9-4 Polar Forms of Conic Sections

Determine the eccentricity, type of conic, and equation of the directrix for each polar equation.

1. $r = \dfrac{20}{4 + 4\sin\theta}$

SOLUTION:

Write the equation in standard form, $r = \dfrac{ed}{1 + e\sin\theta}$.

$$r = \frac{20}{4 + 4\sin\theta}$$
$$r = \frac{4(5)}{4(1 + \sin\theta)}$$
$$r = \frac{5}{1 + \sin\theta}$$

Since $e = 1$, the conic is a parabola. For a polar equation of this form (where $\sin\theta$ is included), the equation of the directrix is $y = d$. From the numerator, we know that $ed = 5$, so $d = 5$. Therefore, the equation of the directrix is $y = 5$.

ANSWER:

$e = 1$; parabola; $y = 5$

2. $r = \dfrac{18}{2 - 6\cos\theta}$

SOLUTION:

Write the equation in standard form, $r = \dfrac{ed}{1 - e\cos\theta}$.

$$r = \frac{18}{2 - 6\cos\theta}$$
$$r = \frac{2(9)}{2(1 - 3\cos\theta)}$$
$$r = \frac{9}{1 - 3\cos\theta}$$

Since $e = 3$, the conic is a hyperbola. For a polar equation of this form (where $\cos\theta$ is included), the equation of the directrix is $x = -d$. From the numerator, we know that $ed = 9$, so $d = 3$. Therefore, the equation of the directrix is $x = -3$.

ANSWER:

$e = 3$; hyperbola; $x = -3$

3. $r = \dfrac{21}{3\cos\theta + 1}$

SOLUTION:

Write the equation in standard form, $r = \dfrac{ed}{1 + e\cos\theta}$.

$$r = \frac{21}{3\cos\theta + 1}$$
$$r = \frac{21}{1 + 3\cos\theta}$$

Since $e = 3$, the conic is a hyperbola. For a polar equation of this form (where $\cos\theta$ is included), the equation of the directrix is $x = d$. From the numerator, we know that $ed = 21$, so $d = 21 \div 3$ or 7. Therefore, the equation of the directrix is $x = 7$.

ANSWER:

$e = 3$; hyperbola; $x = 7$

4. $r = \dfrac{24}{4\sin\theta + 8}$

SOLUTION:

Write the equation in standard form, $r = \dfrac{ed}{1 + e\sin\theta}$.

$$r = \frac{24}{4\sin\theta + 8}$$
$$r = \frac{24}{8 + 4\sin\theta}$$
$$r = \frac{8(3)}{8(1 + 0.5\sin\theta)}$$
$$r = \frac{3}{1 + 0.5\sin\theta}$$

Since $e = 0.5$, the conic is an ellipse. For a polar equation of this form (where $\sin\theta$ is included), the equation of the directrix is $y = d$. From the numerator, we know that $ed = 3$, so $d = 6$. Therefore, the equation of the directrix is $y = 6$.

ANSWER:

$e = 0.5$; ellipse; $y = 6$

5. $r = \dfrac{-12}{6\cos\theta - 6}$

SOLUTION:

Write the equation in standard form, $r = \dfrac{ed}{1 - e\cos\theta}$.

$$r = \frac{-12}{6\cos\theta - 6}$$
$$r = \frac{-6(2)}{-6(1 - \cos\theta)}$$
$$r = \frac{2}{1 - \cos\theta}$$

Since $e = 1$, the conic is a parabola. For a polar equation of this form (where $\cos\theta$ is included), the equation of the directrix is $x = -d$. From the numerator, we know that $ed = 2$, so $d = 2 \div 1$ or 2. Therefore, the equation of the directrix is $x = -2$.

ANSWER:

$e = 1$; parabola; $x = -2$

6. $r = \dfrac{9}{4 - 3\sin\theta}$

SOLUTION:

Write the equation in standard form, $r = \dfrac{ed}{1 - e\sin\theta}$.

$$r = \frac{9}{4 - 3\sin\theta}$$
$$r = \frac{4(2.25)}{4(1 - 0.75\sin\theta)}$$
$$r = \frac{2.25}{1 - 0.75\sin\theta}$$

Since $e = 0.75$, the conic is an ellipse. For a polar equation of this form (where $\sin\theta$ is included), the equation of the directrix is $y = -d$. From the numerator, we know that $ed = 2.25$, so $d = 3$. Therefore, the equation of the directrix is $y = -3$.

ANSWER:

$e = 0.75$; ellipse; $y = -3$

7. $r = \dfrac{-8}{\sin\theta - 0.25}$

SOLUTION:

Write the equation in standard form, $r = \dfrac{ed}{1 - e\sin\theta}$.

$$r = \frac{-8}{\sin\theta - 0.25}$$
$$r = \frac{-0.25(32)}{-0.25(1 - 4\sin\theta)}$$
$$r = \frac{32}{1 - 4\sin\theta}$$

Since $e = 4$, the conic is a hyperbola. For a polar equation of this form (where $\sin\theta$ is included), the equation of the directrix is $y = -d$. From the numerator, we know that $ed = 32$, so $d = 8$. Therefore, the equation of the directrix is $y = -8$.

ANSWER:

$e = 4$; hyperbola; $y = -8$

8. $r = \dfrac{10}{2.5 + 2.5\cos\theta}$

SOLUTION:

Write the equation in standard form, $r = \dfrac{ed}{1 + e\cos\theta}$.

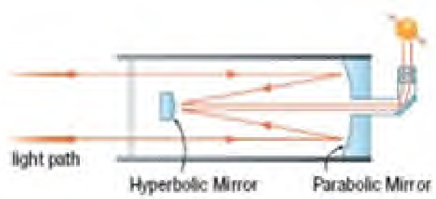
$$r = \frac{10}{2.5 + 2.5\cos\theta}$$
$$r = \frac{2.5(4)}{2.5(1 + \cos\theta)}$$
$$r = \frac{4}{1 + \cos\theta}$$

Since e = 1, the conic is a parabola. For a polar equation of this form (where $\cos\theta$ is included), the equation of the directrix is $x = d$. From the numerator, we know that ed =4, so d = 4. Therefore, the equation of the directrix is y = 4.

ANSWER:

e = 1; parabola; x = 4

9. **TELESCOPES** The Cassegrain Telescope, invented in 1692, produces an image by reflecting light off of parabolic and hyperbolic mirrors. Determine the eccentricity, type of conic, and the equation of the directrix for each equation modeling a mirror in the telescope.



a. $r = \dfrac{7}{2\sin\theta + 2}$

b. $r = \dfrac{28}{12.5\cos\theta + 5}$

SOLUTION:

a. Write the equation in standard form, $r = \dfrac{ed}{1 + e\sin\theta}$.

$$r = \frac{7}{2 + 2\sin\theta}$$
$$r = \frac{2(3.5)}{2(1 + \sin\theta)}$$
$$r = \frac{3.5}{1 + \sin\theta}$$

Since e = 1, the conic is a parabola. For a polar equation of this form, the equation of the directrix is $y = d$. From the numerator, we know that ed = 3.5, so d = 3.5. Therefore, the equation of the directrix is y = 3.5.

b. Write the equation in standard form, $r = \dfrac{ed}{1 + e\cos\theta}$.

$$r = \frac{28}{12.5\cos\theta + 5}$$
$$r = \frac{5(5.6)}{5(1 + 2.5\cos\theta)}$$
$$r = \frac{5.6}{1 + 2.5\cos\theta}$$

Since e = 2.5, the conic is a hyperbola. For a polar equation of this form (where $\cos\theta$ is included), the equation of the directrix is $x = d$. From the numerator, we know that ed = 5.6, so d = 2.24. Therefore, the equation of the directrix is x = 2.24.

ANSWER:

a. e = 1; parabola; y = 3.5
b. e = 2.5; hyperbola; x = 2.24

Write and graph a polar equation and directrix for the conic with the given characteristics.

10. e = 1; directrix: y = 6

SOLUTION:

Because e = 1, the conic is a parabola. The directrix y = 6 is above the pole, so the equation is of the form $r = \dfrac{ed}{1 + e\sin\theta}$ and d = 6. Use the values for e and d to write the equation.

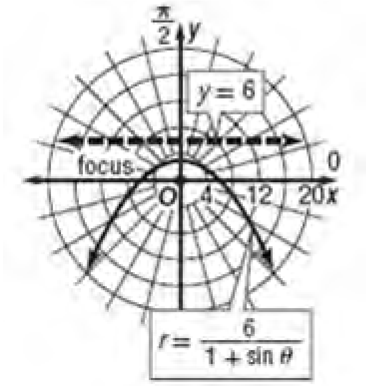
$$r = \frac{ed}{1 + e\sin\theta}$$
$$r = \frac{1(6)}{1 + \sin\theta}$$
$$r = \frac{6}{1 + \sin\theta}$$

Evaluate the function for several θ-values in its domain and use these points to graph the function and its directrix.

Substitute 0 for θ to determine the location of the vertex of the parabola.

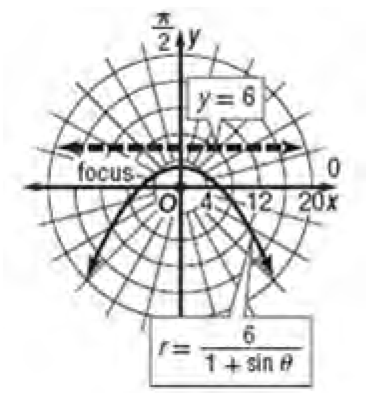
$$r = \frac{6}{1+\sin\theta}$$
$$r = \frac{6}{1+\sin 0}$$
$$r = \frac{6}{1+0}$$
$$r = \frac{6}{1}$$
$$r = 6$$

The graph is a parabola with vertex at (6, 0).



ANSWER:

$$r = \frac{6}{1+\sin\theta}$$



11. $e = 0.75$; directrix: $x = –8$

SOLUTION:

Because $e = 0.75$, the conic is an ellipse. The directrix $x = −8$ is to the left of the pole, so the equation is of the form $r = \frac{ed}{1 - e\cos\theta}$ and $d = 8$. Use the values for e and d to write the equation.

$$r = \frac{ed}{1 - e\cos\theta}$$
$$r = \frac{0.75(8)}{1 - 0.75\cos\theta}$$
$$r = \frac{6}{1 - 0.75\cos\theta}$$

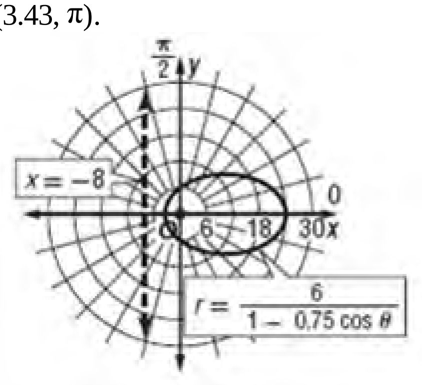
Evaluate the function for several θ-values in its domain and use these points to graph the function and its directrix.

Substitute 0 and π for θ to determine the location of the vertices of the ellipse.

$$r = \frac{6}{1 - 0.75\cos\theta}$$
$$r = \frac{6}{1 - 0.75\cos 0}$$
$$r = \frac{6}{1 - 0.75}$$
$$r = \frac{6}{0.25}$$
$$r = 24$$

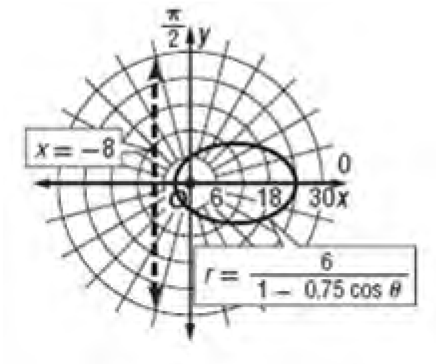
$$r = \frac{6}{1 - 0.75\cos\theta}$$
$$r = \frac{6}{1 - 0.75\cos\pi}$$
$$r = \frac{6}{1 - (-0.75)}$$
$$r = \frac{6}{1.75}$$
$$r \approx 3.43$$

The graph is an ellipse with vertices at (24, 0) and (3.43, π).



ANSWER:

$$r = \frac{6}{1 - 0.75\cos\theta}$$



12. e = 5; directrix: x = 2

SOLUTION:

Because e = 5, the conic is a hyperbola. The directrix x = 2 is to the right of the pole, so the equation is of the form $r = \dfrac{ed}{1 + e\cos\theta}$ and d = 2. Use the values for e and d to write the equation.

$$r = \frac{ed}{1 + e\cos\theta}$$
$$r = \frac{5(2)}{1 + 5\cos\theta}$$
$$r = \frac{10}{1 + 5\cos\theta}$$

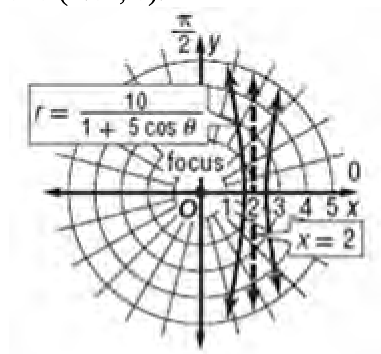
Evaluate the function for several θ-values in its domain and use these points to graph the function and its directrix.

Substitute 0 and π for θ to determine the location of the vertices of the hyperbola.

$$r = \frac{10}{1+5\cos\theta}$$
$$r = \frac{10}{1+5\cos 0}$$
$$r = \frac{10}{1+(5\cdot 1)}$$
$$r = \frac{10}{1+5}$$
$$r = \frac{10}{6}$$
$$r = \frac{5}{3}$$
$$r \approx 1.67$$

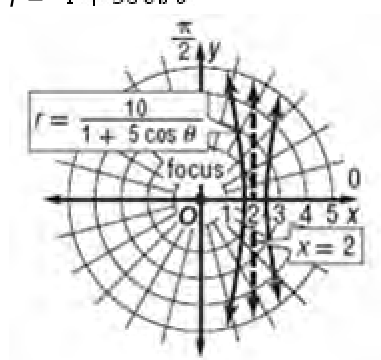
$$r = \frac{10}{1+5\cos\theta}$$
$$r = \frac{10}{1+5\cos\pi}$$
$$r = \frac{10}{1+(5\cdot -1)}$$
$$r = \frac{10}{1-5}$$
$$r = \frac{10}{-4}$$
$$r = -\frac{5}{2}$$
$$r = -2.5$$

The graph is a hyperbola with vertices at (−2.5, π) and (1.67, 0).



ANSWER:

$$r = \frac{10}{1 + 5\cos\theta}$$



13. e = 0.1; directrix: y = 8

SOLUTION:

Because e = 0.1, the conic is an ellipse. The directrix y = 8 is above the pole, so the equation is of the form $r = \dfrac{ed}{1 + e\sin\theta}$ and d = 8. Use the values for e and d to write the equation.

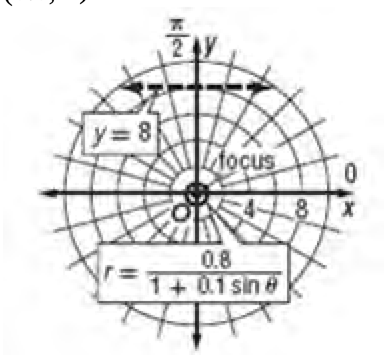
$$r = \frac{ed}{1 + e\sin\theta}$$
$$r = \frac{0.1(8)}{1 + 0.1\sin\theta}$$
$$r = \frac{0.8}{1 + 0.1\sin\theta}$$

Evaluate the function for several θ-values in its domain and use these points to graph the function and its directrix.

Substitute 0 and π for θ to determine the location of the vertices of the ellipse.

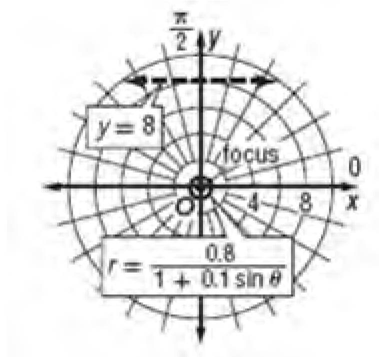
$$r = \frac{0.8}{1+0.1\sin\theta}$$
$$r = \frac{0.8}{1+0.1\sin 0}$$
$$r = \frac{0.8}{1+0.1(0)}$$
$$r = \frac{0.8}{1}$$
$$r = 0.8$$
$$r = \frac{0.8}{1+0.1\sin\theta}$$
$$r = \frac{0.8}{1+0.1\sin\pi}$$
$$r = \frac{0.8}{1+0.1(0)}$$
$$r = \frac{0.8}{1}$$
$$r = 0.8$$

The graph is an ellipse with vertices at (0.8, 0) and (0.8, π).



