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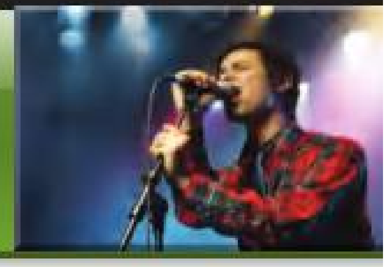
Polar Coordinates and Complex Numbers



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Preliminaries to Calculus

CHAPTER 5

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Chapter Topics

In this chapter, we present a collection of familiar topics, primarily those that we consider *essential* for the study of calculus. While we do not intend this chapter to be a comprehensive review of precalculus mathematics, we have tried to hit the highlights and provide you with some standard notation and language that we will use throughout the text.

As it grows, a chambered nautilus creates a spiral shell. Behind this beautiful geometry is a surprising amount of mathematics. The nautilus grows in such a way that the overall proportions of its shell remain constant. That is, if you draw a rectangle to circumscribe the shell, the ratio of height to width of the rectangle remains nearly constant.

There are several ways to represent this property mathematically. In polar coordinates (which we present in a later chapter), we study logarithmic spirals that have the property that the angle of growth is constant, corresponding to the constant proportions of a nautilus shell. Using basic geometry, you can divide the circumscribing rectangle into a sequence of squares as in the figure. The relative sizes of the squares form the famous Fibonacci sequence 1, 1, 2, 3, 5, 8, . . . , where each number in the sequence is the sum of the preceding two numbers.

The Fibonacci sequence has an amazing list of interesting properties. (Search on the Internet to see what we mean!) Numbers in the sequence have a surprising habit of showing up in nature, such as the number of petals on a lily (3), buttercup (5), marigold (13), black-eyed Susan (21) and pyrethrum (34). Although we have a very simple description of how to generate the Fibonacci sequence, think about how you might describe it as a function. A plot of the first several numbers in the sequence (shown in Figure 5.1 on the following page) should give you the impression of a graph curving up, perhaps a parabola or an exponential curve.

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Two aspects of this problem are important themes throughout the study of calculus. One of these is the importance of looking for patterns to help us better describe the world. A second theme is the interplay between graphs and functions. By connecting the techniques of algebra with the visual images provided by graphs, you will significantly improve your ability to solve real-world problems mathematically.

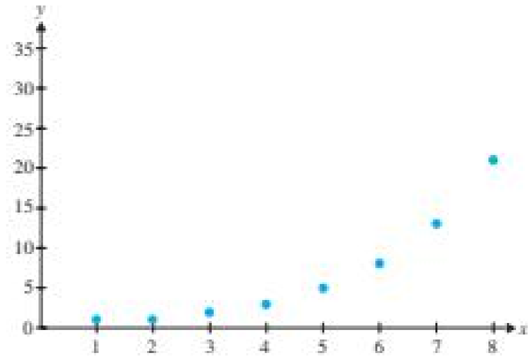


FIGURE 5.1
The Fibonacci sequence

5.1 POLYNOMIALS AND RATIONAL FUNCTIONS

The Real Number System and Inequalities

Our journey into calculus begins with the real number system, focusing on those properties that are of particular interest for calculus.

The set of **integers** consists of the whole numbers and their additive inverses: $0, \pm 1, \pm 2, \pm 3, \ldots$. A **rational number** is any number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. For example, $\frac{2}{3}$, $-\frac{7}{3}$ and $\frac{27}{125}$ are all rational numbers. Notice that every integer n is also a rational number, since we can write it as the quotient of two integers: $n = \frac{n}{1}$.

The **irrational numbers** are all those real numbers that cannot be written in the form $\frac{p}{q}$, where p and q are integers. Recall that rational numbers have decimal expansions that either terminate or repeat. For instance, $\frac{1}{2} = 0.5$, $\frac{1}{3} = 0.3333\overline{3}$, $\frac{1}{8} = 0.125$ and $\frac{1}{6} = 0.16666\overline{6}$ are all rational numbers. By contrast, irrational numbers have decimal expansions that do not repeat or terminate. For instance, three familiar irrational numbers and their decimal expansions are

$$\sqrt{2} = 1.41421\,35623\ldots,$$
$$\pi = 3.14159\,26535\ldots$$

and

$$e = 2.71828\,18284\ldots.$$

We picture the real numbers arranged along the number line displayed in Figure 5.2 (the **real line**). The set of real numbers is denoted by the symbol $\mathbb{R}$.

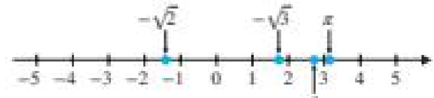


FIGURE 5.2
The real line

FIGURE 5.3
A closed interval

For real numbers a and b, where $a < b$, we define the **closed interval** $[a, b]$ to be the set of numbers between a and b, including a and b (the **endpoints**). That is,

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\},$$

as illustrated in Figure 5.3, where the solid circles indicate that a and b are included in $[a, b]$.

FIGURE 5.4
An open interval

Similarly, the **open interval** (a, b) is the set of numbers between a and b, but *not* including the endpoints a and b; that is,

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\},$$

as illustrated in Figure 5.4, where the open circles indicate that a and b are not included in (a, b). Similarly, we denote the set $\{x \in \mathbb{R} \mid x > a\}$ by the interval notation (a, ∞) and $\{x \in \mathbb{R} \mid x < a\}$ by $(-\infty, a)$. In both of these cases, it is important to recognize that ∞ and $-\infty$ are not real numbers and we are using this notation as a convenience.

You should already be very familiar with the following properties of real numbers.

THEOREM 1.1

If a and b are real numbers and $a < b$, then

(i) For any real number c, $a + c < b + c$.
(ii) For real numbers c and d, if $c < d$, then $a + c < b + d$.
(iii) For any real number $c > 0$, $a \cdot c < b \cdot c$.
(iv) For any real number $c < 0$, $a \cdot c > b \cdot c$.

REMARK 1.1

We need the properties given in Theorem 1.1 to solve inequalities. Notice that (i) says that you can add the same quantity to both sides of an inequality. Part (iii) says that you can multiply both sides of an inequality by a positive number. Finally, (iv) says that if you multiply both sides of an inequality by a negative number, the inequality is reversed.

We illustrate the use of Theorem 1.1 by solving a simple inequality.

EXAMPLE 1.1 Solving a Linear Inequality

Solve the linear inequality $2x + 5 < 13$.

Solution We can use the properties in Theorem 1.1 to solve for x. Subtracting 5 from both sides, we obtain

$$(2x + 5) - 5 < 13 - 5$$

or

$$2x < 8.$$

Dividing both sides by 2, we obtain

$$x < 4.$$

We often write the solution of an inequality in interval notation. In this case, we get the interval $(-\infty, 4)$. ■

You can deal with more complicated inequalities in the same way.

EXAMPLE 1.2 Solving a Two-Sided Inequality

Solve the two-sided inequality $6 < 1 - 3x \le 10$.

Solution First, recognize that this problem requires that we find values of x such that

$$6 < 1 - 3x \quad \text{and} \quad 1 - 3x \le 10.$$

It is most efficient to work with both inequalities simultaneously. First, subtract 1 from each term, to get

$$6 - 1 < (1 - 3x) - 1 \le 10 - 1$$

or $$5 < -3x \le 9.$$

Now, divide by -3, but be careful. Since $-3 < 0$, the inequalities are reversed. We have

$$\frac{5}{-3} > \frac{-3x}{-3} > \frac{9}{-3}$$

or $$-\frac{5}{3} > x \ge -3.$$

We usually write this as $$-3 \le x < -\frac{5}{3},$$

or in interval notation as $[-3, -\frac{5}{3})$. ■

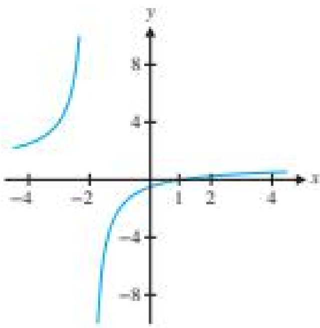
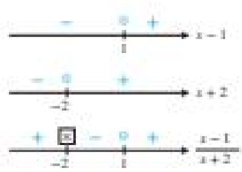


FIGURE 5.5
$y = \dfrac{x-1}{x+2}$



You will often need to solve inequalities involving fractions. We present a typical example in the following.

EXAMPLE 1.3 Solving an Inequality Involving a Fraction

Solve the inequality $\dfrac{x-1}{x+2} \ge 0$.

Solution In Figure 5.5, we show a graph of the function, which appears to indicate that the solution includes all $x < -2$ and $x \ge 1$. Carefully read the inequality and observe that there are only three ways to satisfy this: either both numerator and denominator are positive, both are negative or the numerator is zero. To visualize this, we draw number lines for each of the individual terms, indicating where each is positive, negative or zero, and use these to draw a third number line indicating the value of the quotient, as shown in the margin. In the third number line, we have placed an "⊠" above the -2 to indicate that the quotient is undefined at $x = -2$. From this last number line, you can see that the quotient is non-negative whenever $x < -2$ *or* $x \ge 1$. We write the solution in interval notation as $(-\infty, -2) \cup [1, \infty)$. Note that this solution is consistent with what we see in Figure 5.5. ■

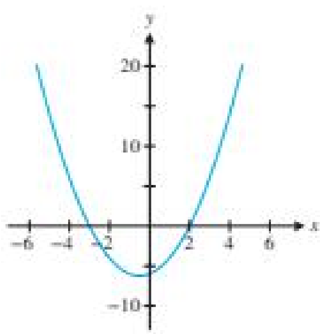
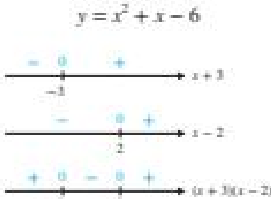


FIGURE 5.6
$y = x^2 + x - 6$



For inequalities involving a polynomial of degree 2 or higher, factoring the polynomial and determining where the individual factors are positive and negative, as in example 1.4, will lead to a solution.

EXAMPLE 1.4 Solving a Quadratic Inequality

Solve the quadratic inequality

$$x^2 + x - 6 > 0. \tag{1.1}$$

Solution In Figure 5.6, we show a graph of the polynomial on the left side of the inequality. Since this polynomial factors, (1.1) is equivalent to

$$(x+3)(x-2) > 0. \tag{1.2}$$

This can happen in only two ways: when both factors are positive or when both factors are negative. As in example 1.3, we draw number lines for both of the individual factors, indicating where each is positive, negative or zero, and use these to draw a number line representing the product. We show these in the margin. Notice that the third number line indicates that the product is positive whenever $x < -3$ *or* $x > 2$. We write this in interval notation as $(-\infty, -3) \cup (2, \infty)$. ■

No doubt, you will recall the following standard definition.

DEFINITION 1.1

The **absolute value** of a real number x is $|x| = \begin{cases} x, & \text{if } x \geq 0. \\ -x, & \text{if } x < 0 \end{cases}$

Make certain that you read Definition 1.1 correctly. If x is negative, then $-x$ is positive. This says that $|x| \geq 0$ for all real numbers x. For instance, using the definition,

$$|-4| = -(-4) = 4.$$

Notice that for any real numbers a and b,

$$|a \cdot b| = |a| \cdot |b|,$$

although

$$|a + b| \neq |a| + |b|,$$

in general. (To verify this, simply take $a = 5$ and $b = -2$ and compute both quantities.)

However, it is always true that

$$|a + b| \leq |a| + |b|.$$

This is referred to as the **triangle inequality**.

NOTES

For any two real numbers a and b, $|a - b|$ gives the *distance* between a and b. (See Figure 5.7.)

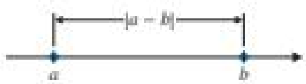


FIGURE 5.7
The distance between a and b

The interpretation of $|a - b|$ as the distance between a and b (see the Notes box) is particularly useful for solving inequalities involving absolute values. Wherever possible, we suggest that you use this interpretation to read what the inequality means, rather than merely following a procedure to produce a solution.

EXAMPLE 1.5 Solving an Inequality Containing an Absolute Value

Solve the inequality

$$|x - 2| < 5. \tag{1.3}$$

Solution First, take a few moments to read what this inequality *says*. Since $|x - 2|$ gives the distance from x to 2, (1.3) says that the *distance* from x to 2 must be *less than 5*. So, find all numbers x whose distance from 2 is less than 5. We indicate the set of all numbers within a distance 5 of 2 in Figure 5.8. You can now read the solution directly from the figure: $-3 < x < 7$ or in interval notation: $(-3, 7)$. ■

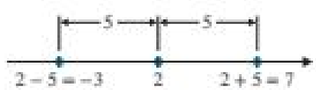


FIGURE 5.8
$|x - 2| < 5$

Many inequalities involving absolute values can be solved simply by reading the inequality correctly, as in example 1.6.

EXAMPLE 1.6 Solving an Inequality with a Sum Inside an Absolute Value

Solve the inequality

$$|x + 4| \leq 7. \tag{1.4}$$

Solution To use our distance interpretation, we must first rewrite (1.4) as

$$|x - (-4)| \leq 7.$$

This now says that the distance from x to -4 is less than or equal to 7. We illustrate the solution in Figure 5.9, from which it follows that $-11 \leq x \leq 3$ or $[-11, 3]$. ■

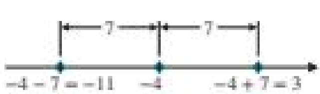


FIGURE 5.9
$|x + 4| \leq 7$

Recall that for any real number $r > 0$, $|x| < r$ is equivalent to the following inequality not involving absolute values:

$$-r < x < r.$$

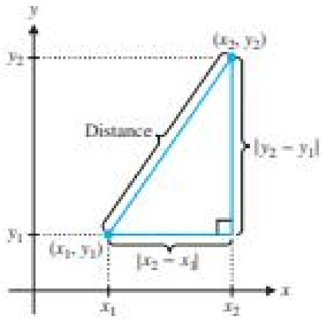


FIGURE 5.10
Distance

In example 1.7, we use this to revisit the inequality from example 1.5.

EXAMPLE 1.7 An Alternative Method for Solving Inequalities

Solve the inequality $|x - 2| < 5$.

Solution This is equivalent to the two-sided inequality

$$-5 < x - 2 < 5.$$

Adding 2 to each term, we get the solution

$$-3 < x < 7,$$

or in interval notation $(-3, 7)$, as before. ■

Recall that the distance between two points (x_1, y_1) and (x_2, y_2) is a simple consequence of the Pythagorean Theorem and is given by

$$d\{(x_1, y_1), (x_2, y_2)\} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

We illustrate this in Figure 5.10.

EXAMPLE 1.8 Using the Distance Formula

Find the distance between the points (1, 2) and (3, 4).

Solution The distance between (1, 2) and (3, 4) is

$$d\{(1, 2), (3, 4)\} = \sqrt{(3 - 1)^2 + (4 - 2)^2} = \sqrt{4 + 4} = \sqrt{8}. \blacksquare$$

Year	*U.S. Population*
1960	179,323,175
1970	203,302,031
1980	226,542,203
1990	248,709,873

x	*y*
0	179
10	203
20	227
30	249

Transformed data

○ Equations of Lines

The federal government conducts a nationwide census every 10 years to determine the population. Population data for several recent decades are shown in the accompanying table.

One difficulty with analyzing these data is that the numbers are so large. This problem is remedied by **transforming** the data. We can simplify the year data by defining x to be the number of years since 1960, so that 1960 corresponds to $x = 0$, 1970 corresponds to $x = 10$ and so on. The population data can be simplified by rounding the numbers to the nearest million. The transformed data are shown in the accompanying table and a scatter plot of these data points is shown in Figure 5.11.

The points in Figure 5.11 may appear to form a straight line. (Use a ruler and see if you agree.) To determine whether the points are, in fact, on the same line (such points are called **collinear**), we might consider the population growth in each of the indicated decades. From 1960 to 1970, the growth was 24 million. (That is, to move from the first point to the second, you increase x by 10 and increase y by 24.) Likewise, from 1970 to 1980, the growth was 24 million. However, from 1980 to 1990, the growth was only 22 million. Since the rate of growth is not constant, the data points do not fall on a line. This argument involves the familiar concept of *slope*.

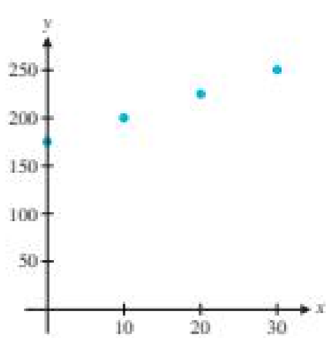


FIGURE 5.11
Population data

DEFINITION 1.2

For $x_1 \neq x_2$, the **slope** of the straight line through the points (x_1, y_1) and (x_2, y_2) is the number

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \qquad (1.5)$$

When $x_1 = x_2$ and $y_1 \neq y_2$, the line through (x_1, y_1) and (x_2, y_2) is **vertical** and the slope is undefined.

We often describe slope as "the change in y divided by the change in x," written $\frac{\Delta y}{\Delta x}$, or more simply as $\frac{\text{Rise}}{\text{Run}}$. (See Figure 5.12a.)

Referring to Figure 5.12b (where the line has positive slope), notice that for any four points A, B, D and E on the line, the two right triangles ΔABC and ΔDEF are similar. Recall that for similar triangles, the ratios of corresponding sides must be the same. In this case, this says that

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y'}{\Delta x'}$$

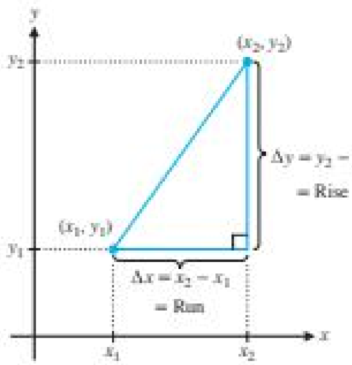


FIGURE 5.12a
Slope

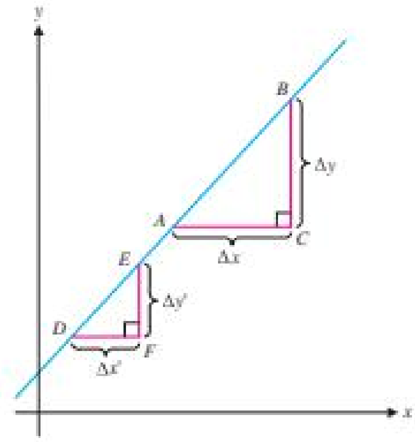


FIGURE 5.12b
Similar triangles and slope

and so, the slope is the same no matter which two points on the line are selected. Notice that a line is horizontal if and only if its slope is zero.

EXAMPLE 1.9 Finding the Slope of a Line

Find the slope of the line through the points (4, 3) and (2, 5).

Solution From (1.5), we get

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{2-4} = \frac{2}{-2} = -1. \blacksquare$$

EXAMPLE 1.10 Using Slope to Determine if Points Are Collinear

Use slope to determine whether the points (1, 2), (3, 10) and (4, 14) are collinear.

Solution First, notice that the slope of the line joining (1, 2) and (3, 10) is

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10-2}{3-1} = \frac{8}{2} = 4.$$

Similarly, the slope through the line joining (3, 10) and (4, 14) is

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{14-10}{4-3} = 4.$$

Since the slopes are the same, the points must be collinear. ■

Recall that if you know the slope and a point through which the line must pass, you have enough information to graph the line. The easiest way to graph a line is to plot two points and then draw the line through them. In this case, you need only to find a second point.

EXAMPLE 1.11 Graphing a Line

If a line passes through the point (2, 1) with slope $\frac{2}{3}$, find a second point on the line and then graph the line.

Solution Since slope is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, we take $m = \frac{2}{3}$, $y_1 = 1$ and $x_1 = 2$, to obtain

$$\frac{2}{3} = \frac{y_2 - 1}{x_2 - 2}.$$

You are free to choose the x-coordinate of the second point. For instance, to find the point at $x_2 = 5$, substitute this in and solve. From

$$\frac{2}{3} = \frac{y_2 - 1}{5 - 2} = \frac{y_2 - 1}{3},$$

we get $2 = y_2 - 1$ or $y_2 = 3$. A second point is then (5, 3). The graph of the line is shown in Figure 5.13a. An alternative method for finding a second point is to use the slope

$$m = \frac{2}{3} = \frac{\Delta y}{\Delta x}.$$

The slope of $\frac{2}{3}$ says that if we move three units to the right, we must move two units up to stay on the line, as illustrated in Figure 5.13b. ■

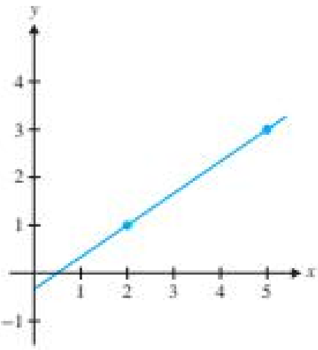


FIGURE 5.13a
Graph of straight line

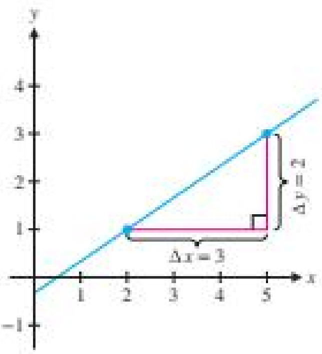


FIGURE 5.13b
Using slope to find a second point

In example 1.11, the choice of $x = 5$ was entirely arbitrary; you can choose any x-value you want to find a second point. Further, since x can be any real number, you can leave x as a variable and write out an equation satisfied by any point (x, y) on the line. In the general case of the line through the point (x_0, y_0) with slope m, we have from (1.5) that

$$m = \frac{y - y_0}{x - x_0}. \qquad (1.6)$$

Multiplying both sides of (1.6) by $(x - x_0)$, we get

$$y - y_0 = m(x - x_0)$$

or

POINT-SLOPE FORM OF A LINE

$$y = m(x - x_0) + y_0. \qquad (1.7)$$

EXAMPLE 1.12 Finding the Equation of a Line Given Two Points

Find an equation of the line through the points (3, 1) and (4, −1), and graph the line.

Solution From (1.5), the slope is $m = \dfrac{-1 - 1}{4 - 3} = \dfrac{-2}{1} = -2$. Using (1.7) with slope $m = -2$, x-coordinate $x_0 = 3$ and y-coordinate $y_0 = 1$, we get the equation of the line:

$$y = -2(x - 3) + 1. \qquad (1.8)$$

To graph the line, plot the points (3, 1) and (4, −1), and you can easily draw the line seen in Figure 5.14. ■

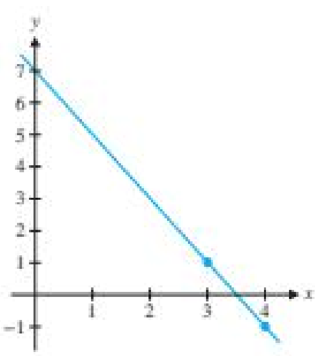


FIGURE 5.14
$y = -2(x - 3) + 1$

Although the point-slope form of the equation is often the most convenient to work with, the **slope-intercept form** is sometimes more convenient. This has the form

$$y = mx + b,$$

where m is the slope and b is the y-intercept (i.e., the place where the graph crosses the y-axis). In example 1.12, you simply multiply out (1.8) to get $y = -2x + 6 + 1$ or

$$y = -2x + 7.$$

As you can see from Figure 5.14, the graph crosses the y-axis at $y = 7$.

Theorem 1.2 presents a familiar result on parallel and perpendicular lines.

THEOREM 1.2

Two (nonvertical) lines are **parallel** if they have the same slope. Further, any two vertical lines are parallel. Two (nonvertical) lines of slope m_1 and m_2 are **perpendicular** whenever the product of their slopes is -1 (i.e., $m_1 \cdot m_2 = -1$). Also, any vertical line and any horizontal line are perpendicular.

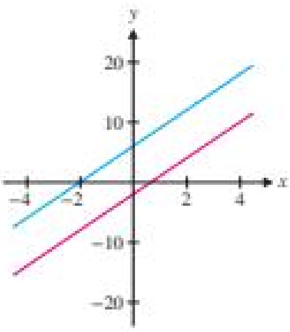


FIGURE 5.15
Parallel lines

Since we can read the slope from the equation of a line, it's a simple matter to determine when two lines are parallel or perpendicular. We illustrate this in examples 1.13 and 1.14.

EXAMPLE 1.13 Finding the Equation of a Parallel Line

Find an equation of the line parallel to $y = 3x - 2$ and through the point $(-1, 3)$.

Solution It's easy to read the slope of the line from the equation: $m = 3$. The equation of the parallel line is then

$$y = 3[x - (-1)] + 3$$

or simply $y = 3x + 6$. We show a graph of both lines in Figure 5.15. ■

EXAMPLE 1.14 Finding the Equation of a Perpendicular Line

Find an equation of the line perpendicular to $y = -2x + 4$ and intersecting the line at the point $(1, 2)$.

Solution The slope of $y = -2x + 4$ is -2. The slope of the perpendicular line is then $-1/(-2) = \frac{1}{2}$. Since the line must pass through the point $(1, 2)$, the equation of the perpendicular line is

$$y = \frac{1}{2}(x - 1) + 2 \qquad \text{or} \qquad y = \frac{1}{2}x + \frac{3}{2}.$$

We show a graph of the two lines in Figure 5.16. ■

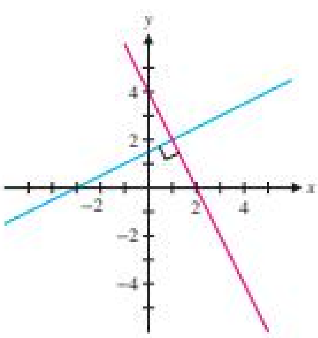


FIGURE 5.16
Perpendicular lines

We now return to this subsection's introductory example and use the equation of a line to estimate the population in the year 2000.

EXAMPLE 1.15 Using a Line to Predict Population

From the population data for the census years 1960, 1970, 1980 and 1990 given above Definition 1.2, predict the population for the year 2000.

Solution We began this subsection by showing that the points in the corresponding table are not collinear. Nonetheless, they are *nearly* collinear. So, why not use the straight line connecting the last two points (20, 227) and (30, 249) (corresponding to the populations in the years 1980 and 1990) to predict the population in 2000? (This is a simple example of a more general procedure called **extrapolation**.) The slope of the line joining the two data points is

$$m = \frac{249 - 227}{30 - 20} = \frac{22}{10} = \frac{11}{5}.$$

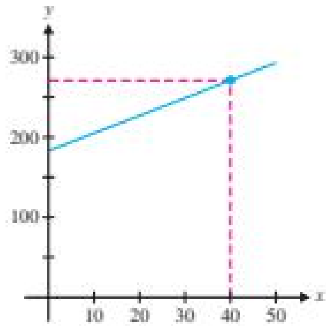
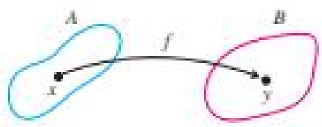


FIGURE 5.17
Population



REMARK 1.2

Functions can be defined by simple formulas, such as $f(x) = 3x + 2$, but in general, any correspondence meeting the requirement of matching exactly *one* y to each x defines a function.

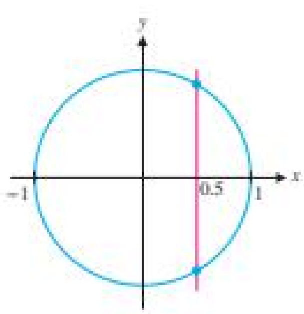


FIGURE 5.19a
Curve fails vertical line test

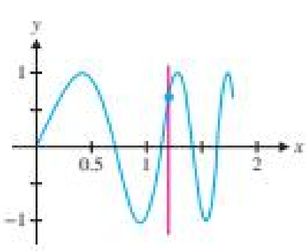


FIGURE 5.19b
Curve passes vertical line test

The equation of the line is then

$$y = \frac{11}{5}(x - 30) + 249.$$

See Figure 5.17 for a graph of the line. If we follow this line to the point corresponding to $x = 40$ (the year 2000), we have the predicted population

$$\frac{11}{5}(40 - 30) + 249 = 271.$$

That is, the predicted population is 271 million people. The actual census figure for 2000 was 281 million, which indicates that the U.S. population grew at a faster rate between 1990 and 2000 than in the previous decade. ■

Functions

For any two subsets A and B of the real line, we make the following familiar definition.

DEFINITION 1.3

A **function** f is a rule that assigns *exactly one* element y in a set B to each element x in a set A. In this case, we write $y = f(x)$.

We call the set A the **domain** of f. The set of all values $f(x)$ in B is called the **range** of f, written $\{y \mid y = f(x), \text{ for some } x \in A\}$. Unless explicitly stated otherwise, whenever a function f is given by a particular expression, the domain of f is the largest set of real numbers for which the expression is defined. We refer to x as the **independent variable** and to y as the **dependent variable**.

By the **graph** of a function f, we mean the graph of the equation $y = f(x)$. That is, the graph consists of all points (x, y), where x is in the domain of f and where $y = f(x)$.

Notice that not every curve is the graph of a function, since for a function, only one y-value can correspond to a given value of x. You can graphically determine whether a curve is the graph of a function by using the **vertical line test**: if any vertical line intersects the graph in more than one point, the curve is not the graph of a function, since in this case, there are two y-values for a given value of x.

EXAMPLE 1.16 Using the Vertical Line Test

Determine which of the curves in Figures 5.18a and 5.18b correspond to functions.

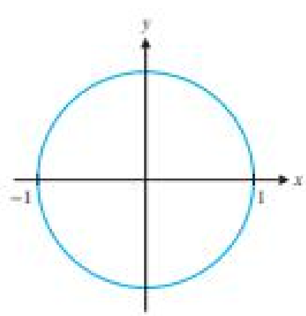


FIGURE 5.18a

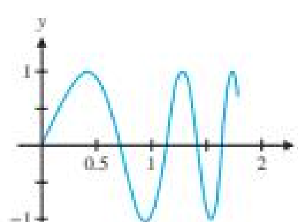


FIGURE 5.18b

Solution Notice that the circle in Figure 5.18a is not the graph of a function, since a vertical line at $x = 0.5$ intersects the circle twice. (See Figure 5.19a.) The graph in Figure 5.18b is the graph of a function, even though it swings up and down repeatedly. Although horizontal lines intersect the graph repeatedly, vertical lines, such as the one at $x = 1.2$, intersect only once. (See Figure 5.19b.) ■

The functions with which you are probably most familiar are *polynomials*. These are the simplest functions to work with because they are defined entirely in terms of arithmetic.

DEFINITION 1.4

A **polynomial** is any function that can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

where $a_0, a_1, a_2, \ldots, a_n$ are real numbers (the **coefficients** of the polynomial) with $a_n \neq 0$ and $n \geq 0$ is an integer (the **degree** of the polynomial).

Note that every polynomial function can be defined for all x's on the entire real line. Further, recognize that the graph of the linear (degree 1) polynomial $f(x) = ax + b$ is a straight line.

EXAMPLE 1.17 Sample Polynomials

The following are all examples of polynomials:

$f(x) = 2$ (polynomial of degree 0 or **constant**),

$f(x) = 3x + 2$ (polynomial of degree 1 or **linear** polynomial),

$f(x) = 5x^2 - 2x + 2/3$ (polynomial of degree 2 or **quadratic** polynomial),

$f(x) = x^3 - 2x + 1$ (polynomial of degree 3 or **cubic** polynomial),

$f(x) = -6x^4 + 12x^2 - 3x + 13$ (polynomial of degree 4 or **quartic** polynomial),

and

$f(x) = 2x^5 + 6x^4 - 8x^2 + x - 3$ (polynomial of degree 5 or **quintic** polynomial).

We show graphs of these six functions in Figures 5.20a–5.20f.

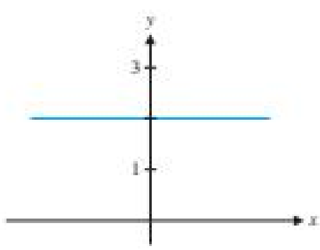


FIGURE 5.20a
$f(x) = 2$

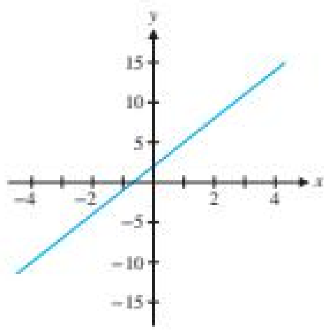


FIGURE 5.20b
$f(x) = 3x + 2$

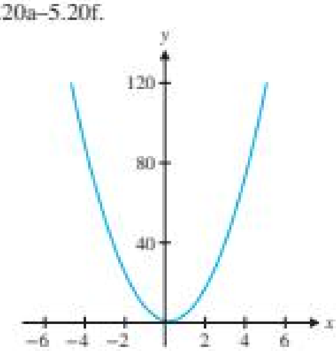


FIGURE 5.20c
$f(x) = 5x^2 - 2x + 2/3$

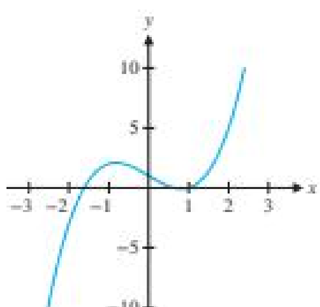


FIGURE 5.20d
$f(x) = x^3 - 2x + 1$

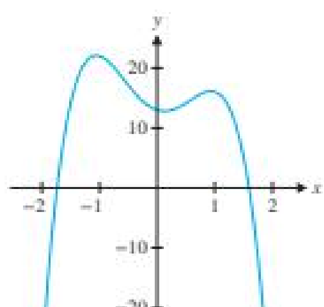


FIGURE 5.20e
$f(x) = -6x^4 + 12x^2 - 3x + 13$

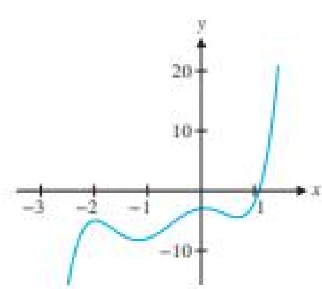


FIGURE 5.20f
$f(x) = 2x^5 + 6x^4 - 8x^2 + x - 3$

DEFINITION 1.5

Any function that can be written in the form

$$f(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomials, is called a **rational** function.

Notice that since $p(x)$ and $q(x)$ are polynomials, they can both be defined for all x, and so, the rational function $f(x) = \frac{p(x)}{q(x)}$ can be defined for all x for which $q(x) \neq 0$.

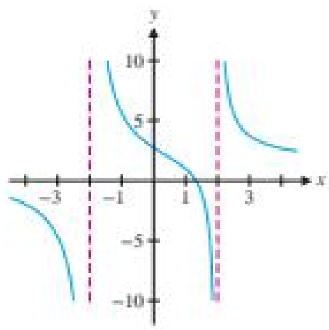


FIGURE 5.21

$f(x) = \frac{x^2 + 7x - 11}{x^2 - 4}$

EXAMPLE 1.18 A Sample Rational Function

Find the domain of the function

$$f(x) = \frac{x^2 + 7x - 11}{x^2 - 4}.$$

Solution Here, $f(x)$ is a rational function. We show a graph in Figure 5.21. Its domain consists of those values of x for which the denominator is non-zero. Notice that

$$x^2 - 4 = (x - 2)(x + 2)$$

and so, the denominator is zero if and only if $x = \pm 2$. This says that the domain of f is

$$\{x \in \mathbb{R} \mid x \neq \pm 2\} = (-\infty, -2) \cup (-2, 2) \cup (2, \infty). \blacksquare$$

The **square root** function is defined in the usual way. When we write $y = \sqrt{x}$, we mean that y is that number for which $y^2 = x$ and $y \geq 0$. In particular, $\sqrt{4} = 2$. Be careful not to write erroneous statements such as $\sqrt{4} = \pm 2$. In particular, be careful to write

$$\sqrt{x^2} = |x|.$$

Since $\sqrt{x^2}$ is asking for the *non-negative* number whose square is x^2, we are looking for $|x|$ and not x. We can say

$$\sqrt{x^2} = x, \textit{ only } \text{for } x \geq 0.$$

Similarly, for any integer $n \geq 2$, $y = \sqrt[n]{x}$ whenever $y^n = x$, where for n even, $x \geq 0$ and $y \geq 0$.

EXAMPLE 1.19 Finding the Domain of a Function Involving a Square Root or a Cube Root

Find the domains of $f(x) = \sqrt{x^2 - 4}$ and $g(x) = \sqrt[3]{x^2 - 4}$.

Solution Since even roots are defined only for non-negative values, $f(x)$ is defined only for $x^2 - 4 \geq 0$. Notice that this is equivalent to having $x^2 \geq 4$, which occurs when $x \geq 2$ or $x \leq -2$. The domain of f is then $(-\infty, -2] \cup [2, \infty)$. On the other hand, odd roots are defined for both positive and negative values. Consequently, the domain of g is the entire real line, $(-\infty, \infty)$. $\blacksquare$

We often find it useful to label intercepts and other significant points on a graph. Finding these points typically involves solving equations. A solution of the equation $f(x) = 0$ is called a **zero** of the function f or a **root** of the equation $f(x) = 0$. Notice that a zero of the function f corresponds to an x-intercept of the graph of $y = f(x)$.

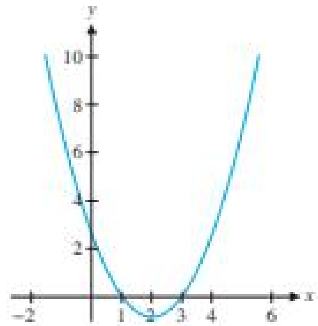


FIGURE 5.22
$y = x^2 - 4x + 3$

EXAMPLE 1.20 Finding Zeros by Factoring

Find all x- and y-intercepts of $f(x) = x^2 - 4x + 3$.

Solution To find the y-intercept, set $x = 0$ to obtain

$$y = 0 - 0 + 3 = 3.$$

To find the x-intercepts, solve the equation $f(x) = 0$. In this case, we can factor to get

$$f(x) = x^2 - 4x + 3 = (x - 1)(x - 3) = 0.$$

You can now read off the zeros: $x = 1$ and $x = 3$, as indicated in Figure 5.22. ■

Unfortunately, factoring is not always so easy. Of course, for the quadratic equation

$$ax^2 + bx + c = 0$$

(for $a \neq 0$), the solution(s) are given by the familiar **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXAMPLE 1.21 Finding Zeros Using the Quadratic Formula

Find the zeros of $f(x) = x^2 - 5x - 12$.

Solution You probably won't have much luck trying to factor this. However, from the quadratic formula, we have

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot (-12)}}{2 \cdot 1} = \frac{5 \pm \sqrt{25 + 48}}{2} = \frac{5 \pm \sqrt{73}}{2}.$$

So, the two solutions are given by $x = \frac{5}{2} + \frac{\sqrt{73}}{2} \approx 6.772$ and $x = \frac{5}{2} - \frac{\sqrt{73}}{2} \approx -1.772$. (No wonder you couldn't factor the polynomial!) ■

Finding zeros of polynomials of degree higher than 2 and other functions is usually trickier and is sometimes impossible. At the least, you can always find an approximation of any zero(s) by using a graph to zoom in closer to the point(s) where the graph crosses the x-axis, as we'll illustrate shortly. A more basic question, though, is to determine *how many* zeros a given function has. In general, there is no way to answer this question without the use of calculus. For the case of polynomials, however, Theorem 1.3 (a consequence of the Fundamental Theorem of Algebra) provides a clue.

THEOREM 1.3

A polynomial of degree n has *at most* n distinct zeros.

REMARK 1.3

Polynomials may also have complex zeros. For instance, $f(x) = x^2 + 1$ has only the complex zeros $x = \pm i$, where i is the imaginary number defined by $i = \sqrt{-1}$. We confine our attention in this text to real zeros.

Notice that Theorem 1.3 does not say how many zeros a given polynomial has, but rather, that the *maximum* number of distinct (i.e., different) zeros is the same as the degree. A polynomial of degree n may have anywhere from 0 to n distinct real zeros. However, polynomials of odd degree must have *at least one* real zero. For instance, for the case of a cubic polynomial, we have one of the three possibilities illustrated in Figures 5.23a, 5.23b and 5.23c. These are the graphs of the functions:

$$f(x) = x^3 - 2x^2 + 3 = (x + 1)(x^2 - 3x + 3),$$
$$g(x) = x^3 - x^2 - x + 1 = (x + 1)(x - 1)^2$$

and

$$h(x) = x^3 - 3x^2 - x + 3 = (x + 1)(x - 1)(x - 3),$$

respectively. Note that you can see from the factored form where the zeros are (and how many there are).

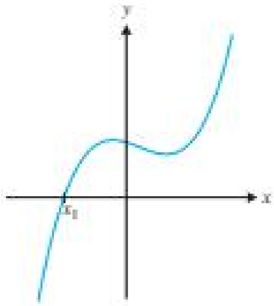


FIGURE 5.23a
One zero

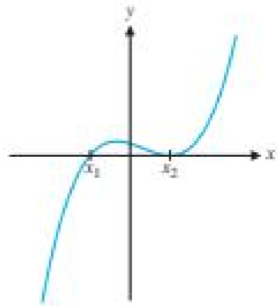


FIGURE 5.23b
Two zeros

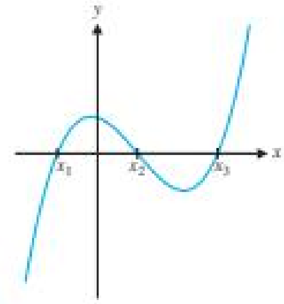


FIGURE 5.23c
Three zeros

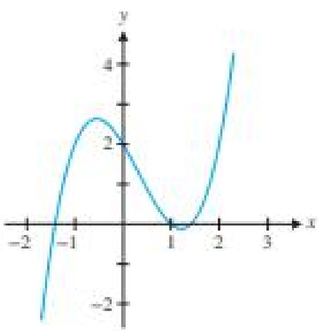


FIGURE 5.24a
$y = x^3 - x^2 - 2x + 2$

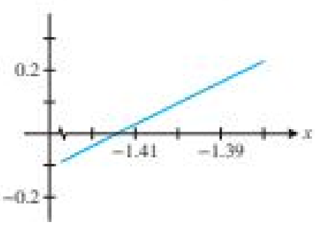


FIGURE 5.24b
Zoomed in on zero near $x = -1.4$

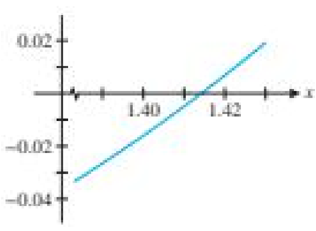


FIGURE 5.24c
Zoomed in on zero near $x = 1.4$

Theorem 1.4 provides an important connection between factors and zeros of polynomials.

THEOREM 1.4 (Factor Theorem)

For any polynomial function f, $f(a) = 0$ if and only if $(x - a)$ is a factor of $f(x)$

EXAMPLE 1.22 Finding the Zeros of a Cubic Polynomial

Find the zeros of $f(x) = x^3 - x^2 - 2x + 2$.

Solution By calculating $f(1)$, you can see that one zero of this function is $x = 1$, but how many other zeros are there? A graph of the function (see Figure 5.24a) shows that there are two other zeros of f, one near $x = -1.5$ and one near $x = 1.5$. You can find these zeros more precisely by using your graphing calculator or computer algebra system to zoom in on the locations of these zeros (as shown in Figures 5.24b and 5.24c). From these zoomed graphs it is clear that the two remaining zeros of f are near $x = 1.41$ and $x = -1.41$. You can make these estimates more precise by zooming in even more closely. Most graphing calculators and computer algebra systems can also find approximate zeros, using a built-in "solve" program. In a later chapter, we present a versatile method (called Newton's method) for obtaining accurate approximations to zeros. The only way to find the exact solutions is to factor the expression (using either long division or synthetic division). Here, we have

$$f(x) = x^3 - x^2 - 2x + 2 = (x - 1)(x^2 - 2) = (x - 1)(x - \sqrt{2})(x + \sqrt{2}),$$

from which you can see that the zeros are $x = 1$, $x = \sqrt{2}$ and $x = -\sqrt{2}$. ■

Recall that to find the points of intersection of two curves defined by $y = f(x)$ and $y = g(x)$, we set $f(x) = g(x)$ to find the x-coordinates of any points of intersection.

EXAMPLE 1.23 Finding the Intersections of a Line and a Parabola

Find the points of intersection of the parabola $y = x^2 - x - 5$ and the line $y = x + 3$.

Solution A sketch of the two curves (see Figure 5.25) shows that there are two intersections, one near $x = -2$ and the other near $x = 4$. To determine these precisely, we set the two functions equal and solve for x:

$$x^2 - x - 5 = x + 3.$$

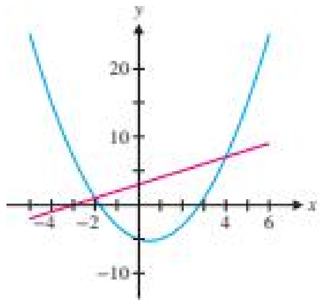


FIGURE 5.25
$y = x + 3$ and $y = x^2 - x - 5$

Subtracting $(x + 3)$ from both sides leaves us with

$$0 = x^2 - 2x - 8 = (x - 4)(x + 2).$$

This says that the solutions are exactly $x = -2$ and $x = 4$. We compute the corresponding y-values from the equation of the line $y = x + 3$ (or the equation of the parabola). The points of intersection are then $(-2, 1)$ and $(4, 7)$. Notice that these are consistent with the intersections seen in Figure 5.25. ■

Unfortunately, you won't always be able to solve equations exactly, as we did in examples 1.20–1.23. More difficult equations require the use of approximation techniques, graphing calculators, or computer algebra programs to find approximate solutions.

EXERCISES 5.1

WRITING EXERCISES

1. If the slope of the line passing through points A and B equals the slope of the line passing through points B and C, explain why the points A, B and C are collinear.

2. If a graph fails the vertical line test, it is not the graph of a function. Explain this result in terms of the definition of a function.

3. You should not automatically write the equation of a line in slope-intercept form. Compare the following forms of the same line: $y = 2.4(x - 1.8) + 0.4$ and $y = 2.4x - 3.92$. Given $x = 1.8$, which equation would you rather use to compute y? How about if you are given $x = 0$? For $x = 8$, is there any advantage to one equation over the other? Can you quickly read off the slope from either equation? Explain why neither form of the equation is "better."

4. To understand Definition 1.1, you must believe that $|x| = -x$ for negative x's. Using $x = -3$ as an example, explain in words why multiplying x by -1 produces the same result as taking the absolute value of x.

In exercises 1–10, solve the inequality.

1. $3x + 2 < 8$
2. $3 - 2x < 7$
3. $1 \le 2 - 3x < 6$
4. $-2 < 2x - 3 \le 5$
5. $\dfrac{x + 2}{x - 4} \ge 0$
6. $\dfrac{2x + 1}{x + 2} < 0$
7. $x^2 + 2x - 3 \ge 0$
8. $x^2 - 5x - 6 < 0$
9. $|x + 5| < 2$
10. $|2x + 1| < 4$

In exercises 11–14, determine if the points are collinear.

11. $(2, 1), (0, 2), (4, 0)$
12. $(3, 1), (4, 4), (5, 8)$
13. $(4, 1), (3, 2), (1, 3)$
14. $(1, 2), (2, 5), (4, 8)$

In exercises 15–18, find (a) the distance between the points, (b) the slope of the line through the given points, and (c) an equation of the line through the points.

15. $(1, 2), (3, 6)$
16. $(1, -2), (-1, -3)$
17. $(0.3, -1.4), (-1.1, -0.4)$
18. $(1.2, 2.1), (3.1, 2.4)$

In exercises 19–22, find a second point on the line with slope m and point P, graph the line and find an equation of the line.

19. $m = 2, P = (1, 3)$
20. $m = 0, P = (-1, 1)$
21. $m = 1.2, P = (2.3, 1.1)$
22. $m = -\frac{1}{4}, P = (-2, 1)$

In exercises 23–28, determine if the lines are parallel, perpendicular, or neither.

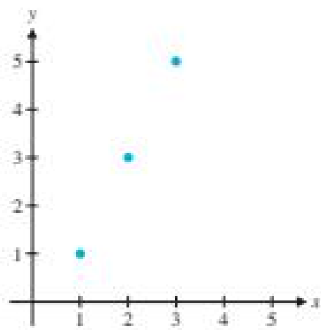
23. $y = 3(x - 1) + 2$ and $y = 3(x + 4) - 1$
24. $y = 2(x - 3) + 1$ and $y = 4(x - 3) + 1$
25. $y = -2(x + 1) - 1$ and $y = \frac{1}{2}(x - 2) + 3$
26. $y = 2x - 1$ and $y = -2x + 2$
27. $y = 3x + 1$ and $y = -\frac{1}{3}x + 2$
28. $x + 2y = 1$ and $2x + 4y = 3$

In exercises 29–32, find an equation of a line through the given point and (a) parallel to and (b) perpendicular to the given line.

29. $y = 2(x + 1) - 2$ at $(2, 1)$
30. $y = 3(x - 2) + 1$ at $(0, 3)$
31. $y = 2x + 1$ at $(3, 1)$
32. $y = 1$ at $(0, -1)$

In exercises 33 and 34, find an equation of the line through the given points and compute the y-coordinate of the point on the line corresponding to $x = 4$.

33.



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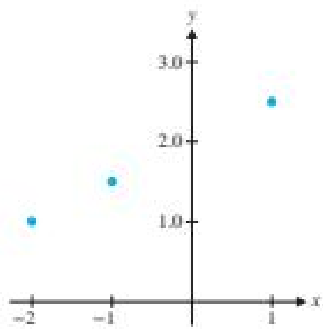
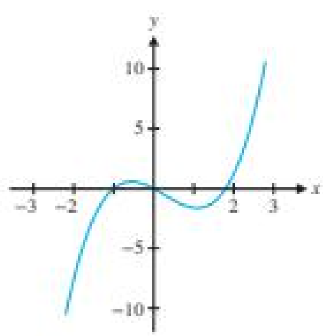


In exercises 35–38, use the vertical line test to determine whether the curve is the graph of a function.

35.



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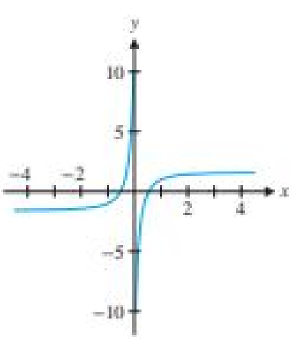

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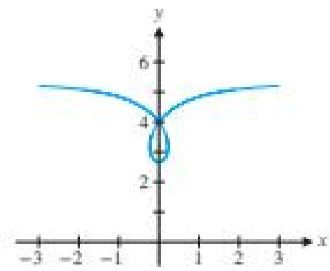

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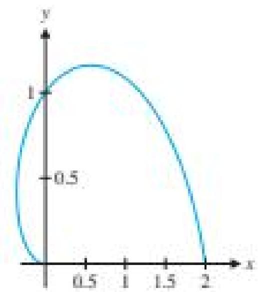


In exercises 39–42, identify the given function as polynomial, rational, both or neither.

39. $f(x) = x^3 - 4x + 1$

40. $f(x) = \dfrac{x^3 + 4x - 1}{x^4 - 1}$

41. $f(x) = \dfrac{x^2 + 2x - 1}{x + 1}$

42. $f(x) = \sqrt{x^2 + 1}$

In exercises 43–48, find the domain of the function.

43. $f(x) = \sqrt{x + 2}$

44. $f(x) = \sqrt[3]{x - 1}$

45. $f(x) = \dfrac{\sqrt{x^2 - x - 6}}{x - 5}$

46. $f(x) = \dfrac{\sqrt{x^2 - 4}}{\sqrt{9 - x^2}}$

47. $f(x) = \dfrac{4}{x^2 - 1}$

48. $f(x) = \dfrac{4x}{x^2 + 2x - 6}$

In exercises 49 and 50, find the indicated function values.

49. $f(x) = x^2 - x - 1$; $f(0), f(2), f(-3), f(1/2)$

50. $f(x) = \dfrac{3}{x}$; $f(1), f(10), f(100), f(1/3)$

In exercises 51 and 52, a brief description is given of a situation. For the indicated variable, state a reasonable domain.

51. A new candy bar is to be sold; x = number of candy bars sold in the first month.

52. A parking deck is to be built on a 200′-by-200′ lot; x = width of deck (in feet).

In exercises 53–56, discuss whether you think y would be a function of x.

53. y = grade you get on an exam, x = number of hours you study

54. y = probability of getting lung cancer, x = number of cigarettes smoked per day

55. y = a person's weight, x = number of minutes exercising per day

56. y = speed at which an object falls, x = weight of object

57. Figure A shows the speed of a bicyclist as a function of time. For the portions of this graph that are flat, what is happening to the bicyclist's speed? What is happening to the bicyclist's speed when the graph goes up? Goes down? Identify the portions of the graph that correspond to the bicyclist going uphill; downhill.

FIGURE A
Bicycle speed

58. Figure B shows the population of a small country as a function of time. During the time period shown, the country experienced two influxes of immigrants, a war and a plague. Identify these important events.

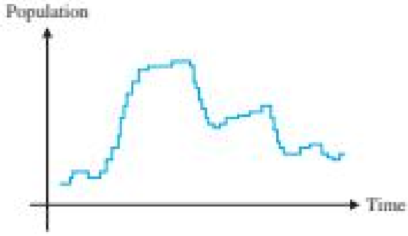


FIGURE B
Population

In exercises 59–64, find all intercepts of the given graph.

59. $y = x^2 - 2x - 8$

60. $y = x^2 + 4x + 4$

61. $y = x^3 - 8$

62. $y = x^3 - 3x^2 + 3x - 1$

63. $y = \dfrac{x^2 - 4}{x + 1}$

64. $y = \dfrac{2x - 1}{x^2 - 4}$

In exercises 65–72, factor and/or use the quadratic formula to find all zeros of the given function.

65. $f(x) = x^2 - 4x + 3$

66. $f(x) = x^2 + x - 12$

67. $f(x) = x^2 - 4x + 2$

68. $f(x) = 2x^2 + 4x - 1$

69. $f(x) = x^3 - 3x^2 + 2x$

70. $f(x) = x^3 - 2x^2 - x + 2$

71. $f(x) = x^6 + x^3 - 2$

72. $f(x) = x^3 + x^2 - 4x - 4$

In exercises 73 and 74, find all points of intersection.

73. $y = x^2 + 2x + 3$ and $y = x + 5$

74. $y = x^2 + 4x - 2$ and $y = 2x^2 + x - 6$

APPLICATIONS

1. The boiling point of water (in degrees Fahrenheit) at elevation h (in thousands of feet above sea level) is given by $B(h) = -1.8h + 212$. Find h such that water boils at 98.6°F. Why would this altitude be dangerous to humans?

2. The spin rate of a golf ball hit with a 9 iron has been measured at 9100 rpm for a 120-compression ball and at 10,000 rpm for a 60-compression ball. Most golfers use 90-compression balls. If the spin rate is a linear function of compression, find the spin rate for a 90-compression ball. Professional golfers often use 100-compression balls. Estimate the spin rate of a 100-compression ball.

3. The chirping rate of a cricket depends on the temperature. A species of tree cricket chirps 160 times per minute at 79°F and 100 times per minute at 64°F. Find a linear function relating temperature to chirping rate.

4. When describing how to measure temperature by counting cricket chirps, most guides suggest that you count the number of chirps in a 15-second time period. Use exercise 3 to explain why this is a convenient period of time.

5. A person has played a computer game many times. The statistics show that she has won 415 times and lost 120 times, and the winning percentage is listed as 78%. How many times in a row must she win to raise the reported winning percentage to 80%?

EXPLORATORY EXERCISES

1. Suppose you have a machine that will proportionally enlarge a photograph. For example, it could enlarge a 4×6 photograph to 8×12 by doubling the width and height. You could make an 8×10 picture by cropping 1 inch off each side. Explain how you would enlarge a $3\frac{1}{2} \times 5$ picture to an 8×10. A friend returns from Scotland with a $3\frac{1}{2} \times 5$ picture showing the Loch Ness monster in the outer $\frac{1}{4}$ inch on the right. If you use your procedure to make an 8×10 enlargement, does Nessie make the cut?

2. Solve the equation $|x - 2| + |x - 3| = 1$. (Hint: It's an unusual solution, in that it's more than just a couple of numbers.) Then, solve the equation $\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1$.

(Hint: If you make the correct substitution, you can use your solution to the previous equation.)

5.2 INVERSE FUNCTIONS

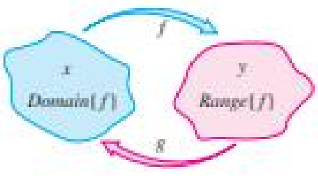


FIGURE 5.26
$g = f^{-1}$

The number of common *inverse* problems is immense. For instance, in an electrocardiogram (EKG), measurements of electrical activity on the surface of the body are used to infer something about the electrical activity on the surface of the heart. This is referred to as an *inverse* problem, since physicians are attempting to determine what *inputs* (i.e., the electrical activity on the surface of the heart) cause an observed *output* (the measured electrical activity on the surface of the chest).

In this section, we introduce the notion of an inverse function. The basic idea is simple enough. Given an output (that is, a value in the range of a given function), we wish to find the input (the value in the domain) that produced that output. That is, given a $y \in Range\{f\}$, find the $x \in Domain\{f\}$ for which $y = f(x)$. (See the illustration of the inverse function g shown in Figure 5.26.)

For instance, suppose that $f(x) = x^3$ and $y = 8$. Can you find an x such that $x^3 = 8$? That is, can you find the x-value corresponding to $y = 8$? (See Figure 5.27.) Of course, the solution of this particular equation is $x = \sqrt[3]{8} = 2$. In general, if $x^3 = y$, then $x = \sqrt[3]{y}$. In light of this, we say that the cube root function is the *inverse* of $f(x) = x^3$.

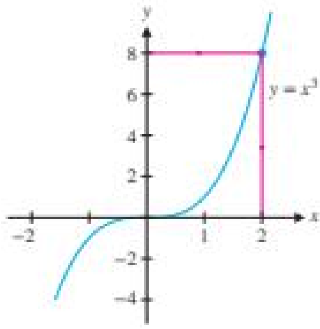


FIGURE 5.27
Finding the x-value corresponding to $y = 8$

EXAMPLE 2.1 Two Functions That Reverse the Action of Each Other

If $f(x) = x^3$ and $g(x) = x^{1/3}$, show that

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x,$$

for all x.

Solution For all real numbers x, we have

$$f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x$$

and

$$g(f(x)) = g(x^3) = (x^3)^{1/3} = x. ■$$

REMARK 2.1

Pay close attention to the notation. Notice that $f^{-1}(x)$ does *not* mean $\frac{1}{f(x)}$. We write the reciprocal of $f(x)$ as

$$\frac{1}{f(x)} = [f(x)]^{-1}.$$

Notice in example 2.1 that the action of f undoes the action of g and vice versa. We take this as the definition of an inverse function. (Again, think of Figure 5.26.)

DEFINITION 2.1

Assume that f and g have domains A and B, respectively, and that $f(g(x))$ is defined for all $x \in B$ and $g(f(x))$ is defined for all $x \in A$. If

$$f(g(x)) = x, \quad \text{for all } x \in B, \quad \text{and}$$
$$g(f(x)) = x, \quad \text{for all } x \in A,$$

we say that g is the **inverse** of f, written $g = f^{-1}$. Equivalently, f is the inverse of g, $f = g^{-1}$.

Observe that many familiar functions have no inverse.

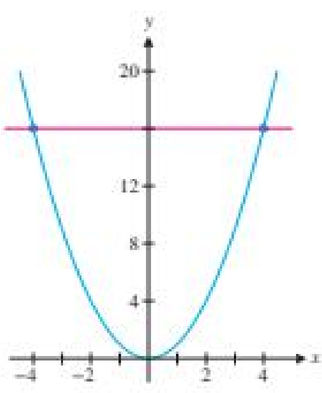


FIGURE 5.28
$y = x^2$

EXAMPLE 2.2 A Function with No Inverse

Show that $f(x) = x^2$ has no inverse on the interval $(-\infty, \infty)$.

Solution Notice that $f(4) = 16$ and $f(-4) = 16$. That is, there are *two* x-values that produce the same y-value. So, if we were to try to define an inverse of f, how would we define $f^{-1}(16)$? Look at the graph of $y = x^2$ (see Figure 5.28) to see what the problem is. For each $y > 0$, there are *two* x-values for which $y = x^2$. Because of this, the function does not have an inverse. ■

For $f(x) = x^2$, it is tempting to jump to the conclusion that $g(x) = \sqrt{x}$ is the inverse of $f(x)$. Notice that although $f(g(x)) = (\sqrt{x})^2 = x$ for all $x \geq 0$ (i.e., for all x in the domain of $g(x)$), it is *not* generally true that $g(f(x)) = \sqrt{x^2} = x$. In fact, this last equality holds *only* for $x \geq 0$. However, for $f(x) = x^2$ restricted to the domain $x \geq 0$, we do have that $f^{-1}(x) = \sqrt{x}$.

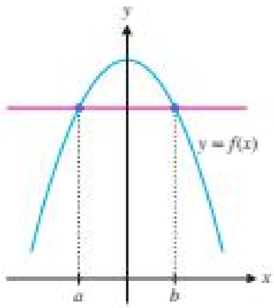


FIGURE 5.29
$f(a) = f(b)$, for $a \neq b$ So, f does not pass the horizontal line test and is not one-to-one.

DEFINITION 2.2

A function f is called **one-to-one** when for every $y \in Range\{f\}$, there is *exactly one* $x \in Domain\{f\}$ for which $y = f(x)$.

REMARK 2.2

Observe that an equivalent definition of one-to-one is the following. A function $f(x)$ is one-to-one if and only if the equality $f(a) = f(b)$ implies $a = b$. This version of the definition is often useful for proofs involving one-to-one functions.

It is helpful to think of the concept of one-to-one in graphical terms. Notice that a function f is one-to-one if and only if every horizontal line intersects the graph in at most one point. This is referred to as the **horizontal line test**. We illustrate this in Figures 5.29 and 5.30. The following result should now make sense.

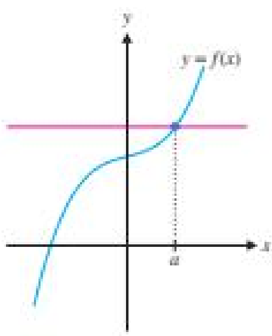


FIGURE 5.30
Every horizontal line intersects the curve in at most one point. So, f passes the horizontal line test and is one-to-one.

THEOREM 2.1

A function f has an inverse if and only if it is one-to-one.

This theorem simply says that every one-to-one function has an inverse and every function that has an inverse is one-to-one. However, it says nothing about how to find an inverse. For very simple functions, we can find inverses by solving equations.

EXAMPLE 2.3 Finding an Inverse Function

Find the inverse of $f(x) = x^3 - 5$.

Solution Note that it is not entirely clear from the graph (see Figure 5.31) whether f passes the horizontal line test. To find the inverse function, write $y = f(x)$ and solve for x (i.e., solve for the input x that produced the observed output y). We have

$$y = x^3 - 5.$$

Adding 5 to both sides and taking the cube root gives us

$$(y + 5)^{1/3} = (x^3)^{1/3} = x.$$

So, $x = f^{-1}(y) = (y + 5)^{1/3}$. Reversing the variables x and y gives us

$$f^{-1}(x) = (x + 5)^{1/3}. \blacksquare$$

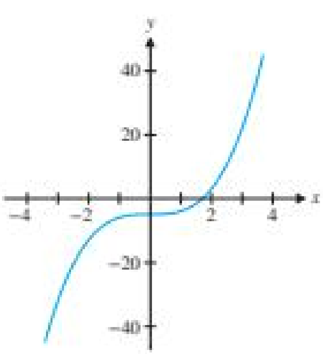


FIGURE 5.31
$y = x^3 - 5$

EXAMPLE 2.4 A Function That Is Not One-to-One

Show that $f(x) = 10 - x^4$ does not have an inverse.

Solution You can see from a graph (see Figure 5.32) that f is not one-to-one; for instance, $f(1) = f(-1) = 9$. Consequently, f does not have an inverse. ■

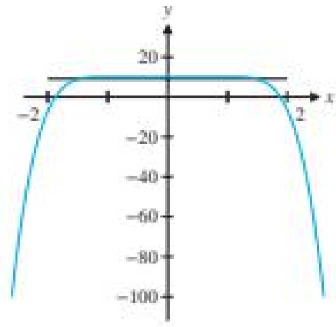


FIGURE 5.32
$y = 10 - x^4$

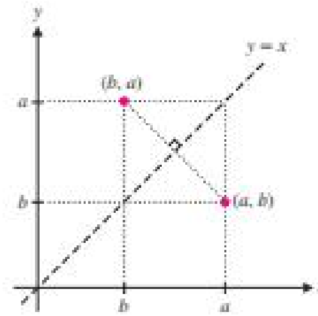


FIGURE 5.33
Reflection through $y = x$

Even when we can't find an inverse function explicitly, we can say something graphically. Notice that if (a, b) is a point on the graph of $y = f(x)$ and f has an inverse, f^{-1}, then since

$$b = f(a),$$

we have that

$$f^{-1}(b) = f^{-1}(f(a)) = a.$$

That is, (b, a) is a point on the graph of $y = f^{-1}(x)$. This tells us a great deal about the inverse function. In particular, we can immediately obtain any number of points on the graph of $y = f^{-1}(x)$, simply by inspection. Further, notice that the point (b, a) is the reflection of the point (a, b) through the line $y = x$. (See Figure 5.33.) It now follows that given the graph of any one-to-one function, you can draw the graph of its inverse simply by reflecting the entire graph through the line $y = x$.

In example 2.5, we illustrate the symmetry of a function and its inverse.

TODAY IN MATHEMATICS

Kim Rossmo (1955–Present) A Canadian criminologist who developed the Criminal Geographic Targeting algorithm that indicates the most probable area of residence for serial murderers, rapists and other criminals. Rossmo served 21 years with the Vancouver Police Department. His mentors were Professors Paul and Patricia Brantingham of Simon Fraser University. The Brantinghams developed Crime Pattern Theory, which predicts crime locations from where criminals live, work and play. Rossmo inverted their model and used the crime sites to determine where the criminal most likely lives. The premiere episode of the television drama *Numbers* was based on Rossmo's work.

EXAMPLE 2.5 The Graph of a Function and its Inverse

Draw a graph of $f(x) = x^3$ and its inverse.

Solution From example 2.1, the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$. Notice the symmetry of their graphs shown in Figure 5.34. ■

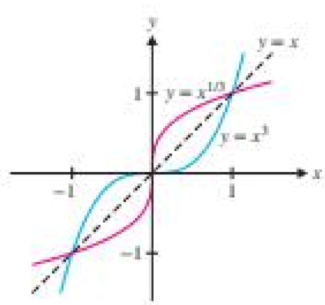


FIGURE 5.34
$y = x^3$ and $y = x^{1/3}$

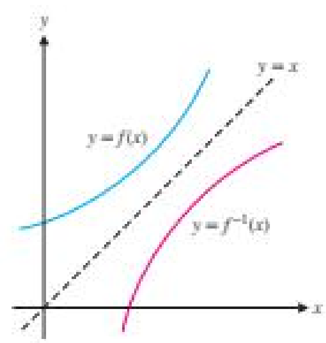


FIGURE 5.35
Graphs of f and f^{-1}

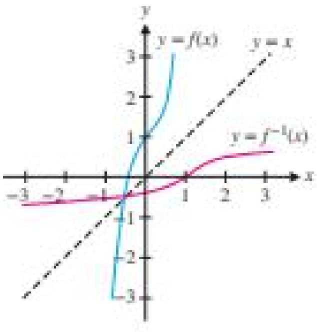


FIGURE 5.36
$y = f(x)$ and $y = f^{-1}(x)$

Most often, we cannot find a formula for an inverse function and must be satisfied with simply knowing that the inverse function exists. Observe that we can use the symmetry principle outlined above to draw the graph of an inverse function, even when we don't have a formula for that function. (See Figure 5.35.)

EXAMPLE 2.6 Drawing the Graph of an Unknown Inverse Function

Draw a graph of $f(x) = x^5 + 8x^3 + x + 1$ and its inverse.

Solution Although we are unable to find a formula for the inverse function, we can draw a graph of f^{-1} with ease. We simply take the graph of $y = f(x)$ and reflect it across the line $y = x$, as shown in Figure 5.36. (When we introduce parametric equations, we will see a clever way to draw this graph with a graphing calculator.)

EXERCISES 5.2

WRITING EXERCISES

1. Explain in words (and a picture) why the following is true: if $f(x)$ is increasing for all x [i.e., if $x_2 < x_1$, then $f(x_2) < f(x_1)$], then f has an inverse.
2. Suppose the graph of a function passes the horizontal line test. Explain why you know that the function has an inverse (defined on the range of the function).
3. Radar works by bouncing a high-frequency electromagnetic pulse off of a moving object, then measuring the disturbance in the pulse as it is bounced back. Explain why this is an inverse problem by identifying the input and output.
4. Each human disease has a set of symptoms associated with it. Physicians attempt to solve an inverse problem: given the symptoms, they try to identify the disease causing the symptoms. Explain why this is not a well-defined inverse problem (i.e., logically it is not always possible to correctly identify diseases from symptoms alone).

In exercises 1–4, show that $f(g(x)) = x$ and $g(f(x)) = x$ for all x:

1. $f(x) = x^5$ and $g(x) = x^{1/5}$
2. $f(x) = 4x^3$ and $g(x) = \left(\frac{1}{4}x\right)^{1/3}$
3. $f(x) = 2x^3 + 1$ or $g(x) = \sqrt[3]{\dfrac{x-1}{2}}$
4. $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{1-2x}{x}$ $(x \neq 0, x \neq -2)$

In exercises 5–12, determine whether the function has an inverse (is one-to-one). If so, find the inverse and graph both the function and its inverse.

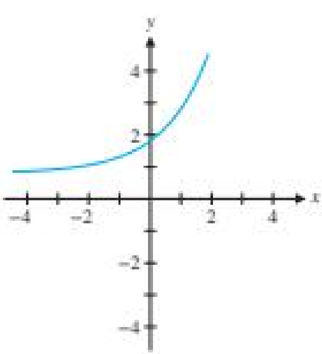
5. $f(x) = x^2 - 2$
6. $f(x) = x^3 + 4$
7. $f(x) = x^5 - 1$
8. $f(x) = x^5 + 4$
9. $f(x) = x^4 + 2$
10. $f(x) = x^4 - 2x - 1$
11. $f(x) = \sqrt{x^3 + 1}$
12. $f(x) = \sqrt{x^2 + 1}$

In exercises 13–18, assume that the function has an inverse. Without solving for the inverse, find the indicated function values.

13. $f(x) = x^3 + 4x - 1$, (a) $f^{-1}(-1)$, (b) $f^{-1}(4)$
14. $f(x) = x^3 + 2x + 1$, (a) $f^{-1}(1)$, (b) $f^{-1}(13)$
15. $f(x) = x^5 + 3x^3 + x$, (a) $f^{-1}(-5)$, (b) $f^{-1}(5)$
16. $f(x) = x^3 + 4x - 2$, (a) $f^{-1}(38)$, (b) $f^{-1}(3)$
17. $f(x) = \sqrt{x^3 + 2x + 4}$, (a) $f^{-1}(4)$, (b) $f^{-1}(2)$
18. $f(x) = \sqrt{x^5 + 4x^3 + 3x + 1}$, (a) $f^{-1}(3)$, (b) $f^{-1}(1)$

In exercises 19–22, use the given graph to graph the inverse function.

19.



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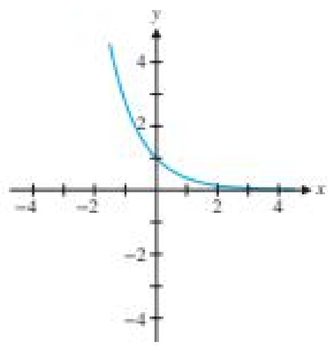

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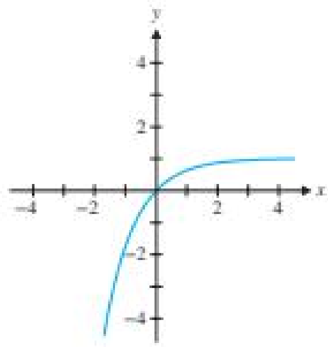

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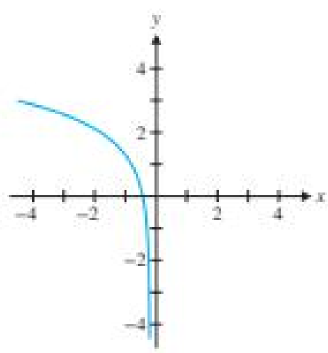


In exercises 23–26, assume that f has an inverse, and explain why the statement is true.

23. If the range of f is all $y > 0$, then the domain of f^{-1} is all $x > 0$.

24. If the graph of f includes the point (a, b), the graph of f^{-1} includes the point (b, a).

25. If the graph of f does not intersect the line $y = 3$, then $f^{-1}(x)$ is undefined at $x = 3$.

26. If the domain of f is all real numbers, then the range of f^{-1} is all real numbers.

In exercises 27–36, use a graph to determine whether the function is one-to-one. If it is, graph the inverse function.

27. $f(x) = x^3 - 5$

28. $f(x) = x^2 - 3$

29. $f(x) = x^3 + 2x - 1$

30. $f(x) = x^3 - 2x - 1$

31. $f(x) = x^5 - 3x^3 - 1$

32. $f(x) = x^5 + 4x^3 - 2$

33. $f(x) = \dfrac{1}{x+1}$

34. $f(x) = \dfrac{4}{x^2+1}$

35. $f(x) = \dfrac{x}{x+4}$

36. $f(x) = \dfrac{x}{\sqrt{x^2+4}}$

Exercises 37–46 involve inverse functions on restricted domains.

37. Show that $f(x) = x^2$ $(x \geq 0)$ and $g(x) = \sqrt{x}$ $(x \geq 0)$ are inverse functions. Graph both functions.

38. Show that $f(x) = x^2 - 1$ $(x \geq 0)$ and $g(x) = \sqrt{x+1}$ $(x \geq -1)$ are inverse functions. Graph both functions.

39. Graph $f(x) = x^2$ for $x \leq 0$ and verify that it is one-to-one. Find its inverse. Graph both functions.

40. Graph $f(x) = x^2 + 2$ for $x \leq 0$ and verify that it is one-to-one. Find its inverse. Graph both functions.

41. Graph $f(x) = (x - 2)^2$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

42. Graph $f(x) = (x + 1)^4$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

43. Graph $f(x) = \sqrt{x^2 - 2x}$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

44. Graph $f(x) = \dfrac{x}{x^2 - 4}$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

45. Graph $f(x) = \sin x$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

46. Graph $f(x) = \cos x$ and find an interval on which it is one-to-one. Find the inverse of the function restricted to that interval. Graph both functions.

APPLICATIONS

In applications 1–6, discuss whether the function described has an inverse.

1. The income of a company varies with time.

2. The height of a person varies with time.

3. For a dropped ball, its height varies with time.

4. For a ball thrown upward, its height varies with time.

5. The shadow made by an object depends on its three-dimensional shape.

6. The number of calories burned depends on how fast a person runs.

7. Suppose that your boss informs you that you have been awarded a 10% raise. The next week, your boss announces that due to circumstances beyond her control, all employees will have their salaries cut by 10%. Are you as well off now as you were two weeks ago? Show that increasing by 10% and decreasing by 10% are not inverse processes. Find the inverse for adding 10%. (Hint: To add 10% to a quantity you can multiply the quantity by 1.10.)

8. Suppose that an employee is offered a 6% raise plus a $500 bonus. Find the inverse of this pay increase if (a) the 6% raise comes before the bonus; (b) the 6% raise comes after the bonus.

EXPLORATORY EXERCISES

1. Find all values of k such that $f(x) = x^3 + kx + 1$ is one-to-one.

2. Find all values of k such that $f(x) = x^3 + 2x^2 + kx - 1$ is one-to-one.

5.3 TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

Many phenomena encountered in your daily life involve *waves.* For instance, music is transmitted from radio stations in the form of electromagnetic waves. Your radio receiver decodes these electromagnetic waves and causes a thin membrane inside the speakers to vibrate, which, in turn, creates pressure waves in the air. When these waves reach your ears, you hear the music from your radio. (See Figure 5.37.) Each of these waves is *periodic,* meaning that the basic shape of the wave is repeated over and over again. The mathematical description of such phenomena involves periodic functions, the most familiar of which are the trigonometric functions. First, we remind you of a basic definition.

Eugenia Gordeeva/Shutterstock

FIGURE 5.37
Sound waves

NOTES
When we discuss the period of a function, we most often focus on the fundamental period.

DEFINITION 3.1

A function f is **periodic** of **period** T if

$$f(x + T) = f(x)$$

for all x such that x and $x + T$ are in the domain of f. The smallest such number $T > 0$ is called the **fundamental period.**

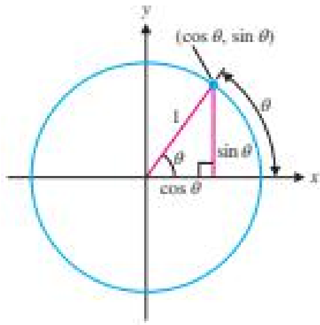


FIGURE 5.38
Definition of $\sin\theta$ and $\cos\theta$: $\cos\theta = x$ and $\sin\theta = y$

There are several equivalent ways of defining the sine and cosine functions. We want to emphasize a simple definition from which you can easily reproduce many of the basic properties of these functions. Referring to Figure 5.38, begin by drawing the unit circle $x^2 + y^2 = 1$. Let θ be the angle measured (counterclockwise) from the positive x-axis to the line segment connecting the origin to the point (x, y) on the circle. Here, we measure θ in **radians**, given by the length of the arc indicated in the figure. Again referring to Figure 5.38, we define $\sin\theta$ to be the y-coordinate of the point on the circle and $\cos\theta$ to be the x-coordinate of the point. From this definition, it follows that $\sin\theta$ and $\cos\theta$ are defined for all values of θ, so that each has domain $-\infty < \theta < \infty$, while the range for each of these functions is the interval $[-1, 1]$.

REMARK 3.1

Unless otherwise noted, we always measure angles in radians.

Note that since the circumference of a circle ($C = 2\pi r$) of radius 1 is 2π, we have that 360° corresponds to 2π radians. Similarly, 180° corresponds to π radians, 90° corresponds to $\pi/2$ radians, and so on. In the accompanying table, we list some common angles as measured in degrees, together with the corresponding radian measures.

Angle in degrees	0°	30°	45°	60°	90°	135°	180°	270°	360°
Angle in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π

THEOREM 3.1

The functions $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$ are periodic, of period 2π.

PROOF

Referring to Figure 5.38, since a complete circle is 2π radians, adding 2π to any angle takes you all the way around the circle and back to the same point (x, y). This says that

$$\sin(\theta + 2\pi) = \sin\theta$$

and

$$\cos(\theta + 2\pi) = \cos\theta,$$

for all values of θ. Furthermore, 2π is the smallest positive angle for which this is true. ■

You are likely already familiar with the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ shown in Figures 5.39a and 5.39b, respectively.

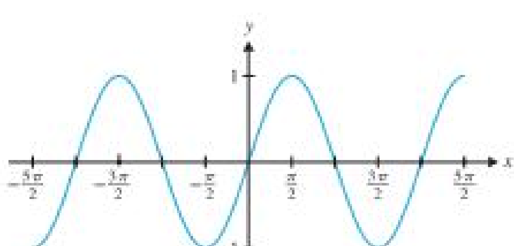


FIGURE 5.39a
$y = \sin x$

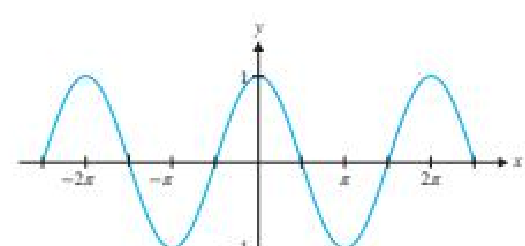


FIGURE 5.39b
$y = \cos x$

x	$\sin x$	$\cos x$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
π	0	-1
$\frac{3\pi}{2}$	-1	0
2π	0	1

Notice that you could slide the graph of $y = \sin x$ slightly to the left or right and get an exact copy of the graph of $y = \cos x$. Specifically, we have the relationship

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x.$$

The accompanying table lists some common values of sine and cosine. Notice that many of these can be read directly from diagrams similar to Figure 5.38.

EXAMPLE 3.1 Solving Equations Involving Sines and Cosines

Find all solutions of the equations (a) $2 \sin x - 1 = 0$ and (b) $\cos^2 x - 3 \cos x + 2 = 0$.

Solution For (a), notice that $2 \sin x - 1 = 0$ if $2 \sin x = 1$ or $\sin x = \frac{1}{2}$. From the unit circle, we find that $\sin x = \frac{1}{2}$ if $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Since $\sin x$ has period 2π, additional solutions are $\frac{\pi}{6} + 2\pi, \frac{5\pi}{6} + 2\pi, \frac{\pi}{6} + 4\pi$ and so on. A convenient way of indicating that *any* integer multiple of 2π can be added to either solution is to write $x = \frac{\pi}{6} + 2n\pi$ or $x = \frac{5\pi}{6} + 2n\pi$, for any integer n. Part (b) may look rather difficult at first. However, notice that it looks like a quadratic equation using $\cos x$ instead of x. With this clue, you can factor the left-hand side to get

$$0 = \cos^2 x - 3 \cos x + 2 = (\cos x - 1)(\cos x - 2),$$

from which it follows that either $\cos x = 1$ or $\cos x = 2$. Since $-1 \le \cos x \le 1$ for all x, the equation $\cos x = 2$ has no solution. However, we get $\cos x = 1$ if $x = 0, 2\pi$ or any integer multiple of 2π. We can summarize all the solutions by writing $x = 2n\pi$, for any integer n. ■

REMARK 3.2

Instead of writing $(\sin \theta)^2$ or $(\cos \theta)^2$, we usually use the notation $\sin^2 \theta$ and $\cos^2 \theta$, respectively. Further, we often suppress parentheses and write, for example, $\sin 2x$ instead of $\sin (2x)$.

We now give definitions of the remaining four trigonometric functions.

DEFINITION 3.2

The **tangent** function is defined by $\tan x = \frac{\sin x}{\cos x}$.

The **cotangent** function is defined by $\cot x = \frac{\cos x}{\sin x}$.

The **secant** function is defined by $\sec x = \frac{1}{\cos x}$.

The **cosecant** function is defined by $\csc x = \frac{1}{\sin x}$.

REMARK 3.3

Most calculators have keys for the functions $\sin x$, $\cos x$ and $\tan x$, but not for the other three trigonometric functions. This reflects the central role that $\sin x$, $\cos x$ and $\tan x$ play in applications. To calculate function values for the other three trigonometric functions, you can simply use the identities

$\cot x = \frac{1}{\tan x}$, $\sec x = \frac{1}{\cos x}$ and $\csc x = \frac{1}{\sin x}$.

We show graphs of these functions in Figures 5.40a, 5.40b, 5.40c and 5.40d. Notice in each graph the locations of the vertical asymptotes. For the "co" functions $\cot x$ and $\csc x$, the division by $\sin x$ causes vertical asymptotes at $0, \pm\pi, \pm2\pi$ and so on (where $\sin x = 0$). For $\tan x$ and $\sec x$, the division by $\cos x$ produces vertical asymptotes at $\pm\pi/2, \pm3\pi/2, \pm5\pi/2$ and so on (where $\cos x = 0$). Once you have determined the vertical asymptotes, the graphs are relatively easy to draw.

Notice that $\tan x$ and $\cot x$ are periodic, of period π, while $\sec x$ and $\csc x$ are periodic, of period 2π.

It is important to learn the effect of slight modifications of these functions. We present a few ideas here and in the exercises.

