\frac{\pi}{6} + \frac{2(2)\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2(2)\pi}{3}\right)$$
$$= \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$$
$$= -i$$

The cube roots of $\boldsymbol{i}$ are $\frac{\sqrt{3}}{2} + \frac{1}{2}\boldsymbol{i}$, $-\frac{\sqrt{3}}{2} + \frac{1}{2}\boldsymbol{i}$, and $\boldsymbol{i}$. Thus, the solutions to the equation are $\frac{\sqrt{3}}{2} + \frac{1}{2}\boldsymbol{i}$, $-\frac{\sqrt{3}}{2} + \frac{1}{2}\boldsymbol{i}$, and $\boldsymbol{i}$.

ANSWER:

$\frac{\sqrt{3}}{2} + \frac{1}{2}\boldsymbol{i}$, $-\frac{\sqrt{3}}{2} + \frac{1}{2}\boldsymbol{i}$, $\boldsymbol{i}$

68. $x^3 + 3 = 128$

SOLUTION:

Solve for x.

$$x^3 + 3 = 128$$
$$x^3 = 125$$
$$x = \sqrt[3]{125}$$

Find the cube roots of 125. First, write 125 in polar form.

$$r = \sqrt{a^2 + b^2} = \sqrt{125^2 + 0^2} = 125 \qquad \theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{0}{125} = 0$$

The polar form of 125 is 125 cos 0 + $\boldsymbol{i}$ sin 0. Now write an expression for the cube roots.

$$125^{\frac{1}{3}}\left(\cos\frac{0 + 2n\pi}{3} + \boldsymbol{i}\sin\frac{0 + 2n\pi}{3}\right)$$
$$= 5\left(\cos\frac{2n\pi}{3} + \boldsymbol{i}\sin\frac{2n\pi}{3}\right)$$

Let n = 0, 1, and 2 successively to find the cube roots.

Let n = 0.

$$5\left(\cos\frac{2(0)\pi}{3} + i\sin\frac{2(0)\pi}{3}\right)$$
$$= 5(\cos 0 + i\sin 0)$$
$$= 5$$

Let n = 1.

$$5\left(\cos\frac{2(1)\pi}{3}+i\sin\frac{2(1)\pi}{3}\right)$$
$$=5\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$
$$=5\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$
$$=-\frac{5}{2}+\frac{5\sqrt{3}}{2}i$$

Let $n = 2$.

$$5\left(\cos\frac{2(2)\pi}{3}+i\sin\frac{2(2)\pi}{3}\right)$$
$$=5\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$$
$$=5\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$$
$$=-\frac{5}{2}-\frac{5\sqrt{3}}{2}i$$

The cube roots of 125 are 5, $-\frac{5}{2}+\frac{5\sqrt{3}}{2}i$, and $-\frac{5}{2}-\frac{5\sqrt{3}}{2}i$. Thus, the solutions to the equation are 5, $-\frac{5}{2}+\frac{5\sqrt{3}}{2}i$, and $-\frac{5}{2}-\frac{5\sqrt{3}}{2}i$.

ANSWER:

5, $-\frac{5}{2}+\frac{5\sqrt{3}}{2}i$, $-\frac{5}{2}-\frac{5\sqrt{3}}{2}i$

69. $x^4 = 81\boldsymbol{i}$

SOLUTION:

Solve for x.

$$x^4 = 81i$$
$$x = \sqrt[4]{81i}$$

Find the fourth roots of 81$\boldsymbol{i}$. First, write 81$\boldsymbol{i}$ in polar form.

$$r=\sqrt{a^2+b^2} \qquad \theta=\tan^{-1}\frac{b}{a}$$
$$=\sqrt{0^2+81^2} \qquad =\tan^{-1}\frac{81}{0}$$
$$=81 \qquad =\frac{\pi}{2}$$

The polar form of 81$\boldsymbol{i}$ is $81\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$. Now write an expression for the fourth roots.

$$81^{\frac{1}{4}}\left(\cos\frac{\frac{\pi}{2}+2n\pi}{4}+i\sin\frac{\frac{\pi}{2}+2n\pi}{4}\right)$$
$$=3\left[\cos\left(\frac{\pi}{8}+\frac{n\pi}{2}\right)+i\sin\left(\frac{\pi}{8}+\frac{n\pi}{2}\right)\right]$$

Let n = 0, 1, 2, and 3 successively to find the fourth roots.

Let $n = 0$.

$$3\left[\cos\left(\frac{\pi}{8}+\frac{(0)\pi}{2}\right)+i\sin\left(\frac{\pi}{8}+\frac{(0)\pi}{2}\right)\right]$$
$$=3\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)$$
$$=2.77+1.15i$$

Let $n = 1$.

$$3\left[\cos\left(\frac{\pi}{8}+\frac{(1)\pi}{2}\right)+i\sin\left(\frac{\pi}{8}+\frac{(1)\pi}{2}\right)\right]$$
$$=3\left(\cos\frac{5\pi}{8}+i\sin\frac{5\pi}{8}\right)$$
$$=-1.15+2.77i$$

Let $n = 2$.

$$3\left[\cos\left(\frac{\pi}{8}+\frac{(2)\pi}{2}\right)+i\sin\left(\frac{\pi}{8}+\frac{(2)\pi}{2}\right)\right]$$
$$=3\left(\cos\frac{9\pi}{8}+i\sin\frac{9\pi}{8}\right)$$
$$=-2.77-1.15i$$

Let $n = 3$.

$$3\left[\cos\left(\frac{\pi}{8}+\frac{(3)\pi}{2}\right)+i\sin\left(\frac{\pi}{8}+\frac{(3)\pi}{2}\right)\right]$$
$$=3\left(\cos\frac{13\pi}{8}+i\sin\frac{13\pi}{8}\right)$$
$$=1.15-2.77i$$

Thus, the solutions to the equation are 2.77 + 1.15$\boldsymbol{i}$, −1.15 + 2.77$\boldsymbol{i}$, −2.77 − 1.15$\boldsymbol{i}$, and 1.15 − 2.77$\boldsymbol{i}$.

ANSWER:

2.77 + 1.15$\boldsymbol{i}$, −1.15 + 2.77$\boldsymbol{i}$, −2.77 − 1.15$\boldsymbol{i}$, 1.15 − 2.77$\boldsymbol{i}$

70. $x^5 - 1 = 1023$

SOLUTION:

Solve for x.

$$\begin{aligned} x^5 - 1 &= 1023 \\ x^5 &= 1024 \\ x &= \sqrt[5]{1024} \end{aligned}$$

Find the fifth roots of 1024. First, write 1024 in polar form.

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{1024^2 + 0^2} \\ &= 1024 \end{aligned} \qquad \begin{aligned} \theta &= \tan^{-1}\frac{b}{a} \\ &= \tan^{-1}\frac{0}{1024} \\ &= 0 \end{aligned}$$

The polar form of 1024 is 1024(cos 0 + $\boldsymbol{i}$ sin 0). Now write an expression for the fourth roots.

$$\begin{aligned} &1024^{\frac{1}{5}}\left(\cos\frac{0 + 2n\pi}{5} + \boldsymbol{i}\sin\frac{0 + 2n\pi}{5}\right) \\ &= 4\left(\cos\frac{2n\pi}{5} + \boldsymbol{i}\sin\frac{2n\pi}{5}\right) \end{aligned}$$

Let n = 0, 1, 2, 3, and 4 successively to find the fifth roots.