ANSWER:

$$r = \frac{0.8}{1+0.1\sin\theta}$$



14. $e = 6$; directrix: $y = -7$

SOLUTION:

Because $e = 6$, the conic is a hyperbola. The directrix $x = -7$ is below the pole, so the equation is of the form $r = \frac{ed}{1 - e\sin\theta}$ and $d = 7$. Use the values for e and d to write the equation.

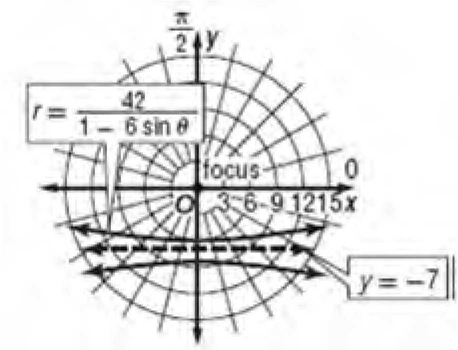
$$r = \frac{ed}{1 - e\sin\theta}$$
$$r = \frac{6(7)}{1 - 6\sin\theta}$$
$$r = \frac{42}{1 - 6\sin\theta}$$

Evaluate the function for several θ-values in its domain and use these points to graph the function and its directrix.

Substitute 0 for θ to determine the location of the vertex of the hyperbola.

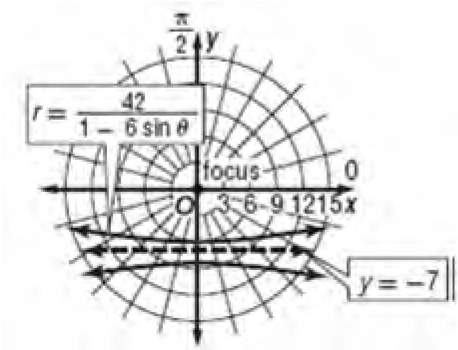
$$r = \frac{42}{1-6\sin\theta}$$
$$r = \frac{42}{1-6\sin 0}$$
$$r = \frac{42}{1-6(0)}$$
$$r = \frac{42}{1}$$
$$r = 42$$

The graph is an hyperbola with a vertex at (42, 0).



ANSWER:

$r = \dfrac{42}{1 - 6\sin \theta}$



15. $e = 1$; directrix: $x = -1.5$

SOLUTION:

Because $e = 1$, the conic is a parabola. The directrix $x = -1.5$ is to the left of the pole, so the equation is of the form $r = \dfrac{ed}{1 - e\cos\theta}$ and $d = 1.5$. Use the values for e and d to write the equation.

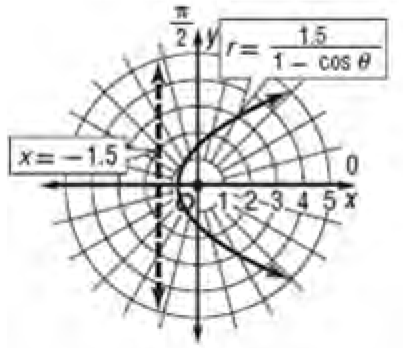
$$r = \frac{ed}{1 - e\cos\theta}$$
$$r = \frac{1(1.5)}{1 - \cos\theta}$$
$$r = \frac{1.5}{1 - \cos\theta}$$

Evaluate the function for several θ-values in its domain and use these points to graph the function and its directrix.

Substitute π for θ to determine the location of the vertex of the parabola.

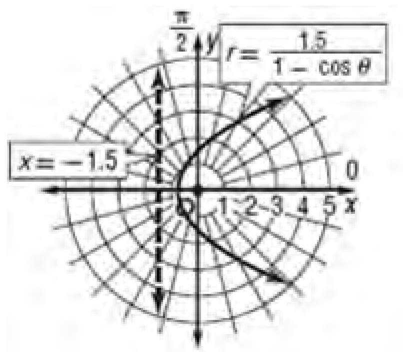
$$r = \frac{1.5}{1 - \cos\theta}$$
$$r = \frac{1.5}{1 - \cos\pi}$$
$$r = \frac{1.5}{1 - (-1)}$$
$$r = \frac{1.5}{2}$$
$$r = 0.75$$

The graph is a parabola with vertex at $(0.75, \pi)$.



ANSWER:

$r = \dfrac{1.5}{1 - \cos\theta}$



16. $e = 0.8$; vertices at $(-36, 0)$ and $(4, 0)$

SOLUTION:

Because $e = 0.8$, the conic is an ellipse. The center of the ellipse is at $(-16, 0)$. This point is to the left of the pole. Therefore, the directirx will be to the right the pole at $x = d$. The polar equation of a conic with this directrix is $r = \dfrac{ed}{1 + e\cos\theta}$. Use the value of e and the polar form of a point on the conic to find the value of d. The vertex point $(4, 0)$ has polar coordinates $(r, \theta) = \left(\sqrt{4^2 + 0^2}, \tan^{-1}\left(\frac{0}{4}\right)\right)$ or $(4, 0)$.

$$r = \frac{ed}{1+e\cos\theta}$$
$$4 = \frac{0.8d}{1+0.8\cos(0)}$$
$$4 = \frac{0.8d}{1.8}$$
$$9 = d$$

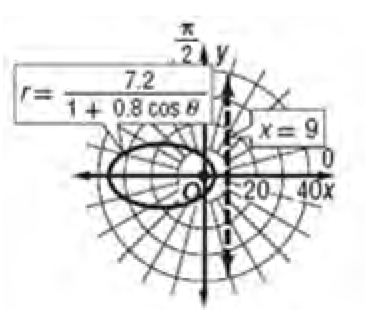
Therefore, the equation for the ellipse is $r = \frac{7.2}{1+0.8\cos\theta}$. Because d = 9, the equation of the directrix is x = 9.

Evaluate the function for several θ-values in its domain and use these points to graph the function and its directrix.

Substitute 0 and π for θ to determine the location of the vertices of the ellipse.

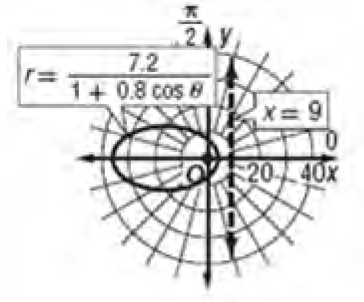
$$r = \frac{7.2}{1+0.8\cos\theta}$$
$$r = \frac{7.2}{1+0.8\cos 0}$$
$$r = \frac{7.2}{1+0.8(1)}$$
$$r = \frac{7.2}{1.8}$$
$$r = 4$$

$$r = \frac{7.2}{1+0.8\cos\theta}$$
$$r = \frac{7.2}{1+0.8\cos\pi}$$
$$r = \frac{7.2}{1+0.8(-1)}$$
$$r = \frac{7.2}{0.2}$$
$$r = 36$$



ANSWER:

$$r = \frac{7.2}{1+0.8\cos\theta}$$



17. e = 1.5; vertices at (–3, 0) and (–15, 0)

SOLUTION:

Because e = 1.5, the conic is a hyperbola. The center of the hyperbola is at (−9, 0), the midpoint of the segment between the given vertices. This point is to the left of the pole. Therefore, the directrix is to the left of the pole at $x = -d$. The polar equation of a conic with this directrix is $r = \frac{ed}{1-e\cos\theta}$. Use the value of e and the polar form of a point on the conic to find the value of d. The vertex point (−3, 0) has polar coordinates (r, θ) = $\left(\sqrt{(-3)^2+0^2}, \tan^{-1}\frac{0}{-3}+\pi\right)$ or (3, π).

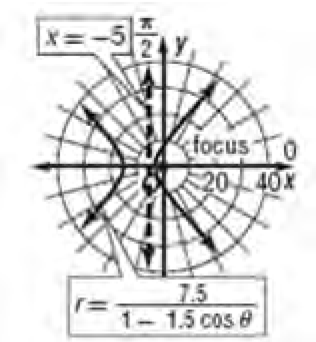
$$r = \frac{ed}{1-e\cos\theta}$$
$$3 = \frac{1.5d}{1-1.5\cos\pi}$$
$$3 = \frac{1.5d}{2.5}$$
$$5 = d$$

Therefore, the equation for the hyperbola is $r = \frac{7.5}{1-1.5\cos\theta}$. Because d = 5, the equation of the directrix is x = −5. Evaluate the function for several θ-values in its domain and use these points to graph the function and its directrix.

Substitute 0 and π for θ to determine the location of the vertices of the hyperbola.

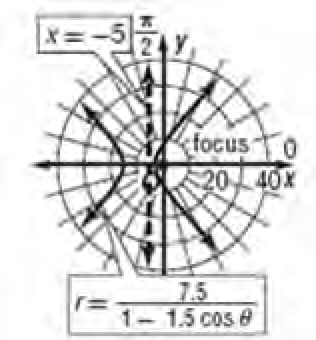
9-4 Polar Forms of Conic Sections

$$r = \frac{7.5}{1 - 1.5\cos\theta}$$
$$r = \frac{7.5}{1 - 1.5\cos 0}$$
$$r = \frac{7.5}{1 - 1.5(1)}$$
$$r = \frac{7.5}{-0.5}$$
$$r = -15$$
$$r = \frac{7.5}{1 - 1.5\cos\theta}$$
$$r = \frac{7.5}{1 - 1.5\cos\pi}$$
$$r = \frac{7.5}{1 - 1.5(-1)}$$
$$r = \frac{7.5}{2.5}$$
$$r = 3$$



ANSWER:

$$r = \frac{7.5}{1 - 1.5\cos\theta}$$



Write each polar equation in rectangular form.

18. $r = \dfrac{4.8}{1 + \sin\theta}$

SOLUTION:

The equation is in standard form.

$$r = \frac{4.8}{1 + \sin\theta}$$

For this equation, e = 1 and d = 4.8. The eccentricity and form of the equation determine that this is a parabola that opens vertically with focus at the pole and a directrix y = 4.8. The general equation of such a parabola in rectangular form is

$(x - h)^2 = -4p(y - k)$.

The vertex lies between the focus F and the directrix of the parabola, occurring when $\theta = \frac{\pi}{2}$.

$$r = \frac{4.8}{1 + \sin\theta}$$
$$r = \frac{4.8}{1 + \sin\frac{\pi}{2}}$$
$$r = \frac{4.8}{1 + 1}$$
$$r = 2.4$$

The vertex lies at polar coordinate $\left(2.4, \frac{\pi}{2}\right)$, which corresponds to rectangular coordinate $(0, 2.4)$. So $(h, k) = (0, 2.4)$.

The distance p from the vertex at $(0, 2.4)$ to the focus at $(0, 0)$ is 2.4.

Substitute the values for h, k, and p into the general equation for rectangular form.

$$(x - h)^2 = -4p(y - k)$$
$$(x - 0)^2 = -4(2.4)(y - 2.4)$$
$$x^2 = -9.6(y - 2.4)$$

ANSWER:

$x^2 = -9.6\,(y - 2.4)$

19. $r = \dfrac{30}{4 + \cos\theta}$

SOLUTION:

Write the equation in standard form.

$$r = \frac{30}{4+\cos\theta}$$
$$r = \frac{4(7.5)}{4(1+0.25\cos\theta)}$$
$$r = \frac{7.5}{1+0.25\cos\theta}$$

For this equation, e = 0.25 and d = 30. The eccentricity and form of the equation determine that this is an ellipse with directrix x = 30.
Therefore, the major axis of the ellipse lies along the polar or x-axis. The general equation of such an ellipse in rectangular form is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

The vertices are the endpoints of the major axis and occur when θ = 0 and π.

$$r = \frac{30}{4+\cos\theta}$$
$$r = \frac{30}{4+\cos 0}$$
$$r = \frac{30}{4+1}$$
$$r = 6$$

$$r = \frac{30}{4+\cos\theta}$$
$$r = \frac{30}{4+\cos\pi}$$
$$r = \frac{30}{4-1}$$
$$r = 10$$

The vertices have polar coordinates (6, 0) and (10, π), which correspond to rectangular coordinates (6, 0) and (−10, 0). The ellipse's center is the midpoint of the segment between the vertices, so (h, k) = (−2, 0).

The distance a between the center and each vertex is 8. The distance c from the center to the focus at (0, 0) is 2.

$$b = \sqrt{8^2 - 2^2} = \sqrt{60}$$

Substitute the values for h, k, a, and b into the standard form of an equation for an ellipse.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
$$\frac{[x-(-2)]^2}{8^2} + \frac{(y-0)^2}{\left(\sqrt{60}\right)^2} = 1$$
$$\frac{(x+2)^2}{64} + \frac{y^2}{60} = 1$$

ANSWER:

$$\frac{(x+2)^2}{64} + \frac{y^2}{60} = 1$$

20. $r = \frac{5}{1-1.5\cos\theta}$

SOLUTION:

Write the equation in standard form.

$$r = \frac{5}{1-1.5\cos\theta}$$
$$r = \frac{1.5(3\frac{1}{3})}{1-1.5\cos\theta}$$
$$r = \frac{5}{1-1.5\cos\theta}$$

For this equation, e = 1.5 and d is about 3.3. The eccentricity and form of the equation determine that this is a hyperbola with directrix x = −3.3.
Therefore, the transverse axis of the hyperbola lies along the polar or x-axis. The general equation of such a hyperbola in rectangular form is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.

The vertices lie on the transverse axis and occur when θ = 0 and π.

$$r = \frac{5}{1-1.5\cos\theta}$$
$$r = \frac{5}{1-1.5\cos 0}$$
$$r = \frac{5}{-0.5}$$
$$r = -10$$

$$r = \frac{5}{1 - 1.5\cos\theta}$$
$$r = \frac{5}{1 - 1.5\cos\pi}$$
$$r = \frac{5}{2.5}$$
$$r = 2$$

The vertices have polar coordinates (-10, 0) and (2, π), which correspond to rectangular coordinates (-10, 0) and (–2, 0). The hyperbola's center is the midpoint of the segment between the vertices, so (h, k) = (-6, 0).

The distance a between the center and each vertex is 4. The distance c from the center to the focus at (0, 0) is 6.

$$b = \sqrt{36 - 16} = \sqrt{20}$$

Substitute the values for h, k, a, and b into the standard form of an equation for an ellipse.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$
$$\frac{[x-(-6)]^2}{4^2} - \frac{(y-0)^2}{\left(\sqrt{20}\right)^2} = 1$$
$$\frac{(x+6)^2}{16} - \frac{y^2}{20} = 1$$

ANSWER:

$$\frac{(x+6)^2}{16} - \frac{y^2}{20} = 1$$

21. $r = \dfrac{5.1}{1 + 0.7\sin\theta}$

SOLUTION:

The equation is in standard form.

$$r = \frac{5.1}{1 + 0.7\sin\theta}$$

For this equation, e = 0.7 and d = 5.1 ÷ 0.7 or about 7.3. The eccentricity and form of the equation determine that this is an ellipse with directrix y = 7.3. The general equation of such an ellipse in rectangular form is $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$.

The vertices are the endpoints of the major axis and occur when $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

$$r = \frac{5.1}{1 + 0.7\sin\theta}$$
$$r = \frac{5.1}{1 + 0.7\sin\frac{\pi}{2}}$$
$$r = \frac{5.1}{1 + 0.7}$$
$$r = 4$$

$$r = \frac{5.1}{1 + 0.7\sin\theta}$$
$$r = \frac{5.1}{1 + 0.7\sin\frac{3\pi}{2}}$$
$$r = \frac{5.1}{1 - 0.7}$$
$$r = 17$$

The vertices lies at polar coordinates $\left(3, \frac{\pi}{2}\right)$ and $\left(17, \frac{3\pi}{2}\right)$, which correspond to rectangular coordinates (0, 3) and (0, –17).

The ellipse's center is the midpoint of the segment between the vertices, so (h, k) = (0, –7). The distance a between the center and each vertex is 10. The distance c from the center to the focus at (0, 0) is 7.

$b = \sqrt{10^2 - 7^2}$ or $\sqrt{51}$.

Substitute the values for h, k, a, and b into the standard form of an equation for an ellipse.

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$
$$\frac{[x-0]^2}{\left(\sqrt{51}\right)^2} + \frac{[y-(-7)]^2}{10^2} = 1$$
$$\frac{x^2}{51} + \frac{(y+7)^2}{100} = 1$$

ANSWER:

$$\frac{x^2}{51} + \frac{(y+7)^2}{100} = 1$$

9-4 Polar Forms of Conic Sections

22. $r = \dfrac{12}{1 - \cos\theta}$

SOLUTION:

The equation is in standard form.

$$r = \frac{12}{1 - \cos\theta}$$

For this equation, $e = 1$ and $d = 12$. The eccentricity and form of the equation determine that this is a parabola that opens horizontally with focus at the pole and directrix $x = -12$.
The general equation of such a parabola in rectangular form is $(y - k)^2 = 4p(x - h)$.
The vertex lies between the focus F and the directrix of the parabola, occurring when $\theta = \pi$.

$$r = \frac{12}{1 - \cos\theta}$$
$$r = \frac{12}{1 - \cos\pi}$$
$$r = \frac{12}{1 - (-1)}$$
$$r = 6$$

The vertex lies at polar coordinate $(6, \pi)$, which correspond to rectangular coordinates $(-6, 0)$. So $(h, k) = (-6, 0)$.
The distance p from the vertex at $(-6, 0)$ to the focus at $(0, 0)$ is 6.