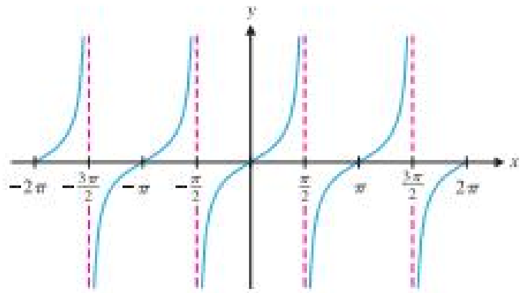


FIGURE 5.40a
$y = \tan x$

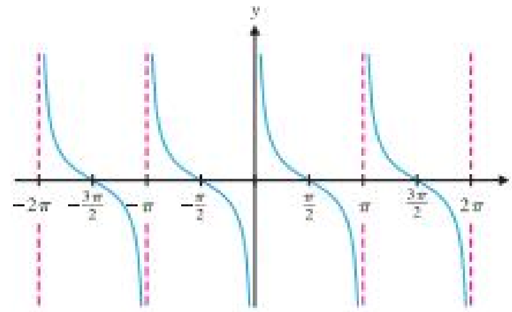


FIGURE 5.40b
$y = \cot x$

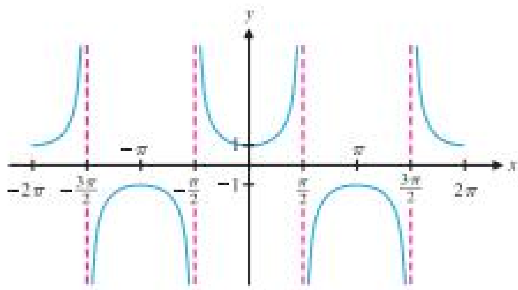


FIGURE 5.40c
$y = \sec x$

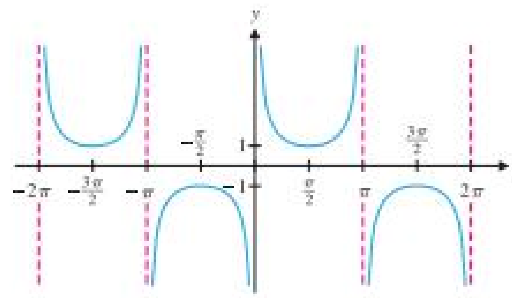


FIGURE 5.40d
$y = \csc x$

EXAMPLE 3.2 Altering Amplitude and Period

Graph $y = 2 \sin x$ and $y = \sin 2x$, and describe how each differs from the graph of $y = \sin x$. (See Figure 5.41a.)

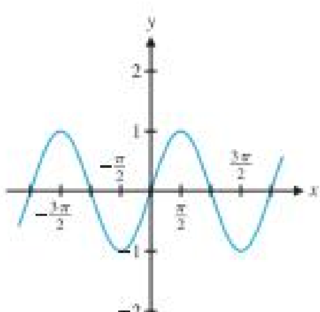


FIGURE 5.41a
$y = \sin x$

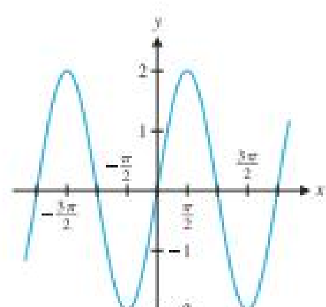


FIGURE 5.41b
$y = 2 \sin x$

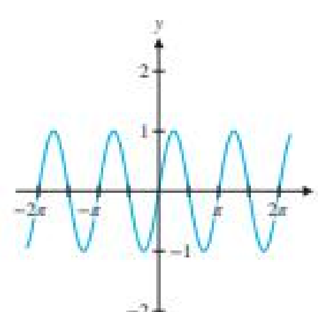


FIGURE 5.41c
$y = \sin (2x)$

Solution The graph of $y = 2 \sin x$ is given in Figure 5.41b. Notice that this graph is similar to the graph of $y = \sin x$, except that the y-values oscillate between -2 and 2 instead of -1 and 1. Next, the graph of $y = \sin 2x$ is given in Figure 5.41c. In this case, the graph is similar to the graph of $y = \sin x$ except that the period is π instead of 2π (so that the oscillations occur twice as fast). ■

The results in example 3.2 can be generalized. For $A > 0$, the graph of $y = A \sin x$ oscillates between $y = -A$ and $y = A$. In this case, we call A the **amplitude** of the sine curve. Notice that for any positive constant c, the period of $y = \sin cx$ is $2\pi/c$. Similarly, for the function $A \cos cx$, the amplitude is A and the period is $2\pi/c$.

The sine and cosine functions can be used to model sound waves. A pure tone (think of a tuning fork note) is a pressure wave described by the sinusoidal function $A \sin ct$. (Here, we are using the variable t, since the air pressure is a function of *time*.) The amplitude A determines how loud the tone is perceived to be and the period determines the pitch of the note. In this setting, it is convenient to talk about the **frequency** $f = c/2\pi$. The higher the frequency is, the higher the pitch of the note will be. (Frequency is measured in hertz, where 1 hertz equals 1 cycle per second.) Note that the frequency is simply the reciprocal of the period.

EXAMPLE 3.3 Finding Amplitude, Period and Frequency

Find the amplitude, period and frequency of (a) $f(x) = 4 \cos 3x$ and (b) $g(x) = 2 \sin(x/3)$.

Solution (a) For $f(x)$, the amplitude is 4, the period is $2\pi/3$ and the frequency is $3/(2\pi)$. (See Figure 5.42a.) (b) For $g(x)$, the amplitude is 2, the period is $2\pi/(1/3) = 6\pi$ and the frequency is $1/(6\pi)$. (See Figure 5.42b.)

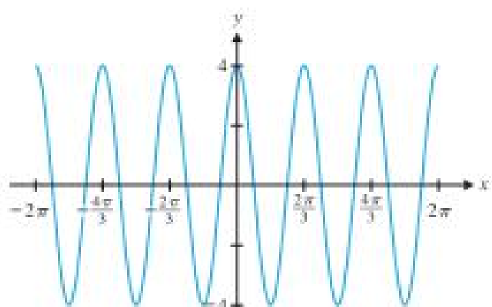


FIGURE 5.42a
$y = 4 \cos 3x$

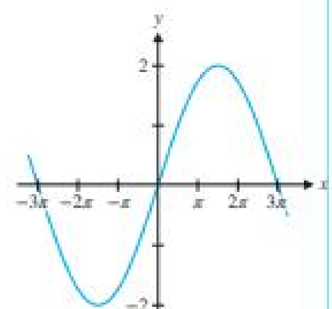


FIGURE 5.42b
$y = 2 \sin (x/3)$

There are numerous formulas or **identities** that are helpful in manipulating the trigonometric functions. You should observe that, from the definition of $\sin \theta$ and $\cos \theta$ (see Figure 5.38), the Pythagorean Theorem gives us the familiar identity

$$\sin^2 \theta + \cos^2 \theta = 1,$$

since the hypotenuse of the indicated triangle is 1. This is true for any angle θ. In addition,

$$\sin(-\theta) = -\sin \theta \quad \text{and} \quad \cos(-\theta) = \cos \theta$$

We list several important identities in Theorem 3.2.

THEOREM 3.2

For any real numbers α and β, the following identities hold:

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha \tag{3.1}$$

$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \tag{3.2}$$

$$\sin^2 \alpha = \tfrac{1}{2}(1 - \cos 2\alpha) \tag{3.3}$$

$$\cos^2 \alpha = \tfrac{1}{2}(1 + \cos 2\alpha). \tag{3.4}$$

From the basic identities summarized in Theorem 3.2, numerous other useful identities can be derived. We derive two of these in example 3.4.

EXAMPLE 3.4 Deriving New Trigonometric Identities

Derive the identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

Solution These can be obtained from formulas (3.1) and (3.2), respectively, by substituting $\alpha = \theta$ and $\beta = \theta$. Alternatively, the identity for $\cos 2\theta$ can be obtained by subtracting equation (3.3) from equation (3.4). ■

The Inverse Trigonometric Functions

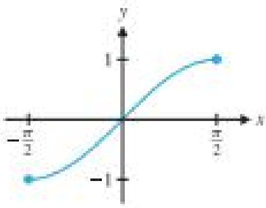


FIGURE 5.43
$y = \sin x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

We now expand the set of functions available to you by defining inverses to the trigonometric functions. To get started, look at a graph of $y = \sin x$. (See Figure 5.41a.) Notice that we cannot define an inverse function, since $\sin x$ is not one-to-one. Although the sine function does not have an inverse function, we can define one by modifying the domain of the sine. We do this by choosing a portion of the sine curve that passes the horizontal line test. If we restrict the domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $y = \sin x$ is one-to-one there (see Figure 5.43) and, hence, has an inverse. We thus define the **inverse sine** function by

$$y = \sin^{-1}x \quad \text{if and only if} \quad \sin y = x \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}. \tag{3.5}$$

Think of this definition as follows: if $y = \sin^{-1}x$, then y is the angle (between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$) for which $\sin y = x$. Note that we could have selected any interval on which $\sin x$ is one-to-one, but $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the most convenient. To verify that these are inverse functions, observe that

$$\sin(\sin^{-1}x) = x, \quad \text{for all } x \in [-1, 1]$$

and

$$\sin^{-1}(\sin x) = x, \quad \text{for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{3.6}$$

REMARK 3.4

Mathematicians often use the notation **arcsin *x*** in place of $\sin^{-1}x$. People read $\sin^{-1}x$ interchangeably as "inverse sine of x" or "arcsine of x."

Read equation (3.6) very carefully. It *does not* say that $\sin^{-1}(\sin x) = x$ for *all* x, but rather, *only* for those in the restricted domain, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For instance, $\sin^{-1}(\sin\pi) \ne \pi$, since

$$\sin^{-1}(\sin\pi) = \sin^{-1}(0) = 0.$$

EXAMPLE 3.5 Evaluating the Inverse Sine Function

Evaluate (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and (b) $\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution For (a), we look for the angle θ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $\sin\theta = \frac{\sqrt{3}}{2}$. Note that since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have that $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$. For (b), note that $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus,

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}. \quad ■$$

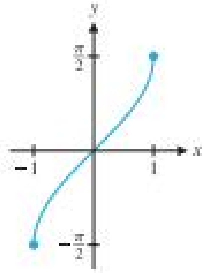


FIGURE 5.44
$y = \sin^{-1}x$

Judging by example 3.5, you might think that (3.5) is a roundabout way of defining a function. If so, you've got the idea exactly. In fact, we want to emphasize that what we know about the inverse sine function is principally through reference to the sine function.

Recall from our discussion in section 5.2 that we can draw a graph of $y = \sin^{-1}x$ simply by reflecting the graph of $y = \sin x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (from Figure 5.43) through the line $y = x$. (See Figure 5.44.)

Turning to $y = \cos x$, observe that restricting the domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, as we did for the inverse sine function, will not work here. (Why not?) The simplest way to

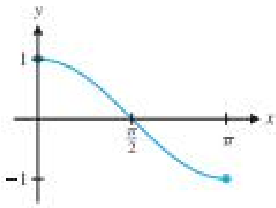


FIGURE 5.45
$y = \cos x$ on $[0, \pi]$

make $\cos x$ one-to-one is to restrict its domain to the interval $[0, \pi]$. (See Figure 5.45.) Consequently, we define the **inverse cosine** function by

$$y = \cos^{-1} x \quad \text{if and only if} \quad \cos y = x \text{ and } 0 \le y \le \pi.$$

Note that here we have

$$\cos(\cos^{-1} x) = x, \quad \text{for all } x \in [-1, 1]$$

and

$$\cos^{-1}(\cos x) = x, \quad \text{for all } x \in [0, \pi].$$

As with the definition of arcsine, it is helpful to think of $\cos^{-1} x$ as that angle θ in $[0, \pi]$ for which $\cos \theta = x$. As with $\sin^{-1} x$, it is common to use $\cos^{-1} x$ and $\arccos x$ interchangeably.

EXAMPLE 3.6 Evaluating the Inverse Cosine Function

Evaluate (a) $\cos^{-1}(0)$ and (b) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$.

Solution For (a), you will need to find that angle θ in $[0, \pi]$ for which $\cos \theta = 0$. It's not hard to see that $\cos^{-1}(0) = \frac{\pi}{2}$. (If you calculate this on your calculator and get 90, your calculator is in degree mode. In this event, you should immediately change it to radian mode.) For (b), look for the angle $\theta \in [0, \pi]$ for which $\cos \theta = -\frac{\sqrt{2}}{2}$. Notice that $\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\frac{3\pi}{4} \in [0, \pi]$. Consequently,

$$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}.$$

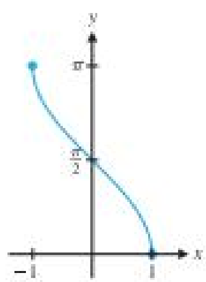


FIGURE 5.46
$y = \cos^{-1} x$

Once again, we obtain the graph of this inverse function by reflecting the graph of $y = \cos x$ on the interval $[0, \pi]$ (seen in Figure 5.45) through the line $y = x$. (See Figure 5.46.)

We can define inverses for each of the four remaining trigonometric functions in similar ways. For $y = \tan x$, we restrict the domain to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Think about why the endpoints of this interval are not included. (See Figure 5.47.) Having done this, you should readily see that we define the **inverse tangent** function by

$$y = \tan^{-1} x \quad \text{if and only if} \quad \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

The graph of $y = \tan^{-1} x$ is then as seen in Figure 5.48, found by reflecting the graph in Figure 5.47 through the line $y = x$.

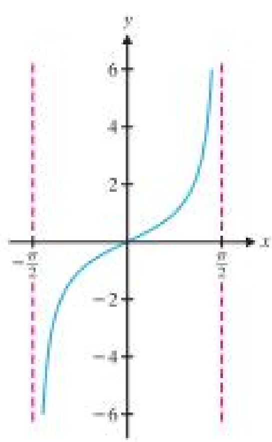


FIGURE 5.47
$y = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

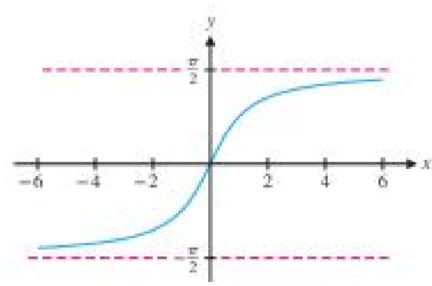


FIGURE 5.48
$y = \tan^{-1} x$

EXAMPLE 3.7 Evaluating an Inverse Tangent

Evaluate $\tan^{-1}(1)$.

Solution You must look for the angle θ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $\tan\theta = 1$. This is easy enough. Since $\tan\left(\frac{\pi}{4}\right) = 1$ and $\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have that $\tan^{-1}(1) = \frac{\pi}{4}$. ■

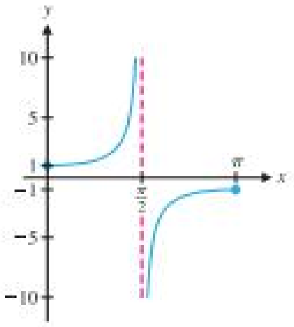


FIGURE 5.49
$y = \sec x$ on $[0, \pi]$

We now turn to defining an inverse for $\sec x$. First, we must issue a disclaimer. There are several reasonable ways in which to suitably restrict the domain and different authors restrict it differently. We have (somewhat arbitrarily) chosen to restrict the domain to be $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. Why not use all of $[0, \pi]$? You need only think about the definition of $\sec x$ to see why we needed to exclude the value $x = \frac{\pi}{2}$. See Figure 5.49 for a graph of $\sec x$ on this domain. (Note the vertical asymptote at $x = \frac{\pi}{2}$). Consequently, we define the **inverse secant** function by

$$y = \sec^{-1}x \quad \text{if and only if} \quad \sec y = x \text{ and } y \in \left[0, \tfrac{\pi}{2}\right) \cup \left(\tfrac{\pi}{2}, \pi\right].$$

A graph of $\sec^{-1}x$ is shown in Figure 5.50.

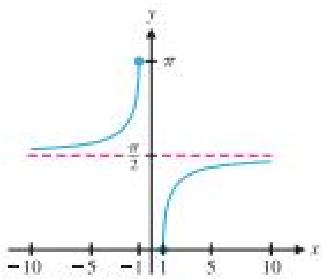


FIGURE 5.50
$y = \sec^{-1}x$

EXAMPLE 3.8 Evaluating an Inverse Secant

Evaluate $\sec^{-1}(-\sqrt{2})$.

Solution You must look for the angle θ with $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, for which $\sec\theta = -\sqrt{2}$. Notice that if $\sec\theta = -\sqrt{2}$, then $\cos\theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$. Since $\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$ and the angle $\frac{3\pi}{4}$ is in the interval $\left(\frac{\pi}{2}, \pi\right]$, we have $\sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}$. ■

REMARK 3.5

We can likewise define inverses to $\cot x$ and $\csc x$. As these functions are used only infrequently, we will omit them here and examine them in the exercises.

Most calculators do not have built-in functions for $\sec x$ or $\sec^{-1}x$. In this case, you must convert the desired secant value to a cosine value and use the inverse cosine function, as we did in example 3.8.

We summarize the domains and ranges of the three main inverse trigonometric functions in the margin.

Function	*Domain*	*Range*
$\sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$[-\infty, \infty]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

In many applications, we need to calculate the length of one side of a right triangle using the length of another side and an **acute** angle (i.e., an angle between 0 and $\frac{\pi}{2}$ radians). We can do this rather easily, as in example 3.9.

EXAMPLE 3.9 Finding the Height of a Tower

A person 100 feet from the base of a tower measures an angle of 60° from the ground to the top of the tower. (See Figure 5.51.) (a) Find the height of the tower. (b) What angle is measured if the person is 200 feet from the base?

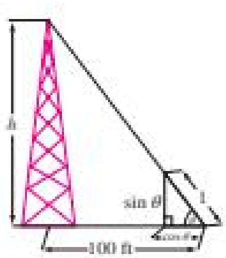


FIGURE 5.51
Height of a tower

Solution For (a), we first convert 60° to radians:

$$60° = 60\frac{\pi}{180} = \frac{\pi}{3} \text{ radians.}$$

We are given that the base of the triangle in Figure 5.51 is 100 fee. We must now compute the height h of the tower. Using the similar triangles indicated in Figure 5.51, we have

$$\frac{\sin\theta}{\cos\theta} = \frac{h}{100},$$

so that the height of the tower is

$$h = 100\frac{\sin\theta}{\cos\theta} = 100\tan\theta = 100\tan\frac{\pi}{3} = 100\sqrt{3} \approx 173 \text{ feet.}$$

For part (b), the similar triangles in Figure 5.51 give us

$$\tan\theta = \frac{h}{200} = \frac{100\sqrt{3}}{200} = \frac{\sqrt{3}}{2}.$$

Since $0 < \theta < \frac{\pi}{2}$, we have

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right) \approx 0.7137 \text{ radians (or about } 41^\circ).$$

In example 3.10, we simplify expressions involving both trigonometric and inverse trigonometric functions.

EXAMPLE 3.10 Simplifying Expressions Involving Inverse Trigonometric Functions

Simplify (a) $\sin(\cos^{-1}x)$ and (b) $\tan(\cos^{-1}x)$.

Solution Do not look for some arcane formula to help you out. *Think* first: $\cos^{-1}x$ is an angle (call it θ) for which $x = \cos\theta$. First, consider the case where $x > 0$. Looking at Figure 5.52, we have drawn a right triangle, with hypotenuse 1 and adjacent angle θ. From the definition of the sine and cosine, then, we have that the base of the triangle is $\cos\theta = x$ and the altitude is $\sin\theta$, which by the Pythagorean Theorem is

$$\sin(\cos^{-1}x) = \sin\theta = \sqrt{1 - x^2}.$$

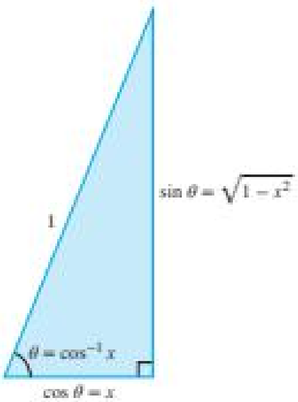


FIGURE 5.52
$\theta = \cos^{-1}x$

Wait! We have not yet finished part (a). Figure 5.52 shows $0 < \theta < \frac{\pi}{2}$, but by definition, $\theta = \cos^{-1}x$ could range from 0 to π. Does our answer change if $\frac{\pi}{2} < \theta < \pi$? To see that it doesn't change, note that if $0 \le \theta \le \pi$, then $\sin\theta \ge 0$. From the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, we get

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta} = \pm\sqrt{1 - x^2}.$$

Since $\sin\theta \ge 0$, we must have

$$\sin\theta = \sqrt{1 - x^2},$$

for *all* values of x.

For part (b), you can read from Figure 5.52 that

$$\tan(\cos^{-1}x) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-x^2}}{x}.$$

Note that this last identity is valid regardless of whether $x = \cos\theta$ is positive or negative. ■

EXERCISES 5.3

WRITING EXERCISES

1. Many students are comfortable using degrees to measure angles and don't understand why they must learn radian measures. As discussed in the text, radians directly measure distance along the unit circle. Distance is an important aspect of many applications. In addition, we will see later that many calculus formulas are simpler in radians form than in degrees. Aside from familiarity, discuss any and all advantages of degrees over radians. On balance, which is better?

2. A student graphs $f(x) = \cos x$ on a graphing calculator and gets what appears to be a straight line at height $y = 1$ instead of the usual cosine curve. Upon investigation, you discover that the calculator has graphing window $-10 \le x \le 10$, $-10 \le y \le 10$ and is in degree mode. Explain what went wrong and how to correct it.

3. Inverse functions are necessary for solving equations. The restricted range we had to use to define inverses of the trigonometric functions also restricts their usefulness in equation solving. Explain how to use $\sin^{-1}x$ to find all solutions of the equation $\sin u = x$.

4. Discuss how to compute $\sec^{-1}x$, $\csc^{-1}x$ and $\cot^{-1}x$ on a calculator that has built-in functions only for $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$.

5. In example 3.3, $f(x) = 4\cos 3x$ has period $2\pi/3$ and $g(x) = 2\sin(x/3)$ has period 6π. Explain why the sum $h(x) = 4\cos 3x + 2\sin(x/3)$ has period 6π.

6. Give a different range for $\sec^{-1}x$ than that given in the text. For which x's would the value of $\sec^{-1}x$ change? Using the calculator discussion in exercise 4, give one reason why we might have chosen the range that we did.

In exercises 1 and 2, convert the given radians measure to degrees.

1. (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{4\pi}{3}$

2. (a) $\frac{3\pi}{5}$ (b) $\frac{\pi}{7}$ (c) 2 (d) 3

In exercises 3 and 4, convert the given degrees measure to radians.

3. (a) 180° (b) 270° (c) 120° (d) 30°

4. (a) 40° (b) 80° (c) 450° (d) 390°

In exercises 5–14, find all solutions of the given equation.

5. $2\cos x - 1 = 0$

6. $2\sin x + 1 = 0$

7. $\sqrt{2}\cos x - 1 = 0$

8. $2\sin x - \sqrt{3} = 0$

9. $\sin^2 x - 4\sin x + 3 = 0$

10. $\sin^2 x - 2\sin x - 3 = 0$

11. $\sin^2 x + \cos x - 1 = 0$

12. $\sin 2x - \cos x = 0$

13. $\cos^2 x + \cos x = 0$

14. $\sin^2 x - \sin x = 0$

In exercises 15–24, sketch a graph of the function.

15. $f(x) = \sin 2x$

16. $f(x) = \cos 3x$

17. $f(x) = \tan 2x$

18. $f(x) = \sec 3x$

19. $f(x) = 3\cos(x - \pi/2)$

20. $f(x) = 4\cos(x + \pi)$

21. $f(x) = \sin 2x - 2\cos 2x$

22. $f(x) = \cos 3x - \sin 3x$

23. $f(x) = \sin x \sin 12x$

24. $f(x) = \sin x \cos 12x$

In exercises 25–32, identify the amplitude, period and frequency.

25. $f(x) = 3\sin 2x$

26. $f(x) = 2\cos 3x$

27. $f(x) = 5\cos 3x$

28. $f(x) = 3\sin 5x$

29. $f(x) = 3\cos(2x - \pi/2)$

30. $f(x) = 4\sin(3x + \pi)$

31. $f(x) = -4\sin x$

32. $f(x) = -2\cos 3x$

In exercises 33–36, prove that the given trigonometric identity is true.

33. $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$

34. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

35. (a) $\cos(2\theta) = 2\cos^2\theta - 1$ (b) $\cos(2\theta) = 1 - 2\sin^2\theta$

36. (a) $\sec^2\theta = \tan^2\theta + 1$ (b) $\csc^2\theta = \cot^2\theta + 1$

In exercises 37–46, evaluate the inverse function by sketching a unit circle, locating the correct angle and evaluating the ordered pair on the circle.

37. $\cos^{-1} 0$

38. $\tan^{-1} 0$

39. $\sin^{-1}(-1)$

40. $\cos^{-1}(1)$

41. $\sec^{-1} 1$

42. $\tan^{-1}(-1)$

43. $\sec^{-1} 2$

44. $\csc^{-1} 2$

45. $\cot^{-1} 1$

46. $\tan^{-1}\sqrt{3}$

47. Prove that, for some constant β,

$$4\cos x - 3\sin x = 5\cos(x + \beta).$$

Then, estimate the value of β.

48. Prove that, for some constant β,

$$2\sin x + \cos x = \sqrt{5}\sin(x + \beta).$$

Then, estimate the value of β.

In exercises 49–52, determine whether the function is periodic. If it is periodic, find the smallest (fundamental) period.

49. $f(x) = \cos 2x + 3\sin \pi x$

50. $f(x) = \sin x - \cos\sqrt{2}x$

51. $f(x) = \sin 2x - \cos 5x$

52. $f(x) = \cos 3x - \sin 7x$

In exercises 53–56, use the range for θ to determine the indicated function value.

53. $\sin\theta = \frac{1}{3}, 0 \le \theta \le \frac{\pi}{2}$; find $\cos\theta$.

54. $\cos\theta = \frac{4}{5}, 0 \le \theta \le \frac{\pi}{2}$; find $\sin\theta$.

55. $\sin\theta = \frac{1}{2}, \frac{\pi}{2} \le \theta \le \pi$; find $\cos\theta$.

56. $\sin\theta = \frac{1}{2}, \frac{\pi}{2} \le \theta \le \pi$; find $\tan\theta$.

In exercises 57–64, use a triangle to simplify each expression. Where applicable, state the range of x's for which the simplification holds.

57. $\cos(\sin^{-1}x)$

58. $\cos(\tan^{-1}x)$

59. $\tan(\sec^{-1}x)$

60. $\cot(\cos^{-1}x)$

61. $\sin(\cos^{-1}\frac{1}{2})$

62. $\cos(\sin^{-1}\frac{1}{2})$

63. $\tan(\cos^{-1}\frac{3}{5})$

64. $\csc(\sin^{-1}\frac{2}{3})$

In exercises 65–68, use a graphing calculator or computer to determine the number of solutions of each equation, and numerically estimate the solutions (x is in radians).

65. $2\cos x = 2 - x$

66. $3\sin x = x$

67. $\cos x = x^2 - 2$

68. $\sin x = x^2$

APPLICATIONS

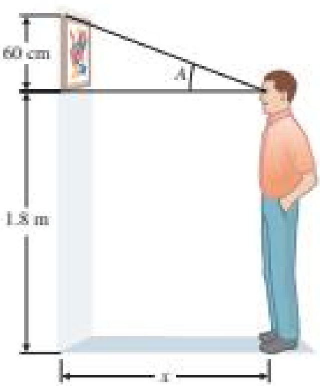
1. A person sitting 2 miles from a rocket launch site measures 20° up to the current location of the rocket. How high up is the rocket?

2. A person who is 6 feet tall stands 4 feet from the base of a light pole and casts a 2-foot-long shadow. How tall is the light pole?

3. A surveyor stands 80 feet from the base of a building and measures an angle of 50° to the top of the steeple on top of the building. The surveyor figures that the center of the steeple lies 20 feet inside the front of the structure. Find the distance from the ground to the top of the steeple.

4. Suppose that the surveyor of exercise 3 estimates that the center of the steeple lies between 20′ and 21′ inside the front of the structure. Determine how much the extra foot would change the calculation of the height of the building.

5. A painting at the Louvre, Abu Dhabi, has a frame 60 centimeters high, and the bottom of the frame is 1.8 meters above the floor. A person whose eyes are 1.8 meters above the floor stands x meters from the wall. Let A be the angle formed by the ray from the person's eye to the bottom of the frame and the ray from the person's eye to the top of the frame. Write A as a function of x and graph $y = A(x)$.



6. In golf, the goal is to hit a ball into a hole of diameter 4.5 inches. Suppose a golfer stands x feet from the hole trying to putt the ball into the hole. A first approximation of the margin of error in a putt is to measure the angle A formed by the ray from the ball to the right edge of the hole and the ray from the ball to the left edge of the hole. Find A as a function of x.

7. In an AC circuit, the voltage is given by $v(t) = v_p \sin(2\pi ft)$, where v_p is the peak voltage and f is the frequency in Hz. A voltmeter actually measures an average (called the **root-mean-square**) voltage, equal to $v_p/\sqrt{2}$. If the voltage has amplitude 170 and period $\pi/30$, find the frequency and meter voltage.

8. An old-style LP record player rotates records at $33\frac{1}{3}$ rpm (revolutions per minute). What is the period (in minutes) of the rotation? What is the period for a 45-rpm record?

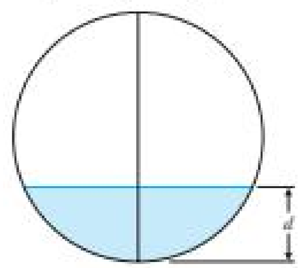
9. Suppose that the ticket sales of an airline (in thousands of dollars) are given by $s(t) = 110 + 2t + 15\sin\left(\frac{1}{6}\pi t\right)$, where t is measured in months. What real-world phenomenon might cause the fluctuation in ticket sales modeled by the sine term? Based on your answer, what month corresponds to $t = 0$? Disregarding seasonal fluctuations, by what amount are the airline's sales increasing annually?

10. Piano tuners sometimes start by striking a tuning fork and then the corresponding piano key. If the tuning fork and piano note each have frequency 8, then the resulting sound is $\sin 8t + \sin 8t$. Graph this. If the piano is slightly out-of-tune at frequency 8.1, the resulting sound is $\sin 8t + \sin 8.1t$. Graph this and explain how the piano tuner can hear the small difference in frequency.

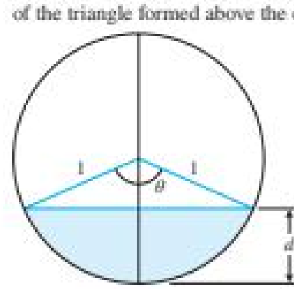
EXPLORATORY EXERCISES

1. In his book and video series *The Ring of Truth,* physicist Philip Morrison[1] performed an experiment to estimate the circumference of the earth. In Nebraska, he measured the angle to a bright star in the sky, then drove 370 miles due south into Kansas and measured the new angle to the star. Some geometry shows that the difference in angles, about 5.02°, equals the angle from the center of the earth to the two locations in Nebraska and Kansas. If the earth is perfectly spherical (it's not) and the circumference of the portion of the circle measured out by 5.02° is 370 miles, estimate the circumference of the earth. This experiment was based on a similar experiment by the ancient Greek scientist Eratosthenes. The ancient Greeks and the Spaniards of Columbus' day knew that the earth was round, they just disagreed about the circumference. Columbus argued for a figure about half of the actual value, since a ship couldn't survive on the water long enough to navigate the true distance.

2. An oil tank with circular cross sections lies on its side. A stick is inserted in a hole at the top and used to measure the depth d of oil in the tank. Based on this measurement, the goal is to compute the percentage of oil left in the tank.



To simplify calculations, suppose the circle is a unit circle with center at (0, 0). Sketch radii extending from the origin to the top of the oil. The area of oil at the bottom equals the area of the portion of the circle bounded by the radii minus the area of the triangle formed above the oil in the figure.



Start with the triangle, which has area one-half base times height. Explain why the height is $1 - d$. Find a right triangle in

[1] Morrison, P. and Morrison, P. (1987). *The Ring of Truth: An Inquiry into How We Know What We Know* (New York: Random House). Video series: Morrison, P. (1987). *The Ring of Truth* (Public Broadcasting Services).

the figure (there are two of them) with hypotenuse 1 (the radius of the circle) and one vertical side of length $1 - d$. The horizontal side has length equal to one-half the base of the larger triangle. Show that this equals $\sqrt{1-(1-d)^2}$. The area of the portion of the circle equals $\pi\theta/2\pi = \theta/2$, where θ is the angle at the top of the triangle. Find this angle as a function of d. (Hint: Go back to the right triangle used above with upper angle $\theta/2$.) Then find the area filled with oil and divide by π to get the portion of the tank filled with oil.

3. Computer graphics can be misleading. This exercise works best using a "disconnected" graph (individual dots, not connected). Graph $y = \sin x^2$ using a graphing window for which each pixel represents a step of 0.1 in the x- or y-direction. You should get the impression of a sine wave that oscillates more and more rapidly as you move to the left and right. Next, change the graphing window so that the middle of the original screen (probably $x = 0$) is at the far left of the new screen. You will likely see what appears to be a random jumble of dots. Continue to change the graphing window by increasing the x-values. Describe the patterns or lack of patterns that you see. You should find one pattern that looks like two rows of dots across the top and bottom of the screen; another pattern looks like the original sine wave. For each pattern that you find, pick adjacent points with x-coordinates a and b. Then change the graphing window so that $a \le x \le b$ and find the portion of the graph that is missing. Remember that, whether the points are connected or not, computer graphs always leave out part of the graph; it is part of your job to know whether or not the missing part is important.

5.4 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Some bacteria reproduce very quickly, as you may have discovered if you have ever had an infected cut or strep throat. Under the right circumstances, the number of bacteria in certain cultures will double in as little as an hour. In this section, we discuss some functions that can be used to model such rapid growth.

Suppose that initially there are 100 bacteria at a given site and the population doubles every hour. Call the population function $P(t)$, where t represents time (in hours) and start the clock running at time $t = 0$. Since the initial population is 100, we have $P(0) = 100$. After 1 hour, the population has doubled to 200, so that $P(1) = 200$. After another hour, the population will have doubled again to 400, making $P(2) = 400$ and so on.

To compute the bacterial population after 10 hours, you could calculate the population at 4 hours, 5 hours and so on, or you could use the following shortcut. To find $P(1)$, double the initial population, so that $P(1) = 2 \cdot 100$. To find $P(2)$, double the population at time $t = 1$, so that $P(2) = 2 \cdot 2 \cdot 100 = 2^2 \cdot 100$. Similarly, $P(3) = 2^3 \cdot 100$. This pattern leads us to

$$P(10) = 2^{10} \cdot 100 = 102{,}400.$$

Observe that the population can be modeled by the function

$$P(t) = 2^t \cdot 100.$$

We call $P(t)$ an **exponential** function because the variable t is in the exponent. There is a subtle question here: what is the domain of this function? We have so far used only integer values of t, but for what other values of t does $P(t)$ make sense? Certainly, rational powers make sense, as in $P(1/2) = 2^{1/2} \cdot 100$, where $2^{1/2} = \sqrt{2}$. This says that the number of bacteria in the culture after a half hour is approximately

$$P(1/2) = 2^{1/2} \cdot 100 = \sqrt{2} \cdot 100 \approx 141.$$

It's a simple matter to interpret fractional powers as roots. For instance,

$$\begin{aligned} x^{1/2} &= \sqrt{x}, \\ x^{1/3} &= \sqrt[3]{x}, \\ x^{2/3} &= \sqrt[3]{x^2} = (\sqrt[3]{x})^2, \\ x^{3.1} &= x^{31/10} = \sqrt[10]{x^{31}} \end{aligned}$$

and so on. But, what about irrational powers? They are harder to define, but they work exactly the way you would want them to. For instance, since π is between 3.14 and 3.15, 2^π is between $2^{3.14}$ and $2^{3.15}$. In this way, we define 2^x for x irrational to fill in the gaps in the graph of $y = 2^x$ for x rational. That is, if x is irrational and $a < x < b$, for rational numbers a and b, then $2^a < 2^x < 2^b$.

If for some reason you wanted to find the bacterial population after π hours, you can use your calculator or computer to obtain the approximate population:

$$P(\pi) = 2^{\pi} \cdot 100 \approx 882.$$

For your convenience, we now summarize the usual rules of exponents.

RULES OF EXPONENTS (FOR x, $y > 0$)

- For any integers m and n ($n \geq 2$),

$$x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m.$$

- For any real number p,

$$x^{-p} = \frac{1}{x^p}, \quad (xy)^p = x^p \cdot y^p \quad \text{and} \quad \left(\frac{x}{y}\right)^p = \frac{x^p}{y^p}.$$

- For any real numbers p and q,

$$(x^p)^q = x^{p \cdot q}.$$

- For any real numbers p and q,

$$x^p \cdot x^q = x^{p+q} \quad \text{and} \quad \frac{x^p}{x^q} = x^{p-q}$$

Throughout your calculus course, you will need to be able to quickly convert back and forth between exponential form and fractional or root form.

EXAMPLE 4.1 Converting Expressions to Exponential Form

Convert each to exponential form: (a) $3\sqrt{x^5}$, (b) $\frac{5}{\sqrt[3]{x}}$, (c) $\frac{3x^2}{2\sqrt{x}}$ and (d) $(2^x \cdot 2^{3+x})^2$.

Solution For (a), simply leave the 3 alone and convert the power:

$$3\sqrt{x^5} = 3x^{5/2}.$$

For (b), use a negative exponent to write x in the numerator:

$$\frac{5}{\sqrt[3]{x}} = 5x^{-1/3}.$$

For (c), first separate the constants from the variables and then simplify:

$$\frac{3x^2}{2\sqrt{x}} = \frac{3}{2}\frac{x^2}{x^{1/2}} = \frac{3}{2}x^{2-1/2} = \frac{3}{2}x^{3/2}.$$

For (d), first work inside the parentheses and then square:

$$(2^x \cdot 2^{3+x})^2 = (2^{x+3+x})^2 = (2^{2x+3})^2 = 2^{4x+6}. \blacksquare$$

In general, we have the following definition.

DEFINITION 4.1

For any constants $a \neq 0$ and $b > 0$, the function $f(x) = a \cdot b^x$ is called an **exponential function**. Here, b is called the **base** and x is the **exponent**.

Be careful to distinguish between algebraic functions such as $f(x) = x^3$ and $g(x) = x^{2/3}$ and exponential functions. For exponential functions such as $h(x) = 2^x$, the variable is in the exponent (hence the name), instead of in the base. Also, notice that the domain of an exponential function is the entire real line, $(-\infty, \infty)$, while the range is the open interval $(0, \infty)$, since $b^x > 0$ for all x.

While any positive real number can be used as a base for an exponential function, three bases are the most commonly used in practice. Base 2 arises naturally when analyzing processes that double at regular intervals (such as the bacteria at the beginning of this section). Our standard counting system is base 10, so this base is commonly used. However, far and away the most useful base is the irrational number e. Like π, the number e has a surprising tendency to occur in important calculations. We define e by

$$\boxed{e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.} \tag{4.1}$$

It suffices for the moment to say that equation (4.1) means that e can be approximated by calculating values of $(1 + 1/n)^n$ for large values of n and that the larger the value of n, the closer the approximation will be to the actual value of e. In particular, if you look at the sequence of numbers $(1 + 1/2)^2$, $(1 + 1/3)^3$, $(1 + 1/4)^4$ and so on, they will get progressively closer and closer to (i.e., home in on) the irrational number e.

To get an idea of the value of e, compute several of these numbers:

$$\left(1 + \frac{1}{10}\right)^{10} = 2.5937\ldots,$$

$$\left(1 + \frac{1}{1000}\right)^{1000} = 2.7169\ldots,$$

$$\left(1 + \frac{1}{10{,}000}\right)^{10{,}000} = 2.7181\ldots$$

and so on. You should compute enough of these values to convince yourself that the first few digits of the decimal representation of e ($e \approx 2.718281828459\ldots$) are correct.

EXAMPLE 4.2 Computing Values of Exponentials

Approximate e^4, $e^{-1/5}$ and e^0.

Solution From a calculator, we find that

$$e^4 = e \cdot e \cdot e \cdot e \approx 54.598.$$

From the usual rules of exponents,

$$e^{-1/5} = \frac{1}{e^{1/5}} = \frac{1}{\sqrt[5]{e}} \approx 0.81873.$$

(On a calculator, it is convenient to replace $-1/5$ with -0.2.) Finally, $e^0 = 1$. ■

The graphs of exponential functions summarize many of their important properties.

EXAMPLE 4.3 Sketching Graphs of Exponentials

Sketch the graphs of the exponential functions $y = 2^x$, $y = e^x$, $y = e^{2x}$, $y = e^{x/2}$, $y = (1/2)^x$ and $y = e^{-x}$.

Solution Using a calculator or computer, you should get graphs similar to those that follow.

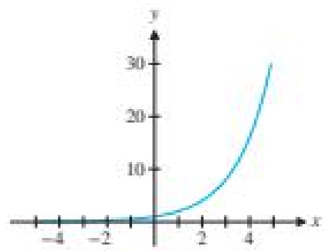


FIGURE 5.53a
$y = 2^x$

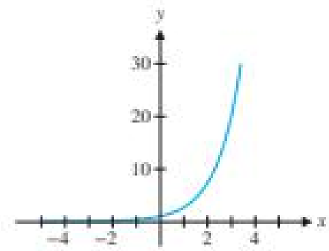


FIGURE 5.53b
$y = e^x$

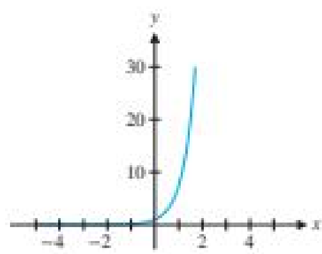


FIGURE 5.54a
$y = e^{2x}$

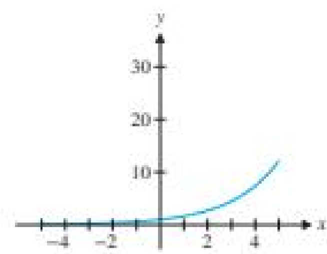


FIGURE 5.54b
$y = e^{x/2}$

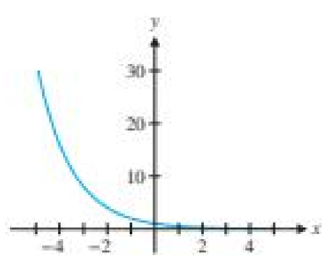


FIGURE 5.55a
$y = (1/2)^x$

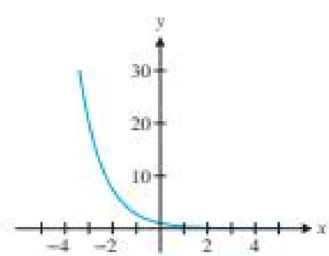


FIGURE 5.55b
$y = e^{-x}$

Notice that each of the graphs in Figures 5.53a, 5.53b, 5.54a and 5.54b starts very near the x-axis (reading left to right), passes through the point (0, 1) and then rises steeply. This is true for all exponentials with base greater than 1 and with a positive coefficient in the exponent. Note that the larger the base ($e > 2$) or the larger the coefficient in the exponent ($2 > 1 > 1/2$), the more quickly the graph rises to the right (and drops to the left). Note that the graphs in Figures 5.55a and 5.55b are the mirror images in the y-axis of Figures 5.53a and 5.53b, respectively. The graphs rise as you move to the left and drop toward the x-axis as you move to the right. It's worth noting that by the rules of exponents, $(1/2)^x = 2^{-x}$ and $(1/e)^x = e^{-x}$. ■

In Figures 5.53–5.55, each exponential function is one-to-one and, hence, has an inverse function. We define the logarithmic functions to be inverses of the exponential functions.

DEFINITION 4.2

For any positive number $b \neq 1$, the **logarithm** function with base b, written $\log_b x$, is defined by

$$y = \log_b x \text{ if and only if } x = b^y.$$

That is, the logarithm $\log_b x$ gives the exponent to which you must raise the base b to get the given number x. For example,

$$\log_{10} 10 = 1 \quad (\text{since } 10^1 = 10),$$
$$\log_{10} 100 = 2 \quad (\text{since } 10^2 = 100),$$
$$\log_{10} 1000 = 3 \quad (\text{since } 10^3 = 1000)$$

and so on. The value of $\log_{10} 45$ is less clear than the preceding three values, but the idea is the same: you need to find the number y such that $10^y = 45$. The answer lies between 1 and 2, but to be more precise, you will need to employ trial and error. You should get $\log_{10} 45 \approx 1.6532$.

Observe from Definition 4.2 that for any base $b > 0$ $(b \neq 1)$, if $y = \log_b x$, then $x = b^y > 0$. That is, the domain of $f(x) = \log_b x$ is the interval $(0, \infty)$. Likewise, the range of f is the entire real line, $(-\infty, \infty)$.

As with exponential functions, the most useful bases turn out to be 2, 10 and e. We usually abbreviate $\log_{10} x$ by $\log x$. Similarly, $\log_e x$ is usually abbreviated $\ln x$ (short for **natural logarithm**).

EXAMPLE 4.4 Evaluating Logarithms

Without using your calculator, determine $\log(1/10)$, $\log(0.001)$, $\ln e$ and $\ln e^3$.

Solution Since $1/10 = 10^{-1}$, $\log(1/10) = -1$. Similarly, since $0.001 = 10^{-3}$, we have that $\log(0.001) = -3$. Since $\ln e = \log_e e^1$, $\ln e = 1$. Similarly, $\ln e^3 = 3$. ■

We want to emphasize the inverse relationship defined by Definition 4.2. That is, b^x and $\log_b x$ are inverse functions for any $b > 0$ $(b \neq 1)$.

In particular, for the base e, we have

$$\boxed{e^{\ln x} = x \quad \text{for any } x > 0 \qquad \text{and} \qquad \ln(e^x) = x \quad \text{for any } x.} \tag{4.2}$$

We demonstrate this as follows. Let

$$y = \ln x = \log_e x.$$

By Definition 4.2, we have that

$$x = e^y = e^{\ln x}.$$

We can use this relationship between natural logarithms and exponentials to solve equations involving logarithms and exponentials, as in examples 4.5 and 4.6.

EXAMPLE 4.5 Solving a Logarithmic Equation

Solve the equation $\ln(x + 5) = 3$ for x.

Solution Taking the exponential of both sides of the equation and writing things backward (for convenience), we have

$$e^3 = e^{\ln(x+5)} = x+5,$$

from (4.2). Subtracting 5 from both sides gives us

$$e^3 - 5 = x. \ ■$$

EXAMPLE 4.6 Solving an Exponential Equation

Solve the equation $e^{x+4} = 7$ for x.

Solution Taking the natural logarithm of both sides and writing things backward (for simplicity), we have from (4.2) that

$$\ln 7 = \ln (e^{x+4}) = x + 4.$$

Subtracting 4 from both sides yields

$$\ln 7 - 4 = x. \blacksquare$$

As always, graphs provide excellent visual summaries of the important properties of a function.

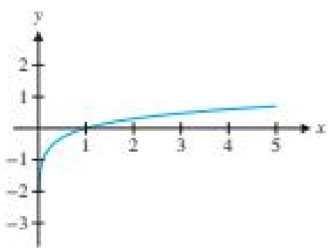


FIGURE 5.56a
$y = \log x$

EXAMPLE 4.7 Sketching Graphs of Logarithms

Sketch graphs of $y = \log x$ and $y = \ln x$, and briefly discuss the properties of each.

Solution From a calculator or computer, you should obtain graphs resembling those in Figures 5.56a and 5.56b. Notice that both graphs appear to have a vertical asymptote at $x = 0$ (why would that be?), cross the x-axis at $x = 1$ and very gradually increase as x increases. Neither graph has any points to the left of the y-axis, since $\log x$ and $\ln x$ are defined only for $x > 0$. The two graphs are very similar, although not identical. ■

The properties just described graphically are summarized in Theorem 4.1.

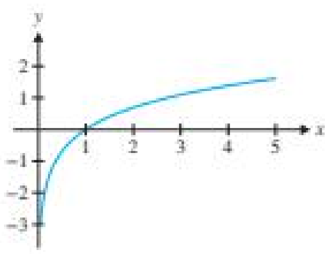


FIGURE 5.56b
$y = \ln x$

THEOREM 4.1

For any positive base $b \neq 1$,

(i) $\log_b x$ is defined only for $x > 0$,
(ii) $\log_b 1 = 0$ and
(iii) if $b > 1$, then $\log_b x < 0$ for $0 < x < 1$ and $\log_b x > 0$ for $x > 1$.

PROOF

(i) Note that since $b > 0$, $b^y > 0$ for any y. So, if $\log_b x = y$, then $x = b^y > 0$.
(ii) Since $b^0 = 1$ for any number $b \neq 0$, $\log_b 1 = 0$ (i.e., the exponent to which you raise the base b to get the number 1 is 0).
(iii) We leave this as an exercise. ■

All logarithms share a set of defining properties, as stated in Theorem 4.2.

THEOREM 4.2

For any positive base $b \neq 1$ and positive numbers x and y, we have

(i) $\log_b (xy) = \log_b x + \log_b y$,
(ii) $\log_b (x/y) = \log_b x - \log_b y$ and
(iii) $\log_b (x^y) = y \log_b x$.

As with most algebraic rules, each one of these properties can dramatically simplify calculations when it applies.

EXAMPLE 4.8 Simplifying Logarithmic Expressions

Write each as a single logarithm: (a) $\log_2 27^x - \log_2 3^x$ and (b) $\ln 8 - 3\ln(1/2)$.

Solution First, note that there is more than one order in which to work each problem. For part (a), we have $27 = 3^3$ and so, $27^x = (3^3)^x = 3^{3x}$. This gives us

$$\begin{aligned}\log_2 27^x - \log_2 3^x &= \log_2 3^{3x} - \log_2 3^x \\ &= 3x\log_2 3 - x\log_2 3 = 2x\log_2 3 = \log_2 3^{2x}.\end{aligned}$$

For part (b), note that $8 = 2^3$ and $1/2 = 2^{-1}$. Then,

$$\begin{aligned}\ln 8 - 3\ln(1/2) &= 3\ln 2 - 3(-\ln 2) \\ &= 3\ln 2 + 3\ln 2 = 6\ln 2 = \ln 2^6 = \ln 64.\end{aligned}$$

In some circumstances, it is beneficial to use the rules of logarithms to expand a given expression, as in example 4.9.

EXAMPLE 4.9 Expanding a Logarithmic Expression

Use the rules of logarithms to expand the expression $\ln\left(\frac{x^3y^4}{z^5}\right)$.

Solution From Theorem 4.2, we have that

$$\begin{aligned}\ln\left(\frac{x^3y^4}{z^5}\right) &= \ln(x^3y^4) - \ln(z^5) = \ln(x^3) + \ln(y^4) - \ln(z^5) \\ &= 3\ln x + 4\ln y - 5\ln z.\end{aligned}$$

Using the rules of exponents and logarithms, we can rewrite any exponential as an exponential with base e, as follows. For any base $a > 0$, we have

$$\boxed{a^x = e^{\ln(a^x)} = e^{x\ln a}.} \tag{4.3}$$

This follows from Theorem 4.2 (iii) and the fact that $e^{\ln y} = y$, for all $y > 0$.

EXAMPLE 4.10 Rewriting Exponentials as Exponentials with Base *e*

Rewrite the exponentials 2^x, 5^x and $(2/5)^x$ as exponentials with base e.

Solution From (4.3), we have

$$2^x = e^{\ln(2^x)} = e^{x\ln 2},$$
$$5^x = e^{\ln(5^x)} = e^{x\ln 5}$$

and

$$\left(\frac{2}{5}\right)^x = e^{\ln[(2/5)^x]} = e^{x\ln(2/5)}.$$

Just as we can rewrite an exponential with any positive base in terms of an exponential with base e, we can rewrite any logarithm in terms of natural logarithms, as follows. We will next show that

$$\boxed{\log_b x = \frac{\ln x}{\ln b}, \text{ if } b > 0,\ b \neq 1 \text{ and } x > 0.} \tag{4.4}$$

Let $y = \log_b x$. Then by Definition 4.2, we have that $x = b^y$. Taking the natural logarithm of both sides of this equation, we get by Theorem 4.2 (iii) that

$$\ln x = \ln(b^y) = y\ln b.$$

Dividing both sides by ln b (since $b \neq 1$, $\ln b \neq 0$) gives us

$$y = \frac{\ln x}{\ln b},$$

establishing (4.4).

Equation (4.4) is useful for computing logarithms with bases other than e or 10. This is important since, more than likely, your calculator has keys only for ln x and log x. We illustrate this idea in example 4.11.

EXAMPLE 4.11 Approximating the Value of Logarithms

Approximate the value of $\log_7 12$.

Solution From (4.4), we have

$$\log_7 12 = \frac{\ln 12}{\ln 7} \approx 1.2769894.$$

Hyperbolic Functions

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There are two special combinations of exponential functions, called the **hyperbolic sine** and **hyperbolic cosine** functions, that have important applications. For instance, the Gateway Arch in Saint Louis was built in the shape of a hyperbolic cosine graph. (See the photograph in the margin.) The hyperbolic sine function [denoted by sinh (x)] and the hyperbolic cosine function [denoted by cosh (x)] are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Graphs of these functions are shown in Figures 5.57a and 5.57b. The hyperbolic functions (including the hyperbolic tangent, tanh x, defined in the expected way) are often convenient to use when solving equations. For now, we verify several basic properties that the hyperbolic functions satisfy in parallel with their trigonometric counterparts.

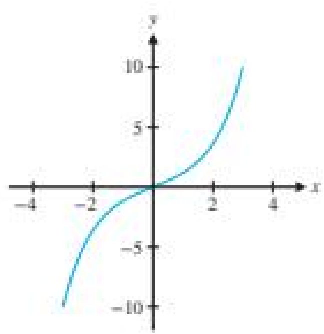


FIGURE 5.57a
$y = \sinh x$

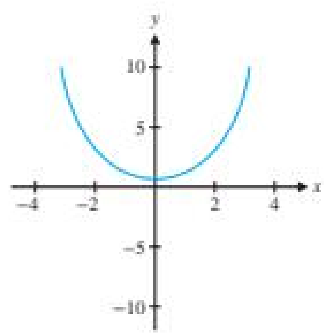


FIGURE 5.57b
$y = \cosh x$

EXAMPLE 4.12 Computing Values of Hyperbolic Functions

Compute $f(0)$, $f(1)$ and $f(-1)$, and determine how $f(x)$ and $f(-x)$ compare for each function: (a) $f(x) = \sinh x$ and (b) $f(x) = \cosh x$.

Solution For part (a), we have $\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$. Note that this means that $\sinh 0 = \sin 0 = 0$. Also, we have $\sinh 1 = \frac{e^1 - e^{-1}}{2} \approx 1.18$, while $\sinh(-1) = \frac{e^{-1} - e^1}{2} \approx -1.18$. Notice that $\sinh(-1) = -\sinh 1$. In fact, for any x,

$$\sinh(-x) = \frac{e^{-x} - e^{x}}{2} = \frac{-(e^{x} - e^{-x})}{2} = -\sinh x.$$

[The same rule holds for the sine function: $\sin(-x) = -\sin x$.] For part (b), we have $\cosh 0 = \frac{e^{0} + e^{-0}}{2} = \frac{1+1}{2} = 1$. Note that this means that $\cosh 0 = \cos 0 = 1$. Also, we have $\cosh 1 = \frac{e^{1} + e^{-1}}{2} \approx 1.54$, while $\cosh(-1) = \frac{e^{-1} + e^{1}}{2} \approx 1.54$. Notice that $\cosh(-1) = \cosh 1$. In fact, for any x,

$$\cosh(-x) = \frac{e^{-x} + e^{x}}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh x.$$

[The same rule holds for the cosine function: $\cos(-x) = \cos x$.] ■

○ Fitting a Curve to Data

You are familiar with the idea that two points determine a straight line. As we see in example 4.13, two points will also determine an exponential function.

EXAMPLE 4.13 Matching Data to an Exponential Curve

Find the exponential function of the form $f(x) = ae^{bx}$ that passes through the points (0, 5) and (3, 9).

Solution We must solve for a and b, using the properties of logarithms and exponentials. First, for the graph to pass through the point (0, 5), this means that

$$5 = f(0) = ae^{b \cdot 0} = a,$$

so that $a = 5$. Next, for the graph to pass through the point (3, 9), we must have

$$9 = f(3) = ae^{3b} = 5e^{3b}.$$

To solve for b, we divide both sides of the equation by 5 and take the natural logarithm of both sides, which yields

$$\ln\left(\frac{9}{5}\right) = \ln e^{3b} = 3b,$$

from (5.2). Finally, dividing by 3 gives us the value for b:

$$b = \frac{1}{3}\ln\left(\frac{9}{5}\right).$$

Thus, $f(x) = 5e^{\frac{1}{3}\ln(9/5)x}$. ■

Year	*Egypt Population*
1960	27,072,397
1970	34,808,599
1980	43,369,552
1990	56,397,273
2000	68,334,905
2010	82,040,994

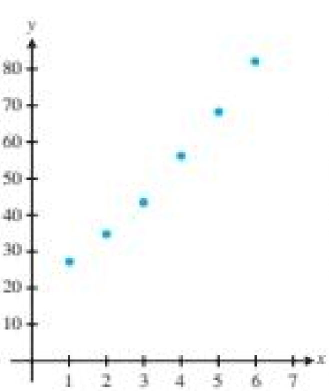


FIGURE 5.58
Egypt Population 1960–2010

Consider the population of Egypt from 1960 to 2010, found in the table at the left. A plot of these data points can be seen in Figure 5.58, where the vertical scale represents the population in millions and the horizontal scale represents the number of decades since 1950. This shows that the population was increasing, with larger and larger increases each decade. If you sketch an imaginary curve through these points, you will probably get the impression of a parabola or perhaps the right half of a cubic or exponential. And that's the question: are these data best modeled by a quadratic function, a cubic function, an exponential function or what?

We can use the properties of logarithms from Theorem 4.2 to help determine whether a given set of data is modeled better by a polynomial or an exponential function, as follows. Suppose that the data actually come from an exponential, say, $y = ae^{bx}$ (i.e., the data points lie on the graph of this exponential). Then,

$$\ln y = \ln(ae^{bx}) = \ln a + \ln e^{bx} = \ln a + bx.$$

If you draw a new graph, where the horizontal axis shows values of x and the vertical axis corresponds to values of $\ln y$, then the graph will be the line $\ln y = bx + c$ (where the constant $c = \ln a$). On the other hand, suppose the data actually came from a polynomial. If $y = bx^n$ (for any n), then observe that

$$\ln y = \ln (bx^n) = \ln b + \ln x^n = \ln b + n \ln x.$$

In this case, a graph with horizontal and vertical axes corresponding to x and $\ln y$, respectively, will look like the graph of a logarithm, $\ln y = n \ln x + c$. Such **semi-log graphs** (i.e., graphs of $\ln y$ versus x) let us distinguish the graph of an exponential from that of a polynomial: graphs of exponentials become straight lines, while graphs of polynomials (of degree ≥ 1) become logarithmic curves. Scientists and engineers frequently use semi-log graphs to help them gain an understanding of physical phenomena represented by some collection of data.

EXAMPLE 4.14 Using a Semi-Log Graph to Identify a Type of Function

Determine whether the population of the Arab Republic of Egypt from 1960 to 2010 was increasing exponentially or as a polynomial.

Solution As already indicated, the trick is to draw a semi-log graph. That is, instead of plotting (1, 27) as the first data point, plot (1, ln 27) and so on. A semi-log plot of this data set is seen in Figure 5.59. Although the points are not exactly collinear (how would you prove this?), the plot is very close to a straight line with ln y-intercept of 3.1 and slope 0.22. You should conclude that the population is well modeled by an exponential function. The exponential model would be $y = P(t) = ae^{bt}$, where t represents the number of decades since 1950. Here, b is the slope and $\ln a$ is the ln y-intercept of the line in the semi-log graph. That is, $b \approx 0.22$ and $\ln a \approx 3.1$, so that $a \approx 22.2$. The population is then modeled by

$$P(t) = 22.2 \cdot e^{0.22t} \text{ million.}$$

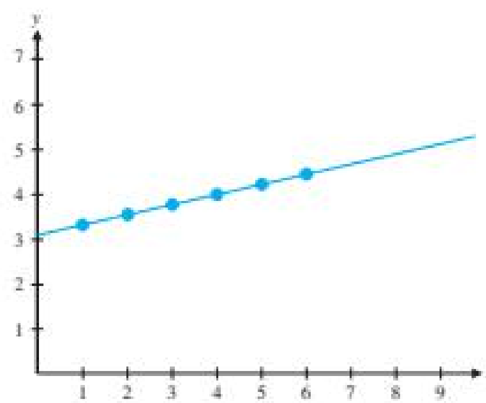


FIGURE 5.59
Semi-log plot of Egypt population

EXERCISES 5.4

WRITING EXERCISES

1. Starting from a single cell, a human being is formed by 50 generations of cell division. Explain why after n divisions there are 2^n cells. Guess how many cells will be present after 50 divisions, then compute 2^{50}. Briefly discuss how rapidly exponential functions increase.

2. Explain why the graphs of $f(x) = 2^{-x}$ and $g(x) = \left(\frac{1}{2}\right)^x$ are the same.

3. Compare $f(x) = x^2$ and $g(x) = 2^x$ for $x = \frac{1}{2}$, $x = 1$, $x = 2$, $x = 3$, and $x = 4$. In general, which function is bigger for large values of x? For small values of x?

4. Compare $f(x) = 2^x$ and $g(x) = 3^x$ for $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$ and $x = 2$. In general, which function is bigger for negative values of x? For positive values of x?

In exercises 1–6, convert each exponential expression into fractional or root form.

1. 2^{-3} 2. 4^{-2} 3. $3^{1/2}$

4. $6^{2/5}$ 5. $5^{2/3}$ 6. $4^{-2/3}$

In exercises 7–12, convert each expression into exponential form.

7. $\frac{1}{x^2}$ 8. $\sqrt[3]{x^2}$ 9. $\frac{2}{x^3}$

10. $\frac{4}{x^2}$ 11. $\frac{1}{2\sqrt{x}}$ 12. $\frac{3}{2\sqrt{x^3}}$

In exercises 13–16, find the integer value of the given expression without using a calculator.

13. $4^{3/2}$ 14. $8^{2/3}$ 15. $\frac{\sqrt{8}}{2^{1/2}}$ 16. $\frac{2}{(1/3)^2}$

In exercises 17–20, use a calculator or computer to estimate each value.

17. $2e^{-1/2}$ 18. $4e^{-2/3}$

19. $\frac{12}{e}$ 20. $\frac{14}{\sqrt{e}}$

In exercises 21–26, sketch graphs of the given functions and compare the graphs.

21. $f(x) = e^{2x}$ and $g(x) = e^{3x}$

22. $f(x) = 2e^{x/4}$ and $g(x) = 4e^{x/2}$

23. $f(x) = 3e^{-2x}$ and $g(x) = 2e^{-3x}$

24. $f(x) = e^{-x^2}$ and $g(x) = e^{-x^2/4}$

25. $f(x) = \ln 2x$ and $g(x) = \ln x^2$

26. $f(x) = e^{2\ln x}$ and $g(x) = x^2$

In exercises 27–36, solve the given equation for x.

27. $e^{2x} = 2$ 28. $e^{4x} = 3$

29. $e^x(x^2 - 1) = 0$ 30. $xe^{-2x} + 2e^{-2x} = 0$

31. $4 \ln x = -8$ 32. $x^2 \ln x - 9 \ln x = 0$

33. $e^{2\ln x} = 4$ 34. $\ln(e^{2x}) = 6$

35. $e^x = 1 + 6e^{-x}$ 36. $\ln x + \ln(x - 1) = \ln 2$

In exercises 37 and 38, use the definition of logarithm to determine the value.

37. (a) $\log_3 9$ (b) $\log_4 64$ (c) $\log_3 \frac{1}{27}$

38. (a) $\log_4 \frac{1}{16}$ (b) $\log_4 2$ (c) $\log_9 3$

In exercises 39 and 40, use equation (5.4) to approximate the value.

39. (a) $\log_3 7$ (b) $\log_4 60$ (c) $\log_3 \frac{1}{24}$

40. (a) $\log_4 \frac{1}{10}$ (b) $\log_4 3$ (c) $\log_9 8$

In exercises 41–46, rewrite the expression as a single logarithm.

41. $\ln 3 - \ln 4$ 42. $2 \ln 4 - \ln 3$

43. $\frac{1}{2} \ln 4 - \ln 2$ 44. $3 \ln 2 - \ln \frac{1}{2}$

45. $\ln \frac{3}{4} + 4 \ln 2$ 46. $\ln 9 - 2 \ln 3$

In exercises 47–50, find a function of the form $f(x) = ae^{bx}$ with the given function values.

47. $f(0) = 2, f(2) = 6$ 48. $f(0) = 3, f(3) = 4$

49. $f(0) = 4, f(2) = 2$ 50. $f(0) = 5, f(1) = 2$

Exercises 51–54 refer to the hyperbolic functions.

51. Show that the range of the hyperbolic cosine is $\cosh x \geq 1$ and the range of the hyperbolic sine is the entire real line.

52. Show that $\cosh^2 x - \sinh^2 x = 1$ for all x.

53. Find all solutions of $\sinh(x^2 - 1) = 0$.

54. Find all solutions of $\cosh(3x + 2) = 0$.

APPLICATIONS

1. A fast-food restaurant gives every customer a game ticket. With each ticket, the customer has a 1-in-10 chance of winning a free meal. If you go 10 times, estimate your chances of winning at least one free meal. The exact probability is $1 - \left(\frac{9}{10}\right)^{10}$. Compute this number and compare it to your guess.

2. In exercise 1, if you had 20 tickets with a 1-in-20 chance of winning, would you expect your probability of winning at least once to increase or decrease? Compute the probability $1 - \left(\frac{19}{20}\right)^{20}$ to find out.

3. In general, if you have n chances of winning with a 1-in-n chance on each try, the probability of winning at least once is $1 - \left(1 - \frac{1}{n}\right)^n$. As n gets larger, what number does this probability approach? (Hint: There is a very good reason that this question is in this section!)

4. If $y = a \cdot x^m$, show that $\ln y = \ln a + m \ln x$. If $v = \ln y$, $u = \ln x$ and $b = \ln a$, show that $v = mu + b$. Explain why the graph of v as a function of u would be a straight line. This graph is called the **log–log plot** of y and x.

5. For the given data (on the next page), compute $v = \ln y$ and $u = \ln x$, and plot points (u, v). Find constants m and b such that $v = mu + b$ and use the results of exercise 4 to find a constant a such that $y = a \cdot x^m$.

x	2.2	2.4	2.6	2.8	3.0	3.2
y	14.52	17.28	20.28	23.52	27.0	30.72

6. Repeat exercise 5 for the given data.

x	2.8	3.0	3.2	3.4	3.6	3.8
y	9.37	10.39	11.45	12.54	13.66	14.81

7. Construct a log-log plot (see exercise 4) of the Egypt population data in example 4.14. Compared to the semi-log plot of the data in Figure 5.59, does the log-log plot look linear? Based on this, are the population data modeled better by an exponential function or a polynomial (power) function?

8. Construct a semi-log plot of the data in exercise 5. Compared to the log–log plot already constructed, does this plot look linear? Based on this, are these data better modeled by an exponential or power function?

9. The concentration $[H^+]$ of free hydrogen ions in a chemical solution determines the solution's pH, as defined by $pH = -\log [H^+]$. Find $[H^+]$ if the pH equals (a) 7, (b) 8 and (c) 9. For each increase in pH of 1, by what factor does $[H^+]$ change?

10. Gastric juice is considered an acid, with a pH of about 2.5. Blood is considered alkaline, with a pH of about 7.5. Compare the concentrations of hydrogen ions in the two substances (see exercise 9).

11. The Richter magnitude M of an earthquake is defined in terms of the energy E in joules released by the earthquake, with $\log_{10} E = 4.4 + 1.5M$. Find the energy for earthquakes with magnitudes (a) 4, (b) 5 and (c) 6. For each increase in M of 1, by what factor does E change?

12. The decibel level of a noise is defined in terms of the intensity I of the noise, with $dB = 10 \log (I/I_0)$. Here, $I_0 = 10^{-12}$ W/m^2 is the intensity of a barely audible sound. Compute the intensity levels of sounds with (a) $dB = 80$, (b) $dB = 90$ and (c) $dB = 100$. For each increase of 10 decibels, by what factor does I change?

13. The Saint Louis Gateway Arch is both 630 feet wide and 630 feet tall. (Most people think that it looks taller than it is wide.) One model for the outline of the arch is $y = 757.7 - 127.7 \cosh\left(\frac{x}{127.7}\right)$ for $y \geq 0$. Use a graphing calculator to approximate the x- and y-intercepts and determine if the model has the correct horizontal and vertical measurements.

14. To model the outline of the Gateway Arch with a parabola, you can start with $y = -c(x + 315)(x - 315)$ for some constant c. Explain why this gives the correct x-intercepts. Determine the constant c that gives a y-intercept of 630. Graph this parabola and the hyperbolic cosine in exercise 13 on the same axes. Are the graphs nearly identical or very different?

15. On a standard piano, the A below middle C produces a sound wave with frequency 220 Hz (cycles per second). The frequency of the A one octave higher is 440 Hz. In general, doubling the frequency produces the same note an octave higher. Find an exponential formula for the frequency f as a function of the number of octaves x above the A below middle C.

16. There are 12 notes in an octave on a standard piano. Middle C is 3 notes above A (see exercise 15). If the notes are tuned equally, this means that middle C is a quarter-octave above A. Use $x = \frac{1}{4}$ in your formula from exercise 15 to estimate the frequency of middle C.

EXPLORATORY EXERCISES

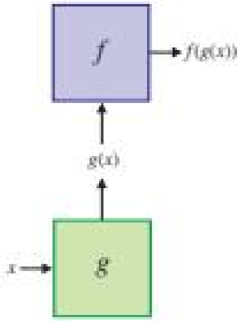
1. Graph $y = x^2$ and $y = 2^x$ and approximate the two positive solutions of the equation $x^2 = 2^x$. Graph $y = x^3$ and $y = 3^x$, and approximate the two positive solutions of the equation $x^3 = 3^x$. Explain why $x = a$ will always be a solution of $x^a = a^x$, $a > 0$. What is different about the role of $x = 2$ as a solution of $x^2 = 2^x$ compared to the role of $x = 3$ as a solution of $x^3 = 3^x$? To determine the a-value at which the change occurs, graphically solve $x^a = a^x$ for $a = 2.1, 2.2, \ldots, 2.9$, and note that $a = 2.7$ and $a = 2.8$ behave differently. Continue to narrow down the interval of change by testing $a = 2.71, 2.72, \ldots, 2.79$. Then guess the exact value of a.

2. Graph $y = \ln x$ and describe the behavior near $x = 0$. Then graph $y = x \ln x$ and describe the behavior near $x = 0$. Repeat this for $y = x^2 \ln x$, $y = x^{1/2} \ln x$ and $y = x^a \ln x$ for a variety of positive constants a. Because the function "blows up" at $x = 0$, we say that $y = \ln x$ has a **singularity** at $x = 0$. The **order** of the singularity at $x = 0$ of a function $f(x)$ is the smallest value of a such that $y = x^a f(x)$ doesn't have a singularity at $x = 0$. Determine the order of the singularity at $x = 0$ for (a) $f(x) = \frac{1}{x}$, (b) $f(x) = \frac{1}{x^2}$ and (c) $f(x) = \frac{1}{x^3}$. The higher the order of the singularity, the "worse" the singularity is. Based on your work, how bad is the singularity of $y = \ln x$ at $x = 0$?

5.5 TRANSFORMATIONS OF FUNCTIONS

You are now familiar with a long list of functions: polynomials, rational functions, trigonometric functions, exponentials and logarithms. One important goal of this course is to more fully understand the properties of these functions. To a large extent, you will build your understanding by examining a few key properties of functions.

We expand on our list of functions by combining them. We begin in a straightforward fashion with Definition 5.1.



$(f \circ g)(x) = f(g(x))$

DEFINITION 5.1

Suppose that f and g are functions with domains D_1 and D_2, respectively. The functions $f + g$, $f - g$ and $f \cdot g$ are defined by

$$(f+g)(x) = f(x) + g(x),$$
$$(f-g)(x) = f(x) - g(x)$$
and
$$(f \cdot g)(x) = f(x) \cdot g(x),$$

for all x in $D_1 \cap D_2$ (i.e., $x \in D_1$, *and* $x \in D_2$). The function $\frac{f}{g}$ is defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

for all x in $D_1 \cap D_2$ such that $g(x) \neq 0$.

In example 5.1, we examine various combinations of several simple functions.

EXAMPLE 5.1 Combinations of Functions

If $f(x) = x - 3$ and $g(x) = \sqrt{x-1}$, determine the functions $f + g$, $3f - g$ and $\frac{f}{g}$, stating the domains of each.

Solution First, note that the domain of f is the entire real line and the domain of g is the set of all $x \geq 1$. Now,

$$(f+g)(x) = x - 3 + \sqrt{x-1}$$

and

$$(3f - g)(x) = 3(x-3) - \sqrt{x-1} = 3x - 9 - \sqrt{x-1}.$$

Notice that the domain of both $(f + g)$ and $(3f - g)$ is $\{x|x \geq 1\}$. For

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x-3}{\sqrt{x-1}},$$

the domain is $\{x \mid x > 1\}$, where we have added the restriction $x \neq 1$ to avoid dividing by 0. ■

Definition 5.1 and example 5.1 show us how to do arithmetic with functions. An operation on functions that does not directly correspond to arithmetic is the *composition* of two functions.

DEFINITION 5.2

The **composition** of functions f and g, written $f \circ g$, is defined by

$$(f \circ g)(x) = f(g(x)),$$

for all x such that x is in the domain of g and $g(x)$ is in the domain of f.

The composition of two functions is a two-step process, as indicated in the margin schematic. Be careful to notice what this definition is saying. In particular, for $f(g(x))$ to be defined, you first need $g(x)$ to be defined, so x must be in the domain of g. Next, f must be defined at the point $g(x)$, so that the number $g(x)$ will need to be in the domain of f.

EXAMPLE 5.2 Finding the Composition of Two Functions

For $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-2}$, find the compositions $f \circ g$ and $g \circ f$ and identify the domain of each.

Solution First, we have

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x-2})$$
$$= (\sqrt{x-2})^2 + 1 = x - 2 + 1 = x - 1.$$

It's tempting to write that the domain of $f \circ g$ is the entire real line, but look more carefully. Note that for x to be in the domain of g, we must have $x \geq 2$. The domain of f is the whole real line, so this places no further restrictions on the domain of $f \circ g$. Even though the final expression $x - 1$ is defined for all x, the domain of $(f \circ g)$ is $\{x|x \geq 2\}$.

For the second composition,

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 1)$$
$$= \sqrt{(x^2+1) - 2} = \sqrt{x^2 - 1}.$$

The resulting square root requires $x^2 - 1 \geq 0$ or $|x| \geq 1$. Since the "inside" function f is defined for all x, the domain of $g \circ f$ is $\{x||x| \geq 1\}$, which we write in interval notation as $(-\infty, -1] \cup [1, \infty)$. ■

As you progress through the calculus, you will often need to recognize that a given function is a composition of simpler functions.

EXAMPLE 5.3 Identifying Compositions of Functions

Identify functions f and g such that the given function can be written as $(f \circ g)(x)$ for each of (a) $\sqrt{x^2+1}$, (b) $(\sqrt{x}+1)^2$, (c) $\sin x^2$ and (d) $\cos^2 x$. Note that more than one answer is possible for each function.

Solution (a) Notice that $x^2 + 1$ is *inside* the square root. So, one choice is to have $g(x) = x^2 + 1$ and $f(x) = \sqrt{x}$.

(b) Here, $\sqrt{x} + 1$ is *inside* the square. So, one choice is $g(x) = \sqrt{x} + 1$ and $f(x) = x^2$.

(c) The function can be rewritten as $\sin(x^2)$, with x^2 clearly *inside* the sine function. Then, $g(x) = x^2$ and $f(x) = \sin x$ is one choice.

(d) The function as written is shorthand for $(\cos x)^2$. So, one choice is $g(x) = \cos x$ and $f(x) = x^2$. ■

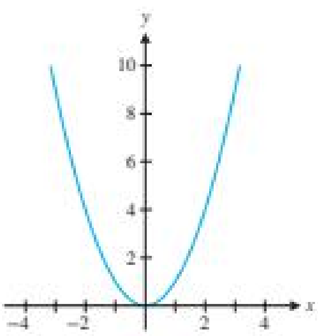


FIGURE 5.60a
$y = x^2$

In general, it is quite difficult to take the graphs of $f(x)$ and $g(x)$ and produce the graph of $f(g(x))$. If one of the functions f and g is linear, however, there is a simple graphical procedure for graphing the composition. Such **linear transformations** are explored in the remainder of this section.

The first case is to take the graph of $f(x)$ and produce the graph of $f(x) + c$ for some constant c. You should be able to deduce the general result from example 5.4.

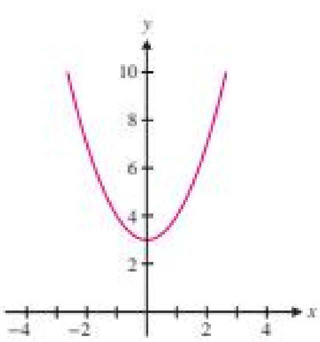


FIGURE 5.60b
$y = x^2 + 3$

EXAMPLE 5.4 Vertical Translation of a Graph

Graph $y = x^2$ and $y = x^2 + 3$; compare and contrast the graphs.

Solution You can probably sketch these by hand. You should get graphs like those in Figures 5.60a and 5.60b. Both figures show parabolas opening upward. The main obvious difference is that x^2 has a y-intercept of 0 and $x^2 + 3$ has a y-intercept of 3. In fact, for *any* given value of x, the point on the graph of $y = x^2 + 3$ will be plotted exactly 3 units higher than the corresponding point on the graph of $y = x^2$. This is shown in Figure 5.61a.

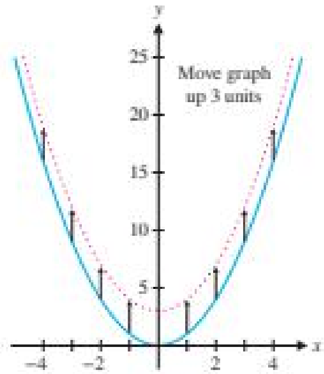


FIGURE 5.61a
Translate graph up

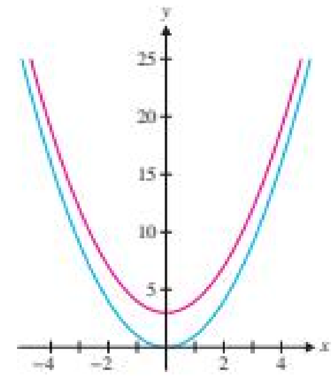


FIGURE 5.61b
$y = x^2$ and $y = x^2 + 3$

In Figure 5.61b, the two graphs are shown on the same set of axes. To many people, it does not look like the top graph is the same as the bottom graph moved up 3 units. This is an unfortunate optical illusion. Humans usually mentally judge distance between curves as the shortest distance between the curves. For these parabolas, the shortest distance is vertical at $x = 0$ but becomes increasingly horizontal as you move away from the y-axis. The distance of 3 between the parabolas is measured *vertically*. ■

REMARK 5.1

In general, the graph of $y = f(x) + c$ is the same as the graph of $y = f(x)$ shifted up (if $c > 0$) or down (if $c < 0$) by $|c|$ units. We usually refer to $f(x) + c$ as a **vertical translation** (up or down, by $|c|$ units).

In example 5.5, we explore what happens if a constant is added to x.

EXAMPLE 5.5 A Horizontal Translation

Compare and contrast the graphs of $y = x^2$ and $y = (x - 1)^2$.

Solution The graphs are shown in Figures 5.62a and 5.62b, respectively.

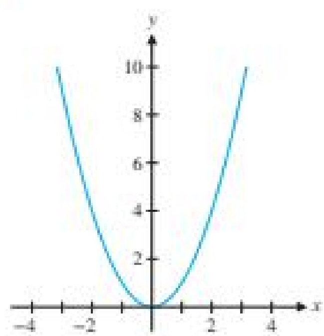


FIGURE 5.62a
$y = x^2$

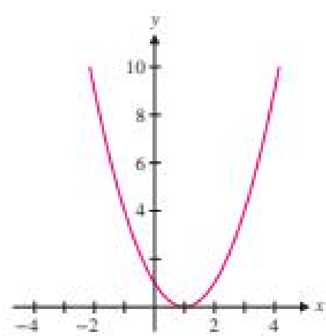


FIGURE 5.62b
$y = (x - 1)^2$

Notice that the graph of $y = (x - 1)^2$ appears to be the same as the graph of $y = x^2$, except that it is shifted 1 unit to the right. This should make sense for the following reason. Pick a value of x, say, $x = 13$. The value of $(x - 1)^2$ at $x = 13$ is 12^2, the same as the value of x^2 at $x = 12$, 1 unit to the left. Observe that this same pattern holds for any x you choose. A simultaneous plot of the two functions (see Figure 5.63) shows this. ■

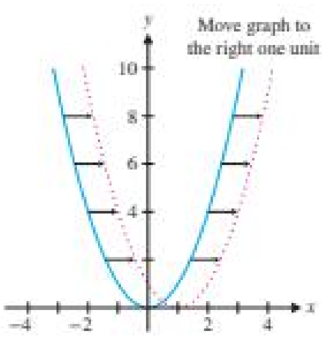


FIGURE 5.63
Translation to the right

REMARK 5.2

In general, for $c > 0$, the graph of $y = f(x - c)$ is the same as the graph of $y = f(x)$ shifted c units to the right. Likewise (again, for $c > 0$), you get the graph of $y = f(x + c)$ by moving the graph of $y = f(x)$ to the left c units. We usually refer to $f(x - c)$ and $f(x + c)$ as **horizontal translations** (to the right and left, respectively, by c units).

To avoid confusion on which way to translate the graph of $y = f(x)$, focus on what makes the argument (the quantity inside the parentheses) zero. For $f(x)$, this is $x = 0$, but for $f(x - c)$ you must have $x = c$ to get $f(0)$ [i.e., the same y-value as $f(x)$ when $x = 0$]. This says that the point on the graph of $y = f(x)$ at $x = 0$ corresponds to the point on the graph of $y = f(x - c)$ at $x = c$.

EXAMPLE 5.6 Comparing Vertical and Horizontal Translations

Given the graph of $y = f(x)$ shown in Figure 5.64a, sketch the graphs of $y = f(x) - 2$ and $y = f(x - 2)$.

Solution To graph $y = f(x) - 2$, simply translate the original graph down 2 units, as shown in Figure 5.64b. To graph $y = f(x - 2)$, simply translate the original graph to the right 2 units (so that the x-intercept at $x = 0$ in the original graph corresponds to an x-intercept at $x = 2$ in the translated graph), as seen in Figure 5.64c.

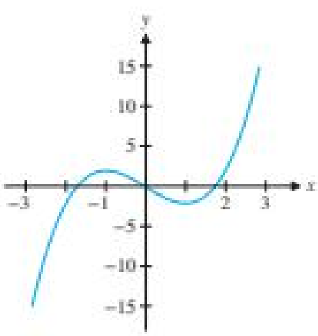


FIGURE 5.64a
$y = f(x)$

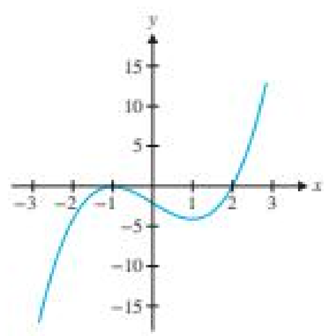


FIGURE 5.64b
$y = f(x) - 2$

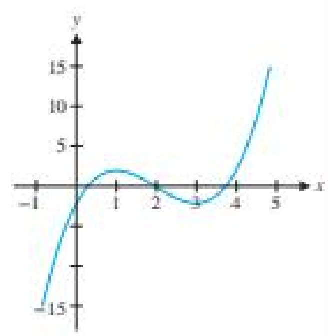


FIGURE 5.64c
$y = f(x - 2)$

Example 5.7 explores the effect of multiplying or dividing x or y by a constant.

EXAMPLE 5.7 Comparing Some Related Graphs

Compare and contrast the graphs of $y = x^2 - 1$, $y = 4(x^2 - 1)$ and $y = (4x)^2 - 1$.

Solution The first two graphs are shown in Figures 5.65a and 5.65b, respectively. These graphs look identical until you compare the scales on the y-axes. The scale in Figure 5.65b is four times as large, reflecting the multiplication of the original function by 4. The effect looks different when the functions are plotted on the same scale, as in Figure 5.65c. Here, the parabola $y = 4(x^2 - 1)$ looks thinner and has a different y-intercept. Note that the x-intercepts remain the same. (Why would that be?)