Let n = 0.

$$\begin{aligned} &4\left(\cos\frac{2(0)\pi}{5} + i\sin\frac{2(0)\pi}{5}\right) \\ &= 4(\cos 0 + i\sin 0) \\ &= 4 \end{aligned}$$

Let n = 1.

$$\begin{aligned} &4\left(\cos\frac{2(1)\pi}{5} + i\sin\frac{2(1)\pi}{5}\right) \\ &= 4\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right) \\ &= 1.24 + 3.80i \end{aligned}$$

Let n = 2.

$$\begin{aligned} &4\left(\cos\frac{2(2)\pi}{5} + i\sin\frac{2(2)\pi}{5}\right) \\ &= 4\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right) \\ &= -3.24 + 2.35i \end{aligned}$$

Let n = 3.

$$\begin{aligned} &4\left(\cos\frac{2(3)\pi}{5} + i\sin\frac{2(3)\pi}{5}\right) \\ &= 4\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right) \\ &= -3.24 - 2.35i \end{aligned}$$

Let n = 4.

$$\begin{aligned} &4\left(\cos\frac{2(4)\pi}{5} + i\sin\frac{2(4)\pi}{5}\right) \\ &= 4\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right) \\ &= 1.24 - 3.80i \end{aligned}$$

Thus, the solutions to the equation are 4, 1.24 + 3.80$\boldsymbol{i}$, −3.24 + 2.35$\boldsymbol{i}$, −3.24 − 2.35$\boldsymbol{i}$, and 1.24 − 3.80$\boldsymbol{i}$.

ANSWER:

4, 1.24 + 3.80$\boldsymbol{i}$, −3.24 + 2.35$\boldsymbol{i}$, −3.24 − 2.35$\boldsymbol{i}$, 1.24 − 3.80$\boldsymbol{i}$

71. $x^3 + 1 = \boldsymbol{i}$

SOLUTION:

Solve for x.

$$\begin{aligned} x^3 + 1 &= \boldsymbol{i} \\ x^3 &= \boldsymbol{i} - 1 \\ x &= \sqrt[3]{\boldsymbol{i} - 1} \end{aligned}$$

Find the cube roots of −1 + $\boldsymbol{i}$. First, write −1 + $\boldsymbol{i}$ in polar form.

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{(-1)^2 + 1^2} \\ &= \sqrt{2} \end{aligned} \qquad \begin{aligned} \theta &= \tan^{-1}\frac{b}{a} + \pi \\ &= \tan^{-1}\frac{1}{-1} + \pi \\ &= \frac{3\pi}{4} \end{aligned}$$

The polar form of −1 + $\boldsymbol{i}$ is $\sqrt{2}\left(\cos\frac{3\pi}{4} + \boldsymbol{i}\sin\frac{3\pi}{4}\right)$.

Now write an expression for the cube roots.

$$\begin{aligned} &(\sqrt{2})^{\frac{1}{3}}\left(\cos\frac{\frac{3\pi}{4} + 2n\pi}{3} + \boldsymbol{i}\sin\frac{\frac{3\pi}{4} + 2n\pi}{3}\right) \\ &= \sqrt[6]{2}\left[\cos\left(\frac{\pi}{4} + \frac{2n\pi}{3}\right) + \boldsymbol{i}\sin\left(\frac{\pi}{4} + \frac{2n\pi}{3}\right)\right] \end{aligned}$$

Let n = 0, 1, and 2 successively to find the cube roots.
Let n = 0.

$$\sqrt[6]{2}\left[\cos\left(\frac{\pi}{4}+\frac{2(0)\pi}{3}\right)+\boldsymbol{i}\sin\left(\frac{\pi}{4}+\frac{2(0)\pi}{3}\right)\right]$$
$$=\sqrt[6]{2}\left(\cos\frac{\pi}{4}+\boldsymbol{i}\sin\frac{\pi}{4}\right)\text{ or }0.79+0.79\boldsymbol{i}$$

Let n = 1.

$$\sqrt[6]{2}\left[\cos\left(\frac{\pi}{4}+\frac{2(1)\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{2(1)\pi}{3}\right)\right]$$
$$=\sqrt[6]{2}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$$
$$=-1.08+0.29i$$

Let n = 2.

$$\sqrt[6]{2}\left[\cos\left(\frac{\pi}{4}+\frac{2(2)\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{2(2)\pi}{3}\right)\right]$$
$$=\sqrt[6]{2}\left(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right)$$
$$=0.29-1.08i$$

Thus, the solutions to the equation are 0.79 + 0.79$\boldsymbol{i}$, −1.08 + 0.29$\boldsymbol{i}$, and 0.29 − 1.08$\boldsymbol{i}$.

ANSWER:

0.79 + 0.79$\boldsymbol{i}$, −1.08 + 0.29$\boldsymbol{i}$, 0.29 − 1.08$\boldsymbol{i}$

72. $x^4 - 2 + \boldsymbol{i} = -1$

SOLUTION:

Solve for x.

$$x^4-2+\boldsymbol{i}=-1$$
$$x^4=1-\boldsymbol{i}$$
$$x=\sqrt[4]{1-\boldsymbol{i}}$$

Find the fourth roots of 1 − $\boldsymbol{i}$. First, write 1 − $\boldsymbol{i}$ in polar form.

$$r=\sqrt{a^2+b^2}=\sqrt{1^2+(-1)^2}=\sqrt{2}$$
$$\theta=\tan^{-1}\frac{b}{a}=\tan^{-1}\frac{-1}{1}=-\frac{\pi}{4}\text{ or }\frac{7\pi}{4}$$

The polar form of 1 − $\boldsymbol{i}$ is $\sqrt{2}\left(\cos\frac{7\pi}{4}+\boldsymbol{i}\sin\frac{7\pi}{4}\right)$.

Now write an expression for the fourth roots.

$$(\sqrt{2})^{\frac{1}{4}}\left(\cos\frac{\frac{7\pi}{4}+2n\pi}{4}+\boldsymbol{i}\sin\frac{\frac{7\pi}{4}+2n\pi}{4}\right)$$
$$=\sqrt[8]{2}\left[\cos\left(\frac{7\pi}{16}+\frac{n\pi}{2}\right)+\boldsymbol{i}\sin\left(\frac{7\pi}{16}+\frac{n\pi}{2}\right)\right]$$

Let n = 0, 1, 2, and 3 successively to find the fourth roots.
Let n = 0.

$$\sqrt[8]{2}\left[\cos\left(\frac{7\pi}{16}+\frac{(0)\pi}{2}\right)+i\sin\left(\frac{7\pi}{16}+\frac{(0)\pi}{2}\right)\right]$$
$$=\sqrt[8]{2}\left(\cos\frac{7\pi}{16}+i\sin\frac{7\pi}{16}\right)$$
$$=0.21+1.07i$$

Let n = 1.

$$\sqrt[8]{2}\left[\cos\left(\frac{7\pi}{16}+\frac{(1)\pi}{2}\right)+i\sin\left(\frac{7\pi}{16}+\frac{(1)\pi}{2}\right)\right]$$
$$=\sqrt[8]{2}\left(\cos\frac{15\pi}{16}+i\sin\frac{15\pi}{16}\right)$$
$$=-1.07+0.21i$$

Let n = 2.

$$\sqrt[8]{2}\left[\cos\left(\frac{7\pi}{16}+\frac{(2)\pi}{2}\right)+i\sin\left(\frac{7\pi}{16}+\frac{(2)\pi}{2}\right)\right]$$
$$=\sqrt[8]{2}\left(\cos\frac{23\pi}{16}+i\sin\frac{23\pi}{16}\right)$$
$$=-0.21-1.07i$$

Let n = 3.

$$\sqrt[8]{2}\left[\cos\left(\frac{7\pi}{16}+\frac{(3)\pi}{2}\right)+i\sin\left(\frac{7\pi}{16}+\frac{(3)\pi}{2}\right)\right]$$
$$=\sqrt[8]{2}\left(\cos\frac{31\pi}{16}+i\sin\frac{31\pi}{16}\right)$$
$$=1.07-0.21i$$

Thus, the solutions to the equation are 0.21 + 1.07$\boldsymbol{i}$, −1.07 + 0.21$\boldsymbol{i}$, −0.21 − 1.07$\boldsymbol{i}$, and 1.07 −0.21$\boldsymbol{i}$.