Substitute the values for h, k, and p into the general equation for rectangular form.

$$(y - k)^2 = 4p(x - h)$$
$$(y - 0)^2 = 4(6)[x - (-6)]$$
$$y^2 = 24(y + 6)$$

ANSWER:

$y^2 = 24(x + 6)$

23. $r = \dfrac{6}{0.25 - 0.75\sin\theta}$

SOLUTION:

Write the equation in standard form.

$$r = \frac{6}{0.25 - 0.75\sin\theta}$$
$$r = \frac{6}{0.25(1 - 3\sin\theta)}$$
$$r = \frac{24}{1 - 3\sin\theta}$$

For this equation, $e = 3$ and $d = 24 \div 3$ or 8. The eccentricity and form of the equation determine that this is a hyperbola with directrix $y = -8$. Therefore, the transverse axis of the hyperbola lies along the line $\theta = \frac{\pi}{2}$ or y-axis.
The general equation of such a hyperbola in rectangular form is $\dfrac{(y - h)^2}{a^2} - \dfrac{(x - k)^2}{b^2} = 1$.
The vertices lie on the transverse axis and occur when $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

$$r = \frac{24}{1 - 3\sin\theta}$$
$$r = \frac{24}{1 - 3\sin\frac{\pi}{2}}$$
$$r = \frac{24}{1 - 3}$$
$$r = -12$$

$$r = \frac{24}{1 - 3\sin\theta}$$
$$r = \frac{24}{1 - 3\sin\frac{3\pi}{2}}$$
$$r = \frac{24}{1 - (-3)}$$
$$r = 6$$

The vertices have polar coordinates $\left(-12, \frac{\pi}{2}\right)$ and $\left(6, \frac{3\pi}{2}\right)$, which correspond to rectangular coordinates (0, –12) and (0, 6). The hyperbola's center is the midpoint of the segment between the vertices, so $(h, k) = (0, -9)$.
The distance a between the center and each vertex is 3. The distance c from the center to the focus at (0, 0) is 9.
$b = \sqrt{9^2 - 3^2}$ or $\sqrt{72}$.

Substitute the values for h, k, a, and b into the standard form of an equation for an ellipse.

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$
$$\frac{[y-(-9)]^2}{3^2}-\frac{(x-0)^2}{(\sqrt{72})^2}=1$$
$$\frac{(y+9)^2}{9}-\frac{x^2}{72}=1$$

ANSWER:

$$\frac{(y+9)^2}{9}-\frac{x^2}{72}=1$$

24. $r=\frac{4.5}{1+1.25\sin\theta}$

SOLUTION:

The equation is in standard form.

$$r=\frac{4.5}{1+1.25\sin\theta}$$

For this equation, $e = 1.25$ and $d = 4.5 \div 1.25$ or 3.6. The eccentricity and form of the equation determine that this is a hyperbola with directrix $y = 3.6$. Therefore, the transverse axis of the hyperbola lies along the line $\theta = \frac{\pi}{2}$ or y-axis.

The general equation of such a hyperbola in rectangular form is $\frac{(y-h)^2}{a^2}-\frac{(x-k)^2}{b^2}=1$.

The vertices lie on the transverse axis and occur when $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

$$r=\frac{4.5}{1+1.25\sin\theta}$$
$$r=\frac{4.5}{1+1.25\sin\frac{\pi}{2}}$$
$$r=\frac{4.5}{1+1.25}$$
$$r=2$$

$$r=\frac{4.5}{1+1.25\sin\theta}$$
$$r=\frac{4.5}{1+1.25\sin\frac{3\pi}{2}}$$
$$r=\frac{4.5}{1-1.25}$$
$$r=-18$$

The vertices have polar coordinates $\left(2, \frac{\pi}{2}\right)$ and $\left(-18, \frac{3\pi}{2}\right)$, which correspond to rectangular coordinates (0, 2) and (0, 18). The hyperbola's center is the midpoint of the segment between the vertices, so $(h, k) = (0, 10)$.

The distance a between the center and each vertex is 8. The distance c from the center to the focus at (0, 0) is 10.

$b = \sqrt{10^2-8^2}$ or 6.

Substitute the values for h, k, a, and b into the standard form of an equation for an ellipse.

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$
$$\frac{(y-10)^2}{8^2}-\frac{(x-0)^2}{6^2}=1$$
$$\frac{(y-10)^2}{64}-\frac{x^2}{36}=1$$

ANSWER:

$$\frac{(y-10)^2}{64}-\frac{x^2}{36}=1$$

25. $r=\frac{8.4}{1-0.4\cos\theta}$

SOLUTION:

The equation is in standard form.

$$r=\frac{8.4}{1-0.4\cos\theta}$$

For this equation, $e = 0.4$ and $d = 8.4 \div 0.4$ or 21. The eccentricity and form of the equation determine that this is an ellipse with directrix $x = -21$.

The general equation of such an ellipse in rectangular form is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$.

The vertices are the endpoints of the major axis and occur when $\theta = 0$ and π.

$$r=\frac{8.4}{1-0.4\cos\theta}$$
$$r=\frac{8.4}{1-0.4\cos 0}$$
$$r=\frac{8.4}{1-0.4}$$
$$r=14$$

$$r = \frac{8.4}{1 - 0.4\cos\theta}$$
$$r = \frac{8.4}{1 - 0.4\cos\pi}$$
$$r = \frac{8.4}{1 + 0.4}$$
$$r = 6$$

The vertices have polar coordinates $(14, 0)$ and $(6, \pi)$, which correspond to rectangular coordinates (14, 0) and (–6, 0). The ellipse's center is the midpoint of the segment between the vertices, so $(h, k) = (4, 0)$. The distance a between the center and each vertex is 10. The distance c from the center to the focus at (0, 0) is 4.
$b = \sqrt{10^2 - 4^2}$ or $\sqrt{84}$.

Substitute the values for h, k, a, and b into the standard form of an equation for an ellipse.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
$$\frac{(x-4)^2}{10^2} + \frac{(y-0)^2}{(\sqrt{84})^2} = 1$$
$$\frac{(x-4)^2}{100} + \frac{y^2}{84} = 1$$

ANSWER:

$$\frac{(x-4)^2}{100} + \frac{y^2}{84} = 1$$

GRAPHING CALCULATOR Determine the type of conic for each polar equation. Then graph each equation.

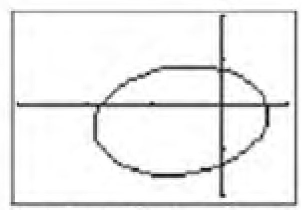
26. $r = \dfrac{2}{2 + \sin\left(\theta + \frac{\pi}{3}\right)}$

SOLUTION:

Write the equation in standard form, $r = \dfrac{ed}{1 + e\sin\theta}$.

$$r = \frac{2}{2 + \sin\left(\theta + \frac{\pi}{3}\right)}$$
$$= \frac{2(1)}{2\left[1 + 0.5\sin\left(\theta + \frac{\pi}{3}\right)\right]}$$
$$= \frac{1}{1 + 0.5\sin\left(\theta + \frac{\pi}{3}\right)}$$

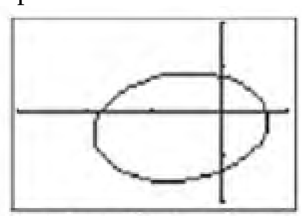
Since $e = 0.5$, the conic is an ellipse. Graph the polar equation using a graphing calculator.



$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-3, 1]$ scl: 1 by $[-2, 2]$ scl: 1

ANSWER:

ellipse



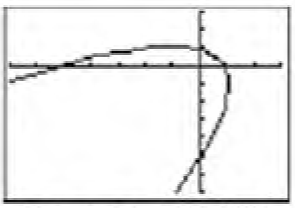
$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-3, 1]$ scl: 1 by $[-2, 2]$ scl: 1

27. $r = \dfrac{3}{1 + \cos\left(\theta - \dfrac{\pi}{4}\right)}$

SOLUTION:

The equation $r = \dfrac{3}{1 + \cos\left(\theta - \dfrac{\pi}{4}\right)}$ is in standard form, $r = \dfrac{ed}{1 + e\cos\theta}$.

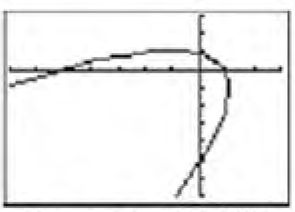
Since $e = 1$, the conic is a parabola. Graph the polar equation using a graphing calculator.



$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-14, 6]$ scl: 2 by $[-14, 6]$ scl: 2

ANSWER:

parabola



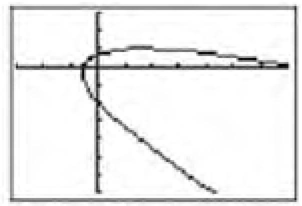
$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-14, 6]$ scl: 2 by $[-14, 6]$ scl: 2

28. $r = \dfrac{2}{1 - \cos\left(\theta + \dfrac{\pi}{6}\right)}$

SOLUTION:

The equation $r = \dfrac{2}{1 - \cos\left(\theta + \dfrac{\pi}{6}\right)}$ is in standard form, $r = \dfrac{ed}{1 - e\cos\theta}$.

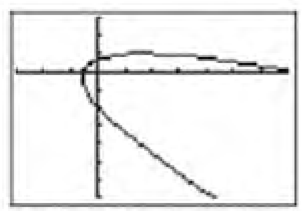
Since $e = 1$, the conic is a parabola. Graph the polar equation using a graphing calculator.



$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-6, 14]$ scl: 2 by $[-14, 6]$ scl: 2

ANSWER:

parabola

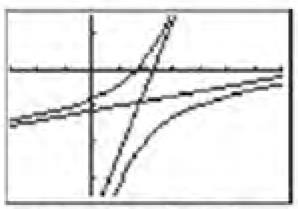


$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-6, 14]$ scl: 2 by $[-14, 6]$ scl: 2

29. $r = \dfrac{4}{1 + 2\sin\left(\theta + \frac{3\pi}{4}\right)}$

SOLUTION:

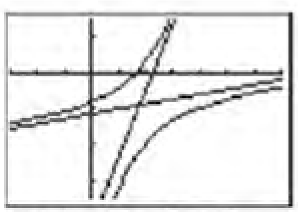
The equation $r = \dfrac{4}{1 + 2\sin\left(\theta + \frac{3\pi}{4}\right)}$ is in standard form, $r = \dfrac{ed}{1 + e\sin\theta}$.
Since e = 2, the conic is a hyperbola. Graph the polar equation using a graphing calculator.



$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-3, 7]$ scl: 1 by $[-7, 3]$ scl: 1

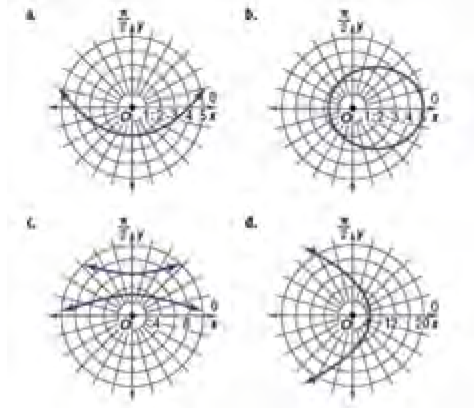
ANSWER:

hyperbola



$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-3, 7]$ scl: 1 by $[-7, 3]$ scl: 1

Match each polar equation with its graph.



30. $r = \dfrac{10}{1 + \cos\theta}$

SOLUTION:

The equation $r = \dfrac{10}{1 + \cos\theta}$ is in standard form, $r = \dfrac{ed}{1 + e\cos\theta}$.
Since e = 1, the conic is a parabola. For a polar equation of this form, the equation of the directrix is $x = d$ and the parabola will open to the left. Substituting 0 for θ yields a vertex of (5, 0).

Therefore, the answer is d.

ANSWER:

d

31. $r = \dfrac{4}{1 - \sin\theta}$

SOLUTION:

The equation $r = \dfrac{4}{1 - \sin\theta}$ is in standard form, $r = \dfrac{ed}{1 - e\sin\theta}$.
Since e = 1, the conic is a parabola. For a polar equation of this form, the equation of the directrix is $y = -d$ and the parabola will open up.

Substituting $\frac{3\pi}{2}$ for θ yields a vertex of $\left(2, \frac{3\pi}{2}\right)$.

Therefore, the answer is a.

ANSWER:

a

32. $r = \dfrac{5}{2 - \cos\theta}$

SOLUTION:

Write the equation in standard form, $r = \dfrac{ed}{1 - e\cos\theta}$.

$$r = \frac{5}{2 - \cos\theta}$$
$$= \frac{2\left(\frac{5}{2}\right)}{2(1 - 0.5\cos\theta)}$$
$$= \frac{\frac{5}{2}}{1 - 0.5\cos\theta}$$

Since $e = 0.5$, the conic is an ellipse. Therefore, the answer is b.

ANSWER:

b

33. $r = \dfrac{12}{1 + 3\sin\theta}$

SOLUTION:

The equation $r = \dfrac{12}{1 + 3\sin\theta}$ is in standard form, $r = \dfrac{ed}{1 + e\sin\theta}$.

Since $e = 3$, the conic is a hyperbola. Therefore, the answer is c.

ANSWER:

c

Determine the eccentricity, type of conic, and equation of the directrix for each polar equation. Then sketch the graph of the equation, and label the directrix.

34. $r = \dfrac{12}{2 - 0.75\cos\theta}$

SOLUTION:

Write the equation in standard form, $r = \dfrac{ed}{1 - e\cos\theta}$.

$$r = \frac{12}{2 - 0.75\cos\theta}$$
$$r = \frac{2(6)}{2(1 - 0.375\cos\theta)}$$
$$r = \frac{6}{1 - 0.375\cos\theta}$$

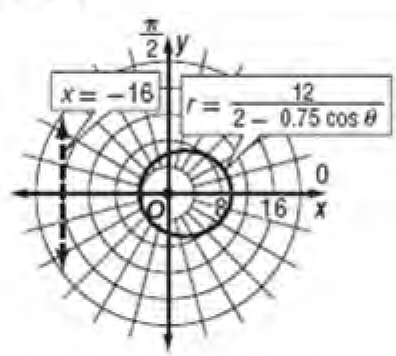
Since $e = 0.375$, the conic is an ellipse. For a polar equation of this form (where $\cos\theta$ is included), the equation of the directrix is $x = -d$. From the numerator, we know that $ed = 6$, so $d = 16$. Therefore, the equation of the directrix is $x = -16$.

Evaluate the function for several θ-values in its domain and use these points to graph the function and its directrix. The graph is an ellipse with vertices at (9.6, 0) and (−4.36, 0).

Substitute 0 and π for θ to determine the location of the vertices of the ellipse.

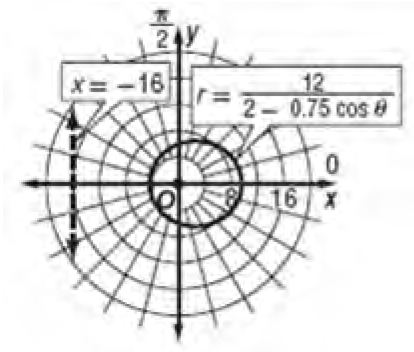
$$r = \frac{6}{1 - 0.375\cos\theta}$$
$$r = \frac{6}{1 - 0.375\cos 0}$$
$$r = \frac{6}{1 - 0.375(1)}$$
$$r = \frac{6}{0.625}$$
$$r = 9.6$$

$$r = \frac{6}{1 - 0.375\cos\theta}$$
$$r = \frac{6}{1 - 0.375\cos\pi}$$
$$r = \frac{6}{1 - 0.375(-1)}$$
$$r = \frac{6}{1.375}$$
$$r \approx 4.36$$



ANSWER:

$e = \frac{3}{8}$; ellipse; $x = -16$



35. $r = \dfrac{1}{0.2 - 0.2\sin\theta}$

SOLUTION:

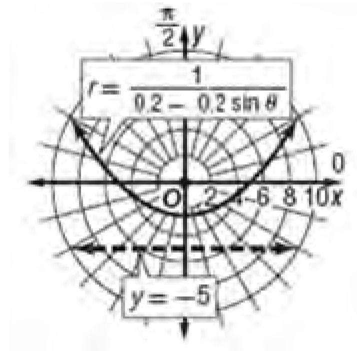
Write the equation in standard form, $r = \dfrac{ed}{1 - e\sin\theta}$.

$$r = \frac{1}{0.2 - 0.2\sin\theta}$$
$$r = \frac{0.2(5)}{2(1 - \sin\theta)}$$
$$r = \frac{5}{1 - \sin\theta}$$

Since e = 1, the conic is a parabola. For a polar equation of this form (where sinθ is included), the equation of the directrix is $y = -d$. From the numerator, we know that ed = 5, so d = 5. Therefore, the equation of the directrix is $y = -5$.