The graphs of $y = x^2 - 1$ and $y = (4x)^2 - 1$ are shown in Figures 5.66a and 5.66b, respectively.

Can you spot the difference here? In this case, the x-scale has now changed, by the same factor of 4 as in the function. To see this, note that substituting $x = 1/4$ into $(4x)^2 - 1$ produces $(1)^2 - 1$, exactly the same as substituting $x = 1$ into the original function. When

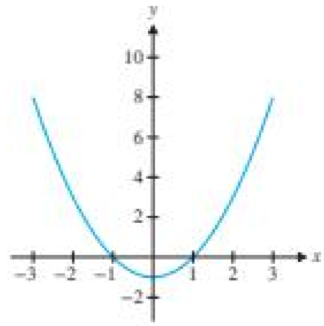


FIGURE 5.65a
$y = x^2 - 1$

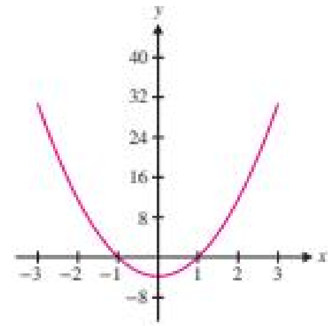


FIGURE 5.65b
$y = 4(x^2 - 1)$

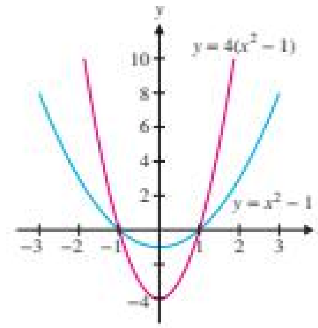


FIGURE 5.65c
$y = x^2 - 1$ and $y = 4(x^2 - 1)$

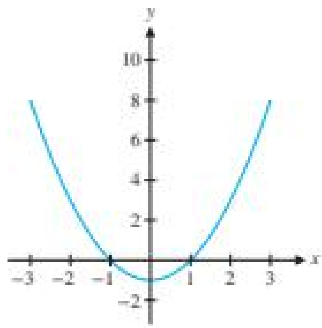


FIGURE 5.66a
$y = x^2 - 1$

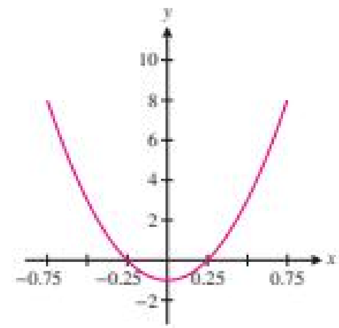


FIGURE 5.66b
$y = (4x)^2 - 1$

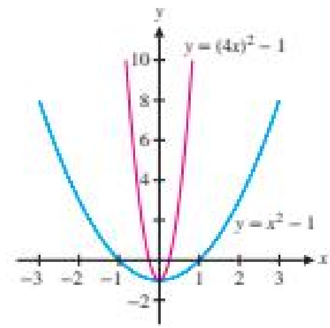


FIGURE 5.66c
$y = x^2 - 1$ and $y = (4x)^2 - 1$

plotted on the same set of axes (as in Figure 5.66c), the parabola $y = (4x)^2 - 1$ looks thinner. Here, the x-intercepts are different, but the y-intercepts are the same. ■

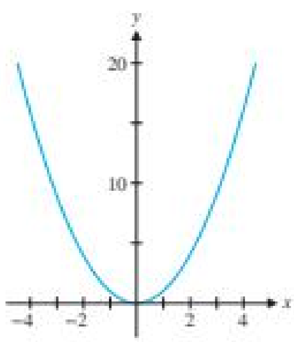


FIGURE 5.67a
$y = x^2$

We can generalize the observations made in example 5.7. Before reading our explanation, try to state a general rule for yourself. How are the graphs of $y = cf(x)$ and $y = f(cx)$ related to the graph of $y = f(x)$?

Based on example 5.7, notice that to obtain a graph of $y = cf(x)$ for some constant $c > 0$, you can take the graph of $y = f(x)$ and multiply the scale on the y-axis by c. To obtain a graph of $y = f(cx)$ for some constant $c > 0$, you can take the graph of $y = f(x)$ and multiply the scale on the x-axis by $1/c$.

These basic rules can be combined to understand more complicated graphs.

EXAMPLE 5.8 A Translation and a Stretching

Describe how to get the graph of $y = 2x^2 - 3$ from the graph of $y = x^2$.

Solution You can get from x^2 to $2x^2 - 3$ by multiplying by 2 and then subtracting 3. In terms of the graph, this has the effect of multiplying the y-scale by 2 and then shifting the graph down by 3 units. (See the graphs in Figures 5.67a and 5.67b.) ■

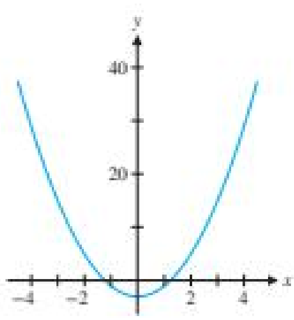


FIGURE 5.67b
$y = 2x^2 - 3$

EXAMPLE 5.9 A Translation in Both x- and y-Directions

Describe how to get the graph of $y = x^2 + 4x + 3$ from the graph of $y = x^2$.

Solution We can again relate this (and the graph of *every* quadratic) to the graph of $y = x^2$. We must first **complete the square**. Recall that in this process, you take the coefficient of $x(4)$, divide by $2(4/2 = 2)$ and square the result $(2^2 = 4)$. Add and subtract this number and then rewrite the x-terms as a perfect square. We have

$$y = x^2 + 4x + 3 = (x^2 + 4x + 4) - 4 + 3 = (x + 2)^2 - 1.$$

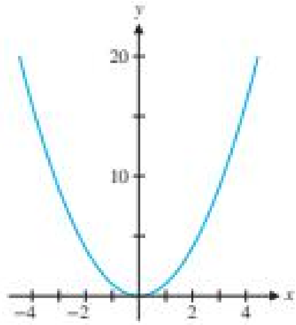


FIGURE 5.68a
$y = x^2$

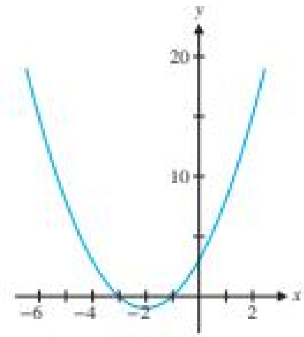


FIGURE 5.68b
$y = (x + 2)^2 - 1$

To graph this function, take the parabola $y = x^2$ (see Figure 5.68a) and translate the graph 2 units to the left and 1 unit down. (See Figure 5.68b.) ■

The following table summarizes our discoveries in this section.

Transformations of $f(x)$

Transformation	*Form*	*Effect on Graph*
Vertical translation	$f(x) + c$	$\lvert c\rvert$ units up ($c > 0$) or down ($c < 0$)
Horizontal translation	$f(x + c)$	$\lvert c\rvert$ units left ($c > 0$) or right ($c < 0$)
Vertical scale	$cf(x)$ ($c > 0$)	multiply vertical scale by c
Horizontal scale	$f(cx)$ ($c > 0$)	divide horizontal scale by c

You will explore additional transformations in the exercises.

EXERCISES 5.5

WRITING EXERCISES

1. The restricted domain of example 5.2 may be puzzling. Consider the following analogy. Suppose you have an airplane flight from Dubai to Rome with a stop in Doha. If bad weather has closed the airport in Doha, explain why your flight will be canceled (or at least rerouted) even if the weather is great in Dubai and Rome.

2. Explain why the graphs of $y = 4(x^2 - 1)$ and $y = (4x)^2 - 1$ in Figures 5.65c and 5.66c appear "thinner" than the graph of $y = x^2 - 1$.

3. As illustrated in example 5.9, completing the square can be used to rewrite any quadratic function in the form $a(x - d)^2 + e$. Using the transformation rules in this section, explain why this means that all parabolas (with $a > 0$) will look essentially the same.

4. Explain why the graph of $y = f(x + 4)$ is obtained by moving the graph of $y = f(x)$ four units to the left, instead of to the right.

In exercises 1–6, find the compositions $f \circ g$ and $g \circ f$, and identify their respective domains.

1. $f(x) = x + 1$, $g(x) = \sqrt{x - 3}$
2. $f(x) = x - 2$, $g(x) = \sqrt{x + 1}$
3. $f(x) = e^x$, $g(x) = \ln x$
4. $f(x) = \sqrt{1 - x}$, $g(x) = \ln x$
5. $f(x) = x^2 + 1$, $g(x) = \sin x$
6. $f(x) = \dfrac{1}{x^2 - 1}$, $g(x) = x^2 - 2$

In exercises 7–16, identify functions $f(x)$ and $g(x)$ such that the given function equals $(f \circ g)(x)$.

7. $\sqrt{x^4 + 1}$
8. $\sqrt[3]{x + 3}$
9. $\dfrac{1}{x^2 + 1}$
10. $\dfrac{1}{x^2} + 1$
11. $(4x + 1)^2 + 3$
12. $4(x + 1)^2 + 3$
13. $\sin^3 x$
14. $\sin x^3$
15. e^{x^2+1}
16. e^{4x-2}

In exercises 17–22, identify functions $f(x)$, $g(x)$ and $h(x)$ such that the given function equals $[f\circ(g\circ h)](x)$.

17. $\dfrac{3}{\sqrt{\sin x+2}}$ **18.** $\sqrt{e^{4x}+1}$

19. $\cos^3(4x-2)$ **20.** $\ln\sqrt{x^2+1}$

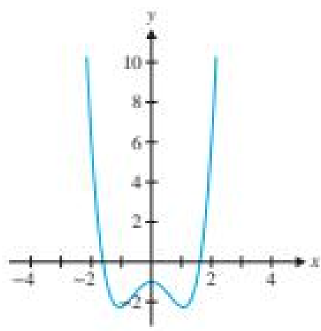
21. $4e^{x^2-1}$ **22.** $\left[\tan^{-1}(3x+1)\right]^2$

In exercises 23–30, use the graph of $y=f(x)$ given in the figure to graph the indicated function.

23. $f(x)-3$ **24.** $f(x+2)$ **25.** $f(x-3)$

26. $f(x)+2$ **27.** $f(2x)$ **28.** $3f(x)$

29. $-3f(x)+2$ **30.** $3f(x+2)$

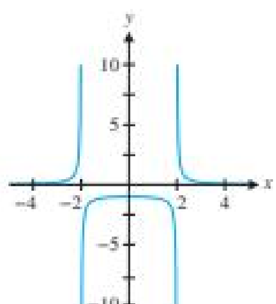


In exercises 31–38, use the graph of $y=f(x)$ given in the figure to graph the indicated function.

31. $f(x-4)$ **32.** $f(x+3)$ **33.** $f(2x)$

34. $f(2x-4)$ **35.** $f(3x+3)$ **36.** $3f(x)$

37. $2f(x)-4$ **38.** $3f(x)+3$



In exercises 39–44, complete the square and explain how to transform the graph of $y=x^2$ into the graph of the given function.

39. $f(x)=x^2+2x+1$ **40.** $f(x)=x^2-4x+4$

41. $f(x)=x^2+2x+4$ **42.** $f(x)=x^2-4x+2$

43. $f(x)=2x^2+4x+4$ **44.** $f(x)=3x^2-6x+2$

In exercises 45–48, graph the given function and compare to the graph of $y=x^2-1$.

45. $f(x)=-2(x^2-1)$

46. $f(x)=-3(x^2-1)$

47. $f(x)=-3(x^2-1)+2$

48. $f(x)=-2(x^2-1)-1$

In exercises 49–52, graph the given function and compare to the graph of $f(x)=(-x)^2-2(-x)$

49. $f(x)=(-x)^2-2(-x)$

50. $f(x)=-(-x)^2+2(-x)$

51. $f(x)=(-x+1)^2+2(-x+1)$

52. $f(x)=(-3x)^2-2(-3x)-3$

53. Based on exercises 45–48, state a rule for transforming the graph of $y=f(x)$ into the graph of $y=cf(x)$ for $c<0$.

54. Based on exercises 49–52, state a rule for transforming the graph of $y=f(x)$ into the graph of $y=f(cx)$ for $c<0$.

55. Sketch the graph of $y=|x|^3$. Explain why the graph of $y=|x|^3$ is identical to that of $y=x^3$ to the right of the y-axis. For $y=|x|^3$, describe how the graph to the left of the y-axis compares to the graph to the right of the y-axis. In general, describe how to draw the graph of $y=f(|x|)$ given the graph of $y=f(x)$.

56. For $y=x^3$, describe how the graph to the left of the y-axis compares to the graph to the right of the y-axis. Show that for $f(x)=x^3$, we have $f(-x)=-f(x)$. In general, if you have the graph of $y=f(x)$ to the right of the y-axis and $f(-x)=-f(x)$ for all x's, describe how to graph $y=f(x)$ to the left of the y-axis.

57. Iterations of functions are important in a variety of applications. To iterate $f(x)$, start with an initial value x_0 and compute $x_1=f(x_0)$, $x_2=f(x_1)$, $x_3=f(x_2)$ and so on. For example, with $f(x)=\cos x$ and $x_0=1$, the **iterates** are $x_1=\cos 1\approx 0.54$, $x_2=\cos x_1\approx\cos 0.54\approx 0.86$, $x_3\approx\cos 0.86\approx 0.65$ and so on. Keep computing iterates and show that they get closer and closer to 0.739085. Then pick your own x_0 (any number you like) and show that the iterates with this new x_0 also converge to 0.739085.

58. Referring to exercise 57, show that the iterates of a function can be written as $x_1=f(x_0)$, $x_2=f(f(x_0))$, $x_3=f(f(f(x_0)))$ and so on. Graph $y=\cos(\cos x)$, $y=\cos(\cos(\cos x))$ and $y=\cos(\cos(\cos(\cos x)))$. The graphs should look more and more like a horizontal line. Use the result of exercise 57 to identify the limiting line.

59. Compute several iterates of $f(x)=\sin x$ (see exercise 57) with a variety of starting values. What happens to the iterates in the long run?

60. Repeat exercise 59 for $f(x)=x^2$.

61. In cases where the iterates of a function (see exercise 57) repeat a single number, that number is called a **fixed point**.

Explain why any fixed point must be a solution of the equation $f(x) = x$. Find all fixed points of $f(x) = \cos x$ by solving the equation $\cos x = x$. Compare your results to that of exercise 57.

62. Find all fixed points of $f(x) = \sin x$ (see exercise 61). Compare your results to those of exercise 59.

EXPLORATORY EXERCISES

1. You have explored how completing the square can transform any quadratic function into the form $y = a(x-d)^2 + e$. We concluded that all parabolas with $a > 0$ look alike. To see that the same statement is not true of cubic polynomials, graph $y = x^3$ and $y = x^3 - 3x$. In this exercise, you will use completing the cube to determine how many different cubic graphs there are. To see what "completing the cube" would look like, first show that $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$. Use this result to transform the graph of $y = x^3$ into the graphs of (a) $y = x^3 - 3x^2 + 3x - 1$ and (b) $y = x^3 - 3x^2 + 3x + 2$. Show that you can't get a simple transformation to $y = x^3 - 3x^2 + 4x - 2$. However, show that $y = x^3 - 3x^2 + 4x - 2$ can be obtained from $y = x^3 + x$ by basic transformations. Show that the following statement is true: any cubic ($y = ax^3 + bx^2 + cx + d$) can be obtained with basic transformations from $y = ax^3 + kx$ for some constant k.

2. In many applications, it is important to take a section of a graph (e.g., some data) and extend it for predictions or other analysis. For example, suppose you have an electronic signal equal to $f(x) = 2x$ for $0 \le x \le 2$. To predict the value of the signal at $x = -1$, you would want to know whether the signal was periodic. If the signal is periodic, explain why $f(-1) = 2$ would be a good prediction. In some applications, you would assume that the function is *even*. That is, $f(x) = f(-x)$ for all x. In this case, you want $f(x) = 2(-x) = -2x$ for $-2 \le x \le 0$. Graph the *even extension* $f(x) = \begin{cases} -2x & \text{if } -2 \le x \le 0 \\ 2x & \text{if } 0 \le x \le 2 \end{cases}$. Find the even extension for (a) $f(x) = x^2 + 2x + 1$, $0 \le x \le 2$ and (b) $f(x) = e^{-x}$, $0 \le x \le 2$.

3. Similar to the even extension discussed in exploratory exercise 2, applications sometimes require a function to be *odd;* that is, $f(-x) = -f(x)$. For $f(x) = x^2$, $0 \le x \le 2$, the odd extension requires that for $-2 \le x \le 0$, $f(x) = -f(-x) = -(-x)^2 = -x^2$ so that $f(x) = \begin{cases} -x^2 & \text{if } -2 \le x \le 0 \\ x^2 & \text{if } 0 \le x \le 2 \end{cases}$. Graph $y = f(x)$ and discuss how to graphically rotate the right half of the graph to get the left half of the graph. Find the odd extension for (a) $f(x) = x^2 + 2x$, $0 \le x \le 2$ and (b) $f(x) = e^{-x} - 1$, $0 \le x \le 2$.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Slope of a line	Parallel lines	Perpendicular lines
Domain	Intercepts	Zeros of a function
Graphing window	Local maximum	Vertical asymptote
Inverse function	One-to-one function	Periodic function
Sine function	Cosine function	Arcsine function
e	Exponential function	Logarithm
Composition		

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. For a graph, you can compute the slope using any two points and get the same value.

2. All graphs must pass the vertical line test.

3. A cubic function has a graph with one local maximum and one local minimum.

4. If a function has no local maximum or minimum, then it is one-to-one.

5. The graph of the inverse of f can be obtained by reflecting the graph of f across the diagonal $y = x$.

6. If f is a trigonometric function, then the solution of the equation $f(x) = 1$ is $f^{-1}(1)$.

7. Exponential and logarithmic functions are inverses of each other.

8. All quadratic functions have graphs that look like the parabola $y = x^2$.

In exercises 1 and 2, find the slope of the line through the given points.

1. (2, 3), (0, 7)

2. (1, 4), (3, 1)

In exercises 3 and 4, determine whether the lines are parallel, perpendicular or neither.

3. $y = 3x + 1$ and $y = 3(x - 2) + 4$

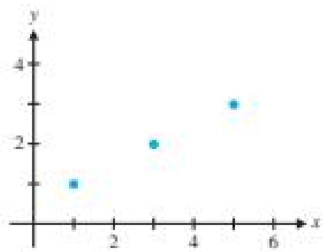
4. $y = -2(x + 1) - 1$ and $y = \frac{1}{2}x + 2$

Review Exercises

5. Determine whether the points (1, 2), (2, 4) and (0, 6) form the vertices of a right triangle.

6. The data represent populations at various times. Plot the points, discuss any patterns and predict the population at the next time: (0, 2100), (1, 3050), (2, 4100) and (3, 5050).

7. Find an equation of the line through the points indicated in the graph that follows and compute the y-coordinate corresponding to $x = 4$.



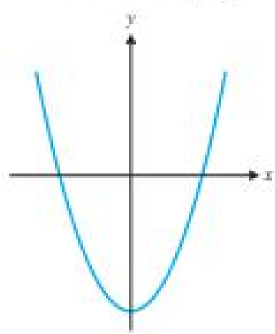
8. For $f(x) = x^2 - 3x - 4$, compute $f(0)$, $f(2)$ and $f(4)$.

In exercises 9 and 10, find an equation of the line with given slope and point.

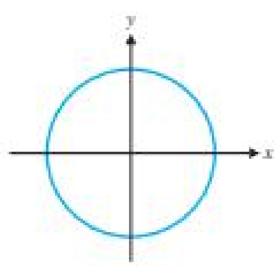
9. $m = -\frac{1}{3}$, $(-1, -1)$ **10.** $m = \frac{1}{4}$, $(0, 2)$

In exercises 11 and 12, use the vertical line test to determine whether the curve is the graph of a function.

11.



12.



In exercises 13 and 14, find the domain of the given function.

13. $f(x) = \sqrt{4 - x^2}$ **14.** $f(x) = \dfrac{x - 2}{x^2 - 2}$

In exercises 15–28, sketch a graph of the function showing extrema, intercepts and asymptotes.

15. $f(x) = x^2 + 2x - 8$ **16.** $f(x) = x^3 - 6x + 1$

17. $f(x) = x^4 - 2x^2 + 1$ **18.** $f(x) = x^5 - 4x^3 + x - 1$

19. $f(x) = \dfrac{4x}{x + 2}$ **20.** $f(x) = \dfrac{x - 2}{x^2 - x - 2}$

21. $f(x) = \sin 3x$ **22.** $f(x) = \tan 4x$

23. $f(x) = \sin x + 2\cos x$ **24.** $f(x) = \sec 2x$

25. $f(x) = 4e^{2x}$ **26.** $f(x) = 3e^{-4x}$

27. $f(x) = \ln 3x$ **28.** $f(x) = e^{\ln 2x}$

29. Determine all intercepts of $y = x^2 + 2x - 8$ (see exercise 15).

30. Determine all intercepts of $y = x^4 - 2x^2 + 1$ (see exercise 17).

31. Find all vertical asymptotes of $y = \dfrac{4x}{x + 2}$.

32. Find all vertical asymptotes of $y = \dfrac{x - 2}{x^2 - x - 2}$.

In exercises 33–36, find or estimate all zeros of the given function.

33. $f(x) = x^2 - 3x - 10$ **34.** $f(x) = x^3 + 4x^2 + 3x$

35. $f(x) = x^3 - 3x^2 + 2$ **36.** $f(x) = x^4 - 3x - 2$

In exercises 37 and 38, determine the number of solutions.

37. $\sin x = x^3$

38. $\sqrt{x^2 + 1} = x^2 - 1$

39. A surveyor stands 50 feet from a telephone pole and measures an angle of 34° to the top. How tall is the pole?

40. Find $\sin \theta$ given that $0 < \theta < \frac{\pi}{2}$ and $\cos \theta = \frac{1}{5}$.

41. Convert to fractional or root form: (a) $5^{-1/2}$ (b) 3^{-2}.

42. Convert to exponential form: (a) $\dfrac{2}{\sqrt{x}}$ (b) $\dfrac{3}{x^2}$.

43. Rewrite $\ln 8 - 2\ln 2$ as a single logarithm.

44. Solve the equation for x: $e^{\ln 4x} = 8$.

In exercises 45 and 46, solve the equation for x.

45. $3e^{2x} = 8$ **46.** $2\ln 3x = 5$

Review Exercises

In exercises 47 and 48, find $f \circ g$ and $g \circ f$, and identify their respective domains.

47. $f(x) = x^2$, $g(x) = \sqrt{x-1}$

48. $f(x) = x^2$, $g(x) = \dfrac{1}{x^2-1}$

In exercises 49 and 50, identify functions $f(x)$ and $g(x)$ such that $(f \circ g)(x)$ equals the given function.

49. e^{3x2+2} **50.** $\sqrt{\sin x + 2}$

In exercises 51 and 52, complete the square and explain how to transform the graph of $y = x^2$ into the graph of the given function.

51. $f(x) = x^2 - 4x + 1$ **52.** $f(x) = x^2 + 4x + 6$

In exercises 53–56, determine whether the function is one-to-one. If so, find its inverse.

53. $x^3 - 1$ **54.** e^{-4x}

55. e^{2x2} **56.** $x^3 - 2x + 1$

In exercises 57–60, graph the inverse without solving for the inverse.

57. $x^5 + 2x^3 - 1$ **58.** $x^3 + 5x + 2$

59. $\sqrt{x^3 + 4x}$ **60.** e^{x3+2x}

In exercises 61–64, evaluate the quantity using the unit circle.

61. $\sin^{-1} 1$ **62.** $\cos^{-1}\left(-\frac{1}{2}\right)$

63. $\tan^{-1}(-1)$ **64.** $\csc^{-1}(-2)$

In exercises 65–68, simplify the expression.

65. $\sin(\sec^{-1} 2)$ **66.** $\tan(\cos^{-1}(4/5))$

67. $\sin^{-1}(\sin(3\pi/4))$ **68.** $\cos^{-1}(\sin(-\pi/4))$

In exercises 69 and 70, find all solutions of the equation.

69. $\sin 2x = 1$ **70.** $\cos 3x = \frac{1}{2}$

EXPLORATORY EXERCISES

1. Sketch a graph of any function $y = f(x)$ that has an inverse. (Your choice.) Sketch the graph of the inverse function $y = f^{-1}(x)$. Then sketch the graph of $y = g(x) = f(x + 2)$. Sketch the graph of $y = g^{-1}(x)$, and use the graph to determine a formula for $g^{-1}(x)$ in terms of $f^{-1}(x)$. Repeat this for $h(x) = f(x) + 3$ and $k(x) = f(x - 4) + 5$.

2. According to the American National Standards Institute,[2] loudness is defined as "*that attribute of auditory sensation in terms of which sounds can be ordered on a scale extending from quiet to loud.*" The loudness of a sound in decibels is given by

$$L = 10 \log\left(\frac{I}{I_0}\right)$$

where I_0 is the intensity of a threshold sound and I is defined in terms of how many times more intense it is than the threshold sound I_0. For example, a normal conversation is approximately 70 decibels, the sound of a diesel truck is around 80 decibels, and the sound of a jet engine (at 15 meters) is around 140 decibels.

a) Use the properties of exponents and logarithms to express I in terms of I_0 and L.

b) The threshold of pain has an intensity of approximately 10^{13} times I_0. What is its loudness reading?

c) Find the intensity of a jet engine relative to I_0. What do you conclude if compared to the threshold of pain reading?

3. The Burj Khalifa in Dubai, United Arab Emirates stands as the tallest building in the world (as of 2020) at 829.8 meters. At 555 meters, the Burj Khalifa Sky is the world's highest outdoor observatory.

a) A tourist is watching the Dubai fountain show from the observatory, which is 200 meters away from the tower. Compute the angle of depression if the tourist is 1.8 meters tall.

b) Suppose that a tourist measures an angle of 70° from the ground to the top of the tower. How far is that tourist from the tower, given that the center of the tower lies 15 meters inside the front of the structure?

4. What follows is a word problem that you will examine in greater detail when you reach the section on Optimization. However, in order to get a glimpse of what's coming your way, we suggest a brief exposure.

A square sheet of cardboard 3 feet on a side is made into a box with an open top by cutting squares of equal size (x) out of each corner.

a) Express the volume of the box in terms of x.

b) Find the volume of the box when $x = \frac{1}{2}$. Would you be able to find a greater value for the volume? Stick around with us to find out why in section 9.7.

[2] American National Standards Institute (1973). "American National Psychoacoustical Terminology", S3.20, American Standards Association.

Limits and Continuity

CHAPTER

6

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Chapter Topics

When you enter a darkened room, your eyes adjust to the reduced level of light by increasing the size of your pupils, allowing more light to enter the eyes and making objects around you easier to see. By contrast, when you enter a brightly lit room, your pupils contract, reducing the amount of light entering the eyes, as too much light would overload your visual system.

Researchers study such mechanisms by performing experiments and trying to find a mathematical description of the results. In this case, you might want to represent the size of the pupils as a function of the amount of light present. Two basic characteristics of such a *mathematical model* would be

1. As the amount of light (x) increases, the pupil size (y) decreases down to a minimum value p; and
2. As the amount of light (x) decreases, the pupil size (y) increases up to a maximum value $\boldsymbol{P}$.

There are many functions with these two properties, but one possible graph of such a function is shown in Figure 6.1. (See example 5.11 on page 351 for more.) In this chapter, we develop the concept of *limit,* which can be used to describe properties such as those listed above. The limit is the fundamental notion of calculus and serves as the thread that binds together virtually all of the calculus you are about to study. An investment in carefully studying limits now will have very significant payoffs throughout the remainder of your calculus experience and beyond.

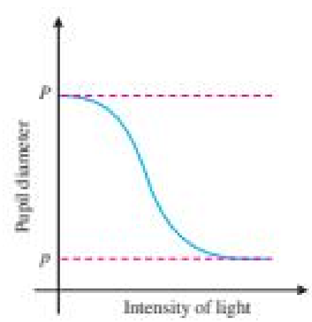


FIGURE 6.1
Size of pupils

6.1 A BRIEF PREVIEW OF CALCULUS: TANGENT LINES AND THE LENGTH OF A CURVE

In this section, we approach the boundary between precalculus mathematics and calculus by investigating several important problems requiring the use of calculus.

Recall that the slope of a straight line is the change in y divided by the change in x. This fraction is the same regardless of which two points you use to compute the slope. For example, the points (0, 1), (1, 4) and (3, 10) all lie on the line $y = 3x + 1$. The slope of 3 can be obtained from any two of the points. For instance,

$$m = \frac{4-1}{1-0} = 3 \qquad \text{or} \qquad m = \frac{10-1}{3-0} = 3.$$

In calculus, we generalize this problem to find the slope of a *curve* at a point. For instance, suppose we wanted to find the slope of the curve $y = x^2 + 1$ at the point (1, 2). You might think of picking a second point on the parabola, say (2, 5). The slope of the line through these two points (called a **secant line**; see Figure 6.2a) is easy enough to compute. We have:

$$m_{sec} = \frac{5-2}{2-1} = 3.$$

However, using the points (0, 1) and (1, 2), we get a different slope (see Figure 6.2b):

$$m_{sec} = \frac{2-1}{1-0} = 1.$$

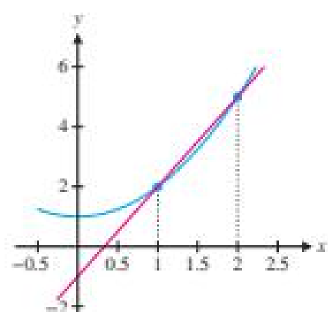


FIGURE 6.2a
Secant line, slope = 3

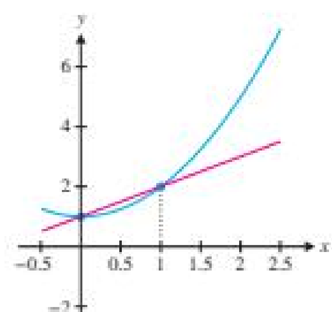


FIGURE 6.2b
Secant line, slope = 1

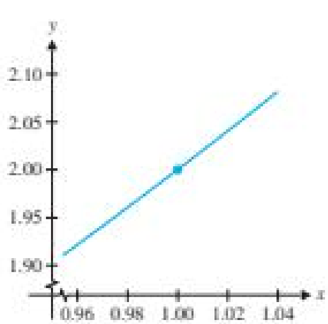


FIGURE 6.3
$y = x^2 + 1$

In general, the slopes of secant lines joining different points on a curve are *not* the same, as seen in Figures 6.2a and 6.2b.

So, in general, what do we mean by the slope of a curve at a point? The answer can be visualized by graphically zooming in on the specified point. In the present case, zooming in tight on the point (1, 2), you should get a graph something like the one in Figure 6.3, which looks very much like a straight line. In fact, the more you zoom in, the straighter the curve appears to be. So, here's the strategy: pick several points on the parabola, each closer to the point (1, 2) than the previous one. Compute the slopes of the lines through (1, 2) and each of the points. The closer the second point gets to (1, 2), the closer the computed slope is to the answer you seek.

For example, the point (1.5, 3.25) is on the parabola fairly close to (1, 2). The slope of the line joining these points is

$$m_{sec} = \frac{3.25-2}{1.5-1} = 2.5.$$

The point (1.1, 2.21) is even closer to (1, 2). The slope of the secant line joining these two points is

$$m_{sec} = \frac{2.21 - 2}{1.1 - 1} = 2.1.$$

Continuing in this way, we obtain successively better estimates of the slope, as illustrated in example 1.1.

EXAMPLE 1.1 Estimating the Slope of a Curve

Estimate the slope of $y = x^2 + 1$ at $x = 1$.

Solution We focus on the point whose coordinates are $x = 1$ and $y = 1^2 + 1 = 2$. To estimate the slope, choose a sequence of points near (1, 2) and compute the slopes of the secant lines joining those points with (1, 2). (We showed sample secant lines in Figures 6.2a and 6.2b.) Choosing points with $x > 1$ (x-values of 2, 1.1 and 1.01) and points with $x < 1$ (x-values of 0, 0.9 and 0.99), we compute the corresponding y-values using $y = x^2 + 1$ and get the slopes shown in the following table.

Second Point	m_{sec}
(2, 5)	$\frac{5-2}{2-1} = 3$
(1.1, 2.21)	$\frac{2.21-2}{1.1-1} = 2.1$
(1.01, 2.0201)	$\frac{2.0201-2}{1.01-1} = 2.01$

Second Point	m_{sec}
(0, 1)	$\frac{1-2}{0-1} = 1$
(0.9, 1.81)	$\frac{1.81-2}{0.9-1} = 1.9$
(0.99, 1.9801)	$\frac{1.9801-2}{0.99-1} = 1.99$

Observe that in both columns, as the second point gets closer to (1, 2), the slope of the secant line gets closer to 2. A reasonable estimate of the slope of the curve at the point (1, 2) is then 2. ■

In the next chapter, we develop a powerful yet simple technique for computing such slopes exactly. We'll see that (under certain circumstances) the secant lines approach a line (the *tangent line*) with the same slope as the curve at that point. Note what distinguishes the calculus problem from the corresponding algebra problem. The calculus problem involves something we call a *limit*. While we presently can only estimate the slope of a curve using a sequence of approximations, the limit allows us to compute the slope exactly.

EXAMPLE 1.2 Estimating the Slope of a Curve

Estimate the slope of $y = \sin x$ at $x = 0$.

Solution This turns out to be a very important problem, one that we will return to later. For now, choose a sequence of points near (0, 0) and compute the slopes of the secant lines joining those points with (0, 0). The following table shows one set of choices.

Second Point	m_{sec}
(1, sin 1)	0.84147
(0.1, sin 0.1)	0.99833
(0.01, sin 0.01)	0.99998

Second Point	m_{sec}
(−1, sin (−1))	0.84147
(−0.1, sin (−0.1))	0.99833
(−0.01, sin (−0.01))	0.99998

Note that as the second point gets closer and closer to (0, 0), the slope of the secant line (m_{sec}) appears to get closer and closer to 1. A good estimate of the slope of the curve at the point (0, 0) would then appear to be 1. Although we presently have no way of computing the slope exactly, this is consistent with the graph of $y = \sin x$ in Figure 6.4. Note that near (0, 0), the graph resembles that of $y = x$, a straight line of slope 1. ■

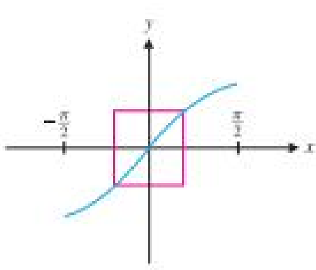


FIGURE 6.4
$y = \sin x$

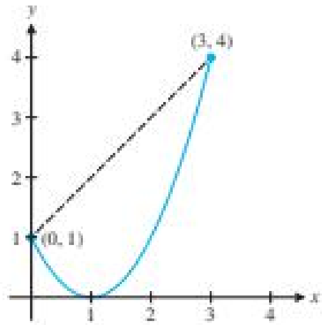


FIGURE 6.5a
$y = (x - 1)^2$

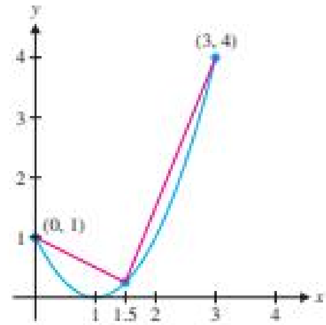


FIGURE 6.5b
Two line segments

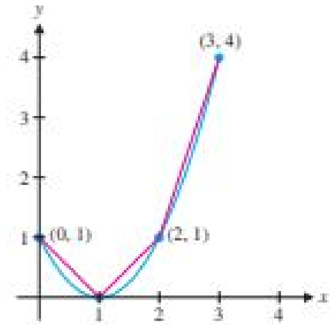


FIGURE 6.5c
Three line segments

A second problem requiring the power of calculus is that of computing distance along a curved path. While this problem is of less significance than our first example (both historically and in the development of the calculus), it provides a good indication of the need for mathematics beyond simple algebra. You should pay special attention to the similarities between the development of this problem and our earlier work with slope.

Recall that the (straight-line) distance between two points (x_1, y_1) and (x_2, y_2) is

$$d\{(x_1, y_1), (x_2, y_2)\} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For instance, the distance between the points (0, 1) and (3, 4) is

$$d\{(0, 1), (3, 4)\} = \sqrt{(3 - 0)^2 + (4 - 1)^2} = 3\sqrt{2} \approx 4.24264.$$

However, this is not the only way we might want to compute the distance between these two points. For example, suppose that you needed to drive a car from (0, 1) to (3, 4) along a road that follows the curve $y = (x - 1)^2$. (See Figure 6.5a.) In this case, you don't care about the straight-line distance connecting the two points, but only about how far you must drive along the curve (the *length* of the curve or *arc length*).

Notice that the distance along the curve must be greater than $3\sqrt{2}$ (the straight-line distance). Taking a cue from the slope problem, we can formulate a strategy for obtaining a sequence of increasingly accurate approximations. Instead of using just one line segment to get the approximation of $3\sqrt{2}$, we could use two line segments, as in Figure 6.5b. Notice that the sum of the lengths of the two line segments appears to be a much better approximation to the actual length of the curve than the straight-line distance of $3\sqrt{2}$. This distance is

$$\begin{aligned} d_2 &= d\{(0, 1), (1.5, 0.25)\} + d\{(1.5, 0.25), (3, 4)\} \\ &= \sqrt{(1.5 - 0)^2 + (0.25 - 1)^2} + \sqrt{(3 - 1.5)^2 + (4 - 0.25)^2} \approx 5.71592. \end{aligned}$$

You're probably way ahead of us by now. If approximating the length of the curve with two line segments gives an improved approximation, why not use three or four or more? Using the three line segments indicated in Figure 6.5c, we get the further improved approximation

$$\begin{aligned} d_3 &= d\{(0, 1), (1, 0)\} + d\{(1, 0), (2, 1)\} + d\{(2, 1), (3, 4)\} \\ &= \sqrt{(1 - 0)^2 + (0 - 1)^2} + \sqrt{(2 - 1)^2 + (1 - 0)^2} + \sqrt{(3 - 2)^2 + (4 - 1)^2} \\ &= 2\sqrt{2} + \sqrt{10} \approx 5.99070. \end{aligned}$$

No. of Segments	*Distance*
1	4.24264
2	5.71592
3	5.99070
4	6.03562
5	6.06906
6	6.08713
7	6.09711

Note that the more line segments we use, the better the approximation appears to be. For now we will list a number of these successively better approximations (produced using points on the curve with evenly spaced x-coordinates) in the table found in the margin. The table suggests that the length of the curve is approximately 6.1 (quite far from the straight-line distance of 4.2). If we continued this process using more and more line segments, the sum of their lengths would approach the actual length of the curve (about 6.126). As in the problem of computing the slope of a curve, the exact arc length is obtained as a limit.

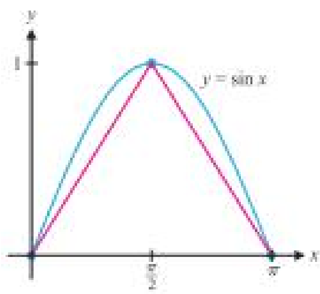


FIGURE 6.6a
Approximating the curve with two line segments

EXAMPLE 1.3 Estimating the Arc Length of a Curve

Estimate the arc length of the curve $y = \sin x$ for $0 \le x \le \pi$. (See Figure 6.6a.)

Solution The endpoints of the curve on this interval are (0, 0) and (π, 0). The distance between these points is $d_1 = \pi$. The point on the graph of $y = \sin x$ corresponding to the midpoint of the interval $[0, \pi]$ is ($\pi/2$, 1). The distance from (0, 0) to ($\pi/2$, 1) plus the distance from ($\pi/2$, 1) to (π, 0) (illustrated in Figure 6.6a) is

$$d_2 = \sqrt{\left(\frac{\pi}{2}\right)^2 + 1} + \sqrt{\left(\frac{\pi}{2}\right)^2 + 1} \approx 3.7242.$$

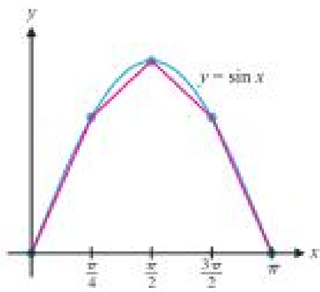


FIGURE 6.6b
Approximating the curve with four line segments

Using the five points (0, 0), ($\pi/4$, $1/\sqrt{2}$), ($\pi/2$, 1), ($3\pi/4$, $1/\sqrt{2}$) and (π, 0) (i.e., four line segments, as indicated in Figure 6.6b), the sum of the lengths of these line segments is

$$d_4 = 2\sqrt{\left(\frac{\pi}{4}\right)^2 + \frac{1}{2}} + 2\sqrt{\left(\frac{\pi}{4}\right)^2 + \left(1 - \frac{1}{\sqrt{2}}\right)^2} \approx 3.7901.$$

Using nine points (i.e., eight line segments), you need a good calculator and some patience to compute the distance of approximately 3.8125. A table showing further approximations is given below. At this stage, it would be reasonable to estimate the length of the sine curve on the interval $[0, \pi]$ as slightly more than 3.8. ■

No. of Line Segments	*Sum of Lengths*
8	3.8125
16	3.8183
32	3.8197
64	3.8201

BEYOND FORMULAS

In the process of estimating both the slope of a curve and the length of a curve, we make some reasonably obvious (straight-line) approximations and then systematically improve on those approximations. In each case, the shorter the line segments are, the closer the approximations are to the desired value. The essence of this is the concept of *limit*, which separates precalculus mathematics from the calculus. At first glance, this limit idea might seem of little practical importance, since in our examples we never compute the exact solution. In the chapters to come, we will find remarkably simple shortcuts to exact answers.

EXERCISES 6.1

WRITING EXERCISES

1. To estimate the slope of $f(x) = x^2 + 1$ at $x = 1$, you would compute the slopes of various secant lines. Note that $y = x^2 + 1$ curves up. Explain why the secant line connecting (1, 2) and (1.1, 2.21) will have a slope greater than the slope of the curve. Discuss how the slope of the secant line between (1, 2) and (0.9, 1.81) compares to the slope of the curve.

2. Explain why each approximation of arc length in example 1.3 is less than the actual arc length.

In exercises 1–6, estimate the slope (as in example 1.1) of $y = f(x)$ at $x = a$.

1. $f(x) = x^2 + 1$, (a) $a = 1$ (b) $a = 2$
2. $f(x) = x^3 + 2$, (a) $a = 1$ (b) $a = 2$
3. $f(x) = \cos x$, (a) $a = 0$ (b) $a = \pi/2$
4. $f(x) = \sqrt{x + 1}$, (a) $a = 0$ (b) $a = 3$
5. $f(x) = e^x$, (a) $a = 0$ (b) $a = 1$
6. $f(x) = \ln x$, (a) $a = 1$ (b) $a = 2$

In exercises 7–12, estimate the length of the curve $y = f(x)$ on the given interval using (a) $n = 4$ and (b) $n = 8$ line segments. (c) If you can program a calculator or computer, use larger n's and conjecture the actual length of the curve.

7. $f(x) = \cos x,\ 0 \le x \le \pi/2$

8. $f(x) = \sin x,\ 0 \le x \le \pi/2$

9. $f(x) = \sqrt{x+1},\ 0 \le x \le 3$

10. $f(x) = 1/x,\ 1 \le x \le 2$

11. $f(x) = x^2 + 1,\ -2 \le x \le 2$

12. $f(x) = x^3 + 2,\ -1 \le x \le 1$

Exercises 13–16 discuss the problem of finding the area of a region.

13. Sketch the parabola $y = 1 - x^2$ and shade in the region above the x-axis between $x = -1$ and $x = 1$. (a) Sketch in the following rectangles:

    1. height $f\left(-\frac{3}{4}\right)$ and width $\frac{1}{2}$ extending from $x = -1$ to $x = -\frac{1}{2}$.
    2. height $f\left(-\frac{1}{4}\right)$ and width $\frac{1}{2}$ extending from $x = -\frac{1}{2}$ to $x = 0$.
    3. height $f\left(\frac{1}{4}\right)$ and width $\frac{1}{2}$ extending from $x = 0$ to $x = \frac{1}{2}$.
    4. height $f\left(\frac{3}{4}\right)$ and width $\frac{1}{2}$ extending from $x = \frac{1}{2}$ to $x = 1$.

    Compute the sum of the areas of the rectangles. (b) Divide the interval $[-1, 1]$ into 8 pieces and construct a rectangle of the appropriate height on each subinterval. Find the sum of the areas of the rectangles. Compared to the approximation in part (a), explain why you would expect this to be a better approximation of the actual area under the parabola.

14. Use a computer or calculator to compute an approximation of the area in exercise 13 using (a) 16 rectangles, (b) 32 rectangles, (c) 64 rectangles. Use these calculations to conjecture the exact value of the area under the parabola.

15. Use the technique of exercise 13 to estimate the area below $y = \sin x$ and above the x-axis between $x = 0$ and $x = \pi$.

16. Use the technique of exercise 13 to estimate the area below $y = x^3$ and above the x-axis between $x = 0$ and $x = 1$.

17. Estimate the length of the curve $y = \sqrt{1 - x^2}$ for $0 \le x \le 1$ with (a) $n = 4$ and (b) $n = 8$ line segments. Explain why the exact length is $\pi/2$. How accurate are your estimates?

18. Estimate the length of the curve $y = \sqrt{9 - x^2}$ for $0 \le x \le 3$ with (a) $n = 4$ and (b) $n = 8$ line segments. Explain why the exact length is $3\pi/2$. How would an estimate of π from part (b) of the exercise compare to that obtained in part (b) of exercise 17?

EXPLORATORY EXERCISE

1. In this exercise, you will learn how to directly compute the slope of a curve at a point. Suppose you want the slope of $y = x^2$ at $x = 1$. You could start by computing slopes of secant lines connecting the point (1, 1) with nearby points. Suppose the nearby point has x-coordinate $1 + h$, where h is a small (positive or negative) number. Explain why the corresponding y-coordinate is $(1 + h)^2$. Show that the slope of the secant line is $\dfrac{(1+h)^2 - 1}{1 + h - 1}$ and show that this simplifies to $2 + h$. As h gets closer and closer to 0, this slope better approximates the slope of the tangent line. Letting h approach 0, show that the slope of the tangent line equals 2. In a similar way, show that the slope of $y = x^2$ at $x = 2$ is 4 and find the slope of $y = x^2$ at $x = 3$. Based on your answers, conjecture a formula for the slope of $y = x^2$ at $x = a$, for any unspecified value of a.

6.2 THE CONCEPT OF LIMIT

In this section, we develop the notion of limit using some common language and illustrate the idea with some simple examples. The notion turns out to be easy to think of intuitively, but a bit harder to pin down in precise terms. We present the precise definition of limit in section 6.6. There, we carefully define limits in considerable detail. The more informal notion of limit that we introduce and work with here and in sections 6.3, 6.4 and 6.5 is adequate for most purposes.

Suppose that a function f is defined for all x in an open interval containing a, except possibly at $x = a$. If we can make $f(x)$ arbitrarily close to some number L (i.e., as close as we'd like to make it) by making x sufficiently close to a (but not equal to a), then we say that L is the *limit of $f(x)$, as x approaches a*, written $\lim_{x \to a} f(x) = L$. For instance, we have $\lim_{x \to 2} x^2 = 4$, since as x gets closer and closer to 2, x^2 gets closer and closer to 4.

Consider the functions

$$f(x) = \frac{x^2 - 4}{x - 2} \quad \text{and} \quad g(x) = \frac{x^2 - 5}{x - 2}.$$

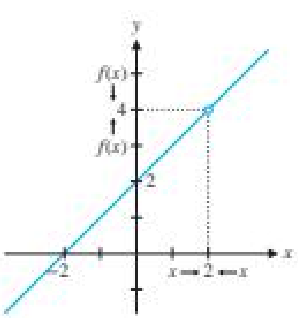


FIGURE 6.7a
$y = \dfrac{x^2 - 4}{x - 2}$

Notice that both functions are undefined at $x = 2$. So, what does this mean, beyond saying that you cannot substitute 2 for x? We often find important clues about the behavior of a function from a graph. (See Figures 6.7a and 6.7b.)

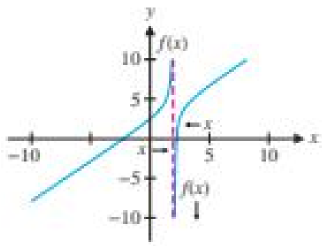


FIGURE 6.7b
$y = \frac{x^2 - 5}{x - 2}$

Notice that the graphs of these two functions look quite different in the vicinity of $x = 2$. Although we can't say anything about the value of these functions at $x = 2$ (since this is outside the domain of both functions), we can examine their behavior in the vicinity of this point. This is what limits will do for us.

EXAMPLE 2.1 Evaluating a Limit

Evaluate $\lim\limits_{x \to 2} \frac{x^2 - 4}{x - 2}$.

Solution First, for $f(x) = \frac{x^2 - 4}{x - 2}$, we compute some values of the function for x close to 2, as in the following tables.

x	$f(x) = \frac{x^2 - 4}{x - 2}$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999

x	$f(x) = \frac{x^2 - 4}{x - 2}$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001

Notice that as you move down the first column of the table, the x-values get closer to 2, but are all less than 2. We use the notation $x \to 2^-$ to indicate that *x approaches 2 from the left side*. Notice that the table and the graph both suggest that as x gets closer and closer to 2 (with $x < 2$), $f(x)$ is getting closer and closer to 4. In view of this, we say that the **limit of $f(x)$ as x approaches 2 from the left** is 4, written

$$\lim_{x \to 2^-} f(x) = 4.$$

Similarly, we use the notation $x \to 2^+$ to indicate that *x approaches 2 from the right side*. We compute some of these values in the second table.

Again, the table and graph both suggest that as x gets closer and closer to 2 (with $x > 2$), $f(x)$ is getting closer and closer to 4. In view of this, we say that the **limit of $f(x)$ as x approaches 2 from the right** is 4, written

$$\lim_{x \to 2^+} f(x) = 4.$$

We call $\lim\limits_{x \to 2^-} f(x)$ and $\lim\limits_{x \to 2^+} f(x)$ **one-sided limits**. Since the two one-sided limits of $f(x)$ are the same, we summarize our results by saying that

$$\lim_{x \to 2} f(x) = 4.$$

The notion of limit as we have described it here is intended to communicate the behavior of a function *near* some point of interest, but not actually *at* that point. We finally observe that we can also determine this limit algebraically, as follows. Notice that since the expression in the numerator of $f(x) = \frac{x^2 - 4}{x - 2}$ factors, we can write

$$\begin{aligned} \lim_{x \to 2} f(x) &= \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} && \text{Cancel the factors of } (x - 2). \\ &= \lim_{x \to 2} (x + 2) = 4, && \text{As } x \text{ approaches 2, } (x + 2) \text{ approaches 4.} \end{aligned}$$

where we can cancel the factors of $(x - 2)$ since in the limit as $x \to 2$, x is *close* to 2, but $x \neq 2$, so that $x - 2 \neq 0$. ■

x	$g(x) = \frac{x^2 - 5}{x - 2}$
1.9	13.9
1.99	103.99
1.999	1003.999
1.9999	10,003.9999

x	$g(x) = \frac{x^2 - 5}{x - 2}$
2.1	−5.9
2.01	−95.99
2.001	−995.999
2.0001	−9995.9999

EXAMPLE 2.2 A Limit That Does Not Exist

Evaluate $\lim_{x\to 2} = \frac{x^2 - 5}{x - 2}$.

Solution As in example 2.1, we consider one-sided limits for $g(x) = \frac{x^2 - 5}{x - 2}$, as $x \to 2$. Based on the graph in Figure 6.7b and the table of approximate function values shown in the margin, observe that as x gets closer and closer to 2 (with $x < 2$), $g(x)$ increases without bound. Since there is no number that $g(x)$ is approaching, we say that the *limit of $g(x)$ as x approaches 2 from the left does not exist*, written

$$\lim_{x\to 2^-} g(x) \textit{ does not exist.}$$

Similarly, the graph and the table of function values for $x > 2$ (shown in the margin) suggest that $g(x)$ decreases without bound as x approaches 2 from the right. Since there is no number that $g(x)$ is approaching, we say that

$$\lim_{x\to 2^+} g(x) \textit{ does not exist.}$$

Finally, since there is no common value for the one-sided limits of $g(x)$ (in fact, neither limit exists), we say that

$$\lim_{x\to 2} g(x) \textit{ does not exist.}$$ ■

Before moving on, we should summarize what we have said about limits.

A limit exists if and only if both corresponding one-sided limits exist and are equal. That is,

$$\lim_{x\to a} f(x) = L, \text{ for some number } L, \text{ if and only if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

In other words, we say that $\lim_{x\to a} f(x) = L$ if we can make $f(x)$ as close as we might like to L, by making x sufficiently close to a (on either side of a), but not equal to a.

Note that we can think about limits from a purely graphical viewpoint, as in example 2.3.

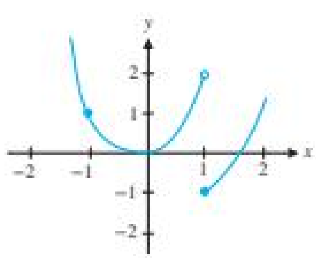


FIGURE 6.8
$y = f(x)$

EXAMPLE 2.3 Determining Limits Graphically

Use the graph in Figure 6.8 to determine $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1} f(x)$ and $\lim_{x\to -1} f(x)$.

Solution For $\lim_{x\to 1^-} f(x)$, we consider the y-values as x gets closer to 1, with $x < 1$. That is, we follow the graph toward $x = 1$ *from the left* ($x < 1$). Observe that the graph dead-ends into the open circle at the point (1, 2). Therefore, we say that $\lim_{x\to 1^-} f(x) = 2$. For $\lim_{x\to 1^+} f(x)$, we follow the graph toward $x = 1$ *from the right* ($x > 1$). In this case, the graph dead-ends into the solid circle located at the point (1, −1). For this reason, we say that $\lim_{x\to 1^+} f(x) = -1$. Because $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$, we say that $\lim_{x\to 1} f(x)$ does not exist. Finally, we have that $\lim_{x\to -1} f(x) = 1$, since the graph approaches a y-value of 1 as x approaches −1 both from the left and from the right. ■

EXAMPLE 2.4 A Limit Where Two Factors Cancel

Evaluate $\lim_{x\to -3} \frac{3x+9}{x^2-9}$.

Solution We examine a graph (see Figure 6.9) and compute some function values for x near -3. Based on this numerical and graphical evidence, it is reasonable to conjecture that

$$\lim_{x\to -3^+} \frac{3x+9}{x^2-9} = \lim_{x\to -3^-} \frac{3x+9}{x^2-9} = -\frac{1}{2}.$$

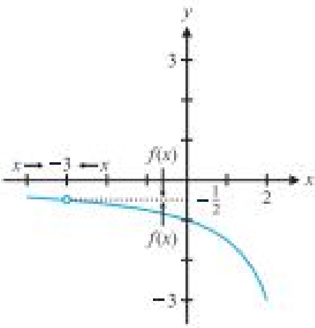


FIGURE 6.9
$\lim_{x\to -3} \frac{3x+9}{x^2-9} = -\frac{1}{2}$

x	$\frac{3x+9}{x^2-9}$
−2.9	−0.508475
−2.99	−0.500835
−2.999	−0.500083
−2.9999	−0.500008

x	$\frac{3x+9}{x^2-9}$
−3.1	−0.491803
−3.01	−0.499168
−3.001	−0.499917
−3.0001	−0.499992

Further, note that

$$\lim_{x\to -3^-} \frac{3x+9}{x^2-9} = \lim_{x\to -3^-} \frac{3(x+3)}{(x+3)(x-3)} \quad \text{Cancel factors of } (x+3).$$
$$= \lim_{x\to -3^-} \frac{3}{x-3} = -\frac{1}{2},$$

since $(x-3) \to -6$ as $x \to -3$. Again, the cancellation of the factors of $(x+3)$ is valid since in the limit as $x \to -3$, x is *close* to -3, but $x \neq -3$, so that $x + 3 \neq 0$. Likewise,

$$\lim_{x\to -3^+} \frac{3x+9}{x^2-9} = -\frac{1}{2}.$$

Finally, since the function approaches the *same* value as $x \to -3$ both from the right and from the left (i.e., the one-sided limits are equal), we write

$$\lim_{x\to -3} \frac{3x+9}{x^2-9} = -\frac{1}{2}. \blacksquare$$

In example 2.4, the limit exists because both one-sided limits exist and are equal. In example 2.5, neither one-sided limit exists.

EXAMPLE 2.5 A Limit That Does Not Exist

Determine whether $\lim_{x\to 3} \frac{3x+9}{x^2-9}$ exists.

Solution We first draw a graph (see Figure 6.10) and compute some function values for x close to 3.

Based on this numerical and graphical evidence, it appears that, as $x \to 3^+$, $\frac{3x+9}{x^2-9}$ is increasing without bound. Thus,

$$\lim_{x\to 3^+} \frac{3x+9}{x^2-9} \text{ does not exist.}$$

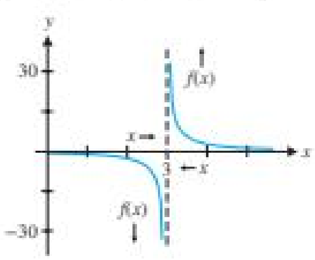


FIGURE 6.10
$y = \frac{3x+9}{x^2-9}$

x	$\frac{3x+9}{x^2-9}$
3.1	30
3.01	300
3.001	3000
3.0001	30,000

x	$\frac{3x+9}{x^2-9}$
2.9	−30
2.99	−300
2.999	−3000
2.9999	−30,000

Similarly, from the graph and the table of values for $x < 3$, we can say that

$$\lim_{x\to 3^-} \frac{3x+9}{x^2-9} \text{ does not exist.}$$

Since neither one-sided limit exists, we say

$$\lim_{x\to 3} \frac{3x+9}{x^2-9} \text{ does not exist.}$$

Here, we considered both one-sided limits for the sake of completeness. Of course, you should keep in mind that if *either* one-sided limit fails to exist, then the limit does not exist. ■

Many limits cannot be resolved using algebraic methods. In these cases, we can approximate the limit using graphical and numerical evidence, as we see in example 2.6.

EXAMPLE 2.6 Approximating the Value of a Limit

Evaluate $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution Unlike some of the limits considered previously, there is no algebra that will simplify this expression. However, we can still draw a graph (see Figure 6.11) and compute some function values.

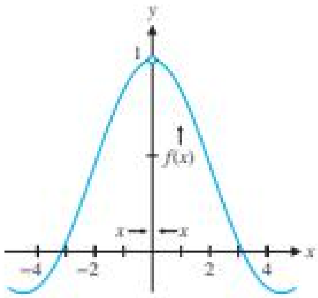


FIGURE 6.11
$\lim_{x\to 0} \frac{\sin x}{x} = 1$

x	$\frac{\sin x}{x}$
0.1	0.998334
0.01	0.999983
0.001	0.99999983
0.0001	0.9999999983
0.00001	0.999999999983

x	$\frac{\sin x}{x}$
−0.1	0.998334
−0.01	0.999983
−0.001	0.99999983
−0.0001	0.9999999983
−0.00001	0.999999999983

The graph and the tables of values lead us to the conjectures:

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^-} \frac{\sin x}{x} = 1,$$

from which we conjecture that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

In Chapter 7, we examine these limits with greater care (and prove that these conjectures are correct). ■

REMARK 2.1

Computer or calculator computation of limits is unreliable. We use graphs and tables of values only as (strong) evidence pointing to what a plausible answer might be. To be certain, we need to obtain careful verification of our conjectures. We explore this in sections 6.3–6.7.

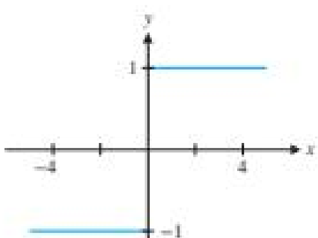


FIGURE 6.12a
$y = \frac{x}{|x|}$

EXAMPLE 2.7 A Case Where One-Sided Limits Disagree

Evaluate $\lim_{x\to 0} \frac{x}{|x|}$.

Solution The computer-generated graph shown in Figure 6.12a is incomplete. Since $\frac{x}{|x|}$ is undefined at $x = 0$, there is no point at $x = 0$. The graph in Figure 6.12b correctly shows open circles at the intersections of the two halves of the graph with the y-axis. We also have

$$\lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} \frac{x}{x} \quad \text{Since } |x| = x, \text{ when } x > 0.$$
$$= \lim_{x\to 0^+} 1$$
$$= 1$$

and

$$\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x} \quad \text{Since } |x| = -x, \text{ when } x < 0.$$
$$= \lim_{x\to 0^-} -1$$
$$= -1.$$

It now follows that $\lim_{x\to 0} \frac{x}{|x|}$ does not exist,

since the one-sided limits are not the same. You should also keep in mind that this observation is entirely consistent with what we see in the graph. ■

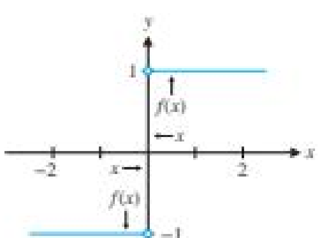


FIGURE 6.12b
$\lim_{x\to 0} \frac{x}{|x|}$ does not exist.

EXERCISES 6.2

WRITING EXERCISES

1. Suppose your professor says, "The limit is a *prediction* of what $f(a)$ will be." Critique this statement. What does it mean? Does it provide important insight? Is there anything misleading about it? Replace the phrase in italics with your own best description of what the limit is.

2. In example 2.6, we conjecture that $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Discuss the strength of the evidence for this conjecture. If it were true that $\frac{\sin x}{x} = 0.998$ for $x = 0.00001$, how much would our case be weakened? Can numerical and graphical evidence ever be completely conclusive?

3. We have observed that $\lim_{x\to a} f(x)$ does not depend on the actual value of $f(a)$, or even on whether $f(a)$ exists. In principle, functions such as $f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 13 & \text{if } x = 2 \end{cases}$ are as "normal" as functions such as $g(x) = x^2$. With this in mind, explain why it is important that the limit concept is independent of how (or whether) $f(a)$ is defined.

4. The most common limit encountered in everyday life is the *speed limit*. Describe how this type of limit is very different from the limits discussed in this section.

In exercises 1–6, use numerical and graphical evidence to conjecture values for each limit. If possible, use factoring to verify your conjecture.

1. $\lim_{x\to 1}\frac{x^2-1}{x-1}$

2. $\lim_{x\to -1}\frac{x^2+x}{x^2-x-2}$

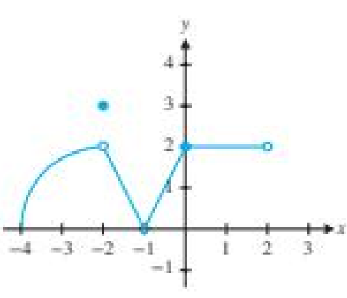
3. $\lim_{x\to 2}\frac{x-2}{x^2-4}$

4. $\lim_{x\to 1}\frac{(x-1)^2}{x^2+2x-3}$

5. $\lim_{x\to 3}\frac{3x-9}{x^2-5x+6}$

6. $\lim_{x\to -2}\frac{2+x}{x^2+2x}$

7. Use the graph of $f(x)$ to answer the following questions.



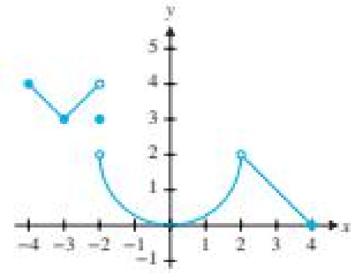
(a) $f(-2)$

(b) $\lim_{x\to -2} f(x)$

(c) $f(-1)$

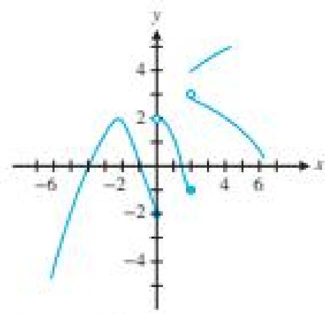
(d) $\lim_{x\to 0^+} f(x)$

8. Use the graph of $f(x)$ to indicate whether the following statements are true or false. Justify your answer.



(a) $f(-2)$ (b) $\lim_{x\to -2^+} f(x)$ (c) $\lim_{x\to -2^-} f(x)$ (d) $\lim_{x\to 2} f(x)$

In exercises 9 and 10, identify each limit or state that it does not exist.



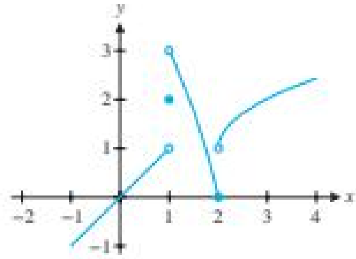
9. (a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$ (c) $\lim_{x\to 0} f(x)$
(d) $\lim_{x\to -2^-} f(x)$ (e) $\lim_{x\to -2^+} f(x)$ (f) $\lim_{x\to -2} f(x)$
(g) $\lim_{x\to -1} f(x)$ (h) $\lim_{x\to 1^-} f(x)$

10. (a) $\lim_{x\to 1^-} f(x)$ (b) $\lim_{x\to 1^+} f(x)$ (c) $\lim_{x\to 1} f(x)$
(d) $\lim_{x\to 2} f(x)$ (e) $\lim_{x\to -2^+} f(x)$ (f) $\lim_{x\to 2} f(x)$
(g) $\lim_{x\to 3} f(x)$ (h) $\lim_{x\to -3} f(x)$

11. Given

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 4 - x^2 & \text{if } 1 < x \le 2 \\ \sqrt{x-2}+1 & \text{if } x > 2 \end{cases}$$

use the graph of $f(x)$ to answer the following



(a) $f(1)$ (b) $\lim_{x\to 1^-} f(x)$ (c) $\lim_{x\to 1^+} f(x)$
(d) $\lim_{x\to 1} f(x)$ (e) $f(2)$ (f) $\lim_{x\to 2^-} f(x)$

12. Sketch the graph of $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$ and identify each limit.

(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$ (c) $\lim_{x\to 2} f(x)$
(d) $\lim_{x\to 1} f(x)$ (e) $\lim_{x\to 3} f(x)$

13. Sketch the graph of $f(x) = \begin{cases} x^3 - 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \sqrt{x+1} - 2 & \text{if } x > 0 \end{cases}$ and identify each limit.

(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$ (c) $\lim_{x\to 0} f(x)$
(d) $\lim_{x\to -1} f(x)$ (e) $\lim_{x\to 1} f(x)$

14. Evaluate $f(1.5)$, $f(1.1)$, $f(1.01)$ and $f(1.001)$, and conjecture a value for $\lim_{x\to 1^+} f(x)$ for $f(x) = \dfrac{x-1}{\sqrt{x}-1}$. Evaluate $f(0.5)$, $f(0.9)$, $f(0.99)$ and $f(0.999)$, and conjecture a value for $\lim_{x\to 1^-} f(x)$ for $f(x) = \dfrac{x-1}{\sqrt{x}-1}$. Does $\lim_{x\to 1} f(x)$ exist?

15. Evaluate $f(-1.5)$, $f(-1.1)$, $f(-1.01)$ and $f(-1.001)$, and conjecture a value for $\lim_{x\to -1^-} f(x)$ for $f(x) = \dfrac{x+1}{x^2-1}$. Evaluate $f(-0.5)$, $f(-0.9)$, $f(-0.99)$ and $f(-0.999)$, and conjecture a value for $\lim_{x\to -1^+} f(x)$ for $f(x) = \dfrac{x+1}{x^2-1}$. Does $\lim_{x\to -1} f(x)$ exist?

In exercises 16–25, use numerical and graphical evidence to conjecture whether the limit at $x = a$ exists. If not, describe what is happening at $x = a$ graphically.

16. $\lim_{x\to 0} \dfrac{x^2+x}{\sin x}$ **17.** $\lim_{x\to 1} \dfrac{x^2-1}{x^2-2x+1}$

18. $\lim_{x\to 0} e^{-1/x^2}$ **19.** $\lim_{x\to 1} \dfrac{x-1}{\ln x}$

20. $\lim_{x\to 0} \dfrac{\tan x}{x}$ **21.** $\lim_{x\to 0} \dfrac{\tan^{-1} x}{x}$

22. $\lim_{x\to 0} \sin\left(\dfrac{1}{x}\right)$ **23.** $\lim_{x\to 1} \dfrac{\sqrt{5-x}-2}{\sqrt{10-x}-3}$

24. $\lim_{x\to 2} \dfrac{x-2}{|x-2|}$ **25.** $\lim_{x\to -1} \dfrac{|x+1|}{x^2-1}$

In exercises 26–29, sketch a graph of a function with the given properties.

26. $f(-1) = 2$, $f(0) = -1$, $f(1) = 3$ and $\lim_{x\to 1} f(x)$ does not exist.

27. $f(x) = 1$ for $-2 \leq x \leq 1$, $\lim_{x\to 1^+} f(x) = 3$ and $\lim_{x\to -2} f(x) = 1$.

28. $f(0) = 1$, $\lim_{x\to 0^-} f(x) = 2$ and $\lim_{x\to 0^+} f(x) = 3$.

29. $\lim_{x\to 0} f(x) = -2$, $f(0) = 1$, $f(2) = 3$ and $\lim_{x\to 2} f(x)$ does not.

30. Compute $\lim_{x\to 1} \dfrac{x^2+1}{x-1}$, $\lim_{x\to 2} \dfrac{x+1}{x^2-4}$ and similar limits to investigate the following. Suppose that $f(x)$ and $g(x)$ are polynomials with $g(a) = 0$ and $f(a) \neq 0$. What can you conjecture about $\lim_{x\to a} \dfrac{f(x)}{g(x)}$?

31. Compute $\lim_{x\to -1} \dfrac{x+1}{x^2+1}$, $\lim_{x\to \pi} \dfrac{\sin x}{x}$ and similar limits to investigate the following. Suppose that $f(x)$ and $g(x)$ are functions with $f(a) = 0$ and $g(a) \neq 0$. What can you conjecture about $\lim_{x\to a} \dfrac{f(x)}{g(x)}$?

32. Consider the following arguments concerning $\lim_{x\to 0^+} \sin \dfrac{\pi}{x}$. First, as $x > 0$, approaches 0, $\dfrac{\pi}{x}$ increases without bound; since $\sin t$ oscillates for increasing t, the limit does not exist. Second: taking $x = 1$, 0.1, 0.01 and so on, we compute $\sin \pi = \sin 10\pi = \sin 100\pi = \ldots = 0$; therefore the limit equals 0. Which argument sounds better to you? Explain. Explore the limit and determine which answer is correct.

33. For $f(x) = \dfrac{x}{x^2 + 0.000001}$, compute $f(0.1)$, $f(0.01)$, and $f(0.001)$. Based on these values, what is a reasonable conjecture for $\lim_{x\to 0} f(x)$? Compute more function values and revise your conjecture.

34. (a) Numerically estimate $\lim_{x\to 0^+} (1+x)^{1/x}$ and $\lim_{x\to 0^-} (1+x)^{1/x}$. Note that the function values for $x > 0$ increase as x decreases, while for $x < 0$ the function values decrease as x increases. Explain why this indicates that, if $\lim_{x\to 0} (1+x)^{1/x}$ exists, it is between function values for positive and negative x's. Approximate this limit correct to eight digits.

(b) Explain what is wrong with the following logic: as $x \to 0$, it is clear that $(1+x) \to 1$. Since 1 raised to any power is 1, $\lim_{x\to 0} (1+x)^{1/x} = \lim_{x\to 0} (1)^{1/x} = 1$.

35. Numerically estimate $\lim_{x\to 0^+} x^{\sec x}$. Try to numerically estimate $\lim_{x\to 0^-} x^{\sec x}$. If your computer has difficulty evaluating the function for negative x's, explain why.

36. Give an example of a function f such that $\lim_{x\to 0} f(x)$ exists but $f(0)$ does not exist. Give an example of a function g such that $g(0)$ exists but $\lim_{x\to 0} g(x)$ does not exist.

37. Give an example of a function f such that $\lim_{x\to 0} f(x)$ exists and $f(0)$ exists, but $\lim_{x\to 0} f(x) \neq f(0)$.

APPLICATIONS

1. As we will see in Chapter 7, the slope of the tangent line to the curve $y = \sqrt{x}$ at $x = 1$ is given by $m = \lim_{h\to 0} \dfrac{\sqrt{1+h}-1}{h}$. Estimate the slope m. Graph $y = \sqrt{x}$ and the line with slope m through the point (1, 1).

2. As we will see in Chapter 7, the velocity of an object that has traveled $\sqrt{x}$ meters in x hours at the $x = 1$ hour mark is given by $v = \lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$. Estimate this limit.

3. A parking lot charges \$2 for each hour or portion of an hour, with a maximum charge of \$12 for all day. If $f(t)$ equals the total parking bill for t hours, sketch a graph of $y = f(t)$ for $0 \leq t \leq 24$. Determine the limits $\lim_{t\to 3.5} f(t)$ and $\lim_{t\to 4} f(t)$, if they exist.

4. For the parking lot in exercise 3, determine all values of a with $0 \le a \le 24$ such that $\lim_{t \to a} f(t)$ does not exist. Briefly discuss the effect this has on your parking strategy (e.g., are there times where you would be in a hurry to move your car or times where it doesn't matter whether you move your car?).

EXPLORATORY EXERCISES

1. In a situation similar to that of example 2.8, the left/right position of a knuckleball pitch in baseball can be modeled by $P = \frac{5}{8\omega^2}(1 - \cos 4\omega t)$, where t is time measured in seconds $(0 \le t \le 0.68)$ and ω is the rotation rate of the ball measured in radians per second. In example 2.8, we chose a specific t-value and evaluated the limit as $\omega \to 0$. While this gives us some information about which rotation rates produce hard-to-hit pitches, a clearer picture emerges if we look at P over its entire domain. Set $\omega = 10$ and graph the resulting function $\frac{1}{160}(1 - \cos 40t)$ for $0 \le t \le 0.68$. Imagine looking at a pitcher from above and try to visualize a baseball starting at the pitcher's hand at $t = 0$ and finally reaching the batter, at $t = 0.68$. Repeat this with $\omega = 5$, $\omega = 1$, $\omega = 0.1$ and whatever values of ω you think would be interesting. Which values of ω produce hard-to-hit pitches?

2. In this exercise, the results you get will depend on the accuracy of your computer or calculator. We will investigate $\lim_{x \to 0} \frac{\cos x - 1}{x^2}$. Start with the calculations presented in the table (your results may vary):

x	$f(x)$
0.1	−0.499583. . .
0.01	−0.49999583. . .
0.001	−0.4999999583. . .

Describe as precisely as possible the pattern shown here. What would you predict for $f(0.0001)$? $f(0.00001)$? Does your computer or calculator give you this answer? If you continue trying powers of 0.1 (0.000001, 0.0000001 etc.) you should eventually be given a displayed result of −0.5. Do you think this is exactly correct or has the answer just been rounded off? Why is rounding off inescapable? It turns out that −0.5 is the exact value for the limit. However, if you keep evaluating the function at smaller and smaller values of x, you will eventually see a reported function value of 0. We discuss this error in section 6.7. For now, evaluate $\cos x$ at the current value of x and try to explain where the 0 came from.

6.3 COMPUTATION OF LIMITS

Now that you have an idea of what a limit is, we need to develop some basic rules for calculating limits of simple functions. We begin with two simple limits.

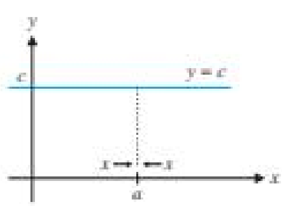


FIGURE 6.13
$\lim_{x \to a} c = c$

> For any constant c and any real number a,
> $$\lim_{x \to a} c = c. \tag{3.1}$$

In other words, the limit of a constant is that constant. This certainly comes as no surprise, since the function $f(x) = c$ does not depend on x and so, stays the same as $x \to a$. (See Figure 6.13.) Another simple limit is the following.

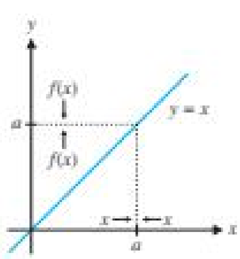


FIGURE 6.14
$\lim_{x \to a} x = a$

> For any real number a,
> $$\lim_{x \to a} x = a. \tag{3.2}$$

Again, this is not a surprise, since as $x \to a$, x will approach a. (See Figure 6.14.) Be sure that you are comfortable enough with the limit notation to recognize how obvious the limits in (3.1) and (3.2) are. As simple as they are, we use them repeatedly in finding more complex limits. We also need the basic rules contained in Theorem 3.1.

THEOREM 3.1

Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and let c be any constant. The following then apply:

(i) $\lim_{x\to a}[c \cdot f(x)] = c \cdot \lim_{x\to a} f(x)$,

(ii) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$,

(iii) $\lim_{x\to a}[f(x) \cdot g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$ and

(iv) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\left(\text{if } \lim_{x\to a} g(x) \neq 0\right)$.

The proof of Theorem 3.1 is found in Appendix A and requires the formal definition of limit discussed in section 6.6. You should think of these rules as sensible results, given your intuitive understanding of what a limit is. Read them in plain English. For instance, part (ii) says that the limit of a sum (or a difference) equals the sum (or difference) of the limits, *provided the limits exist*. Think of this as follows. If as x approaches a, $f(x)$ approaches L and $g(x)$ approaches M, then $f(x) + g(x)$ should approach $L + M$.

Observe that by applying part (iii) of Theorem 3.1 with $g(x) = f(x)$, we get that, whenever $\lim_{x\to a} f(x)$ exists,

$$\begin{aligned}\lim_{x\to a}[f(x)]^2 &= \lim_{x\to a}[f(x) \cdot f(x)] \\ &= \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} f(x)\right] = \left[\lim_{x\to a} f(x)\right]^2.\end{aligned}$$

Likewise, for any positive integer n, we can apply part (iii) of Theorem 3.1 repeatedly, to yield

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n. \tag{3.3}$$

(See exercises 61 and 62.)

Notice that taking $f(x) = x$ in (3.3) gives us that for any integer $n > 0$ and any real number a,

$$\lim_{x\to a} x^n = a^n. \tag{3.4}$$

That is, to compute the limit of any positive power of x, you simply substitute in the value of x being approached.

EXAMPLE 3.1 Finding the Limit of a Polynomial

Apply the rules of limits to evaluate $\lim_{x\to 2}(3x^2 - 5x + 4)$.

Solution We have

$$\begin{aligned}\lim_{x\to 2}(3x^2 - 5x + 4) &= \lim_{x\to 2}(3x^2) - \lim_{x\to 2}(5x) + \lim_{x\to 2} 4 && \text{By Theorem 3.1 (ii).} \\ &= 3\lim_{x\to 2} x^2 - 5\lim_{x\to 2} x + 4 && \text{By Theorem 3.1 (i).} \\ &= 3 \cdot (2)^2 - 5 \cdot 2 + 4 = 6. && \text{By (3.4).}\end{aligned}$$

EXAMPLE 3.2 Finding the Limit of a Rational Function

Apply the rules of limits to evaluate $\lim_{x\to 3} \dfrac{x^3 - 5x + 4}{x^2 - 2}$.

Solution We get

$$\lim_{x\to 3} \frac{x^3 - 5x + 4}{x^2 - 2} = \frac{\lim_{x\to 3}(x^3 - 5x + 4)}{\lim_{x\to 3}(x^2 - 2)} \qquad \text{By Theorem 3.1 (iv).}$$

$$= \frac{\lim_{x\to 3} x^3 - 5\lim_{x\to 3} x + \lim_{x\to 3} 4}{\lim_{x\to 3} x^2 - \lim_{x\to 3} 2} \qquad \text{By Theorem 3.1 (i) and (ii).}$$

$$= \frac{3^3 - 5\cdot 3 + 4}{3^2 - 2} = \frac{16}{7}. \qquad \text{By (3.4).}$$ ■

You may have noticed that in examples 3.1 and 3.2, we simply ended up substituting the value for x, after taking many intermediate steps. In example 3.3, it's not quite so simple.

EXAMPLE 3.3 Finding a Limit by Factoring

Evaluate $\lim_{x\to 1} \frac{x^2 - 1}{1 - x}$.

Solution Notice right away that

$$\lim_{x\to 1} \frac{x^2 - 1}{1 - x} \neq \frac{\lim_{x\to 1}(x^2 - 1)}{\lim_{x\to 1}(1 - x)},$$

since the limit in the denominator is zero. (Recall that the limit of a quotient is the quotient of the limits *only* when both limits exist *and* the limit in the denominator is *not* zero.) We can resolve this problem by observing that

$$\lim_{x\to 1} \frac{x^2 - 1}{1 - x} = \lim_{x\to 1} \frac{(x-1)(x+1)}{-(x-1)} \qquad \text{Factoring the numerator and factoring } -1 \text{ from the denominator.}$$

$$= \lim_{x\to 1} \frac{(x+1)}{-1} = -2, \qquad \text{Simplifying and substituting } x = 1.$$

where the cancellation of the factors of $(x - 1)$ is valid because in the limit as $x \to 1$, x is close to 1, but $x \neq 1$, so that $x - 1 \neq 0$. ■

In Theorem 3.2, we show that the limit of a polynomial is simply the value of the polynomial at that point; that is, to find the limit of a polynomial, we simply substitute in the value that x is approaching.

THEOREM 3.2

For any polynomial $p(x)$ and any real number a,

$$\lim_{x\to a} p(x) = p(a).$$

PROOF

Suppose that $p(x)$ is a polynomial of degree $n \geq 0$,

$$p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0.$$

Then, from Theorem 3.1 and (3.4),

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0)\\ &= c_n \lim_{x\to a} x^n + c_{n-1}\lim_{x\to a} x^{n-1} + \cdots + c_1 \lim_{x\to a} x + \lim_{x\to a} c_0\\ &= c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = p(a).\end{aligned}$$ ■

Evaluating the limit of a polynomial is now easy. Many other limits are evaluated just as easily.

THEOREM 3.3

Suppose that $\lim_{x \to a} f(x) = L$ and n is any positive integer. Then,

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L},$$

where for n even, we must assume that $L > 0$.

The proof of Theorem 3.3 is given in the Appendix. Notice that this result says that we may (under the conditions outlined in the hypotheses) bring limits "inside" nth roots. We can then use our existing rules for computing the limit inside.

EXAMPLE 3.4 Evaluating the Limit of an nth Root of a Polynomial

Evaluate $\lim_{x \to 2} \sqrt[5]{3x^2 - 2x}$.

Solution By Theorems 3.2 and 3.3, we have

$$\lim_{x \to 2} \sqrt[5]{3x^2 - 2x} = \sqrt[5]{\lim_{x \to 2} (3x^2 - 2x)} = \sqrt[5]{8}.$$

REMARK 3.1

In general, in any case where the limits of both the numerator and the denominator are 0, you should try to algebraically simplify the expression, to get a cancellation, as we do in examples 3.3 and 3.5.

EXAMPLE 3.5 Finding a Limit by Rationalizing

Evaluate $\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$.

Solution First, notice that both the numerator and the denominator approach 0 as x approaches 0. Unlike example 3.3, we can't factor the numerator. However, we can rationalize the numerator, as follows:

$$\frac{\sqrt{x+2} - \sqrt{2}}{x} = \frac{(\sqrt{x+2} - \sqrt{2})(\sqrt{x+2} + \sqrt{2})}{x(\sqrt{x+2} + \sqrt{2})} = \frac{x+2-2}{x(\sqrt{x+2} + \sqrt{2})}$$

$$= \frac{x}{x(\sqrt{x+2} + \sqrt{2})} = \frac{1}{\sqrt{x+2} + \sqrt{2}},$$

where the last equality holds if $x \neq 0$ (which is the case in the limit as $x \to 0$). So, we have

$$\lim_{x \to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

So that we are not restricted to discussing only the algebraic functions (i.e. those that can be constructed by using addition, subtraction, multiplication, division, exponentiation and by taking nth roots), we state the following result now, without proof.

THEOREM 3.4

For any real number a, we have

(i) $\lim_{x \to a} \sin x = \sin a$,

(ii) $\lim_{x \to a} \cos x = \cos a$,

(iii) $\lim_{x \to a} e^x = e^a$ and

(iv) $\lim_{x \to a} \ln x = \ln a$, for $a > 0$.