ANSWER:

0.21 + 1.07$\boldsymbol{i}$, −1.07 + 0.21$\boldsymbol{i}$, −0.21 − 1.07$\boldsymbol{i}$, 1.07 −0.21$\boldsymbol{i}$

73. **ERROR ANALYSIS** Alma and Blake are evaluating $\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}\boldsymbol{i}\right)^5$. Alma uses DeMoivre's Theorem and gets an answer of cos $\frac{5\pi}{6}$ + $\boldsymbol{i}$ sin

$\frac{5\pi}{6}$. Blake tells her that she has only completed part of the problem. Is either of them correct? Explain your reasoning.

SOLUTION:

To evaluate $\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}\boldsymbol{i}\right)^5$, first write $-\frac{\sqrt{3}}{2}+\frac{1}{2}\boldsymbol{i}$ in polar form.

$$r = \sqrt{a^2+b^2} = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2} = 1$$

$$\theta = \tan^{-1}\frac{b}{a}+\pi = \tan^{-1}\frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}+\pi = \tan^{-1}-\frac{1}{\sqrt{3}}+\pi = \tan^{-1}-\frac{\sqrt{3}}{3}+\pi = \frac{5\pi}{6}$$

The polar form of $-\frac{\sqrt{3}}{2}+\frac{1}{2}\boldsymbol{i}$ is $\cos\frac{5\pi}{6}+\boldsymbol{i}\sin\frac{5\pi}{6}$.

Now use DeMoivre's Theorem to find the fifth power.

$$\begin{aligned}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}\boldsymbol{i}\right)^5 &= \left[1\left(\cos\frac{5\pi}{6}+\boldsymbol{i}\sin\frac{5\pi}{6}\right)\right]^5\\ &= 1^5\left[\cos 5\left(\frac{5\pi}{6}\right)+\boldsymbol{i}\sin 5\left(\frac{5\pi}{6}\right)\right]\\ &= \cos\frac{25\pi}{6}+\boldsymbol{i}\sin\frac{25\pi}{6}\\ &= \frac{\sqrt{3}}{2}+\frac{1}{2}\boldsymbol{i}\end{aligned}$$

Therefore, $\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}\boldsymbol{i}\right)^5 = \frac{\sqrt{3}}{2}+\frac{1}{2}\boldsymbol{i}$. So, Blake is correct. Sample answer: Alma only converted the expression into polar form. She needed to use DeMoivre's Theorem to find the fifth power.

ANSWER:

Blake; sample answer: Alma only converted the expression into polar form. She needed to use DeMoivre's Theorem to find the fifth power.

74. **REASONING** Suppose $z = a + b\boldsymbol{i}$ is one of the 29th roots of 1.

a. What is the maximum value of a?

b. What is the maximum value of b?

SOLUTION:

To find the 29th roots of unity, first write 1 in polar form.

$$r = \sqrt{a^2+b^2} = \sqrt{1^2+0^2} = 1$$

$$\theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{0}{1} = 0$$

The polar form of 1 is $1 \cdot (\cos 0 + \boldsymbol{i}\sin 0)$. Now write an expression for the 29th roots.

$$1\left(\cos\frac{0+2n\pi}{29}+\boldsymbol{i}\sin\frac{0+2n\pi}{29}\right) = \cos\frac{2n\pi}{29}+\boldsymbol{i}\sin\frac{2n\pi}{29}$$

The value of a is represented by the expression $\cos\frac{2n\pi}{29}$. This expression is evaluated for integer values of n from 0 to 28. The range of the cosine function is $-1 \le y \le 1$. So, the greatest value that this expression can achieve is 1. Thus, the maximum value of a is 1.

b. The expression for the 29th roots of 1 is $\cos\frac{2n\pi}{29}+\boldsymbol{i}\sin\frac{2n\pi}{29}$. The value of b is represented by the expression $\sin\frac{2n\pi}{29}$. This expression is evaluated for integer values of n from 0 to 28. Use a graphing calculator to find the maximum value of $\sin\frac{2n\pi}{29}$. Enter $\sin\frac{2n\pi}{29}$ in the **Y=** menu. Use the **TABLE** function to view the values of y for the different integer values of x.

X	Y1
4	.76216
5	.88351
6	.96355
7	.99853
8	.98603
9	.92898
10	.82769

X=7

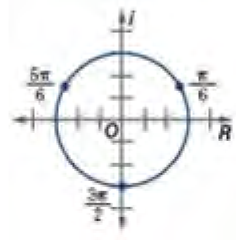
From the table, it appears that y achieves a maximum value of about 0.9985 when $x = 7$. Thus, when $n = 7$, the expression $\sin\frac{2n\pi}{29}$ will achieve a maximum value of $\sin\frac{14\pi}{29}$ or about 0.9985.

ANSWER:

a. 1

b. $\sin \frac{14\pi}{29}$ or about 0.9985

CHALLENGE Find the roots shown on each graph and write them in polar form. Then identify the complex number with the given roots.

75. 

SOLUTION:

Let r be any of the roots depicted. Since r lies on a circle of radius 3, $|r| = 3$. The root r_1 at $\frac{\pi}{6}$ can be represented by $r_1 = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$. The root r_2 at $\frac{5\pi}{6}$ can be represented by $r_2 = 3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$. The root r_3 at $\frac{3\pi}{2}$ can be represented by $r_3 = 3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$. To determine the number whose roots are r_1, r_2, and r_3, use De Moivre's Theorem to cube any one of them.

$$\begin{aligned} r_1^3 &= \left[3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^3 \\ &= 3^3\left[\cos 3\left(\frac{\pi}{6}\right) + i\sin 3\left(\frac{\pi}{6}\right)\right] \\ &= 27\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\ &= 27(0 + i) \\ &= 27i \end{aligned}$$

The roots in polar form are $3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, $3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$, and $3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$. The complex number with the given roots is $27i$.

ANSWER:

$3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, $3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$, $3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$, $27i$

76. 

SOLUTION:

Let r be any of the roots depicted. Since r lies on a circle of radius 2, $|r| = 2$. The root r_1 at $\frac{\pi}{4}$ can be represented by $r_1 = 2\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)$. The root r_2 at $\frac{3\pi}{4}$ can be represented by $r_2 = 2\left(\cos\frac{3\pi}{4} + \boldsymbol{i}\sin\frac{3\pi}{4}\right)$. The root r_3 at $\frac{5\pi}{4}$ can be represented by $r_3 = 2\left(\cos\frac{5\pi}{4} + \boldsymbol{i}\sin\frac{5\pi}{4}\right)$. The root r_4 at $\frac{7\pi}{4}$ can be represented by $r_4 = 2\left(\cos\frac{7\pi}{4} + \boldsymbol{i}\sin\frac{7\pi}{4}\right)$. To determine the number whose roots are r_1, r_2, r_3, and r_4, use De Moivre's Theorem to raise any one of them to the fourth power.

$$
\begin{aligned}
r_1^4 &= \left[2\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)\right]^4 \\
&= 2^4\left[\cos 4\left(\frac{\pi}{4}\right) + \boldsymbol{i}\sin 4\left(\frac{\pi}{4}\right)\right] \\
&= 16(\cos\pi + \boldsymbol{i}\sin\pi) \\
&= 16(-1 + 0\boldsymbol{i}) \\
&= -16
\end{aligned}
$$

The roots in polar form are $2\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right), 2\left(\cos\frac{3\pi}{4} + \boldsymbol{i}\sin\frac{3\pi}{4}\right), 2\left(\cos\frac{5\pi}{4} + \boldsymbol{i}\sin\frac{5\pi}{4}\right)$, and $2\left(\cos\frac{7\pi}{4} + \boldsymbol{i}\sin\frac{7\pi}{4}\right)$. The complex number with the given roots is -16.