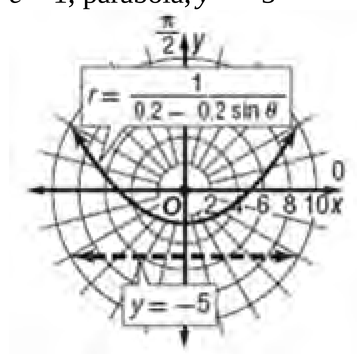
Evaluate the function for several θ-values in its domain and use these points to graph the function and its directrix.

$$r = \frac{1}{0.2 - 0.2\sin\theta}$$
$$r = \frac{1}{0.2 - 0.2\sin 0}$$
$$r = \frac{1}{0.2 - 0.2(0)}$$
$$r = \frac{1}{0.2}$$
$$r = 5$$



ANSWER:

e = 1; parabola; $y = -5$



36. $r = \dfrac{6}{1.2\sin\theta + 0.3}$

SOLUTION:

Write the equation in standard form, $r = \dfrac{ed}{1 + e\sin\theta}$.

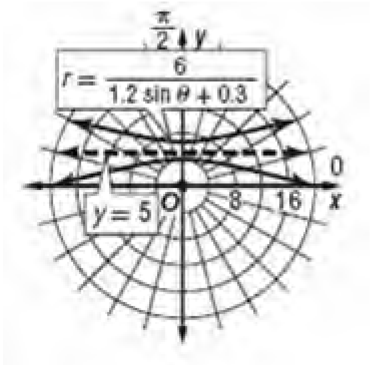
$$r = \frac{6}{1.2\sin\theta + 0.3}$$
$$r = \frac{0.3(20)}{0.3(1 + 4\sin\theta)}$$
$$r = \frac{20}{1 + 4\sin\theta}$$

Since e = 4, the conic is a hyperbola. For a polar equation of this form (where sinθ is included), the equation of the directrix is $y = d$. From the numerator, we know that ed = 20, so d = 5. Therefore, the equation of the directrix is $y = 5$.

Evaluate the function for several θ-values in its domain and use these points to graph the function and its directrix.

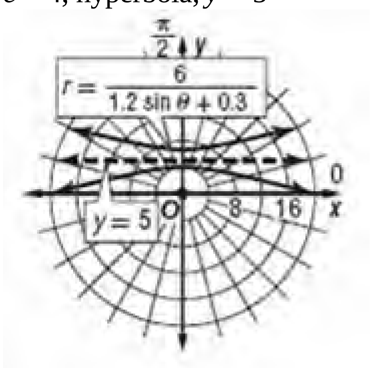
$$r = \frac{20}{1+4\sin\theta} \qquad r = \frac{20}{1+4\sin\theta}$$
$$r = \frac{20}{1+4\sin 0} \qquad r = \frac{20}{1+4\sin\frac{\pi}{2}}$$
$$r = \frac{20}{1+0} \qquad r = \frac{20}{1+4}$$
$$r = 20 \qquad r = 4$$

$$r = \frac{20}{1+4\sin\theta} \qquad r = \frac{20}{1+4\sin\theta}$$
$$r = \frac{20}{1+4\sin\pi} \qquad r = \frac{20}{1+4\sin\frac{3\pi}{2}}$$
$$r = \frac{20}{1+0} \qquad r = \frac{20}{1+(-4)}$$
$$r = 20 \qquad r = -6.67$$



ANSWER:

e = 4; hyperbola; y = 5



37. $r = \frac{8}{\cos\theta + 5}$

SOLUTION:

Write the equation in standard form, $r = \frac{ed}{1+e\cos\theta}$.

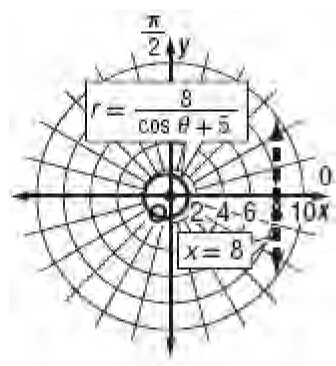
$$r = \frac{8}{\cos\theta + 5}$$
$$r = \frac{5(1.6)}{5(1+0.2\cos\theta)}$$
$$r = \frac{1.6}{1+0.2\cos\theta}$$

Since e = 0.2, the conic is an ellipse. For a polar equation of this form (where cosθ is included), the equation of the directrix is $x = d$. From the numerator, we know that ed = 1.6, so d = 8. Therefore, the equation of the directrix is x = 8.

Evaluate the function for several θ-values in its domain and use these points to graph the function and its directrix.

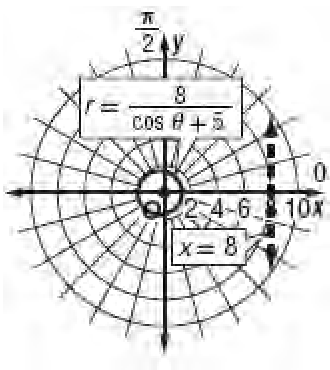
Substitute 0 and π for θ to determine the location of the vertices of the ellipse.

$$r = \frac{1.6}{1+0.2\cos\theta} \qquad r = \frac{1.6}{1+0.2\cos\theta}$$
$$r = \frac{1.6}{1+0.2\cos 0} \qquad r = \frac{1.6}{1+0.2\cos\pi}$$
$$r = \frac{1.6}{1+0.2} \qquad r = \frac{1.6}{1+0.2(-1)}$$
$$r = 1.33 \qquad r = 2$$



ANSWER:

$e = \frac{1}{5}$; ellipse; x = 8



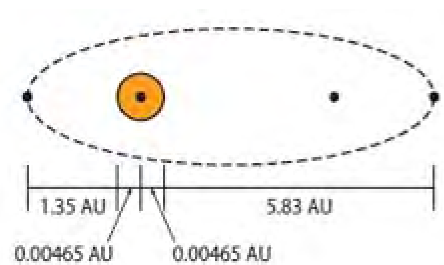
38. **ASTRONOMY** The comet Borrelly travels in an elliptical orbit around the Sun with eccentricity e = 0.624. The point in a comet's orbit nearest to the Sun is defined as the *perihelion*, while the farthest point from the Sun is defined as the *aphelion*. The aphelion occurs at a distance of 5.83 AU (astronomical units, based on the distance between Earth and the Sun) from the Sun and the perihelion occurs at a distance of 1.35 AU. The diameter of the Sun is about 0.0093 AU.

a. Write a polar equation for the elliptical orbit of the comet Borrelly, and graph the equation.

b. Determine the distance in miles between the comet Borrelly and the Sun at the aphelion and perihelion if 1 AU ≈ 93 million miles.

SOLUTION:

a. Sample answer: Let the center of the Sun, located at the origin, be a focus of the elliptical orbit. Since the diameter of the sun is about 0.0093 AU, 0.0093 ÷ 2 or 0.00465 AU must be added to each distance to find the total distance that the comet is from the center of the Sun. Diagram the situation.



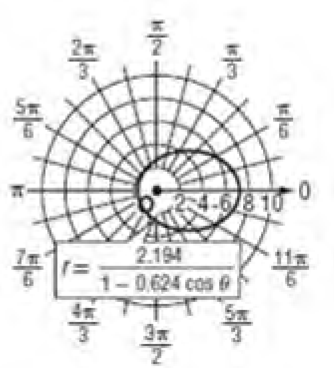
Since the second focus is located to the right of the origin, the equation will be of the form $r = \frac{ed}{1 - e\cos\theta}$. The aphelion occurs at the point (5.83465, 0), which has polar coordinates of (5.83465, 0). Substitute this point and the value of e into the equation to solve for d.

$$r = \frac{ed}{1 - e\cos\theta}$$
$$5.83465 = \frac{0.624d}{1 - 0.624\cos 0}$$
$$5.83465 = \frac{0.624d}{0.376}$$
$$3.516 \approx d$$

Substitute the values for d and e into the polar equation.

$$r = \frac{ed}{1 - e\cos\theta}$$
$$r = \frac{0.624(3.516)}{1 - 0.624\cos\theta}$$
$$r = \frac{2.194}{1 - 0.624\cos\theta}$$

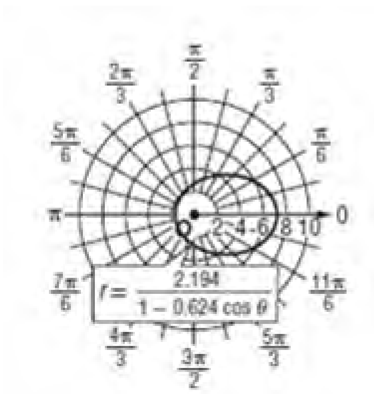
Evaluate the function for several θ-values in its domain and use these points to graph the function.



b. The comet is 5.83 AU from the Sun at the aphelion. To determine the distance in millions of miles, multiply this distance by 93. So, the comet is 5.83 · 93 or 542.19 million miles from the Sun at the aphelion. The comet is 1.35 · 93 or 125.55 million miles from the Sun at the perihelion.

ANSWER:

a. Sample answer: $r = \frac{2.194}{1 - 0.624\cos\theta}$



b. aphelion: 542.19 million miles; perihelion: 125.55 million miles

PROOF Prove each of the following.

39. $b = a\sqrt{1 - e^2}$ for an ellipse

SOLUTION:

b^2	$= a^2 - c^2$	a, b, c relationship
b^2	$= a^2 - (ae)^2$	Since $e = \frac{c}{a}$, $c = ae$.
b^2	$= a^2 - a^2e^2$	Simplify.
b^2	$= a^2(1 - e^2)$	Factor a^2 from each term.
b	$= a\sqrt{1 - e^2}$	Take the square root of each side.

ANSWER:

$$\begin{aligned} b^2 &= a^2 - c^2 \\ b^2 &= a^2 - (ae)^2 \\ b^2 &= a^2 - a^2e^2 \\ b^2 &= a^2(1 - e^2) \\ b &= a\sqrt{1 - e^2} \end{aligned}$$

40. $b = a\sqrt{e^2 - 1}$ for a hyperbola

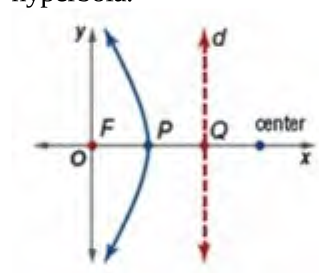
SOLUTION:

b^2	$= c^2 - a^2$	a, b, c relationship
b^2	$= (ae)^2 - a^2$	Since $e = \frac{c}{a}$, $c = ae$.
b^2	$= a^2e^2 - a^2$	Simplify.
b^2	$= a^2(e^2 - 1)$	Factor a^2 from each term.
b	$= a\sqrt{e^2 - 1}$	Take the square root of each side.

ANSWER:

$$\begin{aligned} b^2 &= c^2 - a^2 \\ b^2 &= (ae)^2 - a^2 \\ b^2 &= a^2e^2 - a^2 \\ b^2 &= a^2(e^2 - 1) \\ b &= a\sqrt{e^2 - 1} \end{aligned}$$

41. **PROOF** Use the definition for the eccentricity of a conic, $PF = ePQ$, and the drawing of the hyperbola shown below, to verify that $d = \frac{a(e^2 - 1)}{e}$ for any hyperbola.

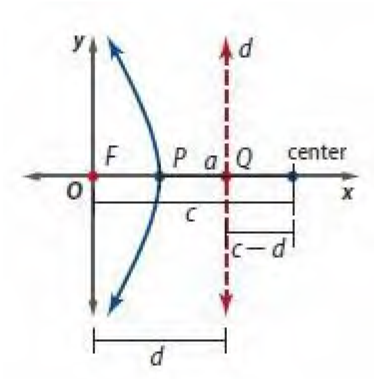


SOLUTION:

a = the distance from a vertex to the center of the hyperbola
c = the the distance from the focus to the center
d = the distance from the origin (in this case, the focus) to the directrix

PF = the distance from the focus to the vertex = $c - a$

$c - d$ = the distance from the directrix to the center
PQ = the distance from the focus to the directrix = $a - (c - d)$



$PF = ePQ$	
$c - a = e[a - (c - d)]$	$PF = c - a$ and $PQ = a - (c - d)$
$c - a = e(a - c + d)$	Simplify.
$ae - a = e(a - ae + d)$	$c = ae$
$ae - a = ae - ae^2 + de$	Distribute.
$-a = -ae^2 + de$	Subtract ae from each side.
$ae^2 - a = de$	Add ae^2 to each side.
$\frac{ae^2 - a}{e} = d$	Divide each side by d.
$\frac{a(e^2 - 1)}{e} = d$	Factor a from the numerator.

ANSWER:

$$\begin{aligned} PF &= ePQ \\ c - a &= e[a - (c - d)] \\ c - a &= e(a - c + d) \\ ae - a &= e(a - ae + d) \\ ae - a &= ae - ae^2 + de \\ -a &= -ae^2 + de \\ ae^2 - a &= de \\ \frac{ae^2 - a}{e} &= d \\ \frac{a(e^2 - 1)}{e} &= d \end{aligned}$$

Write each rectangular equation in polar form.

42. $x^2 = 4y + 4$

SOLUTION:

The equation $x^2 = 4y + 4$ is an equation for a parabola. So, $e = 1$. The equation written in standard form is $x^2 = 4(y + 1)$. The parabola opens up and its vertex is located at $(0, -1)$. Since $p = 1$, the equation for the directrix is $y = -1 - 1$ or $y = -2$. So, the polar equation is of the form $r = \frac{ed}{1 - e\sin\theta}$. Since the equation for the directrix is $y = -2$, $d = 2$. Substitute the values for d and e into the polar equation.

$$\begin{aligned} r &= \frac{ed}{1 - e\sin\theta} \\ &= \frac{(1)(2)}{1 - (1)\sin\theta} \\ &= \frac{2}{1 - \sin\theta} \end{aligned}$$

The polar form of the rectangular equation is $r = \frac{2}{1 - \sin\theta}$.

ANSWER:

$r = \frac{2}{1 - \sin\theta}$

43. $-10y + 25 = x^2$

SOLUTION:

The equation $-10y + 25 = x^2$ is an equation for a parabola. So, $e = 1$. The equation written in standard form is $-10(y - 2.5) = x^2$. The parabola opens down and its vertex is located at (0, 2.5). $p = -10 \div 4$ or -2.5. The equation for the directrix is $y = 2.5 - (-2.5)$ or $y = 5$. Therefore, $d = 5$.

The polar equation is of the form $r = \dfrac{ed}{1 + e\sin\theta}$. Substitute the values for d and e into the polar equation.

$$\begin{aligned} r &= \frac{ed}{1+e\sin\theta} \\ &= \frac{(1)(5)}{1+(1)\sin\theta} \\ &= \frac{5}{1+\sin\theta} \end{aligned}$$

The polar form of the rectangular equation is $r = \dfrac{5}{1+\sin\theta}$.

ANSWER:

$r = \dfrac{5}{1+\sin\theta}$

44. $\dfrac{(x-2)^2}{16} + \dfrac{y^2}{12} = 1$

SOLUTION:

The equation $\dfrac{(x-2)^2}{16} + \dfrac{y^2}{12} = 1$ is an equation for an ellipse. The center of the ellipse is located at (2, 0). Since $a^2 = 16$ and $b^2 = 12$, $c^2 = 4$. So, $c = 2$. Therefore, the foci are located at (4, 0) and (0, 0).

The polar equation is of the form $r = \dfrac{ed}{1 - e\cos\theta}$. Since $a^2 = 16$, $a = 4$. Thus, the vertices are located at (−2, 0) and (6, 0) and the eccentricity of the ellipse is $e = 0.5$.

The point (6, 0) can be represented by the polar coordinate (6, 0). Substitute this point and the value for e into the polar equation and solve for d.

$$\begin{aligned} r &= \frac{ed}{1-e\cos\theta} \\ 6 &= \frac{0.5d}{1-0.5\cos 0} \\ 6 &= \frac{0.5d}{0.5} \\ 6 &= d \end{aligned}$$

Substitute the values for d and e into the polar equation.

$$\begin{aligned} r &= \frac{ed}{1-e\cos\theta} \\ r &= \frac{0.5(6)}{1-0.5\cos\theta} \\ r &= \frac{3}{1-0.5\cos\theta} \end{aligned}$$

The polar form of the rectangular equation is $r = \dfrac{3}{1-0.5\cos\theta}$.