(v) $\lim_{x \to a} \sin^{-1} x = \sin^{-1} a$, for $-1 < a < 1$,

(vi) $\lim_{x \to a} \cos^{-1} x = \cos^{-1} a$, for $-1 < a < 1$,

(vii) $\lim_{x \to a} \tan^{-1} x = \tan^{-1} a$, for $-\infty < a < \infty$ and

(viii) if p is a polynomial and $\lim_{x \to p(a)} f(x) = L$, then $\lim_{x \to a} f(p(x)) = L$.

Notice that Theorem 3.4 says that limits of the sine, cosine, exponential, natural logarithm, inverse sine, inverse cosine and inverse tangent functions are found simply by substitution. A more thorough discussion of functions with this property (called *continuity*) is found in section 6.4.

EXAMPLE 3.6 Evaluating a Limit of an Inverse Trigonometric Function

Evaluate $\lim\limits_{x\to 0} \sin^{-1}\left(\frac{x+1}{2}\right)$.

Solution By Theorem 3.4 parts (v) and (viii), we have

$$\lim_{x\to 0} \sin^{-1}\left(\frac{x+1}{2}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}. \blacksquare$$

So much for limits that we can compute using elementary rules. Many limits can be found only by using more careful analysis, often requiring an indirect approach. For instance, consider the problem in example 3.7.

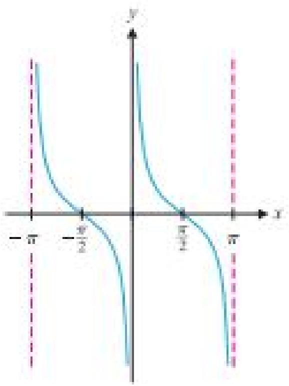


FIGURE 6.15
$y = \cot x$

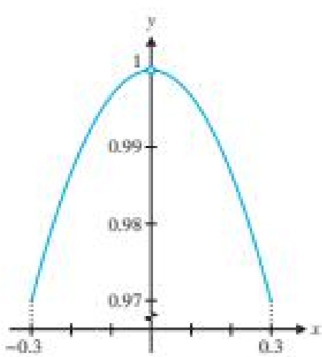


FIGURE 6.16
$y = x \cot x$

x	$x \cot x$
±0.1	0.9967
±0.01	0.999967
±0.001	0.99999967
±0.0001	0.9999999967
±0.00001	0.999999999967

EXAMPLE 3.7 A Limit of a Product That Is Not the Product of the Limits

Evaluate $\lim\limits_{x\to 0} (x \cot x)$.

Solution Your first reaction might be to say that this is a limit of a product and so, must be the product of the limits:

$$\lim_{x\to 0}(x \cot x) = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} \cot x\right) \quad \text{This is incorrect!}$$
$$= 0 \cdot ? = 0, \tag{3.5}$$

where we've written a "?" since you probably don't know what to do with $\lim\limits_{x\to 0} \cot x$. Since the first limit is 0, do we really need to worry about the second limit? The problem here is that we are attempting to apply the result of Theorem 3.1 in a case where the hypotheses are not satisfied. Specifically, Theorem 3.1 says that the limit of a product is the product of the respective limits *when all of the limits exist.* The graph in Figure 6.15 suggests that $\lim\limits_{x\to 0} \cot x$ does not exist. You should compute some function values, as well, to convince yourself that this is in fact the case. Since equation (3.5) does not hold and since none of our rules seem to apply here, we draw a graph (see Figure 6.16) and compute some function values. Based on these, we conjecture that

$$\lim_{x\to 0}(x \cot x) = 1,$$

which is definitely not 0, as you might have initially suspected. You can also think about this limit as follows:

$$\lim_{x\to 0}(x \cot x) = \lim_{x\to 0}\left(x \frac{\cos x}{\sin x}\right) = \lim_{x\to 0}\left(\frac{x}{\sin x} \cos x\right)$$
$$= \left(\lim_{x\to 0} \frac{x}{\sin x}\right)\left(\lim_{x\to 0} \cos x\right)$$
$$= \frac{\lim\limits_{x\to 0} \cos x}{\lim\limits_{x\to 0} \frac{\sin x}{x}} = \frac{1}{1} = 1,$$

since $\lim\limits_{x\to 0} \cos x = 1$ and where we have used the conjecture we made in example 2.6 that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$. (We verify this last conjecture in section 7.6, using the Squeeze Theorem, which follows.) ■

At this point, we introduce a tool that will help us determine a number of important limits.

THEOREM 3.5 (Squeeze Theorem)

Suppose that

$$f(x) \le g(x) \le h(x)$$

for all x in some interval (c, d), except possibly at the point $a \in (c, d)$ and that

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

for some number L. Then, it follows that

$$\lim_{x \to a} g(x) = L, \text{ also.}$$

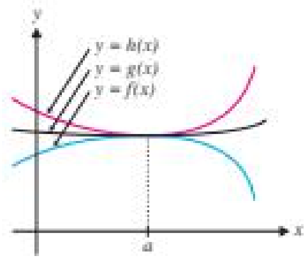


FIGURE 6.17
The Squeeze Theorem

The proof of Theorem 3.5 is given in Appendix A, since it depends on the precise definition of limit found in section 6.6. However, if you refer to Figure 6.17, you should clearly see that if $g(x)$ lies between $f(x)$ and $h(x)$, except possibly at a itself and both $f(x)$ and $h(x)$ have the same limit as $x \to a$, then $g(x)$ gets *squeezed* between $f(x)$ and $h(x)$ and therefore should also have a limit of L. The challenge in using the Squeeze Theorem is in finding appropriate functions f and h that bound a given function g from below and above, respectively, and that have the same limit as $x \to a$.

REMARK 3.2

The Squeeze Theorem also applies to one-sided limits.

EXAMPLE 3.8 Using the Squeeze Theorem to Verify the Value of a Limit

Determine the value of $\lim_{x \to 0}\left[x^2 \cos\left(\frac{1}{x}\right)\right]$.

Solution Your first reaction might be to observe that this is a limit of a product and so might be the product of the limits:

$$\lim_{x \to 0}\left[x^2 \cos\left(\frac{1}{x}\right)\right] \stackrel{?}{=} \left(\lim_{x \to 0} x^2\right)\left[\lim_{x \to 0} \cos\left(\frac{1}{x}\right)\right]. \quad \text{This is incorrect!} \tag{3.6}$$

However, the graph of $y = \cos\left(\frac{1}{x}\right)$ found in Figure 6.18 suggests that $\cos\left(\frac{1}{x}\right)$ oscillates back and forth between -1 and 1. Further, the closer x gets to 0, the more rapid the oscillations become. You should compute some function values, as well, to convince yourself that $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ does not exist. Equation (3.6) then does not hold and since none of our rules seem to apply here, we draw a graph and compute some function values. The graph of $y = x^2 \cos\left(\frac{1}{x}\right)$ appears in Figure 6.19 and a table of function values is shown in the margin.

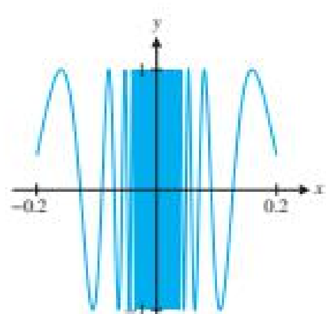


FIGURE 6.18
$y = \cos\left(\frac{1}{x}\right)$

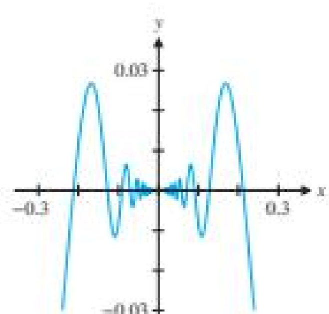


FIGURE 6.19
$y = x^2 \cos\left(\frac{1}{x}\right)$

x	$x^2\cos(1/x)$
± 0.1	-0.008
± 0.01	8.6×10^{-5}
± 0.001	5.6×10^{-7}
± 0.0001	-9.5×10^{-9}
± 0.00001	-9.99×10^{-11}

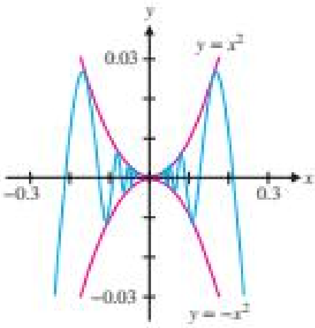


FIGURE 6.20
$y = x^2\cos\left(\frac{1}{x}\right)$, $y = x^2$ and $y = -x^2$

The graph and the table of function values suggest the conjecture

$$\lim_{x\to 0}\left[x^2\cos\left(\frac{1}{x}\right)\right] = 0,$$

which we prove using the Squeeze Theorem. First, we need to find functions f and h such that

$$f(x) \le x^2\cos\left(\frac{1}{x}\right) \le h(x),$$

for all $x \neq 0$ and where $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$. Recall that

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1, \tag{3.7}$$

for all $x \neq 0$. If we multiply (3.7) through by x^2 (notice that since $x^2 \ge 0$, this multiplication preserves the inequalities), we get

$$-x^2 \le x^2\cos\left(\frac{1}{x}\right) \le x^2,$$

for all $x \neq 0$. We illustrate this inequality in Figure 6.20. Further,

$$\lim_{x\to 0}(-x^2) = 0 = \lim_{x\to 0} x^2.$$

So, from the Squeeze Theorem, it now follows that

$$\lim_{x\to 0} x^2\cos\left(\frac{1}{x}\right) = 0,$$

also, as we had conjectured. ■

BEYOND FORMULAS

To resolve the limit in example 3.8, we could not apply the rules for limits contained in Theorem 3.1. So, we used an indirect method to find the limit. This tour de force of graphics plus calculation followed by analysis is sometimes referred to as the **Rule of Three**. (This general strategy for attacking new problems suggests that one look at problems graphically, numerically and analytically.) In the case of example 3.8, the first two elements of this "rule" (the graphics in Figure 6.19 and the accompanying table of function values) suggest a plausible conjecture, while the third element provides us with a careful mathematical verification of the conjecture. In what ways does this sound like the scientific method?

TODAY IN MATHEMATICS

Michael Freedman (1951–Present)
An American mathematician who first solved one of the most famous problems in mathematics, the four-dimensional Poincaré conjecture. A winner of the Fields Medal, the mathematical equivalent of the Nobel Prize, Freedman says, "Much of the power of mathematics comes from combining insights from seemingly different branches of the discipline. Mathematics is not so much a collection of different subjects as a way of thinking. As such, it may be applied to any branch of knowledge." Freedman finds mathematics to be an open field for research, saying that, "It isn't necessary to be an old hand in an area to make a contribution."

Functions are often defined by different expressions on different intervals. Such **piecewise-defined** functions are important and we illustrate such a function in example 3.9.

EXAMPLE 3.9 A Limit for a Piecewise-Defined Function

Evaluate $\lim_{x\to 0} f(x)$, where f is defined by

$$f(x) = \begin{cases} x^2 + 2\cos x + 1, & \text{for } x < 0 \\ e^x - 4, & \text{for } x \ge 0 \end{cases}.$$

Solution Since f is defined by different expressions for $x < 0$ and for $x \ge 0$, we must consider one-sided limits. We have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x^2 + 2\cos x + 1) = 2\cos 0 + 1 = 3,$$

by Theorem 3.4. Also, we have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(e^x - 4) = e^0 - 4 = 1 - 4 = -3.$$

Since the one-sided limits are different, we have that $\lim_{x\to 0} f(x)$ does not exist. ■

We end this section with an example of the use of limits in computing velocity. In section 7.1, we see that for an object moving in a straight line, whose position at time t is given by the function $f(t)$, the instantaneous velocity of that object at time $t = 1$ (i.e., the velocity at the *instant* $t = 1$, as opposed to the average velocity over some period of time) is given by the limit

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}.$$

EXAMPLE 3.10 Evaluating a Limit Describing Velocity

Suppose that the position function for an object at time t (seconds) is given by

$$f(t) = t^2 + 2 \text{ (feet)}.$$

Find the instantaneous velocity of the object at time $t = 1$.

Solution Given what we have just learned about limits, this is now an easy problem to solve. We have

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{[(1+h)^2+2]-3}{h}.$$

While we can't simply substitute $h = 0$ (why not?), we can write

$$\lim_{h\to 0}\frac{[(1+h)^2+2]-3}{h} = \lim_{h\to 0}\frac{(1+2h+h^2)-1}{h}$$

Expanding the squared term.

$$= \lim_{h\to 0}\frac{2h+h^2}{h} = \lim_{h\to 0}\frac{h(2+h)}{h}$$

$$= \lim_{h\to 0}\frac{2+h}{1} = 2.$$

Canceling factors of h.

So, the instantaneous velocity of this object at time $t = 1$ is 2 feet per second. ■

EXERCISES 6.3

WRITING EXERCISES

1. Given your knowledge of the graphs of polynomials, explain why equations (3.1) and (3.2) and Theorem 3.2 are obvious.

2. In one or two sentences, explain the Squeeze Theorem. Use a real-world analogy (e.g., having the functions represent the locations of three people as they walk) to indicate why it is true.

3. Piecewise functions must be carefully interpreted. In example 3.9, explain why $\lim_{x\to 1} f(x) = e - 4$ and $\lim_{x\to -2} f(x) = 5 + 2\cos 2$, but we need one-sided limits to evaluate $\lim_{x\to 0} f(x)$.

4. In example 3.8, explain why it is not good enough to say that since $\lim_{x\to 0} x^2 = 0$, $\lim_{x\to 0} x^2\cos(1/x) = 0$.

In exercises 1–40, evaluate the indicated limit, if it exists. Assume that $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

1. $\lim_{x\to 0}(x^2 - 3x + 1)$

2. $\lim_{x\to 2}\sqrt[3]{2x+1}$

3. $\lim_{x\to 0}\cos^{-1}(x^2)$

4. $\lim_{x\to 2}\frac{x-5}{x^2+4}$

5. $\lim_{x\to 3}\frac{x^2-x-6}{x-3}$

6. $\lim_{x\to 1}\frac{x^2+x-2}{x^2-3x+2}$

7. $\lim_{x\to 2}\frac{x^2-x-2}{x^2-4}$

8. $\lim_{x\to 1}\frac{x^3-1}{x^2+2x-3}$

9. $\lim_{x\to 0}\frac{\sin x}{\tan x}$

10. $\lim_{x\to 0}\frac{\tan x}{x}$

11. $\lim_{x\to 0}\frac{xe^{-2x+1}}{x^2+x}$

12. $\lim_{x\to 0} x^2\csc^2 x$

13. $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$

14. $\lim_{x\to 0}\frac{2x}{3-\sqrt{x+9}}$

15. $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$

16. $\lim_{x\to 4}\frac{x^3-64}{x-4}$

17. $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)$

18. $\lim_{x\to 0}\left(\frac{2}{x}-\frac{2}{|x|}\right)$

19. $\lim_{x\to 0}\frac{1-e^{2x}}{1-e^{x}}$

20. $\lim_{x\to 0}\frac{\sin|x|}{x}$

21. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \ge 2 \end{cases}$

22. $\lim_{x\to -1} f(x)$, where $f(x) = \begin{cases} x^2+1 & \text{if } x < -1 \\ 3x+1 & \text{if } x \ge -1 \end{cases}$

23. $\lim_{x\to -1} f(x)$, where $f(x) = \begin{cases} 2x+1 & \text{if } x < -1 \\ 3 & \text{if } -1 < x < 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$

24. $\lim\limits_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x+1 & \text{if } x < -1 \\ 3 & \text{if } -1 < x < 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$

25. $\lim\limits_{h\to 0} \dfrac{(2+h)^2 - 4}{h}$

26. $\lim\limits_{h\to 0} \dfrac{(1+h)^3 - 1}{h}$

27. $\lim\limits_{x\to 2} \dfrac{\sin(x^2-4)}{x^2-4}$

28. $\lim\limits_{x\to 0} \dfrac{\tan x}{5x}$

29. $\lim\limits_{x\to 0} \left(\dfrac{1}{x} + \dfrac{1}{x^2 - x}\right)$

30. $\lim\limits_{h\to 0} \left(\dfrac{1}{h} - \dfrac{1}{h\sqrt{h+1}}\right)$

31. $\lim\limits_{x\to 0} \left(\dfrac{(x+2)^3 - 8}{x}\right)$

32. $\lim\limits_{u\to -4} \dfrac{\frac{1}{4} + \frac{1}{u}}{4+u}$

33. $\lim\limits_{t\to 0} \dfrac{1 - \sqrt{t^2+1}}{4t^2}$

34. $\lim\limits_{t\to 4} \dfrac{4t}{5 - \sqrt{t^2+9}}$

35. $\lim\limits_{x\to 0} \dfrac{x}{\sin 4x}$

36. $\lim\limits_{x\to 0} \left(\dfrac{\sin 2x}{\sin 4x}\right)$

37. $\lim\limits_{x\to 0} \dfrac{\tan 5x}{\sin 5x}$

38. $\lim\limits_{x\to 4} \left(\dfrac{\sin\sqrt{x-2}}{x-4}\right)$

39. $\lim\limits_{x\to 4} \left(\dfrac{\sin\sqrt{x-2}}{x-4}\right)$

40. $\lim\limits_{x\to 0} \dfrac{x^2}{\sin 4x \cdot \sin \pi x}$

41. Use numerical and graphical evidence to conjecture the value of $\lim\limits_{x\to 0} x^2\sin(1/x)$. Use the Squeeze Theorem to prove that you are correct: identify the functions f and h, show graphically that $f(x) \le x^2 \sin(1/x) \le h(x)$ and justify $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} h(x)$.

42. Why can't you use the Squeeze Theorem as in exercise 41 to prove that $\lim\limits_{x\to 0} x^2\sec(1/x) = 0$? Explore this limit graphically.

43. Use the Squeeze Theorem to prove that $\lim\limits_{x\to 0^+} [\sqrt{x}\cos^2(1/x)] = 0$. Identify the functions f and h, show graphically that $f(x) \le \sqrt{x}\cos^2(1/x) \le h(x)$ for all $x > 0$, and justify $\lim\limits_{x\to 0^+} f(x) = 0$ and $\lim\limits_{x\to 0^+} h(x) = 0$.

44. Suppose that $f(x)$ is bounded: that is, there exists a constant M such that $|f(x)| \le M$ for all x. Use the Squeeze Theorem to prove that $\lim\limits_{x\to 0} x^2 f(x) = 0$.

In exercises 45–48, use the given position function $f(t)$ to find the velocity at time $t = a$.

45. $f(t) = t^2 + 2,\ a = 2$

46. $f(t) = t^2 + 2,\ a = 0$

47. $f(t) = t^3,\ a = 0$

48. $f(t) = t^3,\ a = 1$

49. Given that $\lim\limits_{x\to 0^+} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$, quickly evaluate $\lim\limits_{x\to 0^+} \dfrac{\sqrt{1-\cos x}}{x}$.

50. Given that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, quickly evaluate $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{x^2}$.

51. Suppose $f(x) = \begin{cases} g(x) & \text{if } x < a \\ h(x) & \text{if } x > a \end{cases}$ for polynomials $g(x)$ and $h(x)$. Explain why $\lim\limits_{x\to a^-} f(x) = g(a)$ and determine $\lim\limits_{x\to a^+} f(x)$.

52. Explain how to determine $\lim\limits_{x\to a} f(x)$ if g and h are polynomials and $f(x) = \begin{cases} g(x) & \text{if } x < a \\ c & \text{if } x = a \\ h(x) & \text{if } x > a \end{cases}$.

53. Evaluate each limit and justify each step by citing the appropriate theorem or equation.

(a) $\lim\limits_{x\to 2} (x^2 - 3x + 1)$ (b) $\lim\limits_{x\to 0} \dfrac{x-2}{x^2+1}$

54. Evaluate each limit and justify each step by citing the appropriate theorem or equation.

(a) $\lim\limits_{x\to -1} [(x+1)\sin x]$ (b) $\lim\limits_{x\to 1} \dfrac{xe^x}{\tan x}$

In exercises 55–58, use $\lim\limits_{x\to a} f(x) = 2$, $\lim\limits_{x\to a} g(x) = -3$, and $\lim\limits_{x\to a} h(x) = 0$ to determine the limit, if possible.

55. $\lim\limits_{x\to a} [2f(x) - 3g(x)]$

56. $\lim\limits_{x\to a} [3f(x)g(x)]$

57. $\lim\limits_{x\to a} \dfrac{[f(x)]^2}{g(x)}$

58. $\lim\limits_{x\to a} \dfrac{2f(x)h(x)}{f(x) + h(x)}$

In exercises 59 and 60, compute the limit for $p(x) = x^2 - 1$.

59. $\lim\limits_{x\to 0} p(p(p(p(x))))$

60. $\lim\limits_{x\to 0} p(3 + 2p(x - p(x)))$

61. Find all the errors in the following incorrect string of equalities:

$$\lim_{x\to 0} \frac{1}{x} = \lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} x \lim_{x\to 0} \frac{1}{x^2} = 0 \cdot ? = 0.$$

62. Find all the errors in the following incorrect string of equalities:

$$\lim_{x\to 0} \frac{\sin 2x}{x} = \frac{0}{0} = 1.$$

63. Give an example of functions f and g such that $\lim_{x\to 0} [f(x)+g(x)]$ exists but $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist.

64. Give an example of functions f and g such that $\lim_{x\to 0} [f(x)\cdot g(x)]$ exists but at least one of $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ does not exist.

65. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, is it always true that $\lim_{x\to a} [f(x)+g(x)]$ does not exist? Explain.

66. Is the following true or false? If $\lim_{x\to 0} f(x)$ does not exist, then $\lim_{x\to 0} \frac{1}{f(x)}$ does not exist. Explain.

In exercises 67–72, use numerical evidence to conjecture the value of the limit if it exists. Check your answer with your computer algebra system (CAS). If you disagree, which one of you is correct?

67. $\lim_{x\to 0^+} (1+x)^{1/x}$

68. $\lim_{x\to 0} e^{1/x}$

69. $\lim_{x\to 0^+} x^{-x^2}$

70. $\lim_{x\to 0^+} x^{\ln x}$

71. $\lim_{x\to 0} \tan^{-1}\frac{1}{x}$

72. $\lim_{x\to 0} \ln\left|\frac{1}{x}\right|$

73. Assume that $\lim_{x\to a} f(x) = L$. Use Theorem 3.1 to prove that $\lim_{x\to a} [f(x)]^3 = L^3$. Also, show that $\lim_{x\to a} [f(x)]^4 = L^4$.

74. Use mathematical induction to prove that $\lim_{x\to a} [f(x)]^n = L^n$ for any positive integer n.

75. The **greatest integer function** is denoted by $f(x) = [x]$ and equals the greatest integer that is less than or equal to x. Thus, $[2.3] = 2$, $[-1.2] = -2$ and $[3] = 3$. In spite of this last fact, show that $\lim_{x\to 3} [x]$ does not exist.

76. Investigate the existence of (a) $\lim_{x\to 1} [x]$, (b) $\lim_{x\to 1.5} [x]$, (c) $\lim_{x\to 1.5} [2x]$ and (d) $\lim_{x\to 1} (x-[x])$.

APPLICATIONS

1. Suppose a state's income tax code states the tax liability on x dollars of taxable income is given by

$$T(x) = \begin{cases} 0.14x & \text{if } 0 \le x < 10{,}000 \\ 1500 + 0.21x & \text{if } 10{,}000 \le x \end{cases}.$$

Compute $\lim_{x\to 0^+} T(x)$; why is this good? Compute $\lim_{x\to 10{,}000} T(x)$; why is this bad?

2. Suppose a state's income tax code states that tax liability is 12% on the first \$20,000 of taxable earnings and 16% on the remainder. Find constants a and b for the tax function

$$T(x) = \begin{cases} a + 0.12x & \text{if } x \le 20{,}000 \\ b + 0.16(x-20{,}000) & \text{if } x > 20{,}000 \end{cases}$$

such that $\lim_{x\to 0^+} T(x) = 0$ and $\lim_{x\to 20{,}000} T(x)$ exists. Why is it important for these limits to exist?

EXPLORATORY EXERCISES

1. The value $x = 0$ is called a **zero of multiplicity** n ($n \ge 1$) for the function f if $\lim_{x\to 0} \frac{f(x)}{x^n}$ exists and is non-zero but $\lim_{x\to 0} \frac{f(x)}{x^{n-1}} = 0$. Show that $x = 0$ is a zero of multiplicity 2 for x^2, $x = 0$ is a zero of multiplicity 3 for x^3 and $x = 0$ is a zero of multiplicity 4 for x^4. For polynomials, what does multiplicity describe? The reason the definition is not as straightforward as we might like is so that it can apply to nonpolynomial functions, as well. Find the multiplicity of $x = 0$ for $f(x) = \sin x$; $f(x) = x \sin x$; $f(x) = \sin x^2$. If you know that $x = 0$ is a zero of multiplicity m for $f(x)$ and multiplicity n for $g(x)$, what can you say about the multiplicity of $x = 0$ for $f(x) + g(x)$? $f(x)\cdot g(x)$? $f(g(x))$?

2. We have conjectured that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Using graphical and numerical evidence, conjecture the value of $\lim_{x\to 0} \frac{\sin 2x}{x}$ and $\lim_{x\to 0} \frac{\sin cx}{x}$ for various values of c. Given that $\lim_{x\to 0} \frac{\sin cx}{cx} = 1$ for any constant $c \ne 0$, prove that your conjecture is correct. Then evaluate $\lim_{x\to 0} \frac{\sin cx}{\sin kx}$ and $\lim_{x\to 0} \frac{\tan cx}{\tan kx}$ for numbers c and $k \ne 0$.

6.4 CONTINUITY AND ITS CONSEQUENCES

When told that a machine has been in *continuous* operation for the past 60 hours, most of us would interpret this to mean that the machine has been in operation *all* of that time, without any interruption at all, even for a moment. Likewise, we say that a function is *continuous* on an interval if its graph on that interval can be drawn without interruption, that is, without lifting the pencil from the paper.

First, look at each of the graphs shown in Figures 6.21a–6.21d to determine what keeps the function from being continuous at the point $x = a$.

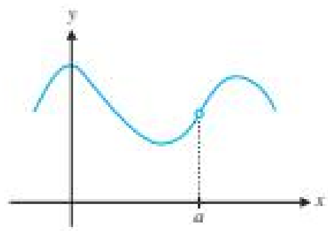


FIGURE 6.21a
$f(a)$ is not defined (the graph has a hole at $x = a$).

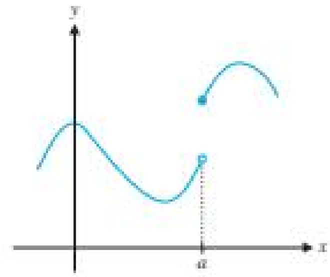


FIGURE 6.21b
$f(a)$ is defined, but $\lim_{x \to a} f(x)$ does not exist (the graph has a jump at $x = a$).

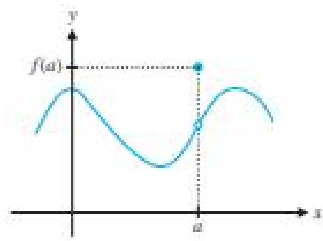


FIGURE 6.21c
$\lim_{x \to a} f(x)$ exists and $f(a)$ is defined, but $\lim_{x \to a} f(x) \neq f(a)$ (the graph has a hole at $x = a$).

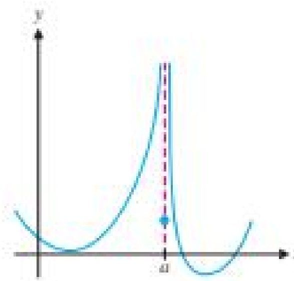


FIGURE 6.21d
$\lim_{x \to a} f(x)$ does not exist (the function "blows up" as x approaches a).

REMARK 4.1

For f to be continuous at $x = a$, the definition says that (i) $f(a)$ must be defined, (ii) the limit $\lim_{x \to a} f(x)$ must exist and (iii) the limit and value of f at the point must be the same.

Further, this says that a function is continuous at a point exactly when you can compute its limit at that point by simply substituting in.

This suggests the following definition of continuity at a point:

DEFINITION 4.1

For a function f defined on an open interval containing $x = a$, we say that f is **continuous** at a when

$$\lim_{x \to a} f(x) = f(a).$$

Otherwise, f is said to be **discontinuous** at $x = a$.

For most purposes, it is best for you to think of the intuitive notion of continuity that we've outlined above. Definition 4.1 should then simply follow from your intuitive understanding of the concept.

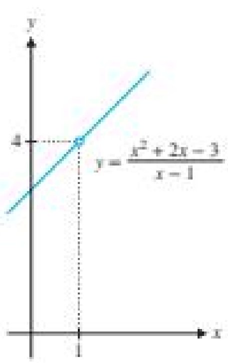


FIGURE 6.22
$y = \dfrac{x^2 + 2x - 3}{x - 1}$

EXAMPLE 4.1 Finding Where a Rational Function Is Continuous

Determine where $f(x) = \dfrac{x^2 + 2x - 3}{x - 1}$ is continuous.

Solution Note that

$$f(x) = \frac{x^2 + 2x - 3}{x - 1} = \frac{(x - 1)(x + 3)}{x - 1} \quad \text{Factoring the numerator.}$$

$$= x + 3, \text{ for } x \neq 1. \quad \text{Canceling common factors.}$$

This says that the graph of f is a straight line, but with a hole in it at $x = 1$, as indicated in Figure 6.22. So, f is continuous for $x \neq 1$. ■

REMARK 4.2

You should be careful not to confuse the continuity of a function at a point with its simply being defined there. A function can be defined at a point without being continuous there. (Look back at Figures 6.21b, 6.21c and 6.21d.)

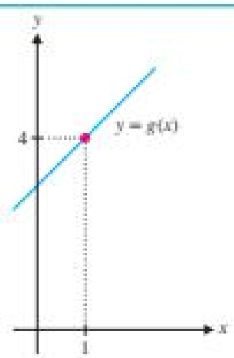


FIGURE 6.23
$y = g(x)$

EXAMPLE 4.2 Removing a Hole in the Graph

Extend the function from example 4.1 to make it continuous everywhere by defining it at a single point.

Solution In example 4.1, we saw that the function is continuous for $x \neq 1$ and it is undefined at $x = 1$. So, suppose we just go ahead and define it, as follows. Let

$$g(x) = \begin{cases} \dfrac{x^2 + 2x - 3}{x - 1}, & \text{if } x \neq 1 \\ a, & \text{if } x = 1, \end{cases}$$

for some real number a.

Notice that $g(x)$ is defined for all x and equals $f(x)$ for all $x \neq 1$. Here, we have

$$\lim_{x\to 1} g(x) = \lim_{x\to 1} \frac{x^2 + 2x - 3}{x - 1}$$
$$= \lim_{x\to 1} (x + 3) = 4.$$

Observe that if we choose $a = 4$, we now have that

$$\lim_{x\to 1} g(x) = 4 = g(1)$$

and, so, g is continuous at $x = 1$.

Note that the graph of g is the same as the graph of f seen in Figure 6.22, except that we now include the point (1, 4). (See Figure 6.23.) Also, note that there's a very simple way to write $g(x)$. (Think about this.) ■

Note that in example 4.2, for any choice of a other than $a = 4$, g is discontinuous at $x = 1$. When we can remove a discontinuity by simply redefining the function at that point, we call the discontinuity **removable**. Not all discontinuities are removable, however. Carefully examine Figures 6.21b and 6.21c and convince yourself that the discontinuity in Figure 6.21c is removable, while the one in Figures 6.21b and 6.21d are nonremovable. Briefly, a function f has a nonremovable discontinuity at $x = a$ if $\lim_{x\to a} f(x)$ does not exist.

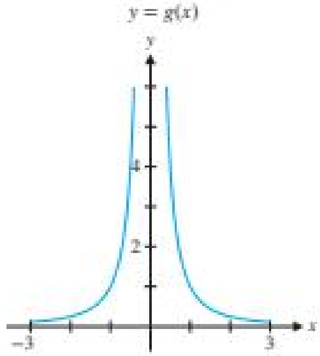


FIGURE 6.24a
$y = \dfrac{1}{x^2}$

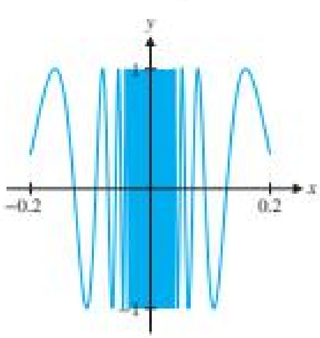


FIGURE 6.24b
$y = \cos(1/x)$

EXAMPLE 4.3 Functions That Cannot be Extended Continuously

Show that (a) $f(x) = \dfrac{1}{x^2}$ and (b) $g(x) = \cos\left(\dfrac{1}{x}\right)$ cannot be extended to a function that is continuous everywhere.

Solution (a) Observe from Figure 6.24a (also, construct a table of function values) that

$$\lim_{x\to 0} \frac{1}{x^2} \text{ does not exist.}$$

Hence, regardless of how we might define $f(0)$, f will not be continuous at $x = 0$.

(b) Similarly, observe that $\lim_{x\to 0} \cos(1/x)$ does not exist, due to the endless oscillation of $\cos(1/x)$ as x approaches 0. (See Figure 6.24b.) Again, notice that since the limit does not exist, there is no way to redefine the function at $x = 0$ to make it continuous there. ■

From your experience with the graphs of some common functions, the following result should come as no surprise:

THEOREM 4.1

All polynomials are continuous everywhere. Additionally, $\sin x$, $\cos x$, $\tan^{-1} x$ and e^x are continuous everywhere, $\sqrt[n]{x}$ is continuous for all x, when n is odd and for $x > 0$, when n is even. We also have that $\ln x$ is continuous for $x > 0$ and $\sin^{-1} x$ and $\cos^{-1} x$ are continuous for $-1 < x < 1$.

PROOF

We have already established (in Theorem 3.2) that for any polynomial $p(x)$ and any real number a,

$$\lim_{x \to a} p(x) = p(a),$$

from which it follows that p is continuous at $x = a$. The rest of the theorem follows from Theorems 3.3 and 3.4 in a similar way.

From these very basic continuous functions, we can build a large collection of continuous functions, using Theorem 4.2. ■

> **THEOREM 4.2**
>
> Suppose that f and g are continuous at $x = a$. Then all of the following are true:
>
> i. $(f \pm g)$ is continuous at $x = a$,
> ii. $(f \cdot g)$ is continuous at $x = a$ and
> iii. (f/g) is continuous at $x = a$ if $g(a) \neq 0$.

Simply put, Theorem 4.2 says that a sum, difference or product of continuous functions is continuous, while the quotient of two continuous functions is continuous at any point at which the denominator is non-zero.

PROOF

(i) If f and g are continuous at $x = a$, then

$$\begin{aligned} \lim_{x \to a} [f(x) \pm g(x)] &= \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) && \text{From Theorem 3.1.} \\ &= f(a) \pm g(a) && \text{Since } f \text{ and } g \text{ are continuous at } a. \\ &= (f \pm g)(a), \end{aligned}$$

by the usual rules of limits. Thus, $(f \pm g)$ is also continuous at $x = a$.

Parts (ii) and (iii) are proved in a similar way and are left as exercises. ■

EXAMPLE 4.4 Continuity for a Rational Function

Determine where f is continuous, for $f(x) = \dfrac{x^4 - 3x^2 + 2}{x^2 - 3x - 4}$.

Solution Here, f is a quotient of two polynomial (hence continuous) functions. The computer-generated graph of the function indicated in Figure 6.25 suggests a vertical asymptote at around $x = 4$, but doesn't indicate any discontinuity. From Theorem 4.2, f will be continuous at all x where the denominator is not zero, that is, where

$$x^2 - 3x - 4 = (x + 1)(x - 4) \neq 0.$$

Thus, f is continuous for $x \neq -1, 4$. (Think about why you didn't see anything peculiar about the graph at $x = -1$.) ■

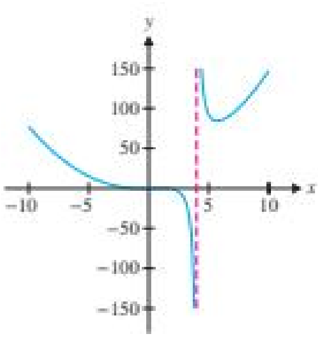


Figure 6.25

$y = \dfrac{x^4 - 3x^2 + 2}{x^2 - 3x - 4}$

With the addition of the result in Theorem 4.3, we will have all the basic tools needed to establish the continuity of most elementary functions.

THEOREM 4.3

Suppose that $\lim_{x \to a} g(x) = L$ and f is continuous at L. Then,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(L).$$

A proof of Theorem 4.3 is given in the Appendix.

Notice that this says that if f is continuous, then we can bring the limit "inside." This should make sense, since as $x \to a$, $g(x) \to L$ and so, $f(g(x)) \to f(L)$, since f is continuous at L.

COROLLARY 4.1

Suppose that g is continuous at a and f is continuous at $g(a)$. Then, the composition $f \circ g$ is continuous at a.

PROOF

From Theorem 4.3, we have

$$\begin{aligned}\lim_{x \to a} (f \circ g)(x) &= \lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) \\ &= f(g(a)) = (f \circ g)(a). \quad \text{Since } g \text{ is continuous at } a. \blacksquare\end{aligned}$$

EXAMPLE 4.5 Continuity for a Composite Function

Determine where $h(x) = \cos(x^2 - 5x + 2)$ is continuous.

Solution Note that

$$h(x) = f(g(x)),$$

where $g(x) = x^2 - 5x + 2$ and $f(x) = \cos x$. Since both f and g are continuous for all x, h is continuous for all x, by Corollary 4.1. ■

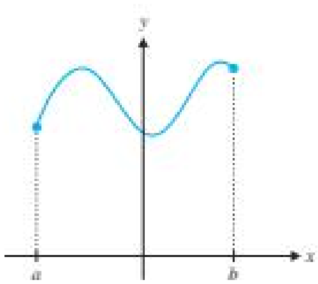


Figure 6.26
f continuous on $[a, b]$

DEFINITION 4.2

If f is continuous at every point on an open interval (a, b), we say that f is **continuous on** (a, b). Following Figure 6.26, we say that f is **continuous on the closed interval** $[a, b]$, if f is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a) \text{ and } \lim_{x \to b^-} f(x) = f(b).$$

Finally, if f is continuous on all of $(-\infty, \infty)$, we simply say that f is **continuous**. (That is, when we don't specify an interval, we mean continuous everywhere.)

For many functions, it's a simple matter to determine the intervals on which the function is continuous. We illustrate this in example 4.6.

EXAMPLE 4.6 Continuity on a Closed Interval

Determine the interval(s) where f is continuous, for $f(x) = \sqrt{4 - x^2}$.

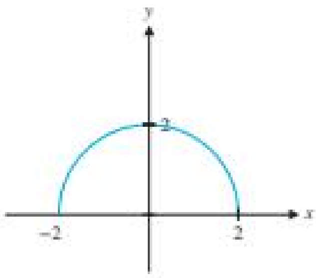


FIGURE 6.27
$y = \sqrt{4 - x^2}$

Solution First, observe that f is defined only for $-2 \le x \le 2$. Next, note that f is the composition of two continuous functions and, hence, is continuous for all x for which $4 - x^2 > 0$. We show a graph of the function in Figure 6.27. Since

$$4 - x^2 > 0$$

for $-2 < x < 2$, we have that f is continuous for all x in the interval $(-2, 2)$, by Theorem 4.1 and Corollary 4.1. Finally, we test the endpoints to see that $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0 = f(2)$ and $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 = f(-2)$ so that f is continuous on the closed interval $[-2, 2]$. ■

EXAMPLE 4.7 Interval of Continuity for a Logarithm

Determine the interval(s) where $f(x) = \ln(x - 3)$ is continuous.

Solution It follows from Theorem 4.1 and Corollary 4.1 that f is continuous whenever $(x - 3) > 0$ (i.e., for $x > 3$). Thus, f is continuous on the interval $(3, \infty)$. ■

The Internal Revenue Service presides over some of the most despised functions in existence. The first few lines of a recent Tax Rate Schedule (for single taxpayers) looked like:

For taxable amount over	*but not over*	*your tax liability is*	*minus*
$0	$6000	10%	0
$6000	$27,950	15%	300
$27,950	$67,700	27%	$3654

Where do the numbers $300 and $3654 come from? If we write the tax liability $T(x)$ as a function of the taxable amount x (assuming that x can be any real number and not just a whole dollar amount), we have

$$T(x) = \begin{cases} 0.10x & \text{if } 0 < x \le 6000 \\ 0.15x - 300 & \text{if } 6000 < x \le 27{,}950 \\ 0.27x - 3654 & \text{if } 27{,}950 < x \le 67{,}700. \end{cases}$$

Be sure you understand our translation so far. Note that it is important that this be a continuous function: think of the fairness issues that would arise if it were not!

EXAMPLE 4.8 Continuity of Federal Tax Tables

Verify that the federal tax rate function T is continuous at the "joint" $x = 27, 950$. Then, find a to complete the table. (You will find b and c as exercises.)

For taxable amount over	*but not over*	*your tax liability is*	*minus*
$67,700	$141,250	30%	*a*
$141,250	$307,050	35%	*b*
$307,050	—	38.6%	*c*

Solution For T to be continuous at $x = 27{,}950$, we must have

$$\lim_{x \to 27{,}950^-} T(x) = \lim_{x \to 27{,}950^+} T(x).$$

Since both functions $0.15x - 300$ and $0.27x - 3654$ are continuous, we can compute the one-sided limits by substituting $x = 27{,}950$. Thus,

$$\lim_{x \to 27{,}950^-} T(x) = 0.15(27{,}950) - 300 = 3892.50$$

and

$$\lim_{x \to 27{,}950^+} T(x) = 0.27(27{,}950) - 3654 = 3892.50.$$

Since the one-sided limits agree and equal the value of the function at that point, $T(x)$ is continuous at $x = 27{,}950$. We leave it as an exercise to establish that $T(x)$ is also continuous at $x = 6000$. (Note that the function could be written with equal signs on all of the inequalities; this would be incorrect if the function were discontinuous.) To complete the table, we choose a to get the one-sided limits at $x = 67{,}700$ to match. We have

$$\lim_{x \to 67{,}700^-} T(x) = 0.27(67{,}700) - 3654 = 14{,}625,$$

while

$$\lim_{x \to 67{,}700^+} T(x) = 0.30(67{,}700) - a = 20{,}310 - a.$$

So, we set the one-sided limits equal, to obtain

$$14{,}625 = 20{,}310 - a$$

or

$$a = 20{,}310 - 14{,}625 = 5685. \blacksquare$$

HISTORICAL NOTES

Karl Weierstrass (1815–1897)
A German mathematician who proved the Intermediate Value Theorem and several other fundamental results of the calculus. Weierstrass was known as an excellent teacher whose students circulated his lecture notes throughout Europe, because of their clarity and originality. Also known as a superb fencer, Weierstrass was one of the founders of modern mathematical analysis.

Theorem 4.4 should seem an obvious consequence of our intuitive definition of continuity.

THEOREM 4.4 (Intermediate Value Theorem)

Suppose that f is continuous on the closed interval $[a, b]$ and W is any number between $f(a)$ and $f(b)$. Then, there is a number $c \in [a, b]$ for which $f(c) = W$.

Theorem 4.4 says that if f is continuous on $[a, b]$, then f must take on *every* value between $f(a)$ and $f(b)$ at least once. That is, a continuous function cannot skip over any numbers between its values at the two endpoints. To do so, the graph would need to leap across the horizontal line $y = W$, something that continuous functions cannot do. (See Figure 6.28a.) Of course, a function may take on a given value W more than once. (See Figure 6.28b.) Although these graphs make this result seem reasonable, the proof is more complicated than you might imagine and we must refer you to an advanced calculus text.

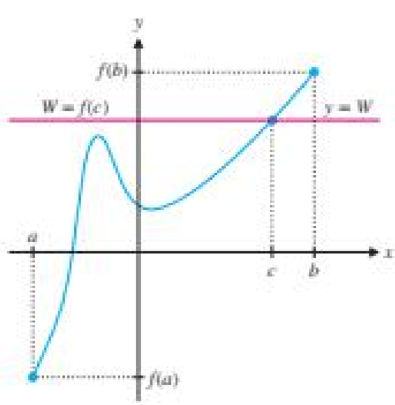


FIGURE 6.28a
An illustration of the Intermediate Value Theorem

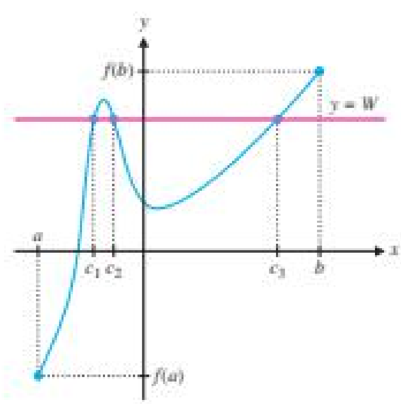


FIGURE 6.28b
More than one value of c

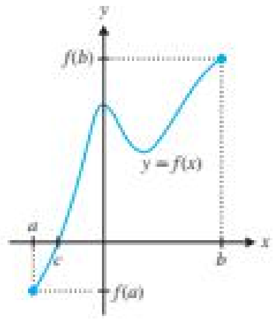


FIGURE 6.29
Intermediate Value Theorem where c is a zero of f

In Corollary 4.2, we see an important application of the Intermediate Value Theorem.

COROLLARY 4.2

Suppose that f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs [i.e., $f(a) \cdot f(b) < 0$]. Then, there is at least one number $c \in (a, b)$ for which $f(c) = 0$. (Recall that c is then a *zero* of f.)

Notice that Corollary 4.2 is simply the special case of the Intermediate Value Theorem where $W = 0$. (See Figure 6.29.) The Intermediate Value Theorem and Corollary 4.2 are examples of *existence theorems;* they tell you that there *exists* a number c satisfying some condition, but they do *not* tell you what c is.

The Method of Bisections

In example 4.9, we see how Corollary 4.2 can help us locate the zeros of a function.

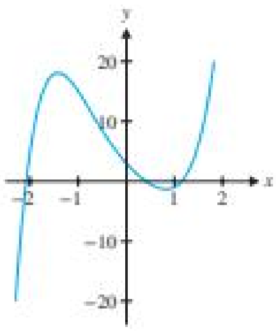


FIGURE 6.30
$y = x^5 + 4x^2 - 9x + 3$

EXAMPLE 4.9 Finding Zeros by the Method of Bisections

Find the zeros of $f(x) = x^5 + 4x^2 - 9x + 3$.

Solution Since f is a polynomial of degree 5, we don't have any formulas for finding its zeros. The only alternative, then, is to approximate the zeros. A good starting place would be to draw a graph of $y = f(x)$ like the one in Figure 6.30. There are three zeros visible on the graph. Since f is a polynomial, it is continuous everywhere and so Corollary 4.2 says that there must be a zero on any interval on which the function changes sign. From the graph, you can see that there must be zeros between -3 and -2, between 0 and 1 and between 1 and 2. We could also conclude this by noting the function's change of sign between these x-values. For instance, $f(0) = 3$ and $f(1) = -1$.

While a rootfinding program can provide an accurate approximation of the zeros, the issue here is not so much to get an answer as it is to understand how to find one. Corollary 4.2 suggests a simple yet effective method, called the **method of bisections**.

Taking the midpoint of the interval [0,1], since $f(0.5) \approx -0.469 < 0$ and $f(0) = 3 > 0$, there must be a zero between 0 and 0.5. Next, the midpoint of [0, 0.5] is 0.25 and $f(0.25) \approx 1.001 > 0$, so that the zero is in the interval (0.25, 0.5). We continue in this way to narrow down the interval in which there's a zero, as shown in the following table.

a	b	$f(a)$	$f(b)$	Midpoint	f(midpoint)
0	1	3	−1	0.5	−0.469
0	0.5	3	−0.469	0.25	1.001
0.25	0.5	1.001	−0.469	0.375	0.195
0.375	0.5	0.195	−0.469	0.4375	−0.156
0.375	0.4375	0.195	−0.156	0.40625	0.015
0.40625	0.4375	0.015	−0.156	0.421875	−0.072
0.40625	0.421875	0.015	−0.072	0.4140625	−0.029
0.40625	0.4140625	0.015	−0.029	0.41015625	−0.007
0.40625	0.41015625	0.015	−0.007	0.408203125	0.004

Continuing this process through 20 more steps leads to the approximate zero $x = 0.40892288$, which is accurate to at least eight decimal places. The other zeros can be found in a similar fashion. ■

Although the method of bisections is a tedious process, it's a reliable, yet simple method for finding approximate zeros.

EXERCISES 6.4

WRITING EXERCISES

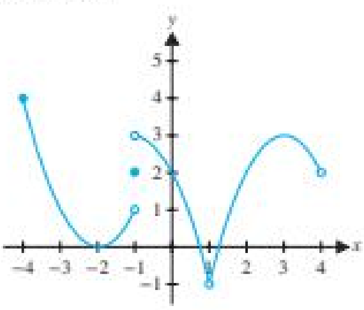
1. Think about the following "real-life" functions, each of which is a function of the independent variable time: the height of a falling object, the amount of money in a bank account, the cholesterol level of a person, the amount of a certain chemical present in a test tube and a machine's most recent measurement of the cholesterol level of a person. Which of these are continuous functions? Explain your answers.

2. Whether a process is continuous or not is not always clear-cut. When you watch television or a movie, the action seems to be continuous. This is an optical illusion, since both movies and television consist of individual "snapshots" that are played back at many frames per second. Where does the illusion of continuous motion come from? Given that the average person blinks several times per minute, is our perception of the world actually continuous?

3. When you sketch the graph of the parabola $y = x^2$ with pencil or pen, is your sketch (at the molecular level) actually the graph of a continuous function? Is your calculator or computer actually the graph of a continuous function? Do we ever have problems correctly interpreting a graph due to these limitations?

4. For each of the graphs in Figures 6.21a–6.21d, describe (with an example) what the formula for $f(x)$ might look like to produce the given graph.

5. Discussing continuity graphically.
Indicate whether the following statements are true or false. Justify your answer.



(a) $f(4)$ *is undefined*
(b) $f(-1) = 1$
(c) $\lim_{x\to-1^-} f(x) = 1$
(d) $f(x)$ *is continuous at* $x = -1$
(e) $f(x)$ *is continuous at* $x = 0$
(f) $f(x)$ *is continuous at* $x = 1$

In exercises 1–14, determine where f is continuous. If possible, extend f as in example 4.2 to a new function that is continuous on a larger domain.

1. $f(x) = \dfrac{x^2 + x - 2}{x + 2}$

2. $f(x) = \dfrac{x^2 - x - 6}{x - 3}$

3. $f(x) = \dfrac{x - 1}{x^2 - 1}$

4. $f(x) = \dfrac{4x}{x^2 + x - 2}$

5. $f(x) = \dfrac{4x}{x^2 + 4}$

6. $f(x) = \dfrac{3x}{x^2 - 2x - 4}$

7. $f(x) = x^2 \tan x$

8. $f(x) = x \cot x$

9. $f(x) = \ln x^2$

10. $f(x) = 3/\ln x^2$

11. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$

12. $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

13. $f(x) = \begin{cases} 3x - 1 & \text{if } x \leq -1 \\ x^2 + 5x & \text{if } -1 < x < 1 \\ 3x^3 & \text{if } x \geq 1 \end{cases}$

14. $f(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ \sin x & \text{if } 0 < x \leq \pi \\ x - \pi & \text{if } x > \pi \end{cases}$

In exercises 15–20, explain why each function fails to be continuous at the given x-value by indicating which of the three conditions in Definition 4.1 are not met.

15. $f(x) = \dfrac{x}{x - 1}$ at $x = 1$

16. $f(x) = \dfrac{x^2 - 1}{x - 1}$ at $x = 1$

17. $f(x) = \sin\dfrac{1}{x}$ at $x = 0$

18. $f(x) = \dfrac{e^{x-1}}{e^x - 1}$ at $x = 0$

19. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$ at $x = 2$

20. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$ at $x = 2$

In exercises 21–28, determine the intervals on which f is continuous.

21. $f(x) = \sqrt{x + 3}$

22. $f(x) = \sqrt{x^2 - 4}$

23. $f(x) = \sqrt[3]{x + 2}$

24. $f(x) = (x - 1)^{3/2}$

25. $f(x) = \sin^{-1}(x + 2)$

26. $f(x) = \ln(\sin x)$

27. $f(x) = \dfrac{\sqrt{x + 1} + e^x}{x^2 - 2}$

28. $f(x) = \dfrac{\ln(x^2 - 1)}{\sqrt{x^2 - 2x}}$

29. Suppose that a state's income tax code states that the tax liability on x dollars of taxable income is given by

$$T(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 0.14x & \text{if } 0 < x < 10{,}000 \\ c + 0.21x & \text{if } 10{,}000 \leq x. \end{cases}$$

Determine the constant c that makes this function continuous for all x. Give a rationale why such a function should be continuous.

30. Suppose a state's income tax code states that tax liability is 12% on the first \$20,000 of taxable earnings and 16% on the remainder. Find constants a and b for the tax function

$$T(x) = \begin{cases} 0 & \text{if } x \le 0 \\ a + 0.12x & \text{if } 0 < x \le 20{,}000 \\ b + 0.16(x - 20{,}000) & \text{if } x > 20{,}000 \end{cases}$$

such that $T(x)$ is continuous for all x.

31. In example 4.8, find b and c to complete the table.

32. In example 4.8, show that $T(x)$ is continuous for $x = 6000$.

In exercises 33–36, use the Intermediate Value Theorem to verify that $f(x)$ has a zero in the given interval. Then use the method of bisections to find an interval of length 1/32 that contains the zero.

33. $f(x) = x^2 - 7$, (a) $[2, 3]$; (b) $[-3, -2]$

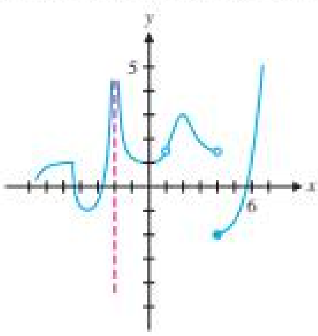
34. $f(x) = x^3 - 4x - 2$, (a) $[2, 3]$; (b) $[-1, 0]$

35. $f(x) = \cos x - x$, $[0, 1]$

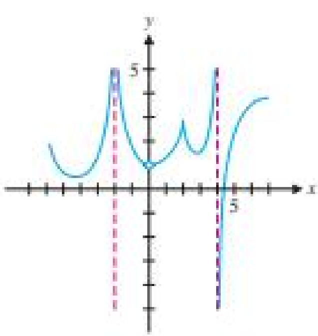
36. $f(x) = e^x + x$, $[-1, 0]$

In exercises 37 and 38, use the given graph to identify all intervals on which the function is continuous.

37.



38.



In exercises 39–41, determine values of a and b that make the given function continuous.

39. $f(x) = \begin{cases} \dfrac{2 \sin x}{x} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ b \cos x & \text{if } x > 0 \end{cases}$

40. $f(x) = \begin{cases} ae^x + 1 & \text{if } x < 0 \\ \sin^{-1} \dfrac{x}{2} & \text{if } 0 \le x \le 2 \\ x^2 - x + b & \text{if } x > 2 \end{cases}$

41. $f(x) = \begin{cases} a(\tan^{-1} x + 2) & \text{if } x < 0 \\ 2e^{bx} + 1 & \text{if } 0 \le x \le 3 \\ \ln(x - 2) + x^2 & \text{if } x > 3 \end{cases}$

42. Prove Corollary 4.1.

A function is continuous from the right at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$. In exercises 43 and 44, determine whether $f(x)$ is continuous from the right at $x = 2$.

43. $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 3x - 3 & \text{if } x > 2 \end{cases}$

44. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 3x - 3 & \text{if } x > 2 \end{cases}$

45. Define what it means for a function to be **continuous from the left** at $x = a$ and determine which of the functions in exercises 43 and 44 are continuous from the left at $x = 2$.

46. Suppose that $f(x) = \dfrac{g(x)}{h(x)}$ and $h(a) = 0$. Determine whether each of the following statements is always true, always false, or maybe true/maybe false. Explain. (a) $\lim_{x \to a} f(x)$ does not exist. (b) $f(x)$ is not continuous at $x = a$.

47. Suppose that $f(x)$ is continuous at $x = 0$. Prove that $\lim_{x \to 0} xf(x) = 0$.

48. The **converse** of exercise 47 is not true. That is, the fact $\lim_{x \to 0} xf(x) = 0$ does not guarantee that $f(x)$ is continuous at $x = 0$. Find a counterexample; that is, find a function f such that $\lim_{x \to 0} xf(x) = 0$ and $f(x)$ is not continuous at $x = 0$.

49. If $f(x)$ is continuous at $x = a$, prove that $g(x) = |f(x)|$ is continuous at $x = a$.

50. Determine whether the converse of exercise 49 is true. That is, if $|f(x)|$ is continuous at $x = a$, is it necessarily true that $f(x)$ must be continuous at $x = a$?

51. Let $f(x)$ be a continuous function for $x \ge a$ and define $h(x) = \max_{a \le t \le x} f(t)$. Prove that $h(x)$ is continuous for $x \ge a$. Would this still be true without the assumption that $f(x)$ is continuous?

52. If $f(x) = \begin{cases} x^2, & \text{if } x \ne 0 \\ 4, & \text{if } x = 0 \end{cases}$ and $g(x) = 2x$, show that $\lim_{x \to 0} f(g(x)) \ne f\left(\lim_{x \to 0} g(x)\right)$.

53. Suppose that $f(x)$ is a continuous function with consecutive zeros at $x = a$ and $x = b$; that is, $f(a) = f(b) = 0$ and $f(x) \ne 0$ for $a < x < b$. Further, suppose that $f(c) > 0$ for some number c between a and b. Use the Intermediate Value Theorem to argue that $f(x) > 0$ for all $a < x < b$.

54. For $f(x) = 2x - \dfrac{400}{x}$, we have $f(-1) > 0$ and $f(2) < 0$. Does the Intermediate Value Theorem guarantee a zero of $f(x)$ between $x = -1$ and $x = 2$? What happens if you try the method of bisections?

55. Prove that if f is continuous on an interval $[a, b]$, $f(a) > a$ and $f(b) < b$, then f has a fixed point [a solution of $f(x) = x$] in the interval (a, b).

56. Prove the final two parts of Theorem 4.2.

57. Graph $f(x) = \dfrac{\sin|x^3 - 3x^2 + 2x|}{x^3 - 3x^2 + 2x}$ and determine all real numbers x for which f is not continuous.

58. Use the method of bisections to estimate the other two zeros in example 4.9.

APPLICATIONS

1. If you push gently on a heavy box resting on the ground, at first nothing will happen because of the static friction force that opposes motion. If you push hard enough, the box will start sliding, although there is again a friction force that opposes the motion. Suppose you are given the following description of the friction force. Up to 100 newtons, friction matches the force you apply to the box. Over 100 newtons, the box will move and the friction force will equal 80 newtons. Sketch a graph of friction as a function of your applied force based on this description. Where is this graph discontinuous? What is significant physically about this point? Do you think the friction force actually ought to be continuous? Modify the graph to make it continuous while still retaining most of the characteristics described.

2. Suppose a worker's salary starts at \$40,000 with \$2000 raises every 3 months. Graph the salary function $s(t)$; why is it discontinuous? How does the function $f(t) = 40{,}000 + \frac{2000}{3}t$ (t in months) compare? Why might it be easier to do calculations with $f(t)$ than $s(t)$?

3. On Monday morning, a saleswoman leaves on a business trip at 7:13 A.M. and arrives at her destination at 2:03 P.M. The following morning, she leaves for home at 7:17 A.M. and arrives at 1:59 P.M. The woman notices that at a particular stoplight along the way, a nearby bank clock changes from 10:32 A.M. to 10:33 A.M. on both days. Therefore, she must have been at the same location at the same time on both days. Her boss doesn't believe that such an unlikely coincidence could occur. Use the Intermediate Value Theorem to argue that it *must* be true that at some point on the trip, the saleswoman was at exactly the same place at the same time on both Monday and Tuesday.

4. Suppose you ease your car up to a stop sign at the top of a hill. Your car rolls back a half meter and then you drive through the intersection. A police officer pulls you over for not coming to a complete stop. Use the Intermediate Value Theorem to argue that there was an instant in time when your car was stopped (in fact, there were at least two). What is the difference between this stopping and the stopping that the police officer wanted to see?

5. The sex of newborn Mississippi alligators is determined by the temperature of the eggs in the nest. The eggs fail to develop unless the temperature is between 26°C and 36°C. All eggs between 26°C and 30°C develop into females, and eggs between 34°C and 36°C develop into males. The percentage of females decreases from 100% at 30°C to 0% at 34°C. If $f(T)$ is the percentage of females developing from an egg at T°C, then

$$f(T) = \begin{cases} 100 & \text{if } 26 \le T \le 30 \\ g(T) & \text{if } 30 < T < 34 \\ 0 & \text{if } 34 \le T \le 36, \end{cases}$$

for some function $g(T)$. Explain why it is reasonable that $f(T)$ be continuous. Determine a function $g(T)$ such that $0 \le g(T) \le 100$ for $30 \le T \le 34$ and the resulting function $f(T)$ is continuous. [Hint: It may help to draw a graph first and make $g(T)$ linear.]

EXPLORATORY EXERCISES

1. In the text, we discussed the use of the method of bisections to find an approximate solution of equations such as $f(x) = x^3 + 5x - 1 = 0$. We can start by noticing that $f(0) = -1$ and $f(1) = 5$. Since $f(x)$ is continuous, the Intermediate Value Theorem tells us that there is a solution between $x = 0$ and $x = 1$. For the method of bisections, we guess the midpoint, $x = 0.5$. Is there any reason to suspect that the solution is actually closer to $x = 0$ than to $x = 1$? Using the function values $f(0) = -1$ and $f(1) = 5$, devise your own method of guessing the location of the solution. Generalize your method to using $f(a)$ and $f(b)$, where one function value is positive and one is negative. Compare your method to the method of bisections on the problem $x^3 + 5x - 1 = 0$; for both methods, stop when you are within 0.001 of the solution, $x \approx 0.1984$. Which method performed better? Before you get overconfident in your method, compare the two methods again on $x^3 + 5x^2 - 1 = 0$. Does your method get close on the first try? See if you can determine graphically why your method works better on the first problem.

2. Determine all values of x for which each function is continuous.

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ x & \text{if } x \text{ is rational} \end{cases}$$

$$g(x) = \begin{cases} x^2 + 3 & \text{if } x \text{ is irrational} \\ 4x & \text{if } x \text{ is rational and} \end{cases}$$

$$h(x) = \begin{cases} \cos 4x & \text{if } x \text{ is irrational} \\ \sin 4x & \text{if } x \text{ is rational} \end{cases}$$

6.5 LIMITS INVOLVING INFINITY; ASYMPTOTES

In this section, we revisit some old limit problems to give more informative answers and examine some related questions.

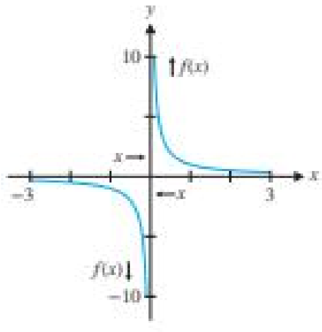


FIGURE 6.31
$\lim_{x\to 0^+}\frac{1}{x}=\infty$ and $\lim_{x\to 0^-}\frac{1}{x}=-\infty$

x	$\frac{1}{x}$
0.1	10
0.01	100
0.001	1000
0.0001	10,000
0.00001	100,000

x	$\frac{1}{x}$
−0.1	−10
−0.01	−100
−0.001	−1000
−0.0001	−10,000
−0.00001	−100,000

EXAMPLE 5.1 A Simple Limit Revisited

Examine $\lim_{x\to 0}\frac{1}{x}$.

Solution Of course, we can draw a graph (see Figure 6.31) and compute a table of function values easily, by hand. (See the tables that follow.)

While we say that the limits $\lim_{x\to 0^+}\frac{1}{x}$ and $\lim_{x\to 0^-}\frac{1}{x}$ do not exist, they do so for different reasons. Specifically, as $x\to 0^+$, $\frac{1}{x}$ *increases* without bound, while as $x\to 0^-$, $\frac{1}{x}$ *decreases* without bound. To indicate this, we write

$$\lim_{x\to 0^+}\frac{1}{x}=\infty \tag{5.1}$$

and

$$\lim_{x\to 0^-}\frac{1}{x}=-\infty. \tag{5.2}$$

Graphically, this says that the graph of $y=\frac{1}{x}$ approaches the vertical line $x=0$, as $x\to 0$, as seen in Figure 6.31. When this occurs, we say that the line $x=0$ is a **vertical asymptote.** It is important to note that while the limits (5.1) and (5.2) *do not exist,* we say that they "equal" ∞ and −∞, respectively, only to be specific as to *why* they do not exist. Finally, in view of the one-sided limits (5.1) and (5.2), we say (as before) that

$$\lim_{x\to 0}\frac{1}{x} \text{ does not exist.}$$

REMARK 5.1

It may at first seem contradictory to say that $\lim_{x\to 0^+}\frac{1}{x}$ does not exist and then to write $\lim_{x\to 0^+}\frac{1}{x}=\infty$. However, since ∞ is *not* a real number, there is no contradiction here. We say that $\lim_{x\to 0^+}\frac{1}{x}=\infty$ to indicate that as $x\to 0^+$, the function values are increasing without bound.

EXAMPLE 5.2 A Function Whose One-Sided Limits Are Both Infinite

Evaluate $\lim_{x\to 0}\frac{1}{x^2}$.

Solution The graph (in Figure 6.32) seems to indicate a vertical asymptote at $x=0$. From this and the accompanying tables, we can see that

x	$\frac{1}{x^2}$
0.1	100
0.01	10,000
0.001	1×10^6
0.0001	1×10^8
0.00001	1×10^{10}

x	$\frac{1}{x^2}$
−0.1	100
−0.01	10,000
−0.001	1×10^6
−0.0001	1×10^8
−0.00001	1×10^{10}

$$\lim_{x\to 0^+}\frac{1}{x^2}=\infty \quad\text{and}\quad \lim_{x\to 0^-}\frac{1}{x^2}=\infty.$$

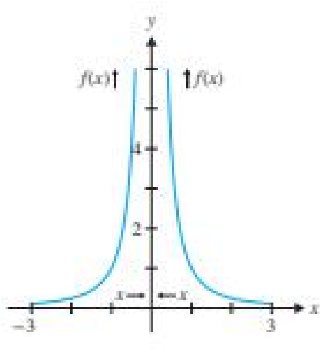


FIGURE 6.32
$\lim_{x\to 0}\frac{1}{x^2}=\infty$

REMARK 5.2

Mathematicians try to convey as much information as possible with as few symbols as possible. For instance, we prefer to say $\lim_{x\to 0}\frac{1}{x^2}=\infty$ rather than $\lim_{x\to 0}\frac{1}{x^2}$ does not exist, since the first statement not only says that the limit does not exist, but also says that $\frac{1}{x^2}$ increases without bound as x approaches 0, with $x>0$ or $x<0$.

Since both one-sided limits agree (i.e., both tend to ∞), we say that

$$\lim_{x\to 0}\frac{1}{x^2}=\infty .$$

This one concise statement says that the limit does not exist, but also that there is a vertical asymptote at $x=0$, where $f(x)\to\infty$ as $x\to 0$ from either side. ■

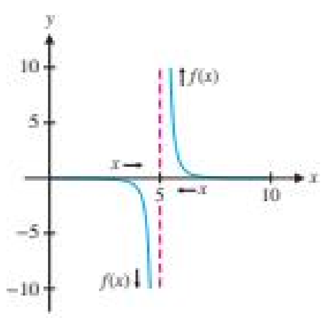


Figure 6.33
$\lim_{x\to 5^+}\frac{1}{(x-5)^3}=\infty$ and $\lim_{x\to 5^-}\frac{1}{(x-5)^3}=-\infty$

EXAMPLE 5.3 A Case Where Infinite One-Sided Limits Disagree

Evaluate $\lim_{x\to 5}\frac{1}{(x-5)^3}$.

Solution From the graph of the function in Figure 6.33, you should get a pretty clear idea that there's a vertical asymptote at $x=5$. We can verify this behavior algebraically, by noticing that as $x\to 5$, the denominator approaches 0, while the numerator approaches 1. This says that the fraction grows large in absolute value, without bound as $x\to 5$. Specifically,

We indicate the sign of each factor by printing a small "+" or "−" sign above or below each one. This enables you to see the signs of the various terms at a glance. In this case, we have

$$\lim_{x\to 5^+}\frac{\overset{+}{1}}{\underset{+}{(x-5)^3}}=\infty. \quad \text{Since } (x-5)^3>0, \text{ for } x>5.$$

Likewise, as $x\to 5^-$, $(x-5)^3\to 0$ and $(x-5)^3<0$.

In this case, we have

$$\lim_{x\to 5^-}\frac{\overset{+}{1}}{\underset{-}{(x-5)^3}}=-\infty. \quad \text{Since } (x-5)^3<0, \text{ for } x<5.$$

Finally, we say that $\lim_{x\to 5}\frac{1}{(x-5)^3}$ does not exist,

since the one-sided limits are different. ■

Based on examples 5.1, 5.2 and 5.3, recognize that if the denominator tends to 0 and the numerator does not, then the limit in question does not exist. In this event, we determine whether the limit tends to ∞ or $-\infty$ by carefully examining the signs of the various factors.

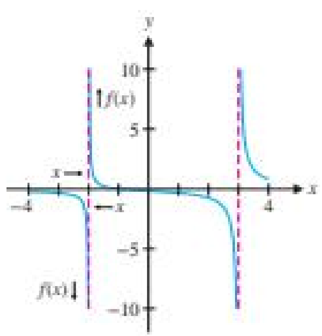


Figure 6.34
$\lim_{x\to -2}\frac{x+1}{(x-3)(x+2)}$ does not exist.

EXAMPLE 5.4 Another Case Where Infinite One-Sided Limits Disagree

Evaluate $\lim_{x\to -2}\frac{x+1}{(x-3)(x+2)}$.

Solution First, notice from the graph of the function shown in Figure 6.34 that there appears to be a vertical asymptote at $x=-2$. Further, the function appears to tend to ∞ as $x\to -2^+$, and to $-\infty$ as $x\to -2^-$. You can verify this behavior, by observing that

$$\lim_{x\to -2^+}\frac{\overset{-}{x+1}}{\underset{-}{(x-3)}\,\underset{+}{(x+2)}}=\infty \quad \text{Since } (x+1)<0,\ (x-3)<0 \text{ and } (x+2)>0, \text{ for } -2<x<-1.$$

and

$$\lim_{x\to -2^-}\frac{\overset{-}{x+1}}{\underset{-}{(x-3)}\,\underset{-}{(x+2)}}=-\infty \quad \text{Since } (x+1)<0,\ (x-3)<0 \text{ and } (x+2)<0, \text{ for } x<-2.$$

So, there is indeed a vertical asymptote at $x=-2$ and

$$\lim_{x\to -2}\frac{x+1}{(x-3)(x+2)} \text{ does not exist.}$$ ■

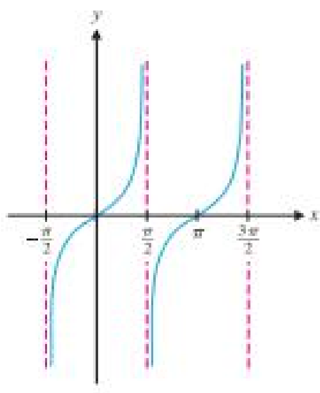


Figure 6.35
$y = \tan x$

EXAMPLE 5.5 A Limit Involving a Trigonometric Function

Evaluate $\lim\limits_{x \to \frac{\pi}{2}} \tan x$.

Solution The graph of the function shown in Figure 6.35 suggests that there is a vertical asymptote at $x = \frac{\pi}{2}$. We verify this behavior by observing that

$$\lim_{x \to \frac{\pi}{2}^-} \tan x = \lim_{x \to \frac{\pi}{2}^-} \frac{\overset{+}{\sin x}}{\underset{+}{\cos x}} = \infty$$

Since $\sin x > 0$ and $\cos x > 0$ for $0 < x < \frac{\pi}{2}$.

and

$$\lim_{x \to \frac{\pi}{2}^+} \tan x = \lim_{x \to \frac{\pi}{2}^+} \frac{\overset{+}{\sin x}}{\underset{-}{\cos x}} = -\infty.$$

Since $\sin x > 0$ and $\cos x < 0$ for $\frac{\pi}{2} < x < \pi$.

So, the line $x = \frac{\pi}{2}$ is indeed a vertical asymptote and

$$\lim_{x \to \frac{\pi}{2}} \tan x \text{ does not exist.}$$

Limits at Infinity

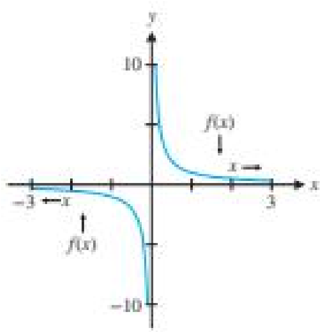


FIGURE 6.36
$\lim\limits_{x \to \infty} \frac{1}{x} = 0$ and $\lim\limits_{x \to -\infty} \frac{1}{x} = 0$

We are also interested in examining the limiting behavior of functions as x increases without bound (written $x \to \infty$) or as x decreases without bound (written $x \to -\infty$). Returning to $f(x) = \frac{1}{x}$, we can see that as $x \to \infty$, $\frac{1}{x} \to 0$. In view of this, we write

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

Similarly,

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

Notice that in Figure 6.36, the graph appears to approach the horizontal line $y = 0$, as $x \to \infty$ and as $x \to -\infty$. In this case, we call $y = 0$ a **horizontal asymptote**.

EXAMPLE 5.6 Finding Horizontal Asymptotes

Find any horizontal asymptotes to the graph of $f(x) = 2 - \frac{1}{x}$.

Solution We show a graph of $y = f(x)$ in Figure 6.37. Since as $x \to \pm\infty$, $\frac{1}{x} \to 0$, we get that

$$\lim_{x \to \infty} \left(2 - \frac{1}{x}\right) = 2$$

and

$$\lim_{x \to -\infty} \left(2 - \frac{1}{x}\right) = 2.$$

Thus, the line $y = 2$ is a horizontal asymptote.

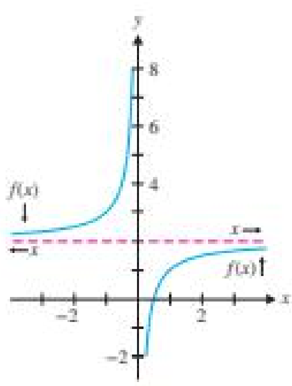


Figure 6.37
$\lim\limits_{x \to \infty} \left(2 - \frac{1}{x}\right) = 2$ and $\lim\limits_{x \to -\infty} \left(2 - \frac{1}{x}\right) = 2$

As you can see in Theorem 5.1, the behavior of $\frac{1}{x^t}$, for any positive rational power t, as $x \to \pm\infty$, is largely the same as we observed for $f(x) = \frac{1}{x}$.

THEOREM 5.1

For any rational number $t > 0$,

$$\lim_{x \to \pm\infty} \frac{1}{x^t} = 0,$$

where for the case where $x \to -\infty$, we assume that $t = \frac{p}{q}$, where q is odd.

REMARK 5.3

All of the usual rules for limits stated in Theorem 3.1 also hold for limits as $x \to \pm\infty$.

A proof of Theorem 5.1 is given in the Appendix. Be sure that the following argument makes sense to you: for $t > 0$, as $x \to \infty$, we also have $x^t \to \infty$, so that $\frac{1}{x^t} \to 0$.

In Theorem 5.2, we see that the behavior of a polynomial at infinity is easy to determine.

THEOREM 5.2

For a polynomial of degree $n > 0$, $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, we have

$$\lim_{x\to\infty} p_n(x) = \begin{cases} \infty, & \text{if } a_n > 0 \\ -\infty, & \text{if } a_n < 0 \end{cases}$$

PROOF

We have

$$\begin{aligned} \lim_{x\to\infty} p_n(x) &= \lim_{x\to\infty} (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0) \\ &= \lim_{x\to\infty} \left[x^n\left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right)\right] \\ &= \infty, \end{aligned}$$

if $a_n > 0$, since

$$\lim_{x\to\infty}\left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right) = a_n$$

and $\lim_{x\to\infty} x^n = \infty$. The result is proved similarly for $a_n < 0$. ■

Observe that you can make similar statements regarding the value of $\lim_{x\to-\infty} p_n(x)$, but be careful: the answer will change depending on whether n is even or odd. (We leave this as an exercise.)

In example 5.7, we again see the need for caution when applying our basic rules for limits (Theorem 3.1), which also apply to limits as $x \to \infty$ or as $x \to -\infty$.

EXAMPLE 5.7 A Limit of a Quotient That Is Not the Quotient of the Limits

Evaluate $\lim_{x\to\infty} \frac{5x-7}{4x+3}$.

Solution You might be tempted to write

$$\lim_{x\to\infty} \frac{5x-7}{4x+3} = \frac{\lim_{x\to\infty}(5x-7)}{\lim_{x\to\infty}(4x+3)}$$

This is an incorrect use of Theorem 3.1, since the limits in the numerator and the denominator do not exist.

$$= \frac{\infty}{\infty} = 1. \qquad (5.3)$$

This is incorrect!

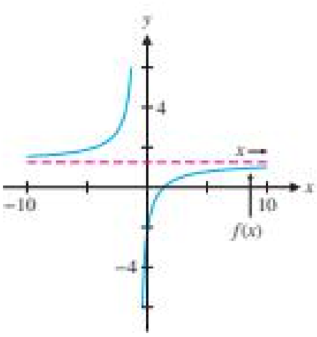


Figure 6.38
$\lim_{x\to\infty} \frac{5x-7}{4x+3} = \frac{5}{4}$

The graph in Figure 6.38 and the accompanying table suggest that the conjectured value of 1 is incorrect. Recall that the limit of a quotient is the quotient of the limits only when *both* limits exist (and the limit in the denominator is non-zero). Since both the limit in the denominator and that in the numerator are infinite, these limits *do not exist.*

It turns out that, when a limit has the form $\frac{\infty}{\infty}$, the actual value of the limit can be anything at all. For this reason, we call $\frac{\infty}{\infty}$ an **indeterminate form**, meaning that the value of the limit cannot be determined solely by noticing that both numerator and denominator tend to ∞.

x	$\frac{5x-7}{4x+3}$
10	1
100	1.223325
1000	1.247315
10,000	1.249731
100,000	1.249973

Rule of Thumb: When faced with the indeterminate form $\frac{\infty}{\infty}$ in calculating the limit of a rational function, divide numerator and denominator by the highest power of x appearing in the *denominator.*

Here, we have

$$\lim_{x\to\infty} \frac{5x-7}{4x+3} = \lim_{x\to\infty}\left[\frac{5x-7}{4x+3}\cdot\frac{(1/x)}{(1/x)}\right]$$

Multiply numerator and denominator by $\frac{1}{x}$.

$$= \lim_{x\to\infty} \frac{5-7/x}{4+3/x}$$

Multiply through by $\frac{1}{x}$.

$$= \frac{\lim_{x\to\infty}(5 - 7/x)}{\lim_{x\to\infty}(4 + 3/x)} \quad \text{By Theorem 3.1 (iv).}$$

$$= \frac{5}{4} = 1.25,$$

which is consistent with what we observed both graphically and numerically. ■

In example 5.8, we apply our rule of thumb to a common limit problem.

EXAMPLE 5.8 Finding Slant Asymptotes

Evaluate $\lim_{x\to\infty} \frac{4x^3 + 5}{-6x^2 - 7x}$ and find any slant asymptotes.

Solution As usual, we first examine a graph. (See Figure 6.39a.) Note that here, the graph appears to tend to $-\infty$ as $x \to \infty$. Further, observe that outside of the interval $[-2, 2]$, the graph looks very much like a straight line. If we look at the graph in a somewhat larger window, this linearity is even more apparent. (See Figure 6.39b.)

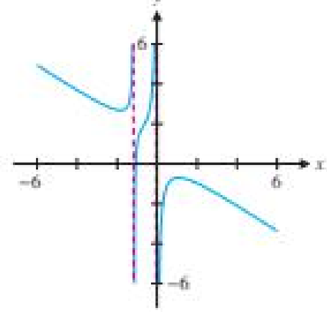


Figure 6.39a
$y = \frac{4x^3 + 5}{-6x^2 - 7x}$

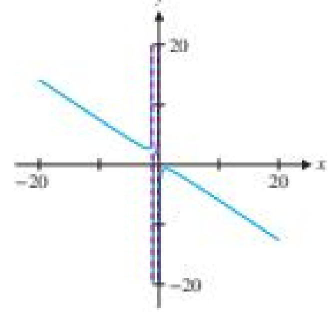


Figure 6.39b
$y = \frac{4x^3 + 5}{-6x^2 - 7x}$

Using our rule of thumb, we have

$$\lim_{x\to\infty} \frac{4x^3 + 5}{-6x^2 - 7x} = \lim_{x\to\infty}\left[\frac{4x^3 + 5}{-6x^2 - 7x} \cdot \frac{(1/x^2)}{(1/x^2)}\right] \quad \text{Multiply numerator and denominator by } \frac{1}{x^2}.$$

$$= \lim_{x\to\infty} \frac{4x + 5/x^2}{-6 - 7/x} \quad \text{Multiply through by } \frac{1}{x^2}.$$

$$= -\infty,$$

since as $x \to \infty$, the numerator tends to ∞ and the denominator tends to -6.

To further explain the behavior seen in Figure 6.39b, we perform a long division:

$$\frac{4x^3 + 5}{-6x^2 - 7x} = -\frac{2}{3}x + \frac{7}{9} + \frac{5 + 49/9x}{-6x^2 - 7x}.$$

Since the third term in this expansion tends to 0 as $x \to \infty$, the function values approach those of the linear function

$$-\frac{2}{3}x + \frac{7}{9},$$

as $x \to \infty$. For this reason, we say that the graph has a **slant (or oblique) asymptote**. That is, instead of approaching a vertical or horizontal line, as happens with vertical or horizontal asymptotes, the graph is approaching the slanted straight line $y = -\frac{2}{3}x + \frac{7}{9}$. (This is the behavior we're seeing in Figure 6.39b.) ■

Limits involving exponential functions are very important in many applications.

EXAMPLE 5.9 Two Limits of an Exponential Function

Evaluate $\lim_{x\to 0^-} e^{1/x}$ and $\lim_{x\to 0^+} e^{1/x}$.

Solution A computer-generated graph is shown in Figure 6.40a. Although it is an unusual looking graph, it appears that the function values are approaching 0, as x approaches 0 from the left and tend to infinity as x approaches 0 from the right. To verify this, recall that $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to -\infty} e^x = 0$. (See Figure 6.40b for a graph of $y = e^x$.) Combining these results, we get

$$\lim_{x\to 0^-} e^{1/x} = 0.$$

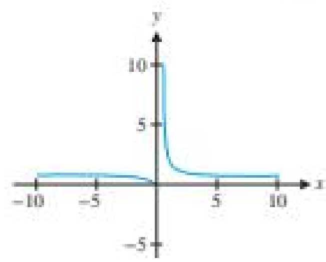


Figure 6.40a
$y = e^{1/x}$

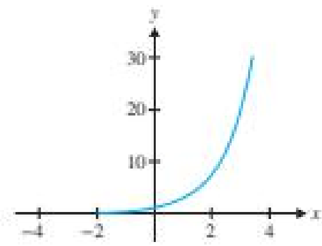


Figure 6.40b
$y = e^x$

Similarly, $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to \infty} e^x = \infty$, so that

$$\lim_{x\to 0^+} e^{1/x} = \infty.$$

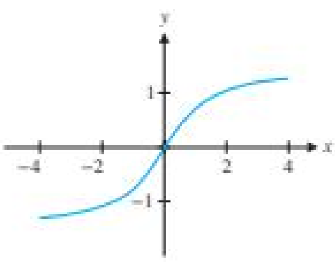


Figure 6.41a
$y = \tan^{-1} x$

As we see in example 5.10, the graphs of some inverse trigonometric functions have horizontal asymptotes.

EXAMPLE 5.10 Two Limits of an Inverse Trigonometric Function

Evaluate $\lim_{x\to \infty} \tan^{-1} x$ and $\lim_{x\to -\infty} \tan^{-1} x$.