ANSWER:

$2\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right), 2\left(\cos\frac{3\pi}{4} + \boldsymbol{i}\sin\frac{3\pi}{4}\right), 2\left(\cos\frac{5\pi}{4} + \boldsymbol{i}\sin\frac{5\pi}{4}\right), 2\left(\cos\frac{7\pi}{4} + \boldsymbol{i}\sin\frac{7\pi}{4}\right); -16$

77. **PROOF** Given $z_1 = r_1(\cos\theta_1 + \boldsymbol{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \boldsymbol{i}\sin\theta_2)$, where $r_2 \neq 0$, prove that $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$.

SOLUTION:

Given: $z_1 = r_1(\cos\theta_1 + \boldsymbol{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \boldsymbol{i}\sin\theta_2)$

Prove: $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + \boldsymbol{i}\sin(\theta_1 - \theta_2)]$

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\
&= \frac{r_1}{r_2}\left(\frac{\cos\theta_1 + \boldsymbol{i}\sin\theta_1}{\cos\theta_2 + \boldsymbol{i}\sin\theta_2}\right)\cdot\left(\frac{\cos\theta_2 - \boldsymbol{i}\sin\theta_2}{\cos\theta_2 - \boldsymbol{i}\sin\theta_2}\right) \\
&= \frac{r_1}{r_2}\left(\frac{\cos\theta_1\cos\theta_2 - \boldsymbol{i}\sin\theta_2\cos\theta_1 + \boldsymbol{i}\sin\theta_1\cos\theta_2 - \boldsymbol{i}^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2 - \boldsymbol{i}\sin\theta_2\cos\theta_2 + \boldsymbol{i}\sin\theta_2\cos\theta_2 - \boldsymbol{i}^2\sin^2\theta_2}\right) \\
&= \frac{r_1}{r_2}\left(\frac{\cos\theta_1\cos\theta_2 - \boldsymbol{i}\sin\theta_2\cos\theta_1 + \boldsymbol{i}\sin\theta_1\cos\theta_2 - \boldsymbol{i}^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}\right) \\
&= \frac{r_1}{r_2}(\cos\theta_1\cos\theta_2 - \boldsymbol{i}\sin\theta_2\cos\theta_1 + \boldsymbol{i}\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\
&= \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + \boldsymbol{i}(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)] \\
&= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + \boldsymbol{i}\sin(\theta_1 - \theta_2)]
\end{aligned}
$$

ANSWER:

Given: $z_1 = r_1(\cos\theta_1 + \boldsymbol{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \boldsymbol{i}\sin\theta_2)$

Prove: $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + \boldsymbol{i}\sin(\theta_1 - \theta_2)]$

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + \boldsymbol{i}\sin\theta_1)}{r_2(\cos\theta_2 + \boldsymbol{i}\sin\theta_2)} \\
&= \frac{r_1}{r_2}\left(\frac{\cos\theta_1 + \boldsymbol{i}\sin\theta_1}{\cos\theta_2 + \boldsymbol{i}\sin\theta_2}\right)\cdot\left(\frac{\cos\theta_2 - \boldsymbol{i}\sin\theta_2}{\cos\theta_2 - \boldsymbol{i}\sin\theta_2}\right) \\
&= \frac{r_1}{r_2}\left(\frac{\cos\theta_1\cos\theta_2 - \boldsymbol{i}\sin\theta_2\cos\theta_1 + \boldsymbol{i}\sin\theta_1\cos\theta_2 - \boldsymbol{i}^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2 - \boldsymbol{i}\sin\theta_2\cos\theta_2 + \boldsymbol{i}\sin\theta_2\cos\theta_2 - \boldsymbol{i}^2\sin^2\theta_2}\right) \\
&= \frac{r_1}{r_2}\left(\frac{\cos\theta_1\cos\theta_2 - \boldsymbol{i}\sin\theta_2\cos\theta_1 + \boldsymbol{i}\sin\theta_1\cos\theta_2 - \boldsymbol{i}^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}\right) \\
&= \frac{r_1}{r_2}(\cos\theta_1\cos\theta_2 - \boldsymbol{i}\sin\theta_2\cos\theta_1 + \boldsymbol{i}\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\
&= \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + \boldsymbol{i}(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)]
\end{aligned}
$$

$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + \boldsymbol{i}\sin(\theta_1 - \theta_2)]$

REASONING Determine whether each statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

78. The p th roots of a complex number z are equally spaced around the circle centered at the origin with radius $r^{\frac{1}{p}}$.

SOLUTION:

The expression for the distinct roots of a complex number is

$$r^{\frac{1}{p}}\left(\cos\frac{\theta+2n\pi}{p}+i\sin\frac{\theta+2n\pi}{p}\right) \text{ or }$$
$$r^{\frac{1}{p}}\left[\cos\left(\frac{\theta}{p}+\frac{2n\pi}{p}\right)+i\sin\left(\frac{\theta}{p}+\frac{2n\pi}{p}\right)\right].$$

Each root will have the same modulus, $r^{\frac{1}{p}}$. Since the modulus is the distance from the root to the origin, this acts as the radius of a circle on which the points are located. The arguments of each successive root are found by repeatedly adding $\frac{2\pi}{\pi}$. Thus, they are equally spaced from one another.

So, the statement is always true. Sample answer: The p th roots of a complex number all have the same modulus, which is determine by $r^{\frac{1}{p}}$, that acts as the radius of the circle on which the roots are located. The arguments of each successive root are found by repeatedly adding $\frac{2\pi}{\pi}$. Thus, the roots are equally spaced around the circle.

ANSWER:

Always; sample answer: The p th roots of a complex number all have the same modulus, which is determined by $r^{\frac{1}{p}}$, that acts as the radius of the circle on which the roots are located. The arguments of each successive root are found by repeatedly adding $\frac{2\pi}{p}$. Thus, the roots are equally spaced around the circle.

79. The complex conjugate of $z = a + bi$ is $\bar{z} = a - bi$. For any z, $z + \bar{z}$ and $z \cdot \bar{z}$ are real numbers.

SOLUTION:

Let $z = a + b\boldsymbol{i}$ and $\bar{z} = a - b\boldsymbol{i}$. Find $z + \bar{z}$ and $z \cdot \bar{z}$.

$$\begin{aligned} z+\bar{z} &= (a+b\boldsymbol{i})+(a-b\boldsymbol{i}) \\ &= a+b\boldsymbol{i}+a-b\boldsymbol{i} \\ &= 2a \end{aligned}$$

$$\begin{aligned} z\cdot\bar{z} &= (a+b\boldsymbol{i})(a-b\boldsymbol{i}) \\ &= a^2-ab\boldsymbol{i}+ab\boldsymbol{i}-b^2\boldsymbol{i}^2 \\ &= a^2-b^2\boldsymbol{i}^2 \\ &= a^2-b^2(-1) \\ &= a^2+b^2 \end{aligned}$$

So, the statement is always true. Sample answer: If $z = a + b\boldsymbol{i}$ and $\bar{z} = a - b\boldsymbol{i}, z + \bar{z} = a + b\boldsymbol{i} + a - b\boldsymbol{i}$ or $2a$ and $z \cdot \bar{z} = (a + b\boldsymbol{i})(a - b\boldsymbol{i})$ or $a^2 + b^2$.

ANSWER:

Always; sample answer: If $z = a + b\boldsymbol{i}$ and $\bar{z} = a - b\boldsymbol{i}, z + \bar{z} = a + b\boldsymbol{i} + a - b\boldsymbol{i}$ or $2a$ and $z \cdot \bar{z} = (a + b\boldsymbol{i})(a - b\boldsymbol{i})$ or $a^2 + b^2$.

80. **OPEN ENDED** Find two complex numbers $a + b\boldsymbol{i}$ in which $a \neq 0$ and $b \neq 0$ with an absolute value of $\sqrt{17}$.