ANSWER:

$r = \dfrac{3}{1-0.5\cos\theta}$

45. $\dfrac{(x+4)^2}{64}+\dfrac{y^2}{48}=1$

SOLUTION:

The equation $\dfrac{(x+4)^2}{64}+\dfrac{y^2}{48}=1$ is an equation for an ellipse. The center of the ellipse is located at (−4, 0). Since $a^2 = 64$ and $b^2 = 48$, $c^2 = 16$ and $c = 4$. Therefore, the foci are located at (0, 0) and (−8, 0).

The polar equation is of the form $r = \dfrac{ed}{1+e\cos\theta}$. Since $a^2 = 64$, $a = 8$. Thus, the vertices are located at (4, 0) and (−12, 0) and the eccentricity of the ellipse is $e = 0.5$.

The point (4, 0) can be represented by the polar coordinate (4, 0). Substitute this point and the value for e into the polar equation and solve for d.

$$r=\frac{ed}{1+e\cos\theta}$$
$$4=\frac{0.5d}{1+0.5\cos 0}$$
$$4=\frac{0.5d}{1.5}$$
$$12=d$$

Substitute the values for d and e into the polar equation.

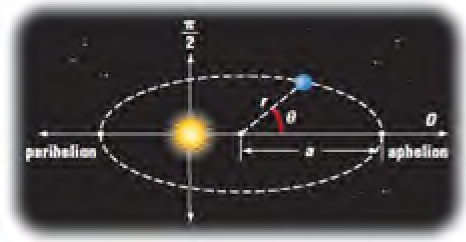
$$r=\frac{ed}{1+e\cos\theta}$$
$$r=\frac{0.5(12)}{1+0.5\cos\theta}$$
$$r=\frac{6}{1+0.5\cos\theta}$$

The polar form of the rectangular equation is $r = \dfrac{6}{1+0.5\cos\theta}$.

ANSWER:

$r = \dfrac{6}{1+0.5\cos\theta}$

46. **ASTRONOMY** The planets travel around the Sun in approximately elliptical orbits with the Sun at one focus as shown below.



a. Show that the polar equation of the planets' orbit can be written as $r = \dfrac{a(1-e^2)}{(1-e\cos\theta)}$.

b. Prove that the perihelion distance of any planet is $a(1 - e)$ and the aphelion distance is $a(1 + e)$.

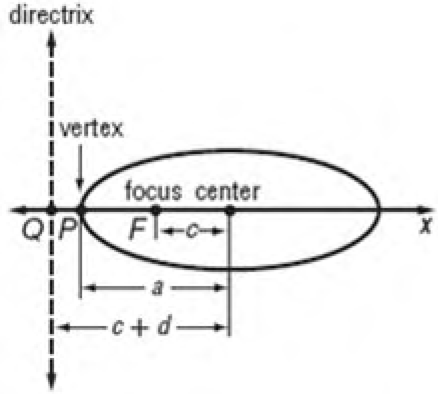
c. Use the formulas from part **a** to find the perihelion and aphelion distances for each of the planets.

Planet	a	e	Planet	a	e
Earth	1.000	0.017	Neptune	30.06	0.009
Jupiter	5.203	0.048	Saturn	9.539	0.056
Mars	1.524	0.093	Uranus	19.18	0.047
Mercury	0.387	0.206	Venus	0.723	0.007

d. For which planet is the distance between the perihelion and aphelion the smallest? The greatest?

SOLUTION:

a. Diagram the situation.



PF	$= ePQ$	Definition for the eccentricity of a conic
$a - c$	$= e(c + d - a)$	$PF = a - c$ and $PQ = c + d - a$
$a - ae$	$= e(ae + d - a)$	Since $e = \frac{c}{a}$, $c = ae$
$a - ae$	$= ae^2 + de - ae$	Distribute.
$ae^2 + de$	$= a$	Subtract $-ae$ from each side.

$de = a - ae^2$ Subtract ae^2 from each side.

$de = a(1 - e^2)$ Factor a.

Since the Sun is at one focus and the other focus is located to the right of the origin, the equation is of th form $r = \frac{ed}{1 - \cos\theta}$. So, upon substituting $de = a(1$ $e^2)$ into $r = \frac{de}{1 - e\cos\theta}$, the equation becomes $r = \frac{a(1-e^2)}{1 - e\cos\theta}$.

b. Sample answer: The distance of the perihelion to t center of the ellipse is a. The distance of the Sun to t center of the ellipse is c. So, the perihelion distance i $- c$. The eccentricity of the ellipse is $e = \frac{c}{a}$. Thus, c ea. Using substitution, the perihelion distance can be written as $a - ea$ or $a(1 - e)$. The distance of the aphelion to the center of the ellipse is also a. Since th aphelion is located on the opposite side of the center the ellipse than the Sun, the aphelion distance is a + Using substitution, this can be written as $a + ea$ or $a($ $+ e)$.

c. To find the perihelion and aphelion distances for Earth, substitute $a = 1$ and $e = 0.017$ into the express found in part **b**.

perihelion	aphelion
$a(1-e) = 1(1-0.017)$	$a(1+e) = 1(1+0.017)$
$= 0.983$	$= 1.017$

To find the perihelion and aphelion distances for Jupit substitute $a = 5.203$ and $e = 0.048$ into the expressio found in part **b**.

perihelion	aphelion
$a(1-e) = 5.203(1-0.048)$	$a(1+e) = 5.203(1+0.04$
≈ 4.953	≈ 5.453

To find the perihelion and aphelion distances for Mar substitute $a = 1.524$ and $e = 0.093$ into the expressio found in part **b**.

perihelion	aphelion
$a(1-e) = 1.524(1-0.093)$	$a(1+e) = 1.524(1+0.09$
≈ 1.382	≈ 1.666

To find the perihelion and aphelion distances for Mercury, substitute $a = 0.387$ and $e = 0.206$ into the expression found in part **b**.

perihelion	aphelion
$a(1-e) = 0.387(1-0.206)$	$a(1+e) = 0.387(1+0.2($
≈ 0.307	≈ 0.467

To find the perihelion and aphelion distances for Neptune, substitute $a = 30.06$ and $e = 0.009$ into the expression found in part **b**.

perihelion	aphelion
$a(1-e) = 30.06(1-0.009)$	$a(1+e) = 30.06(1+0.0($
≈ 29.789	≈ 30.331

To find the perihelion and aphelion distances for Satu substitute $a = 9.539$ and $e = 0.056$ into the expressio found in part **b**.

perihelion	aphelion
$a(1-e) = 9.539(1-0.056)$	$a(1+e) = 9.539(1+0.0$
≈ 9.005	≈ 10.073

To find the perihelion and aphelion distances for Uranus, substitute $a = 19.18$ and $e = 0.047$ into the expression found in part **b**.

perihelion	aphelion
$a(1-e) = 19.18(1-0.047)$	$a(1+e) = 19.18(1+0.04$
≈ 18.279	≈ 20.081

To find the perihelion and aphelion distances for Uranus, substitute $a = 0.723$ and $e = 0.007$ into the expression found in part **b**.

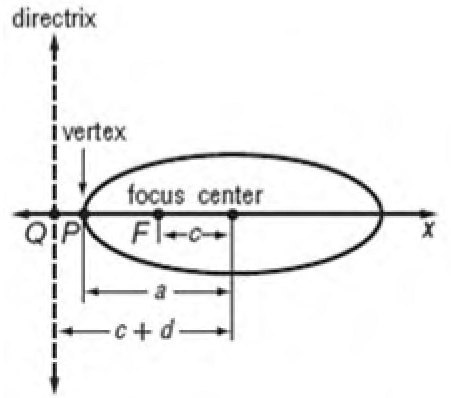
perihelion	aphelion
$a(1-e) = 0.723(1-0.007)$	$a(1+e) = 0.723(1+0.0($
≈ 0.718	≈ 0.728

Planet	Perihelion Distance	Aphelion Distance
Earth	0.983	1.017
Jupiter	4.953	5.453
Mars	1.382	1.666
Mercury	0.307	0.467
Neptune	29.789	30.331
Saturn	9.005	10.073
Uranus	18.279	20.081
Venus	0.718	0.728

d. Since the perihelion and aphelion are located on opposite sides of the Sun, to find the distance betwee them, you must add the two distances together. Thus the planet with the smallest distance between the perihelion and aphelion is Mercury, 0.307 + 0.467 or 0.774. The planet with the largest distance between t perihelion and aphelion is Neptune, 29.789 + 30.331 60.12.

ANSWER:

a.



$$\begin{aligned} PF &= ePQ \\ a - c &= e(c + d - a) \\ a - ae &= e(ae + d - a) \\ a - ae &= ae^2 + de - ae \\ ae^2 + de &= a \\ de &= a - ae^2 \\ de &= a(1 - e^2) \end{aligned}$$

So, upon substituting $de = a(1 - e^2)$ into $r = \frac{de}{1 - e\cos\theta}$, the equation becomes $r = \frac{a(1-e^2)}{1 - e\cos\theta}$

b. Sample answer: The distance of the perihelion to t center of the ellipse is a. The distance of the Sun to t center of the ellipse is c. So, the perihelion distance i $-c$. The eccentricity of the ellipse is $e = \frac{c}{a}$. Thus, c ea. The perihelion distance can be written as $a - ea$ $a(1 - e)$. The aphelion distance is $a + c$, which can b written as $a + ea$ or $a(1 + e)$.

c.

Planet	Perihelion Distance	Aphelion Distance
Earth	0.983	1.017
Jupiter	4.953	5.453
Mars	1.382	1.666
Mercury	0.307	0.467
Neptune	29.789	30.331
Saturn	9.005	10.073
Uranus	18.279	20.081
Venus	0.718	0.728

d. Mercury; Neptune

Write each equation in polar form. (*Hint*: Translate each conic so that a focus lies on the pole.)

47. $\frac{(x-2)^2}{64} - \frac{y^2}{36} = 1$

SOLUTION:

The equation $\frac{(x-2)^2}{64} - \frac{y^2}{36} = 1$ is a hyperbola with center (2, 0), $a = 8$, $b = 6$, and $c = 10$. The foci of this hyperbola are located at (–8, 0) and (12, 0). To place the right focus at the origin, translate the hyperbola left 12 units.

$$\begin{aligned} \frac{(x-2)^2}{64} - \frac{y^2}{36} &= 1 \\ \frac{[x' - 2 - (-12)]^2}{64} - \frac{(y')^2}{36} &= 1 \\ \frac{(x' + 10)^2}{64} - \frac{(y')^2}{36} &= 1 \end{aligned}$$

The hyperbola is horizontal, so the directrix is vertical. The center of the translated hyperbola is located at (–10, 0). Since $c = 10$, the foci are located at (–20, 0) and (0, 0) and the directrix is of the form $x = -d$.

We now know the polar equation is of the form $r = \frac{ed}{1 - e\cos\theta}$. Since $a = 8$, the vertices are located at (–18, 0) and (–2, 0) and the eccentricity of the hyperbola is $e = 1.25$.

The point (–2, 0) can be represented by the polar coordinate $(2, \pi)$. Substitute this point and the value for e into the polar equation and solve for d.

$$\begin{aligned} r &= \frac{ed}{1 - e\cos\theta} \\ 2 &= \frac{1.25d}{1 - 1.25\cos\pi} \\ 2 &= \frac{1.25d}{2.25} \\ 4.5 &= 1.25d \\ 3.6 &= d \end{aligned}$$

Substitute the values for d and e into the polar equation.

$$\begin{aligned} r' &= \frac{ed}{1 - e\cos\theta} \\ r' &= \frac{1.25(3.6)}{1 - 1.25\cos\theta} \\ r' &= \frac{4.5}{1 - 1.25\cos\theta} \end{aligned}$$

The polar form of the translated rectangular equation is $r' = \dfrac{4.5}{1 - 1.25\cos\theta'}$.

ANSWER:

$r' = \dfrac{4.5}{1 - 1.25\cos\theta'}$

48. $3(x + 5)^2 + 4y^2 = 192$

SOLUTION:

$$\begin{aligned} 3(x+5)^2 + 4y^2 &= 192 \\ \frac{(x+5)^2}{64} + \frac{y^2}{48} &= 1 \end{aligned}$$

This is the equation of an ellipse with center $(-5, 0)$, $a = 8$, $b = 4\sqrt{3}$, and $c = 4$. The foci of this ellipse are located at $(-9, 0)$ and $(-1, 0)$. To place the right focus at the origin, translate the hyperbola right 1 unit.

$$\begin{aligned} \frac{(x+5)^2}{64} + \frac{y^2}{48} &= 1 \\ \frac{[x' + 5 - 1]^2}{64} + \frac{(y')^2}{48} &= 1 \\ \frac{(x' + 4)^2}{64} + \frac{(y')^2}{48} &= 1 \end{aligned}$$

The center of the translated ellipse is located at $(-4, 0)$. Since $c = 4$, the foci are located at $(-8, 0)$ and $(0, 0)$. The ellipse is horizontal and the foci are left of the pole, so the directrix is right of the pole and is of the form $x = d$.

The polar equation is of the form $r = \dfrac{ed}{1 + e\cos\theta}$. Since $a = 8$, the vertices are located at $(-12, 0)$ and $(4, 0)$ and the eccentricity of the ellipse is $e = 0.5$.

The point $(4, 0)$ can be represented by the polar coordinate $(4, 0)$. Substitute this point and the value for e into the polar equation and solve for d.

$$\begin{aligned} r &= \frac{ed}{1 + e\cos\theta} \\ 4 &= \frac{0.5d}{1 + 0.5\cos 0} \\ 4 &= \frac{0.5d}{1.5} \\ 6 &= 0.5d \\ 12 &= d \end{aligned}$$

Substitute the values for d and e into the polar equation.

$$\begin{aligned} r &= \frac{ed}{1 + e\cos\theta} \\ r &= \frac{0.5(12)}{1 + 0.5\cos\theta} \\ r &= \frac{6}{1 + 0.5\cos\theta} \end{aligned}$$

The polar form of the translated rectangular equation is $r' = \dfrac{6}{1 + 0.5\cos\theta'}$.

ANSWER:

$r' = \dfrac{6}{1 + 0.5\cos\theta'}$

49. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the effects of varying the eccentricity and the directrix on graphs of conic sections.
a. NUMERICAL Write an equation for a conic section with focus (0, 0) and directrix $x = 3$ for $e = 0.4, 0.6, 1, 1.6$, and 2. Then identify the type of conic that each equation represents.
b. GRAPHICAL Graph and label the eccentricity f

each of the equations that you found in part **a** on the same coordinate plane.
c. VERBAL Describe the changes in the graphs fro part **b** as e approaches 2.
d. NUMERICAL Write an equation for a conic section with focus (0, 0) and eccentricity e = 0.5 for d = 0.25, 1, and 4.
e. GRAPHICAL Graph each of the equations on the same coordinate plane.
f. VERBAL Describe the relationship between the value of d and the distances between the vertices and the foci of the graphs from part **e**.

SOLUTION:

a. Sample answer: Use the polar equation $r = \dfrac{ed}{1+e\cos\theta}$. Substitute d = 3 and each value for e into the equation.

e = 0.4

$$r=\frac{ed}{1+e\cos\theta}$$
$$r=\frac{0.4(3)}{1+0.4\cos\theta}$$
$$r=\frac{1.2}{1+0.4\cos\theta}$$

Since e = 0.4, $r = \dfrac{1.2}{1+0.4\cos\theta}$ is an equation for an ellipse.

e = 0.6

$$r=\frac{ed}{1+e\cos\theta}$$
$$r=\frac{0.6(3)}{1+0.6\cos\theta}$$
$$r=\frac{1.8}{1+0.6\cos\theta}$$

Since e = 0.6, $r = \dfrac{1.8}{1+0.6\cos\theta}$ is an equation for an ellipse.

e = 1

$$r=\frac{ed}{1+e\cos\theta}$$
$$r=\frac{1(3)}{1+1\cos\theta}$$
$$r=\frac{3}{1+\cos\theta}$$

Since e = 1, $r = \dfrac{3}{1+\cos\theta}$ parabola.

e = 1.6

$$r=\frac{ed}{1+e\cos\theta}$$
$$r=\frac{1.6(3)}{1+1.6\cos\theta}$$
$$r=\frac{4.8}{1+1.6\cos\theta}$$

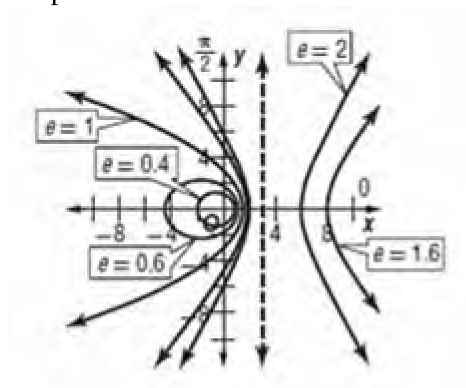
Since e = 1.6, $r = \dfrac{4.8}{1+1.6\cos\theta}$ is an equation for a hyperbola.

e = 2

$$r=\frac{ed}{1+e\cos\theta}$$
$$r=\frac{2(3)}{1+2\cos\theta}$$
$$r=\frac{6}{1+2\cos\theta}$$

Since e = 2, $r = \dfrac{6}{1+2\cos\theta}$ is an equation for a hyperbola.

b. Evaluate each equation for several points in their domains. Use these points to graph each equation. Graph the directrix x = 3.



c. Sample answer: The ellipse with the smaller eccentricity is much more circular than the ellipse with greater eccentricity. As e approaches 1, the ellipses open farther to the left, approaching the parabola with e = 1. For values where e > 1, the hyperbola with the greater eccentricity approaches its asymptotes much more rapidly than the hyperbola with the smaller eccentricity. Also, as e approaches 1, one branch of the hyperbola moves farther and farther away from its asymptotes, approaching the parabola with e = 1.

d. Sample answer: Use the polar equation r =

$\frac{ed}{1+e\cos\theta}$. Substitute $e = 0.5$ and each value for d into the equation.