Solution The graph of $y = \tan^{-1} x$ (shown in Figure 6.41a) suggests a horizontal asymptote of about $y = -1.5$ as $x \to -\infty$ and about $y = 1.5$ as $x \to \infty$. We can be more precise with this, as follows. For $\lim_{x\to\infty} \tan^{-1} x$, we are looking for the angle that θ must approach, with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, such that $\tan\theta$ tends to ∞. Referring to the graph of $y = \tan x$ in Figure 6.41b, we see that $\tan x$ tends to ∞ as x approaches $\frac{\pi-}{2}$. Likewise, $\tan x$ tends to $-\infty$ as x approaches $-\frac{\pi^+}{2}$, so that

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \quad \text{and} \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}.$$

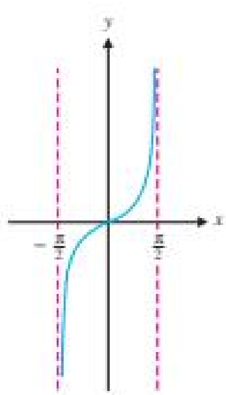


Figure 6.41b
$y = \tan x$

In example 5.11, we consider a model of the size of an animal's pupils. Recall that in bright light, pupils shrink to reduce the amount of light entering the eye, while in dim light, pupils dilate to allow in more light. (See the chapter introduction.)

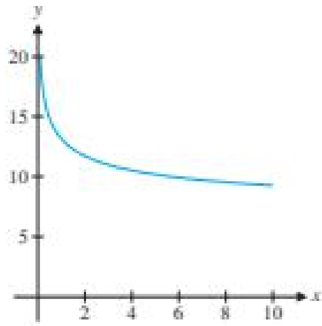


Figure 6.42a
$y = f(x)$

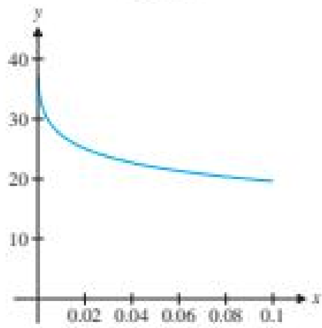


Figure 6.42b
$y = f(x)$

EXAMPLE 5.11 Finding the Size of an Animal's Pupils

Suppose that the diameter in millimeters of an animal's pupils is given by $f(x)$, where x is the intensity of light on the pupils. If $f(x) = \dfrac{160x^{-0.4} + 90}{4x^{-0.4} + 15}$, find the diameter of the pupils with (a) minimum light and (b) maximum light.

Solution Since $f(0)$ is undefined, we consider the limit of $f(x)$ as $x \to 0^+$ (since x cannot be negative). A computer-generated graph of $y = f(x)$ with $0 \le x \le 10$ is shown in Figure 6.42a. It appears that the y-values approach 20 as x approaches 0. We multiply numerator and denominator by $x^{0.4}$, to eliminate the negative exponents, so that

$$\lim_{x\to 0^+} \frac{160x^{-0.4} + 90}{4x^{-0.4} + 15} = \lim_{x\to 0^+} \frac{160x^{-0.4} + 90}{4x^{-0.4} + 15} \cdot \frac{x^{0.4}}{x^{0.4}}$$
$$= \lim_{x\to 0^+} \frac{160 + 90x^{0.4}}{4 + 15x^{0.4}} = \frac{160}{4} = 40 \text{ mm.}$$

This limit does not seem to match our graph. However, in Figure 6.42b, we have zoomed in so that $0 \le x \le 0.1$, making a limit of 40 look more reasonable.

For part (b), we consider the limit as x tends to ∞. From Figure 6.42a, it appears that the graph has a horizontal asymptote at a value somewhat below $y = 10$. We compute the limit

$$\lim_{x\to\infty} \frac{160x^{-0.4} + 90}{4x^{-0.4} + 15} = \frac{90}{15} = 6 \text{ mm.}$$

So, the pupils have a limiting size of 6 mm, as the intensity of light tends to ∞. ■

EXERCISES 6.5

WRITING EXERCISES

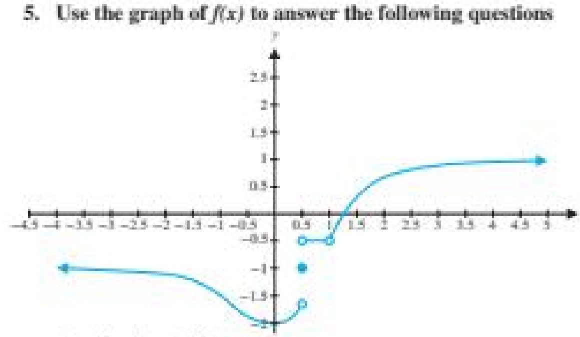
1. It may seem odd that we use ∞ in describing limits but do not count ∞ as a real number. Discuss the existence of ∞: is it a number or a concept?

2. In example 5.7, we dealt with the "indeterminate form" $\frac{\infty}{\infty}$. Thinking of a limit of ∞ as meaning "getting very large" and a limit of 0 as meaning "getting very close to 0," explain why the following are indeterminate forms: $\frac{\infty}{\infty}$, $\frac{0}{0}$, $\infty - \infty$, and $\infty \cdot 0$. Determine what the following non-indeterminate forms represent: $\infty + \infty$, $-\infty - \infty$, $\infty + 0$ and $0/\infty$.

3. On your computer or calculator, graph $y = 1/(x - 2)$ and look for the horizontal asymptote $y = 0$ and the vertical asymptote $x = 2$. Many computers will draw a vertical line at $x = 2$ and will show the graph completely flattening out at $y = 0$ for large x's. Is this accurate? Or misleading? Most computers will compute the locations of points for adjacent x's and try to connect the points with a line segment. Why might this result in a vertical line at the location of a vertical asymptote?

4. Many students learn that asymptotes are lines that the graph gets closer and closer to without ever reaching. This is true for many asymptotes, but not all. Explain why vertical asymptotes are never crossed. Explain why horizontal or slant asymptotes may, in fact, be crossed any number of times; draw one example.

5. **Use the graph of $f(x)$ to answer the following questions**



(a) Evaluate $f(0)$.
(b) Evaluate $\lim_{x\to 0.5^-} f(x)$.
(c) Is the function continuous $f(x)$ at $x = 0.5$? Justify.
(d) Is the function continuous $f(x)$ at $x = 1$? Justify.
(e) Evaluate $\lim_{x\to -\infty} f(x)$.
(f) Evaluate $\lim_{x\to +\infty} f(x)$.

In exercises 1–4, determine (a) $\lim_{x\to a^-} f(x)$ (b) $\lim_{x\to a^+} f(x)$ and (c) $\lim_{x\to a} f(x)$ (answer as appropriate, with a number, ∞, $-\infty$ or does not exist).

1. $f(x) = \dfrac{1 - 2x}{x^2 - 1}$, $a = 1$
2. $f(x) = \dfrac{1 - 2x}{x^2 - 1}$, $a = -1$
3. $f(x) = \dfrac{x - 4}{x^2 - 4x + 4}$, $a = 2$
4. $f(x) = \dfrac{1 - x}{(x + 1)^2}$, $a = -1$

In exercises 5–22, determine each limit (answer as appropriate, with a number, ∞, −∞ or does not exist).

5. $\lim_{x\to -2} \frac{x^2+2x-1}{x^2-4}$

6. $\lim_{x\to -1^+} (x^2-2x-3)^{-2/3}$

7. $\lim_{x\to 0} \cot x$

8. $\lim_{x\to \pi/2} x \sec^2 x$

9. $\lim_{x\to \infty} \frac{x^2+3x-2}{3x^2+4x-1}$

10. $\lim_{x\to \infty} \frac{2x^2-x+1}{4x^2-3x-1}$

11. $\lim_{x\to -\infty} \frac{-x}{\sqrt{4+x^2}}$

12. $\lim_{x\to \infty} \frac{2x^2-1}{4x^3-5x-1}$

13. $\lim_{x\to \infty} \ln\left(\frac{x^2+1}{x-3}\right)$

14. $\lim_{x\to 0^+} \ln(x \sin x)$

15. $\lim_{x\to 0^+} e^{-2/x^3}$

16. $\lim_{x\to \infty} e^{-(x+1)/(x^2+2)}$

17. $\lim_{x\to \infty} \cot^{-1} x$

18. $\lim_{x\to \infty} \sec^{-1} \frac{x^2+1}{x+1}$

19. $\lim_{x\to 0} \sin(e^{-1/x^2})$

20. $\lim_{x\to \infty} \sin(\tan^{-1} x)$

21. $\lim_{x\to \pi/2} e^{-\tan x}$

22. $\lim_{x\to 0^+} \tan^{-1}(\ln x)$

In exercises 23–28, determine all horizontal and vertical asymptotes. For each side of each vertical asymptote, determine whether $f(x)\to\infty$ or $f(x)\to-\infty$.

23. (a) $f(x)=\frac{x}{4-x^2}$ (b) $f(x)=\frac{x^2}{4-x^2}$

24. (a) $f(x)=\frac{x}{\sqrt{4+x^2}}$ (b) $f(x)=\frac{x}{\sqrt{4-x^2}}$

25. $f(x)=\frac{3x^2+1}{x^2-2x-3}$

26. $f(x)=\frac{1-x}{x^2+x-2}$

27. $f(x)=4\tan^{-1}x-1$

28. $f(x)=\ln(1-\cos x)$

In exercises 29–32, determine all vertical and slant asymptotes.

29. $y=\frac{x^3}{4-x^2}$

30. $y=\frac{x^2+1}{x-2}$

31. $y=\frac{x^3}{x^2+x-4}$

32. $y=\frac{x^4}{x^3+2}$

33. Suppose that the size of the pupil, in millimeters, of a certain animal is given by $f(x)$, where x is the intensity of the light on the pupil. If $f(x)=\frac{80x^{-0.3}+60}{2x^{-0.3}+5}$, find the size of the pupil with no light and the size of the pupil with an infinite amount of light.

34. Repeat exercise 33 with $f(x)=\frac{80x^{-0.3}+60}{8x^{-0.3}+15}$.

35. Modify the function in exercise 33 to find a function f such that $\lim_{x\to 0^+} f(x)=8$ and $\lim_{x\to\infty} f(x)=2$.

36. Find a function of the form $f(x)=\frac{20x^{-0.4}+16}{g(x)}$ such that $\lim_{x\to 0^+} f(x)=5$ and $\lim_{x\to\infty} f(x)=4$.

37. Suppose that the velocity of a skydiver t seconds after jumping is given by $v(t)=-\sqrt{\frac{32}{k}}\frac{1-e^{-2t\sqrt{32k}}}{1+e^{-2t\sqrt{32k}}}$. Find the limiting velocity with $k=0.00064$ and $k=0.00128$. By what factor does a skydiver have to change the value of k to cut the limiting velocity in half?

38. Graph the velocity function in exercise 37 with $k=0.00016$ (representing a headfirst dive) and estimate how long it takes for the diver to reach a speed equal to 90% of the limiting velocity. Repeat with $k=0.001$ (representing a spread-eagle position).

In exercises 39–48, use graphical and numerical evidence to conjecture a value for the indicated limit.

39. $\lim_{x\to\infty} \frac{\ln(x+2)}{\ln(x^2+3x+3)}$

40. $\lim_{x\to\infty} \frac{\ln(2+e^{2x})}{\ln(1+e^x)}$

41. $\lim_{x\to\infty} \frac{x^2-4x+7}{2x^2+x\cos x}$

42. $\lim_{x\to -\infty} \frac{2x^3+7x^2+1}{x^3-x\sin x}$

43. $\lim_{x\to\infty} \frac{x^3+4x+5}{e^{x/2}}$

44. $\lim_{x\to\infty} (e^{x/3}-x^4)$

45. $\lim_{x\to 0} \frac{e^x-1}{x}$

46. $\lim_{x\to 0} \frac{\ln x^2}{x^2}$

47. $\lim_{x\to 0^+} x^{1/\ln x}$

48. $\lim_{x\to 0^+} x^{1/x}$

In exercises 49 and 50, use graphical and numerical evidence to conjecture the value of the limit. Then, verify your conjecture by finding the limit exactly.

49. $\lim_{x\to\infty} (\sqrt{4x^2-2x+1}-2x)$ (Hint: Multiply and divide by the conjugate expression: $\sqrt{4x^2-2x+1}+2x$ and simplify.)

50. $\lim_{x\to\infty} (\sqrt{5x^2+4x+7}-\sqrt{5x^2+x+3})$ (See the hint for exercise 49.)

51. Suppose that $f(x)$ is a rational function $f(x)=\frac{p(x)}{q(x)}$ with the degree of $p(x)$ greater than the degree of $q(x)$. Determine whether $y=f(x)$ has a horizontal asymptote.

52. Suppose that $f(x)$ is a rational function $f(x)=\frac{p(x)}{q(x)}$ with the degree (largest exponent) of $p(x)$ less than the degree of $q(x)$. Determine the horizontal asymptote of $y=f(x)$.

53. Suppose that $f(x)$ is a rational function $f(x)=\frac{p(x)}{q(x)}$. If $y=f(x)$ has a slant asymptote $y=x+2$, how does the degree of $p(x)$ compare to the degree of $q(x)$?

54. Suppose that $f(x)$ is a rational function $f(x)=\frac{p(x)}{q(x)}$. If $y=f(x)$ has a horizontal asymptote $y=2$, how does the degree of $p(x)$ compare to the degree of $q(x)$?

55. Find a quadratic function $q(x)$ such that $f(x)=\frac{x^2-4}{q(x)}$ has one horizontal asymptote $y=-\frac{1}{2}$ and exactly one vertical asymptote $x=3$.

56. Find a quadratic function $q(x)$ such that $f(x)=\frac{x^2-4}{q(x)}$ has one horizontal asymptote $y=2$ and two vertical asymptotes $x=\pm 3$.

57. Find a function $g(x)$ such that $f(x)=\frac{x^3-3}{g(x)}$ has no vertical asymptotes and has a slant asymptote $sy=x$.

58. Find a function $g(x)$ such that $f(x)=\frac{x-4}{g(x)}$ has two horizontal asymptotes $y=\pm 1$ and no vertical asymptotes.

In exercises 59–64, label the statement as true or false (not always true) for real numbers a and b.

59. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$, then $\lim_{x\to\infty} [f(x) + g(x)] = a + b$.

60. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$, then $\lim_{x\to\infty} \left[\frac{f(x)}{g(x)}\right] = \frac{a}{b}$.

61. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} [f(x) - g(x)] = 0$.

62. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} [f(x) + g(x)] = \infty$.

63. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} \left[\frac{f(x)}{g(x)}\right] = 0$.

64. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} \left[\frac{f(x)}{g(x)}\right] = 1$.

65. It is very difficult to find simple statements in calculus that are always true; this is one reason that a careful development of the theory is so important. You may have heard the simple rule: to find the vertical asymptotes of $f(x) = \frac{g(x)}{h(x)}$, simply set the denominator equal to 0 [i.e., solve $h(x) = 0$]. Give an example where $h(a) = 0$ but there is *not* a vertical asymptote at $x = a$.

66. (a) State and prove a result analogous to Theorem 5.2 for $\lim_{x\to-\infty} p_n(x)$, for n odd.

(b) State and prove a result analogous to Theorem 5.2 for $\lim_{x\to-\infty} p_n(x)$, for n even.

APPLICATIONS

1. Suppose that the length of a small animal t days after birth is $h(t) = \frac{300}{1 + 9(0.8)^t}$ millimeters. What is the length of the animal at birth? What is the eventual length of the animal (i.e., the length as $t \to \infty$)?

2. Suppose that the length of a small animal t days after birth is $h(t) = \frac{100}{2 + 3(0.4)^t}$ millimeters. What is the length of the animal at birth? What is the eventual length of the animal (i.e., the length as $t \to \infty$)?

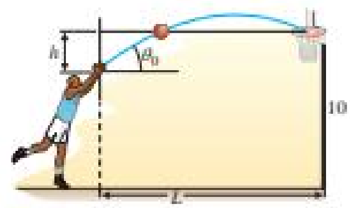
3. Suppose an object with initial velocity $v_0 = 0$ m/s and (constant) mass m kg is accelerated by a constant force of F newtons for t seconds. According to Newton's laws of motion, the object's speed will be $v_N = Ft/m$. According to Einstein's theory of relativity, the object's speed will be $v_E = Fct/\sqrt{m^2c^2 + F^2t^2}$, where c is the speed of light. Compute $\lim_{t\to\infty} v_N$ and $\lim_{t\to\infty} v_E$.

4. After an injection, the concentration of a drug in a muscle varies according to a function of time $f(t)$. Suppose that t is measured in hours and $f(t) = e^{-0.02t} - e^{-0.42t}$. Find the limit of $f(t)$ both as $t \to 0$ and $t \to \infty$, and interpret both limits in terms of the concentration of the drug.

5. Ignoring air resistance, the maximum height reached by a rocket launched with initial velocity v_0 is $h = \frac{v_0^2 R}{19.6R - v_0^2}$ m/s, where R is the radius of the earth. In this exercise, we interpret this as a function of v_0. Explain why the domain of this function must be restricted to $v_0 \ge 0$. There is an additional restriction. Find the (positive) value v_e such that h is undefined. Sketch a possible graph of h with $0 \le v_0 < v_e$ and discuss the significance of the vertical asymptote at v_e. Explain why v_e is called the **escape velocity.**

EXPLORATORY EXERCISES

1. Suppose you are shooting a basketball from a (horizontal) distance of L feet, releasing the ball from a location h feet below the basket. To get a perfect swish, it is necessary that the initial velocity v_0 and initial release angle θ_0 satisfy the equation $v_0 = \sqrt{gL}/\sqrt{2\cos^2\theta_0(\tan\theta_0 - h/L)}$.



For a free throw, take $L = 15$ ft, $h = 2$ ft, and $g = 32$ ft/s^2, and graph v_0 as a function of θ_0. What is the significance of the two vertical asymptotes? Explain in physical terms what type of shot corresponds to each vertical asymptote. Estimate the minimum value of v_0 (call it v_{min}). Explain why it is easier to shoot a ball with a small initial velocity. There is another advantage to this initial velocity. Assume that the basket is 2 feet in diameter and the ball is 1 foot in diameter. For a free throw, $L = 15$ ft is perfect. What is the maximum horizontal distance the ball could travel and still go in the basket (without bouncing off the backboard)? What is the minimum horizontal distance? Call these numbers L_{max} and L_{min}. Find the angle θ_1 corresponding to v_{min} and L_{min} and the angle θ_2 corresponding to v_{min} and L_{max}. The difference $|\theta_2 - \theta_1|$ is the angular margin of error. Brancazio has shown that the angular margin of error for v_{min} is larger than for any other initial velocity.

2. In applications, it is common to compute $\lim_{x\to\infty} f(x)$ to determine the "stability" of the function $f(x)$. Consider the function $f(x) = xe^{-x}$. As $x \to \infty$, the first factor in $f(x)$ goes to ∞, but the second factor goes to 0. What does the product do when one term is getting smaller and the other term is getting larger? It depends on which one is changing faster. What we want to know is which term "dominates." Use graphical and numerical evidence to conjecture the value of $\lim_{x\to\infty}(xe^{-x})$. Which term dominates? In the limit $\lim_{x\to\infty}(x^2e^{-x})$, which term dominates? Also, try $\lim_{x\to\infty}(x^3e^{-x})$. Based on your investigation, is it always true that exponentials dominate polynomials? Try to determine which type of function, polynomials or logarithms, dominates.

6.6 FORMAL DEFINITION OF THE LIMIT

HISTORICAL NOTES

Augustin-Louis Cauchy (1789–1857) A French mathematician who brought rigor to mathematics, including a modern definition of limit. (The ε-δ formulation shown in this section is due to Weierstrass.) Cauchy was one of the most prolific mathematicians in history, making important contributions to number theory, linear algebra, differential equations, astronomy, optics and complex variables. A difficult man to get along with, a colleague wrote, "Cauchy is mad and there is nothing that can be done about him, although right now, he is the only one who knows how mathematics should be done."

Recall that we write

$$\lim_{x\to a} f(x) = L,$$

if $f(x)$ gets closer and closer to L as x gets closer and closer to a. Although intuitive, this description is imprecise, since we do not have a precise definition for what it means to be "close." In this section, however, we will make this more precise and you will begin to see how **mathematical analysis** (that branch of mathematics of which the calculus is the most elementary study) works.

Studying more advanced mathematics without an understanding of the precise definition of limit is somewhat akin to studying brain surgery without bothering with all that background work in chemistry and biology. In medicine, it has only been through a careful examination of the microscopic world that a deeper understanding of our own macroscopic world has developed. Likewise, in mathematical analysis, it is through an understanding of the microscopic behavior of functions (such as the precise definition of limit) that a deeper understanding of the mathematics will come about.

We begin with the careful examination of an elementary example. You should certainly believe that

$$\lim_{x\to 2} (3x + 4) = 10.$$

If asked to explain the meaning of this particular limit to a fellow student, you would probably repeat the intuitive explanation we have used so far: that as x gets closer and closer to 2, $(3x + 4)$ gets arbitrarily close to 10. That is, we should be able to make $(3x + 4)$ as close as we like to 10, just by making x sufficiently close to 2. But can we actually do this? For instance, can we force $(3x + 4)$ to be within distance 1 of 10? To see what values of x will guarantee this, we write an inequality that says that $(3x + 4)$ is within 1 unit of 10:

$$|(3x + 4) - 10| < 1.$$

Eliminating the absolute values, we see that this is equivalent to

$$-1 < (3x + 4) - 10 < 1$$

or

$$-1 < 3x - 6 < 1.$$

Since we need to determine how close x must be to 2, we want to isolate $x - 2$, instead of x. So, dividing by 3, we get

$$-\frac{1}{3} < x - 2 < \frac{1}{3}$$

or

$$|x - 2| < \frac{1}{3}. \tag{6.1}$$

Reversing the steps that lead to inequality (6.1), we see that if x is within distance $\frac{1}{3}$ of, 2, then $(3x + 4)$ will be within the specified distance (1) of 10. (See Figure 6.43 for a graphical interpretation of this.) So, does this convince you that you can make $(3x + 4)$ as close as you want to 10? Probably not, but if you used a smaller distance, perhaps you'd be more convinced.

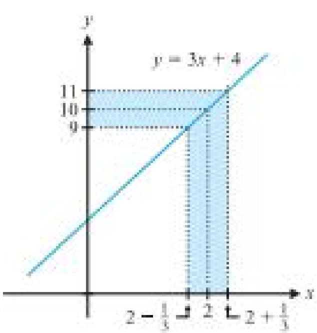


FIGURE 6.43
$2 - \frac{1}{3} < x < 2 + \frac{1}{3}$ guarantees that $|(3x + 4) - 10| < 1$

EXAMPLE 6.1 Exploring a Simple Limit

Find the values of x for which $(3x + 4)$ is within distance $\frac{1}{100}$ of 10.

Solution We want

$$|(3x + 4) - 10| < \frac{1}{100}.$$

Eliminating the absolute values, we get

$$-\frac{1}{100} < (3x+4) - 10 < \frac{1}{100}$$

or $$-\frac{1}{100} < 3x - 6 < \frac{1}{100}.$$

Dividing by 3 yields $$-\frac{1}{300} < x - 2 < \frac{1}{300},$$

which is equivalent to $$|x-2| < \frac{1}{300}.$$ ■

While we showed in example 6.1 that we can make $(3x + 4)$ reasonably close to 10, how close do we need to be able to make it? The answer is *arbitrarily close*, as close as anyone would ever demand. We can accomplish this by repeating the arguments in example 6.1, this time for an unspecified distance, call it ε (*epsilon*, where $\varepsilon > 0$).

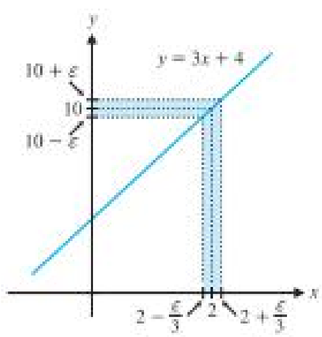


Figure 6.44
The range of x-values that keep $|(3x+4) - 10| < \varepsilon$

EXAMPLE 6.2 Verifying a Limit

Show that we can make $(3x + 4)$ within any specified distance $\varepsilon > 0$ of 10 (no matter how small ε is), just by making x sufficiently close to 2.

Solution The objective is to determine the range of x-values that will guarantee that $(3x + 4)$ stays within ε of 10. (See Figure 6.44 for a sketch of this range.) We have

$$|(3x+4) - 10| < \varepsilon,$$

which is equivalent to $$-\varepsilon < (3x+4) - 10 < \varepsilon$$

or $$-\varepsilon < 3x - 6 < \varepsilon.$$

Dividing by 3, we get $$-\frac{\varepsilon}{3} < x - 2 < \frac{\varepsilon}{3}$$

or $$|x-2| < \frac{\varepsilon}{3}.$$

Notice that each of the preceding steps is reversible, so that $|x - 2| < \frac{\varepsilon}{3}$ also implies that $|(3x + 4) - 10| < \varepsilon$. This says that as long as x is within distance $\frac{\varepsilon}{3}$ of 2, $(3x + 4)$ will be within the required distance ε of 10. That is,

$$|(3x+4) - 10| < \varepsilon \text{ whenever } |x-2| < \frac{\varepsilon}{3}.$$ ■

Take a moment or two to recognize what we've done in example 6.2. By using an *unspecified* distance, ε, we have verified that we can indeed make $(3x + 4)$ as close to 10 as might be demanded (i.e., arbitrarily close; just name whatever $\varepsilon > 0$ you would like), simply by making x sufficiently close to 2. Further, we have explicitly spelled out what "sufficiently close to 2" means in the context of the present problem. Thus, no matter how close we are asked to make $(3x + 4)$ to 10, we can accomplish this simply by taking x to be in the specified interval.

Next, we examine this more precise notion of limit in the case of a function that is not defined at the point in question.

EXAMPLE 6.3 Proving That a Limit Is Correct

Prove that $\lim_{x \to 1} \frac{2x^2 + 2x - 4}{x - 1} = 6$.

Solution It is easy to use the usual rules of limits to establish this result. It is yet another matter to verify that this is correct using our new and more precise notion of limit. In this case, we want to know how close x must be to 1 to ensure that

$$f(x) = \frac{2x^2 + 2x - 4}{x - 1}$$

is within an unspecified distance $\varepsilon > 0$ of 6.

First, notice that f is undefined *at* $x = 1$. So, we seek a distance δ (*delta*, $\delta > 0$), such that if x is within distance δ of 1, but $x \neq 1$ (i.e., $0 < |x - 1| < \delta$), then this guarantees that $|f(x) - 6| < \varepsilon$.

Notice that we have specified that $0 < |x - 1|$ to ensure that $x \neq 1$. Further, $|f(x) - 6| < \varepsilon$ is equivalent to

$$-\varepsilon < \frac{2x^2 + 2x - 4}{x - 1} - 6 < \varepsilon.$$

Finding a common denominator and subtracting in the middle term, we get

$$-\varepsilon < \frac{2x^2 + 2x - 4 - 6(x - 1)}{x - 1} < \varepsilon \text{ or } -\varepsilon < \frac{2x^2 - 4x + 2}{x - 1} < \varepsilon.$$

Since the numerator factors, this is equivalent to

$$-\varepsilon < \frac{2(x - 1)^2}{x - 1} < \varepsilon.$$

Since $x \neq 1$, we can cancel two of the factors of $(x - 1)$ to yield

$$-\varepsilon < 2(x - 1) < \varepsilon$$

or

$$-\frac{\varepsilon}{2} < x - 1 < \frac{\varepsilon}{2}, \quad \text{Dividing by 2.}$$

which is equivalent to $|x - 1| < \varepsilon/2$. So, taking $\delta = \varepsilon/2$ and working backward, we see that requiring x to satisfy

$$0 < |x - 1| < \delta = \frac{\varepsilon}{2}$$

will guarantee that

$$\left| \frac{2x^2 + 2x - 4}{x - 1} - 6 \right| < \varepsilon.$$

We illustrate this graphically in Figure 6.45. ■

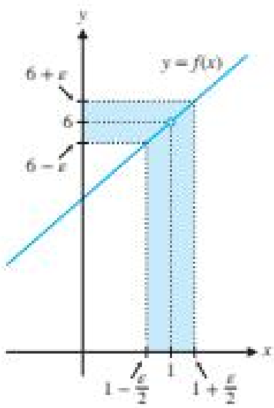


Figure 6.45
$0 < |x - 1| < \frac{\varepsilon}{2}$ guarantees that $6 - \varepsilon < \frac{2x^2 + 2x - 4}{x - 1} < 6 + \varepsilon$.

What we have seen so far motivates us to make the following general definition, illustrated in Figure 6.46.

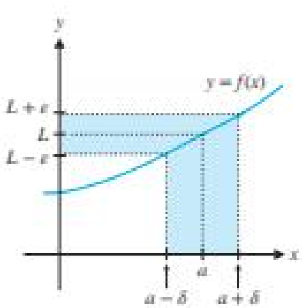


FIGURE 6.46
$a - \delta < x < a + \delta$ guarantees that $L - \varepsilon < f(x) < L + \varepsilon$.

DEFINITION 6.1 (Precise Definition of Limit)

For a function f defined in some open interval containing a (but not necessarily *at* a itself), we say

$$\lim_{x \to a} f(x) = L,$$

if given any number $\varepsilon > 0$, there is another number $\delta > 0$, such that $0 < |x - a| < \delta$ guarantees that $|f(x) - L| < \varepsilon$.

Notice that example 6.2 amounts to an illustration of Definition 6.1 for $\lim_{x\to 2}(3x+4)$. There, we found that $\delta = \varepsilon/3$ satisfies the definition.

REMARK 6.1

We want to emphasize that this formal definition of limit is not a new idea. Rather, it is a more precise mathematical statement of the intuitive notion of limit that we introduced in Section 6.2. Also, we should in all honesty point out that it is rather difficult to explicitly find δ as a function of ε, for all but a few simple examples. Despite this, learning how to work through the definition, even for a small number of problems, will shed considerable light on a deep concept.

Example 6.4, although only slightly more complex than the last several problems, provides an unexpected challenge.

EXAMPLE 6.4 Using the Precise Definition of Limit

Use Definition 6.1 to prove that $\lim_{x\to 2} x^2 = 4$.

Solution If this limit is correct, then given any $\varepsilon > 0$, there must be a $\delta > 0$ for which $0 < |x-2| < \delta$ guarantees that

$$|x^2 - 4| < \varepsilon.$$

Notice that

$$|x^2 - 4| = |x+2||x-2|. \tag{6.2}$$

Factoring the difference of two squares

Since we're interested only in what happens near $x = 2$, we assume that x lies in the interval $[1, 3]$. In this case, we have

$$|x+2| \le 5,$$

Since $x \in [1, 3]$

and so, from (6.2),

$$\begin{aligned} |x^2 - 4| &= |x+2||x-2| \\ &\le 5|x-2|. \end{aligned} \tag{6.3}$$

Finally, if we require that

$$5|x-2| < \varepsilon, \tag{6.4}$$

then we will also have from (6.3) that

$$|x^2 - 4| \le 5|x-2| < \varepsilon.$$

Of course, (6.4) is equivalent to

$$|x-2| < \frac{\varepsilon}{5}.$$

In view of this, we now have two restrictions: that $|x-2| < 1$ *and* that $|x-2| < \frac{\varepsilon}{5}$. To ensure that both restrictions are met, we choose $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$ $\left(\text{i.e., the minimum of 1 and } \frac{\varepsilon}{5}\right)$. Working backward (see the margin), we get that for this choice of δ,

$$0 < |x-2| < \delta$$

will guarantee that

$$|x^2 - 4| < \varepsilon,$$

as desired. We illustrate this in Figure 6.47. ■

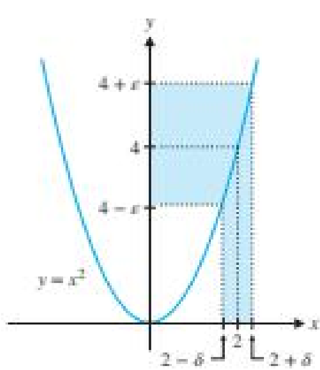


Figure 6.47
$0 < |x-2| < \delta$ guarantees that $|x^2 - 4| < \varepsilon$.

PROOF

Let $\varepsilon > 0$ be arbitrary. Define $\delta = \min\{1, \frac{\varepsilon}{5}\}$. If $0 < |x-2| < \delta$, then $|x-2| < 1$, $-1 < x < 3$ and $|x+2| < 5$. Also, $|x-2| < \frac{\varepsilon}{5}$. Then $|x^2-4| = |x-2| \cdot |x+2| < \frac{\varepsilon}{5}(5) = \varepsilon$. ■

The work presented in the text above shows how to determine a value for δ. The formal proof for the limit should follow the steps shown in the margin.

Exploring the Definition of Limit Graphically

As you can see from example 6.4, finding a δ for a given ε is not always easily accomplished. However, we can explore the definition graphically for any function. First, we reexamine example 6.4 graphically.

EXAMPLE 6.5 Exploring the Precise Definition of Limit Graphically

Explore the precise definition of limit graphically, for $\lim_{x\to 2} x^2 = 4$.

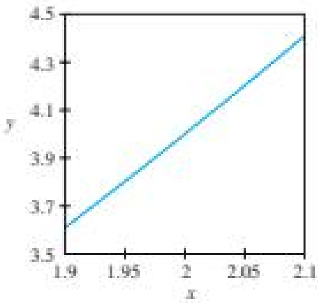


Figure 6.48
$y = x^2$

Solution In example 6.4, we discovered that for $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$,

$$0 < |x - 2| < \delta \text{ implies that } |x^2 - 4| < \varepsilon.$$

This says that (for $\varepsilon \le 5$) if we draw a graph of $y = x^2$ and restrict the x-values to lie in the interval $\left(2 - \frac{\varepsilon}{5}, 2 + \frac{\varepsilon}{5}\right)$, then the y-values will lie in the interval $(4 - \varepsilon, 4 + \varepsilon)$. Take $\varepsilon = \frac{1}{2}$, for instance. If we draw the graph in the window defined by $2 - \frac{1}{10} \le x \le 2 + \frac{1}{10}$ and $3.5 \le y \le 4.5$, then the graph will not run off the top or bottom of the screen. (See Figure 6.48.) Of course, we can draw virtually the same picture for any given value of ε, since we have an explicit formula for finding δ given ε. For most limit problems, we are not so fortunate. ■

EXAMPLE 6.6 Exploring the Definition of Limit for a Trigonometric Function

Graphically find a $\delta > 0$ corresponding to (a) $\varepsilon = \frac{1}{2}$ and (b) $\varepsilon = 0.1$ for $\lim_{x\to 2} \sin\frac{\pi x}{2} = 0$.

Solution This limit seems plausible enough. After all, $\sin\frac{2\pi}{2} = 0$ and $f(x) = \sin x$ is a continuous function. However, the point is to verify this carefully. Given any $\varepsilon > 0$, we want to find a $\delta > 0$, for which

$$0 < |x - 2| < \delta \text{ guarantees that } \left|\sin\frac{\pi x}{2} - 0\right| < \varepsilon.$$

Note that since we have no algebra for simplifying $\sin\frac{\pi x}{2}$, we cannot accomplish this symbolically. Instead, we'll try to graphically find δ's corresponding to the specific ε's given. First, for $\varepsilon = \frac{1}{2}$, we would like to find a $\delta > 0$ for which if $0 < |x - 2| < \delta$, then

$$-\frac{1}{2} < \sin\frac{\pi x}{2} - 0 < \frac{1}{2}.$$

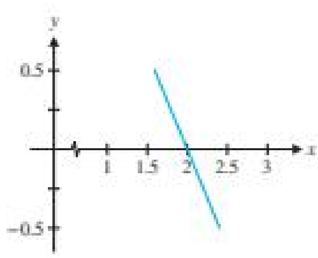


Figure 6.49a
$y = \sin\frac{\pi x}{2}$

Drawing the graph of $y = \sin\frac{\pi x}{2}$ with $1 \le x \le 3$ and $-\frac{1}{2} \le y \le \frac{1}{2}$, we get Figure 6.49a.

If you trace along a calculator or computer graph, you will notice that the graph stays on the screen (i.e., the y-values stay in the interval $[-0.5, 0.5]$) for $x \in [1.666667, 2.333333]$. Thus, we have determined experimentally that for $\varepsilon = \frac{1}{2}$,

$$\delta = 2.333333 - 2 = 2 - 1.666667 = 0.333333$$

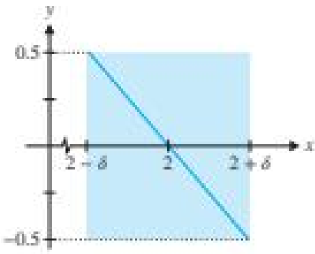


Figure 6.49b

$y = \sin \frac{\pi x}{2}$

will work. (Of course, any value of δ smaller than 0.333333 will also work.) To illustrate this, we redraw the last graph, but restrict x to lie in the interval [1.67, 2.33]. (See Figure 6.49b.) In this case, the graph stays in the window over the entire range of displayed x-values. Taking $\varepsilon = 0.1$, we look for an interval of x-values that will guarantee that $\sin \frac{\pi x}{2}$ stays between -0.1 and 0.1. We redraw the graph from Figure 6.49a, with the y-range restricted to the interval $[-0.1, 0.1]$. (See Figure 6.50a.) Again, tracing along the graph tells us that the y-values will stay in the desired range for $x \in [1.936508, 2.063492]$. Thus, we have experimentally determined that

$$\delta = 2.063492 - 2 = 2 - 1.936508 = 0.063492$$

will work here. We redraw the graph using the new range of x-values (see Figure 6.50b), since the graph remains in the window for all values of x in the indicated interval.

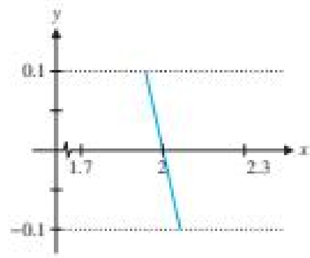


Figure 6.50a

$y = \sin \frac{\pi x}{2}$

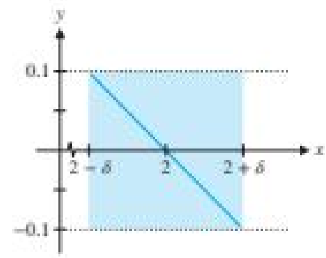


Figure 6.50b

$y = \sin \frac{\pi x}{2}$

It is important to recognize that we are not *proving* that the above limit is correct. To prove this requires us to symbolically find a δ for *every* $\varepsilon > 0$. The idea here is to use these graphical illustrations to become more familiar with the definition and with what δ and ε represent. ■

x	$\frac{x^2 + 2x}{\sqrt{x^3 + 4x^2}}$
0.1	1.03711608
0.01	1.0037461
0.001	1.00037496
0.0001	1.0000375

EXAMPLE 6.7 Exploring the Definition of Limit Where the Limit Does Not Exist

Determine whether or not $\lim_{x \to 0} \frac{x^2 + 2x}{\sqrt{x^3 + 4x^2}} = 1$.

Solution We first construct a table of function values. From the table alone, we might be tempted to conjecture that the limit is 1. However, we would be making a *huge* error, as we have not considered negative values of x or drawn a graph. Figure 6.51a shows the default graph drawn by our computer algebra system. In this graph, the function values do not quite look like they are approaching 1 as $x \to 0$ (at least as $x \to 0^-$). We now investigate the limit graphically for $\varepsilon = \frac{1}{2}$. Here, we need to find a $\delta > 0$ for which $0 < |x| < \delta$ guarantees that

$$1 - \frac{1}{2} < \frac{x^2 + 2x}{\sqrt{x^3 + 4x^2}} < 1 + \frac{1}{2}$$

or

$$\frac{1}{2} < \frac{x^2 + 2x}{\sqrt{x^3 + 4x^2}} < \frac{3}{2}.$$

TODAY IN MATHEMATICS

Paul Halmos (1916–2006)
A Hungarian-born mathematician who earned a reputation as one of the best mathematical writers ever. For Halmos, calculus did not come easily, with understanding coming in a flash of inspiration only after a long period of hard work. "I remember standing at the blackboard in Room 213 of the mathematics building with Warren Ambrose and suddenly I understood epsilons. I understood what limits were, and all of that stuff that people had been drilling into me became clear. . . . I could prove the theorems. That afternoon I became a mathematician."[1]

[1] Albers, D.J. and Alexanderson, G.L. (1985). *Mathematical People: Profiles and Interviews* (Chicago: Contemporary Books).

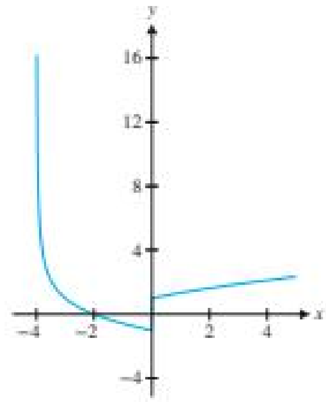


Figure 6.51a
$y = \dfrac{x^2+2x}{\sqrt{x^3+4x^2}}$

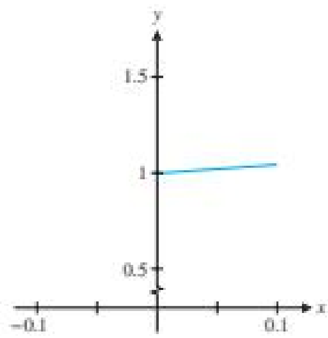


Figure 6.51b
$y = \dfrac{x^2+2x}{\sqrt{x^3+4x^2}}$

We try $\delta = 0.1$ to see if this is sufficiently small. So, we set the x-range to the interval $[-0.1, 0.1]$ and the y-range to the interval $[0.5, 1.5]$ and redraw the graph in this window. (See Figure 6.51b.) Notice that no points are plotted in the window for any $x < 0$. According to the definition, the y-values must lie in the interval $(0.5, 1.5)$ for *all* x in the interval $(-\delta, \delta)$. Further, you can see that $\delta = 0.1$ clearly does not work since $x = -0.05$ lies in the interval $(-\delta, \delta)$, but $f(-0.05) \approx -0.981$ is not in the interval $(0.5, 1.5)$. You should convince yourself that no matter how small you make δ, there is an x in the interval $(-\delta, \delta)$ such that $f(x) \notin (0.5, 1.5)$. (In fact, notice that for all x's in the interval $(-1, 0)$, $f(x) < 0$.) That is, there is no choice of δ that makes the defining inequality true for $\varepsilon = \frac{1}{2}$. Thus, the conjectured limit of 1 is incorrect.

You should note here that, while we've only shown that the limit is not 1, it's somewhat more complicated to show that the limit does not exist. ■

Limits Involving Infinity

Recall that we write

$$\lim_{x \to a} f(x) = \infty,$$

whenever the function increases without bound as $x \to a$. That is, we can make $f(x)$ as large as we like, simply by making x sufficiently close to a. So, given any large positive number, M, we must be able to make $f(x) > M$, for x sufficiently close to a. This leads us to the following definition:

DEFINITION 6.2

For a function f defined in some open interval containing a (but not necessarily at a itself), we say

$$\lim_{x \to a} f(x) = \infty,$$

if given any number $M > 0$, there is another number $\delta > 0$, such that $0 < |x - a| < \delta$ guarantees that $f(x) > M$. (See Figure 6.52 for a graphical interpretation of this.)

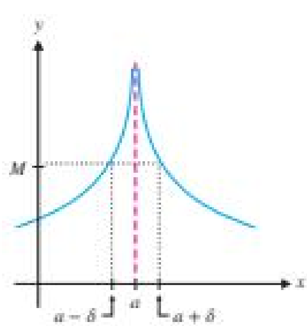


FIGURE 6.52
$\lim_{x \to a} f(x) = \infty$

Similarly, we had said that if $f(x)$ decreases without bound as $x \to a$, then $\lim_{x \to a} f(x) = -\infty$. Think of how you would make this more precise and then consider the following definition:

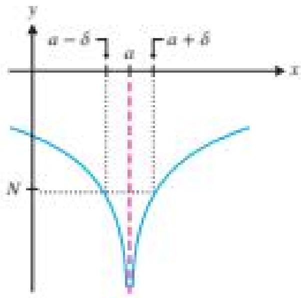


FIGURE 6.53
$\lim_{x \to a} f(x) = -\infty$

DEFINITION 6.3

For a function f defined in some open interval containing a (but not necessarily at a itself), we say

$$\lim_{x \to a} f(x) = -\infty,$$

if given any number $N < 0$, there is another number $\delta > 0$, such that $0 < |x - a| < \delta$ guarantees that $f(x) < N$. (See Figure 6.53 for a graphical interpretation of this.)

It's easy to keep these definitions straight if you think of their meaning. Don't simply memorize them.

EXAMPLE 6.8 Using the Definition of Limit Where the Limit Is Infinite

Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution Given any (large) number $M > 0$, we need to find a distance $\delta > 0$ such that if x is within δ of 0 (but not equal to 0) then

$$\frac{1}{x^2} > M. \tag{6.5}$$

Since both M and x^2 are positive, (6.5) is equivalent to

$$x^2 < \frac{1}{M}.$$

Taking the square root of both sides and recalling that $\sqrt{x^2} = |x|$, we get

$$|x| < \sqrt{\frac{1}{M}}.$$

So, for any $M > 0$, if we take $\delta = \sqrt{\frac{1}{M}}$ and work backward, we have that $0 < |x - 0| < \delta$ guarantees that

$$\frac{1}{x^2} > M,$$

as desired. Note that this says, for instance, that for $M = 100, \frac{1}{x^2} > 100$, whenever $0 < |x| < \sqrt{\frac{1}{100}} = \frac{1}{10}$. (Verify that this works, as an exercise.) ■

There are two remaining limits that we have yet to place on a careful footing. Before reading on, try to figure out for yourself what appropriate definitions would look like.

If we write $\lim_{x \to \infty} f(x) = L$, we mean that as x increases without bound, $f(x)$ gets closer and closer to L. That is, we can make $f(x)$ as close to L as we like, by choosing x sufficiently large. More precisely, we have the following definition.

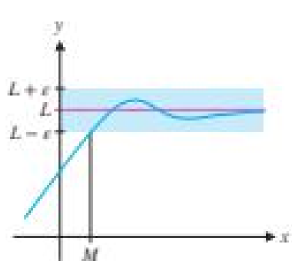


FIGURE 6.54
$\lim_{x \to \infty} f(x) = L$

DEFINITION 6.4

For a function f defined on an interval (a, ∞), for some $a > 0$, we say

$$\lim_{x \to \infty} f(x) = L,$$

if given any $\varepsilon > 0$, there is a number $M > 0$ such that $x > M$ guarantees that

$$|f(x) - L| < \varepsilon.$$

(See Figure 6.54 for a graphical interpretation of this.)

Similarly, we have said that $\lim_{x\to-\infty} f(x) = L$ means that as x decreases without bound, $f(x)$ gets closer and closer to L. So, we should be able to make $f(x)$ as close to L as desired, just by making x sufficiently large in absolute value and negative. We have the following definition.

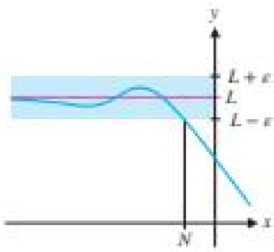


FIGURE 6.55
$\lim_{x\to-\infty} f(x) = L$

DEFINITION 6.5

For a function f defined on an interval $(-\infty, a)$, for some $a < 0$, we say

$$\lim_{x\to-\infty} f(x) = L,$$

if given any $\varepsilon > 0$, there is a number $N < 0$ such that $x < N$ guarantees that

$$|f(x) - L| < \varepsilon.$$

(See Figure 6.55 for a graphical interpretation of this.)

We use Definitions 6.4 and 6.5 essentially the same as we do Definitions 6.1–6.3, as we see in example 6.9.

REMARK 6.2

You should take care to note the commonality among the definitions of the five limits we have given. All five deal with a precise description of what it means to be "close." It is of considerable benefit to work through these definitions until you can provide your own words for each. Don't just memorize the formal definitions as stated here. Rather, work toward understanding what they mean and come to appreciate the exacting language that mathematicians use.

EXAMPLE 6.9 Using the Definition of Limit Where x Is Becoming Infinite

Prove that $\lim_{x\to-\infty} \frac{1}{x} = 0$.

Solution Here, we must show that given any $\varepsilon > 0$, we can make $\frac{1}{x}$ within ε of 0, simply by making x sufficiently large in absolute value and negative. So, we need to determine those x's for which

$$\left|\frac{1}{x} - 0\right| < \varepsilon$$

or

$$\left|\frac{1}{x}\right| < \varepsilon. \tag{6.6}$$

Since $x < 0$, $|x| = -x$, and so (6.6) becomes

$$\frac{1}{-x} < \varepsilon.$$

Dividing both sides by ε and multiplying by x (remember that $x < 0$ and $\varepsilon > 0$, so that this will change the direction of the inequality), we get

$$-\frac{1}{\varepsilon} > x.$$

So, if we take $N = -\frac{1}{\varepsilon}$ and work backward, we have satisfied the definition and thereby proved that the limit is correct. ■

We don't use the limit definitions to prove each and every limit that comes along. Actually, we use them to prove only a few basic limits and to prove the limit theorems that we've been using for some time without proof. Further use of these theorems then provides solid justification of new limits. As an illustration, we now prove the rule for a limit of a sum.

THEOREM 6.1

Suppose that for a real number a, $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$. Then,

$$\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2.$$

PROOF

Since $\lim_{x\to a} f(x) = L_1$, we know that given any number $\varepsilon_1 > 0$, there is a number $\delta_1 > 0$ for which

$$0 < |x - a| < \delta_1 \text{ guarantees that } |f(x) - L_1| < \varepsilon_1. \tag{6.7}$$

Likewise, since $\lim_{x\to a} g(x) = L_2$, we know that given any number $\varepsilon_2 > 0$, there is a number $\delta_2 > 0$ for which

$$0 < |x - a| < \delta_2 \text{ guarantees that } |g(x) - L_2| < \varepsilon_2. \tag{6.8}$$

Now, in order to get

$$\lim_{x\to a} [f(x) + g(x)] = (L_1 + L_2),$$

we must show that, given any number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \text{ guarantees that } |[f(x) + g(x)] - (L_1 + L_2)| < \varepsilon.$$

Notice that

$$\begin{aligned} |[f(x) + g(x)] - (L_1 + L_2)| &= |[f(x) - L_1] + [g(x) - L_2]| \\ &\le |f(x) - L_1| + |g(x) - L_2|, \end{aligned} \tag{6.9}$$

by the triangle inequality. Of course, both terms on the right-hand side of (6.9) can be made arbitrarily small, from (6.7) and (6.8). In particular, if we take $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$, then as long as

$$0 < |x - a| < \delta_1 \text{ and } 0 < |x - a| < \delta 2,$$

we get from (6.7), (6.8) and (6.9) that

$$\begin{aligned} |[f(x) + g(x)] - (L_1 + L_2)| &\le |f(x) - L_1| + |g(x) - L_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

as desired. Of course, this will happen if we take

$$0 < |x - a| < \delta = \min\{\delta_1, \delta_2\}. \blacksquare$$

The other rules for limits are proven similarly. We show these in the Appendix.

EXERCISES 6.6

WRITING EXERCISES

1. In his 1687 masterpiece *Mathematical Principles of Natural Philosophy*, which introduces many of the fundamentals of calculus, Sir Isaac Newton described the important limit $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ (which we study at length in Chapter 7) as "the limit to which the ratios of quantities decreasing without limit do always converge, and to which they approach nearer than by any given difference, but never go beyond, nor ever reach until the quantities vanish." If you ever get weary of all the notation that we use in calculus, think of what it would look like in words! Critique Newton's definition of limit, addressing the following questions in the process. What restrictions do the phrases "never go beyond" and "never reach" put on the limit process? Give an example of a simple limit, not necessarily of the form $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$, that violates these restrictions. Give your own (English-language) description of the limit, avoiding restrictions such as Newton's. Why do mathematicians consider the $\varepsilon - \delta$ definition simple and elegant?
2. You have computed numerous limits before seeing the definition of limit. Explain how this definition changes and/or improves your understanding of the limit process.
3. Each word in the $\varepsilon - \delta$ definition is carefully chosen and precisely placed. Describe what is wrong with each of the following slightly incorrect "definitions" (use examples!):
   (a) There exists $\varepsilon > 0$ such that there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.
   (b) For all $\varepsilon > 0$ and for all $\delta > 0$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.
   (c) For all $\delta > 0$ there exists $\varepsilon > 0$ such that $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.
4. In order for the limit to exist, given every $\varepsilon > 0$, we must be able to find a $\delta > 0$ such that the if/then inequalities are true. To prove that the limit does not exist, we must find a particular $\varepsilon > 0$ such that the if/then inequalities are not true for any choice of $\delta > 0$. To understand the logic behind the

swapping of the "for every" and "there exists" roles, draw an analogy with the following situation. Suppose the statement, "Everybody loves somebody" is true. If you wanted to verify the statement, why would you have to talk to every person on earth? But, suppose that the statement is not true. What would you have to do to disprove it?

In exercises 1–3, use the given to find a value for $\delta > 0$ that works for $\varepsilon = 0.1$

1. $\lim_{x\to 10} \sqrt{x-1} = 3$
2. $\lim_{x\to 2} \sqrt{4x+1} = 3$
3. $\lim_{x\to 1} \sqrt{17-x} = 4$

In exercises 4–15, symbolically find δ in terms of ε.

4. $\lim_{x\to 0} 3x = 0$
5. $\lim_{x\to 1} 3x = 3$
6. $\lim_{x\to 2} (3x+2) = 8$
7. $\lim_{x\to 1} (3x+2) = 5$
8. $\lim_{x\to 1} (3-4x) = -1$
9. $\lim_{x\to -1} (3-4x) = 7$
10. $\lim_{x\to 1} \dfrac{x^2+x-2}{x-1} = 3$
11. $\lim_{x\to -1} \dfrac{x^2-1}{x+1} = -2$
12. $\lim_{x\to 1} (x^2-1) = 0$
13. $\lim_{x\to 1} (x^2-x+1) = 1$
14. $\lim_{x\to 2} (x^2-1) = 3$
15. $\lim_{x\to 0} (x^3+1) = 1$

16. Determine a formula for δ in terms of ε for $\lim_{x\to a} (mx+b)$. (Hint: Use exercises 4–9.) Does the formula depend on the value of a? Try to explain this answer graphically.

17. Based on exercises 12 and 14, does the value of δ depend on the value of a for $\lim_{x\to a} (x^2+b)$? Try to explain this graphically.

In exercises 18–21, numerically and graphically determine a δ corresponding to (a) $\varepsilon = 0.1$ and (b) $\varepsilon = 0.05$. Graph the function in the $\varepsilon - \delta$ window [x-range is $(a-\delta, a+\delta)$ and y-range is $(L-\varepsilon, L+\varepsilon)$] to verify that your choice works.

18. $\lim_{x\to 0} (x^2+1) = 1$
19. $\lim_{x\to 0} \cos x = 1$
20. $\lim_{x\to 1} \sqrt{x+3} = 2$
21. $\lim_{x\to 1} \dfrac{x+2}{x^2} = 3$

22. Modify the $\varepsilon - \delta$ definition to define the one-sided limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$.

23. Symbolically find the largest δ corresponding to $\varepsilon = 0.1$ in the definition of $\lim_{x\to 1^-} 1/x = 1$. Symbolically find the largest δ corresponding to $\varepsilon = 0.1$ in the definition of $\lim_{x\to 1^+} 1/x = 1$. Which δ could be used in the definition of $\lim_{x\to 1} 1/x = 1$? Briefly explain. Then prove that $\lim_{x\to 1} 1/x = 1$.

In exercises 24 and 25, find a δ corresponding to $M = 100$ or $N = -100$ (as appropriate) for each limit.

24. (a) $\lim_{x\to 1^+} \dfrac{2}{x-1} = \infty$ (b) $\lim_{x\to 1^-} \dfrac{2}{x-1} = -\infty$
25. (a) $\lim_{x\to 0^+} \cot x = \infty$ (b) $\lim_{x\to \pi^-} \cot x = -\infty$

In exercises 26–29, find an M or N corresponding to $\varepsilon = 0.1$ for each limit at infinity.

26. $\lim_{x\to\infty} \dfrac{x^2-2}{x^2+x+1} = 1$
27. $\lim_{x\to\infty} \dfrac{e^x+x}{e^x-x^2} = 1$
28. $\lim_{x\to-\infty} \dfrac{x^2+3}{4x^2-4} = 0.25$
29. $\lim_{x\to-\infty} \dfrac{3x^2-2}{x^2+1} = 3$

In exercises 30–35, prove that the limit is correct using the appropriate definition (assume that k is an integer).

30. $\lim_{x\to\infty} \left(\dfrac{1}{x^2+2} - 3\right) = -3$
31. $\lim_{x\to\infty} \dfrac{1}{(x-7)^2} = 0$
32. $\lim_{x\to -3} \dfrac{-2}{(x+3)^4} = -\infty$
33. $\lim_{x\to 7} \dfrac{3}{(x-7)^2} = \infty$
34. $\lim_{x\to\infty} \dfrac{1}{x^k} = 0$, for $k > 0$
35. $\lim_{x\to-\infty} \dfrac{1}{x^{2k}} = 0$, for $k > 0$

In exercises 36–39, identify a specific $\varepsilon > 0$ for which no $\delta > 0$ exists to satisfy the definition of limit.

36. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2+3 & \text{if } x > 1 \end{cases}$, $\lim_{x\to 1} f(x) \neq 2$
37. $f(x) = \begin{cases} x^2-1 & \text{if } x < 0 \\ -x-2 & \text{if } x > 0 \end{cases}$, $\lim_{x\to 0} f(x) \neq -2$
38. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 5-x^2 & \text{if } x > 1 \end{cases}$, $\lim_{x\to 1} f(x) \neq 2$
39. $f(x) = \begin{cases} x-1 & \text{if } x < 2 \\ x^2 & \text{if } x > 2 \end{cases}$, $\lim_{x\to 2} f(x) \neq 1$

40. Prove Theorem 3.1 (i).

41. Prove Theorem 3.1 (ii).

42. Prove the Squeeze Theorem, as stated in Theorem 3.5.

43. Given that $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$, prove that $\lim_{x\to a} f(x) = L$.

44. A metal washer of (outer) radius r cm weighs $2r^2$ ounces. A company manufactures 2-inch washers for different customers who have different error tolerances. If the customer demands a washer of weight $8 \pm \varepsilon$ ounces, what is the error tolerance for the radius? That is, find δ such that a radius of r within the interval $(2-\delta, 2+\delta)$ guarantees a weight within $(8-\varepsilon, 8+\varepsilon)$.

45. A fiberglass company ships its glass as spherical marbles. If the volume of each marble must be within ε of $\pi/6$, how close does the radius need to be to 1/2?

EXPLORATORY EXERCISES

1. In this section, we have not yet solved any problems we could not already solve in previous sections. We do so now, while investigating an unusual function. Recall that rational numbers can be written as fractions p/q, where p and q are integers. We will assume that p/q has been simplified by dividing out common factors (e.g., 1/2 and not 2/4). Define $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/q & \text{if } x = \dfrac{p}{q} \text{ is rational} \end{cases}$. We will try to show that

$\lim_{x\to 2/3} f(x)$ exists. Without graphics, we need a good definition to answer this question. We know that $f(2/3) = 1/3$, but recall that the limit is independent of the actual function value. We need to think about x's close to 2/3. If such an x is irrational, $f(x) = 0$. A simple hypothesis would then be $\lim_{x\to 2/3} f(x) = 0$. We'll try this out for $\varepsilon = 1/6$. We would like to guarantee that $|f(x)| < 1/6$ whenever $0 < |x - 2/3| < \delta$. Well, how many x's have a function value greater than 1/6? The only possible function values are 1/5, 1/4, 1/3, 1/2 and 1. The x's with function value 1/5 are 1/5, 2/5, 3/5, 4/5 and so on. The closest of these x's to 2/3 is 3/5. Find the closest x (not counting $x = 2/3$) to 2/3 with function value 1/4. Repeat for $f(x) = 1/3$, $f(x) = 1/2$ and $f(x) = 1$. Out of all these closest x's, how close is the absolute closest? Choose δ to be this number, and argue that if $0 < |x - 2/3| < \delta$, we are guaranteed that $|f(x)| < 1/6$. Argue that a similar process can find a δ for any ε.

2. State a definition for "$f(x)$ is continuous at $x = a$" using Definition 6.1. Use it to prove that the function in exploratory exercise 1 is continuous at every irrational number and discontinuous at every rational number.

6.7 LIMITS AND LOSS-OF-SIGNIFICANCE ERRORS

"Pay no attention to that man behind the curtain . . ." (from *The Wizard of Oz*)

Things are not always what they appear to be. Even so, people tend to accept a computer's answer as a fact not subject to debate. However, when we use a computer (or calculator), we must always keep in mind that these devices perform most computations only approximately. Most of the time, this will cause us no difficulty whatsoever. Occasionally, however, the results of round-off errors in a string of calculations are disastrous. In this section, we briefly investigate these errors and learn how to recognize and avoid some of them.

We first consider a relatively tame-looking example.

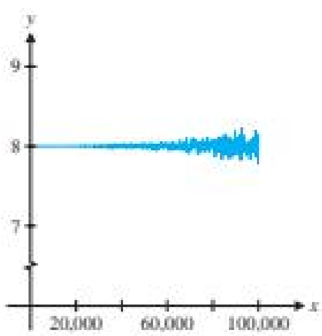


FIGURE 6.56a

$y = \dfrac{(x^3+4)^2 - x^6}{x^3}$

EXAMPLE 7.1 A Limit with Unusual Graphical and Numerical Behavior

Evaluate $\displaystyle\lim_{x\to\infty} \frac{(x^3+4)^2 - x^6}{x^3}$.

Solution At first glance, the numerator looks like $\infty - \infty$, which is indeterminate, while the denominator tends to ∞. Algebraically, the only reasonable step is to multiply out the first term in the numerator. First, we draw a graph and compute some function values. (Not all computers and software packages will produce these identical results, but for large values of x, you should see results similar to those shown here.) In Figure 6.56a, the function appears nearly constant, until it begins oscillating around $x = 40,000$. Notice that the accompanying table of function values is inconsistent with Figure 6.56a.

The last two values in the table may surprise you. Up until that point, the function values seem to be settling down to 8.0 very nicely. So, what is happening here and what is the correct value of the limit? To answer this, we look carefully at function values in the interval between $x = 1 \times 10^4$ and $x = 1 \times 10^5$. Have a look at the more detailed table.

Incorrect calculated values

x	$\frac{(x^3+4)^2 - x^6}{x^3}$
10	8.016
100	8.000016
1×10^3	8.0
1×10^4	8.0
1×10^5	0.0
1×10^6	0.0

x	$\frac{(x^3+4)^2 - x^6}{x^3}$
2×10^4	8.0
3×10^4	8.14815
4×10^4	7.8125
5×10^4	0

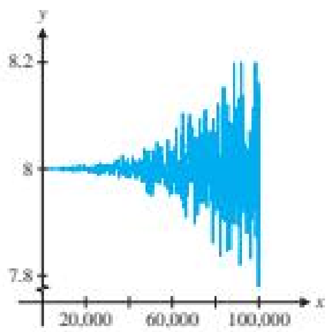


Figure 6.56b
$y = \frac{(x^3+4)^2 - x^6}{x^3}$

In Figure 6.56b, we have blown up the graph to enhance the oscillation observed between $x = 1 \times 10^4$ and $x = 1 \times 10^5$. The deeper we look into this limit, the more erratically the function appears to behave. We use the word *appears* because all of the oscillatory behavior we are seeing is an illusion, created by the finite precision of the computer used to perform the calculations and draw the graph. ■

○ Computer Representation of Real Numbers

The reason for the unusual behavior seen in example 7.1 boils down to the way in which computers represent real numbers. Without getting into all of the intricacies of computer arithmetic, it suffices to think of computers and calculators as storing real numbers internally in scientific notation. For example, the number 1,234,567 would be stored as 1.234567×10^6. The number preceding the power of 10 is called the **mantissa** and the power is called the **exponent**. Thus, the mantissa here is 1.234567 and the exponent is 6.

All computing devices have finite memory and consequently have limitations on the size mantissa and exponent that they can store. (This is called **finite precision**.) Many calculators carry a 14-digit mantissa and a 3-digit exponent. On a 14-digit computer, this would suggest that the computer retains only the first 14 digits in the decimal expansion of any given number.

EXAMPLE 7.2 Computer Representation of a Rational Number

Determine how $\frac{1}{3}$ is stored internally on a 10-digit computer and how $\frac{2}{3}$ is stored internally on a 14-digit computer.

Solution On a 10-digit computer, $\frac{1}{3}$ is stored internally as $\underbrace{3.333333333}_{10\text{ digits}} \times 10^{-1}$. On a 14-digit computer, $\frac{2}{3}$ is stored internally as $\underbrace{6.6666666666667}_{14\text{ digits}} \times 10^{-1}$. ■

For most purposes, such finite precision presents no problem. However, this occasionally leads to a disastrous error. In example 7.3, we subtract two relatively close numbers and examine the resulting catastrophic error.

EXAMPLE 7.3 A Computer Subtraction of Two "Close" Numbers

Compare the exact value of

$$1.\underbrace{0000000000000}_{13\text{ zeros}}4 \times 10^{18} - 1.\underbrace{0000000000000}_{13\text{ zeros}}1 \times 10^{18}$$

with the result obtained from a calculator or computer with a 14-digit mantissa.

Solution Notice that

$$1.\underbrace{0000000000000}_{13\text{ zeros}}4 \times 10^{18} - 1.\underbrace{0000000000000}_{13\text{ zeros}}1 \times 10^{18} = 0.\underbrace{0000000000000}_{13\text{ zeros}}3 \times 10^{18}$$

$$= 30{,}000. \tag{7.1}$$

However, if this calculation is carried out on a computer or calculator with a 14-digit (or smaller) mantissa, both numbers on the left-hand side of (7.1) are stored by the computer as 1×10^{18} and hence, the difference is calculated as 0. Try this calculation for yourself now. ■

EXAMPLE 7.4 Another Subtraction of Two "Close" Numbers

Compare the exact value of

$$1.\underbrace{0000000000000}_{13\text{ zeros}}6 \times 10^{20} - 1.\underbrace{0000000000000}_{13\text{ zeros}}4 \times 10^{20}$$

with the result obtained from a calculator or computer with a 14-digit mantissa.

Solution Notice that

$$\begin{aligned} 1.\underbrace{0000000000000}_{13\text{ zeros}}6 \times 10^{20} - 1.\underbrace{0000000000000}_{13\text{ zeros}}4 \times 10^{20} &= 0.\underbrace{0000000000000}_{13\text{ zeros}}2 \times 10^{20} \\ &= 2{,}000{,}000. \end{aligned}$$

However, if this calculation is carried out on a calculator with a 14-digit mantissa, the first number is represented as $1.0000000000001 \times 10^{20}$, while the second number is represented by 1.0×10^{20}, due to the finite precision and rounding. The difference between the two values is then computed as $0.0000000000001 \times 10^{20}$ or 10,000,000, which is, again, a very serious error. ■

In examples 7.3 and 7.4, we witnessed a gross error caused by the subtraction of two numbers whose significant digits are very close to one another. This type of error is called a **loss-of-significant-digits error** or simply a **loss-of-significance error**. These are subtle, often disastrous computational errors. Returning now to example 7.1, we will see that it was this type of error that caused the unusual behavior noted.

EXAMPLE 7.5 A Loss-of-Significance Error

In example 7.1, we considered the function $f(x) = \dfrac{(x^3+4)^2 - x^6}{x^3}$.

Follow the calculation of $f(5 \times 10^4)$ one step at a time, as a 14-digit computer would do it.

Solution We have

$$\begin{aligned} f(5 \times 10^4) &= \frac{[(5 \times 10^4)^3 + 4]^2 - (5 \times 10^4)^6}{(5 \times 10^4)^3} \\ &= \frac{(1.25 \times 10^{14} + 4)^2 - 1.5625 \times 10^{28}}{1.25 \times 10^{14}} \\ &= \frac{(125{,}000{,}000{,}000{,}000 + 4)^2 - 1.5625 \times 10^{28}}{1.25 \times 10^{14}} \\ &= \frac{(1.25 \times 10^{14})^2 - 1.5625 \times 10^{28}}{1.25 \times 10^{14}} = 0, \end{aligned}$$

since 125,000,000,000,004 is rounded off to 125,000,000,000,000.

Note that the real culprit here was not the rounding of 125,000,000,000,004, but the fact that this was followed by a subtraction of a nearly equal value. Further, note that this is not a problem unique to the numerical computation of limits. ■

REMARK 7.1

If at all possible, avoid subtractions of nearly equal values. Sometimes this can be accomplished by some algebraic manipulation of the function.

In the case of the function from example 7.5, we can avoid the subtraction and hence, the loss-of-significance error by rewriting the function as follows:

$$\begin{aligned} f(x) &= \frac{(x^3+4)^2 - x^6}{x^3} \\ &= \frac{(x^6 + 8x^3 + 16) - x^6}{x^3} \\ &= \frac{8x^3 + 16}{x^3}, \end{aligned}$$

where we have eliminated the subtraction. Using this new (and equivalent) expression for the function, we can compute a table of function values reliably. Notice, too, that if we

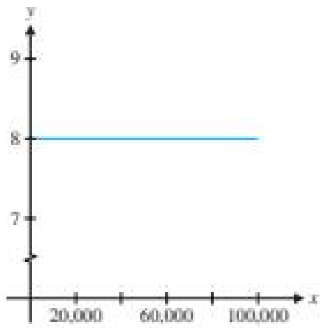


FIGURE 6.57
$y = \dfrac{8x^3 + 16}{x^3}$

x	$\dfrac{8x^3 + 16}{x^3}$
10	8.016
100	8.000016
1×10^3	8.000000016
1×10^4	8.00000000002
1×10^5	8.0
1×10^6	8.0
1×10^7	8.0

redraw the graph in Figure 6.56a using the new expression (see Figure 6.57), we no longer see the oscillation present in Figures 6.56a and 6.56b.

From the rewritten expression, we easily obtain

$$\lim_{x\to\infty} \frac{(x^3+4)^2 - x^6}{x^3} = 8,$$

which is consistent with Figure 6.57 and the corrected table of function values.

In example 7.6, we examine a loss-of-significance error that occurs for x close to 0.

EXAMPLE 7.6 Loss-of-Significance Involving a Trigonometric Function

Evaluate $\displaystyle\lim_{x\to 0} \frac{1 - \cos x^2}{x^4}$.

Solution As usual, we look at a graph (see Figure 6.58) and some function values.

x	$\dfrac{1 - \cos x^2}{x^4}$
0.1	0.499996
0.01	0.5
0.001	0.5
0.0001	0.0
0.00001	0.0

x	$\dfrac{1 - \cos x^2}{x^4}$
−0.1	0.499996
−0.01	0.5
−0.001	0.5
−0.0001	0.0
0.00001	0.0

As in example 7.1, note that the function values seem to be approaching 0.5, but then suddenly take a jump down to 0.0. Once again, we are seeing a loss-of-significance error. In this particular case, this occurs because we are subtracting nearly equal values ($\cos x^2$ and 1). We can again avoid the error by eliminating the subtraction. Note that

$$\begin{aligned}\frac{1-\cos x^2}{x^4} &= \frac{1-\cos x^2}{x^4}\cdot\frac{1+\cos x^2}{1+\cos x^2} && \text{Multiply numerator and denominator by } (1+\cos x^2).\\ &= \frac{1-\cos^2(x^2)}{x^4(1+\cos x^2)} && 1-\cos^2(x^2) = \sin^2(x^2).\\ &= \frac{\sin^2(x^2)}{x^4(1+\cos x^2)}.\end{aligned}$$

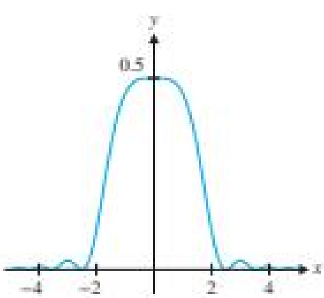


Figure 6.58
$y = \dfrac{1 - \cos x^2}{x^4}$

x	$\dfrac{\sin^2(x^2)}{x^4(1 + \cos x^2)}$
±0.1	0.499996
±0.01	0.4999999996
±0.001	0.5
±0.0001	0.5
±0.00001	0.5

Since this last (equivalent) expression has no subtraction indicated, we should be able to use it to reliably generate values without the worry of loss-of-significance error. Using this to compute function values, we get the accompanying table.

Using the graph and the new table, we conjecture that

$$\lim_{x\to 0} \frac{1 - \cos x^2}{x^4} = \frac{1}{2}. \blacksquare$$

We offer one final example where a loss-of-significance error occurs, even though no subtraction is explicitly indicated.

EXAMPLE 7.7 A Loss-of-Significance Error Involving a Sum

Evaluate $\displaystyle\lim_{x\to -\infty} x[(x^2+4)^{1/2} + x]$.

Solution Initially, you might think that since there is no subtraction (explicitly) indicated, there will be no loss-of-significance error. We first draw a graph (see Figure 6.59) and compute a table of values.

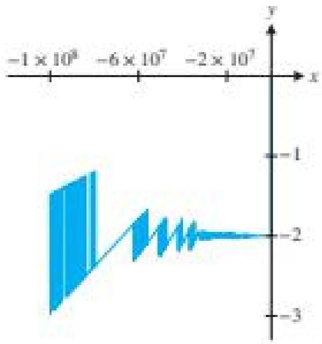


Figure 6.59
$y = x[(x^2+4)^{1/2} + x]$

x	$x[(x^2+4)^{1/2} + x]$
-100	-1.9998
-1×10^3	-1.999998
-1×10^4	-2.0
-1×10^5	-2.0
-1×10^6	-2.0
-1×10^7	0.0
-1×10^8	0.0

You should notice the sudden jump in values in the table and the wild oscillation visible in the graph. Although a subtraction is not explicitly indicated, there is indeed a subtraction here, since we have $x < 0$ and $(x^2+4)^{1/2} > 0$. We can again remedy this with some algebraic manipulation, as follows.

$$x[(x^2+4)^{1/2} + x] = x[(x^2+4)^{1/2} + x]\frac{[(x^2+4)^{1/2} - x]}{[(x^2+4)^{1/2} - x]}$$

Multiply numerator and denominator by the conjugate.

$$= x\frac{[(x^2+4) - x^2]}{[(x^2+4)^{1/2} - x]}$$

Simplify the numerator.

$$= \frac{4x}{[(x^2+4)^{1/2} - x]}.$$

We use this last expression to generate a graph in the same window as that used for Figure 6.59 and to generate the accompanying table of values. In Figure 6.60, we can see none of the wild oscillation observed in Figure 6.60 and the graph appears to be a horizontal line.

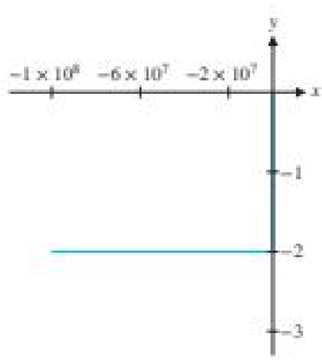


Figure 6.60
$y = \frac{4x}{[(x^2+4)^{1/2} - x]}$

x	$\frac{4x}{[(x^2+4)^{1/2} - x]}$
-100	-1.9998
-1×10^3	-1.999998
-1×10^4	-1.99999998
-1×10^5	-1.9999999998
-1×10^6	-2.0
-1×10^7	-2.0
-1×10^8	-2.0

Further, the values displayed in the table no longer show the sudden jump indicative of a loss-of-significance error. We can now confidently conjecture that

$$\lim_{x \to -\infty} x[(x^2+4)^{1/2} + x] = -2. \blacksquare$$

BEYOND FORMULAS

In examples 7.5–7.7, we demonstrated calculations that suffered from catastrophic loss-of-significance errors. In each case, we showed how we could rewrite the expression to avoid this error. We have by no means exhibited a general procedure for recognizing and repairing such errors. Rather, we hope that by seeing a few of these subtle errors, and by seeing how to fix even a limited number of them, you will become a more skeptical and intelligent user of technology.

EXERCISES 6.7

WRITING EXERCISES

1. Caution is important in using technology. Equally important is redundancy. This property is sometimes thought to be a negative (wasteful, unnecessary), but its positive role is one of the lessons of this section. By redundancy, we mean investigating a problem using graphical, numerical and symbolic tools. Why is it important to use multiple methods?

2. When should you do each of the following: look at a graph; compute function values; do symbolic work; do an $\varepsilon - \delta$ proof; prioritize the techniques in this chapter?

3. The limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ is important in Chapter 7. For a specific function and specific a, we could compute a table of values of the fraction for smaller values of h. Why should we be wary of loss-of-significance errors?

4. We rationalized the numerator in example 7.7. The old rule of rationalizing the denominator is intended to minimize computational errors. To see why you might want the square root in the numerator, suppose that you can get only one decimal place of accuracy, so that $\sqrt{3} \approx 1.7$. Compare $\frac{6}{1.7}$ to $\frac{6}{\sqrt{3}}$ and then compare $2(1.7)$ to $\frac{6}{\sqrt{3}}$. Which of the approximations could you do in your head?

In exercises 1–12, (a) use graphics and numerics to conjecture a value of the limit. (b) Find a computer or calculator graph showing a loss-of-significance error. (c) Rewrite the function to avoid the loss-of-significance error.

1. $\lim_{x\to\infty} x\left(\sqrt{4x^2+1}-2x\right)$

2. $\lim_{x\to-\infty} x\left(\sqrt{4x^2+1}+2x\right)$

3. $\lim_{x\to\infty} \sqrt{x}\left(\sqrt{x+4}-\sqrt{x+2}\right)$

4. $\lim_{x\to\infty} x^2\left(\sqrt{x^4+8}-x^2\right)$

5. $\lim_{x\to\infty} x\left(\sqrt{x^2+4}-\sqrt{x^2+2}\right)$

6. $\lim_{x\to\infty} x\left(\sqrt{x^3+8}-x^{3/2}\right)$

7. $\lim_{x\to 0} \frac{1-\cos 2x}{12x^2}$

8. $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

9. $\lim_{x\to 0} \frac{1-\cos x^3}{x^6}$

10. $\lim_{x\to 0} \frac{1-\cos x^4}{x^8}$

11. $\lim_{x\to\infty} x^{4/3}\left(\sqrt[3]{x^2+1}-\sqrt[3]{x^2-1}\right)$

12. $\lim_{x\to\infty} x^{2/3}\left(\sqrt[3]{x+4}-\sqrt[3]{x-3}\right)$

In exercises 13 and 14, compare the limits to show that small errors can have disastrous effects.

13. $\lim_{x\to 1} \frac{x^2+x-2}{x-1}$ and $\lim_{x\to 1} \frac{x^2+x-2.01}{x-1}$

14. $\lim_{x\to 2} \frac{x-2}{x^2-4}$ and $\lim_{x\to 2} \frac{x-2}{x^2-4.01}$

15. Compare $f(x) = \sin \pi x$ and $g(x) = \sin 3.14x$ for $x = 1$ (radian), $x = 10$, $x = 100$ and $x = 1000$.

16. If you have access to a CAS, test it on the limits of examples 7.1, 7.6 and 7.7. Based on these results, do you think that your CAS does precise calculations or numerical estimates?

In exercises 17 and 18, compare the exact answer to one obtained by a computer with a six-digit mantissa.

17. $(1.000003 - 1.000001) \times 10^7$

18. $(1.000006 - 1.000001) \times 10^7$

EXPLORATORY EXERCISE

1. Just as we are subject to round-off error in using calculations from a computer, so are we subject to errors in a computer-generated graph. After all, the computer has to compute function values before it can decide where to plot points. On your computer or calculator, graph $y = \sin x^2$ (a disconnected graph—or point plot—is preferable). You should see the oscillations you expect from the sine function, but with the oscillations getting faster as x gets larger. Shift your graphing window to the right several times. At some point, the plot will become very messy and almost unreadable. Depending on your technology, you may see patterns in the plot. Are these patterns real or an illusion? To explain what is going on,

recall that a computer graph is a finite set of pixels, with each pixel representing one x and one y. Suppose the computer is plotting points at $x = 0$, $x = 0.1$, $x = 0.2$ and so on. The y-values would then be $\sin 0^2$, $\sin 0.1^2$, $\sin 0.2^2$ and so on. Investigate what will happen between $x = 15$ and $x = 16$. Compute all the points $(15, \sin 15^2)$, $(15.1, \sin 15.1^2)$ and so on. If you were to graph these points, what pattern would emerge? To explain this pattern, argue that there is approximately half a period of the sine curve missing between each point plotted. Also, investigate what happens between $x = 31$ and $x = 32$.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Secant line	Limit	Infinite limit
One-sided limit	Continuous	Loss-of-significance error
Removable discontinuity	Horizontal asymptote	Slant asymptote
Vertical asymptote	Squeeze Theorem	Intermediate Value Theorem
Method of bisections	Length of line segment	
Slope of curve		

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to make a new statement that is true.

1. In calculus, problems are often solved by first approximating the solution and then improving the approximation.
2. If $f(1.1) = 2.1$, $f(1.01) = 2.01$ and so on, then $\lim_{x\to 1} f(x) = 2$.
3. $\lim_{x\to a} [f(x) \cdot g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
5. If $f(2) = 1$ and $f(4) = 2$, then there exists an x between 2 and 4 such that $f(x) = 0$.
6. For any polynomial $p(x)$, $\lim_{x\to\infty} p(x) = \infty$.
7. If $f(x) = \frac{p(x)}{q(x)}$ for polynomials p and q with $q(a) = 0$, then f has a vertical asymptote at $x = a$.
8. Small round-off errors typically have only small effects on a calculation.
9. $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a} \sqrt{f(x)} = \sqrt{L}$.

In exercises 1 and 2, numerically estimate the slope of $y = f(x)$ at $x = a$.

1. $f(x) = x^2 - 2x$, $a = 2$
2. $f(x) = \sin 2x$, $a = 0$

In exercises 3 and 4, numerically estimate the length of the curve using (a) $n = 4$ and (b) $n = 8$ line segments and evenly spaced x-coordinates.

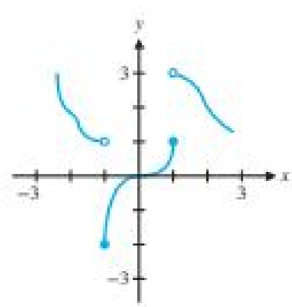
3. $f(x) = \sin x, 0 \le x \le \frac{\pi}{4}$
4. $f(x) = x^2 - x$, $0 \le x \le 2$

In exercises 5–10, use numerical and graphical evidence to conjecture the value of the limit.

5. $\lim_{x\to 0} \frac{\tan^{-1} x^2}{x^2}$
6. $\lim_{x\to 1} \frac{x^2 - 1}{\ln x^2}$
7. $\lim_{x\to -2} \frac{x + 2}{|x + 2|}$
8. $\lim_{x\to 0} (1 + 2x)^{1/x}$
9. $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x$
10. $\lim_{x\to\infty} x^{2/x}$

In exercises 11 and 12, identify the limits from the graph of f.

11. (a) $\lim_{x\to -1^-} f(x)$ (b) $\lim_{x\to -1^+} f(x)$
    (c) $\lim_{x\to -1} f(x)$ (d) $\lim_{x\to 0} f(x)$
12. (a) $\lim_{x\to 1^-} f(x)$ (b) $\lim_{x\to 1^+} f(x)$
    (c) $\lim_{x\to 1} f(x)$ (d) $\lim_{x\to 2} f(x)$



13. Identify the discontinuities in the function graphed above.
14. Sketch a graph of a function f with $f(-1) = 0$, $f(0) = 0$, $\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = -1$.

Review Exercises

In exercises 15–36, evaluate the limit. Answer with a number, ∞, −∞ or does not exist.