SOLUTION:

Since the absolute value of a complex number is determined by $\sqrt{a^2+b^2}$, $\sqrt{a^2+b^2} = \sqrt{17}$. Squaring each side results in $a^2 + b^2 = 17$. Choose a such that a^2 is as close to but less than 17. Let $a = 4$. Solve for b.

$$\begin{aligned} a^2+b^2 &= 17 \\ 4^2+b^2 &= 17 \\ 16+b^2 &= 17 \\ b^2 &= 1 \\ b &= \pm 1 \end{aligned}$$

So, $4 + \boldsymbol{i}$ and $4 - \boldsymbol{i}$ are complex numbers with an absolute value of $\sqrt{17}$.

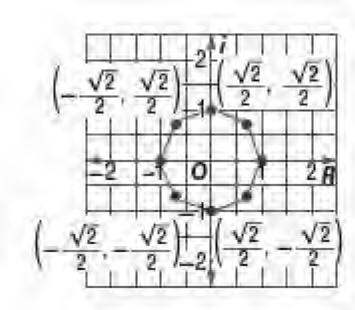
ANSWER:

Sample answers: $4 + \boldsymbol{i}$, $4 - \boldsymbol{i}$

81. **WRITING IN MATH** Explain why the sum of the imaginary parts of the p distinct p th roots of any positive real number must be zero. (*Hint*: The roots are the vertices of a regular polygon.)

SOLUTION:

Sample answer: Consider the polygon created by the 8 distinct 8th roots of 1.



The vertices of the polygon in rectangular form are 1, $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\boldsymbol{i}$, $\boldsymbol{i}$, $-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\boldsymbol{i}$, -1, $-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\boldsymbol{i}$, $-\boldsymbol{i}$, and $\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\boldsymbol{i}$.

$$\frac{\sqrt{2}}{2}i+i+\frac{\sqrt{2}}{2}i+\left(-\frac{\sqrt{2}}{2}i\right)+(-i)+\left(-\frac{\sqrt{2}}{2}i\right)=0$$

Since the roots are evenly spaced around the polygon and a vertex of the polygon lies on the positive real axis, the polygon is symmetric about the real axis and the non-real complex roots occur in conjugate pairs. Since the imaginary part of the sum of two complex conjugates is 0, the imaginary part of the sum of all of the roots must be 0. This will always occur when one of the roots is a positive real number.

ANSWER:

Sample answer: Since one of the roots must be a positive real number, a vertex of the polygon lies on the positive real axis and the polygon is symmetric about the real axis. This means that the non-real complex roots occur in conjugate pairs. Since the imaginary part of the sum of two complex conjugates is 0, the imaginary part of the sum of all of the roots must be 0.

Write each polar equation in rectangular form.

82. $r = \frac{15}{1+4\cos\theta}$

SOLUTION:

Write the equation in standard form.

$$r=\frac{15}{1+4\cos\theta}$$
$$r=\frac{4(3.75)}{1+4\cos\theta}$$

For this equation, $e = 4$ and $d = 15 \div 4$ or 3.75. The eccentricity and form of the equation determine that this is a hyperbola with directrix $x = 3.75$. Therefore, the transverse axis of the hyperbola lies along the polar or x-axis.

The general equation of such a hyperbola in rectangular form is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.

The vertices lie on the transverse axis and occur when $\theta = 0$ and π.

$$r=\frac{15}{1+4\cos\theta}$$
$$r=\frac{15}{1+4\cos 0}$$
$$r=\frac{15}{1+4}$$
$$r=3$$

$$r=\frac{15}{1+4\cos\theta}$$
$$r=\frac{15}{1+4\cos\pi}$$
$$r=\frac{15}{1-4}$$
$$r=-5$$

The vertices have polar coordinates $(3, 0)$ and $(-5, \pi)$, which correspond to rectangular coordinates $(3, 0)$ and $(5, 0)$ .The hyperbola's center is the midpoint of the segment between the vertices, so $(h, k) = (4, 0)$.

The distance a between the center and each vertex is 1. The distance c from the center to the focus at $(0, 0)$ is 4.

$b = \sqrt{4^2-1^2}$ or $\sqrt{15}$.

Substitute the values for h, k, a, and b into the standard form of an equation for an ellipse.

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$
$$\frac{(x-4)^2}{1^2}-\frac{(y-0)^2}{\left(\sqrt{15}\right)^2}=1$$
$$\frac{(x-4)^2}{1}-\frac{y^2}{15}=1$$

ANSWER:

$$(x-4)^2-\frac{y^2}{15}=1$$

83. $r=\dfrac{14}{2\cos\theta+2}$

SOLUTION:

Write the equation in standard form.

$$r=\frac{14}{2\cos\theta+2}$$
$$r=\frac{2(7)}{2(1+\cos\theta)}$$
$$r=\frac{7}{1+\cos\theta}$$

For this equation, e = 1 and d = 7. The eccentricity and form of the equation determine that this is a parabola that opens horizontally with focus at the pole and a directrix x = 7.

The general equation of such a parabola in rectangular form is $(y-k)^2=-4p(x-h)$. The vertex lies between the focus F and the directrix of the parabola, occurring when $\theta=0$.

$$r=\frac{7}{1+\cos\theta}$$
$$r=\frac{7}{1+\cos 0}$$
$$r=\frac{7}{1+1}$$
$$r=3.5$$

The vertex lies at polar coordinate $(3.5, 0)$, which correspond to rectangular coordinates $(3.5, 0)$. So (h,k) = $(3.5, 0)$. The distance p from the vertex at $(3.5, 0)$ to the focus at $(0,0)$ is 3.5.

Substitute the values for h, k, and p into the general equation for rectangular form.

$$(y-k)^2=-4p(x-h)$$
$$(y-0)^2=-4(3.5)(x-3.5)$$
$$y^2=-14(x-3.5)$$

ANSWER:

y^2 = −14(x – 3.5)

84. $r=\dfrac{-6}{\sin\theta-2}$

SOLUTION:

The equation is in standard form.

$$r = \frac{-6}{\sin\theta - 2}$$
$$r = \frac{-2(3)}{-2(1 - 0.5\sin\theta)}$$
$$r = \frac{3}{1 - 0.5\sin\theta}$$

For this equation, $e = 0.5$ and $d = 3 \div 0.5$ or 6. The eccentricity and form of the equation determine that this is an ellipse with directrix $y = -6$.

The general equation of such an ellipse in rectangular form is $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$.

The vertices are the endpoints of the major axis and occur when $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

$$r = \frac{3}{1 - 0.5\sin\theta}$$
$$r = \frac{3}{1 - 0.5\sin\frac{\pi}{2}}$$
$$r = \frac{3}{1 - 0.5}$$
$$r = 6$$

$$r = \frac{3}{1 - 0.5\sin\theta}$$
$$r = \frac{3}{1 - 0.5\sin\frac{3\pi}{2}}$$
$$r = \frac{3}{1 - (-0.5)}$$
$$r = 2$$

The vertices have polar coordinates $\left(6, \frac{\pi}{2}\right)$ and $\left(2, \frac{3\pi}{2}\right)$, which correspond to rectangular coordinates (0, 6) and (0, –2). The ellipse's center is the midpoint of the segment between the vertices, so (h, k) = (0, 2). The distance a between the center and each vertex is 4. The distance c from the center to the focus at (0, 0) is 2.

$b = \sqrt{4^2 - 2^2}$ or $\sqrt{12}$.

Substitute the values for h, k, a, and b into the standard form of an equation for an ellipse.

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$
$$\frac{(x-0)^2}{(\sqrt{12})^2} + \frac{(y-2)^2}{4^2} = 1$$
$$\frac{x^2}{12} + \frac{(y-2)^2}{16} = 1$$
$$\frac{(y-2)^2}{16} + \frac{x^2}{12} = 1$$

ANSWER:

$$\frac{(y-2)^2}{16} + \frac{x^2}{12} = 1$$

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation.