$d = 0.25$

$$r = \frac{ed}{1+e\cos\theta}$$
$$r = \frac{0.5(0.25)}{1+0.5\cos\theta}$$
$$r = \frac{0.125}{1+0.5\cos\theta}$$

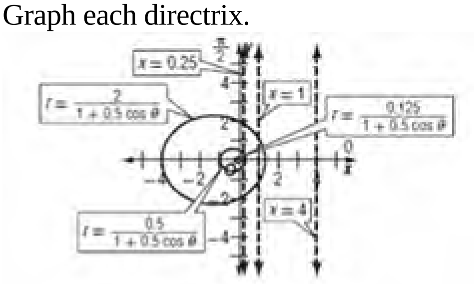
$d = 1$

$$r = \frac{ed}{1+e\cos\theta}$$
$$r = \frac{0.5(1)}{1+0.5\cos\theta}$$
$$r = \frac{0.5}{1+0.5\cos\theta}$$

$d = 4$

$$r = \frac{ed}{1+e\cos\theta}$$
$$r = \frac{0.5(4)}{1+0.5\cos\theta}$$
$$r = \frac{2}{1+0.5\cos\theta}$$

e. Evaluate each equation for several points in their domains. Use these points to graph each equation. Graph each directrix.

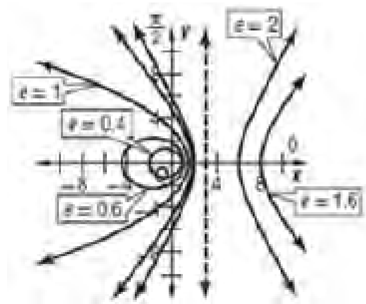


f. Sample answer: As the value of the directrix increases from 0.25 to 4, the ellipses' axes expand and thus, larger ellipses are created. Therefore, the distances between the vertices and the foci of the ellipses increase.

ANSWER:

a. Sample answer: $r = \frac{1.2}{1+0.4\cos\theta}$, ellipse; $r = \frac{1.8}{1+0.6\cos\theta}$, ellipse; $r = \frac{3}{1+\cos\theta}$, parabola; $r = \frac{4.8}{1+1.6\cos\theta}$, hyperbola; $r = \frac{6}{1+2\cos\theta}$, hyperbola

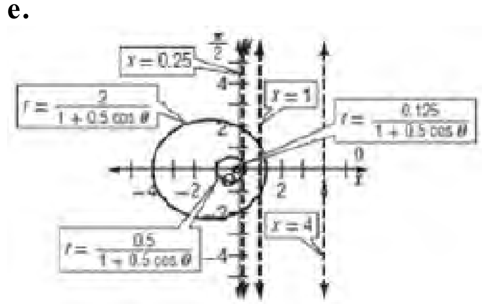
b.



c. Sample answer: The ellipse with the smaller eccentricity is much more circular than the ellipse with greater eccentricity. As e approaches 1, the ellipses open farther to the left, approaching the parabola with $e = 1$. For values where $e > 1$, the hyperbola with the greater eccentricity approaches its asymptotes much more rapidly than the hyperbola with the smaller eccentricity. Also, as e approaches 1, one branch of the hyperbola moves farther and farther away from its asymptotes, approaching the parabola with $e = 1$.

d. Sample answer: $r = \frac{0.125}{1+0.5\cos\theta}$; $r = \frac{0.5}{1+0.5\cos\theta}$; $r = \frac{2}{1+0.5\cos\theta}$

e.



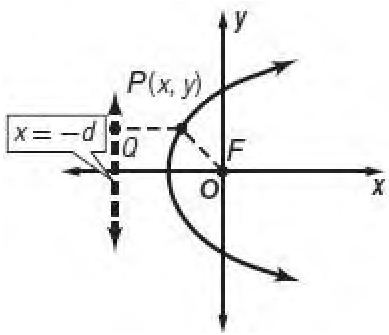
f. Sample answer: As the value of the directrix increases from 0.25 to 4, the distances between the vertices and the foci of the ellipses increase.

Derive each of the following polar equations of conics as shown on page 562 for the equation $r = \frac{ed}{1+e\cos\theta}$. Include a diagram with each derivation.

50. $r = \frac{ed}{1-e\cos\theta}$

SOLUTION:

Since there is a negative and a cosine in the denominator, draw the conic with a vertical directrix left of the pole. The conic opens to the right



Use the definition of a conic section to start the proof. Then use the distance formula, the definition of r and substitution to complete the proof.

<u>Statements   (Reasons)</u>

1. $PF = ePQ$ (Definition of a conic section)

2. $\sqrt{x^2 + y^2} = e[x - (-d)]$ ($PF = \sqrt{x^2 + y^2}$ and $PQ = x - (-d)$)

3. $\sqrt{x^2 + y^2} = e(d + x)$ ($x - (-d) = x + d$ or $d + x$)

4. $r = e(d + r\cos\theta)$ ($r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta$)

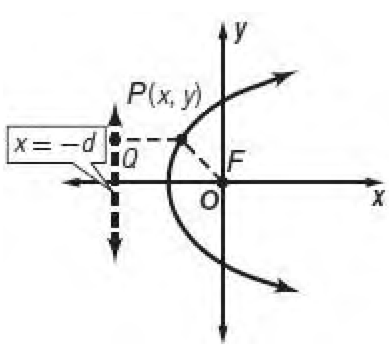
5. $r\ ed + er\cos\theta$ (Distributive Property)

6. $r - er\cos\theta = ed$ (Isolate r-terms.)

7. $r(1 - e\cos\theta) = ed$ (Factor.)

8. $r = \dfrac{ed}{1 - e\cos\theta}$ (Solve for r.)

ANSWER:



<u>Statements   (Reasons)</u>

1. $PF = ePQ$ (Definition of a conic section)

2. $\sqrt{x^2 + y^2} = e[x - (-d)]$ ($PF = \sqrt{x^2 + y^2}$ and $PQ = x - (-d)$)

3. $\sqrt{x^2 + y^2} = e(d + x)$ ($x - (-d) = x + d$ or $d + x$)

4. $r = e(d + r\cos\theta)$ ($r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta$)

5. $r = ed + er\cos\theta$ (Distributive Property)

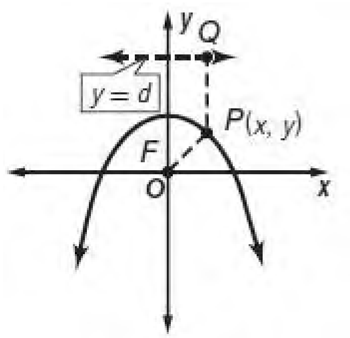
6. $r - er\cos\theta = ed$ (Isolate r-terms.)

7. $r(1 - e\cos\theta) = ed$ (Factor.)

8. $r = \dfrac{ed}{1 - e\cos\theta}$ (Solve for r.)

51. $r = \dfrac{ed}{1 + e\sin\theta}$

SOLUTION:



<u>Statements</u> <u>(Reasons)</u>

1. $PF = ePQ$ (Definition of a conic section)

2. $\sqrt{x^2 + y^2} = e(d - y)$ ($PF = \sqrt{x^2 + y^2}$ and $PQ = d - y$)

3. $r = e(d - r\sin\theta)$ ($r = \sqrt{x^2 + y^2}$ and $y = r\sin\theta$)

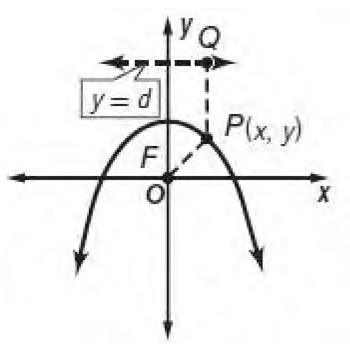
4. $r = ed - er\sin\theta$ (Distributive Property)

5. $r + er\sin\theta = ed$ (Isolate r-terms.)

6. $r(1 + e\sin\theta) = ed$ (Factor.)

7. $r = \dfrac{ed}{1 + e\sin\theta}$ (Solve for r.)

ANSWER:



<u>Statements</u> <u>(Reasons)</u>

1. $PF = ePQ$ (Definition of a conic section)

2. $\sqrt{x^2 + y^2} = e(d - y)$ ($PF = \sqrt{x^2 + y^2}$ and $PQ = d - y$)

3. $r = e(d - r\sin\theta)$ ($r = \sqrt{x^2 + y^2}$ and $y = r\sin\theta$)

4. $r = ed - er\sin\theta$ (Distributive Property)

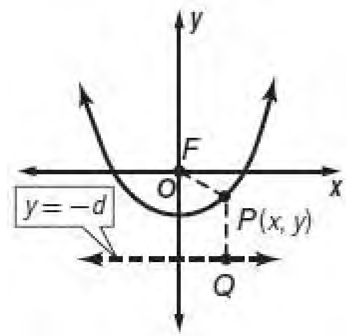
5. $r + er\sin\theta = ed$ (Isolate r-terms.)

6. $r(1 + e\sin\theta) = ed$ (Factor.)

7. $r = \dfrac{ed}{1 + e\sin\theta}$ (Solve for r.)

52. $r = \dfrac{ed}{1 - e\sin\theta}$

SOLUTION:



<u>Statements</u> <u>(Reasons)</u>

1. $PF = ePQ$ (Definition of a conic section)

2. $\sqrt{x^2 + y^2} = e[y - (-d)]$ ($PF = \sqrt{x^2 + y^2}$ and $PQ = y - (-d)$)

3. $\sqrt{x^2 + y^2} = e(d + y)$ ($y - (-d) = y + d$ or $d + y$)

4. $r = e(d + r\sin\theta)$ ($r = \sqrt{x^2 + y^2}$ and $y = r\sin\theta$)

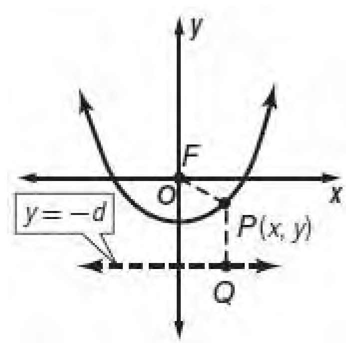
5. $r = ed + er\sin\theta$ (Distributive Property)

6. $r - er\sin\theta = ed$ (Isolate r-terms.)

7. $r(1 - e\sin\theta) = ed$ (Factor.)

8. $r = \dfrac{ed}{1 - e\sin\theta}$ (Solve for r.)

ANSWER:



<u>Statements</u> <u>(Reasons)</u>

1. $PF = ePQ$ (Definition of a conic section)

2. $\sqrt{x^2 + y^2} = e[y - (-d)]$ ($PF = \sqrt{x^2 + y^2}$ and $PQ = y - (-d)$)

3. $\sqrt{x^2 + y^2} = e(d + y)$ ($y - (-d) = y + d$ or $d + y$)

4. $r = e(d + r\sin\theta)$ ($r = \sqrt{x^2 + y^2}$ and $y = r\sin\theta$)

5. $r = ed + er \sin \theta$ (Distributive Property)
6. $r - er \sin \theta = ed$ (Isolate r-terms.)
7. $r(1 - e \sin \theta) = ed$ (Factor.)
8. $r = \dfrac{ed}{1 - e\sin \theta}$ (Solve for r.)

53. **WRITING IN MATH** Describe two definitions that can be used to define a conic section.

SOLUTION:

Sample answer: A conic section can be defined as a figure that is formed when a plane intersects a double-napped right cone. Conic sections can also be defined in terms of their eccentricity as the locus of points such that the distance from a point to the focus and the distance from the point to the directrix is a constant ratio.

ANSWER:

Sample answer: A conic section can be defined as a figure that is formed when a plane intersects a double-napped right cone or as the locus of points such that the distance from a point to the focus and the distance from the point to the directrix is a constant ratio.

54. **REASONING** Explain why $r = \dfrac{ed}{1 + e\sin \theta}$ does not produce a true circle for any value of e.

SOLUTION:

Sample answer: The eccentricity of a circle is 0. Substitute $e = 0$ into the equation and simplify.

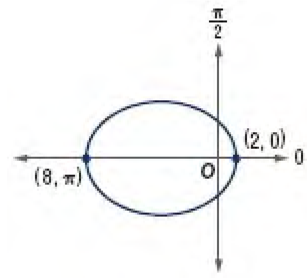
$$r = \frac{ed}{1 + e\sin\theta}$$
$$r = \frac{(0)d}{1 + (0)\sin\theta}$$
$$r = \frac{0}{1} \text{ or } 0$$

Since $e = 0$ for a circle, the equation would simplify to $r = 0$, which is a point.

ANSWER:

Sample answer: Since $e = 0$ for a circle, the equation would simplify to $r = 0$, which is a point.

CHALLENGE Determine a polar equation for the ellipse with the given vertices if one focus is at the pole.

55. 

SOLUTION:

Since one focus is at the pole and the other focus appears to be to the left of the pole, the equation is of the form $r = \dfrac{ed}{1 + e\cos \theta}$. Substitute the points (2, 0) and (8, π) and solve for e in the resulting system of equations.

$$2 = \frac{ed}{1 + e\cos 0} \qquad 8 = \frac{ed}{1 + e\cos \pi}$$
$$2 = \frac{ed}{1 + e} \qquad 8 = \frac{ed}{1 - e}$$
$$2(1 + e) = ed \qquad 8(1 - e) = ed$$

Now use substitution.

$$2(1 + e) = 8(1 - e)$$
$$2 + 2e = 8 - 8e$$
$$10e = 6$$
$$e = \frac{3}{5}$$

Substitute $e = \frac{3}{5}$ into $2(1 + e) = ed$ and solve for d.

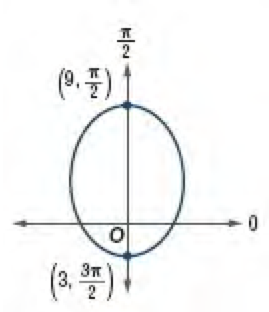
$$2\left(1 + \frac{3}{5}\right) = \frac{3}{5}d$$
$$\frac{16}{5} = \frac{3}{5}d$$
$$d = \frac{16}{3}$$

$$r = \frac{\frac{3}{5} \cdot \frac{16}{3}}{1 + \frac{3}{5}\cos\theta}$$

$$r = \frac{16}{5 + 3\cos\theta}$$

ANSWER:

$$r = \frac{16}{5 + 3\cos \theta}$$

56. 

SOLUTION:

Since one focus is at the pole and the other focus appears to be to above the pole, the equation is of the form $r = \dfrac{ed}{1 - e\sin\theta}$. Substitute the points $\left(9, \frac{\pi}{2}\right)$ and $\left(3, \frac{3\pi}{2}\right)$ and solve for e in the resulting system of equations.

$$9 = \frac{ed}{1 - e\sin\frac{\pi}{2}} \qquad 3 = \frac{ed}{1 - e\sin\frac{3\pi}{2}}$$

$$9 = \frac{ed}{1 - e} \qquad 3 = \frac{ed}{1 + e}$$

$$9(1 - e) = ed \qquad 3(1 + e) = ed$$

Now use substitution.

$$\begin{aligned} 9(1 - e) &= 3(1 + e) \\ 9 - 9e &= 3 + 3e \\ 6 &= 12e \\ e &= \frac{1}{2} \end{aligned}$$

Substitute $e = \frac{1}{2}$ into $9(1 - e) = ed$ and solve for d.

$$\begin{aligned} 9\left(1 - \frac{1}{2}\right) &= \frac{1}{2}d \\ \frac{9}{2} &= \frac{1}{2}d \\ d &= 9 \end{aligned}$$

$$r = \frac{\frac{1}{2} \cdot 9}{1 - \frac{1}{2}\sin\theta}$$

$$r = \frac{9}{2 - \sin\theta}$$

ANSWER:

$$r = \frac{9}{2 - \sin\theta}$$

57. **WRITING IN MATH** Explain how you can find the distance from the focus at (0, 0) to any point on a conic when the rectangular coordinates, polar coordinates, or θ is provided.