15. $\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4}$

16. $\lim_{x\to 1} \frac{x^2 - 1}{x^2 + x - 2}$

17. $\lim_{x\to 0} \frac{x^2 + x}{\sqrt{x^4 + 2x^2}}$

18. $\lim_{x\to 0} e^{-\cos x}$

19. $\lim_{x\to 0} (2 + x)\sin(1/x)$

20. $\lim_{x\to 0} \frac{\sin x^2}{x^2}$

21. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} 3x - 1 & \text{if } x < 2 \\ x^2 + 1 & \text{if } x \geq 2 \end{cases}$

22. $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ x^2 + 1 & \text{if } x \geq 1 \end{cases}$

23. $\lim_{x\to 0} \frac{\sqrt[3]{1 + 2x} - 1}{x}$

24. $\lim_{x\to 1} \frac{x - 1}{\sqrt{10 - x} - 3}$

25. $\lim_{x\to 0} \cot(x^2)$

26. $\lim_{x\to 1} \tan^{-1}\left(\frac{x}{x^2 - 2x + 1}\right)$

27. $\lim_{x\to \infty} \frac{x^2 - 4}{3x^2 + x + 1}$

28. $\lim_{x\to \infty} \frac{2x}{\sqrt{x^2 + 4}}$

29. $\lim_{x\to \pi/2} e^{-\tan^2 x}$

30. $\lim_{x\to -\infty} e^{-x^2}$

31. $\lim_{x\to \infty} \ln 2x$

32. $\lim_{x\to 0^+} \ln 3x$

33. $\lim_{x\to -\infty} \frac{2x}{x^2 + 3x - 5}$

34. $\lim_{x\to -2} \frac{2x}{x^2 + 3x + 2}$

35. $\lim_{x\to 0} (1 - 3x)^{2/x}$

36. $\lim_{x\to 0} \frac{2x - |x|}{|3x| - 2x}$

37. Use the Squeeze Theorem to prove that $\lim_{x\to 0} \frac{2x^3}{x^2 + 1} = 0$.

38. Use the Intermediate Value Theorem to verify that $f(x) = x^3 - x - 1$ has a zero in the interval [1, 2]. Use the method of bisections to find an interval of length 1/32 that contains a zero.

In exercises 39–42, find all discontinuities and determine which are removable.

39. $f(x) = \frac{x - 1}{x^2 + 2x - 3}$

40. $f(x) = \frac{x + 1}{x^2 - 4}$

41. $f(x) = \begin{cases} \sin x & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x \leq 2 \\ 4x - 3 & \text{if } x > 2 \end{cases}$

42. $f(x) = x \cot x$

In exercises 43–46, find all intervals of continuity.

43. $f(x) = \frac{x + 2}{x^2 - x - 6}$

44. $f(x) = \ln(3x - 4)$

45. $f(x) = \sin(1 + e^x)$

46. $f(x) = \sqrt{x^2 - 4}$

In exercises 47–54, determine all vertical, horizontal and slant asymptotes.

47. $f(x) = \frac{x + 1}{x^2 - 3x + 2}$

48. $f(x) = \frac{x + 2}{x^2 - 2x - 8}$

49. $f(x) = \frac{x^2}{x^2 - 1}$

50. $f(x) = \frac{x^3}{x^2 - x - 2}$

51. $f(x) = 2e^{1/x}$

52. $f(x) = 3\tan^{-1} 2x$

53. $f(x) = \frac{3}{e^x - 2}$

54. $f(x) = 3\ln(x - 2)$

In exercises 55 and 56, (a) use graphical and numerical evidence to conjecture a value for the indicated limit. (b) Find a computer or calculator graph showing a loss-of-significance error. (c) Rewrite the function to avoid the loss-of-significance error.

55. $\lim_{x\to 0} \frac{1 - \cos x}{2x^2}$

56. $\lim_{x\to \infty} x\left(\sqrt{x^2 + 1} - x\right)$

APPLICATIONS

1. A tanker's internal reservoir is leaking at a rate $r(t)$ in liters per minute as modeled by

$$r(t) = \begin{cases} \frac{20t - 80}{t^2 - 4t} & 1 \leq t \leq 4 \\ \frac{20(\sqrt{t} - 2)}{t - 4} & t > 4 \end{cases}$$

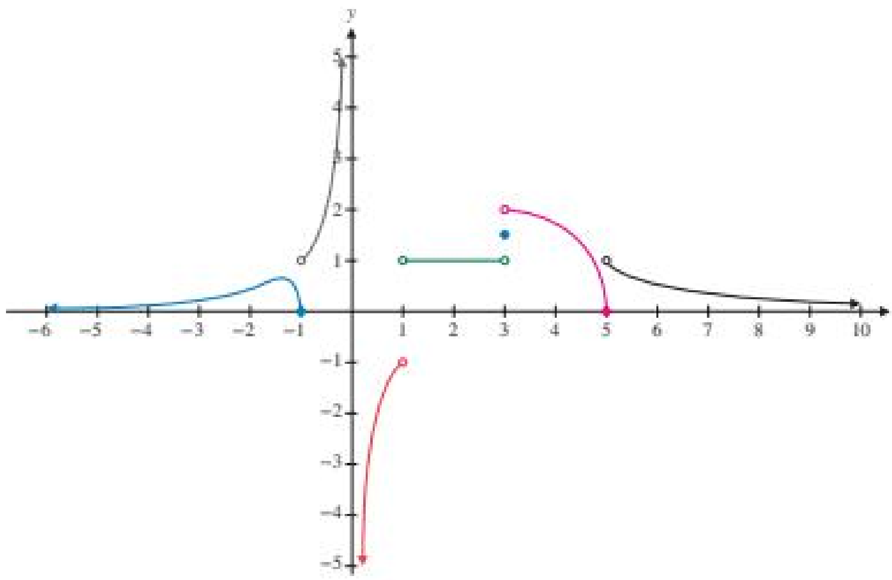
a. What is the rate at $t = 3$?

b. Is the function $r(t)$ continuous at $t = 4$?

2. The power dissipated in a resistor can be obtained through the equation $P = I^2R$, where P is the power in watts, I is the current in amperes, and R is the resistance in ohms. In what interval must I lie if the power is 200 watts and R is 6 ± 0.2 ohms?

Review Exercises

3. Use the graph of $f(x)$ to answer the following questions:



(a) $\lim_{x \to -1^-} f(x)$	(b) $\lim_{x \to -1^+} f(x)$	(c) $\lim_{x \to -1} f(x)$
(d) $\lim_{x \to 0^-} f(x)$	(e) $\lim_{x \to 0^+} f(x)$	(f) $\lim_{x \to 0} f(x)$
(g) $\lim_{x \to +\infty} f(x)$	(h) $\lim_{x \to -\infty} f(x)$	(i) Is $f(x)$ continuous at $x = 0$?
(j) Is $f(x)$ continuous at $x = 2$?	(k) $f(5)$	(l) Is $f(x)$ continuous at $x = 5$?

EXPLORATORY EXERCISES

1. For $f(x) = \frac{2x^2 - 2x - 4}{x^2 - 5x + 6}$, do the following. (a) Find all values of x at which f is not continuous. (b) Determine which value in (a) is a removable discontinuity. For this value, find the limit of f as x approaches this value. Sketch a portion of the graph of f near this x-value showing the behavior of the function. (c) For the value in part (a) that is not removable, find the two one-sided infinite limits and sketch the graph of f near this x-value. (d) Find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ and sketch the portion of the graph of f corresponding to these values. (e) Connect the pieces of your graph as simply as possible. If available, compare your graph to a computer-generated graph.

2. Let $f(t)$ represent the price of an autograph of a famous person at time t (years after 2000). Interpret each of the following (independently) in financial terms: (a) horizontal asymptote $y = 1000$, (b) vertical asymptote at $t = 10$, (c) $\lim_{t \to 4^-} f(t) = 500$ and $\lim_{t \to 4^+} f(t) = 800$ and (d) $\lim_{t \to 8} f(t) = 950$.

Differentiation: Definition and Fundamental Properties

CHAPTER

7

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The marathon is one of the most famous running events, covering 42.195 kilometers. The 2016 Olympic marathon in Rio de Janeiro was won by Eliud Kipchoge of Kenya in a time of 2:04:44. Using the familiar formula "rate equals distance divided by time," we can compute Eliud's average speed of

$$\frac{42.195}{2+\frac{4}{60}+\frac{44}{3600}} \approx 20.3 \text{ km/h}$$

This says that Eliud averaged less than 3 minutes per kilometer for over 42 kilometers! However, the 100-meter and 200-meter sprint were both won by the fabulous Usain Bolt of Jamaica in 9.81 seconds and 19.78 seconds. The average speeds for Bolt were

$$\frac{0.1}{\frac{9.81}{3600}} \approx 36.7 \text{ km/h} \quad \text{and} \quad \frac{0.2}{\frac{19.78}{3600}} \approx 36.4 \text{ km/h}$$

Chapter Topics

Since these speeds are much faster than that of the marathon runner, the winners of these events are often called the "World's Fastest Human." An interesting connection can be made with a thought experiment. If Bolt ran 200 meters in 19.78 seconds with the first 100 meters covered in 9.81 seconds, compare the average speeds for the first and second 100 meters. In the second 100 meters, the distance run is $200 - 100 = 100$ meters and the time is $19.78 - 9.81 = 9.97$ seconds. The average speed is then

$$\frac{200-100}{19.78-9.81} \approx 10.03 \text{ m/s} \approx 36.1 \text{ km/h}$$

Notice that the speed calculation in m/s is the same calculation we would use for the slope between the points (9.81, 100) and (19.78, 200). The connection between slope and speed (and other quantities of interest) is explored in this chapter.

7.1 TANGENT LINES AND VELOCITY

A traditional slingshot is essentially a rock on the end of a string, which you rotate around in a circular motion and then release. When you release the string, in which direction will the rock travel? An overhead view of this is illustrated in

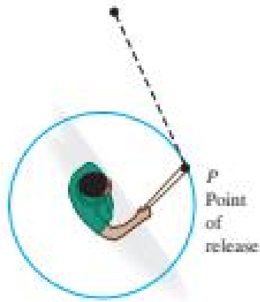


FIGURE 7.1
Path of rock

Figure 7.1. Many people mistakenly believe that the rock will follow a curved path, but Newton's first law of motion tells us that the path as viewed from above is straight. In fact, the rock follows a path along the tangent line to the circle at the point of release. Our aim in this section is to extend the notion of tangent line to more general curves.

To make our discussion more concrete, suppose that we want to find the tangent line to the curve $y = x^2 + 1$ at the point (1, 2). (See Figure 7.2.) The tangent line hugs the curve near the point of tangency. In other words, like the tangent line to a circle, this tangent line has the same direction as the curve at the point of tangency. Observe that, if we zoom in sufficiently far, the graph appears to approximate that of a straight line. In Figure 7.3, we show the graph of $y = x^2 + 1$ zoomed in on the small rectangular box indicated in Figure 7.2. We now choose two points from the curve—for example, (1, 2) and (3, 10)—and compute the slope of the line joining these two points. Such a line is called a **secant** line and we denote its slope by m_{sec}:

$$m_{sec} = \frac{10 - 2}{3 - 1} = 4.$$

CAUTION

Be aware that the "axes" indicated in Figure 7.3 do not intersect at the origin. We provide them only as a guide as to the scale used to produce the figure.

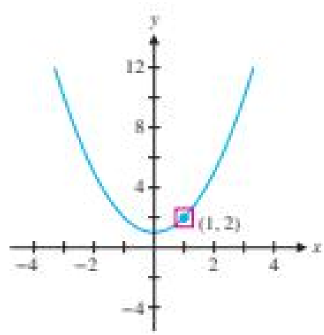


FIGURE 7.2
$y = x^2 + 1$

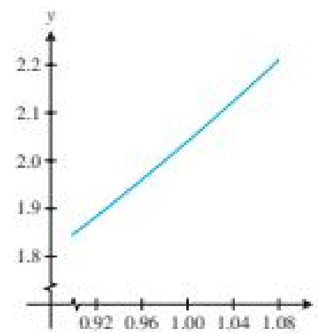


FIGURE 7.3
$y = x^2 + 1$

An equation of the secant line is then determined by

$$\frac{y - 2}{x - 1} = 4,$$

so that

$$y = 4(x - 1) + 2.$$

As can be seen in Figure 7.4a, the secant line doesn't look very much like a tangent line.

Refining this procedure, we take the second point a little closer to the point of tangency, say (2, 5). This gives the slope of the secant line as

$$m_{sec} = \frac{5 - 2}{2 - 1} = 3$$

and an equation of this secant line as $y = 3(x - 1) + 2$. As seen in Figure 7.4b, this looks much more like a tangent line, but it's still not quite there. Choosing our second point much closer to the point of tangency, say (1.05, 2.1025), should give us an even better approximation. In this case, we have

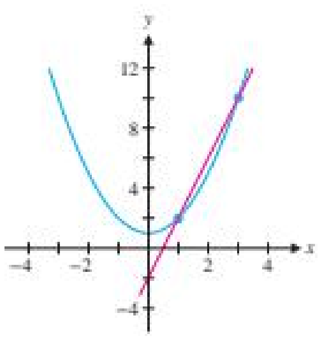


FIGURE 7.4a
Secant line joining (1, 2) and (3,10)

$$m_{sec} = \frac{2.1025 - 2}{1.05 - 1} = 2.05$$

and an equation of this secant line is $y = 2.05(x - 1) + 2$. As can be seen in Figure 7.4c, the secant line looks very much like a tangent line, even when zoomed in quite far, as in Figure 7.4d. We continue this process by computing the slope of the secant line joining (1, 2) and the unspecified point $(1 + h, f(1 + h))$, for some value of h close to 0 (but $h \neq 0$). The slope of this secant line is

$$\begin{aligned} m_{sec} &= \frac{f(1+h)-2}{(1+h)-1} = \frac{[(1+h)^2+1]-2}{h} \\ &= \frac{(1+2h+h^2)-1}{h} = \frac{2h+h^2}{h} && \text{Multiply out and cancel.} \\ &= \frac{h(2+h)}{h} = 2+h. && \text{Factor out common } h \text{ and cancel.} \end{aligned}$$

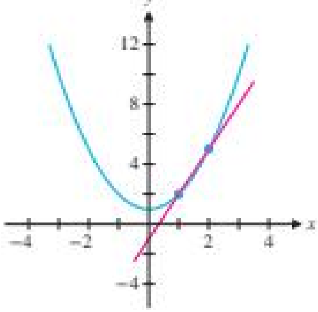


FIGURE 7.4b
Secant line joining (1, 2) and (2, 5)

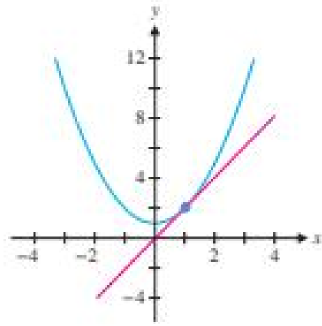


FIGURE 7.4c
Secant line joining (1, 2) and (1.05, 2.1025)

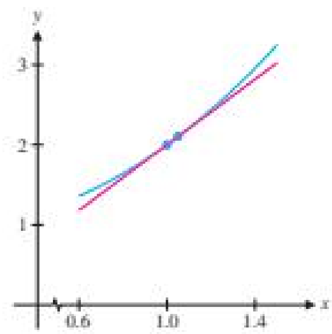


FIGURE 7.4d
Close-up of secant line

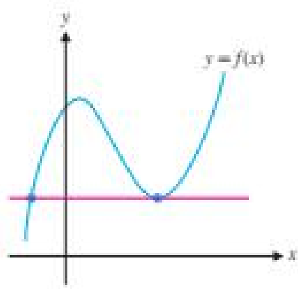


FIGURE 7.5
Tangent line intersecting a curve at more than one point

Notice that as h approaches 0, the slope of the secant line approaches 2, which we define to be the slope of the tangent line.

REMARK 1.1

We should make one more observation before moving on to the general case of tangent lines. Unlike the case for a circle, tangent lines may intersect a curve at more than one point, as seen in Figure 7.5.

The General Case

To find the slope of the tangent line to $y = f(x)$ at $x = a$, first pick two points on the curve. One point is the point of tangency, $(a, f(a))$. Call the x-coordinate of the second point $x = a + h$, for some small number h ($h \neq 0$); the corresponding y-coordinate is then $y = f(a + h)$. It is natural to think of h as being positive, as shown in Figure 7.6a, although h can also be negative, as shown in Figure 7.6b.

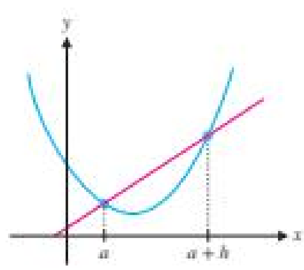


FIGURE 7.6a
Secant line ($h > 0$)

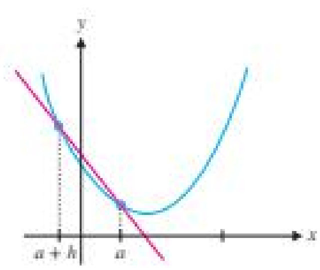


FIGURE 7.6b
Secant line ($h < 0$)

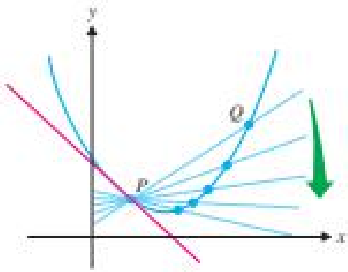


FIGURE 7.7
Secant lines approaching the tangent line at the point P

The slope of the secant line through the points $(a, f(a))$ and $(a + h, f(a + h))$ is given by

$$m_{sec} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}. \tag{1.1}$$

Notice that the expression in (1.1) (called a **difference quotient**) gives the slope of the secant line for any second point we might choose (i.e., for any $h \neq 0$). Recall that in order to obtain an improved approximation to the tangent line, we take the second point closer to the point of tangency, which in turn makes h closer to 0. We illustrate this process in Figure 7.7, where we have plotted a number of secant lines for $h > 0$. Notice that as the point Q approaches the point P (i.e., as $h \to 0$), the secant lines approach the tangent line at P.

We define the slope of the tangent line to be the limit of the slopes of the secant lines in (1.1) as h tends to 0, whenever this limit exists.

DEFINITION 1.1

The **slope** m_{tan} of the tangent line to $y = f(x)$ at $x = a$ is given by

$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \tag{1.2}$$

provided the limit exists.

The tangent line is then the line passing through the point $(a, f(a))$ with slope m_{tan}, with equation given by $\frac{y - f(a)}{x - a} = m_{tan}$ or

Equation of tangent line
$$y = m_{tan}(x - a) + f(a).$$

EXAMPLE 1.1 Finding the Equation of a Tangent Line

Find an equation of the tangent line to $y = x^2 + 1$ at $x = 1$.

Solution We compute the slope using (1.2):

$$\begin{aligned} m_{tan} &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{[(1+h)^2 + 1] - (1+1)}{h} \\ &= \lim_{h \to 0} \frac{1 + 2h + h^2 + 1 - 2}{h} && \text{Multiply out and cancel.} \\ &= \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} \frac{h(2+h)}{h} && \text{Factor out common } h \text{ and cancel.} \\ &= \lim_{h \to 0} (2 + h) = 2. \end{aligned}$$

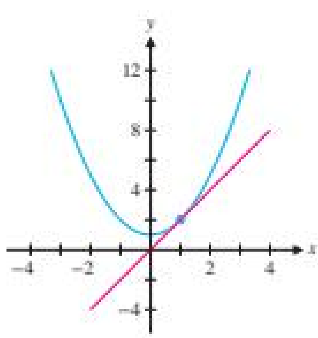


FIGURE 7.8
$y = x^2 + 1$ and the tangent line at $x = 1$

Notice that the point corresponding to $x = 1$ is $(1, 2)$ and the line with slope 2 through the point $(1, 2)$ has equation

$$y = 2(x - 1) + 2 \quad \text{or} \quad y = 2x.$$

Note how closely this corresponds to the secant lines computed earlier. We show a graph of the function and this tangent line in Figure 7.8. ■

EXAMPLE 1.2 Tangent Line to the Graph of a Rational Function

Find an equation of the tangent line to $y = \dfrac{2}{x}$ at $x = 2$.

Solution From (1.2), we have

$$m_{\tan} = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\dfrac{2}{2+h} - 1}{h} \qquad \text{Since } f(2+h) = \frac{2}{2+h}$$

$$= \lim_{h\to 0} \frac{\left[\dfrac{2 - (2+h)}{(2+h)}\right]}{h} = \lim_{h\to 0} \frac{\left[\dfrac{2 - 2 - h}{(2+h)}\right]}{h} \qquad \text{Add fractions and multiply out.}$$

$$= \lim_{h\to 0} \frac{-h}{(2+h)h} = \lim_{h\to 0} \frac{-1}{2+h} = -\frac{1}{2}. \qquad \text{Cancel } h\text{'s}$$

The point corresponding to $x = 2$ is (2, 1), since $f(2) = 1$. An equation of the tangent line is then

$$y = -\frac{1}{2}(x - 2) + 1.$$

We show a graph of the function and this tangent line in Figure 7.9. ■

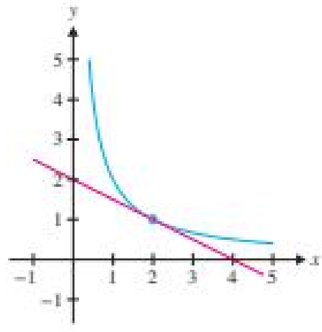


FIGURE 7.9
$y = \dfrac{2}{x}$ and tangent line at (2, 1)

In cases where we cannot (or cannot easily) evaluate the limit for the slope of the tangent line, we can approximate the limit numerically. We illustrate this in example 1.3.

EXAMPLE 1.3 Graphical and Numerical Approximation of Tangent Lines

Graphically and numerically approximate the slope of the tangent line to $y = \dfrac{x-1}{x+1}$ at $x = 0$.

Solution A graph of $y = \dfrac{x-1}{x+1}$ is shown in Figure 7.10a. We sketch the tangent line at the point (0, −1) in Figure 7.10b, where we have zoomed in to provide better detail. To approximate the slope, we estimate the coordinates of one point on the tangent line other than (0, −1). In Figure 7.10b, it appears that the tangent line passes through the point (1, 1). An estimate of the slope is then $m_{\tan} \approx \dfrac{1-(-1)}{1-0} = 2$. To approximate the slope numerically, we choose several points near (0, −1) and compute the slopes of the secant lines. For example, rounding the y-values to four decimal places, we have

Second Point	m_{sec}	*Second Point*	m_{sec}
(1, 0)	$\dfrac{0-(-1)}{1-0} = 1$	(−0.5, −3)	$\dfrac{-3-(-1)}{-0.5-0} = 4.0$
(0.1, −0.8182)	$\dfrac{-0.8182-(-1)}{0.1-0} = 1.818$	(−0.1, −1.2222)	$\dfrac{-1.2222-(-1)}{-0.1-0} = 2.222$
(0.01, −0.9802)	$\dfrac{-0.9802-(-1)}{0.01-0} = 1.98$	(−0.01, −1.0202)	$\dfrac{-1.0202-(-1)}{-0.01-0} = 2.02$

In both columns, as the second point gets closer to (0, −1), the slope of the secant line gets closer to 2. A reasonable estimate of the slope of the tangent line at the point (0, −1) is then 2. ■

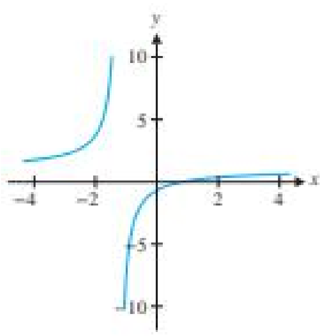


FIGURE 7.10a
$y = \dfrac{x-1}{x+1}$

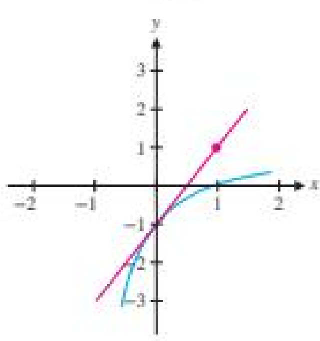


FIGURE 7.10b
Tangent line

Velocity

We often describe *velocity* as a quantity determining the speed *and direction* of an object. Observe that if your car did not have a speedometer, you could determine your speed using the familiar formula

$$\text{distance} = \text{rate} \times \text{time}. \tag{1.3}$$

Using (1.3), you can find the rate (speed) by simply dividing the distance by the time. While the rate in (1.3) refers to *average* speed over a period of time, we are interested in the speed at a specific instant. The following story should indicate the difference.

During traffic stops, police officers frequently ask drivers if they know how fast they were going. Consider the following response from an overzealous student, who might answer that during the past, say, 3 years, 2 months, 7 days, 5 hours and 45 minutes, they've driven exactly 45,259.7 miles, so that their speed was

$$\text{rate} = \frac{\text{distance}}{\text{time}} = \frac{45{,}259.7 \text{ miles}}{27{,}917.75 \text{ hours}} \approx 1.62118 \text{ mph}.$$

Of course, most police officers would not be impressed with this analysis, but, *why* is it wrong? While there's nothing wrong with formula (1.3) or the arithmetic, it's reasonable to argue that unless they were in their car during this entire 3-year period, the results are invalid.

Suppose that the driver substitutes the following argument instead: "I left home at 6:17 P.M. and by the time you pulled me over at 6:43 P.M., I had driven exactly 17 miles. Therefore, my speed was

$$\text{rate} = \frac{17 \text{ miles}}{26 \text{ minutes}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \approx 39.2 \text{ mph},$$

well under the posted 45 mph speed limit."

While this is a much better estimate of the velocity than the 1.6 mph computed previously, it's still an average velocity using too long of a time period.

More generally, suppose that the function $s(t)$ gives the position at time t of an object moving along a straight line. That is, $s(t)$ gives the displacement (**signed distance**) from a fixed reference point, so that $s(t) < 0$ means that the object is located $|s(t)|$ away from the reference point, but in the negative direction. Then, for two times, a and b (where $a < b$), $s(b) - s(a)$ gives the signed distance between positions $s(a)$ and $s(b)$. The **average velocity** v_{avg} is then given by

$$\boxed{v_{avg} = \frac{\text{signed distance}}{\text{time}} = \frac{s(b) - s(a)}{b - a}.} \tag{1.4}$$

EXAMPLE 1.4 Finding Average Velocity

The position of a car after t minutes driving in a straight line is given by

$$s(t) = \frac{1}{2}t^2 - \frac{1}{12}t^3, \quad 0 \le t \le 4,$$

where s is measured in miles and t is measured in minutes. Approximate the velocity at time $t = 2$.

Solution Averaging over the 2 minutes from $t = 2$ to $t = 4$, we get from (1.4) that

$$\begin{aligned} v_{avg} &= \frac{s(4) - s(2)}{4 - 2} \approx \frac{2.6667 - 1.3333}{2} \\ &\approx 0.6667 \text{ mile/minute} \\ &\approx 40 \text{ mph}. \end{aligned}$$

Of course, a 2-minute-long interval is rather long, given that cars can speed up and slow down a great deal in 2 minutes. We get an improved approximation by averaging over just one minute:

$$v_{avg} = \frac{s(3) - s(2)}{3 - 2} \approx \frac{2.25 - 1.3333}{1}$$
$$\approx 0.91667 \text{ mile/minute}$$
$$\approx 55 \text{ mph.}$$

While this latest estimate is certainly better than the first one, we can do better. As we make the time interval shorter and shorter, the average velocity should be getting closer and closer to the velocity at the instant $t = 2$. It stands to reason that, if we compute the average velocity over the time interval $[2, 2 + h]$ (where $h > 0$) and then let $h \to 0$, the resulting average velocities should be getting closer and closer to the velocity at the instant $t = 2$.

We have $$v_{avg} = \frac{s(2 + h) - s(2)}{(2 + h) - 2} = \frac{s(2 + h) - s(2)}{h}.$$

h	$\frac{s(2+h)-s(2)}{h}$
1.0	0.9166666667
0.1	0.9991666667
0.01	0.9999916667
0.001	0.999999917
0.0001	1.0
0.00001	1.0

A sequence of these average velocities is displayed in the accompanying table, for $h > 0$, with similar results if we allow h to be negative. It appears that the average velocity is approaching 1 mile/minute (60 mph), as $h \to 0$. ■

This leads us to make the following definition:

NOTES

(i) Notice that if (for example) t is measured in seconds and $s(t)$ is measured in meters, then velocity (average or instantaneous) is measured in meters per second (m/s).
(ii) When used without qualification, the term *velocity* refers to instantaneous velocity.

DEFINITION 1.2

If $s(t)$ represents the position of an object relative to some fixed location at time t as the object moves along a straight line, then the **instantaneous velocity** at time $t = a$ is given by

$$v(a) = \lim_{h \to 0} \frac{s(a + h) - s(a)}{(a + h) - a} = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}, \tag{1.5}$$

provided the limit exists. The **speed** is the absolute value of the velocity.

EXAMPLE 1.5 Finding Average and Instantaneous Velocity

Suppose that the height of a falling object t seconds after being dropped from a height of 64 feet is given by $s(t) = 64 - 16t^2$ feet. Find the average velocity between times $t = 1$ and $t = 2$; the average velocity between times $t = 1.5$ and $t = 2$; the average velocity between times $t = 1.9$ and $t = 2$; and the instantaneous velocity at time $t = 2$.

Solution The average velocity between times $t = 1$ and $t = 2$ is

$$v_{avg} = \frac{s(2) - s(1)}{2 - 1} = \frac{64 - 16(2)^2 - [64 - 16(1)^2]}{1} = -48 \text{ (ft/s).}$$

The average velocity between times $t = 1.5$ and $t = 2$ is

$$v_{avg} = \frac{s(2) - s(1.5)}{2 - 1.5} = \frac{64 - 16(2)^2 - [64 - 16(1.5)^2]}{0.5} = -56 \text{ (ft/s).}$$

The average velocity between times $t = 1.9$ and $t = 2$ is

$$v_{avg} = \frac{s(2) - s(1.9)}{2 - 1.9} = \frac{64 - 16(2)^2 - [64 - 16(1.9)^2]}{0.1} = -62.4 \text{ (ft/s).}$$

The instantaneous velocity is the limit of such average velocities. From (1.5), we have

$$
\begin{aligned}
v(2) &= \lim_{h\to 0} \frac{s(2+h)-s(2)}{(2+h)-2} \\
&= \lim_{h\to 0} \frac{[64 - 16(2+h)^2] - [64 - 16(2)^2]}{h} \\
&= \lim_{h\to 0} \frac{[64 - 16(4 + 4h + h^2)] - [64 - 16(2)^2]}{h} \qquad \text{Multiply out and cancel.} \\
&= \lim_{h\to 0} \frac{-64h - 16h^2}{h} = \lim_{h\to 0} \frac{-16h(h+4)}{h} \qquad \text{Factor out common } h \text{ and cancel.} \\
&= \lim_{h\to 0} [-16(h+4)] = -64 \text{ ft/s.}
\end{aligned}
$$

Recall that velocity indicates both speed and direction. In this problem, $s(t)$ measures the height above the ground. So, the negative velocity indicates that the object is moving in the negative (or downward) direction. The speed of the object at the 2-second mark is then 64 ft/s. ■

Observe that the formulas for instantaneous velocity (1.5) and for the slope of a tangent line (1.2) are identical. To make this connection stronger, we graph the position function $s(t) = 64 - 16t^2$ for $0 \le t \le 3$, from example 1.5. The average velocity between $t = 1$ and $t = 2$ corresponds to the slope of the secant line between the points at $t = 1$ and $t = 2$. (See Figure 7.11a.) Similarly, the average velocity between $t = 1.5$ and $t = 2$ gives the slope of the corresponding secant line. (See Figure 7.11b.) Finally, the instantaneous velocity at time $t = 2$ corresponds to the slope of the tangent line at $t = 2$. (See Figure 7.11c.)

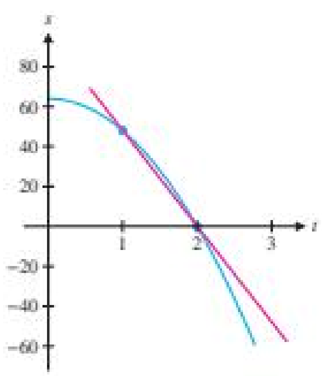


FIGURE 7.11a
Secant line between $t = 1$ and $t = 2$

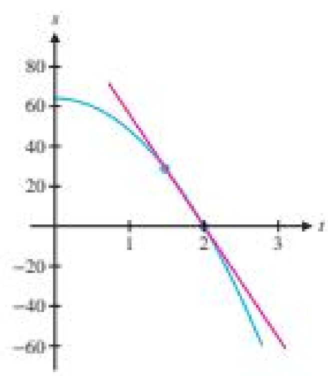


FIGURE 7.11b
Secant line between $t = 1.5$ and $t = 2$

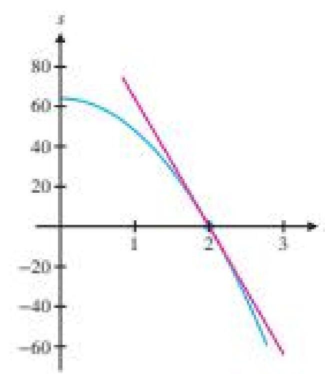


FIGURE 7.11c
Tangent line at $t = 2$

Velocity is a *rate* (more precisely, the instantaneous rate of change of position with respect to time). In general, the **average rate of change** of a function $f(x)$ between $x = a$ and $x = b$ $(a \ne b)$ is given by

$$\frac{f(b) - f(a)}{b - a}.$$

The **instantaneous rate of change** of $f(x)$ at $x = a$ is given by

$$\lim_{h\to\infty} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists. The units of the instantaneous rate of change are the units of f divided by (or "per") the units of x. You should recognize this limit as the slope of the tangent line to $y = f(x)$ at $x = a$.

EXAMPLE 1.6 Interpreting Rates of Change

If the function $f(t)$ gives the population of a city in millions of people t years after January 1, 2000, interpret each of the following quantities, assuming that they equal the given numbers. (a) $\dfrac{f(2)-f(0)}{2}=0.34$, (b) $f(2)-f(1)=0.31$ and (c) $\displaystyle\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=0.3$.

Solution Since $\dfrac{f(b)-f(a)}{b-a}$ is the average rate of change of the function f between a and b, expression (a) tells us that the average rate of change of f between $a=0$ and $b=2$ is 0.34. That is, the city's population grew at an average rate of 0.34 million people per year between 2000 and 2002. Similarly, expression (b) is the average rate of change between $a=1$ and $b=2$, so that the city's population grew at an average rate of 0.31 million people per year in 2001. Finally, expression (c) gives the instantaneous rate of change of the population at time $t=2$. As of January 1, 2002, the city's population was growing at a rate of 0.3 million people per year. ■

You hopefully noticed that we tacked the phrase "provided the limit exists" onto the end of the definitions of the slope of a tangent line, the instantaneous velocity and the instantaneous rate of change. This was important, since these defining limits do not always exist, as we see in example 1.7.

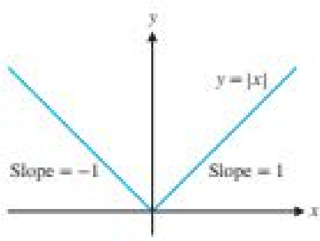


FIGURE 7.12
$y=|x|$

EXAMPLE 1.7 A Graph with No Tangent Line at a Point

Determine whether there is a tangent line to $y=|x|$ at $x=0$.

Solution From the graph in Figure 7.12, notice that, no matter how far we zoom in on (0, 0), the graph continues to look like Figure 7.12. (This is one reason why we left off the scale on Figure 7.12.) This indicates that the tangent line does not exist. Further, if h is any positive number, the slope of the secant line through (0, 0) and $(h, |h|)$ is 1. However, the secant line through (0, 0) and $(h, |h|)$ for any negative number h has slope -1. Defining $f(x)=|x|$ and considering one-sided limits, if $h>0$, then $|h|=h$, so that

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{|h|-0}{h}=\lim_{h\to 0^+}\frac{h}{h}=1.$$

On the other hand, if $h<0$, then $|h|=-h$ (remember that if $h<0$, $-h>0$), so that

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{|h|-0}{h}=\lim_{h\to 0^-}\frac{-h}{h}=-1.$$

Since the one-sided limits are different, we conclude that

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}\text{ does not exist}$$

and hence, the tangent line does not exist. ■

EXERCISES 7.1

WRITING EXERCISES

1. What does the phrase "off on a tangent" mean? Relate the common meaning of the phrase to the image of a tangent to a circle. In what way does Figure 7.4d promote a different view of the relationship between a curve and its tangent?

2. In general, the instantaneous velocity of an object cannot be computed directly; the limit process is the only way to compute velocity *at an instant* from its position function. Given this, how does a car's speedometer compute speed? (Hint: Look this up in a reference book or on the internet.)

3. Look in the news media and find references to at least five different *rates*. We have defined a rate of change as the limit of the difference quotient of a function. For your five examples, state as precisely as possible what the original function is. Is the rate given as a percentage or a number? In calculus, we usually compute rates as numbers; is this in line with the standard usage?

4. Sketch the graph of a function that is discontinuous at $x = 1$. Then sketch the graph of a function that is continuous at $x = 1$ but has no tangent line at $x = 1$. In both cases, explain why there is no tangent line at $x = 1$.

In exercises 1–8, use Definition 1.1 to find an equation of the tangent line to $y = f(x)$ at $x = a$. Graph $y = f(x)$ and the tangent line to verify that you have the correct equation.

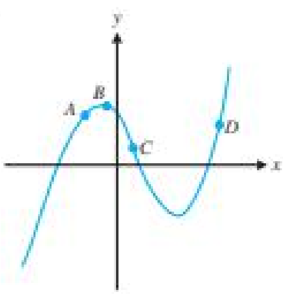
1. $f(x) = x^2 - 2,\ a = 1$
2. $f(x) = x^2 - 2,\ a = 0$
3. $f(x) = x^2 - 3x,\ a = -2$
4. $f(x) = x^3 + x,\ a = 1$
5. $f(x) = \dfrac{2}{x+1},\ a = 1$
6. $f(x) = \dfrac{x}{x-1},\ a = 0$
7. $f(x) = \sqrt{x+3},\ a = -2$
8. $f(x) = \sqrt{x+3},\ a = 1$

In exercises 9–12, compute the slope of the secant line between the points at (a) $x = 1$ and $x = 2$, (b) $x = 2$ and $x = 3$, (c) $x = 1.5$ and $x = 2$, (d) $x = 2$ and $x = 2.5$, (e) $x = 1.9$ and $x = 2$, (f) $x = 2$ and $x = 2.1$, and (g) use parts (a)–(f) and other calculations as needed to estimate the slope of the tangent line at $x = 2$.

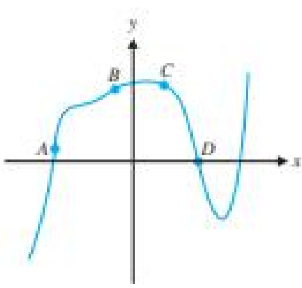
9. $f(x) = x^3 - x$
10. $f(x) = \sqrt{x^2 + 1}$
11. $f(x) = \dfrac{x-1}{x+1}$
12. $f(x) = e^x$

In exercises 13 and 14, list the points A, B, C and D in order of increasing slope of the tangent line.

13.



14.



In exercises 15–18, use the position function s (in meters) to find the velocity at time $t = a$ seconds.

15. $s(t) = -4.9t^2 + 5$, (a) $a = 1$; (b) $a = 2$
16. $s(t) = 4t - 4.9t^2$, (a) $a = 0$; (b) $a = 1$
17. $s(t) = \sqrt{t + 16}$, (a) $a = 0$; (b) $a = 2$
18. $s(t) = 4/t$, (a) $a = 2$; (b) $a = 4$

In exercises 19–22, the function represents the position in feet of an object at time t seconds. Find the average velocity between (a) $t = 0$ and $t = 2$, (b) $t = 1$ and $t = 2$, (c) $t = 1.9$ and $t = 2$, (d) $t = 1.99$ and $t = 2$, and (e) estimate the instantaneous velocity at $t = 2$.

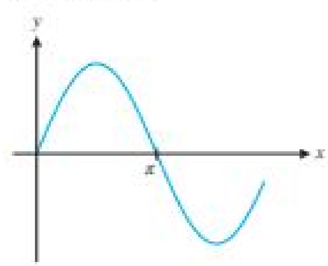
19. $s(t) = 16t^2 + 10$
20. $s(t) = 3t^3 + t$
21. $s(t) = \sqrt{t^2 + 8t}$
22. $s(t) = 3\sin(t - 2)$

In exercises 23–26, use graphical and numerical evidence to explain why a tangent line to the graph of $y = f(x)$ at $x = a$ does not exist.

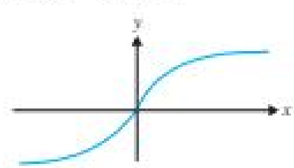
23. $f(x) = |x - 1|$ at $a = 1$
24. $f(x) = \dfrac{4x}{x-1}$ at $a = 1$
25. $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$ at $a = 0$
26. $f(x) = \begin{cases} -2x & \text{if } x < 0 \\ x^2 - 4x & \text{if } x > 0 \end{cases}$ at $a = 0$

In exercises 27–30, sketch in a plausible tangent line at the given point, or state that there is no tangent line.

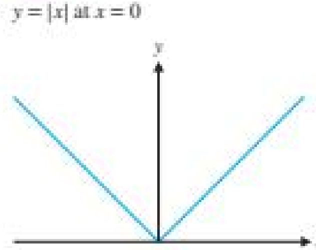
27. $y = \sin x$ at $x = \pi$



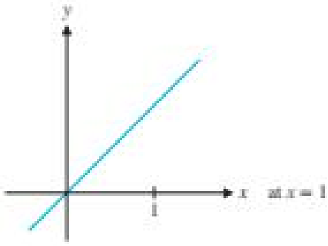
28. $y = \tan^{-1} x$ at $x = 0$



29. $y = |x|$ at $x = 0$



30. $y = x$ at $x = 1$



In exercises 31 and 32, interpret (a)–(c), as in example 1.6.

31. Suppose that $f(t)$ represents the balance in dollars of a bank account t years after January 1, 2000. (a) $\dfrac{f(4) - f(2)}{2} = 21{,}034$, (b) $2[f(4) - f(3.5)] = 25{,}036$ and (c) $\lim\limits_{h \to 0} \dfrac{f(4 + h) - f(4)}{h} = 30{,}000$.

32. Suppose that $f(m)$ represents the value of a car in dollars that has been driven m thousand miles. (a) $\dfrac{f(40) - f(38)}{2} = -2103$, (b) $f(40) - f(39) = -2040$ and (c) $\lim\limits_{h \to 0} \dfrac{f(40 + h) - f(40)}{h} = 2000$.

33. Sometimes an incorrect method accidentally produces a correct answer. For quadratic functions (but definitely *not* most other functions), the average velocity between $t = r$ and $t = s$ equals the average of the velocities at $t = r$ and $t = s$. To show this, assume that $f(t) = at^2 + bt + c$ is the distance function. Show that the average velocity between $t = r$ and $t = s$ equals $a(s + r) + b$. Show that the velocity at $t = r$ is $2ar + b$ and the velocity at $t = s$ is $2as + b$. Finally, show that $a(s + r) + b = \dfrac{(2ar + b) + (2as + b)}{2}$.

34. Find a cubic function [try $f(t) = t^3 + \cdots$] and numbers r and s such that the average velocity between $t = r$ and $t = s$ is different from the average of the velocities at $t = r$ and $t = s$.

35. (a) Find all points at which the slope of the tangent line to $y = x^3 + 3x + 1$ equals 5.

(b) Show that the slope of the tangent line to $y = x^3 + 3x + 1$ cannot equal 1 at any point.

36. (a) Show that the graphs of $y = x^2 + 1$ and $y = x$ do not intersect.

(b) Find the value of x such that the tangent lines to $y = x^2 + 1$ and $y = x$ are parallel.

37. (a) Find an equation of the tangent line to $y = x^3 + 3x + 1$ at $x = 1$.

(b) Show that the tangent line in part (a) intersects $y = x^3 + 3x + 1$ at more than one point.

(c) Show that for any number c the tangent line to $y = x^2 + 1$ at $x = c$ only intersects $y = x^2 + 1$ at one point.

38. Show that $\lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h} = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$. (Hint: Let $h = x - a$.)

APPLICATIONS

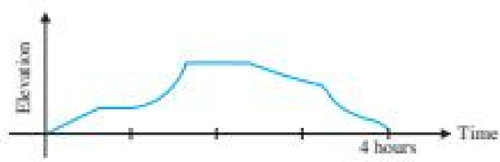
1. The table shows the freezing temperature of water in degrees Celsius at various pressures. Estimate the slope of the tangent line at $p = 1$ and interpret the result. Estimate the slope of the tangent line at $p = 3$ and interpret the result.

p (atm)	0	1	2	3	4
°C	0	−7	−20	−16	−11

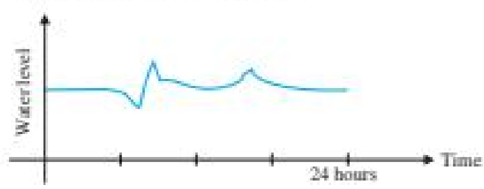
2. The table shows the range of a soccer kick launched at 30° above the horizontal at various initial speeds. Estimate the slope of the tangent line at $v = 50$ and interpret the result.

Distance (yd)	19	28	37	47	58
Speed (mph)	30	40	50	60	70

3. The graph shows the elevation of a person on a climb up a cliff as a function of time. When did the climber reach the top? When was the hiker going the fastest on the way up? When was the hiker going the fastest on the way down? What do you think occurred at places where the graph is level?



4. The graph shows the amount of water in a city water tank as a function of time. When was the tank the fullest? the emptiest? When was the tank filling up at the fastest rate? When was the tank emptying at the fastest rate? What time of day do you think the level portion represents?



5. Suppose a hot cup of coffee is left in a room for 2 hours. Sketch a reasonable graph of what the temperature would look like as a function of time. Then sketch a graph of what the rate of change of the temperature would look like.

6. Sketch a graph representing the height of a bungee-jumper. Sketch the graph of the person's velocity (use + for upward velocity and − for downward velocity).

EXPLORATORY EXERCISES

1. A car moves on a road that takes the shape of $y = x^2$. The car moves from left to right, and its headlights illuminate a deer standing at the point $\left(1, \frac{3}{4}\right)$. Find the location of the car. If the car moves from right to left, how does the answer change? Is there a location (x, y) such that the car's headlights would never illuminate (x, y)?

2. What is the peak speed for a human being? It has been estimated that Carl Lewis reached a peak speed of 28 mph while winning a gold medal in the 1992 Olympics. Suppose that we have the following data for a sprinter.

Meters	*Seconds*
50	5.16666
56	5.76666
58	5.93333
60	6.1

Meters	*Seconds*
62	6.26666
64	6.46666
70	7.06666

We want to estimate peak speed. We could start by computing $\frac{\text{distance}}{\text{time}} = \frac{100\text{ m}}{10\text{ s}} = 10$ m/s, but this is the average speed over the entire race, not the peak speed. Argue that we want to compute average speeds only using adjacent measurements (e.g., 50 and 56 meters). Do this for all 6 adjacent pairs and find the largest speed (if you want to convert to mph, divide by 0.447).

Notice that all times are essentially multiples of 1/30, indicating a video capture rule of 30 frames per second. Given this, why is it suspicious that all the distances are whole numbers? To get an idea of how much this might affect your calculations, change some of the distances. For instance, if you change 60 (meters) to 59.8, how much do your average velocity calculations change? One way to identify where mistakes have been made is to look at the pattern of average velocities: does it seem reasonable? In places where the pattern seems suspicious, try adjusting the distances and produce a more realistic pattern. Try to quantify your error analysis: what is the highest (and lowest) the peak speed could be?

7.2 THE DERIVATIVE

In section 7.1, we investigated two seemingly unrelated concepts: slopes of tangent lines and velocity, both of which are expressed in terms of the *same* limit. This is an indication of the power of mathematics, that otherwise unrelated notions are described by the *same* mathematical expression. This particular limit turns out to be so useful that we give it a special name.

DEFINITION 2.1

The **derivative** of the function f at the point $x = a$ is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \tag{2.1}$$

provided the limit exists. If the limit exists, we say that f is **differentiable** at $x = a$.

An alternative form of (2.1) is

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}. \tag{2.2}$$

(See exercise 38 in section 7.1.)

EXAMPLE 2.1 Finding the Derivative at a Point

Compute the derivative of $f(x) = 3x^3 + 2x - 1$ at $x = 1$.

Solution From (2.1), we have

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{[3(1+h)^3 + 2(1+h) - 1] - (3 + 2 - 1)}{h} \\ &= \lim_{h \to 0} \frac{3(1 + 3h + 3h^2 + h^3) + (2 + 2h) - 1 - 4}{h} && \text{Multiply out and cancel.} \\ &= \lim_{h \to 0} \frac{11h + 9h^2 + 3h^3}{h} && \text{Factor out common } h \text{ and cancel.} \\ &= \lim_{h \to 0} (11 + 9h + 3h^2) = 11. \end{aligned}$$

■

Suppose that in example 2.1 we had also needed to find $f'(2)$ and $f'(3)$. Rather than repeat the same long limit calculation to find each of $f'(2)$ and $f'(3)$ in example 2.2, we compute the derivative without specifying a value for x, leaving us with a function from which we can calculate $f'(a)$ for any a, simply by substituting a for x.

EXAMPLE 2.2 Finding the Derivative at an Unspecified Point

Find the derivative of $f(x) = 3x^3 + 2x - 1$ at an unspecified value of x. Then, evaluate the derivative at $x = 1$, $x = 2$ and $x = 3$.

Solution Replacing a with x in the definition of the derivative (2.1), we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[3(x+h)^3 + 2(x+h) - 1] - (3x^3 + 2x - 1)}{h} \\ &= \lim_{h\to 0} \frac{3(x^3 + 3x^2h + 3xh^2 + h^3) + (2x + 2h) - 1 - 3x^3 - 2x + 1}{h} && \text{Multiply out and cancel.} \\ &= \lim_{h\to 0} \frac{9x^2h + 9xh^2 + 3h^3 + 2h}{h} && \text{Factor out common } h \text{ and cancel.} \\ &= \lim_{h\to 0} (9x^2 + 9xh + 3h^2 + 2) \\ &= 9x^2 + 0 + 0 + 2 = 9x^2 + 2. \end{aligned}$$

Notice that in this case, we have derived a new *function*, $f'(x) = 9x^2 + 2$. Simply substituting in for x, we get $f'(1) = 9 + 2 = 11$ (the same as we got in example 2.1!), $f'(2) = 9(4) + 2 = 38$ and $f'(3) = 9(9) + 2 = 83$. ■

Example 2.2 leads us to the following definition:

DEFINITION 2.2

The **derivative** of the function f is the function f' given by

$$f'(x) = \lim_{h\to\infty} \frac{f(x+h) - f(x)}{h}. \tag{2.3}$$

The domain of f' is the set of all x's for which this limit exists. The process of computing a derivative is called **differentiation**. Further, f is **differentiable on an open interval** I if it is differentiable at every point in I.

In examples 2.3 and 2.4, observe that finding a derivative involves writing down the defining limit and then finding some way of evaluating that limit (which initially has the indeterminate form $\frac{0}{0}$).

EXAMPLE 2.3 Finding the Derivative of a Simple Rational Function

If $f(x) = \frac{1}{x}$ $(x \neq 0)$, find $f'(x)$.

Solution We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\left(\dfrac{1}{x+h} - \dfrac{1}{x}\right)}{h} && \text{Since } f(x+h) = \frac{1}{x+h}. \end{aligned}$$

$$= \lim_{h\to 0} \frac{\left[\dfrac{x-(x+h)}{x(x+h)}\right]}{h} \quad \text{Add fractions and cancel.}$$

$$= \lim_{h\to 0} \frac{-h}{hx(x+h)} \quad \text{Cancel } h\text{'s.}$$

$$= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2},$$

so that $f'(x) = -x^{-2}$. ■

EXAMPLE 2.4 The Derivative of the Square Root Function

If $f(x) = \sqrt{x}$ (for $x \ge 0$), find $f'(x)$.

Solution We have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) \quad \text{Multiply numerator and denominator by the conjugate: } \sqrt{x+h}+\sqrt{x}$$

$$= \lim_{h\to 0} \frac{(x+h)-x}{h\left[\sqrt{x+h}+\sqrt{x}\right]} \quad \text{Multiply out and cancel.}$$

$$= \lim_{h\to 0} \frac{h}{h\left[\sqrt{x+h}+\sqrt{x}\right]} \quad \text{Cancel common } h\text{'s.}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}.$$

Notice that $f'(x)$ is defined only for $x > 0$, even though $f(x)$ is defined for $x \ge 0$. ■

The benefits of having a derivative *function* go well beyond simplifying the computation of a derivative at multiple points. As we'll see, the derivative function tells us a great deal about the original function.

Keep in mind that the value of the derivative function at a point is the slope of the tangent line at that point. In Figures 7.13a–7.13c, we have graphed a function along with its tangent lines at three different points. The slope of the tangent line in Figure 7.13a

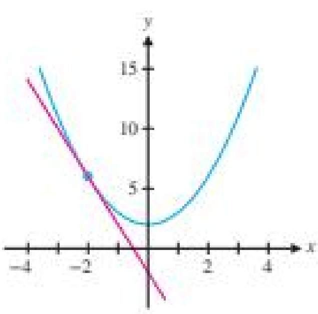


FIGURE 7.13a
$m_{\tan} < 0$

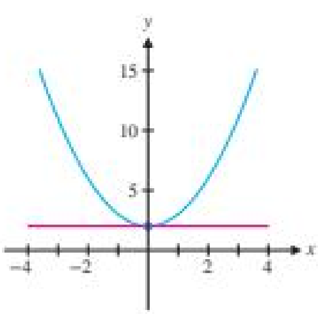


FIGURe 7.13b
$m_{\tan} = 0$

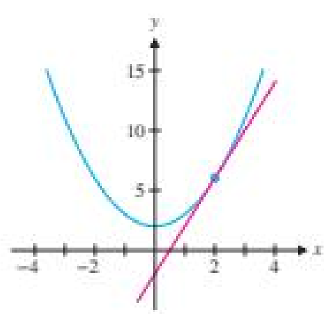


FIGURe 7.13c
$m_{\tan} > 0$

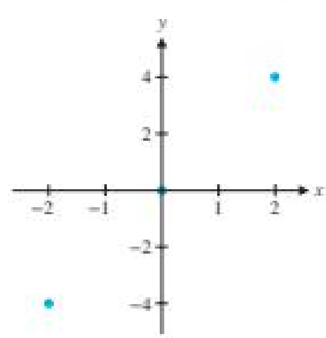


FIGURE 7.13d
$y = f'(x)$ (three points)

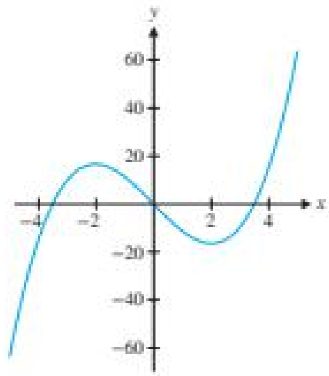


Figure 7.14
$y = f(x)$

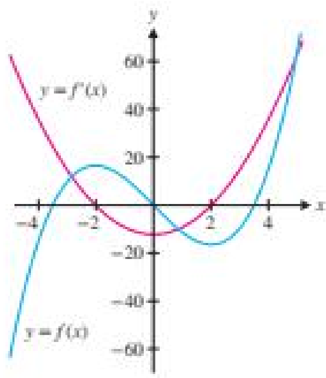


Figure 7.15
$y = f(x)$ and $y = f'(x)$

is negative; the slope of the tangent line in Figure 7.13c is positive and the slope of the tangent line in Figure 7.13b is zero. These three tangent lines give us three points on the graph of the derivative function (see Figure 7.13d), by estimating the value of $f'(x)$ at the three points.

EXAMPLE 2.5 Sketching the Graph of f' Given the Graph of f

Given the graph of f in Figure 7.14, sketch a plausible graph of f'.

Solution Rather than worrying about exact values of $f'(x)$, we only wish to find the general shape of its graph. As in Figures 7.13a–7.13d, pick a few important points to analyze carefully. You should focus on any discontinuities and any places where the graph of f turns around.

The graph of $y = f(x)$ levels out at approximately $x = -2$ and $x = 2$. At these points, the derivative is 0. As we move from left to right, the graph rises for $x < -2$, drops for $-2 < x < 2$ and rises again for $x > 2$. This means that $f'(x) > 0$ for $x < -2$, $f'(x) < 0$ for $-2 < x < 2$ and finally $f'(x) > 0$ for $x > 2$. We can say even more. As x approaches -2 from the left, observe that the tangent lines get less steep. Therefore, $f'(x)$ becomes less positive as x approaches -2 from the left. Moving to the right from $x = -2$, the graph gets steeper until about $x = 0$, then gets less steep until it levels out at $x = 2$. Thus, $f'(x)$ gets more negative until $x = 0$, then less negative until $x = 2$. Finally, the graph gets steeper as we move to the right from $x = 2$. Putting this all together, we have the possible graph of f' shown in red in Figure 7.15, superimposed on the graph of f. ■

It is even more interesting to ask what the graph of $y = f(x)$ looks like given the graph of $y = f'(x)$. We explore this in example 2.6.

EXAMPLE 2.6 Sketching the Graph of f Given the Graph of f'

Given the graph of f' in Figure 7.16, sketch a plausible graph of f.

Solution Again, do not worry about getting exact values of the function, but rather only the general shape of the graph. Notice from the graph of $y = f'(x)$ that $f'(x) < 0$ for $x < -2$, so that on this interval, the slopes of the tangent lines to $y = f(x)$ are negative and the graph is falling. On the interval $(-2, 1)$, $f'(x) > 0$, indicating that the tangent lines to the graph of $y = f(x)$ have positive slope and the graph is rising. Further, this says that the graph turns around (i.e., goes from falling to rising) at $x = -2$. Further, $f'(x) < 0$ on the interval $(1, 3)$, so that the graph falls here. Finally, for $x > 3$, we have that $f'(x) > 0$, so that the graph is rising here. We show a graph exhibiting all of these behaviors superimposed on the graph of $y = f(x)$ in Figure 7.17. We have

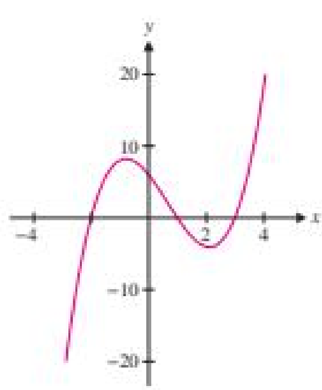


Figure 7.16
$y = f'(x)$

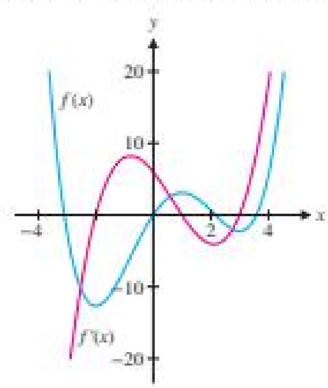


Figure 7.17
$y = f'(x)$ and a plausible graph of $y = f(x)$

drawn the graph of f so that the small "valley" on the right side of the y-axis is not as deep as the one on the left side of the y-axis for a reason. Look carefully at the graph of $f'(x)$ and notice that $|f'(x)|$ gets much larger on $(-2, 1)$ than on $(1, 3)$. This says that the tangent lines and hence, the graph will be much steeper on the interval $(-2, 1)$ than on $(1, 3)$. ■

Alternative Derivative Notations

We have denoted the derivative function by f'. There are other commonly used notations for f', each with advantages and disadvantages. One of the coinventors of calculus, Gottfried Leibniz, used the notation $\frac{df}{dx}$ (**Leibniz notation**) for the derivative. If we write $y = f(x)$, the following are all alternatives for denoting the derivative:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx}\frac{d}{dx}f(x).$$

The expression $\frac{d}{dx}$ is called a **differential operator** and tells you to take the derivative of whatever expression follows.

In section 7.1, we observed that $f(x) = |x|$ does not have a tangent line at $x = 0$ (i.e., it is not differentiable at $x = 0$), although it is continuous everywhere. Thus, there are continuous functions that are not differentiable. You might have already wondered whether the reverse is true. That is, are there differentiable functions that are not continuous? The answer is "no," as provided by Theorem 2.1.

HISTORICAL NOTES

Gottfried Wilhelm Leibniz (1646–1716) A German mathematician and philosopher who introduced much of the notation and terminology in calculus and who is credited (together with Sir Isaac Newton) with inventing the calculus. Leibniz was a prodigy who had already received his law degree and published papers on logic and jurisprudence by age 20. A true Renaissance man, Leibniz made important contributions to politics, philosophy, theology, engineering, linguistics, geology, architecture and physics, while earning a reputation as the greatest librarian of his time. Mathematically, he derived many fundamental rules for computing derivatives and helped promote the development of calculus through his extensive communications. The simple and logical notation he invented made calculus accessible to a wide audience and has only been marginally improved upon in the intervening 300 years. He wrote,

"We observe in signs an advantage in discovery that is greatest when they express and, as it were, depict the innermost nature of a thing in a few signs, for then the labor of thought is wonderfully diminished."[1]

THEOREM 2.1

If f is differentiable at $x = a$, then f is continuous at $x = a$.

PROOF

For f to be continuous at $x = a$, we need only show that $\lim_{x\to a} f(x) = f(a)$. We consider

$$\begin{aligned}\lim_{x\to a}[f(x) - f(a)] &= \lim_{x\to a}\left[\frac{f(x) - f(a)}{x - a}(x - a)\right] && \text{Multiply and divide by } (x - a).\\ &= \lim_{x\to a}\left[\frac{f(x) - f(a)}{x - a}\right]\lim_{x\to a}(x - a) && \text{By Theorem 3.1 (iii) from section 6.3.}\\ &= f'(a)(0) = 0, && \text{Since } f \text{ is differentiable at } x = a.\end{aligned}$$

where we have used the alternative definition of derivative (2.2) discussed earlier. By Theorem 3.1 in section 6.3, it now follows that

$$\begin{aligned}0 &= \lim_{x\to a}[f(x) - f(a)] = \lim_{x\to a} f(x) - \lim_{x\to a} f(a)\\ &= \lim_{x\to a} f(x) - f(a),\end{aligned}$$

which gives us the result. ■

Note that Theorem 2.1 says that if a function is *not* continuous at a point, then it *cannot* have a derivative at that point. It also turns out that functions are not differentiable at any point where their graph has a "sharp" corner, as is the case for $f(x) = |x| = 0$. (See example 1.7 in the previous section.)

[1] Letter from Leibniz to Ehrenfried Walther von Tschirnhaus. Reproduced in *Briefwechsel von G. W. Leibniz mit Mathematikern. Erster Band*, ed. C. I. Gerhardt, vol. I, p. 375 (Berlin: Mayer & Müller, 1899). Available at: http://quod.lib.umich.edu/u/umhistmath/aax2762.0001.001?view=toc

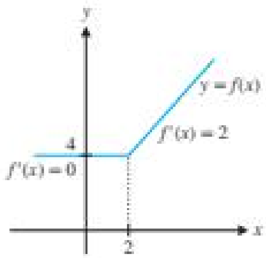


Figure 7.18
A sharp corner

EXAMPLE 2.7 Showing That a Function is Not Differentiable at a Point

Show that $f(x) = \begin{cases} 4 & \text{if } x < 2 \\ 2x & \text{if } x \geq 2 \end{cases}$ is not differentiable at $x = 2$.

Solution The graph (see Figure 7.18) indicates a sharp corner at $x = 2$, so you might expect that the derivative does not exist. To verify this, we investigate the derivative by evaluating one-sided limits. For $h > 0$, note that $(2 + h) > 2$ and so, $f(2 + h) = 2(2 + h)$. This gives us

$$\lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^+} \frac{2(2+h) - 4}{h}$$

$$= \lim_{h\to 0^+} \frac{4 + 2h - 4}{h} \quad \text{Multiply out and cancel.}$$

$$= \lim_{h\to 0^+} \frac{2h}{h} = 2. \quad \text{Cancel common } h\text{'s.}$$

Likewise, if $h < 0$, $(2 + h) < 2$ and so, $f(2 + h) = 4$. Thus, we have

$$\lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^-} \frac{4 - 4}{h} = 0.$$

Since the one-sided limits do not agree $(0 \neq 2)$, $f'(2)$ does not exist (i.e., f is not differentiable at $x = 2$). ■

Figures 7.19a–7.19d show a variety of functions for which $f'(a)$ does not exist. In each case, convince yourself that the derivative does not exist.

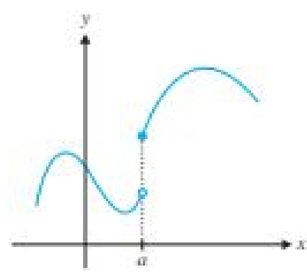


FIGURE 7.19a
A jump discontinuity

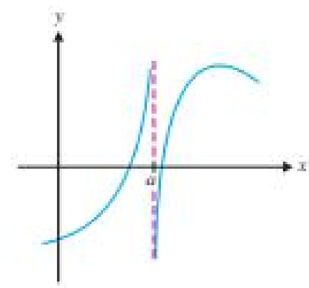


FIGURE 7.19b
A vertical asymptote

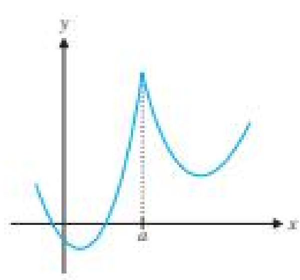


FIGURE 7.19c
A cusp

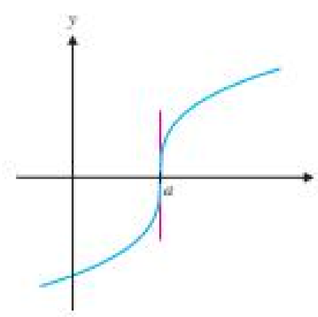


FIGURE 7.19d
A vertical tangent line

Numerical Differentiation

There are many times in applications when it is not possible or practical to compute derivatives symbolically. This is frequently the case where we have only some data (i.e., a table of values) representing an otherwise *unknown* function.

EXAMPLE 2.8 Approximating a Derivative Numerically

Numerically estimate the derivative of $f(x) = x^2\sqrt{x^3 + 2}$ at $x = 1$.

Solution Although working through the limit definition of derivative for this function is a challenge, the definition tells us that the derivative at $x = 1$ is the limit of slopes of secant lines. We compute some of these below:

h	$\frac{f(1+h)-f(1)}{h}$
0.1	4.7632
0.01	4.3715
0.001	4.3342

h	$\frac{f(1+h)-f(1)}{h}$
−0.1	3.9396
−0.01	4.2892
−0.001	4.3260

Notice that the slopes seem to be converging to approximately 4.33 as h approaches 0. Thus, we make the approximation $f'(1) \approx 4.33$. ■

EXAMPLE 2.9 Estimating Velocity Numerically

Suppose that a sprinter reaches the following distances in the given times. Estimate the velocity of the sprinter at the 6-second mark.

t (sec)	5.0	5.5	5.8	5.9	6.0	6.1	6.2	6.5	7.0
$f(t)$ (ft)	123.7	141.01	151.41	154.90	158.40	161.92	165.42	175.85	193.1

Time Interval	*Average Velocity*
(5.9, 6.0)	35.0 m/s
(6.0, 6.1)	35.2 m/s

Time Interval	*Average Velocity*
(5.5, 6.0)	34.78 m/s
(5.8, 6.0)	34.95 m/s
(5.9, 6.0)	35.00 m/s
(6.0, 6.1)	35.20 m/s
(6.0, 6.2)	35.10 m/s
(6.0, 6.5)	34.90 m/s

Solution The instantaneous velocity is the limit of the average velocity as the time interval shrinks. We first compute the average velocities over the shortest intervals given, from 5.9 to 6.0 and from 6.0 to 6.1.

Since these are the best individual estimates available from the data, we could just split the difference and estimate a velocity of 35.1 ft/s. However, there is useful information in the rest of the data. Based on the accompanying table, we can conjecture that the sprinter was reaching a peak speed at about the 6-second mark. Thus, we might accept the higher estimate of 35.2 ft/s. We should emphasize that there is not a single correct answer to this question, since the data are incomplete (i.e., we know the distance only at fixed times, rather than over a continuum of times). ■

BEYOND FORMULAS

In the next sections, we derive numerous formulas for computing derivatives. As you learn these formulas, keep in mind the reasons that we are interested in the derivative. Careful studies of the slope of the tangent line to a curve and the velocity of a moving object led us to the same limit, which we named the *derivative*. In general, the derivative represents the rate of change of one quantity with respect to another quantity. The study of change in a quantifiable way has led directly to countless advances in modern science and engineering.

EXERCISES 7.2

WRITING EXERCISES

1. The derivative is important because of its many different uses and interpretations. Describe four aspects of the derivative: graphical (think of tangent lines), symbolic, numerical and applications.
2. Mathematicians often use the word "smooth" to describe functions with certain properties. Graphically, how are differentiable functions smoother than functions that are continuous but not differentiable, or functions that are not continuous?
3. Briefly describe what the derivative tells you about the original function. In particular, if the derivative is positive at a point, what do you know about the trend of the function at that point? What is different if the derivative is negative at that point?
4. The derivative of $f(x) = 3x - 5$ is $f'(x) = 3$. Explain in terms of slope why this is true.

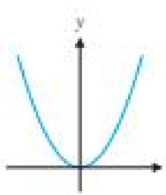
In exercises 1–4, compute $f'(a)$ using the limits (2.1) and (2.2).

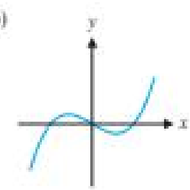
1. $f(x) = 3x + 1, a = 1$
2. $f(x) = 3x^2 + 1, a = 1$
3. $f(x) = \sqrt{3x+1}, a = 1$
4. $f(x) = \dfrac{3}{x+1}, a = 2$

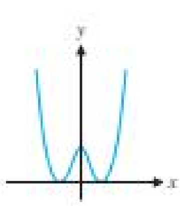
In exercises 5–12, compute the derivative function f' using (2.1) or (2.2).

5. $f(x) = 3x^2 + 1$
6. $f(x) = x^2 - 2x + 1$
7. $f(x) = x^3 + 2x - 1$
8. $f(x) = x^4 - 2x^2 + 1$
9. $f(x) = \dfrac{3}{x+1}$
10. $f(x) = \dfrac{2}{2x-1}$
11. $f(t) = \sqrt{3t+1}$
12. $f(t) = \sqrt{2t+4}$

In exercises 13–16, use the graph of f to sketch a graph of f'.

13. (a)


(b)


14. (a)


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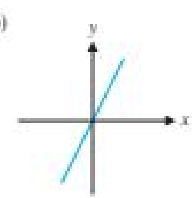
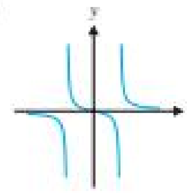
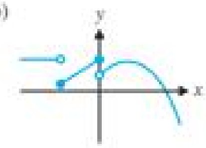
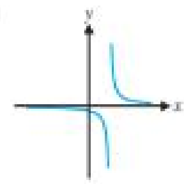
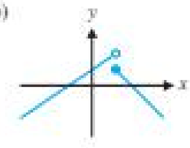


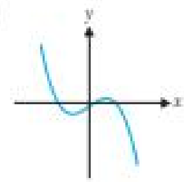
15. (a)


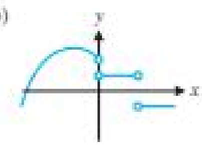
(b)


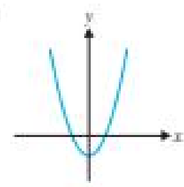
16. (a)


(b)


In exercises 17 and 18, use the given graph of f' to sketch a plausible graph of a continuous function f.

17. (a)


(b)


18. (a)


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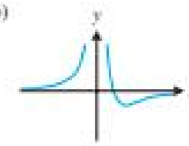


In exercises 19–22, compute the right-hand derivative. $D_+f(0) = \lim_{h\to 0^+} \dfrac{f(h) - f(0)}{h}$ and the left-hand derivative $D_-f(0) = \lim_{h\to 0^-} \dfrac{f(h) - f(0)}{h}$. Does $f'(0)$ exist?

19. $f(x) = \begin{cases} 2x + 1 & \text{if } x < 0 \\ 3x + 1 & \text{if } x \ge 0 \end{cases}$

20. $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x & \text{if } x \ge 0 \end{cases}$

21. $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x^3 & \text{if } x \ge 0 \end{cases}$

22. $f(x) = \begin{cases} 2x & \text{if } x < 0 \\ x^2 + 2x & \text{if } x \ge 0 \end{cases}$

In exercises 23–26, numerically estimate the derivative.

23. $f'(1)$ for $f(x) = \dfrac{x}{\sqrt{x^2+1}}$

24. $f'(2)$ for $f(x) = xe^{x^2}$

25. $f'(0)$ for $f(x) = \cos 3x$

26. $f'(2)$ for $f(x) = \ln 3x$

In exercises 27 and 28, use the distances $f(t)$ to estimate the velocity at $t = 2$.

27.

t	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$f(t)$	3.1	3.9	4.8	5.8	6.8	7.7	8.5

28.

t	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$f(t)$	4.6	5.3	6.1	7.0	7.8	8.6	9.3

29. Graph and identify all x-values at which f is not differentiable.

(a) $f(x) = |x| + |x-2|$, (b) $f(x) = |x^2 - 4x|$

30. Graph and identify all x-values at which g is not differentiable.

(a) $g(x) = e^{-2|x|}$, (b) $g(x) = e^{-2|x^3-x|}$

31. For $f(x) = x^p$ find all real numbers p such that $f'(0)$ exists.

32. For $f(x) = \begin{cases} x^2 + 2x & x < 0 \\ ax + b & x \ge 0 \end{cases}$ find all real numbers a and b such that $f'(0)$ exists.

33. Give an example showing that the following is not true for all functions f: if $f(x) \le x$, then $f'(x) \le 1$ for all x.

34. Determine whether the following is true for all functions f: if $f(0) = 0$, $f'(x)$ exists for all x and $f(x) \le x$ then $f'(x) \le 1$ for all x.

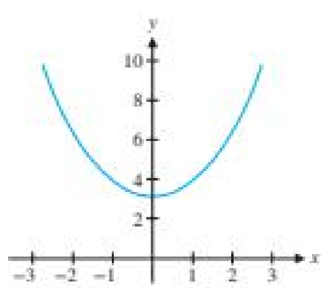
35. If f is differentiable at $x = a \ne 0$, evaluate

$$\lim_{x \to a} \frac{[f(x)]^2 - [f(a)]^2}{x^2 - a^2}$$

36. Prove that if f is differentiable at $x = a$, then

$$\lim_{h \to 0} \frac{f(a+ch) - f(a)}{h} = cf'(a).$$

37. Use the graph to list the following in increasing order: $f(1)$, $f(2) - f(1)$, $\dfrac{f(1.5) - f(1)}{0.5}$, $f'(1)$.



38. Use the graph to list the following in increasing order:

$f(0)$, $f(0) - f(-1)$, $\dfrac{f(0) - f(-0.5)}{0.5}$, $f'(0)$

39. Sketch the graph of a function with the following properties: $f(0) = 1$, $f(1) = 0$, $f(3) = 6$, $f'(0) = 0$, $f'(1) = -1$ and $f'(3) = 4$.

40. Sketch the graph of a function with the following properties: $f(-2) = 4$, $f(0) = -2$, $f(2) = 1$, $f'(-2) = -2$, $f'(0) = 2$ and $f'(2) = 1$.

41. Compute the derivative function for x^2, x^3 and x^4. Based on your results, identify the pattern and conjecture a general formula for the derivative of x^n.

42. Test your conjecture from exercise 41 on the functions $\sqrt{x} = x^{1/2}$ and $1/x = x^{-1}$.

APPLICATIONS

1. The table shows the margin of error in degrees for tennis serves hit from a height of x meters. (Data from Jake Bennett, Roanoke College.) Estimate the value of the derivative of the margin of error at $x = 2.5$ and interpret it in terms of the benefit of hitting the ball from larger heights.

x meters	2.39	2.5	2.7	2.85	3
Margin of error	1.11	1.29	1.62	1.87	2.12

2. Use the table in exercise 1 to estimate the derivative at $x = 2.85$. Compare your estimate to that of exercise 1.

3. Suppose that an environmental protection agency uses the measurement of ton-MPG to evaluate the power-train efficiency of vehicles. The ton-MPG rating of a vehicle is given by the weight of the vehicle (in tons) multiplied by a rating of the vehicle's fuel efficiency in miles per gallon. Several years of data for new cars are given in the table. Estimate the rate of change of ton-MPG in (a) 2010 and (b) 2016. Do your estimates imply that cars are becoming more or less efficient?

Year	2008	2010	2012	2014	2016
Ton-MPG	44.9	45.7	46.5	47.3	47.7

4. The fuel efficiencies in miles per gallon of cars from 2008 to 2016 are shown in the following table. Estimate the rate of change in MPG in (a) 2008 and (b) 2016. Do your estimates imply that cars are becoming more or less fuel efficient? Comparing your answers to exercise 3, what must be happening to the average weight of cars? If weight had remained constant, what do you expect would have happened to MPG?

Year	2008	2010	2012	2014	2016
MPG	28.0	28.1	28.3	28.5	28.1

In exercises 5 and 6, give the units for the derivative function.

5. (a) $f(t)$ represents position, measured in meters, at time t seconds.

(b) $f(x)$ represents the demand, in number of items, of a product when the price is x dollars.

6. (a) $c(t)$ represents the amount of a chemical present, in grams, at time t minutes.

(b) $p(x)$ represents the mass, in kilograms, of the first x meters of a pipe.

7. Let $f(t)$ represent the trading value of a stock at time t days. If $f'(t) < 0$, what does that mean about the stock? If you held some shares of this stock, should you sell what you have or buy more?

8. Suppose that there are two stocks with trading values $f(t)$ and $g(t)$, where $f(t) > g(t)$ and $0 < f'(t) < g'(t)$. Based on this information, which stock should you buy? Briefly explain.

9. One model for the spread of a disease assumes that at first the disease spreads very slowly, gradually the infection rate increases to a maximum and then the infection rate decreases back to zero, marking the end of the epidemic. If $I(t)$ represents the number of people infected at time t, sketch a graph of both $I(t)$ and $I'(t)$, assuming that those who get infected do not recover.

10. One model for urban population growth assumes that, at first, the population is growing very rapidly, then the growth rate decreases until the population starts decreasing. If $P(t)$ is the population at time t, sketch a graph of both $P(t)$ and $P'(t)$.

11. A phone company charges one dollar for the first 20 minutes of a call, then 10 cents per minute for the next 60 minutes and 8 cents per minute for each additional minute (or partial minute). Let $f(t)$ be the price in cents of a t-minute phone call, $t > 0$. Determine $f'(t)$ as completely as possible.

12. A state charges 10% income tax on the first \$20,000 of income and 16% on income over \$20,000. Let $f(t)$ be the state tax on \$$t$ of income. Determine $f'(t)$ as completely as possible.

EXPLORATORY EXERCISES

1. Suppose there is a function $f(x)$ such that $F(1) = 1$ and $F(0) = f_0$, where $0 < f_0 < 1$. If $f'(1) > 1$, show graphically that the equation $f(x) = x$ has a solution q where $0 < q < 1$. (Hint: Graph $y = x$ and a plausible $f(x)$ and look for intersections.) Sketch a graph where $f'(1) < 1$ and there are no solutions to the equation $f(x) = x$ with $0 < x < 1$. Solutions have a connection with the probability of the extinction of animals or family names. Suppose you and your descendants have children according to the following probabilities: $f_0 = 0.2$ is the probability of having no children, $f_1 = 0.3$ is the probability of having exactly one child, and $f_2 = 0.5$ is the probability of having two children. Define $f(x) = 0.2 + 0.3x + 0.5x^2$ and show that $f'(1) > 1$. Find the solution of $f(x) = x$ between $x = 0$ and $x = 1$; this number is the probability that your "line" will go extinct some time into the future. Find non-zero values of f_0, f_1 and f_2 such that the corresponding $f(x)$ satisfies $f'(1) < 1$ and hence the probability of your line going extinct is 1.

2. The **symmetric difference quotient** of a function f centered at $x = a$ has the form $\frac{f(a+h) - f(a-h)}{2h}$. If $f(x) = x^2 + 1$ and $a = 1$, illustrate the symmetric difference quotient as a slope of a secant line for $h = 1$ and $h = 0.5$. Based on your picture, conjecture the limit of the symmetric difference quotient as h approaches 0. Then compute the limit and compare to the derivative $f'(1)$ found in example 1.1. For $h = 1$, $h = 0.5$ and $h = 0.1$, compare the actual values of the symmetric difference quotient and the usual difference quotient $\frac{f(a+h) - f(a)}{h}$. In general which difference quotient provides a better estimate of the derivative? Next, compare the values of the difference quotients with $h = 0.5$ and $h = -0.5$ to the derivative $f'(1)$. Explain graphically why one is smaller and one is larger. Compare the average of these two difference quotients to the symmetric difference quotient with $h = 0.5$. Use this result to explain why the symmetric difference quotient might provide a better estimate of the derivative. Next, compute several symmetric difference quotients of $f(x) = \begin{cases} 4 & x < 2 \\ 2x & \text{if } x \geq 2 \end{cases}$ centered at $a = 2$. Recall that in example 2.7 we showed that the derivative $f'(2)$ does not exist. Given this, discuss one major problem with using the symmetric difference quotient to approximate derivatives. Finally, show that if $f'(a)$ exists, then $\lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h} = f'(a)$.

7.3 COMPUTATION OF DERIVATIVES: THE POWER RULE

You have now computed numerous derivatives using the limit definition. In fact, you may have computed enough that you have started taking some shortcuts. We continue that process in this section, by developing some basic rules.

The Power Rule

We first revisit the limit definition of derivative to compute two very simple derivatives.

For any constant c,
$$\frac{d}{dx}c = 0. \tag{3.1}$$

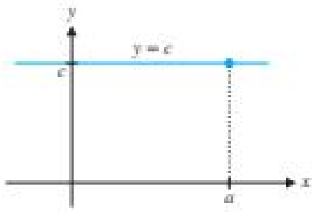


FIGURE 7.20
A horizontal line

Notice that (3.1) says that for any constant c, the horizontal line $y = c$ has a tangent line with zero slope. That is, the tangent line to a horizontal line is the same horizontal line. (See Figure 7.20.)

To prove equation (3.1), let $f(x) = c$, for all x. From the limit definition, we have

$$\frac{d}{dx}c = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

Similarly, we have

$$\frac{d}{dx}x = 1. \quad (3.2)$$

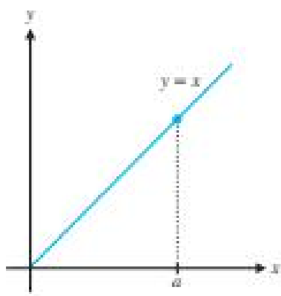


FIGURE 7.21
Tangent line to $y = x$

Notice that (3.2) says that the tangent line to the line $y = x$ is a line of slope one (i.e., $y = x$; see Figure 7.21), which is not surprising.

To verify equation (3.2), we let $f(x) = x$. From the limit definition, we have

$$\frac{d}{dx}x = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x+h) - x}{h}$$
$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1.$$

$f(x)$	$f'(x)$
$1 = x^0$	0
$x = x^1$	$1x^0 = 1$
x^2	$2x^1$
x^3	$3x^2$
x^4	$4x^3$

The table shown in the margin presents a short list of derivatives calculated previously either as examples or in the exercises using the limit definition. Note that the power of x in the derivative is always one less than the power of x in the original function. Further, the coefficient of x in the derivative is the same as the power of x in the original function. This suggests the following result.

THEOREM 3.1 (Power Rule)

For any integer $n > 0$, $$\frac{d}{dx}x^n = nx^{n-1}.$$

PROOF

From the limit definition of derivative given in equation (2.3), if $f(x) = x^n$, then

$$\frac{d}{dx}x^n = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}. \quad (3.3)$$

To evaluate the limit, we will need to simplify the expression in the numerator. Recall that $(x+h)^2 = x^2 + 2xh + h^2$ and $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$. More generally, you may recall from the binomial theorem that for any positive integer n,

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n. \quad (3.4)$$

Substituting (3.4) into (3.3), we get

$$f'(x) = \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n - x^n}{h}$$

Cancel x^n terms

$$= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$

$$= \lim_{h\to 0} \frac{h\left[nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^1 + \cdots + nxh^{n-2} + h^{n-1}\right]}{h}$$

Factor out common h and cancel.

$$= \lim_{h\to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h^1 + \cdots + nxh^{n-2} + h^{n-1}\right] = nx^{n-1},$$

since every term but the first has a factor of h. ■

The power rule is very easy to apply, as we see in example 3.1.