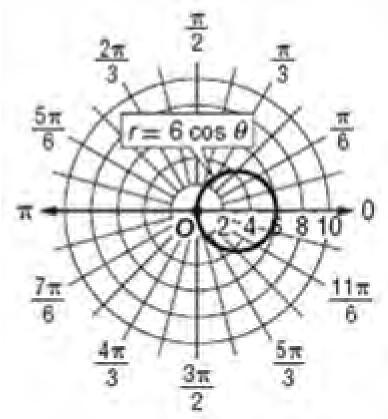
85. $(x-3)^2+y^2=9$

SOLUTION:

The graph of $(x-3)^2+y^2=9$ is a circle with radius 3 centered at (3, 0). To find the polar form of this equation, replace x with $r\cos\theta$ and y with $r\sin\theta$. Then simplify.

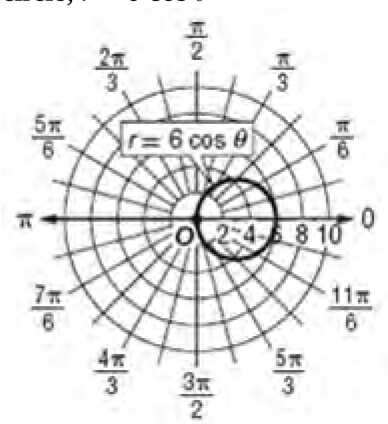
$$\begin{aligned}(x-3)^2+y^2&=9\\(r\cos\theta-3)^2+(r\sin\theta)^2&=9\\r^2\cos^2\theta-6r\cos\theta+9+r^2\sin^2\theta&=9\\r^2\cos^2\theta-6r\cos\theta+r^2\sin^2\theta&=0\\r^2\cos^2\theta+r^2\sin^2\theta&=6r\cos\theta\\r^2(\cos^2\theta+\sin^2\theta)&=6r\cos\theta\\r^2(1)&=6r\cos\theta\\r&=6\cos\theta\end{aligned}$$

Evaluate the function for several θ-values in its domain and use these points to graph the function. The graph of this polar equation is a circle.



ANSWER:

circle; $r = 6\cos\theta$



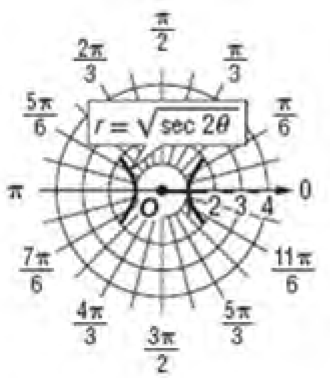
86. $x^2-y^2=1$

SOLUTION:

The graph of $x^2-y^2=1$ is a hyperbola. To find the polar form of this equation, replace x with $r\cos\theta$ and y with $r\sin\theta$. Then simplify.

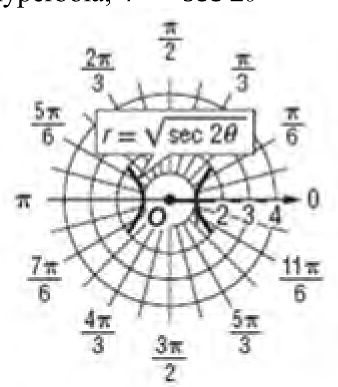
$$\begin{aligned}x^2-y^2&=1\\(r\cos\theta)^2-(r\sin\theta)^2&=1\\r^2\cos^2\theta-r^2\sin^2\theta&=1\\r^2(\cos^2\theta-\sin^2\theta)&=1\\r^2&=\frac{1}{\cos^2\theta-\sin^2\theta}\\r^2&=\frac{1}{\cos 2\theta}\\r^2&=\sec 2\theta\\r&=\sqrt{\sec 2\theta}\end{aligned}$$

Evaluate the function for several θ-values in its domain and use these points to graph the function. The graph of this polar equation is a circle.



ANSWER:

hyperbola; $r^2 = \sec 2\theta$



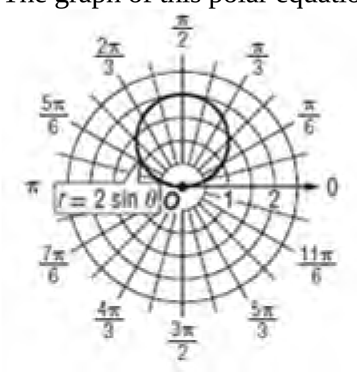
87. $x^2 + y^2 = 2y$

SOLUTION:

The graph of $x^2 + y^2 = 2y$ is a circle. To find the polar form of this equation, replace x with $r \cos \theta$ and y with $r \sin \theta$. Then simplify.

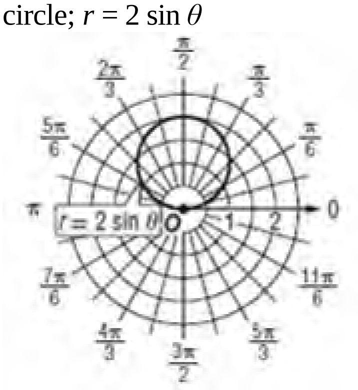
$$\begin{aligned} x^2 + y^2 &= 2y \\ (r\cos\theta)^2 + (r\sin\theta)^2 &= 2(r\sin\theta) \\ r^2\cos^2\theta + r^2\sin^2\theta &= 2r\sin\theta \\ r^2(\cos^2\theta + \sin^2\theta) &= 2r\sin\theta \\ r^2 &= 2r\sin\theta \\ r &= 2\sin\theta \end{aligned}$$

Evaluate the function for several θ-values in its domain and use these points to graph the function. The graph of this polar equation is a circle.



ANSWER:

circle; $r = 2 \sin \theta$

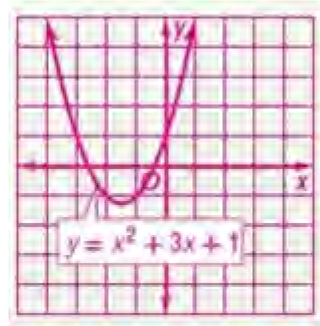


Graph the conic given by each equation.

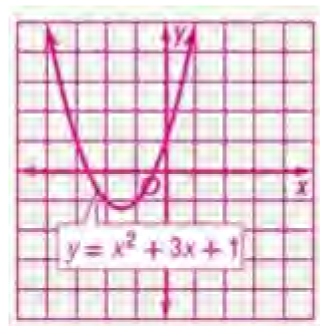
88. $y = x^2 + 3x + 1$

SOLUTION:

Graph the equation $y = x^2 + 3x + 1$.



ANSWER:

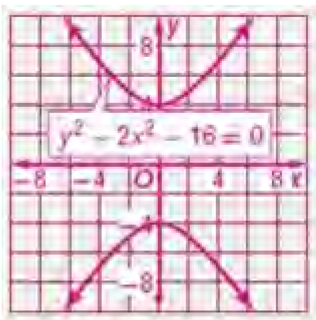


89. $y^2 - 2x^2 - 16 = 0$

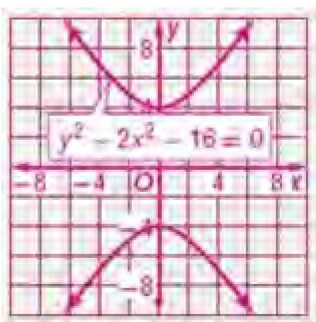
SOLUTION:

Graph the equation by solving for y.

$$y^2 - 2x^2 - 16 = 0$$
$$y^2 = 2x^2 + 16$$
$$y = \pm\sqrt{2x^2 + 16}$$



ANSWER:

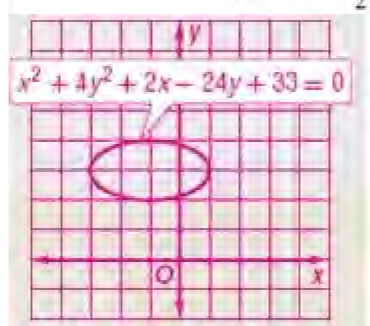


90. $x^2 + 4y^2 + 2x - 24y + 33 = 0$

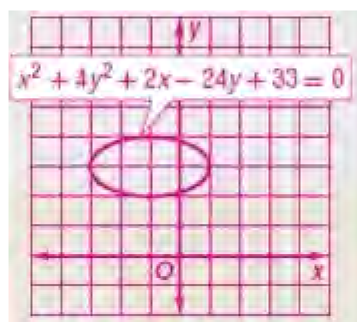
SOLUTION:

Graph the equation by solving for y.

$$x^2 + 4y^2 + 2x - 24y + 33 = 0$$
$$4y^2 - 24y = -x^2 - 2x - 33$$
$$4(y^2 - 6y + 9) = -x^2 - 2x - 33 + 36$$
$$(y-3)^2 = \frac{1}{4}\left(-x^2 - 2x + 3\right)$$
$$(y-3) = \sqrt{\frac{1}{4}\left(-x^2 - 2x + 3\right)}$$
$$y = \frac{\pm\sqrt{\left(-x^2 - 2x + 3\right)}}{2} + 3$$



ANSWER:



Find the center, foci, and vertices of each ellipse.

91. $\dfrac{(x+8)^2}{9}+\dfrac{(y-7)^2}{81}=1$

SOLUTION:

The ellipse is in standard form, where h = –8 and k = 7. So, the center is located at (–8, 7). The ellipse has a vertical orientation, so a^2 = 81, a = 9, and b^2 = 9.

Use the values of a and b to find c.

$$c^2 = a^2 - b^2$$
$$c^2 = 81 - 6$$
$$c^2 = 72$$
$$c = \sqrt{72} \text{ or } 6\sqrt{2}$$

The foci are c units from the center, so they are located at (–8, 7 ± 6$\sqrt{2}$). The vertices are a units from the center, so they are located at (–8, 16), (–8, –2).

ANSWER:

center: (–8, 7); foci: (–8, 7 ± 6$\sqrt{2}$); vertices: (–8, 16), (–8, –2)

92. $25x^2 + 4y^2 + 150x + 24y = -161$

SOLUTION:

First, write the equation in standard form.

$$25x^2+4y^2+150x+24y=-161$$
$$25(x^2+6x)+4(y^2+6y)=-161$$
$$25(x^2+6x+9)+4(y^2+6y+9)=-161+225+36$$
$$25(x+3)^2+4(y+3)^2=100$$
$$\frac{(x+3)^2}{4}+\frac{(y+3)^2}{25}=1$$

The equation is now in standard form, where h = –3 and k = –3. So, the center is located at (–3, –3). The ellipse has a vertical orientation, so a^2 = 25, a = 5, and b^2 = 4.

Use the values of a and b to find c.

$$c^2 = a^2 - b^2$$
$$c^2 = 25 - 4$$
$$c^2 = 21$$
$$c = \sqrt{21}$$

The foci are c units from the center, so they are located at (–3, –3 ± $\sqrt{21}$). The vertices are a units from the center, so they are located at (–3, 2), (–3, –8).

ANSWER:

center: (–3, –3); foci: (–3, –3 ± $\sqrt{21}$); vertices: (–3, 2), (–3, –8)

93. $4x^2 + 9y^2 - 56x + 108y = -484$

SOLUTION:

First, write the equation in standard form.

$$4x^2 + 9y^2 - 56x + 108y = -484$$
$$4(x^2 - 14x) + 9(y^2 + 12y) = -484$$
$$4(x^2 - 14x + 49) + 9(y^2 + 12y + 36) = -484 + 196 + 324$$
$$4(x-7)^2 + 9(y+6)^2 = 36$$
$$\frac{(x-7)^2}{9} + \frac{(y+6)^2}{4} = 1$$

The equation is now in standard form, where $h = 7$ and $k = -6$. So, the center is located at (7, –6). The ellipse has a horizontal orientation, so $a^2 = 9$, $a = 3$, and $b^2 = 4$.

Use the values of a and b to find c.

$$c^2 = a^2 - b^2$$
$$c^2 = 9 - 4$$
$$c^2 = 5$$
$$c = \sqrt{5}$$

The foci are c units from the center, so they are located at $(7 \pm \sqrt{5}, -6)$. The vertices are a units from the center, so they are located at (10, –6), (4, –6).

ANSWER:

center: (7, –6); foci: $(7 \pm \sqrt{5}, -6)$; vertices: (10, –6), (4, –6)

Solve each system of equations using Gauss-Jordan elimination.

94. $x + y + z = 12$
$6x - 2y - z = 16$
$3x + 4y + 2z = 28$

SOLUTION:

Write the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 6 & -2 & -1 & 16 \\ 3 & 4 & 2 & 28 \end{array}\right]$$

Apply elementary row operations to obtain reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 & 1 & 12 \\ 6 & -2 & -1 & 16 \\ 3 & 4 & 2 & 28 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & -8 & -7 & -56 \\ 3 & 4 & 2 & 28 \end{bmatrix} \xleftarrow{R_2 - 6R_1}$$

$$\begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & -8 & -7 & -56 \\ 0 & 1 & -1 & -8 \end{bmatrix} \xleftarrow{R_3 - 3R_1}$$

$$\begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & \frac{7}{8} & 7 \\ 0 & 1 & -1 & -8 \end{bmatrix} \xleftarrow{R_3 \div (-8)}$$

$$\begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & \frac{7}{8} & 7 \\ 0 & 0 & -\frac{15}{8} & -15 \end{bmatrix} \xleftarrow{R_3 - R_2}$$

$$\begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & \frac{7}{8} & 7 \\ 0 & 0 & 1 & 8 \end{bmatrix} \xleftarrow{R_3 \div (-15/8)}$$

$$\begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8 \end{bmatrix} \xleftarrow{R_2 - (7/8)R_3}$$

$$\begin{bmatrix} 1 & 0 & 1 & 12 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8 \end{bmatrix} \xleftarrow{R_1 - R_2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8 \end{bmatrix} \xleftarrow{R_1 - R_3}$$

The solution is (4, 0, 8).

ANSWER:

(4, 0, 8)

95. $9g + 7h = -30$
$8h + 5j = 11$
$-3g + 10j = 73$

SOLUTION:

Write the augmented matrix.

$$\left[\begin{array}{ccc|c} 9 & 7 & 0 & -30 \\ 0 & 8 & 5 & 11 \\ -3 & 0 & 10 & 73 \end{array}\right]$$

Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 9 & 7 & 0 & -30 \\ 0 & 8 & 5 & 11 \\ -3 & 0 & 10 & 73 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 9 & 7 & 0 & -30 \\ 0 & 8 & 5 & 11 \\ 0 & 7 & 30 & 189 \end{array}\right] \begin{array}{l} \\ \\ \xleftarrow{3R_1+R_3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{7}{9} & 0 & -\frac{10}{3} \\ 0 & 8 & 5 & 11 \\ 0 & 7 & 30 & 189 \end{array}\right] \begin{array}{l} \xleftarrow{R_1\div(9)} \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{7}{9} & 0 & -\frac{10}{3} \\ 0 & 1 & \frac{5}{8} & \frac{11}{8} \\ 0 & 7 & 30 & 189 \end{array}\right] \begin{array}{l} \\ \xleftarrow{R_2\div(8)} \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{7}{9} & 0 & -\frac{10}{3} \\ 0 & 1 & \frac{5}{8} & \frac{11}{8} \\ 0 & 0 & \frac{205}{8} & \frac{1435}{8} \end{array}\right] \begin{array}{l} \\ \\ \xleftarrow{R_3-7R_2} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{7}{9} & 0 & -\frac{10}{3} \\ 0 & 1 & \frac{5}{8} & \frac{11}{8} \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} \\ \\ \xleftarrow{R_3\div(205/8)} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{7}{9} & 0 & -\frac{10}{3} \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} \\ \xleftarrow{R_2-(5/8)R_3} \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} \xleftarrow{R_1-(7/9)R_2} \\ \\ \\ \end{array}$$

The solution is $(-1, -3, 7)$.