SOLUTION:

Sample answer: When the point is given in rectangular coordinates, you can use the distance formula to find the distance between the point at $P(a, b)$ and the focus at (0, 0). When the point is given in polar coordinates, it is of the form $P(r, \theta)$ where r is the given distance from the point to the pole at (0, 0). Therefore, the distance between the point at $P(a, b)$ and the focus at (0, 0) is r. When θ is provided, substitute this value into the polar equation and solve for r. This will provide the distance between the point at θ and the focus.

ANSWER:

Sample answer: When the point is given in rectangular coordinates, you can use the distance formula to find the distance between the point at $P(a, b)$ and the focus at (0, 0). When the point is given in polar coordinates, it is of the form $P(r, \theta)$ where r is the given distance from the point to the pole at (0, 0). Therefore, the distance between the point at $P(a, b)$ and the focus at (0, 0) is r. When θ is provided, substitute this value into the polar equation and solve for r. This will provide the distance between the point at θ and the focus.

Find two pairs of polar coordinates for each point with the given rectangular coordinates if $0 \le \theta \le 2\pi$. If necessary, round to the nearest hundredth.

58. $\left(-\sqrt{2}, \sqrt{2}\right)$

SOLUTION:

For $\left(-\sqrt{2}, \sqrt{2}\right)$, $x = -\sqrt{2}$ and $y = \sqrt{2}$.

Since $x < 0$, use $\tan^{-1}\frac{y}{x} + \pi$ to find θ.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{\left(-\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\frac{y}{x} + \pi \\ &= \tan^{-1}\frac{\sqrt{2}}{-\sqrt{2}} + \pi \\ &= -\frac{\pi}{4} + \pi \\ &= \frac{3\pi}{4} \end{aligned}$$

One set of polar coordinates is $\left(2, \frac{3\pi}{4}\right)$. Another representation that uses a negative r-value is $\left(-2, \frac{7\pi}{4}\right)$.

ANSWER:

$\left(2, \frac{3\pi}{4}\right), \left(-2, \frac{7\pi}{4}\right)$

59. $(-2, -5)$

SOLUTION:

For $(-2, -5)$, $x = -2$ and $y = -5$.

Since $x < 0$, use $\tan^{-1}\frac{y}{x} + \pi$ to find θ.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-2)^2 + (-5)^2} \\ &= \sqrt{29} \text{ or about } 5.39 \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\frac{y}{x} + \pi \\ &= \tan^{-1}\frac{-5}{-2} + \pi \\ &\approx 4.33 \end{aligned}$$

One set of polar coordinates is (5.39, 4.33). Another representation that uses a negative r-value is (−5.39, 4.33 – π) or (−5.39, 1.19).

ANSWER:

(5.39, 4.33), (–5.39, 1.19)

60. (8, −12)

SOLUTION:

For (8, −12), $x = 8$ and $y = –12$.

Since $x > 0$, use $\tan^{-1}\frac{y}{x}$ to find θ.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(8)^2 + (-12)^2} \\ &= \sqrt{208} \text{ or about } 14.42 \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\frac{y}{x} \\ &= \tan^{-1}\frac{-12}{8} \\ &\approx -0.98 \end{aligned}$$

One set of polar coordinates is (14.42, –0.98). Since this set is not in the required domain, two more sets have to be found. A representation that uses a positive r-value is (14.42, −0.98 + 2π) or (14.42, 5.30). A representation that uses a negative r-value is (−14.42, −0.98 + π) or (−14.42, 2.16).

ANSWER:

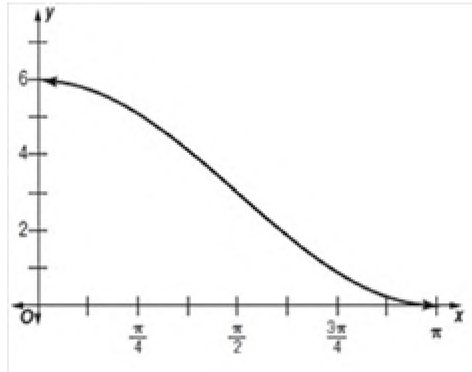
(14.42, 5.30), (−14.42, 2.16)

Identify and graph each classic curve.

61. $r = 3 + 3 \cos \theta$

SOLUTION:

The equation $r = 3 + 3 \cos \theta$ is of the form $r = a \pm b\cos \theta$ where $a = b = 3$, so its graph is a cardioid. Because this polar equation is a function of the cosine function, it is symmetric with respect to the polar axis. Sketch the graph of the rectangular function $y = 3 + 3 \cos x$ on the interval $[0, \pi]$. From the graph, you can see that y has a maximum value of 6 when $x = 0$ and $y = 0$ when $x = \pi$.

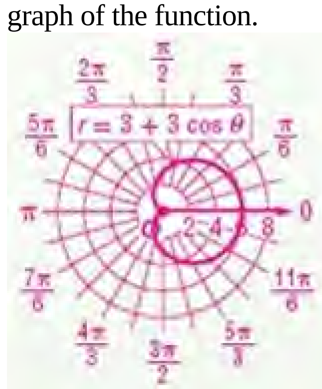


Interpreting these results in terms of the polar equation $r = 3 + 3 \cos \theta$, we can say that $|r|$ has a maximum value of 6 when $\theta = 0$ and $r = 0$ when $\theta = \pi$

Since the function is symmetric with respect to the polar axis, make a table and calculate the values of r on $[0, \pi]$.

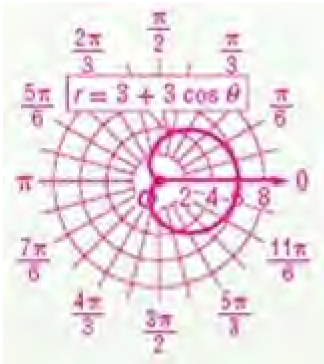
θ	$r = 3 + 3 \cos \theta$
0	6
$\frac{\pi}{6}$	5.6
$\frac{\pi}{4}$	5.1
$\frac{\pi}{3}$	4.5
$\frac{\pi}{2}$	3
$\frac{2\pi}{3}$	1.5
$\frac{3\pi}{4}$	0.9
$\frac{5\pi}{6}$	0.4
π	0

Use these and a few additional points to sketch the graph of the function.



ANSWER:

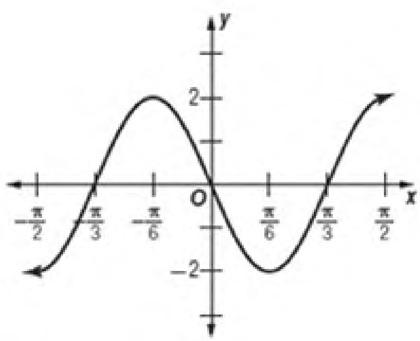
cardioid



62. $r = -2 \sin 3\theta$

SOLUTION:

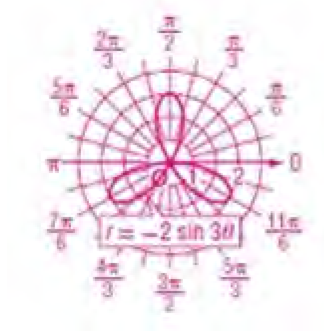
The equation $r = -2 \sin 3\theta$ is of the form $r = a \sin n\theta$ where n is 3 and odd. Because this polar equation is a function of the sine function, it is symmetric with respect to the line $\theta = \frac{\pi}{2}$. Sketch the graph of the rectangular function $y = -2 \sin 3x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. From the graph, you can see that $|y|$ has a maximum value of 2 when $x = -\frac{\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{6}$, and $\frac{\pi}{2}$, and $y = 0$ when $x = -\frac{\pi}{3}$, 0, and $\frac{\pi}{3}$.



Interpreting these results in terms of the polar equation $y = -2 \sin 3x$, we can say that $|r|$ has a maximum value of 2 when $\theta = -\frac{\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{6}$, and $\frac{\pi}{2}$, and $r = 0$ when $\theta = -\frac{\pi}{3}$, 0, and $\frac{\pi}{3}$.
Since the function is symmetric with respect to the polar axis, make a table and calculate the values of r on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

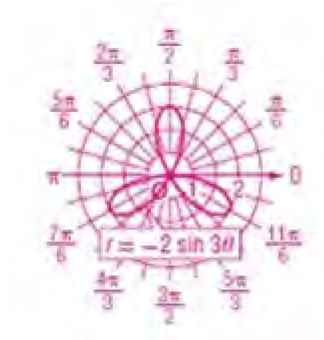
θ	$r = -2 \sin 3\theta$
$-\frac{\pi}{2}$	-2
$-\frac{\pi}{3}$	0
$-\frac{\pi}{4}$	1.4
$-\frac{\pi}{6}$	2
0	0
$\frac{\pi}{6}$	-2
$\frac{\pi}{4}$	-1.4
$\frac{\pi}{3}$	0
$\frac{\pi}{2}$	2

Use these and a few additional points to sketch the graph of the function.



ANSWER:

rose



63. $r = \frac{5}{2}\theta, \theta \geq 0$

SOLUTION:

The equation is of the form $r = a\theta + b$, so its graph is a spiral of Archimedes. Replacing (r, θ) with $(-r, -\theta)$ yields $(-r) = \frac{5}{2}(-\theta)$ or $r = \frac{5}{2}\theta$. Therefore, the function has symmetry with respect to the line $\theta = \frac{\pi}{2}$. However, $\theta \geq 0$, the symmetry will not be seen in the graph.
Spirals are unbounded. Therefore, the function has

no maximum r-values and only one zero when $\theta = 0$. Use points on the interval $[0, 2\pi]$ to sketch the graph of the function.

θ	$r = \frac{5}{2}\theta$
0	0
$\frac{\pi}{4}$	2.0
$\frac{\pi}{2}$	3.9
π	7.9
$\frac{3\pi}{2}$	11.8
2π	15.7

ANSWER:

spiral of Archimedes

Determine an equation of an ellipse with each set of characteristics

64. co-vertices (5,8), (5, 0);
foci (8, 4), (2, 4)

Because the y coordinates of the vertices are the same, the major axis is horizontal, and the standard form of the equation is $\frac{(x-h)^2}{a^2}$ $\frac{(y-k)^2}{b^2}$ 1.

The center is located at the midpoint of the co vertices, or (5, 4). So, $h = 5$ and $k = 4$. The distance between the co vertices is equal to $2b$ units. So, $2b = 8$, $b = 4$, and $b^2 = 16$. The length of the major axis is equal to $2a$ units. So, $2a = 10$, $a = 5$, and $a^2 = 25$.

The distance between the foci is equal to $2c$ units. So, $2c = 6$ and $c = 3$.

Use the values of a and c to find b.

$$c^2 = a^2 - b^2$$
$$3^2 = 5^2 - b^2$$
$$b^2 = 25 - 9$$
$$b^2 = 16$$
$$b = 4$$

Using the values of h, k, a, and b, the equation for the ellipse is $\frac{(x-5)^2}{25} + \frac{(y-4)^2}{16}$.

$$\frac{(x-5)^2}{25} + \frac{(y-4)^2}{16}$$

major axis (2, 4) to (8, 4);
minor axis (3, 1) to (3, 7)

Because the y coordinates of the major axis are the same, the major axis is vertical, and the standard form of the equation is $\dfrac{(x-h)^2}{a^2} \quad \dfrac{(y-k)^2}{b^2}$ 1.

The length of the major axis is equal to 2a units. So, 2a = 10, a = 5, and a^2
axis is equal to 2b units. So, 2b = 6, b = 3, and b^2 = 9. The center of the ellipse is at the midpoint of the major axis, or (3, 4). So, h = 3 and k = 4.

Using the values of h, k, a, and b, the equation for the ellipse is $\dfrac{(x-3)^2}{25} + \dfrac{(y-4)^2}{9} = 1$.

ANSWER:

$$\frac{(x-3)^2}{25} + \frac{(y-4)^2}{9} = 1$$

66. foci (1, −1), (9, −1);
length of minor axis equals 6

SOLUTION:

foci (1, −1), (9, −1);
length of minor axis equals 6
Because the y–coordinates of the foci are the same, the major axis is horizontal, and the standard form of the equation is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$.

The midpoint is the distance between the foci, or (5, –1). So, h = 5 and k = –1. The distance between the foci is equal to 2c units. So, 2c = 8 and c = 4. The length of the minor axis is equal to 2b units. So, 2b = 6, b = 3, and b^2 = 9.
Use the values of b and c to find a.

$$\begin{aligned} c^2 &= a^2 - b^2 \\ 4^2 &= a^2 - 3^2 \\ 16 + 9 &= a^2 \\ 25 &= a^2 \\ 5 &= a \end{aligned}$$

Using the values of h, k, a, and b, the equation for the ellipse is $\dfrac{(x-5)^2}{25} + \dfrac{(y+1)^2}{9} = 1$.

ANSWER:

$$\frac{(x-5)^2}{25} + \frac{(y+1)^2}{9} = 1$$

67. **OLYMPICS** In the Olympic Games, team standings are determined according to each team's total points. Each type of Olympic medal earns a team a given number of points. Use the information to determine which Olympics the United States earned the most points.

Olympics	Gold	Silver	Bronze
1996	44	32	25
2000	37	24	31
2004	35	39	29
2008	36	38	36

Medal	Points
gold	3
silver	2
bronze	1

SOLUTION:

Let matrix *X* represent the total amount of each medal earned during each Olympic games, and let matrix *Y* represent the points awarded for each type of medal. Then find the product *XY*.

$$XY = \begin{bmatrix} 44 & 32 & 25 \\ 37 & 24 & 31 \\ 35 & 39 & 29 \\ 36 & 38 & 36 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 221 \\ 190 \\ 212 \\ 220 \end{bmatrix}$$

During the 1996 Olympics, the United States earned 221 points.

ANSWER:

1996 Olympics

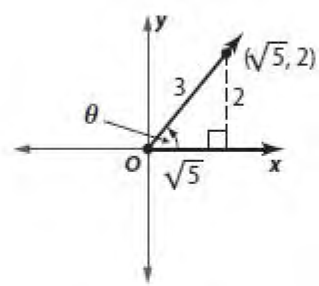
Find the values of sin 2θ, cos 2θ, and tan 2θ for the given value and interval.

68. sin $\theta = \frac{2}{3}$, (0°, 90°)

SOLUTION:

sin $\theta = \frac{2}{3}$, (0°, 90°)

Since $\sin\theta = \frac{2}{3}$ on the interval (0°, 90°), one point on the terminal side of θ has *y*-coordinate 2 and a distance of 3 units from the origin as shown. The *x*-coordinate of this point is $\sqrt{5}$.



Using this point, we find that $\cos\theta = \frac{\sqrt{5}}{3}$ and $\tan\theta = \frac{2\sqrt{5}}{5}$. Now use the double-angle identities for sine and cosine to find sin 2θ and cos 2θ.

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ &= 2\left(\frac{2}{3}\right)\left(\frac{\sqrt{5}}{3}\right) \\ &= \frac{4\sqrt{5}}{9} \end{aligned}$$

$$\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ &= 2\left(\frac{\sqrt{5}}{3}\right)^2 - 1 \\ &= 2\left(\frac{5}{9}\right) - 1 \\ &= \frac{10}{9} - \frac{9}{9} \\ &= \frac{1}{9} \end{aligned}$$

Use the definition of tangent to find tan 2θ.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{2\sqrt{5}}{5}\right)}{1-\left(-\frac{2\sqrt{5}}{5}\right)^2} = \frac{\frac{4\sqrt{5}}{5}}{1-\left(\frac{20}{25}\right)} = \frac{\frac{4\sqrt{5}}{5}}{\frac{1}{5}} = 4\sqrt{5}$$

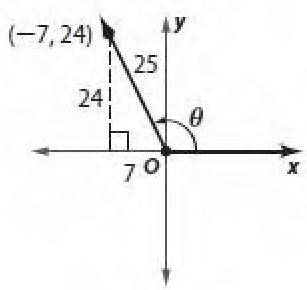
ANSWER:

$\frac{4\sqrt{5}}{9}; \frac{1}{9}; 4\sqrt{5}$

69. $\tan\theta = -\frac{24}{7}, \left(\frac{\pi}{2}, \pi\right)$

SOLUTION:

Since $\tan\theta = -\frac{24}{7}$ on the interval $\left(\frac{\pi}{2}, \pi\right)$, one point on the terminal side of θ has x-coordinate –7 and y-coordinate 24 as shown. The distance from the point to the origin is 25.



Using this point, we find that $\sin\theta = \frac{24}{25}$ and $\cos\theta = -\frac{7}{25}$. Now use the double-angle identities for sine, cosine, and tangent to find sin 2θ, cos 2θ, and tan 2θ.