EXAMPLE 3.1 Using the Power Rule

Find the derivative of (a) $f(x) = x^8$ and (b) $g(t) = t^{107}$.

Solution (a) We have

$$f'(x) = \frac{d}{dx}x^8 = 8x^{8-1} = 8x^7.$$

(b) Similarly, $$g'(t) = \frac{d}{dt}t^{107} = 107t^{107-1} = 107t^{106}.$$ ■

Recall that in section 7.2, we showed that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}. \tag{3.5}$$

Notice that we can rewrite (3.5) as

$$\frac{d}{dx}x^{-1} = (-1)x^{-2}.$$

That is, the derivative of x^{-1} follows the same pattern as the power rule that we just stated and proved for *positive* integer exponents.

Likewise, in section 7.2, we used the limit definition to show that

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}. \tag{3.6}$$

We can also rewrite equation (3.6) as $\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}$,

so that the derivative of this rational power of x also follows the same pattern as the power rule that we proved for positive *integer* exponents.

REMARK 3.1

As we will see, the power rule holds for *any* power of x. We will not be able to prove this fact for some time now, as the proof of Theorem 3.1 does *not* generalize, since the expansion in equation (3.4) holds only for positive integer exponents. Even so, we will use the rule freely for any power of x. We state this in Theorem 3.2.

THEOREM 3.2 (General Power Rule)

For any real number $r \neq 0$, $$\frac{d}{dx}x^r = rx^{r-1}. \tag{3.7}$$

The power rule is simple to use, as we see in example 3.2.

CAUTION

Be careful here to avoid a common error:

$$\frac{d}{dx}x^{-19} \neq -19x^{-18}.$$

The power rule says to *subtract* 1 from the exponent (even if the exponent is negative).

EXAMPLE 3.2 Using the General Power Rule

Find the derivative of (a) $f(x) = \frac{1}{x^{19}}$, (b) $g(x) = \sqrt[3]{x^2}$ and (c) $h(x) = x^{\pi}$.

Solution (a) From (3.7), we have

$$f'(x) = \frac{d}{dx}\left(\frac{1}{x^{19}}\right) = \frac{d}{dx}x^{-19} = -19x^{-19-1} = -19x^{-20}.$$

(b) If we rewrite $\sqrt[3]{x^2}$ as a fractional power of x, we can use (3.7) to compute the derivative, as follows.

$$g'(x) = \frac{d}{dx}\sqrt[3]{x^2} = \frac{d}{dx}x^{2/3} = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3}.$$

(c) Finally, we have

$$h'(x) = \frac{d}{dx}x^{\pi} = \pi x^{\pi-1}. \blacksquare$$

Notice that there is the additional conceptual problem in example 3.2 of deciding what x^{π} means. Since the exponent isn't rational, what exactly do we mean when we raise a number to the irrational power π?

General Derivative Rules

The power rule gives us a large class of functions whose derivatives we can quickly compute without using the limit definition. The following rules for combining derivatives further expand the number of derivatives we can compute without resorting to the definition. Keep in mind that a derivative is a limit; the differentiation rules in Theorem 3.3 then follow immediately from the corresponding rules for limits (found in Theorem 3.1 in Chapter 6).

THEOREM 3.3

If $f(x)$ and $g(x)$ are differentiable at x and c is any constant, then

(i) $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$,

(ii) $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$ and

(iii) $\frac{d}{dx}[cf(x)] = cf'(x)$.

PROOF

We prove only part (i). The proofs of parts (ii) and (iii) are left as exercises. Let $k(x) = f(x) + g(x)$. Then, from the limit definition of the derivative (2.3), we get

$$\frac{d}{dx}[f(x) + g(x)] = k'(x) = \lim_{h\to 0}\frac{k(x+h) - k(x)}{h}$$

$$= \lim_{h\to 0}\frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$ By definition of $k(x)$.

$$= \lim_{h\to 0}\frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h}$$ Grouping the f terms together and the g terms together.

$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}$$ By Theorem 3.1 in Chapter 6.

$$= f'(x) + g'(x).$$ Recognizing the derivatives of f and of g. $\blacksquare$

We illustrate Theorem 3.3 by working through the calculation of a derivative step by step, showing all of the details.

EXAMPLE 3.3 Finding the Derivative of a Sum

Find the derivative of $f(x) = 2x^6 + 3\sqrt{x}$

Solution We have

$$\begin{aligned} f'(x) &= \frac{d}{dx}(2x^6) + \frac{d}{dx}(3\sqrt{x}) && \text{By Theorem 3.3 (i).} \\ &= 2\frac{d}{dx}(x^6) + 3\frac{d}{dx}(x^{1/2}) && \text{By Theorem 3.3 (iii).} \\ &= 2(6x^5) + 3\left(\frac{1}{2}x^{-1/2}\right) && \text{By the power rule.} \\ &= 12x^5 + \frac{3}{2\sqrt{x}}. && \text{Simplifying.} \end{aligned}$$

EXAMPLE 3.4 Rewriting a Function before Computing the Derivative

Find the derivative of $f(x) = \dfrac{4x^2 - 3x + 2\sqrt{x}}{x}$.

Solution Since we don't yet have any rule for computing the derivative of a quotient, we first rewrite $f(x)$ by dividing out the x in the denominator. We have

$$f(x) = \frac{4x^2}{x} - \frac{3x}{x} + \frac{2\sqrt{x}}{x} = 4x - 3 + 2x^{-1/2}.$$

From Theorem 3.3 and the power rule (3.7), we get

$$f'(x) = 4\frac{d}{dx}(x) - 3\frac{d}{dx}(1) + 2\frac{d}{dx}(x^{-1/2}) = 4 - 0 + 2\left(-\frac{1}{2}x^{-3/2}\right) = 4 - x^{-3/2}.$$

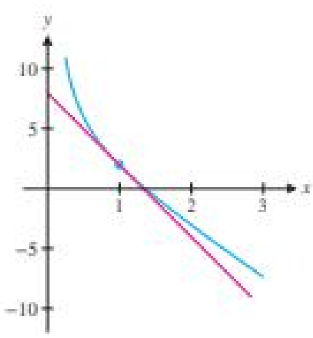


Figure 7.22
$y = f(x)$ and the tangent line at $x = 1$

EXAMPLE 3.5 Finding an Equation of the Tangent Line

Find an equation of the tangent line to the graph of $f(x) = 4 - 4x + \dfrac{2}{x}$ at $x = 1$.

Solution First, notice that $f(x) = 4 - 4x + 2x^{-1}$. From Theorem 3.3 and the power rule, we have

$$f'(x) = 0 - 4 - 2x^{-2} = -4 - 2x^{-2}.$$

At $x = 1$, the slope of the tangent line is then $f'(1) = -4 - 2 = -6$. The line with slope -6 through the point $(1, 2)$ has equation

$$y - 2 = -6(x - 1).$$

We show a graph of $y = f(x)$ and the tangent line at $x = 1$ in Figure 7.22.

○ Higher Order Derivatives

One consequence of having the derivative function is that we can compute the derivative of a derivative. It turns out that such **higher order** derivatives have important applications.

Suppose we start with a function f and compute its derivative f'. We can then compute the derivative of f', called the **second derivative** of f and written f''. We can then compute the derivative of f'', called the **third derivative** of f, written f'''. We can continue to take derivatives indefinitely. Next, we show common notations for the first five derivatives of f [where we assume that $y = f(x)$]. Note that we use primes only for the first three derivatives.

For fourth and higher derivatives, we indicate the order of the derivative in parentheses. Be careful to distinguish these from exponents.

Order	Prime Notation	Leibniz Notation
1	$y' = f'(x)$	$\frac{df}{dx}$
2	$y'' = f''(x)$	$\frac{d^2f}{dx^2}$
3	$y''' = f'''(x)$	$\frac{d^3f}{dx^3}$
4	$y^{(4)} = f^{(4)}(x)$	$\frac{d^4f}{dx^4}$
5	$y^{(5)} = f^{(5)}(x)$	$\frac{d^5f}{dx^5}$

Computing higher order derivatives is done by simply computing several first derivatives, as we see in example 3.6.

EXAMPLE 3.6 Computing Higher Order Derivatives

If $f(x) = 3x^4 - 2x^2 + 1$, compute as many derivatives as possible.

Solution We have

$$f'(x) = \frac{df}{dx} = \frac{d}{dx}(3x^4 - 2x^2 + 1) = 12x^3 - 4x.$$

Then,

$$f''(x) = \frac{d^2f}{dx^2} = \frac{d}{dx}(12x^3 - 4x) = 36x^2 - 4,$$

$$f'''(x) = \frac{d^3f}{dx^3} = \frac{d}{dx}(36x^2 - 4) = 72x,$$

$$f^{(4)}(x) = \frac{d^4f}{dx^4} = \frac{d}{dx}(72x) = 72,$$

$$f^{(5)}(x) = \frac{d^5f}{dx^5} = \frac{d}{dx}(72) = 0$$

and so on. It follows that

$$f^{(n)}(x) = \frac{d^nf}{dx^n} = 0, \text{ for } n \geq 5.$$

Acceleration

What information does the second derivative of a function give us? Graphically, we get a property called *concavity*, which we develop in Chapter 9. One important application of the second derivative is acceleration, which we briefly discuss now.

You are probably familiar with the term **acceleration**, which is the **instantaneous rate of change of velocity**. Consequently, if the velocity of an object at time t is given by $v(t)$, then the acceleration is

$$a(t) = v'(t) = \frac{dv}{dt}.$$

EXAMPLE 3.7 Computing the Acceleration of a Skydiver

Suppose that the height of a skydiver t seconds after jumping from an airplane is given by $f(t) = 200 - 6t - 4.9t^2$ meters. Find the person's acceleration at time t.

Solution Since acceleration is the derivative of velocity, we first compute velocity:

$$v(t) = f'(t) = 0 - 6 - 9.8t = -6 - 9.8t \text{ m/s}.$$

Computing the derivative of this function gives us

$$a(t) = v'(t) = -9.8 \text{ m/s}^2.$$

Since the distance here is measured in meters and time is measured in seconds, the units of the velocity are meters per second, so that the units of acceleration are meters per second per second, written m/s/s, or more commonly m/s^2 (meters per second squared). This indicates that the velocity changes by -9.8 m/s every second and the speed in the downward (negative) direction increases by 9.8 m/s every second due to gravity. ■

BEYOND FORMULAS

The power rule gives us a much-needed shortcut for computing many derivatives. Mathematicians always seek the shortest, most efficient computations. By skipping unnecessary lengthy steps and saving brain power, mathematicians free themselves to tackle complex problems with creativity. It is important to remember, however, that shortcuts such as the power rule require careful proof.

EXERCISES 7.3

WRITING EXERCISES

1. Explain to a non-calculus-speaking friend how to (mechanically) use the power rule. Decide whether it is better to give separate explanations for positive and negative exponents; integer and noninteger exponents; other special cases.

2. In the 1700s, mathematical "proofs" were, by modern standards, a bit fuzzy and lacked rigor. In 1734, the Irish metaphysician Bishop Berkeley wrote *The Analyst* to an "infidel mathematician" (thought to be Edmund Halley of Halley's comet fame). The accepted proof at the time of the power rule may be described as follows. If x is incremented to $x + h$, then x^n is incremented to $(x + h)^n$. It follows that $\dfrac{(x+h)^n - x^n}{(x+h) - x} = nx^{n-1} + \dfrac{n^2 - n}{2}hx^{n-2} + \cdots$.
Now, let the increment h vanish, and the derivative is nx^{n-1}.
Bishop Berkeley objected to this argument:
"But it should seem that the reasoning is not fair or conclusive. For when it is said, 'let the increments vanish,' the former supposition that the increments were something, or that there were increments, is destroyed, and yet a consequence of that supposition is retained. Which . . . is a false way of reasoning. Certainly, when we suppose the increments to vanish, we must suppose . . . everything derived from the supposition of their existence to vanish with them."
Do you think Berkeley's objection is fair? Is it logically acceptable to assume that something exists to draw one conclusion, and then assume that the same thing does not exist to avoid having to accept other consequences? Mathematically speaking, how does the limit avoid Berkeley's objection of the increment h both existing and not existing?

3. The historical episode in exercise 2 is just one part of an ongoing conflict between people who blindly use mathematical techniques without proof and those who insist on a full proof before permitting anyone to use the technique. To which side are you sympathetic? Defend your position in an essay. Try to anticipate and rebut the other side's arguments.

4. Now that you know the "easy" way to compute the derivative of $f(x) = x^4$, you might wonder why we wanted you to learn the "hard" way. To provide one answer, discuss how you would find the derivative of a function for which you had not learned a shortcut.

In exercises 1–14, differentiate each function.

1. $f(x) = x^3 - 2x + 1$
2. $f(x) = x^9 - 3x^5 + 4x^2 - 4x$
3. $f(t) = 3t^3 - 2\sqrt{t}$
4. $f(s) = 5\sqrt{s} - 4s^2 + 3$
5. $f(w) = \dfrac{3}{w} - 8w + 1$
6. $f(y) = \dfrac{2}{y^4} - y^3 + 2$
7. $h(x) = \dfrac{10}{\sqrt[3]{x}} - 2x + \pi$
8. $h(x) = 12x - x^2 - \dfrac{3}{\sqrt{x^2}}$
9. $f(s) = 2s^{3/2} - 3s^{-1/3}$
10. $f(t) = 3t^{\pi} - 2t^{1.3}$
11. $f(x) = \dfrac{3x^2 - 3x + 1}{2x}$
12. $f(x) = \dfrac{4x^2 - x + 3}{\sqrt{x}}$
13. $f(x) = x\left(3x^2 - \sqrt{x}\right)$
14. $f(x) = (x + 1)(3x^2 - 4)$

In exercises 15–20, compute the indicated derivative.

15. $f''(t)$ for $f(t) = t^4 + 3t^2 - 2$
16. $f'''(t)$ for $f(t) = 4t^2 - 12 + \dfrac{4}{t^2}$

17. $\frac{d^2f}{dx^2}$ for $f(x) = 2x^4 - \frac{3}{\sqrt{x}}$

18. $\frac{d^2f}{dx^2}$ for $f(x) = x^6 - \sqrt{x}$

19. $f^{(4)}(x)$ for $f(x) = x^4 + 3x^2 - 2/\sqrt{x}$

20. $f^{(5)}(x)$ for $f(x) = x^{10} - 3x^4 + 2x - 1$

In exercises 21–24, use the given position function to find the velocity and acceleration functions.

21. $s(t) = -16t^2 + 40t + 10$

22. $s(t) = -4.9t^2 + 12t - 3$

23. $s(t) = \sqrt{t} + 2t^2$

24. $s(t) = 10 - \frac{10}{t}$

In exercises 25 and 26, the given function represents the height of an object. Compute the velocity and acceleration at time $t = t_0$. Is the object going up or down?

25. $h(t) = -16t^2 + 40t + 5$, (*a*) $t_0 = 1$ (*b*) $t_0 = 2$

26. $h(t) = 10t^2 - 24t$, (*a*) $t_0 = 2$ (*b*) $t_0 = 1$

In exercises 27–30, find an equation of the tangent line to $y = f(x)$ at $x = a$.

27. $f(x) = x^2 - 2, a = 2$

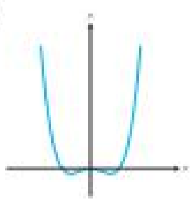
28. $f(x) = x^2 - 2x + 1, a = 2$

29. $f(x) = 4\sqrt{x} - 2x, a = 4$

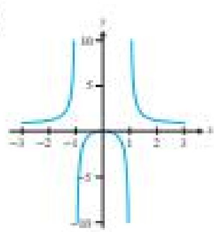
30. $f(x) = 3\sqrt{x} + 4, a = 2$

In exercises 31 and 32, use the graph of f to sketch a graph of f''. (Hint: Sketch f' first.)

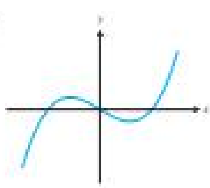
31. (a)



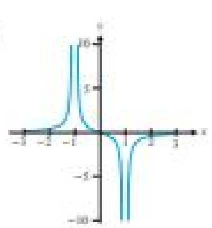
(b)



32. (a)



(b)



In exercises 33 and 34, (a) determine the value(s) of x for which the tangent line to $y = f(x)$ is horizontal. (b) Graph the function and determine the graphical significance of each such point. (c) Determine the value(s) of x for which the tangent line to $y = f(x)$ intersects the x-axis at a 45° angle.

33. $f(x) = x^3 - 3x + 1$

34. $f(x) = x^4 - 4x + 2$

In exercises 35 and 36, (a) determine the value(s) of x for which the slope of the tangent line to $y = f(x)$ does not exist. (b) Graph the function and determine the graphical significance of each such point.

35. (a) $f(x) = x^{2/3}$ (b) $f(x) = |x - 5|$
(c) $f(x) = |x^2 - 3x - 4|$

36. (a) $f(x) = x^{1/3}$ (b) $f(x) = |x + 2|$
(c) $f(x) = |x^2 + 5x + 4|$

37. Find all values of x for which the tangent line to $y = x^3 - 3x + 1$ is (a) at an angle of 45° with the x-axis; (b) at an angle of 30° with the x-axis, assuming that the angles are measured counterclockwise.

38. Find all values of x for which the tangent lines to $y = x^3 + 2x + 1$ and $y = x^4 + x^3 + 3$ are (a) parallel; (b) perpendicular.

39. Find a second-degree polynomial (of the form $ax^2 + bx + c$) such that (a) $f(0) = -2$, $f'(0) = 2$ and $f''(0) = 3$. (b) $f(0) = 0$, $f'(0) = 5$ and $f''(0) = 1$.

40. Find a general formula for the nth derivative $f^{(n)}(x)$ for

(a) $f(x) = \sqrt{x}$ (b) $f(x) = \frac{2}{x}$

41. Find the area of the triangle bounded by $x = 0$, $y = 0$ and the tangent line to $y = \frac{1}{x}$ at $x = 1$. Repeat with the triangle bounded by $x = 0$, $y = 0$ and the tangent line to $y = \frac{1}{x}$ at $x = 2$. Show that you get the same area using the tangent line to $y = \frac{1}{x}$ at any $x = a > 0$.

42. Show that the result of exercise 41 does not hold for $y = \frac{1}{x^2}$. That is, the area of the triangle bounded by $x = 0$, $y = 0$ and the tangent line to $y = \frac{1}{x^2}$ at $x = a > 0$ does depend on the value of a.

43. Assume that a is a real number, f is differentiable for all $x \ge a$ and $g(x) = \max_{a \le t \le x} f(t)$ for $x \ge a$. Find $g'(x)$ in the cases (a) $f'(x) > 0$ and (b) $f'(x) < 0$.

44. Assume that a is a real number, f is differentiable for all $x \ge a$ and $g(x) = \min_{a \le t \le x} f(t)$ for $x \ge a$. Find $g'(x)$ in the cases (a) $f'(x) > 0$ and (b) $f'(x) < 0$.

In exercises 45–48, find a function with the given derivative.

45. $f'(x) = 4x^3$

46. $f'(x) = 5x^4$

47. $f'(x) = \sqrt{x}$

48. $f'(x) = \frac{1}{x^2}$

APPLICATIONS

1. For most land animals, the relationship between leg width w and body length b follows an equation of the form $w = cb^{3/2}$ for some constant $c > 0$. Show that if b is large enough, $w'(b) > 1$. Conclude that for larger animals, leg width (necessary for support) increases faster than body length. Why does this put a limitation on the size of land animals?

2. Suppose the function $v(d)$ represents the average speed in m/s of the world record running time for d meters. For example, if the fastest 200-meter time ever is 19.32 seconds, then $v(200) = 200/19.32 \approx 10.35$. Explain what $v'(d)$ represents.

3. Let $f(t)$ equal the gross domestic product (GDP) in billions of dollars for the Kingdom of Saudi Arabia in year t. Several values are given in the table. Estimate and interpret $f'(2015)$ and $f''(2015)$. [Hint: To estimate the second derivative, estimate $f'(2013)$ and $f'(2014)$ and look for a trend.]

t	2011	2012	2013	2014	2015	2016
$f(t)$	669.51	733.96	744.34	753.83	646.00	637.79

4. Suppose that $f(t)$ equal the average weight of a midsize SUV in year t. Several values are given in the table below. Estimate and interpret $f'(2017)$ and $f''(2017)$.

t	2014	2015	2016	2017
$f(t)$	1974	1936	1900	1840

5. If the position x of an object at time t is given by $f(t)$, then $f'(t)$ represents velocity and $f''(t)$ gives acceleration. By Newton's second law, acceleration is proportional to the net force on the object (causing it to accelerate). Interpret the third derivative $f'''(t)$ in terms of force. The term **jerk** is sometimes applied to $f'''(t)$. Explain why this is an appropriate term.

6. A public official solemnly proclaims, "We have achieved a reduction in the rate at which the national debt is increasing." If $d(t)$ represents the national debt at time t years, which derivative of $d(t)$ is being reduced? What can you conclude about the size of $d(t)$ itself?

EXPLORATORY EXERCISES

1. A plane is cruising at an altitude of 2 miles at a distance of 10 miles from an airport. The airport is at the point (0, 0), and the plane starts its descent at the point (10, 2) to land at the airport. Sketch a graph of a reasonable flight path $y = f(x)$, where y represents altitude and x gives the ground distance from the airport. (Think about it as you draw!) Explain what the derivative $f'(x)$ represents. (Hint: It's not velocity.) Explain why it is important and/or necessary to have $f(0) = 0$, $f(10) = 2$, $f'(0) = 0$, and $f'(10) = 0$. The simplest polynomial that can meet these requirements is a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$. Find values of the constants a, b, c and d to fit the flight path. [Hint: Start by setting $f(0) = 0$ and then set $f'(0) = 0$. You may want to use your CAS to solve the equations.] Graph the resulting function; does it look right? Suppose that airline regulations prohibit a derivative of $\frac{4}{10}$ or larger. Why might such a regulation exist? Show that the flight path you found is illegal. Argue that in fact all flight paths meeting the four requirements are illegal. Therefore, the descent needs to start farther away than 10 miles. Find a flight path with descent starting at 20 miles away that meets all requirements.

2. In the enjoyable book *Surely You're Joking Mr. Feynman*, physicist Richard Feynman tells the story of a contest he had pitting his brain against the technology of the day (an abacus).[2] The contest was to compute the cube root of 1729.03. Feynman came up with 12.002 before the abacus expert gave up. Feynman admits to some luck in the choice of the number 1729.03: he knew that a cubic foot contains 1728 cubic inches. Explain why this told Feynman that the answer is slightly greater than 12. How did he get three digits of accuracy? "I had learned in calculus that for small fractions, the cube root's excess is one-third of the number's excess. The excess, 1.03, is only one part in nearly 2000. So all I had to do is find the fraction 1/1728, divide by 3 and multiply by 12." To see what he did, find an equation of the tangent line to $y = x^{1/3}$ at $x = 1728$ and find the y-coordinate of the tangent line at $x = 1729.03$.

[2] Feynman, R. P. and Sackett, P. D. (1985). "Surely You're Joking Mr. Feynman!" Adventures of a Curious Character. *American Journal of Physics*, 53(12), 1214–1216.

7.4 THE PRODUCT AND QUOTIENT RULES

We have now developed rules for computing the derivatives of a variety of functions, including general formulas for the derivative of a sum or difference of two functions. Given this, you might wonder whether the derivative of a product of two functions is the same as the product of the derivatives. We test this conjecture with a simple example.

Product Rule

Consider $\frac{d}{dx}[(x^2)(x^5)]$. Combining the two terms, we have

$$\frac{d}{dx}[(x^2)(x^5)] = \frac{d}{dx}x^7 = 7x^6.$$

However,

$$\left(\frac{d}{dx}x^2\right)\left(\frac{d}{dx}x^5\right) = (2x)(5x^4)$$
$$= 10x^5 \neq 7x^6 = \frac{d}{dx}[(x^2)(x^5)]. \tag{4.1}$$

You can now plainly see from (4.1) that the derivative of a product is *not* generally the product of the corresponding derivatives. The correct rule is given in Theorem 4.1.

THEOREM 4.1 (Product Rule)

Suppose that f and g are differentiable. Then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x). \tag{4.2}$$

PROOF

Since we are proving a general rule, we have only the limit definition of derivative to use. For $p(x) = f(x)g(x)$, we have

$$\frac{d}{dx}[f(x)g(x)] = p'(x) = \lim_{h\to 0}\frac{p(x+h)-p(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}. \tag{4.3}$$

Notice that the elements of the derivatives of f and g are present, but we need to get them into the right form. Adding and subtracting $f(x)g(x+h)$ in the numerator, we have

$$p'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h} + \lim_{h\to 0}\frac{f(x)g(x+h)-f(x)g(x)}{h} \quad \text{Break into two pieces.}$$
$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}g(x+h)\right] + \lim_{h\to 0}\left[f(x)\frac{g(x+h)-g(x)}{h}\right]$$
$$= \left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right] + \left[\lim_{h\to 0} g(x+h)\right] + f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$
$$= f'(x)g(x) + f'(x)g'(x). \quad \text{Recognize the derivative of } f \text{ and the derivative of } g.$$

There is a subtle technical detail in the last step: since g is differentiable at x, recall that it must also be continuous at x, so that $g(x+h) \to g(x)$ as $h \to 0$. ■

In example 4.1, notice that the product rule saves us from multiplying out a messy product.

EXAMPLE 4.1 Using the Product Rule

Find $f'(x)$ if $f(x) = (2x^4 - 3x + 5)\left(x^2 - \sqrt{x} + \frac{2}{x}\right)$.

Solution Although we could first multiply out the expression, the product rule will simplify our work:

$$f'(x) = \left[\frac{d}{dx}(2x^4 - 3x + 5)\right]\left(x^2 - \sqrt{x} + \frac{2}{x}\right) + (2x^4 - 3x + 5)\frac{d}{dx}\left(x^2 - \sqrt{x} + \frac{2}{x}\right)$$
$$= (8x^3 - 3)\left(x^2 - \sqrt{x} + \frac{2}{x}\right) + (2x^4 - 3x + 5)\left(2x - \frac{1}{2\sqrt{x}} + \frac{2}{x^2}\right).$$ ■

EXAMPLE 4.2 Finding the Equation of the Tangent Line

Find an equation of the tangent line to

$$y = (x^4 - 3x^2 + 2x)(x^3 - 2x + 3)$$

at $x = 0$.

Solution From the product rule, we have

$$y' = (4x^3 - 6x + 2)(x^3 - 2x + 3) + (x^4 - 3x^2 + 2x)(3x^2 - 2).$$

Evaluating at $x = 0$, we have $y'(0) = (2)(3) + (0)(-2) = 6$. The line with slope 6 and passing through the point (0, 0) [why (0, 0)?] has equation $y = 6x$. ■

○ Quotient Rule

Given our experience with the product rule, you probably have no expectation that the derivative of a quotient will turn out to be the quotient of the derivatives. Just to be sure, let's try a simple experiment. Note that

$$\frac{d}{dx}\left(\frac{x^5}{x^2}\right) = \frac{d}{dx}(x^3) = 3x^2,$$

while

$$\frac{\frac{d}{dx}(x^5)}{\frac{d}{dx}(x^2)} = \frac{5x^4}{2x^1} = \frac{5}{2}x^3 \neq 3x^2 = \frac{d}{dx}\left(\frac{x^5}{x^2}\right).$$

Since these are obviously not the same, we know that the derivative of a quotient is generally not the quotient of the corresponding derivatives. The correct rule is given in Theorem 4.2.

THEOREM 4.2 (Quotient Rule)

Suppose that f and g are differentiable. Then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \tag{4.4}$$

provided $g(x) \neq 0$.

PROOF

For $Q(x) = \frac{f(x)}{g(x)}$, we have from the limit definition of derivative that

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = Q'(x) = \lim_{h\to 0}\frac{Q(x+h) - Q(x)}{h}$$
$$= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{\left[\dfrac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}\right]}{h} \quad \text{Add the fractions.}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}. \quad \text{Simplify.}$$

As in the proof of the product rule, we look for the right term to add and subtract in the numerator, so that we can isolate the limit definitions of $f'(x)$ and $g'(x)$. Adding and subtracting $f(x)g(x)$, we get

$$Q'(x) = \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}$$

$$= \lim_{h \to 0} \frac{\dfrac{f(x+h) - f(x)}{h} g(x) - f(x) \dfrac{g(x+h) - g(x)}{h}}{g(x+h)g(x)} \quad \text{Group first two and last two terms together and factor out common terms.}$$

$$= \frac{\lim\limits_{h \to 0} \left[\dfrac{f(x+h) - f(x)}{h}\right] g(x) - f(x) \lim\limits_{h \to 0} \left[\dfrac{g(x+h) - g(x)}{h}\right]}{\lim\limits_{h \to 0} g(x+h)g(x)}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}. \quad \text{Recognize the derivatives of } f \text{ and } g.$$

where we have used the fact that g is differentiable to imply that g is continuous, so that $g(x+h) \to g(x)$, as $h \to 0$. ■

Notice that the numerator in the quotient rule looks very much like the product rule, but with a minus sign between the two terms. For this reason, you need to be very careful with the order.

EXAMPLE 4.3 Using the Quotient Rule

Compute the derivative of $f(x) = \dfrac{x^2 - 2}{x^3 + 1}$.

Solution Using the quotient rule, we have

$$f'(x) = \frac{\left[\dfrac{d}{dx}(x^2 - 2)\right](x^3 + 1) - (x^2 - 2)\dfrac{d}{dx}(x^3 + 1)}{(x^3 + 1)^2}$$

$$= \frac{2x(x^3 + 1) - (x^2 - 2)(3x^2)}{(x^3 + 1)^2}$$

$$= \frac{-x^4 + 6x^2 2x}{(x^3 + 1)^2}.$$

In this case, we rewrote the numerator because it simplified significantly. This often occurs with the quotient rule. ■

Now that we have the quotient rule, we can justify the use of the power rule for negative integer exponents. (Recall that we have been using this rule without proof since section 7.3.)

THEOREM 4.3 (Power Rule)

For any integer exponent n, $\dfrac{d}{dx}x^n = nx^{n-1}$.

PROOF

We have already proved this for *positive* integer exponents. So, suppose that $n < 0$ and let $M = -n > 0$. Then, using the quotient rule, we get

$$\frac{d}{dx}x^n = \frac{d}{dx}x^{-M} = \frac{d}{dx}\left(\frac{1}{x^M}\right) \qquad \text{Since } x^{-M} = \frac{1}{x^M}.$$

$$= \frac{\left[\frac{d}{dx}(1)\right]x^M - (1)\frac{d}{dx}(x^M)}{(x^M)^2} \qquad \text{By the quotient rule.}$$

$$= \frac{(0)x^M - (1)Mx^{M-1}}{x^{2M}} \qquad \text{By the power rule, since } M > 0.$$

$$= \frac{-Mx^{M-1}}{x^{2M}} = Mx^{M-1-2M} \qquad \text{By the usual rules of exponents.}$$

$$= (-M)x^{-M-1} = nx^{n-1}. \qquad \text{Since } n = -M. \blacksquare$$

As we see in example 4.4, it is sometimes preferable to rewrite a function, instead of automatically using the product or quotient rule.

EXAMPLE 4.4 A Case Where the Product and Quotient Rules Are Not Needed

Compute the derivative of $f(x) = x\sqrt{x} + \frac{2}{x^2}$.

Solution Although it may be tempting to use the product rule for the first term and the quotient rule for the second term, notice that it's simpler to first rewrite the function. We can combine the two powers of x in the first term. Since the second term is a fraction with a constant numerator, we can more simply write it using a negative exponent. We have

$$f(x) = x\sqrt{x} + \frac{2}{x^2} = x^{3/2} + 2x^{-2}.$$

Using the power rule, we have simply

$$f'(x) = \frac{3}{2}x^{1/2} - 4x^{-3}. \blacksquare$$

○ APPLICATIONS

You will see important uses of the product and quotient rules throughout your mathematical and scientific studies. We start you off with a couple of simple applications now.

EXAMPLE 4.5 Investigating the Rate of Change of Revenue

Suppose that a product currently sells for \$25, with the price increasing at the rate of \$2 per year. At this current price, consumers will buy 150 thousand items, but the number sold is decreasing at the rate of 8 thousand per year. At what rate is the total revenue changing? Is the total revenue increasing or decreasing?

Solution To answer these questions, we need the basic relationship

revenue = quantity × price

(e.g., if you sell 10 items at \$4 each, you earn \$40). Since these quantities are changing in time, we write $R(t) = Q(t)\,P(t)$, where $R(t)$ is revenue, $Q(t)$ is quantity sold and $P(t)$ is the price, all at time t. We don't have formulas for any of these functions, but from the product rule, we have

$$R'(t) = Q'(t)P(t) + Q(t)P'(t).$$

We have information about each of these terms: the initial price, $P(0)$, is 25 (dollars); the rate of change of the price is $P'(0) = 2$ (dollars per year); the initial quantity, $Q(0)$, is 150 (thousand items) and the rate of change of quantity is $Q'(0) = -8$ (thousand items per year). Note that the negative sign of $Q'(0)$ denotes a decrease in Q. Thus,

$$R'(0) = (-8)(25) + (150)(2) = 100 \text{ thousand dollars per year.}$$

Since the rate of change is positive, the revenue is increasing. ■

EXAMPLE 4.6 Using the Derivative to Analyze a Golf Shot

A golf ball of mass 0.05 kg struck by a golf club of mass m kg with speed 50 m/s will have an initial speed of $u(m) = \dfrac{83m}{m + 0.05}$ m/s. Show that $u'(m) > 0$ and interpret this result in golf terms. Compare $u'(0.15)$ and $u'(0.20)$.

Solution From the quotient rule, we have

$$u'(m) = \frac{83(m + 0.05 - 83m)}{(m + 0.05)^2} = \frac{4.15}{(m + 0.05)^2}.$$

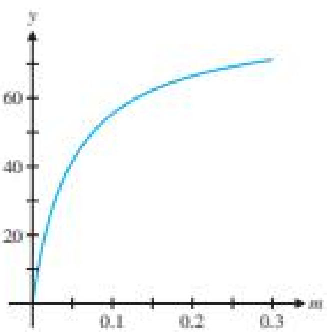


Figure 7.23
$u(m) = \dfrac{83m}{m + 0.05}$

Both the numerator and denominator are positive, so $u'(m) > 0$. A positive slope for all tangent lines indicates that the graph of $u(m)$ should rise from left to right. (See Figure 7.23.) Said a different way, $u(m)$ increases as m increases. In golf terms, this says that (all other things being equal) the greater the mass of the club, the greater the velocity of the ball will be. Finally, we compute $u'(0.15) = 103.75$ and $u'(0.20) = 66.4$. This says that the rate of increase in ball speed is much less for the heavier club than for the lighter one. Since heavier clubs can be harder to control, the relatively small increase in ball speed obtained by making the heavy club even heavier may not compensate for the decrease in control. ■

EXERCISES 7.4

WRITING EXERCISES

1. The product and quotient rules give you the ability to symbolically calculate the derivative of a wide range of functions. However, many calculators and almost every computer algebra system (CAS) can do this work for you. Discuss why you should learn these basic rules anyway. (Keep example 4.5 in mind.)

2. Gottfried Wilhelm Leibniz is recognized, along with Sir Isaac Newton, as a coinventor of calculus. Many of the fundamental methods and notation of calculus are due to Leibniz. The product rule was worked out by Leibniz in 1675, in the form $d(xy) = (dx)y + x(dy)$. His "proof," as given in a letter written in 1699, follows. "If we are to differentiate xy we write:

$$(x + dx)(y + dy) - xy = x\,dy + y\,dx + dx\,dy.$$

But here $dx\,dy$ is to be rejected as incomparably less than $x\,dy + y\,dx$. Thus, in any particular case the error is less than any finite quantity." Answer Leibniz' letter with one describing your own "discovery" of the product rule for $d(xyz)$.

3. You may have noticed that in example 4.1, we did not multiply out the terms of the derivative. If you want to compute $f'(a)$ for some number a, discuss whether it would be easier to substitute $x = a$ first and then simplify or multiply out all terms and then substitute $x = a$.

4. Many students prefer the product rule to the quotient rule. Many computer algebra systems actually use the product rule to compute the derivative of $f(x)[g(x)]^{-1}$ instead of using the quotient rule on $\dfrac{f(x)}{g(x)}$. (See exercise 34.) Given the simplifications in problems like example 4.3, explain why the quotient rule can be preferable.

In exercises 1–16, find the derivative of each function.

1. $f(x) = (x^2 + 3)(x^3 - 3x + 1)$
2. $f(x) = (x^3 - 2x^2 + 5)(x^4 - 3x^2 + 2)$
3. $f(x) = (\sqrt{x} + 3x)\left(5x^2 - \dfrac{3}{x}\right)$
4. $f(x) = (x^{3/2} - 4x)\left(x^4 - \dfrac{3}{x^2} + 2\right)$

5. $g(t) = \dfrac{3t-2}{5t+1}$

6. $g(t) = \dfrac{t^2+2t+5}{t^2+5t+1}$

7. $f(x) = \dfrac{3x-6\sqrt{x}}{5x^2-2}$

8. $f(x) = \dfrac{6x-2/x}{x^2+\sqrt{x}}$

9. $f(u) = \dfrac{(u+1)(u-2)}{u^2-5u+1}$

10. $f(u) = \dfrac{2u}{u^2+1}(u+3)$

11. $f(x) = \dfrac{x^2+3x-2}{\sqrt{x}}$

12. $f(x) = \dfrac{x^2-2x}{x^2+5x}$

13. $h(t) = t(\sqrt[3]{t}+3)$

14. $h(t) = \dfrac{t^2}{3} + \dfrac{5}{t^2}$

15. $f(x) = (x^2-1)\dfrac{x^3+3x^2}{x^2+2}$

16. $f(x) = (x+2)\dfrac{x^2-1}{x^2+x}$

In exercises 17–20, find an equation of the tangent line to the graph of $y = f(x)$ at $x = a$.

17. $f(x) = (x^2+2x)(x^4+x^2+1)$, $a = 0$

18. $f(x) = (x^3+x+1)(3x^2+2x-1)$, $a = 1$

19. $f(x) = \dfrac{x+1}{x+2}$, $a = 0$

20. $f(x) = \dfrac{x+3}{x^2+1}$, $a = 1$

In exercises 21–24, assume that f and g are differentiable with $f(0) = -1$, $f(1) = -2$, $f'(0) = -1$, $f'(1) = 3$, $g(0) = 3$, $g(1) = 1$, $g'(0) = -1$ and $g'(1) = -2$. Find an equation of the tangent line to the graph of $y = h(x)$ at $x = a$.

21. $h(x) = f(x)g(x)$; (a) $a = 0$; (b) $a = 1$

22. $h(x) = \dfrac{f(x)}{g(x)}$; (a) $a = 1$; (b) $a = 0$

23. $h(x) = x^2 f(x)$; (a) $a = 1$; (b) $a = 0$

24. $h(x) = \dfrac{x^2}{g(x)}$; (a) $a = 1$; (b) $a = 0$

25. Suppose that for some toy, the quantity sold $Q(t)$ at time t years decreases at a rate of 4%; explain why this translates to $Q'(t) = -0.04Q(t)$. Suppose also that the price increases at a rate of 3%; write out a similar equation for $P'(t)$ in terms of $P(t)$. The revenue for the toy is $R(t) = Q(t)P(t)$. Substituting the expressions for $Q'(t)$ and $P'(t)$ into the product rule $R'(t) = Q'(t)P(t) + Q(t)P'(t)$, show that the revenue decreases at a rate of 1%. Explain why this is "obvious."

26. As in exercise 25, suppose that the quantity sold decreases at a rate of 4%. By what rate must the price be increased to keep the revenue constant?

27. Suppose the price of an object is \$20 and 20,000 units are sold. If the price increases at a rate of \$1.25 per year and the quantity sold increases at a rate of 2000 per year, at what rate will revenue increase?

28. Suppose the price of an object is \$14 and 12,000 units are sold. The company wants to increase the quantity sold by 1200 units per year, while increasing the revenue by \$20,000 per year. At what rate would the price have to be increased to reach these goals?

29. A cricket ball with mass 0.15 kg and speed 45 m/s is struck by a cricket bat of mass m kg and speed 40 m/s (in the opposite direction of the ball's motion). After the collision, the ball has initial speed $u(m) = \dfrac{82.5m - 6.75}{m + 0.15}$ m/s. Show that $u'(m) > 0$ and interpret this in cricket terms. Compare $u'(1)$ and $u'(1.2)$.

30. In exercise 29, if the cricket ball has mass M kg at speed 45 m/s and the bat has mass 1.05 kg and speed 40 m/s, the ball's initial speed is $u(M) = \dfrac{86.625 - 45M}{M + 1.05}$ m/s. Compute $u'(M)$ and interpret its sign (positive or negative) in cricket terms.

31. In example 4.6, it is reasonable to assume that the speed of the golf club at impact decreases as the mass of the club increases. If, for example, the speed of a club of mass m is $v = 8.5/m$ m/s at impact, then the initial speed of the golf ball is $u(m) = \dfrac{14.11}{M + 0.05}$ m/s. Show that $u'(m) < 0$ and interpret this in golf terms.

32. In example 4.6, if the golf club has mass 0.17 kg and strikes the ball with speed v m/s, the ball has initial speed $u(v) = \dfrac{0.2822v}{0.217}$ m/s. Compute and interpret the derivative $u'(v)$.

33. Write out the product rule for the function $f(x)g(x)h(x)$. (Hint: Group the first two terms together.) Describe the general product rule: for n functions, what is the derivative of the product $f_1(x)\, f_2(x)\, f_3(x) \cdots f_n(x)$? How many terms are there? What does each term look like?

34. Use the quotient rule to show that the derivative of $[g(x)]^{-1}$ is $-g'(x)[g(x)]^{-2}$. Then use the product rule to compute the derivative of $f(x)[g(x)]^{-1}$.

In exercises 35 and 36, find the derivative of each function using the general product rule developed in exercise 33.

35. $f(x) = x^{2/3}(x^2-2)(x^3-x+1)$

36. $f(x) = (x+4)(x^3-2x^2+1)(3-2/x)$

37. Assume that g is continuous at $x = 0$ and define $f(x) = xg(x)$. Show that f is differentiable at $x = 0$. Illustrate the result with $g(x) = |x|$.

38. In exercise 37, if $x = 0$ is replaced with $x = a \neq 0$, how must you modify the definition of $f(x)$ to guarantee that f is differentiable?

39. For $f(x) = \dfrac{x}{x^2+1}$, show that the slope m of the tangent line to $y = f(x)$ satisfies $\dfrac{-1}{8} \leq m \leq 1$. Graph the function and identify points of maximum and minimum slope.

40. For $f(x) = \dfrac{x}{\sqrt{x^2+1}}$, show that the slope m of the tangent line to $y = f(x)$ satisfies $0 < m \leq 1$. Graph the function and identify the point of maximum slope.

41. Repeat example 4.4 with your CAS. If its answer is not in the same form as ours in the text, explain how the CAS computed its answer.

42. Use your CAS to sketch the derivative of $\sin x$. What function does this look like? Repeat with $\sin 2x$ and $\sin 3x$. Generalize to conjecture the derivative of $\sin kx$ for any constant k.

43. Find the derivative of $f(x) = \dfrac{\sqrt{3x^3 + x^2}}{x}$ on your CAS. Compare its answer to $\dfrac{3}{2\sqrt{3x+1}}$ for $x > 0$ and $\dfrac{-3}{2\sqrt{3x+1}}$ for $x < 0$. Explain how to get this answer and your CAS's answer, if it differs.

44. Find the derivative of $f(x) = \dfrac{x^2 - x - 2}{x - 2}\left(2x - \dfrac{2x^2}{x+1}\right)$ on your CAS. Compare its answer to 2. Explain how to get this answer and your CAS's answer, if it differs.

45. Suppose that $F(x) = f(x)g(x)$ for infinitely differentiable functions f and g (that is, $f'(x)$, $f''(x)$, etc. exist for all x). Show that $F'' = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$. Compute $F'''(x)$. Compare $F''(x)$ to the binomial formula for $(a + b)^2$ and compare $F'''(x)$ to the formula for $(a + b)^3$.

46. With $f(x)$ defined as in exercise 45, compute $F^{(4)}(x)$ using the fact that $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

47. Use the product rule to show that if $g(x) = [f(x)]^2$ and $f(x)$ is differentiable, then $g'(x) = 2f(x)f'(x)$. This can also be obtained using the *chain rule*, to be discussed in section 7.5.

48. Use the result from exercise 47 and the product rule to show that if $g(x) = [f(x)]^3$ and $f(x)$ is differentiable, then $g'(x) = 3f(x)]^2f'(x)$. Hypothesize the derivative of $[f(x)]^n$.

APPLICATIONS

1. The amount of an **allosteric enzyme** is affected by the presence of an activator. If x is the amount of activator and f is the amount of enzyme, then one model of an allosteric activation is $f(x) = \dfrac{x^{2.7}}{1 + x^{2.7}}$. Find and interpret $\lim_{x\to 0} f(x)$ and $\lim_{x\to\infty} f(x)$. Compute and interpret $f'(x)$.

2. Enzyme production can also be inhibited. In this situation, the amount of enzyme as a function of the amount of inhibitor is modeled by $f(x) = \dfrac{1}{1 + x^{2.7}}$. Find and interpret $\lim_{x\to 0} f(x)$, $\lim_{x\to\infty} f(x)$ and $f'(x)$.

3. Most cars are rated for fuel efficiency by estimating miles per gallon in city driving (c) and miles per gallon in highway driving (h). The Environmental Protection Agency uses the formula $r = \dfrac{1}{0.55/c + 0.45/h}$ as its overall rating of gas usage.

(a) Think of c as the variable and h as a constant, and show that $\dfrac{dr}{dc} > 0$. Interpret this result in terms of gas mileage.

(b) Think of h as the variable and c as a constant, and show that $\dfrac{dr}{dc} > 0$.

(c) Show that if $c = h$, then $r = c$.

(d) Show that if $c < h$, then $c < r < h$. To do this, assume that c is a constant and $c < h$. Explain why the results of parts (b) and (c) imply that $r > c$. Next, show that $\dfrac{dr}{dh} < 0.45$. Explain why this result along with the result of part (c) implies that $r < h$.

Explain why the results of parts (a)–(d) must be true if the EPA's combined formula is a reasonable way to average the ratings c and h. To get some sense of how the formula works, take $c = 20$ and graph r as a function of h. Comment on why the EPA might want to use a function whose graph flattens out as this one does.

EXPLORATORY EXERCISES

1. In many sports, the collision between a ball and a striking implement is central to the game. Suppose the ball has weight w and velocity v before the collision and the striker (bat, tennis racket, golf club, etc.) has weight W and velocity $-V$ before the collision (the negative indicates the striker is moving in the opposite direction from the ball). The velocity of the ball after the collision will be $u = \dfrac{WV(1 + c) + v(cW - w)}{W + w}$, where the parameter c, called the **coefficient of restitution**, represents the "bounciness" of the ball in the collision. Treating W as the independent variable (like x) and the other parameters as constants, compute the derivative and verify that $\dfrac{du}{dW} = \dfrac{V(1 + c)w + cvw + vw}{(W + w)^2} \geq 0$ since all parameters are non-negative. Explain why this implies that if the athlete uses a bigger striker (bigger W) with all other things equal, the speed of the ball increases. Does this match your intuition? What is doubtful about the assumption of all other things being equal? Similarly compute and interpret $\dfrac{du}{dw}$, $\dfrac{du}{dv}$, $\dfrac{du}{dV}$ and $\dfrac{du}{dc}$. (Hint: c is between 0 and 1, with 0 representing a dead ball and 1 the liveliest ball possible.)

2. Suppose that a soccer player strikes the ball with enough energy that a stationary ball would have initial speed 80 mph. Show that the same energy kick on a ball moving directly to the player at 40 mph will launch the ball at approximately 100 mph. (Use the general collision formula in exploratory exercise 1 with $c = 0.5$ and assume that the ball's mass is much less than the soccer player's mass.) In general, what proportion of the ball's incoming speed is converted by the kick into extra speed in the opposite direction?

7.5 THE CHAIN RULE

We currently have no way to compute the derivative of a function such as $P(t) = \sqrt{100 + 8t}$, except by the limit definition. However, observe that $P(t)$ is the composition of the two functions $f(t) = \sqrt{t}$ and $g(t) = 100 + 8t$, so that $P(t) = f(g(t))$, where both $f'(t)$ and $g'(t)$ are easily computed. We now develop a general rule for the derivative of a composition of two functions.

The following simple examples will help us to identify the form of the chain rule. Notice that from the product rule

$$\begin{aligned} \frac{d}{dx}[(x^2+1)^2] &= \frac{d}{dx}[(x^2+1)(x^2+1)] \\ &= 2x(x^2+1) + (x^2+1)2x \\ &= 2(x^2+1)2x. \end{aligned}$$

Of course, we can write this as $4x(x^2+1)$, but the unsimplified form helps us to understand the form of the chain rule. Using this result and the product rule, notice that

$$\begin{aligned} \frac{d}{dx}[(x^2+1)^3] &= \frac{d}{dx}[(x^2+1)(x^2+1)^2] \\ &= 2x(x^2+1) + (x^2+1)2x(x^2+1)2x \\ &= 3(x^2+1)^2 2x. \end{aligned}$$

We leave it as a straightforward exercise to extend this result to

$$\frac{d}{dx}[(x^2+1)^4] = 4(x^2+1)^3 2x.$$

You should observe that, in each case, we have brought the exponent down, lowered the power by one and then multiplied by $2x$, the derivative of $x^2 + 1$. Notice that we can write $(x^2+1)^4$ as the composite function $f(g(x)) = (x^2+1)^4$, where $g(x) = x^2 + 1$ and $f(x) = x^4$. Finally, observe that the derivative of the composite function is

$$\frac{d}{dx}[f(g(x))] = \frac{d}{dx}[(x^2+1)^4] = 4(x^2+1)^3 2x = f'(g(x))g'(x).$$

This is an example of the *chain rule*, which has the following general form.

THEOREM 5.1 (Chain Rule)

If g is differentiable at x and f is differentiable at $g(x)$, then

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

PROOF

At this point, we can prove only the special case where $g'(x) \neq 0$. Let $f(x) = f(g(x))$. Then,

$$\begin{aligned} \frac{d}{dx}[f(g(x))] = F'(x) &= \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x)}{h}\,\frac{g(x+h) - g(x)}{g(x+h) - g(x)} && \text{Since } F(x) = f(g(x)). \\ &= \lim_{h\to 0}\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} && \text{Multiply numerator and denominator by } g(x+h) - g(x). \\ &= \lim_{g(x+h)\to g(x)}\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} && \text{Regroup terms.} \\ &= f'(g(x))g'(x). \end{aligned}$$

REMARK 5.1

The chain rule should make sense intuitively as follows. We think of $\frac{dy}{dx}$ as the (instantaneous) rate of change of y with respect to x, $\frac{dy}{du}$ as the (instantaneous) rate of change of y with respect to u and $\frac{du}{dx}$ as the (instantaneous) rate of change of u with respect to x. So, if $\frac{dy}{du} = 2$ (i.e., y is changing at twice the rate of u) and $\frac{du}{dx} = 5$ (i.e., u is changing at five times the rate of x), it should make sense that y is changing at $2 \times 5 = 10$ times the rate of x. That is, $\frac{dy}{dx} = 10$, which is precisely what equation (5.1) says.

where the next to the last line is valid since as $h \to 0$, $g(x+h) \to g(x)$, by the continuity of g. (Recall that since g is differentiable, it is also continuous.) You will be asked in exercise 44 to fill in some of the gaps in this argument. In particular, you should identify why we need $g'(x) \neq 0$ in this proof. ■

It is often helpful to think of the chain rule in Leibniz notation. If $y = f(u)$ and $u = g(x)$, then $y = f(g(x))$ and the chain rule says that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}, \tag{5.1}$$

where it looks like we are canceling the du's, even though these are not fractions.

EXAMPLE 5.1 Using the Chain Rule

Differentiate $y = (x^3 + x - 1)^5$.

Solution For $u = x^3 + x - 1$, note that $y = u^5$. From (5.1), we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}(u^5)\frac{du}{dx} \qquad \text{Since } y = u^5.\\ &= 5u^4 \frac{d}{dx}(x^3 + x - 1)\\ &= 5(x^3 + x - 1)^4(3x^2 + 1). \ ■\end{aligned}$$

For the composition $f(g(x))$, f is often referred to as the *outside* function and g is referred to as the *inside* function. The chain rule derivative $f'(g(x))g'(x)$ can then be viewed as the derivative of the outside function times the derivative of the inside function. In example 5.1, the inside function is $x^3 + x - 1$ (the expression inside the parentheses) and the outside function is u^5.

EXAMPLE 5.2 Using the Chain Rule with a Square Root Function

Find $\frac{d}{dt}(\sqrt{100 + 8t})$.

Solution Let $u = 100 + 8t$ and note that $\sqrt{100 + 8t} = u^{1/2}$. Then, from (5.1),

$$\begin{aligned}\frac{d}{dt}(\sqrt{100 + 8t}) &= \frac{d}{dt}(u^{1/2}) = \frac{1}{2}u^{-1/2}\frac{du}{dt}\\ &= \frac{1}{2\sqrt{100 + 8t}}\frac{d}{dt}(100 + 8t) = \frac{4}{\sqrt{100 + 8t}}.\end{aligned}$$

Notice that the derivative of the *inside* here is the derivative of the expression under the square root sign. ■

You are now in a position to calculate the derivative of a very large number of functions, by using the chain rule in combination with other differentiation rules.

EXAMPLE 5.3 Derivatives Involving the Chain Rule and Other Rules

Compute the derivative of $f(x) = x^3\sqrt{4x + 1}$, $g(x) = \frac{8x}{(x^3 + 1)^2}$ and $h(x) = \frac{8x}{(x^3 + 1)^2}$.

Solution Notice the differences in these three functions. The first function $f(x)$ is a product of two functions, $g(x)$ is a quotient of two functions and $h(x)$ is a constant divided by a function. This tells us to use the product rule for $f(x)$, the quotient rule for $g(x)$ and simply the chain rule for $h(x)$. For the first function, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^3\sqrt{4x+1}\right) = 3x^2\sqrt{4x+1} + x^3\frac{d}{dx}\sqrt{4x+1} && \text{By the product rule.} \\ &= 3x^2\sqrt{4x+1} + x^3\frac{1}{2}(4x+1)^{-1/2}\underbrace{\frac{d}{dx}(4x+1)}_{\text{derivative of the inside}} && \text{By the chain rule.} \\ &= 3x^2\sqrt{4x+1} + 2x^3(4x+1)^{-1/2}. && \text{Simplifying.} \end{aligned}$$

Next, we have

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left[\frac{8x}{(x^3+1)^2}\right] = \frac{8(x^3+1)^2 - 8x\frac{d}{dx}[(x^3+1)^2]}{(x^3+1)^4} && \text{By the quotient rule.} \\ &= \frac{8(x^3+1)^2 - 8x\left[2(x^3+1)\underbrace{\frac{d}{dx}(x^3+1)}_{\text{derivative of the inside}}\right]}{(x^3+1)^4} && \text{By the chain rule.} \\ &= \frac{8(x^3+1)^2 - 16x(x^3+1)3x^2}{(x^3+1)^4} \\ &= \frac{8(x^3+1) - 48x^3}{(x^3+1)^3} = \frac{8-40x^3}{(x^3+1)^3}. && \text{Simplifying.} \end{aligned}$$

For $h(x)$, notice that instead of using the quotient rule, it is simpler to rewrite the function as $h(x) = 8(x^3+1)^{-2}$. Then

$$\begin{aligned} h'(x) &= \frac{d}{dx}[8(x^3+1)^{-2}] = -16(x^3+1)^{-3}\underbrace{\frac{d}{dx}(x^3+1)}_{\text{derivative of the inside}} = -16(x^3+1)^{-3}(3x^2) \\ &= -48x^2(x^3+1)^{-3}. \blacksquare \end{aligned}$$

In example 5.4, we apply the chain rule to a composition of a function with a composition of functions.

EXAMPLE 5.4 A Derivative Involving Multiple Chain Rules

Find the derivative of $f(x) = \left(\sqrt{x^2+4} - 3x^2\right)^{3/2}$.

Solution We have

$$\begin{aligned} f'(x) &= \frac{3}{2}\left(\sqrt{x^2+4} - 3x^2\right)^{1/2}\frac{d}{dx}\left(\sqrt{x^2+4} - 3x^2\right) && \text{By the chain rule.} \\ &= \frac{3}{2}\left(\sqrt{x^2+4} - 3x^2\right)^{1/2}\left[\frac{1}{2}(x^2+4)^{-1/2}\frac{d}{dx}(x^2+4) - 6x\right] && \text{By the chain rule.} \\ &= \frac{3}{2}\left(\sqrt{x^2+4} - 3x^2\right)^{1/2}\left[\frac{1}{2}(x^2+4)^{-1/2}(2x) - 6x\right] \\ &= \frac{3}{2}\left(\sqrt{x^2+4} - 3x^2\right)^{1/2}\left[x(x^2+4)^{-1/2} - 6x\right]. && \text{Simplifying.} \end{aligned}$$

■

We now use the chain rule to compute the derivative of an inverse function in terms of the original function. Recall that we write $g(x) = f^{-1}(x)$ if $g(f(x)) = x$ for all x in the domain of f and $f(g(x)) = x$ for all x in the domain of g. From this last equation, assuming that f and g are differentiable, it follows that

$$\frac{d}{dx}[f(g(x))] = \frac{d}{dx}(x).$$

From the chain rule, we now have

$$f'(g(x))g'(x) = 1.$$

Solving this for $g'(x)$, we get $$g'(x) = \frac{1}{f'(g(x))},$$

assuming we don't divide by zero. We now state this result as Theorem 5.2.

THEOREM 5.2

If f is differentiable everywhere on its domain and has an inverse function $g = f^{-1}$, then

$$g'(x) = \frac{1}{f'(g(x))},$$

for all x in the domain of g, provided $f'(g(x)) \neq 0$.

As we see in example 5.5, in order to use Theorem 5.2, we must be able to compute values of the inverse function.

TODAY IN MATHEMATICS

Fan Chung (1949–Present) A Taiwanese mathematician with a highly successful career in American industry and academia. She says, "As an undergraduate in Taiwan, I was surrounded by good friends and many women mathematicians. . . . A large part of education is learning from your peers, not just the professors."* Collaboration has been a hallmark of her career. "Finding the right problem is often the main part of the work in establishing the connection. Frequently a good problem from someone else will give you a push in the right direction and the next thing you know, you have another good problem."†

*Hoffman P. (1998) The Man Who Loved Only Numbers (London)

†Albers D. (1995) Making Connections: A Profile of Fan Chung, Mathematical Horizon, 14–18.

EXAMPLE 5.5 The Derivative of an Inverse Function

Given that the function $f(x) = x^5 + 3x^3 + 2x + 1$ has an inverse function g, compute $g'(7)$.

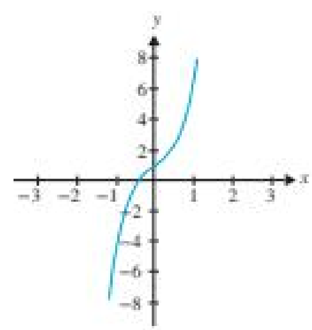


FIGURE 7.24
$y = x^5 + 3x^3 + 2x + 1$

Solution First, notice from Figure 7.24 that f appears to be one-to-one and so, will have an inverse. From Theorem 5.2, we have

$$g'(7) = \frac{1}{f'(g(7))}. \tag{5.2}$$

It's easy to compute $f'(x) = 5x^4 + 9x^2 + 2$, but to use Theorem 5.2 we also need the value of $g(7)$. If we write $x = g(7)$, then $x = f^{-1}(7)$, so that $f(x) = 7$. In general, solving the equation $f(x) = 7$ may be beyond our algebraic abilities. (Try solving $x^5 + 3x^3 + 2x + 1 = 7$ to see what we mean.) By trial and error, however, it is not hard to see that $f(1) = 7$, so that $g(7) = 1$. [Keep in mind that for inverse functions, $f(x) = y$ and $g(y) = x$ are equivalent statements.] Returning to equation (5.2), we now have

$$g'(7) = \frac{1}{f'(1)} = \frac{1}{16}.$$ ■

THE BIG PICTURE

When you are asked to differentiate a function using the chain rule, it would be helpful to think of the big picture. Do you see a power, a product or a quotient? This will lead to the right choice of rule to follow in determining a quantity that is not directly computable. Maintain the "BIG PICTURE" and be attentive to the smaller details.

Notice that our solution in example 5.5 is dependent on finding an x for which $f(x) = 7$. This particular example was workable by trial and error, but finding most other values would have been quite difficult or impossible to solve exactly.

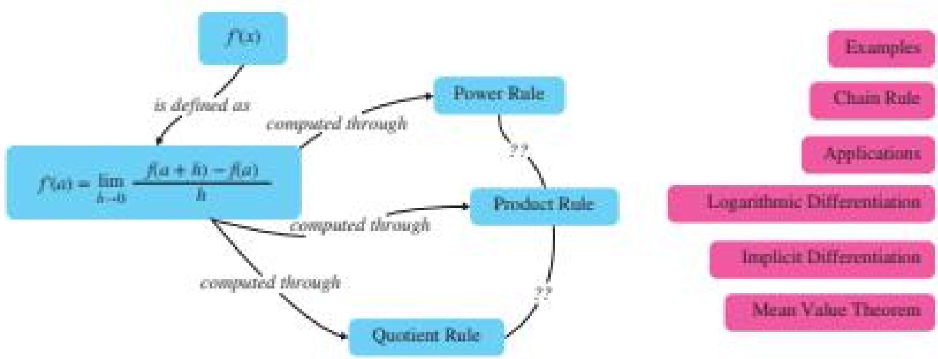
BEYOND FORMULAS

If you think that the method used in example 5.5 is roundabout, then you have the right idea. The chain rule in particular and calculus in general give us methods for determining quantities that are not directly computable. In the case of example 5.5, we use the properties of one function to determine properties of another function. The key to our ability to do this is understanding the theory behind the chain rule.

Concept Mapping

A concept map is an important organizational tool that we will develop in different sections of the book. We consider this as an activity suitable for students to showcase their understanding of the different topics presented in this calculus series. The connections between the different concepts will strengthen your understanding of the bigger picture underlying the approaches being studied. Your concept map may start small, but will develop as you expand and deepen your knowledge.

The concept map that you see below is far from being finished, and it is better classified as work in progress. We hope that you will be able to expand, modify or change it as you see fit throughout your progression in this course. To the right-hand side of the map, we left you some concepts that you might incorporate when you progress through the different sections in this chapter.



EXERCISES 7.5

WRITING EXERCISES

1. If gear 1 rotates at 10 rpm and gear 2 rotates twice as fast as gear 1, how fast does gear 2 rotate? The answer is obvious for most people. Formulate this simple problem as a chain rule calculation and conclude that the chain rule (in this context) is obvious.

2. The biggest challenge in computing the derivatives of $\sqrt{(x^2+4)(x^3-x+1)}$, $(x^2+4)\sqrt{x^3-x+1}$ and x^2+4 $\sqrt{x^3-x+1}$ is knowing which rule (product, chain etc.) to use when. Discuss how you know which rule to use when. (Hint: Think of the order in which you would perform operations to compute the value of each function for a specific choice of x.)

3. One simple implication of the chain rule is: if $g(x) = f(x-a)$, then $g'(x) = f'(x-a)$. Explain this derivative graphically: how does $g(x)$ compare to $f(x)$ graphically and why do the slopes of the tangent lines relate as the formula indicates?

4. Another simple implication of the chain rule is: if $h(x) = f(2x)$, then $h'(x) = 2f'(2x)$. Explain this derivative graphically: how does $h(x)$ compare to $f(x)$ graphically and why do the slopes of the tangent lines relate as the formula indicates?

In exercises 1–4, find the derivative with and without using the chain rule.

1. $f(x) = (x^3-1)^2$
2. $f(x) = (x^2+2x+1)^2$
3. $f(x) = (x^2+1)^3$
4. $f(x) = (2x+1)^4$

In exercises 5–16, differentiate each function.

5. (a) $f(x) = (x^3 - x)^3$ (b) $f(x) = \sqrt{x^2 + 4}$

6. (a) $f(x) = (x^3 + x - 1)^3$ (b) $f(x) = \sqrt{4x - 1/x}$

7. (a) $f(t) = t^5\sqrt{t^3 + 2}$ (b) $f(t) = (t^3 + 2)\sqrt{t}$

8. (a) $f(t) = (t^4 + 2)\sqrt{t^2 + 1}$ (b) $f(t) = \sqrt{t}(t^{4/3} + 3)$

9. (a) $f(u) = \dfrac{u^2 + 1}{u + 4}$ (b) $f(u) = \dfrac{u^3}{(u^2 + 4)^2}$

10. (a) $f(v) = \dfrac{v^2 - 1}{v^2 + 1}$ (b) $f(v) = \dfrac{v^2 + 4}{(v^3)^2}$

11. (a) $g(x) = \dfrac{x}{\sqrt{x^2 + 1}}$ (b) $g(x) = \sqrt{\dfrac{x}{x^2 + 1}}$

12. (a) $g(x) = x^2\sqrt{x + 1}$ (b) $g(x) = \sqrt{(x^2 + 1)(\sqrt{x} + 1)^3}$

13. (a) $h(\omega) = \dfrac{6}{\sqrt{\omega^2 + 4}}$ (b) $h(\omega) = \dfrac{\sqrt{\omega^2 + 4}}{6}$

14. (a) $h(\omega) = \dfrac{(\omega^3 + 4)^5}{8}$ (b) $h(\omega) = \dfrac{8}{(\omega^3 + 4)^5}$

15. (a) $f(x) = \left(\sqrt{x^3 + 2} + 2x\right)^{-2}$ (b) $f(x) = \sqrt{x^3 + 2 + 2x^{-2}}$

16. (a) $f(x) = \sqrt{4x^2 + (8 + x^2)^2}$ (b) $f(x) = \left(\sqrt{4x^2 + 8} - x^2\right)^2$

In exercises 17–22, f has an inverse g. Use Theorem 5.2 to find $g'(a)$.

17. $f(x) = x^3 + 4x - 1,\ a = -1$

18. $f(x) = x^3 + 4x - 2,\ a = -2$

19. $f(x) = x^5 + 3x^3 + x,\ a = 5$

20. $f(x) = x^3 + 2x + 1,\ a = -2$

21. $f(x) = \sqrt{x^3 + 2x + 4},\ a = 2$

22. $f(x) = \sqrt{x^5 + 4x^3 + 3x + 1},\ a = 3$

In exercises 23–26, name the method (chain rule, product rule, quotient rule) that you would use first to find the derivative of the function. Then list any other rule(s) that you would use, in order. Do not compute the derivative.

23. $f(x) = \sqrt[3]{x\sqrt{x^4 + 2x\sqrt[4]{\dfrac{8}{x + 2}}}}$

24. $f(x) = \dfrac{3x^2 + 2\sqrt{x^3 + 4/x^4}}{(x^3 - 4)\sqrt{x^2 + 2}}$

25. $f(t) = \sqrt{t^2 + 4/t^3}\left(\dfrac{8t + 5}{2t - 1}\right)^3$

26. $f(t) = \left(3t + \dfrac{4\sqrt{t^2 + 1}}{t - 5}\right)^3$

In exercises 27 and 28, find an equation of the tangent line to the graph of $y = f(x)$ at $x = a$.

27. $f(x) = \sqrt{x^2 + 16},\ a = 3$

28. $f(x) = \dfrac{6}{x^2 + 4},\ a = -2$

In exercises 29 and 30, use the position function to find the velocity at time $t = 2$. (Assume units of meters and seconds.)

29. $s(t) = \sqrt{t^2 + 8}$

30. $s(t) = \dfrac{60t}{\sqrt{t^2 + 1}}$

In exercises 31 and 32, use the relevant information to compute the derivative for $h(x) = f(g(x))$.

31. $h'(1)$, where $f(1) = 3$, $g(1) = 2$, $f'(1) = 4$, $f'(2) = 3$, $g'(1) = -2$ and $g'(3) = 5$

32. $h'(2)$, where $f(2) = 1$, $g(2) = 3$, $f'(2) = -1$, $f'(3) = -3$, $g'(1) = 2$ and $g'(2) = 4$

33. A function f is an **even function** if $f(-x) = f(x)$ for all x and is an **odd function** if $f(-x) = -f(x)$ for all x. Prove that the derivative of an even function is odd and the derivative of an odd function is even.

34. If the graph of a differentiable function f is symmetric about the line $x = a$, what can you say about the symmetry of the graph of f'?

In exercises 35–38, find the derivative where f is an unspecified differentiable function.

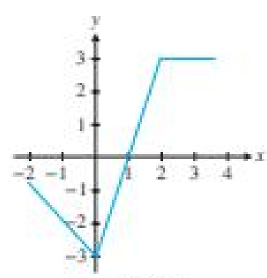
35. (a) $f(x^2)$ (b) $[f(x)]^2$ (c) $f(f(x))$

36. (a) $f(\sqrt{x})$ (b) $\sqrt{f(x)}$ (c) $f(x\,f(x))$

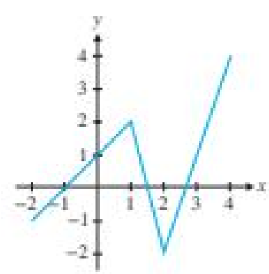
37. (a) $f(1/x)$ (b) $1/f(x)$ (c) $f\left(\dfrac{x}{f(x)}\right)$

38. (a) $1 + f(x^2)$ (b) $[1 + f(x)]^2$ (c) $f(1 + f(x))$

In exercises 39 and 40, use the graphs to find the derivative of the composite function at the point, if it exists.



$y = f(x)$



$y = g(x)$

39. $f(g(x))$ at (a) $x = 0$, (b) $x = 1$ and (c) $x = 3$

40. $g(f(x))$ at (a) $x = 0$, (b) $x = 1$ and (c) $x = 3$

In exercises 41 and 42, find the second derivative of each function.

41. (a) $f(x) = \sqrt{x^2 + 4}$ (b) $f(t) = \dfrac{2}{\sqrt{t^2 + 4}}$

42. (a) $h(t) = (t^3 + 3)^2$ (b) $g(x) = \dfrac{3}{(5^2 + 1)^2}$

43. (a) Determine all values of x such that $f(x) = \sqrt[3]{x^3 - 3x^2 + 2x}$ is not differentiable. Describe the graphical property that prevents the derivative from existing.

(b) Repeat part (a) for $f(x) = \sqrt{x^4 - 3x^3 + 3x^2 - x}$

44. Which steps in our outline of the proof of the chain rule are not well-documented? Where do we use the assumption that $g'(x) \neq 0$?

In exercises 45–48, find a function g such that $g'(x) = f(x)$.

45. $f(x) = (x^2 + 3)^2 (2x)$

46. $f(x) = x^2(x^3 + 4)^{2/3}$

47. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$

48. $f(x) = \dfrac{x}{(x^2 + 1)^2}$

EXPLORATORY EXERCISES

1. Newton's second law of motion is $F = ma$, where m is the mass of the object that undergoes an acceleration a due to an applied force F. This law is accurate at low speeds. At high speeds, we use the corresponding formula from Einstein's theory of relativity, $F = m\dfrac{d}{dt}\left(\dfrac{v(T)}{\sqrt{1 - v^2(t)/c^2}}\right)$, where $v(t)$ is the velocity function and c is the speed of light. Compute $\dfrac{d}{dt}\left(\dfrac{v(T)}{\sqrt{1 - v^2(t)/c^2}}\right)$. What has to be "ignored" to simplify this expression to the acceleration $a = v'(t)$ in Newton's second law?

2. Suppose that f is a function such that $f(1) = 0$ and $f'(x) = \dfrac{1}{x}$ for all $x > 0$.

(a) If $g_1(x) = f(x^n)$ and $g_2(x) = n\, f(x)$ for $x > 0$, show that $g_1'(x) = g_2'(x)$. Since $g_1(1) = g_2(1) = 0$, can you conclude that $g_1(x) = g_2(x)$ for all $x > 0$?

(b) For positive differentiable functions h_1 and h_2, define $g_3(x) = f(h_1(x)h_2(x))$ and $g_4(x) = f(h_1(x)) + f(h_2(x))$. Show that $g_3'(x) = g_4'(x)$. Can you conclude that $g_3(x) = g_4(x)$ for all x?

(c) If f has an inverse g, find $g'(x)$.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Tangent line	Velocity	Average velocity
Derivative	Power rule	Acceleration
Product rule	Quotient rule	Chain rule

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to make a new statement that is true.

1. If a function is continuous at $x = a$, then it has a tangent line at $x = a$.

2. The average velocity between $t = a$ and $t = b$ is the average of the velocities at $t = a$ and $t = b$.

3. The derivative of a function gives its slope.

4. Given the graph of $f'(x)$, you can construct the graph of $f(x)$.

5. The power rule gives the rule for computing the derivative of any polynomial.

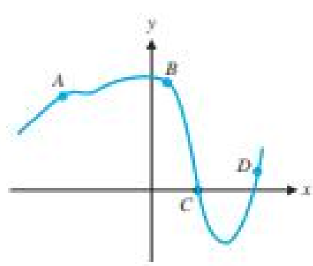
6. If a function is written as a quotient, use the quotient rule to find its derivative.

7. The chain rule gives the derivative of the composition of two functions. The order does not matter.

1. Estimate the value of $f'(1)$ from the given data.

x	0	0.5	1	1.5	2
$f(x)$	2.0	2.6	3.0	3.4	4.0

2. List the points A, B, C and D in order of increasing slope of the tangent line.



Review Exercises

In exercises 3–8, use the limit definition to find the indicated derivative.

3. $f'(2)$ for $f(x) = x^2 - 2x$
4. $f'(1)$ for $f(x) = 1 + \frac{1}{x}$
5. $f'(1)$ for $f(x) = \sqrt{x}$
6. $f'(0)$ for $f(x) = x^3 - 2x$
7. $f'(x)$ for $f(x) = x^3 + x$
8. $f'(x)$ for $f(x) = \frac{3}{x}$

In exercises 9–10, find an equation of the tangent line.

9. $y = x^4 - 2x + 1$ at $x = 1$
10. $y = \sqrt{x^2 + 1}$ at $x = 0$

In exercises 15–18, use the given position function to find velocity and acceleration.

11. $s(t) = -16t^2 + 40t + 10$
12. $s(t) = -9.8t^2 - 22t + 6$
13. $s(t) = \sqrt{4t + 16} - 4$

14. In exercise 11, $s(t)$ gives the height of a ball at time t. Find the ball's velocity at $t = 1$; is the ball going up or down? Find the ball's velocity at $t = 2$; is the ball going up or down?

In exercise 15, compute the slope of the secant line between (a) $x = 1$ and $x = 2$, (b) $x = 1$ and $x = 1.5$, (c) $x = 1$ and $x = 1.1$, and (d) estimate the slope of the tangent line at $x = 1$.

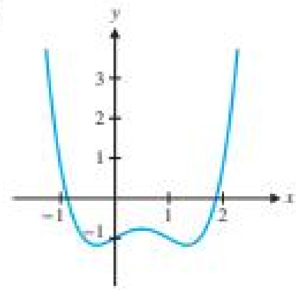
15. $f(x) = \sqrt{x + 1}$

In exercises 16–25, find the derivative of the given function.

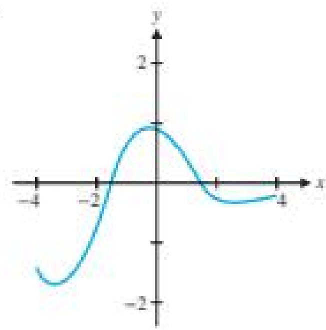
16. $f(x) = x^4 - 3x^3 + 2x - 1$
17. $f(x) = x^{2/3} - 4x^2 + 5$
18. $f(x) = \frac{3}{\sqrt{x}} + \frac{5}{x^2}$
19. $f(x) = \frac{2 - 3x + x^2}{\sqrt{x}}$
20. $f(t) = t^2(t + 2)^3$
21. $f(t) = (t^2 + 1)(t^3 - 3t + 2)$
22. $g(x) = \frac{x}{3x^2 - 1}$
23. $g(x) = \frac{3x^2 - 1}{x}$
24. $f(x) = \left(\frac{x + 1}{x - 1}\right)^2$
25. $f(x) = \frac{6x}{(x - 1)^2}$

In exercises 26 and 27, use the graph of $y = f(x)$ to sketch the graph of $y = f'(x)$.

26.



27.



In exercises 28–31, find the indicated derivative.

28. $f''(x)$ for $f(x) = x^4 - 3x^3 + 2x^2 - x - 1$
29. $f'''(x)$ for $f(x) = \sqrt{x + 1}$
30. $f''(x)$ for $f(x) = \frac{4}{x + 1}$
31. $f^{(4)}(x)$ for $f(x) = (x^6 - 3x^4 + 2x^3 - 7x + 1)^2$

32. Revenue equals price times quantity. Suppose that the current price is \$2.40 and 12,000 items are sold at that price. If the price is increasing at the rate of 10 cents per year and the quantity sold decreases at the rate of 1500 items per year, at what rate is the revenue changing?

Review Exercises

In exercises 33–36, find all points at which the tangent line to the curve is (a) horizontal and (b) vertical.

33. $y = x^3 - 6x^2 + 1$

34. $y = x^{2/3}$

35. $x^2y - 4y = x^2$

36. $y = x^4 - 2x^2 + 3$

37. Prove that the equation $x^3 + 7x - 1 = 0$ has exactly one solution.

38. Prove that the equation $x^5 + 3x^3 - 2 = 0$ has exactly one solution.

In exercise 39, do both parts without solving for the inverse: (a) find the derivative of the inverse at $x = 2$, and (b) graph the inverse.

39. $x^5 + 2x^3 - 1$

40. If $f(x)$ is differentiable at $x = a$, show that $g(x)$ is continuous at $x = a$ where $g(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a} & \text{if } x \neq a \\ f'(a) & \text{if } x \neq a \end{cases}$.

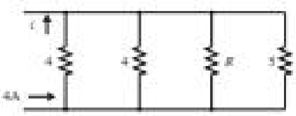
41. If f is differentiable at $x = a$ and $T(x) = f(a) + f'(a)(x - a)$ is the tangent line to $f(x)$ at $x = a$, prove that $f(x) - T(x) = e(x)(x - a)$ for some error function $e(x)$ with $\lim_{x \to a} e(x) = 0$.

42. A polynomial $f(x)$ has a **double root** at $x = a$ if $(x - a)^2$ is a factor of $f(x)$ but $(x - a)^3$ is not. The line through the point (1, 2) with slope m has equation $y = m(x - 1) + 2$. Find m such that $f(x) = x^3 + 1 - [m(x - 1) + 2]$ has a double root at $x = 1$. Show that $y = m(x - 1) + 2$ is the tangent line to $y = x^3 + 1$ at the point (1, 2).

43. Repeat exercise 42 for $f(x) = x^3 + 2x$ and the point (2, 12).

44. A guitar string of length L, density p and tension T will vibrate at the frequency $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{p}}$. Compute the derivative $\dfrac{df}{dT}$, where we think of T as the independent variable and treat p and L as constants. Interpret this derivative in terms of a guitarist tightening or loosening the string to "tune" it. Compute the derivative $\dfrac{df}{dL}$ and interpret it in terms of a guitarist playing notes by pressing the string against a fret.

45. In the accompanying figure, a constant current of 4 A passes through the current divider parallel resistors. The current i is given by $i = \dfrac{80R}{56R + 4}$. Find $\dfrac{di}{dR}$.



46. The Colpitts oscillator frequency is determined by the function $f = \dfrac{1}{2\pi\sqrt{L\left(\dfrac{C_1C_2}{C_1 + C_2}\right)}}$. C_1 and C_2 are capacitors and L is an inductance. Find $\dfrac{df}{dL}$.

47. The surface area of a right circular cone can be expressed as $A = \pi r(r + \sqrt{h^2 + r^2})$, where r is the radius of the base and h is the height.

(a) Find $\dfrac{dA}{dr}$ if the height is constant.

(b) Find $\dfrac{dA}{dh}$ if the radius is constant.

Appendix A

ANSWERS TO ODD-NUMBERED EXERCISES

CHAPTER 5

Exercises 5.1

1. $x < 2$ **3.** $-\frac{4}{3} < x \le \frac{1}{3}$

5. $x > 4$ or $x \le -2$ **7.** $x \ge 1$ or $x \le -3$

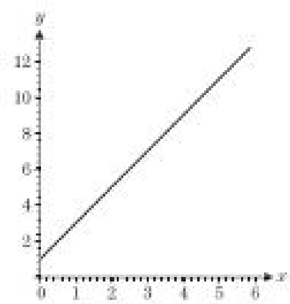
9. $-7 < x < -3$

11. yes **13.** no

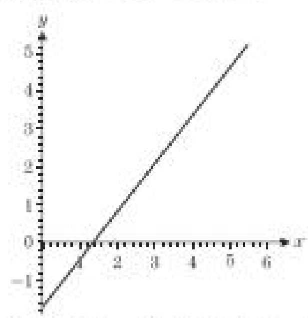
15. (a) $\sqrt{20}$ (b) 2 (c) $y = 2x$

17. (a) $\sqrt{2.96}$ (b) $-\frac{5}{7}$ (c) $1.4y = -x - 1.66$

19. $(2, 5)$, $y = 2(x - 1) + 3$



21. $(3.3, 2.3)$, $y = 1.2(x - 2.3) + 1.1$



23. perpendicular **25.** perpendicular **27.** parallel

29. (a) $y = 2(x - 2) + 1$ (b) $y = -\frac{1}{2}(x - 2) + 1$

31. (a) $y = 2(x - 3) + 1$ (b) $y = -\frac{1}{2}(x - 3) + 1$

33. $y = 2(x - 1) + 1$; 7

35. yes **37.** no **39.** both **41.** rational

43. $x \ge -2$ **45.** $(-\infty, -2) \cup (3, 5) \cup (5, \infty)$ **47.** $x \ne \pm 1$

49. $-1, 1, 11, -\frac{5}{4}$

51. $0 \le x \le$ number made, x an integer

53. no: many y's for one x

55. no: many y's for one x

57. constant, increasing, decreasing; graph going down; graph going up

59. x-intercepts: −2, 4; y-intercept: −8

61. x-intercept: 2; y-intercept: −8

63. x-intercepts: ±2; y-intercept: −4

65. 1, 3 **67.** $2 + \sqrt{2}, 2 - \sqrt{2}$ **69.** 0, 1, 2

71. $1, -\sqrt[3]{2}$ **73.** $(-2, 3), (1, 6)$

Applications

1. 63,000 feet **3.** $y = 4x - 156$ **5.** 51

Exercises 5.2

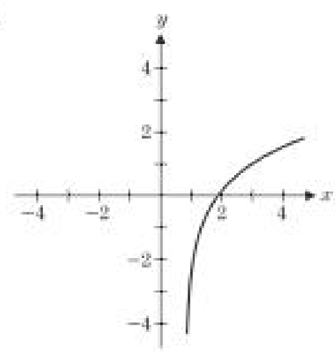
5. $f^{-1}(x) = \sqrt[3]{x + 2}$ **7.** $f^{-1}(x) = \sqrt[3]{x + 1}$

9. not one-to-one

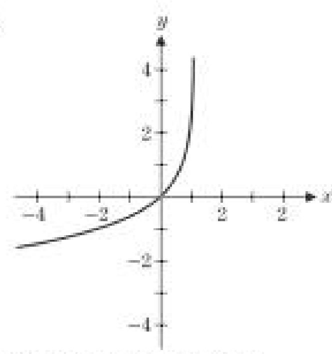
11. $f^{-1}(x) = \sqrt[3]{x^2 - 1}, x \ge 0$

13. (a) 0 (b) 1 **15.** (a) −1 (b) 1 **17.** (a) 2 (b) 0

19.



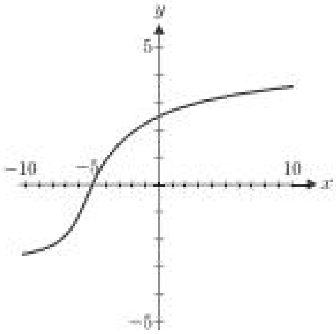
21.



23. If $f^{-1}(x) = y$, then $f(y) = x > 0$

25. If $f(x) \ne 3$, then $f^{-1}(3) \ne x$

27.