ANSWER:

$(-1, -3, 7)$

96. $2k - n = 2$
$3p = 21$
$4k + p = 19$

SOLUTION:

$2k - n = 2$
$3p = 21$
$4k + p = 19$

Write the augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 2 \\ 0 & 0 & 3 & 21 \\ 4 & 0 & 1 & 19 \end{array}\right]$$

Apply elementary row operations to obtain reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 2 \\ 0 & 0 & 3 & 21 \\ 4 & 0 & 1 & 19 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 2 \\ 4 & 0 & 1 & 19 \\ 0 & 0 & 3 & 21 \end{array}\right] \begin{array}{l} \\ \leftarrow R_3 \\ \leftarrow R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 2 \\ 4 & 0 & 1 & 19 \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow R_3 \div (3) \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 2 \\ 0 & 2 & 1 & 15 \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} \\ \leftarrow R_2 + -2R_1 \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & 1 \\ 0 & 2 & 1 & 15 \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} \leftarrow R_1 \div (2) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{15}{2} \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} \\ \leftarrow R_2 \div (2) \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} \\ \leftarrow R_2 - (1/2)R_3 \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} \leftarrow R_1 + (1/2)R_2 \\ \\ \end{array}$$

The solution is (3, 4, 7).

ANSWER:

(3, 4, 7)

97. **POPULATION** In the beginning of 2008, the world's population was about 6.7 billion. If the world's population grows continuously at a rate of 2%, the future population P, in billions, can be predicted by $P = 6.5e^{0.02t}$, where t is the time in years since 2008.
a. According to this model, what will be the world's population in 2018?
b. Some experts have estimated that the world's food supply can support a population of at most 18 billion people. According to this model, for how many more years will the food supply be able to support the trend in world population growth?

SOLUTION:

a. Let $t = 10$ since 2018 is 10 years since 2008.
Substitute $t = 10$ into $P = 6.5e^{0.02t}$ and solve for P.

$$\begin{aligned} P &= 6.5e^{0.02t} \\ &= 6.5e^{0.02(10)} \\ &\approx 7.94 \end{aligned}$$

According to this model, the world's population will be about 7.94 billion.

b. Substitute $P = 18$ into $P = 6.5e^{0.02t}$ and solve for t.

$$\begin{aligned} P &= 6.5e^{0.02t} \\ 18 &= 6.5e^{0.02t} \\ \frac{18}{6.5} &= e^{0.02t} \\ \ln\frac{18}{6.5} &= 0.02t \\ \frac{\ln\frac{18}{6.5}}{0.02} &= t \\ 50.93 &\approx t \end{aligned}$$

According to this model, the food supply will be able to support population for about 51 more years.

ANSWER:

a. about 7.94 billion
b. about 51 years

98. **SAT/ACT** The graph on the *xy*-plane of the quadratic function g is a parabola with vertex at (3, −2). If $g(0) = 0$, then which of the following must also equal 0?
A $g(2)$
B $g(3)$
C $g(4)$
D $g(6)$
E $g(7)$

SOLUTION:

Since g is a quadratic function, it is a parabola that opens either up or down. The vertex is located at (3, −2). Since $g(0) = 0$, the point (0, 0) is also on the parabola. Thus, the parabola opens up. The parabola has an axis of symmetry at $x = 3$. Points to the left of the axis of symmetry will have corresponding points to the right of the axis. The x-coordinate of (0, 0) is 0. 0 is 3 units to the left of the vertex. Thus, the point 3 units to the right of the vertex will have the same y-coordinate. Therefore, $g(3 + 3)$ or $g(6)$ will have the same y-coordinate at $g(0)$.
The correct answer is D.

ANSWER:

D

99. Which of the following expresses the complex number $20 - 21\boldsymbol{i}$ in polar form?
F $29(\cos 5.47 + \boldsymbol{i} \sin 5.47)$
G $29(\cos 5.52 + \boldsymbol{i} \sin 5.52)$
H $32(\cos 5.47 + \boldsymbol{i} \sin 5.47)$
J $32(\cos 5.52 + \boldsymbol{i} \sin 5.52)$

SOLUTION:

Find the modulus r and argument θ.

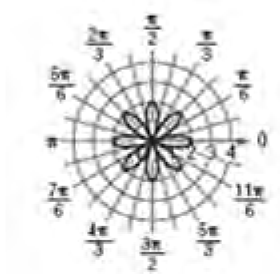
$$r = \sqrt{a^2 + b^2} = \sqrt{20^2 + (-21)^2} = 29 \qquad \theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{-21}{20} \approx -0.81$$

To find a positive value for θ, add 2π. So, $\theta = -0.81 + 2\pi$ or about 5.47.
The polar form of $20 - 21\boldsymbol{i}$ is $29(\cos 5.47 + \boldsymbol{i} \sin 5.47)$.
The correct answer is F.

ANSWER:

F

100. **FREE RESPONSE** Consider the graph.
a. Give a possible equation for the function.
b. Describe the symmetries of the graph.
c. Give the zeroes of the function over the domain $0 \le \theta \le 2\pi$.
d. What is the minimum value of r over the domain $0 \le \theta \le 2\pi$?

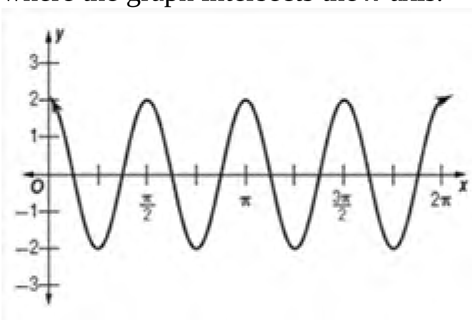


SOLUTION:

a. Sample answer: The graph is of a rose with 8 petals. Since the graph contains the point (2, 0), which lies on the x-axis, the equation will be of the form $r = a \cos n\theta$. The petals extend 2 units, so $a = 2$. Since the number of petals is even, the number of petals is equal to $2n$. Thus, $2n = 8$ or $n = 4$. So, a possible equation for the function is $r = 2 \cos 4\theta$.

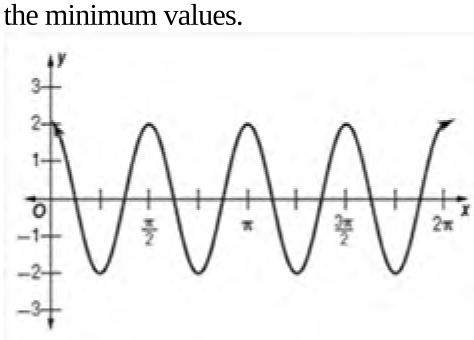
b. Sample answer: Since the equation found in part **a** is a function of cos θ, the function is symmetric with respect to the polar axis. It is also symmetric with respect to the line $\theta = \frac{\pi}{2}$ and the pole.

c. Sample answer: Graph the rectangular function $y = 2 \cos 4x$ over the domain $0 \leq x \leq 2\pi$ and look for where the graph intersects the x-axis.



The graph has zeros when $\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{15\pi}{9}$, and $\frac{17\pi}{9}$.

d. Sample answer: Graph the rectangular function $y = 2 \cos 4x$ over the domain $0 \leq x \leq 2\pi$ and look for the minimum values.



The minimum value of r is −2. This occurs when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

ANSWER:

a. Sample answer: $r = 2 \cos 4\theta$

b. Sample answer: The function is symmetric with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, and the pole.

c. Sample answer: The graph has zeros when $\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{15\pi}{9}$, and $\frac{17\pi}{9}$.

d. Sample answer: The minimum value of r is −2. This occurs when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$.