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{24}{25}\right)\left(-\frac{7}{25}\right) = -\frac{336}{625}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(-\frac{7}{25}\right)^2 - 1 = 2\left(\frac{49}{625}\right) - 1 = \frac{98}{625} - \frac{625}{625} = -\frac{527}{625}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(-\frac{24}{7}\right)}{1-\left(-\frac{24}{7}\right)^2} = \frac{-\frac{48}{7}}{1-\frac{576}{49}} = \frac{-\frac{48}{7}}{-\frac{527}{49}} = \frac{336}{527}$$

Then $\sin 2\theta = -\frac{336}{625}$, $\cos 2\theta = -\frac{527}{625}$, and tan $2\theta = \frac{336}{527}$.

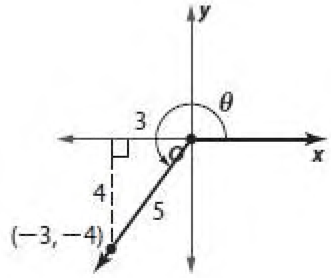
ANSWER:

$-\frac{336}{625}; -\frac{527}{625}; \frac{336}{527}$

70. $\sin\theta = -\frac{4}{5}, \left(\pi, \frac{3\pi}{2}\right)$

SOLUTION:

Since $\sin\theta = -\frac{4}{5}$ on the interval $\left(\pi, \frac{3\pi}{2}\right)$, one point on the terminal side of θ has y-coordinate −4 and a distance of 5 units from the origin as shown. The x-coordinate of this point is –3.



Using this point, we find that $\cos\theta = -\frac{3}{5}$ and $\tan\theta = \frac{4}{3}$. Now use the double-angle identities for sine and cosine to find sin 2θ and cos 2θ.

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2\left(-\frac{4}{5}\right)\left(-\frac{3}{5}\right)\\ &= \frac{24}{25}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 2\left(-\frac{3}{5}\right)^2 - 1\\ &= 2\left(\frac{9}{25}\right) - 1\\ &= \frac{18}{25} - \frac{25}{25}\\ &= -\frac{7}{25}\end{aligned}$$

Use the definition of tangent to find tan 2θ.

$$\begin{aligned}\tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta}\\ &= \frac{\frac{24}{25}}{-\frac{7}{25}}\\ &= -\frac{24}{7}\end{aligned}$$

ANSWER:

$\frac{24}{25}; -\frac{7}{25}; -\frac{24}{7}$

Locate the vertical asymptotes and sketch the graph of each function.

71. $y = \sec\left(x + \frac{\pi}{3}\right)$

SOLUTION:

The graph of $y = \sec\left(x + \frac{\pi}{3}\right)$ is the graph of y = sec x shifted $\frac{\pi}{3}$ units to the left. The period is $\frac{2\pi}{|1|}$ or 2π. Find the location of two vertical asymptotes.

$$\begin{aligned}bx + c &= -\frac{\pi}{2}\\ (1)x + \frac{\pi}{3} &= -\frac{\pi}{2}\\ x &= -\frac{5\pi}{6}\end{aligned} \quad \text{and} \quad \begin{aligned}bx + c &= \frac{3\pi}{2}\\ (1)x + \frac{\pi}{3} &= \frac{3\pi}{2}\\ x &= \frac{7\pi}{6}\end{aligned}$$

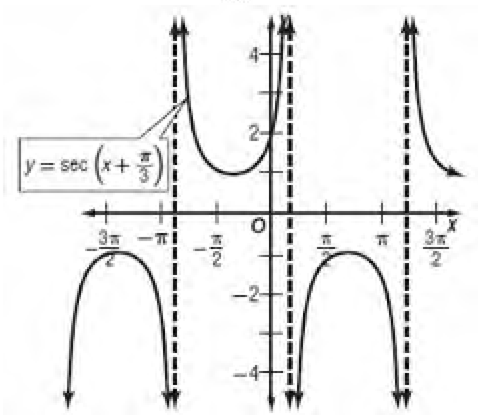
Create a table listing the coordinates of key points for $y = \sec\left(x + \frac{\pi}{3}\right)$ for one period on $\left[-\frac{5\pi}{6}, \frac{7\pi}{6}\right]$.

Function	y = sec x	$y = \sec\left(x + \frac{\pi}{3}\right)$
Vertical Asymptote	$x = -\frac{\pi}{2}$	$x = -\frac{5\pi}{6}$
Intermediate Point	(0, 1)	$\left(-\frac{\pi}{3}, 1\right)$
Vertical Asymptote	$x = \frac{\pi}{2}$	$x = \frac{\pi}{6}$
Intermediate Point	$(\pi, -1)$	$\left(\frac{2\pi}{3}, -2\right)$
Vertical Asymptote	$x = \frac{3\pi}{2}$	$x = \frac{7\pi}{6}$

Sketch the curve through the indicated key points for the function. Then repeat the pattern.

ANSWER:

asymptotes at $x = \frac{\pi}{6} + n\pi$



72. $y = 4 \cot \frac{x}{2}$

SOLUTION:

The graph of $y = 4 \cot \frac{x}{2}$ is the graph of $y = \cot x$ expanded horizontally. The period is $\frac{\pi}{\left|\frac{1}{2}\right|}$ or 2π. Find the location of two consecutive vertical asymptotes.

$$bx + c = 0 \qquad\qquad bx + c = \pi$$

$$\frac{x}{2} + 0 = 0 \quad \text{and} \quad \frac{x}{2} + 0 = \pi$$

$$x = 0 \qquad\qquad x = 2\pi$$

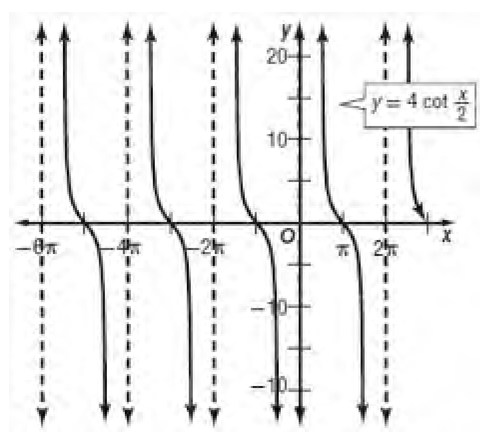
Create a table listing the coordinates of key points for $y = 4 \cot \frac{x}{2}$ for one period on $[0, 2\pi]$.

Function	y = cot x	$y = 4 \cot \frac{x}{2}$
Vertical Asymptote	$x = 0$	$x = 0$
Intermediate Point	$\left(\frac{\pi}{4}, 1\right)$	$\left(\frac{\pi}{2}, 4\right)$
***x*-int**	$\left(\frac{\pi}{2}, 0\right)$	$(\pi, 0)$
Intermediate Point	$\left(\frac{3\pi}{4}, -1\right)$	$\left(\frac{3\pi}{2}, -4\right)$
Vertical Asymptote	$x = \pi$	$x = 2\pi$

Sketch the curve through the indicated key points for the function. Then repeat the pattern.

ANSWER:

asymptotes at $x = 2\pi n$



73. $y = 2\cot\left[\frac{2}{3}\left(x - \frac{\pi}{2}\right)\right] + 0.75$

SOLUTION:

The graph of $y = 2\cot\left[\frac{2}{3}\left(x - \frac{\pi}{2}\right)\right] + 0.75$ is the graph of $y = \cot x$ expanded horizontally, shifted $\frac{\pi}{3}$ units to the right, and shifted 0.75 units up. The period is $\frac{\pi}{\left|\frac{2}{3}\right|} = \frac{3\pi}{2}$. Find the location of two consecutive vertical asymptotes.

$$\begin{aligned} bx + c &= 0 \\ \frac{2x}{3} - \frac{\pi}{3} &= 0 \\ \frac{2x}{3} &= \frac{\pi}{3} \\ x &= \frac{\pi}{2} \end{aligned} \quad \text{and} \quad \begin{aligned} bx + c &= \pi \\ \frac{2x}{3} - \frac{\pi}{3} &= \pi \\ \frac{2x}{3} &= \frac{4\pi}{3} \\ x &= 2\pi \end{aligned}$$

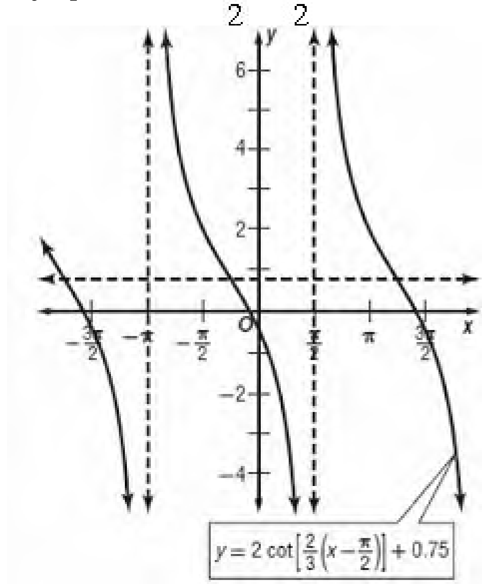
Create a table listing the coordinates of key points for $y = 2\cot\left[\frac{2}{3}\left(x - \frac{\pi}{2}\right)\right] + 0.75$ for one period on $\left[\frac{\pi}{2}, 2\pi\right]$.

Function	$y = \cot x$	$y = 2\cot\left[\frac{2}{3}\left(x - \frac{\pi}{2}\right)\right] + 0.75$
Vertical Asymptote	$x = 0$	$x = \frac{\pi}{2}$
Intermediate Point	$\left(\frac{\pi}{4}, 1\right)$	$\left(\frac{7\pi}{8}, 2.75\right)$
***x*-int**	$\left(\frac{\pi}{2}, 0\right)$	$\left(\frac{5\pi}{4}, 0\right)$
Intermediate Point	$\left(\frac{3\pi}{4}, -1\right)$	$\left(\frac{13\pi}{8}, -1.25\right)$
Vertical Asymptote	$x = \pi$	$x = 2\pi$

Sketch the curve through the indicated key points for the function. Then repeat the pattern.

ANSWER:

asymptotes at $x = \frac{\pi}{2} + \frac{3\pi}{2}n$



Find the exact values of the five remaining trigonometric functions of θ.

74. sec θ = 2, where sin θ > 0 and cos θ > 0

SOLUTION:

To find the other function values, you must find the coordinates of a point on the terminal side of θ. You know that sin θ is positive and cos θ is positive, so θ must lie in Quadrant I. This means that x is positive and y is positive.

Because sec θ = $\frac{r}{x}$ or $\frac{2}{1}$, use the point $(1, y)$ and $r = 2$ to find y.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ (2)^2 &= \left(\sqrt{1^2 + y^2}\right)^2 \\ 4 &= 1 + y^2 \\ y^2 &= 3 \\ y &= \sqrt{3} \end{aligned}$$

Use $x = 1$, $y = \sqrt{3}$, and $r = 2$ to write the five remaining trigonometric ratios.

$$\sin\theta = \frac{y}{r} \text{ or } \frac{\sqrt{3}}{2} \qquad \csc\theta = \frac{r}{y} = \frac{2}{\sqrt{3}} \text{ or } \frac{2\sqrt{3}}{3}$$
$$\cos\theta = \frac{x}{r} \text{ or } \frac{1}{2} \qquad \tan = \frac{y}{x} = \frac{\sqrt{3}}{1} \text{ or } \sqrt{3}$$
$$\cot\theta = \frac{x}{y} = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}$$

ANSWER:

sin θ = $\frac{\sqrt{3}}{2}$, cos θ = $\frac{1}{2}$, tan θ = $\sqrt{3}$, csc θ = $\frac{2\sqrt{3}}{3}$, cot θ = $\frac{\sqrt{3}}{3}$

75. csc θ = $\sqrt{5}$, where sin θ > 0 and cos θ > 0

SOLUTION:

To find the other function values, you must find the coordinates of a point on the terminal side of θ. You know that sin θ is positive cos θ are positive, so θ must lie in Quadrant I. This means that x is positive and y is positive.

Because csc θ = $\frac{r}{y}$ or $\frac{\sqrt{5}}{1}$, use the point $(x, 1)$ and $r = \sqrt{5}$ to find x.

$$\begin{aligned} \sqrt{5} &= \sqrt{x^2 + 1^2} \\ 5 &= \left(\sqrt{x^2 + 1}\right)^2 \\ 5 &= x^2 + 1 \\ 4 &= x^2 \\ x &= 2 \end{aligned}$$

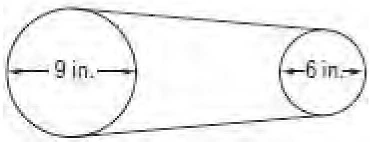
Use $x = 2$, $y = 1$, and $r = \sqrt{5}$ to write the five remaining trigonometric ratios.

$$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{5}} \text{ or } \frac{\sqrt{5}}{5} \qquad \tan\theta = \frac{y}{x} \text{ or } \frac{1}{2}$$
$$\cos\theta = \frac{x}{r} = \frac{2}{\sqrt{5}} \text{ or } \frac{2\sqrt{5}}{5} \qquad \sec\theta = \frac{r}{x} \text{ or } \frac{\sqrt{5}}{2}$$
$$\cot\theta = \frac{x}{y} \text{ or } 2$$

ANSWER:

sin θ = $\frac{\sqrt{5}}{5}$, cos θ = $\frac{2\sqrt{5}}{5}$, tan θ = $\frac{1}{2}$, sec θ = $\frac{\sqrt{5}}{2}$, cot θ = 2

76. **SAT/ACT** A pulley with a 9-inch diameter is belted to a pulley with a 6-inch diameter, as shown in the figure. If the larger pulley runs at 120 rpm (revolutions per minute), how fast does the smaller pulley run?



A 80 rpm
B 120 rpm
C 160 rpm
D 180 rpm
E 200 rpm

SOLUTION:

The pulley with a 9-inch diameter has a circumference of 9π. If it spins 120 rpm, then the total amount of belt that will pass along the pulley in one minute is 9π · 120 or 1080π. The pulley with a 6-inch diameter has a circumference of 6π. If 1080π of belt must pass along this pulley in one minute, then the pulley is moving at a speed of 1080π ÷ 6π or 180 rpm.
The correct answer is D.

ANSWER:

D

77. What type of conic is given by $\frac{3}{2 - 0.5\cos\theta}$?
F circle
G ellipse
H parabola
J hyperbola

SOLUTION:

Write the equation in standard form, $r = \frac{ed}{1 - e\cos\theta}$.

$$r = \frac{3}{2 - 0.5\cos\theta}$$
$$r = \frac{2\left(\frac{3}{2}\right)}{2(1 - 0.25\cos\theta)}$$
$$r = \frac{\frac{3}{2}}{1 - 0.25\cos\theta}$$

Since $e = 0.25$, the conic is an ellipse.
The correct answer is G.

ANSWER:

G

78. **REVIEW** Which of the following includes the component form and magnitude of $\overrightarrow{AB}$ with initial point $A(3, 4, -2)$ and terminal point $B(-5, 2, 1)$?
A $\langle -8, -2, 3\rangle$, $\sqrt{77}$
B $\langle 8, -2, 3\rangle$, $\sqrt{77}$
C $\langle -8, -2, 3\rangle$, $\sqrt{109}$
D $\langle 8, -2, 3\rangle$ $\sqrt{109}$

SOLUTION:

Find the component form of $\overrightarrow{AB}$.

$$\begin{aligned}\overrightarrow{AB} &= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle \\ &= \langle -5 - 3, 2 - 4, 1 - (-2)\rangle \\ &= \langle -8, -2, 3\rangle\end{aligned}$$

Use the component form to find the magnitude of $\overrightarrow{AB}$.

$$\begin{aligned}\left|\overrightarrow{AB}\right| &= \sqrt{(-8)^2 + (-2)^2 + 3^2} \\ &= \sqrt{77}\end{aligned}$$

The correct answer is A.

ANSWER:

A

79. **REVIEW** What is the eccentricity of the ellipse described by $\frac{y^2}{47} + \frac{(x-12)^2}{34} = 1$?

F 0.38
G 0.41
H 0.53
J 0.62

SOLUTION:

For the ellipse described by $\frac{y^2}{47} + \frac{(x-12)^2}{34} = 1$, $a^2 = 47$ and $b^2 = 34$. Use $c^2 = a^2 - b^2$ to find c.

$$c^2 = a^2 - b^2$$
$$c^2 = 47 - 34$$
$$c^2 = 13$$
$$c = \sqrt{13}$$

Since $a^2 = 47$, $a = \sqrt{47}$. Use $e = \frac{c}{a}$ to find the eccentricity of the ellipse.

$$e = \frac{c}{a}$$
$$e = \frac{\sqrt{13}}{\sqrt{47}}$$
$$e \approx 0.53$$

The correct answer is H.

ANSWER:

H