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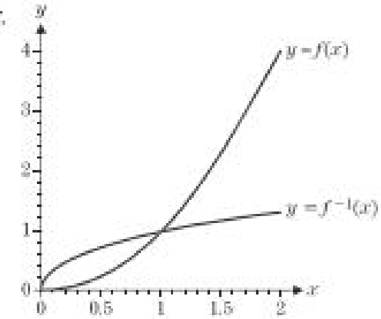

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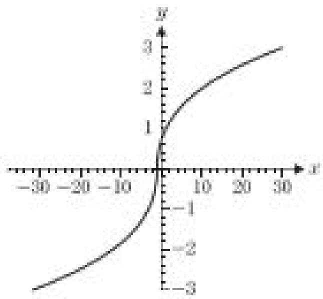
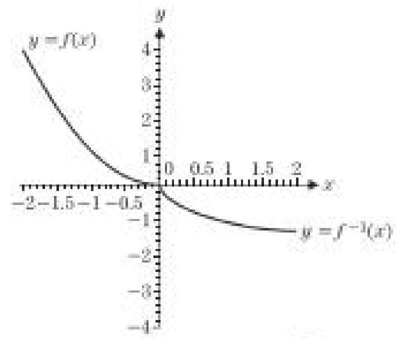
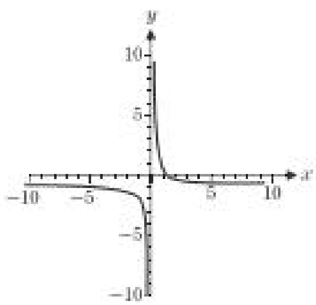


31. not one-to-one

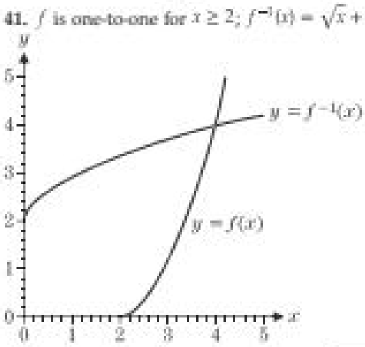
39. $f^{-1}(x) = -\sqrt{x}, x \geq 0$



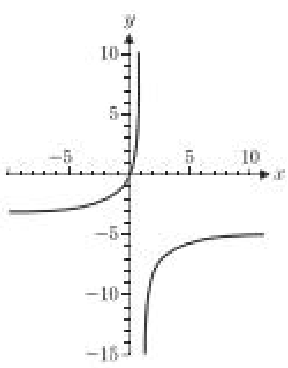
33.



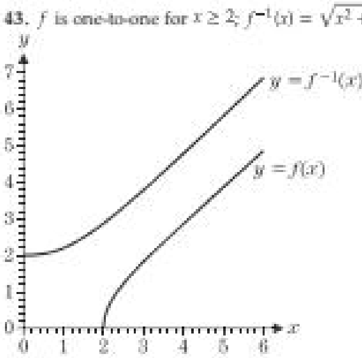
41. f is one-to-one for $x \geq 2$; $f^{-1}(x) = \sqrt{x} + 2$



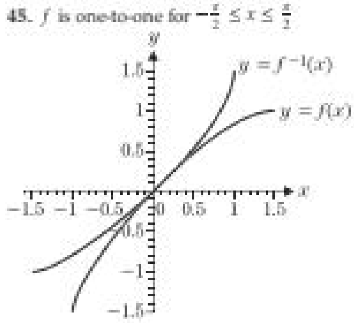
35.



43. f is one-to-one for $x \geq 2$; $f^{-1}(x) = \sqrt{x^2 + 1} + 1$



45. f is one-to-one for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$



Applications

7. no; subtract $\frac{100}{11}\%$

Exercises 5.3

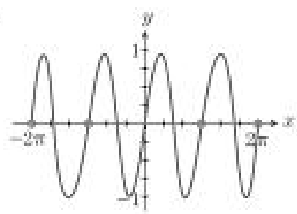
1. (a) $45°$ (b) $60°$ (c) $30°$ (d) $240°$

3. (a) π (b) $\frac{3\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{6}$

5. $-\frac{\pi}{3} + 2n\pi; \frac{\pi}{3} + 2n\pi$ 7. $-\frac{\pi}{4} + 2n\pi; \frac{\pi}{4} + 2n\pi$

9. $\frac{\pi}{2} + 2n\pi$ 11. $\frac{\pi}{2} + n\pi; 2n\pi$ 13. $\pi + 2n\pi; \frac{\pi}{2} + n\pi$

15.



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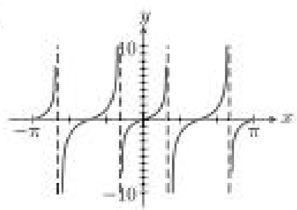

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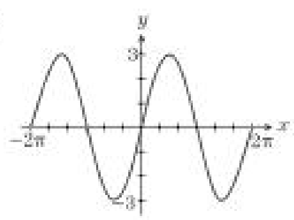

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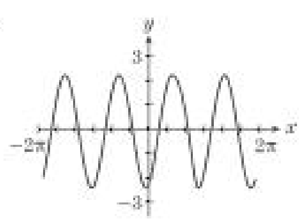
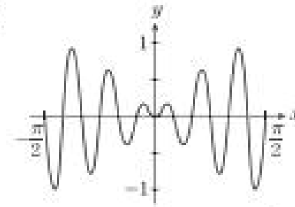


23.



25. $A = 3$, period $= \pi$, frequency $= \frac{1}{\pi}$

27. $A = 5$, period $= \frac{2\pi}{3}$, $f = \frac{3}{2\pi}$

29. $A = 3$, period $= \pi$, $f = \frac{1}{\pi}$

31. $A = 4$, period $= 2\pi$, $f = \frac{1}{2\pi}$

37. $\frac{\pi}{2}$ 39. $-\pi/2$ 41. 0 43. $\frac{\pi}{3}$ 45. $\frac{\pi}{4}$

47. $\beta \approx 0.6435$

49. no 51. yes, 2π

53. $\frac{2\sqrt{2}}{3}$

55. $-\frac{\sqrt{3}}{2}$

57. $\sqrt{1 - x^2}$; $-1 \le x \le 1$

59. $\sqrt{x^2 - 1}, x \ge 1$ or $-\sqrt{x^2 - 1}, x \le -1$

61. $\frac{\sqrt{3}}{2}$ 63. $\frac{4}{3}$

65. 3; $x = 0$, $x \approx 1.109$, $x \approx 3.698$

67. 2; $x \approx -1.455$, $x \approx 1.455$

Applications

1. $2\tan 20° \approx 0.73$ mile

3. $100\tan 50° \approx 119$ feet

5. $A(x) = \tan^{-1}\left(\frac{0.6}{x}\right)$

7. $f = \frac{30}{\pi}, \frac{170}{\sqrt{2}} \approx 120.2$ volts

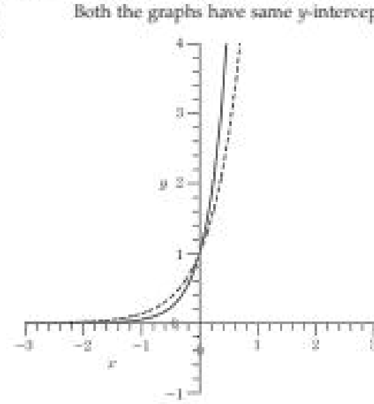
9. $24,000 per year

Exercises 5.4

1. $\frac{1}{8}$ 3. $\sqrt{3}$ 5. $\sqrt[3]{25}$ 7. x^{-2} 9. $2x^{-3}$

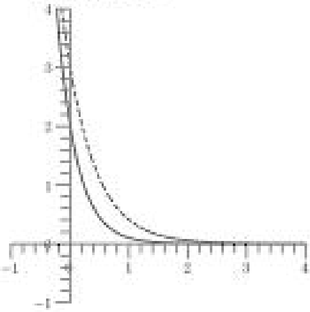
11. $\frac{1}{2}x^{-1/2}$ 13. 8 15. 2 17. 1.213 19. 4.415

21. Both the graphs have same y-intercept.



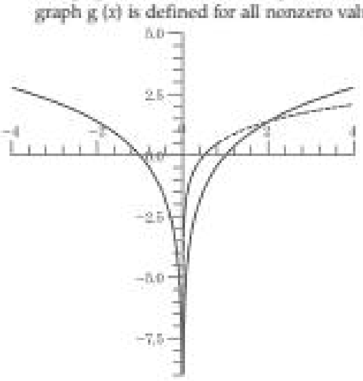
Graph of f (x): Dotted line.
Graph of g(x): Solid line.

23. For the graph f (x), y-intercept is 3 and for the graph g(x), y-intercept is 2.



Graph of f(x): Dotted line.
Graph of g(x): Solid line

25. The graph f (*x*), is defined for positive values of *x* only and the graph g (*x*) is defined for all nonzero value of *x*.



Graph of f (x): Dotted line.
Graph of g(x): Solid line

27. $\frac{1}{2}\ln 2$ **29.** $x = -1, x = 1$

31. e^{-2} **33.** 2 **35.** $\ln 3$

37. (a) 2 (b) 3 (c) -3

39. (a) 1.771 (b) 2.953 (c) -2.893

41. $\ln\frac{3}{4}$ **43.** $\ln 1 = 0$ **45.** $\ln 12$

47. $2e^{(1/2\ln 3)x}$ **49.** $4e^{(1/2\ln 1/2)x}$

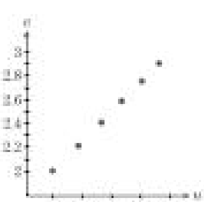
53. $x = -1, x = 1$

Applications

1. 0.651

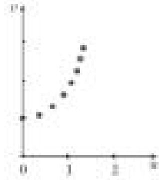
3. $1 - e^{-1} \approx 0.632$

5.



$m = 2,\ b \approx 1.0986,$
$a = e^b = 3$

7.



no, exponential

9. $10^{-7}, 10^{-8}, 10^{-9}, \frac{1}{10}$

11. $10^{10.4}, 10^{11.9}, 10^{13.4}, 10^{1.5} \approx 31.6$

13. yes

15. $f = (220)\,2^x$

Exercises 5.5

1. $(f \circ g)(x) = \sqrt{x-3} + 1, x \ge 3$
$(g \circ f)(x) = \sqrt{x-2}, x \ge 2$

3. $(f \circ g)(x) = x, x > 0$
$(g \circ f)(x) = x$, all reals

5. $(f \circ g)(x) = \sin^2 x + 1$, all reals
$(g \circ f)(x) = \sin(x^2 + 1)$, all reals

7. possible answer: $f(x) = \sqrt{x}, g(x) = x^4 + 1$

9. possible answer: $f(x) = \frac{1}{x}, g(x) = x^2 + 1$

11. possible answer: $f(x) = x^2 + 3, g(x) = 4x + 1$

13. possible answer: $f(x) = x^3, g(x) = \sin x$

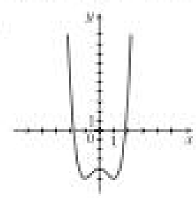
15. possible answer: $f(x) = e^x, g(x) = x^2 + 1$

17. possible answer: $f(x) = \frac{3}{x}, g(x) = \sqrt{x}, h(x) = \sin x + 2$

19. possible answer: $f(x) = x^3, g(x) = \cos x, h(x) = 4x - 2$

21. possible answer: $f(x) = 4x - 5, g(x) = e^x, h(x) = x^2$

23.



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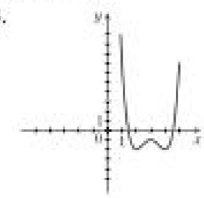

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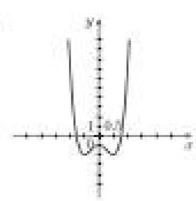
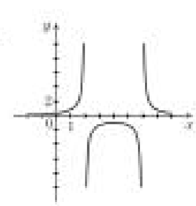


29.

31.



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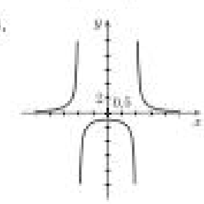

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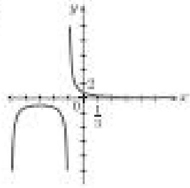
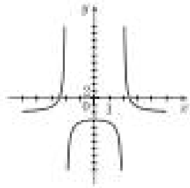


37.



39. y = (x + 1)2 , shift left one

41. y = (x + 1)2 + 3, shift left one, up three

43. y = 2[(x + 1)2 + 1], shift left one, up one, double vertical scale

45. reflect across x-axis, double vertical scale

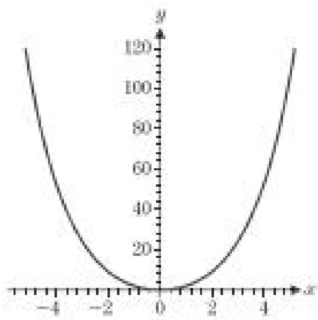
47. reflect across x-axis, triple vertical scale, shift up two

49. reflect across y-axis

51. shift right one.

53. reflect across x-axis, vertical scale times |*c*|

55.



59. go to 0 **61.** 0.739085

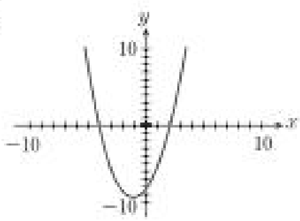
Chapter 5 Review Exercises

1. −2 **3.** parallel **5.** no

7. $y = \frac{1}{2}(x-1)+1, y = \frac{5}{2}$ **9.** $y = -\frac{1}{3}(x+1)-1$

11. yes **13.** $-2 \le x \le 2$

15.



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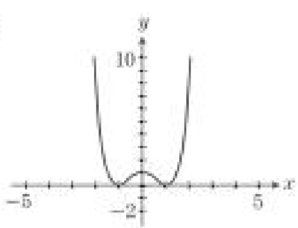

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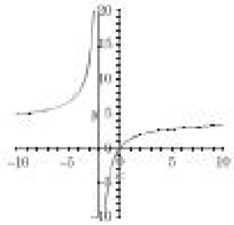

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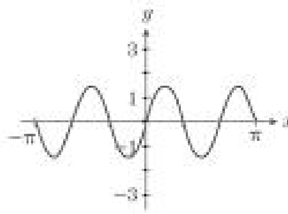

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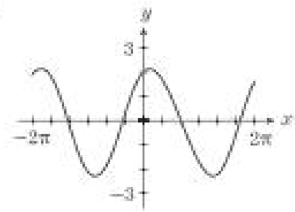

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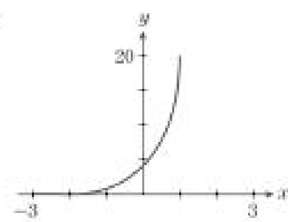
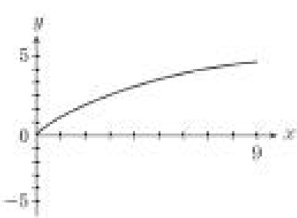


27.



29. $x = -4, x = 2, y = -8$ **31.** $x = -2$

33. $-2, 5$ **35.** $1, 1+\sqrt{3}, 1-\sqrt{3}$ **37.** 3

39. $50 \tan 34° \approx 33.7$ feet

41. (a) $\frac{1}{\sqrt{5}}$ (b) $\frac{1}{9}$

43. $\ln 2$ **45.** $\frac{1}{2}\ln\frac{8}{5}$

47. $(f \circ g)(x) = x - 1, x \ge 1$

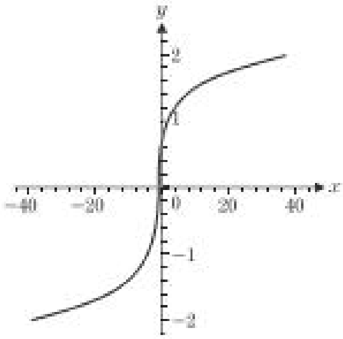
$(g \circ f)(x) = \sqrt{x^2 - 1}, x \le -1$ or $x \ge 1$

49. $f(x) = e^x, g(x) = 3x^2 + 2$

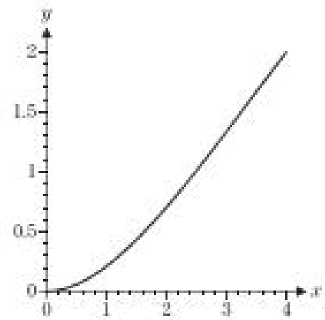
51. $(x-2)^2-3$, shift two right and three down

53. yes; $f^{-1}(x)=\sqrt[3]{x+1}$ **55.** no

57.



59.



61. $\frac{\pi}{2}$ **63.** $-\frac{\pi}{4}$ **65.** $\frac{\sqrt{3}}{2}$

67. $\frac{\pi}{4}$ **69.** $\frac{\pi}{4}+n\pi$

CHAPTER 6

Exercises 6.1

1. (a) 2 (b) 4 **3.** (a) 0 (b) −1

5. (a) 1 (b) 2.7

7. (a) 1.90626 (b) 1.90913 (c) 1.91010

9. (a) 3.16732 (b) 3.16771 (c) 3.16784

11. (a) 9.15298 (b) 9.25345 (c) 9.29357

13. (a) $\frac{11}{8}$ (b) $\frac{43}{32}$ **15.** (a) 2.05 (b) 2.01

17. (a) 1.55 (b) 1.56; quarter circle

Exercises 6.2

1. $$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)}=\lim_{x\to 1}(x+1)=2$$

3. $$\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{(x+2)}=\frac{1}{4}$$

5. $$\lim_{x\to 3}\frac{3x-9}{x^2-5x+6}=\lim_{x\to 3}\frac{3(x-3)}{(x-3)(x-2)}=\lim_{x\to 3}\frac{3}{x-2}=3$$

7. (a) $f(-2)=3$

(b) $\lim_{x\to -2}f(x)=2$

(c) $f(-1)$ is undefined.

(d) $\lim_{x\to 0^+}f(x)=2$

9. (a) $\lim_{x\to 0^-}f(x)=-2$

(b) $\lim_{x\to 0^+}f(x)=2$

(c) $\lim_{x\to 0}f(x)$ does not exist.

(d) $\lim_{x\to -2^-}f(x)=2$

(e) $\lim_{x\to -2^+}f(x)=2$

(f) $\lim_{x\to -2}f(x)=2$

(g) $\lim_{x\to -1}f(x)=0$

(h) $\lim_{x\to -3}f(x)=1$

11. (a) $f(1)=2$

(b) $\lim_{x\to 1^-}f(x)=1$

(c) $\lim_{x\to 1^+}f(x)=3$

(d) $\lim_{x\to 1}f(x)$ does not exist.

(e) $f(2)=0$

(f) $\lim_{x\to 2}f(x)$ does not exist.

13. (a) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^3 - 1 = -1$

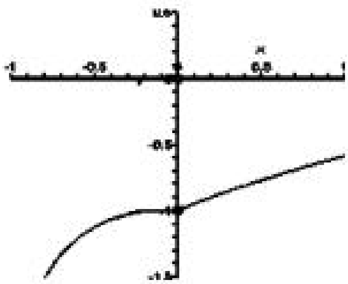
(b) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{x+1} - 2$

$= -1$

(c) $\lim_{x\to 0} f(x) = -1$

(d) $\lim_{x\to -1} f(x) = \lim_{x\to -1} x^3 - 1 = -2$

(e) $\lim_{x\to 3} f(x) = \lim_{x\to 3} \sqrt{x+1} - 2 = 0$



15. $f(-1.5) = -0.4$

$f(-1.1) = -0.4762$

$f(-1.01) = -0.4975$

$f(-1.001) = -0.4998$

The values of $f(x)$ seem to be approaching −0.5 as x approaches −1 from the left.

$f(-0.5) = -0.6667$

$f(-0.9) = -0.5263$

$f(-0.99) = -0.5025$

$f(-0.999) = -0.5003$

The values of $f(x)$ seem to be approaching −0.5 as x approaches −1 from the right. Since the limits from the left and right exist and are the same, the limit exists.

17. The numerical evidence suggests that the function the function blows up at $x = 1$. From the graph we see that the function has a vertical asymptote at $x = 1$.

19.

x	$y = f(x)$
0.9	0.949122
0.99	0.994991
0.999	0.999500
1.001	1.000500
1.01	1.005000
1.1	1.049206

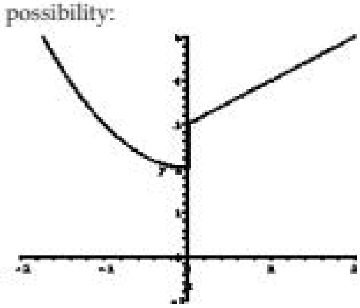
By inspecting the graph, and using a sequence of values, we see that the limit is approximately 1.

21. The limit exists and equals 1.

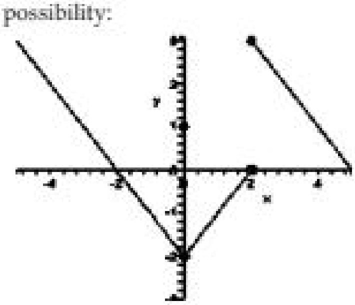
23. By inspecting the graph, and using a sequence of values (as in exercises 11 and 12), we see that the limit is approximately 3/2.

25. The function approaches 1/2 from the left, and −1/2 from the right. Since these are not equal, the limit does not exist.

27. One possibility:



29. One possibility:



31. $\lim_{x\to -1} \frac{x+1}{x^2+1} = 0$ and $\lim_{x\to \pi} \frac{\sin x}{x} = 0$. If the numerator f(a) = 0, and the denominator g(a) ≠ 0, then the limit $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \left(\frac{x}{x^2 + 0.000001}\right) = 0$$

31.

x	$x^{\sec x}$
0.1	0.099
0.01	0.010
0.001	0.001

$$\lim_{x\to 0^+} x^{\sec x} = 0$$

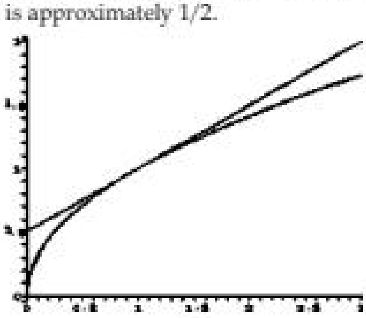
For negative x the values of $x^{\sec x}$ are usually not real numbers, so $\lim_{x\to 0^-} x^{\sec x} = 0$ does not exist.

37. There are many possibilities. Here is a simple one

$$f(x) = \begin{cases} -x & x < 0 \\ 3 & x = 0 \\ x & x > 0 \end{cases}$$

Applications

1. By inspecting the graph, and using a sequence of values (as in exercises 11 and 12), we see that the limit is approximately 1/2.



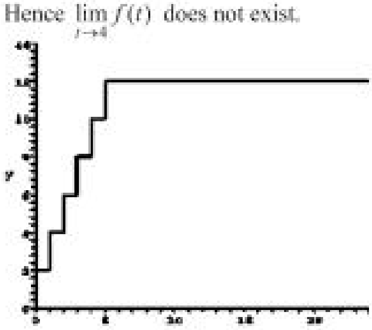
3. For $3 \le t \le 4$, $f(t) = 8$, so $\lim_{t\to 3.5} f(t) = 8$.

Also $\lim_{t\to 4^-} f(t) = 8$.

On the other hand, for $4 \le t \le 5$, $f(t) = 10$,

so $\lim_{t\to 4^+} f(t) = 10$.

Hence $\lim_{t\to 4} f(t)$ does not exist.



Exercises 6.3

1. $\lim_{x\to 0}(x^2 - 3x + 1) = 0^2 - 3(0) + 1 = 1$

3. $\lim_{x\to 0} \cos^{-1}(x^2) = \cos^{-1} 0 = \frac{\pi}{2}$

5. $\lim_{x\to 3} \frac{x^2 - x - 6}{x - 3}$

$= \lim_{x\to 3} \frac{(x-3)(x+2)}{x-3}$

$= \lim_{x\to 3}(x+2) = 3 + 2 = 5$

7. $\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4}$

$= \lim_{x\to 2} \frac{(x-2)(x+1)}{(x+2)(x-2)}$

$= \lim_{x\to 2} \frac{x+1}{x+2} = \frac{2+1}{2+2} = \frac{3}{4}$

9. $\lim_{x\to 0} \frac{\sin x}{\tan x} = \lim_{x\to 0} \frac{\sin x}{\frac{\sin x}{\cos x}}$

$= \lim_{x\to 0} \cos x = \cos 0 = 1$

11. $\lim_{x\to 0} \frac{xe^{-2x+1}}{x^2 + x}$

$= \lim_{x\to 0} \frac{x(e^{-2x+1})}{x(x+1)}$

$= \lim_{x\to 0} \frac{e^{-2x+1}}{x+1} = \frac{e^{-2(0)+1}}{0+1} = e$

13. $\lim_{x\to 0} \frac{\sqrt{x+4} - 2}{x}$

$= \lim_{x\to 0} \frac{\sqrt{x+4} - 2}{x}\left(\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right)$

$= \lim_{x\to 0} \frac{x + 4 - 4}{x(\sqrt{x+4}+2)}$

$= \lim_{x\to 0} \frac{x}{x(\sqrt{x+4}+2)}$

$= \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2}$

$= \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$

15. $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$

$=\lim_{x\to 1}\frac{(\sqrt{x}+1)(\sqrt{x}-1)}{\sqrt{x}-1}$

$=\lim_{x\to 1}(\sqrt{x}+1)=\sqrt{1}+1=2$

17. $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)$

$=\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{(x-1)(x+1)}\right)$

$=\lim_{x\to 1}\left(\frac{x+1}{(x-1)(x+1)}-\frac{2}{(x-1)(x+1)}\right)$

$=\lim_{x\to 1}\left(\frac{x-1}{(x-1)(x+1)}\right)$

$=\lim_{x\to 1}\left(\frac{1}{x+1}\right)=\frac{1}{2}$

19. $\lim_{x\to 0}\frac{1-e^{2x}}{1-e^x}$

$=\lim_{x\to 0}\frac{(1-e^x)(1+e^x)}{1-e^x}$

$=\lim_{x\to 0}(1+e^x)=2$

21. $\lim_{x\to 2^-}f(x)=\lim_{x\to 2^-}2x=2(2)=4$

$\lim_{x\to 2^+}f(x)=\lim_{x\to 2^+}x^2=2^2=4$

$\lim_{x\to 2}f(x)=4$

23. $\lim_{x\to -1^-}f(x)=\lim_{x\to -1^-}(2x+1)$

$=2(-1)+1=-1$

$\lim_{x\to -1^+}f(x)=\lim_{x\to -1^+}3=3$

Therefore $\lim_{x\to -1}f(x)$ does not exist.

25. $\lim_{h\to 0}\frac{(2+h)^2-4}{h}$

$=\lim_{h\to 0}\frac{(4+4h+h^2)-4}{h}$

$=\lim_{h\to 0}\frac{4h+h^2}{h}=\lim_{h\to 0}4+h=4$

27. Consider $f(x)=\frac{\sin x}{x}$ and a polynomial $p(x)=x^2-4$ such that $p(2)=0$.

Also $\lim_{x\to 0}\frac{\sin x}{x}=1$.

Therefore by the theorem 3.4(viii),

$\lim_{x\to a}f(p(x))=L$

$\Rightarrow\lim_{x\to 2}\frac{\sin(x^2-4)}{x^2-4}=1.$

29. $\lim_{x\to 0}\left(\frac{1}{x}+\frac{1}{x^2-x}\right)=\lim_{x\to 0}\left(\frac{x-1+1}{x(x-1)}\right)$

$=\lim_{x\to 0}\left(\frac{x}{x(x-1)}\right)=\lim_{x\to 0}\left(\frac{1}{(x-1)}\right)=-1$

31. $\lim_{x\to 0}\frac{(x+2)^3-8}{x}$

$=\lim_{x\to 0}\frac{[x+2-2]\left[(x+2)^2+2(x+2)+4\right]}{x}$

$=\lim_{x\to 0}\left[(x+2)^2+2(x+2)+4\right]=12$

33. $\lim_{t\to 0}\frac{1-\sqrt{t^2+1}}{4t^2}$

$=\lim_{t\to 0}\frac{1-\sqrt{t^2+1}}{4t^2}\cdot\frac{1+\sqrt{t^2+1}}{1+\sqrt{t^2+1}}=\lim_{t\to 0}\frac{1-(t^2+1)}{4t^2\left(1+\sqrt{t^2+1}\right)}$

$=\lim_{t\to 0}\frac{-t^2}{4t^2\left(1+\sqrt{t^2+1}\right)}=\lim_{t\to 0}\frac{-1}{4\left(1+\sqrt{t^2+1}\right)}=-\frac{1}{8}$

35. $\lim_{x\to 0}\frac{x}{\sin 4x}$

$=\lim_{x\to 0}\frac{x}{\sin 4x}\cdot\frac{4}{4}=\frac{1}{4}$

37. $\lim_{x\to 0}\frac{\tan 5x}{\sin 5x}$

$=\lim_{x\to 0}\frac{\frac{\sin 5x}{\cos 5x}}{\sin 5x}=\lim_{x\to 0}\frac{\sin 5x}{\cos 5x\sin 5x}$

$=\lim_{x\to 0}\frac{1}{\cos 5x}=1$

39. $\lim_{x\to 4}\dfrac{\sin\left(\sqrt{x}-2\right)}{x-4}$

$$=\lim_{x\to 4}\frac{\sin\left(\sqrt{x}-2\right)}{\left(\sqrt{x}+2\right)\left(\sqrt{x}-2\right)}=\lim_{x\to 4}\frac{1}{\left(\sqrt{x}+2\right)}=\frac{1}{4}$$

41.

x^2	$x^2\sin(1/x)$
−0.1	0.0054
−0.01	5×10^{-5}
−0.001	-8×10^{-7}
0.1	−0.005
0.01	-5×10^{-5}
0.001	8×10^{-7}

Conjecture: $\lim_{x\to 0} x^2\sin(1/x)=0$.

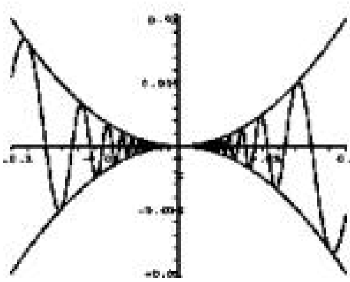
Let $f(x)=-x^2$, $h(x)=x^2$.

Then $f(x)\le x^2\sin(\frac{1}{x})\le h(x)$

$\lim_{x\to 0}(-x^2)=0,\ \lim_{x\to 0}(x^2)=0$

Therefore, by the Squeeze Theorem,

$\lim_{x\to 0} x^2\sin(\frac{1}{x})=0.$



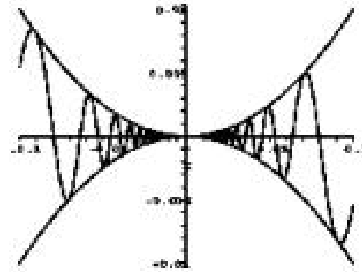
43. Let $f(x)=0$, $h(x)=\sqrt{x}$. We see that

$f(x)\le\sqrt{x}\cos^2(1/x)\le h(x)$,

$\lim_{x\to 0^+}0=0,\ \lim_{x\to 0^+}\sqrt{x}=0$

Therefore, by the Squeeze Theorem,

$\lim_{x\to 0^+}\sqrt{x}\cos^2(\frac{1}{x})=0.$



45. Velocity is given by the limit

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$=\lim_{h\to 0}\frac{(2+h)^2+2-(2^2+2)}{h}$$
$$=\lim_{h\to 0}\frac{4h+h^2}{h}$$
$$=\lim_{h\to 0}4+h=4.$$

47. Velocity is given by the limit

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$
$$=\lim_{h\to 0}\frac{(0+h)^3-(0)^3}{h}$$
$$=\lim_{h\to 0}\frac{h^3}{h}$$
$$=\lim_{h\to 0}h^2=0.$$

49. $\lim_{x\to 0^+}\dfrac{\sqrt{1-\cos x}}{x}=\sqrt{\dfrac{1}{2}}=\dfrac{\sqrt{2}}{2}$

51. $\lim_{x\to a^-}f(x)=\lim_{x\to a^-}g(x)=g(a)$ because $g(x)$ is a polynomial. Similarly,

$\lim_{x\to a^+}f(x)=\lim_{x\to a^+}h(x)=h(a).$

53. (a) $\lim_{x\to 2}(x^2-3x+1)$

$= 2^2 - 3(2) + 1$ (Theorem 3.2)

$= -1$

(b) $\lim_{x\to 0}\frac{x-2}{x^2+1}$

$= \frac{\lim_{x\to 0}(x-2)}{\lim_{x\to 0}(x^2+1)}$ (Theorem 3.1(iv))

$= \frac{\lim_{x\to 0}x - \lim_{x\to 0}2}{\lim_{x\to 0}x^2 + \lim_{x\to 0}1}$ (Theorem 3.1(ii))

$= \frac{0-2}{0+1}$ (Equations 3.1, 3.2, and 3.5)

$= -2$

55. $\lim_{x\to a}[2f(x)-3g(x)]$

$= 2\lim_{x\to a}f(x) - 3\lim_{x\to a}g(x)$

$= 2(2) - 3(-3) = 13$

57. $\lim_{x\to a}\frac{[f(x)]^2}{g(x)} = \frac{\left[\lim_{x\to a}f(x)\right]^2}{\lim_{x\to a}g(x)} = \frac{(2)^2}{-3} = -\frac{4}{3}$

59. $\lim_{x\to 0}p(p(p(p(x))))$

$= \lim_{x\to 0}p(p(p(x^2-1)))$

$= \lim_{x\to 0}p\left(p\left((x^2-1)^2-1\right)\right)$

$= \lim_{x\to 0}p(p(x^4-2x^2))$

$= \lim_{x\to 0}p\left((x^4-2x^2)^2-1\right)$

$= \lim_{x\to 0}p(x^8-4x^6+4x^4-1)$

$= \lim_{x\to 0}(x^8-4x^6+4x^4-1)^2-1$

$= (-1)^2 - 1 = 0$

61. We can't split the limit of a product into a product of limits unless we know that both limits exist; the limit of the product of a term tending toward 0 and a term with an unknown limit is not necessarily 0 but instead is unknown.

63. One possibility is $f(x) = \frac{1}{x}, g(x) = -\frac{1}{x}$.

65. Yes. If $\lim_{x\to a}[f(x)+g(x)]$ exists, then, it would also be true that

$\lim_{x\to a}[f(x)+g(x)] - \lim_{x\to a}f(x)$

exists. But by Theorem **3.1** (ii)

$\lim_{x\to a}[f(x)+g(x)] - \lim_{x\to a}f(x)$

$= \lim_{x\to a}[[f(x)+g(x)]-[f(x)]]$

$= \lim_{x\to a}g(x)$

so $\lim_{x\to a}g(x)$ would exist, but we are given that $\lim_{x\to a}g(x)$ does not exist.

67. $\lim_{x\to 0^+}(1+x)^{1/x} = e \approx 2.71828$

69. $\lim_{x\to 0^+}x^{-x^2} = 1$

71. When x is small and positive, $1/x$ is large and positive, so $\tan^{-1}(1/x)$ approaches $\pi/2$. But when x is small and negative, $1/x$ is large and negative, so $\tan^{-1}(1/x)$ approaches $-\pi/2$. So the limit does not exist.

73. $\lim_{x\to a}[f(x)]^3$

$= \left[\lim_{x\to a}f(x)\right]\left[\lim_{x\to a}f(x)\right]\left[\lim_{x\to a}f(x)\right]$

$= L \cdot L \cdot L = L^3$

$\lim_{x\to a}[f(x)]^4 = \left[\lim_{x\to a}f(x)\right]\left[\lim_{x\to a}[f(x)]^3\right]$

$= L \cdot L^3 = L^4$

75. $\lim_{x\to 3^-}[x] = 2;\ \lim_{x\to 3^+}[x] = 3$

Therefore $\lim_{x\to 3}[x]$ does not exist.

Application

1. $\lim_{x\to 0^+}T(x) = \lim_{x\to 0^+}(0.14x) = 0 = T(0)$.

$\lim_{x\to 10{,}000^-}T(x) = 0.14(10{,}000) = 1400$

$\lim_{x\to 10{,}000^+}T(x) = 1500 + 0.21(10{,}000) = 3600$

Therefore $\lim_{x\to 10{,}000}T(x)$ does not exist.

A small change in income should result in a small change in tax liability. This is true near $x = 0$ but is not true near $x = 10{,}000$. As your income grows past \$10,000 your tax liability jumps enormously.

Exercises 6.4

1. $x \neq -2; g(x) = x - 1$

3. $x \neq \pm 1, g(x) = \dfrac{1}{x+1}$

5. all real numbers **7.** $x \neq \frac{n\pi}{2}$ for odd integers n

9. $x \neq 0$ **11.** $x \neq 1$ **13.** $x \neq 1$

15. $f(1)$ is not defined and $\lim_{x\to 1} f(x)$ does not exist

17. $f(0)$ is not defined and $\lim_{x\to 0} f(x)$ does not exist

19. $\lim_{x\to 2} f(x) \neq f(2)$

21. $[-3, \infty)$ **23.** $(-\infty, \infty)$

25. $[-3, -1]$. **27.** $[-1, \sqrt{2}), (\sqrt{2}, \infty)$ **29.** -700

31. $b = \$12{,}747.50$, $c = \$23{,}801.30$

33. (a) $\left[2\frac{20}{32}, 2\frac{21}{32}\right]$ (b) $\left[-2\frac{21}{32}, -2\frac{20}{32}\right]$

35. $\left[\frac{23}{32}, \frac{24}{32}\right]$ **37.** $(-7, -2), (-2, -1), (1, 4), (4, 7)$

j

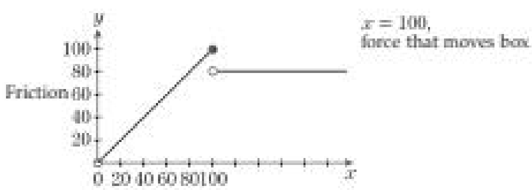
39. $a = b = 2$ **41.** $a = \frac{3}{2}, b = \frac{1}{3}\ln 4$

43. no **45.** #43 is **51.** No

57. $x = -2,\ x = -1, x = 0$

Applications

1.



5. One answer: $g(T) = 100 - 25(T - 30)$

Exercises 6.5

1. (a) ∞ (b) $-\infty$ (c) does not exist

3. (a) $-\infty$ (b) $-\infty$ (c) $-\infty$

5. does not exist **7.** does not exist **9.** $\frac{1}{3}$

11. 1 **13.** ∞ **15.** 0 **17.** 0 **19.** 0

21. does not exist

23. (a) vertical asymptotes at $x = \pm 2$; horizontal asymptote at $y = 0$
(b) vertical asymptotes at $x = \pm 2$; horizontal asymptote at $y = -1$

25. vertical asymptotes at $x = -1$ and $x = 3$; horizontal asymptote at $y = 3$

27. horizontal asymptotes at $y = \pm 2\pi - 1$

29. vertical asymptotes at $x = \pm 2$; slant asymptote at $y = -x$

31. vertical asymptotes at $x = -\frac{1}{2} \pm \sqrt{\frac{37}{4}}$; slant asymptote at $y = x - 1$

33. with no light, 40 mm; with an infinite amount of light, 12 mm

35. $f(x) = \dfrac{80x^{-0.3} + 60}{10x^{-0.3} + 30}$

37. -223.6 ft/s; -158.1 ft/s; 4

39. $\frac{1}{2}$ **41.** $\frac{1}{2}$ **43.** 0 **45.** 1 **47.** 2.7183

49. $-\frac{1}{2}$ **51.** no **53.** one larger

55. $-2(x-3)^2$ **57.** $x^2 + 1$

59. true **61.** false **63.** true

65. $g(x) = \sin x, h(x) = x$

Applications

1. 30 mm, 300 mm

2. ∞, c **3.** $ve = \sqrt{19.6R}$

Exercises 6.6

1. $\delta = \varepsilon/3$.

3. $\delta = \varepsilon/3$.

5. $\delta = \varepsilon/4$.

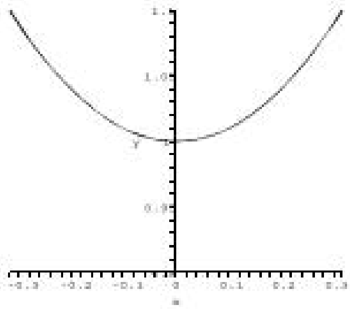
7. $\delta = \varepsilon$.

9. $\delta = \min\{1, \varepsilon/3\}$.

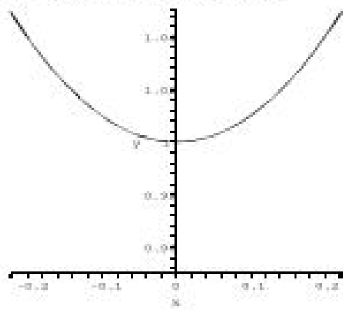
11. $\delta = \min\{1, \varepsilon/5\}$.

13. Let $f(x) = mx + b$. Since $f(x)$ is continuous, we know that $\lim_{x\to a} f(x) = ma + b$. So, we want to find a δ which forces $|mx + b - (ma + b)| < \varepsilon$.
But
$$|mx + b - (ma + b)| = |mx - ma| = |m||x - a|.$$
So as long as $|x - a| < \delta = \varepsilon/|m|$, we will have $|f(x) - (ma + b)| < \varepsilon$. This δ clearly does not depend on a. This is due to the fact that $f(x)$ is a linear function, so the slope is constant, which means that the ratio of the change in y to the change in x is constant.

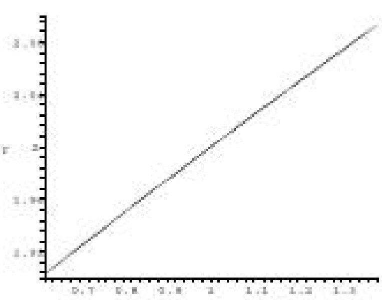
15. (a) From the graph, we determine that we can take δ = 0:316, as shown below.



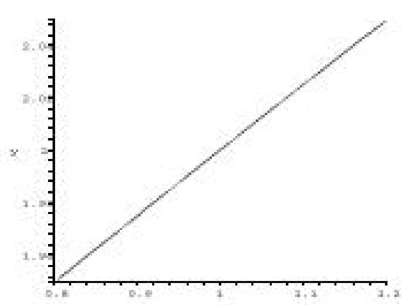
(b) From the graph, we determine that we can take δ = 0:223, as shown below.



17. (a) From the graph, we determine that we can take δ = 0:38, as shown below.



(b) From the graph, we determine that we can take δ = 0:2, as shown below.



19. For a function $f(x)$ defined on some open interval $(c; a)$ we say

$$\lim_{x \to a^-} f(x) = L$$

if, given any number $\varepsilon > 0$, there is another number $\delta > 0$ such that whenever $x \in (c;\ a)$ and $a - \delta < x < a$, we have $f(x) - L| < \varepsilon$.

For a function f(x) defined on some open interval $(a; c)$ we say

$$\lim_{x \to a^+} f(x) = L$$

if, given any number $\varepsilon > 0$, there is another number $\delta >$ such that whenever $x \in (a;\ c)$ and $a < x < a + \delta$, we have $f(x) - L| < \varepsilon$.

21. (a) As $x \to 1^+$, $x - 1 > 0$ so we compute

$$\frac{2}{x-1} > 100$$
$$2 > 100(x-1)$$
$$\frac{2}{100} > x - 1$$

So take $\delta = 2/100$.

(b) As $x \to 1^-$, $x - 1 < 0$ so we compute

$$\frac{2}{x-1} < -100$$
$$2 > -100(x-1)$$
$$-\frac{2}{100} < x - 1$$
$$\frac{2}{100} > -x + 1 = |x - 1|$$

So take $\delta = 2/100$.

23. We want M such that if $x > M$,

$$\left|\frac{x^2-2}{x^2+x+1}-1\right| < 0.1$$

We have

$$\left|\frac{x^2-2}{x^2+x+1}-1\right| = \left|\frac{x^2-2-(x^2+x+1)}{x^2+x+1}\right| = \left|\frac{-x-3}{x^2+x+1}\right| = \left|\frac{x+3}{x^2+x+1}\right|$$

Now, as long as $x > 3$, we have

$$\left|\frac{x+3}{x^2+x+1}\right| < \left|\frac{2x}{x^2+x}\right| = \left|\frac{2}{x+1}\right|$$

We want $\left|\frac{2}{x+1}\right| < 0.1$. Since $x \to \infty$, we cantake $x > 0$, so we solve $\frac{2}{x+1} < 0.1$ to get $x > 19$, i.e., $M = 19$.

25. We have

$$\left|\frac{x^2+3}{4x^2-4}-\frac{1}{4}\right| = \left|\frac{x^2+3-(x^2-1)}{4x^2-4}\right| = \left|\frac{4}{4x^2-4}\right| = \left|\frac{1}{x^2-1}\right|$$

Since $x \to -\infty$, we may take $x < -1$ so that $x^2-1 > 1 > 0$. We now need $\frac{1}{x^2-1} < 0.1$. Solving for x gives $x| > \sqrt{11} \approx$11 3:3166. So we can take $N = -4$.

27. Let $\varepsilon > 0$ be given and assume $\varepsilon \le 1/2$. Let $N = -(\frac{1}{\varepsilon}-2)^{1/2}$ Then if $x < N$,

$$\left|\frac{1}{x^2+2}-3-(-3)\right| = \left|\frac{1}{x^2+2}\right| < \left|\frac{1}{\left(-(\frac{1}{\varepsilon}-2)^{1/2}\right)^2+2}\right| = \varepsilon$$

29. Let $N < 0$ be given and let $\delta = \sqrt[4]{-2/N}$. Then for any x such that $x+3| < \delta$,

$$\frac{-2}{(x+3)^4}\Bigg| > \left|\frac{-2}{(\sqrt[4]{-2/N})^4}\right| = |N|$$

31. Let $\varepsilon > 0$ be given and let $M = \varepsilon^{-1/k}$. Then if $x > M$,

$$\frac{1}{x^k}\Bigg| < \left|\frac{1}{\left(\varepsilon^{-1/k}\right)^k}\right| = \varepsilon$$

33. We observe that $\lim_{x\to 1^-} f(x) = 2$ and $\lim_{x\to 1^+} f(x) = 4$. For any $x \in (1,2)$, $|f(x)-2| = |x^2+3-2| = |x^2+1| > 2$. So if $\varepsilon \le 2$, there is no $\delta > 0$ to satisfy the denition of limit.

35. We observe that lim $\lim_{x\to 1^-} f(x) = 2$ and $\lim_{x\to 1^+} f(x) =$ 4. For any $x \in (1, \sqrt{2})$,

$$|f(x)-2| = |5-x^2-2| = |3-x^2| > |3-(\sqrt{2})^2| = 1.$$

So $\varepsilon \le 1$, there is no >0 to satisfy the definition of limit.

37. Let $L = \lim_{x\to a} f(x)$. Given any $\varepsilon > 0$, we know there exists $\delta > 0$ such that whenever $0 < |x-a| < \delta$, we have

$$f(x) - L| < \frac{\varepsilon}{|c|}.$$

Here, we can take $\varepsilon/|c|$ instead of ε because there is such a δ for any ε, including $\varepsilon/|c|$. But now we have

$$|c\cdot f(x) - c\cdot L| = |c|\cdot|f(x)-L| < |c|\cdot\frac{\varepsilon}{|c|} = \varepsilon.$$

Therefore, $\lim_{x\to a} c\cdot f(x) = c\cdot L$ as desired.

39. Let $\varepsilon > 0$ be given. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 >$ 1 such that whenever $0 < |x-a| < \delta_1$, we have

$|f(x) - L| < \varepsilon$.

In particular, we know that

$L - \varepsilon < f(x)$.

Similarly, since $\lim_{x\to a} h(x) = L$, there exists $\delta_2 > 0$ such that whenever $0 < x-a| < \delta_2$, we have

$h(x) - L| < \varepsilon$.

In particular, we know that $h(x) < L + \varepsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

Then whenever $0 < x - a| < \delta$, we have

$L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon$.

Therefore

$g(x) - L| < \varepsilon$

and so $\lim_{x\to a} g(x) = L$ as desired.

41. We want to find, for any given $\varepsilon > 0$, a $\delta > 0$ such that whenever $0 < |r-2| < \delta$, we have $|2r^2-8| < \varepsilon$. We see that

$|2r^2-8| = 2|r^2-4| = 2|r-2||r+2|$.

Since we want a radius close to 2, we may take $|r-2| < 1$ which implies $|r+2| < 5$ and so

$|2r^2-8| < 10|r-2|$

whenever $|r-2| < 1$. If we then take $\delta = \min\{1, \varepsilon/10\}$, we see that whenever

$0 < |r-2| < \delta$, we have

$$|2r^2-8| < 10\cdot\delta \le 10\cdot\frac{\varepsilon}{10} = \varepsilon.$$

$$\left|\frac{4}{3}\pi r^3 - \frac{\pi}{6}\right| < \frac{7\pi}{3}\left|r-\frac{1}{2}\right| < \frac{7\pi}{3}\cdot\frac{3\varepsilon}{7\pi} = \varepsilon.$$

Exercises 6.7

1. $\frac{1}{4}; \frac{x}{\sqrt{4x^2+1}+2x}$ **3.** $1; \frac{2\sqrt{x}}{\sqrt{x+4}+\sqrt{x+2}}$

5. $1; \frac{2x}{\sqrt{x^2+4}+\sqrt{x^2+2}}$ **7.** $\frac{1}{6}; \frac{\sin^2 2x}{12x^2(1+\cos 2x)}$

9. $\frac{1}{2}; \frac{\sin^2(x^3)}{x^6(1+\cos(x^3))}$

11. $\frac{2}{3}; \frac{2x^{4/3}}{\sqrt[3]{(x^2+1)^2}+\sqrt[3]{(x^2+1)(x^2-1)}+\sqrt[3]{(x^2-1)^2}}$

13. 3, does not exist

15. $f(x) = 0, g(x) = 0.0016, -0.0159, -0.1586, -0.9998$

17. 20, 0

Chapter 6 Review Exercises

1. 2 **3.** (a) 1.05799 (b) 1.05807 **5.** 1

7. does not exist **9.** 7.39 (exact: e^2)

11. (a) 1 (b) −2 (c) does not exist (d) 0

13. $x = -1, x = 1$ **15.** $\frac{3}{4}$ **17.** does not exist

19. does not exist **21.** 5 **23.** $\frac{2}{3}$ **25.** ∞

27. $\frac{1}{3}$ **29.** 0 **31.** ∞ **33.** 0 **35.** e^{-6}

39. $x = -3, x = 1$ (removable)

41. $x = 2$ **43.** $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$

45. $(-\infty, \infty)$ **47.** $x = 1, x = 2, y = 0$

49. $x = -1, x = 1, y = 1$ **51.** $x = 0, y = 2$

53. $x = \ln 2$, $y = -1.5$ and $y = 0$

55. $\frac{1}{4}; \frac{\sin^2 x}{2x^2(1+\cos x)}$

CHAPTER 7

Exercises 7.1

1. $y = 2(x-1) - 1$ **3.** $y = -7(x+2) + 10$

5. $y = -\frac{1}{2}(x-1) + 1$ **7.** $y = \frac{1}{2}(x+2) + 1$

9. (a) 6 (b) 18 (c) 8.25 (d) 14.25 (e) 10.41 (f) 11.61 (g) 11

11. (a) 0.33 (b) 0.17 (c) 0.27 (d) 0.19 (e) 0.23 (f) 0.22 (g) 0.22

13. C, B, A, D

15. (a) −9.8 m/s (b) −19.6 m/s

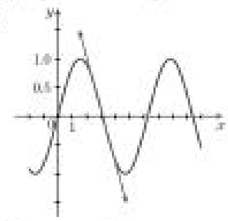
19. (a) 32 ft/s (b) 48 ft/s (c) 62.4 ft/s (d) 63.84 ft/s (e) 64 ft/s

21. (a) 2.236 ft/s (b) 1.472 ft/s (c) 1.351 ft/s (d) 1.343 ft/s (e) 1.342 ft/s

23. sharp corner

25. jump discontinuity

27.



29. No tangent line

31. (a) from 2002 to 2004, the balance increased at an average rate of AED 21,034 per year

35. (a) $\left(\sqrt{2/3}, 5\sqrt{2/3}+1\right), \left(-\sqrt{2/3}, -5\sqrt{2/3}+1\right)$

37. (a) $y = 6(x-1) + 5$ (b) $x = -2, x = 1$

Applications

1. −10; −4.5

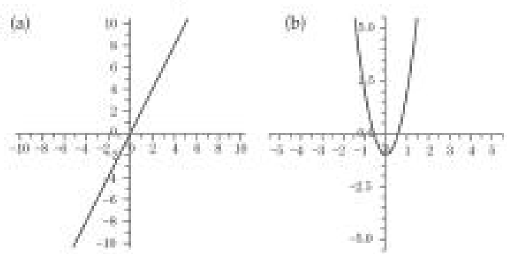
3. about 1.75 hours; 1.5 hours; 4 hours; rest

Exercises 7.2

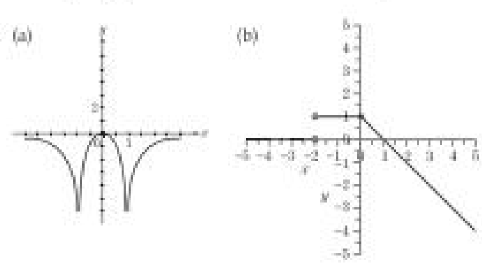
1. 3 **3.** $\frac{3}{4}$ **5.** $6x$ **7.** $3x^2 + 2$

9. $\frac{-3}{(x+1)^2}$ **11.** $\frac{3}{2\sqrt{3x+1}}$

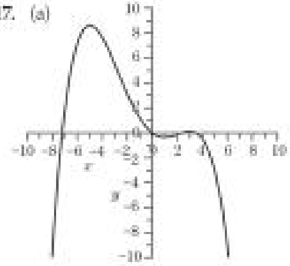
13. (a) (b)



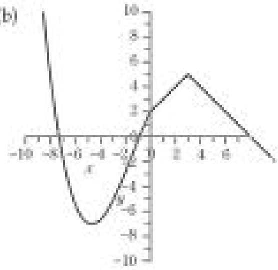
15. (a) (b)



17. (a)



(b)



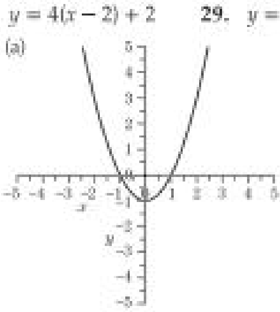
19. $D_+f(0) = 3$, $D_-f(0) = 2$; no
21. $D_+f(0) = D_-f(0) = 0$; yes
23. 0.35 **25.** 0 **27.** 10
29. (a) $x = 0, x = 2$ (b) $x = 0, x = 4$
31. $p \geq 1$ **33.** $f(x) = -1 - x^2$
35. $\frac{f(a)f'(a)}{a}$ **37.** $f'(1), \frac{f(1.5) - f(1)}{0.5}, f(2) - f(1), f(1)$
41. $2x, 3x^2, 4x^3, nx^{n-1}$

Applications

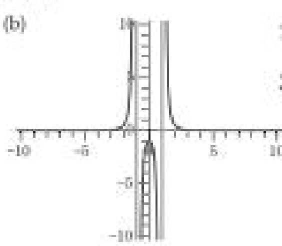
1. 1.64 degrees per meter
3. (a) 0.4 ton per year (b) 0.2 ton per year
5. (a) meters per second (b) items per dollar
7. losing value; sell
11. $f'(t) = \begin{cases} 0, & 0 < t < 20 \\ 10, & 20 < t < 80 \\ 8, & t > 80 \end{cases}$

Exercises 7.3

1. $3x^2 - 2$ **3.** $9t^2 - \frac{1}{\sqrt{t}}$ **5.** $-\frac{3}{w^2} - 8$
7. $\frac{-10}{3}x^{-4/3} - 2$ **9.** $3s^{1/2} + s^{-4/3}$
11. $\frac{3}{2} - \frac{1}{2}x^{-2}$ **13.** $9x^2 - \frac{3}{2}x^{1/2}$
15. $12t^2 + 6$ **17.** $24x^2 - \frac{9}{4}x^{-5/2}$ **19.** 24
21. $v(t) = -32t + 40, a(t) = -32$
23. $v(t) = \frac{1}{2}t^{-1/2} + 4t, a(t) = -\frac{1}{4}t^{-3/2} + 4$
25. (a) $v(1) = 8$ (going up); $a(1) = -32$
(b) $v(2) = -24$ (going down); $a(2) = -32$
27. $y = 4(x - 2) + 2$ **29.** $y = -x + 4$
31. (a)



(b)



33. $x = -1$ (peak); $x = 1$ (trough); $x = \sqrt[4]{\frac{2}{3}}, \sqrt[4]{\frac{4}{3}}$
35. (a) $x = 0$; vertical tangent (b) $x = 5$; sharp corner
(c) $x = -1, x = 4$; sharp corners
37. (a) $x = \pm\sqrt{\frac{4}{3}}$ (b) $x = \pm\sqrt{1 + 3^{-3/2}}$
39. (a) $\frac{3}{2}x^2 + 2x - 2$ (b) $\frac{1}{2}x^2 + 5x$
41. 2 **43.** (a) $f'(x)$ (b) 0 **45.** x^4 **47.** $\frac{2}{3}x^{3/2}$

Applications

1. $b > \frac{4}{9c^2}$ **3.** $f'(2000) \approx 174.4, f''(2000) \approx -160$

Exercises 7.4

1. $2x(x^3 - 3x + 1) + (x^2 + 3)(3x^2 - 3)$
3. $\left(\frac{1}{2}x^{-1/2} + 3\right)\left(5x^2 - \frac{3}{x}\right) + (\sqrt{x} + 3x)(10x + 3x^{-2})$
5. $\frac{3(5t + 1) - (3t - 2)5}{(5t + 1)^2} = \frac{13}{(5t + 1)^2}$
7. $\frac{(3 - 3x^{-1/2})(5x^2 - 2) - (3x - 6\sqrt{x})10x}{(5x^2 - 2)^2}$
9. $\frac{(2u - 1)(u^2 - 5u + 1) - (u^2 - u - 2)(2u - 5)}{(u^2 - 5u + 1)^2}$
11. $\frac{3}{2}x^{1/2} + \frac{3}{2}x^{-1/2} + x^{-3/2}$ **13.** $\frac{4}{3}t^{1/3} + 3$
15. $2x\frac{x^3 + 3x^2}{x^2 + 2} + (x^2 - 1)\frac{(3x^2 + 6x)(x^2 + 2) - (x^3 + 3x^2)(2x)}{(x^2 + 2)^2}$
17. $y = 2x$ **19.** $y = \frac{1}{4}x + \frac{1}{2}$
21. (a) $y = -2x - 3$ (b) $y = 7(x - 1) - 2$
23. (a) $y = -(x - 1) - 2$ (b) $y = 0$
25. $P'(t) = 0.03P(t)$; $3 - 4 = -1$ **27.** $65,000 per year
29. $\frac{19.125}{(m + 0.15)^2}$; bigger bat gives greater speed
31. $\frac{-14.11}{(m + 0.05)^2}$; heavier club gives less speed
33. $f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$
35. $\frac{2}{3}x^{-1/3}(x^2 - 2)(x^3 - x + 1) + x^{2/3}(2x)(x^3 - x + 1) +$
$x^{2/3}(x^2 - 2)(3x^2 - 1)$
39. maximum slope at $x = 0$; minimum slope at $x = \pm\sqrt{3}$
45. $F'''(x) = f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + f(x)g'''(x)$

Applications

1. 0; 1; $\frac{2.7x^{1.7}}{\left(1 + x^{2.7}\right)^2}$
3. (a) if c increases, r increases
(d) increase in r is less than increase in h

Exercises 7.5

1. $6x^2(x^3-1)$ **3.** $6x(x^2+1)^2$

5. (a) $(9x^2-3)(x^3-x)^2$ (b) $\dfrac{x}{\sqrt{x^2+4}}$

7. (a) $5t^4\sqrt{t^5+2}+\dfrac{3t^7}{2\sqrt{t^5+2}}$ (b) $\frac{7}{2}t^{5/2}+t^{-1/2}$

9. (a) $\dfrac{u^2+8u-1}{(u+4)^2}$ (b) $\dfrac{12u^2-u^4}{(u^2+4)^3}$

11. (a) $\dfrac{1}{(x^2+1)^{3/2}}$ (b) $\frac{1}{2}(1-x^2)x^{-1/2}(x^2+1)^{-3/2}$

13. (a) $-6w(w^2+4)^{-3/2}$ (b) $\dfrac{w}{6\sqrt{w^2+4}}$

15. (a) $-2(\sqrt{x^3+2}+2x)^{-3}\left(\frac{3x^2}{2\sqrt{x^3+2}}+2\right)$

(b) $\frac{3x^2-4x^{-3}}{2\sqrt{x^3+2+2x^{-2}}}$

17. $\dfrac{1}{f'(0)}=\dfrac{1}{4}$ **19.** $\dfrac{1}{f'(1)}=\dfrac{1}{15}$ **21.** $\dfrac{1}{f'(0)}=2$

23. chain rule; product

25. product rule; chain, quotient

27. $y=\frac{3}{5}(x-3)+5$ **29.** $\dfrac{1}{\sqrt{3}}$ **31.** -6

35. (a) $2xf'(x^2)$ (b) $2f(x)f'(x)$ (c) $f'(f(x))f'(x)$

37. (a) $-\frac{f'(1/x)}{x^2}$ (b) $-\frac{f'(x)}{[f(x)]^2}$ (c) $f'\left(\frac{x}{f(x)}\right)\frac{f(x)-xf'(x)}{[f(x)]^2}$

39. (a) 3 (b) does not exist (c) 9

41. (a) $4(x^2+4)^{-3/2}$ (b) $\dfrac{8-4t^2}{(t^2+4)^{5/2}}$

43. (a) $x=0, x=1, x=2$; vertical tangents
(b) $x=0$; vertical tangent

45. (a) $\frac{1}{3}(x^2+3)^3$ **47.** $\sqrt{x^2+1}$

Chapter 7 Review Exercises

1. 0.8 **3.** 2 **5.** $\frac{1}{2}$ **7.** $3x^2+1$

9. $y=2x-2$

11. $v(t)=-32t+40$; $a(t)=-32$

13. $v(t)=\dfrac{2}{\sqrt{4t+16}}$ $a(t)=-\dfrac{4}{(4t+16)^{3/2}}$

15. (a) 0.3178 (b) 0.3339 (c) 0.3492 (d) 0.35

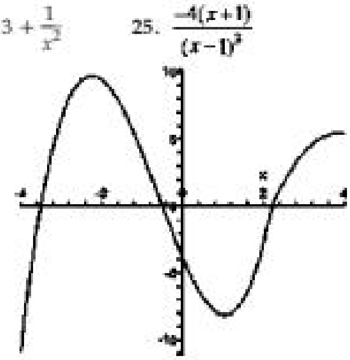
17. $\frac{2}{3}x^{-1/3}-8x$

19. $\dfrac{\sqrt{x}(-3+2x)}{x}$ $\dfrac{(2-3x+x^2)\frac{1}{2\sqrt{x}}}{x}$

21. $2t(t^3-3t+2)+(t^2+1)(3t^2-3)$

23. $3+\dfrac{1}{x^2}$ **25.** $\dfrac{-4(x+1)}{(x-1)^3}$

27.



29. $\dfrac{3}{8}(x+1)^{-5/2}$

31. $11{,}880x^8-30{,}240x^6+12{,}096x^5+15{,}120x^4-21{,}840x^3+2160x^2+5040x-816$

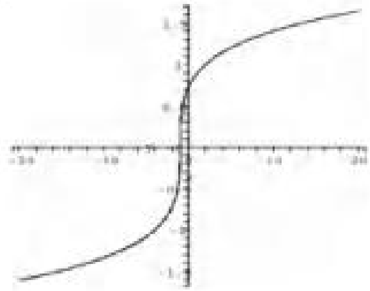
33. (a) (0, 1) and (4, −31) (b) none

35. (a) (0,0) (b) none

37. $f(x)$ is continuous and differentiable for all x, and $f'(x)=3x^2+7$, which is positive for all x. By Theorem 9.2, if the equation $f(x)=0$ has two solutions, then $f'(x)=0$ would have at least one solution, but it has none. We discussed at length (Section 7.9) why every odd degree polynomial has at least one root, so in this case there is exactly one root.

39. (a) 1/11

(b)



41. We have

$$\begin{aligned}&f(x)-T(x)\\&=f(x)-f(a)-f'(a)(x-a)\\&=\left(\frac{f(x)-f(a)}{x-a}-f'(a)\right)(x-a)\end{aligned}$$

Letting $e(x)=\dfrac{f(x)-f(a)}{x-a}-f'(a)$, we obtain the desired form. Since $f(x)$ is differentiable at $x=a$, we know that

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)\text{, so}$$

$$\lim_{x\to a}e(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}-f'(a)=0$$

43. We are asked to find m so that
$x^3+2x-[m(x-2)+12]=x^3+(2-m)x+(2m-12)$
has a double root. A cubic with a double root factors as
$(x-a)^2(x-b)=x^3-(2a+b)x^2+(2ab+a^2)x-a^2b$.
Equating like coefficients gives a system of three equations
$2a+b=0$,
$2ab+a^2=2-m$, and
$-a^2b=2m-12$.
The first equation gives $b=-2a$. Substituting this into the second equation gives $m=2+3a^2$.
Substituting these results into the third equation gives a cubic polynomial in a with zeros $a=-1$ and $a=2$. This gives two solutions: $m=5$ and $m=14$.
$f'(x)=3x^2+2$, so $f'(2)=14$. The tangent line at (2, 12) is $y=14(x-2)+12$.
The second solution corresponds to the tangent line to $f(x)$ at $x=-1$, which happens to pass through (2, 12).

Student Handbook

Symbols, Formulas, and Key Concepts

Symbols

Algebra	
$\neq$	is not equal to
$\approx$	is approximately equal to
$\sim$	is similar to
$>, \geq$	is greater than, is greater than or equal to
$<, \leq$	is less than, is less than or equal to
$-a$	opposite or additive inverse of a
$\lvert a \rvert$	absolute value of a
$\sqrt{a}$	principal square root of a
$a : b$	ratio of a to b
(x, y)	ordered pair
(x, y, z)	ordered triple
i	the imaginary unit
$b^{\frac{1}{n}} = \sqrt[n]{b}$	nth root of b
$\mathbb{Q}$	rational numbers
$\mathbb{I}$	irrational numbers
$\mathbb{Z}$	integers
$\mathbb{W}$	whole numbers
$\mathbb{N}$	natural numbers
∞	infinity
$-\infty$	negative infinity
$[\ ]$	endpoint included
$(\ )$	endpoints not included
$\log_b x$	logarithm base b of x
$\log x$	common logarithm of x
$\ln x$	natural logarithm of x
ω	omega, angular speed
α	alpha, angle measure
β	beta, angle measure
γ	gamma, angle measure
θ	theta, angle measure
λ	lambda, wavelength
ϕ	phi, angle measure
a	vector **a**
$\lvert \mathbf{a} \rvert$	magnitude of vector **a**

Sets and Logic	
$\in$	is an element of
$\subset$	is a subset of
$\cap$	intersection
$\cup$	union
$\varnothing$	empty set
$\sim p$	negation of p, not p
$p \wedge q$	conjunction of p and q
$p \vee q$	disjunction of p and q
$p \longrightarrow q$	conditional statement, if p then q
$p \longleftrightarrow q$	biconditional statement, p if and only if q

Geometry	
$\angle$	angle
$\triangle$	triangle
$^\circ$	degree
π	pi
$\measuredangle$	angles
$m\angle A$	degree measure of $\angle A$
$\overleftrightarrow{AB}$	line containing points A and B
$\overline{AB}$	segment with endpoints A and B
$\overrightarrow{AB}$	ray with endpoint A containing B
AB	measure of $\overline{AB}$, distance between points A and B
$\parallel$	is parallel to
$\nparallel$	is not parallel to
$\perp$	is perpendicular to
$\triangle$	triangle
$\square$	parallelogram
n-gon	polygon with n sides
$\vec{a}$	vector a
$\overrightarrow{AB}$	vector from A to B
$\lvert \overrightarrow{AB} \rvert$	magnitude of the vector from A to B
A'	the image of preimage A
$\rightarrow$	is mapped onto
$\odot A$	circle with center A
$\overset{\frown}{AB}$	minor arc with endpoints A and B
$\overset{\frown}{ABC}$	major arc with endpoints A and C
$m\overset{\frown}{AB}$	degree measure of arc AB

Trigonometry	
$\sin x$	sine of x
$\cos x$	cosine of x
$\tan x$	tangent of x
$\sin^{-1} x$	Arcsin x
$\cos^{-1} x$	Arccos x
$\tan^{-1} x$	Arctan x

Symbols

Functions

$f(x)$	f of x, the value of f at x
$f(x) = \{$	piecewise-defined function
$f(x) = \lvert x \rvert$	absolute value function
$f(x) = [\![x]\!]$	function of greatest integer not greater than x
$f(x, y)$	f of x and y, a function with two variables, x and y
$[f \circ g](x)$	f of g of x, the composition of functions f and g
$f^{-1}(x)$	inverse of $f(x)$

Calculus

$\lim\limits_{x \to c}$	limit as x approaches c
m_{sec}	slope of a secant line
$f'(x)$	derivative of $f(x)$
Δ	delta, change
$\int$	indefinite integral
$\int_a^b$	definite integral
$F(x)$	antiderivative of $f(x)$

Probability and Statistics

$P(a)$	probability of a
$P(n, r)$ or ${}_nP_r$	permutation of n objects taken r at a time
$C(n, r)$ or ${}_nC_r$	combination of n objects taken r at a time
$P(A)$	probability of A
$P(A\|B)$	the probability of A given that B has already occurred
$n!$	Factorial of n (n being a natural number)
Σ	sigma (uppercase), summation
μ	mu, population mean
σ	sigma (lowercase), population standard deviation
σ^2	population variance
s	sample standard deviation
s^2	sample variance
$\sum\limits_{n=1}^{k}$	summation from $n = 1$ to k
$\bar{x}$	x-bar, sample mean
H_0	null hypothesis
H_a	alternative hypothesis

Measures

Metric	Customary
Length	
1 kilometer (km) = 1000 meters (m) 1 meter = 100 centimeters (cm) 1 centimeter = 10 millimeters (mm)	1 mile (mi) = 1760 yards (yd) 1 mile = 5280 feet (ft) 1 yard = 3 feet 1 foot = 12 inches (in) 1 yard = 36 inches
Volume and Capacity	
1 liter (L) = 1000 milliliters (mL) 1 kiloliter (kL) = 1000 liters	1 gallon (gal) = 4 quarts (qt) 1 gallon = 128 fluid ounces (fl oz) 1 quart = 2 pints (pt) 1 pint = 2 cups (c) 1 cup = 8 fluid ounces
Weight and Mass	
1 kilogram (kg) = 1000 grams (g) 1 gram = 1000 milligrams (mg) 1 metric ton (t) = 1000 kilograms	1 ton (T) = 2000 pounds (lb) 1 pound = 16 ounces (oz)

Arithmetic Operations and Relations

Identity	For any number a, $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
Substitution (=)	If $a = b$, then a may be replaced by b.
Reflexive (=)	$a = a$
Symmetric (=)	If $a = b$, then $b = a$.
Transitive (=)	If $a = b$ and $b = c$, then $a = c$.
Commutative	For any numbers a and b, $a + b = b + a$ and $a \cdot b = b \cdot a$.
Associative	For any numbers a, b, and c, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
Distributive	For any numbers a, b, and c, $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$.
Additive Inverse	For any number a, there is exactly one number $-a$ such that $a + (-a) = 0$.
Multiplicative Inverse	For any number $\frac{a}{b}$, where $a, b \neq 0$, there is exactly one number $\frac{b}{a}$ such that $\frac{a}{b} \cdot \frac{b}{a} = 1$.
Multiplicative (0)	For any number a, $a \cdot 0 = 0 \cdot a = 0$.
Addition (=)	For any numbers a, b, and c, if $a = b$, then $a + c = b + c$.
Subtraction (=)	For any numbers a, b, and c, if $a = b$, then $a - c = b - c$.
Multiplication and Division (=)	For any numbers a, b, and c, with $c \neq 0$, if $a = b$, then $ac = bc$ and $\frac{a}{c} = \frac{b}{c}$.
Addition (>)*	For any numbers a, b, and c, if $a > b$, then $a + c > b + c$.
Subtraction (>)*	For any numbers a, b, and c, if $a > b$, then $a - c > b - c$.
Multiplication and Division (>)*	For any numbers a, b, and c, 1. if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$. 2. if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
Zero Product	For any real numbers a and b, if $ab = 0$, then $a = 0$, $b = 0$, or both a and b equal 0.

** These properties are also true for $<$, $\geq$, and $\leq$.*

Algebraic Formulas and Key Concepts

Matrices

Adding	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$	**Multiplying by a Scalar**	$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$
Subtracting	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$	**Multiplying**	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$

Polynomials

Quadratic Formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$	**Square of a Difference**	$(a - b)^2 = (a - b)(a - b)$ $= a^2 - 2ab + b^2$
Square of a Sum	$(a + b)^2 = (a + b)(a + b)$ $= a^2 + 2ab + b^2$	**Product of Sum and Difference**	$(a + b)(a - b) = (a - b)(a + b)$ $= a^2 - b^2$

Logarithms

Product Property	$\log_x ab = \log_x a + \log_x b$	**Power Property**	$\log_b m^p = p \log_b m$
Quotient Property	$\log_x \frac{a}{b} = \log_x a - \log_x b, b \neq 0$	**Change of Base**	$\log_a n = \frac{\log_b n}{\log_b a}$

Algebraic Formulas and Key Concepts

Exponential and Logarithmic Functions

Compound Interest	$A = P\left(1 + \frac{r}{n}\right)^{nt}$	**Exponential Growth or Decay**	$N = N_0(1 + r)^t$
Continuous Compound Interest	$A = Pe^{rt}$	**Continuous Exponential Growth or Decay**	$N = N_0e^{kt}$
Product Property	$\log_b xy = \log_b x + \log_b y$	**Power Property**	$\log_b x^p = p \log_b x$
Quotient Property	$\log_b \frac{x}{y} = \log_b x - \log_b y$	**Change of Base**	$\log_b x = \frac{\log_a x}{\log_a b}$
Logistic Growth	$f(t) = \frac{c}{1 + ae^{-bt}}$		

Sequences and Series

***n*th term, Arithmetic**	$a_n = a_1 + (n - 1)d$	***n*th term, Geometric**	$a_n = a_1r^{n-1}$
Sum of Arithmetic Series	$S_n = n\left(\frac{a_1 + a_2}{2}\right)$ or $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$	**Sum of Geometric Series**	$S_n = \frac{a_1 - a_1r^n}{1 - r}$ or $S_n = \frac{a_1 - a_nr}{1 - r}, r \neq 1$
Sum of Infinite Geometric Series	$S = \frac{a_1}{1 - r}, \lvert r \rvert < 1$	**Euler's Formula**	$e^{i\theta} = \cos\theta + i\sin\theta$
Power Series	$\sum_{n=0}^{\infty} a_nx^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$	**Exponential Series**	$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
Binomial Theorem	$(a + b)^n = {}_nC_0\, a^nb^0 + {}_nC_1\, a^{n-1}b^1 + {}_nC_2\, a^{n-2}b^2 + \dots + {}_nC_r\, a^{n-r}b^r + \dots + {}_nC_n\, a^0b^n$		
Cosine and Sine Power Series	$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^nx^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$		$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^nx^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Vectors

Addition in Plane	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$	**Addition in Space**	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
Subtraction in Plane	$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$	**Subtraction in Space**	$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$
Scalar Multiplication in Plane	$k\mathbf{a} = \langle ka_1, ka_2 \rangle$	**Scalar Multiplication in Space**	$k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$
Dot Product in Plane	$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$	**Dot Product in Space**	$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
Angle Between Two Vectors	$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$	**Projection of u onto v**	$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\lvert\mathbf{v}\rvert^2}\right)\mathbf{v}$
Magnitude of a Vector	$\lvert\mathbf{v}\rvert = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	**Triple Scalar Product**	$\mathbf{t} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} t_1 & t_2 & t_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

Equations of a Line on a Coordinate Plane

Slope-intercept form of a line	$y = mx + b$
Point-slope form of a line	$y - y_1 = m(x - x_1)$

Algebraic Formulas and Key Concepts

Conic Sections			
Parabola	$(x-h)^2 = 4p(y-k)$ or $(y-k)^2 = 4p(x-h)$	**Circle**	$x^2 + y^2 = r^2$ or $(x-h)^2 + (y-k)^2 = r^2$
Ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$	**Hyperbola**	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Rotation of Conics	$x' = x\cos\theta + y\sin\theta$ and $y' = y\cos\theta - x\sin\theta$		

Parametric Equations			
Vertical Position	$y = tv_0\sin\theta - \frac{1}{2}gt^2 + h_0$	**Horizontal Distance**	$x = tv_0\cos\theta$

Complex Numbers			
Product Formula	$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + \boldsymbol{i}\sin(\theta_1+\theta_2)]$	**Quotient Formula**	$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + \boldsymbol{i}\sin(\theta_1-\theta_2)]$
Distinct Roots Formula	$r^{\frac{1}{p}}\left(\cos\frac{\theta+2n\pi}{p} + \boldsymbol{i}\sin\frac{\theta+2n\pi}{p}\right)$	**De Moivre's Theorem**	$z^n = [r(\cos\theta + \boldsymbol{i}\sin\theta)]^n$ or $r^n(\cos n\theta + \boldsymbol{i}\sin n\theta)$

Geometric Formulas and Key Concepts

Coordinate Geometry			
Slope	$m = \frac{y_2-y_1}{x_2-x_1}, x_2 \neq x_1$	**Distance on a number line**	$d = \|a-b\|$
Distance on a coordinate plane	$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$	**Arc length**	$\ell = \frac{x}{360} \cdot 2\pi r$
Midpoint on a coordinate plane	$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$	**Midpoint on a number line**	$M = \frac{a+b}{2}$
Midpoint in space	$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$	**Pythagorean Theorem**	$a^2 + b^2 = c^2$
Distance in space	$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$		

Perimeter and Circumference					
Square	$P = 4s$	**Rectangle**	$P = 2\ell + 2w$	**Circle**	$C = 2\pi r$ or $C = \pi d$

Lateral Surface Area			
Prism	$L = Ph$	**Pyramid**	$L = \frac{1}{2}P\ell$
Cylinder	$L = 2\pi rh$	**Cone**	$L = \pi r\ell$

Total Surface Area					
Prism	$S = Ph + 2B$	**Cone**	$S = \pi r\ell + \pi r^2$	**Cylinder**	$S = 2\pi rh + 2\pi r^2$
Pyramid	$S = \frac{1}{2}P\ell + B$	**Sphere**	$S = 4\pi r^2$	**Cube**	$S = 6s^2$

Volume					
Prism	$V = Bh$	**Cone**	$V = \frac{1}{3}\pi r^2h$	**Cylinder**	$V = \pi r^2h$
Pyramid	$V = \frac{1}{3}Bh$	**Sphere**	$V = \frac{4}{3}\pi r^3$	**Cube**	$V = s^3$
Rectangular prism	$V = \ell wh$				

Trigonometric Functions and Identities

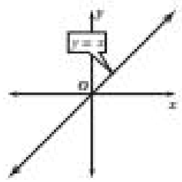
Trigonometric Functions			
Trigonometric Functions	$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin\theta}$	$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\cos\theta}$	$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\cos\theta}{\sin\theta}$
Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos A$	$b^2 = a^2 + c^2 - 2ac\cos B$	$c^2 = a^2 + b^2 - 2ab\cos C$
Law of Sines	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	**Heron's Formula**	Area $= \sqrt{s(s-a)(s-b)(s-c)}$
Linear Speed	$v = \frac{s}{t}$	**Angular Speed**	$\omega = \frac{\theta}{t}$

Trigonometric Identities			
Reciprocal	$\sin\theta = \frac{1}{\csc\theta}$ $\csc\theta = \frac{1}{\sin\theta}$	$\cos\theta = \frac{1}{\sec\theta}$ $\sec\theta = \frac{1}{\cos\theta}$	$\tan\theta = \frac{1}{\cot\theta}$ $\cot\theta = \frac{1}{\tan\theta}$
Pythagorean	$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$\cot^2\theta + 1 = \csc^2\theta$
Cofunction	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$	$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$	$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$
Odd-Even	$\sin(-\theta) = -\sin\theta$ $\csc(-\theta) = -\csc\theta$	$\cos(-\theta) = \cos\theta$ $\sec(-\theta) = \sec\theta$	$\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$
Sum & Difference	$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$	$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$	
Double-Angle	$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = 2\cos^2\theta - 1$ $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$	$\cos 2\theta = 1 - 2\sin^2\theta$
Power-Reducing	$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$	$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$	$\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$
Half-Angle	$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$ $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$	$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$ $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$	$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$
Product-to-Sum	$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$ $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$	$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$ $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$	
Sum-to-Product	$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$	$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$	

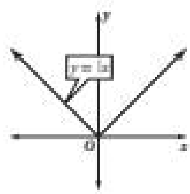
Parent Functions and Function Operations

Parent Functions

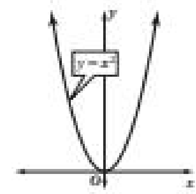
Linear Functions



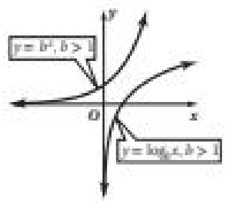
Absolute Value Functions



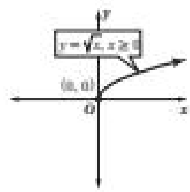
Quadratic Functions



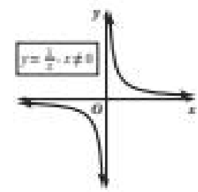
Exponential and Logarithmic Functions



Square Root Functions



Reciprocal and Rational Functions



Function Operations

Addition	$(f + g)(x) = f(x) + g(x)$	**Multiplication**	$(f \cdot g)(x) = f(x) \cdot g(x)$
Subtraction	$(f - g)(x) = f(x) - g(x)$	**Division**	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$

Calculus

Limits

Sum	$\lim_{x\to c} [f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$	**Difference**	$\lim_{x\to c} [f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
Scalar Multiple	$\lim_{x\to c} [k\, f(x)] = k \lim_{x\to c} f(x)$	**Product**	$\lim_{x\to c} [f(x) \cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
Quotient	$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, if $\lim_{x\to c} g(x) \neq 0$	**Power**	$\lim_{x\to c} [f(x)^n] = \left[\lim_{x\to c} f(x)\right]^n$
***n*th Root**	$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$, if $\lim_{x\to c} f(x) > 0$ when n is even	**Velocity**	**Average** $v_{avg} = \frac{f(b) - f(a)}{b - a}$ **Instantaneous** $v(t) = \lim_{h\to 0} \frac{f(t + h) - f(t)}{h}$

Derivatives

Power Rule	If $f(x) = x^n$, $f'(x) = nx^{n-1}$	**Sum or Difference**	If $f(x) = g(x) \pm h(x)$, then $f'(x) = g'(x) \pm h'(x)$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$	**Quotient Rule**	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Integrals

Indefinite Integral	$\int f(x)\,dx = F(x) + C$	**Fundamental Theorem of Calculus**	$\int_a^b f(x)\,dx = F(b) - F(a)$

Statistics Formulas and Key Concepts

z-Values	$z = \frac{X - \mu}{\sigma}$	**z-Value of a Sample Mean**	$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$
Binomial Probability	$P(X) = {}_nC_x\, p^x q^{n-x} = \frac{n!}{(n-x)!\,x!} p^x q^{n-x}$	**Maximum Error of Estimate**	$E = z \cdot \sigma_{\bar{x}}$ or $z \cdot \frac{\sigma}{\sqrt{n}}$
Confidence Interval, Normal Distribution	$CI = \bar{x} \pm E$ or $\bar{x} \pm z \cdot \frac{\sigma}{\sqrt{n}}$	**Confidence Interval, t-Distribution**	$CI = \bar{x} \pm t \cdot \frac{s}{\sqrt{n}}$
Correlation Coefficient	$r = \frac{1}{n-1}\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$	**t-Test for the Correlation Coefficient**	$t = r\sqrt{\frac{n-2}{1-r^2}}$, degrees of freedom: $n - 2$

Cover label guide

Cycle 03 Color