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2. Application of the Definite Integral, from Calculus: Early Transcendental Functions 5e Chapter 5 © 2018
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2 Applications of the Definite Integral

3 Integration Techniques

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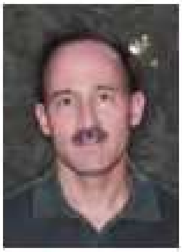
6 Parametric Equations and Polar Coordinates

7 Vectors and Vector-Valued Functions

Student Handbook

Authors



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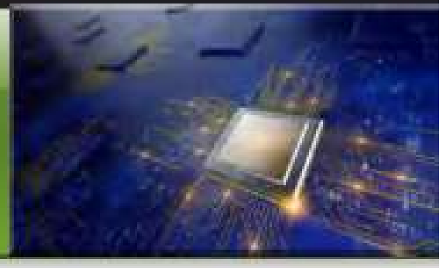

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Glossary is available in the electronic version

Integration

CHAPTER

1

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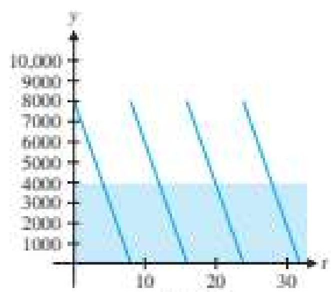
Chapter Topics

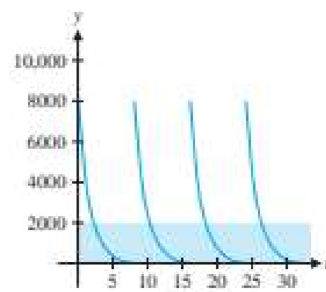
1.1 Antiderivatives
1.2 Sums and Sigma Notation
1.3 Area Under a Curve and Integration
1.4 The Definite Integral
1.5 The Fundamental Theorem of Calculus
1.6 Integration by Substitution
1.7 Numerical Integration
1.8 The Natural Logarithm as an Integral

In the modern business world, companies must find the most cost-efficient method of handling their inventory. One method is **just-in-time inventory**, where new inventory arrives just as existing stock is running out. As a simplified example of this, suppose that a heating oil company's terminal receives shipments of 8000 barrels of oil at a time and orders are shipped out to customers at a constant rate of 1000 barrels per day, where each shipment of oil arrives just as the last barrel on hand is shipped out. Inventory costs are determined based on the average number of barrels held at the terminal. So, how would we calculate this average?

To translate this into a calculus problem, let $f(t)$ represent the number of barrels of oil at the terminal at time t (days), where a shipment arrives at time $t = 0$. In this case, $f(0) = 8000$. Further, for $0 < t < 8$, there is no oil coming in, but oil is leaving at the rate of 1000 barrels per day. Since "rate" means derivative, we have $f'(t) = -1000$, for $0 < t < 8$. This tells us that the graph of $y = f(t)$ has slope -1000 until time $t = 8$, at which point another shipment arrives to refill the terminal, so that $f(8) = 8000$. Continuing in this way, we generate the graph of $f(t)$ shown here at the left.



$y = f(t)$



$y = g(t)$

Since the inventory ranges from 0 barrels to 8000 barrels, you might guess that the average inventory of oil is 4000 barrels. However, look at the graph at the right, showing a different inventory function $g(t)$, where the oil is not shipped out at a constant rate. Although the inventory again ranges from 0 to 8000, the drop in inventory is so rapid immediately following each delivery that the average number of barrels on hand is well below 4000.

As we will see in this chapter, our usual way of averaging a set of numbers is analogous to an area problem. Specifically, the average value of a function is the height of the rectangle that has the same area as the area between the graph of the function and the x-axis. For our original $f(t)$, notice that 4000 appears to work well, while for $g(t)$, an average of 2000 appears to be better, as you can see in the graphs.

Notice that we have introduced several new problems: finding a function from its derivative, finding the average value of a function and finding the area under a curve. We will explore these problems in this chapter.

1.1 ANTIDERIVATIVES

NOTES

For a realistic model of a system as complex as a space shuttle, we must consider much more than the simple concepts discussed here. For a very interesting presentation of this problem, see the article by Long and Weiss in the February 1999 issue of *The American Mathematical Monthly.*[1]

[1] Long, L.N. and Weiss, H. (1999) The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians. *American Mathematical Monthly,* 106(2), 127–135.

Calculus provides us with a powerful set of tools for understanding the world around us. Initial designs of the space shuttle included aircraft engines to power its flight through the atmosphere after re-entry. In order to cut costs, the aircraft engines were scrapped and the space shuttle became a huge glider. NASA engineers use calculus to provide precise answers to flight control problems. While we are not in a position to deal with the vast complexities of a space shuttle flight, we can consider an idealized model.

As we often do with real-world problems, we begin with a physical principle(s) and use this to produce a *mathematical model* of the physical system. We then solve the mathematical problem and interpret the solution in terms of the physical problem.

If we consider only the vertical motion of an object falling toward the ground, the physical principle governing the motion is Newton's second law of motion:

$$\text{Force} = \text{mass} \times \text{acceleration} \quad \text{or} \quad F = ma.$$

This says that the sum of all the forces acting on an object equals the product of its mass and acceleration. Two forces that you might identify here are gravity pulling downward and air drag pushing in the direction opposite the motion. From experimental evidence, we know that the force due to air drag, F_d, is proportional to the square of the speed of the object and acts in the direction opposite the motion. So, for the case of a falling object,

Space shuttle *Endeavor*
fergregory/123RF.com

$$F_d = kv^2,$$

for some constant $k > 0$.

The force due to gravity is simply the weight of the object, $W = -mg$, where the gravitational constant g is approximately 9.8 m/s^2. (The minus sign indicates that the force of gravity acts downward.) Putting this together, Newton's second law of motion gives us

$$F = ma = -mg + kv^2.$$

Recognizing that $a = v'(t)$, we have

$$mv'(t) = -mg + kv^2(t). \tag{1.1}$$

Notice that equation (1.1) involves both the unknown function $v(t)$ and its derivative $v'(t)$. Such an equation is called a **differential equation**. To get started now, we simplify the problem by assuming that gravity is the only force acting on the object. Taking $k = 0$ in (1.1) gives us

$$mv'(t) = -mg \quad \text{or} \quad v'(t) = -g.$$

Now, let $y(t)$ be the position function, giving the altitude of the object in feet t seconds after the start of re-entry. Since $v(t) = y'(t)$ and $a(t) = v'(t)$, we have

$$y''(t) = -9.8$$

From this, we'd like to determine $y(t)$. More generally, we need to find a way to *undo* differentiation. That is, given a function, f, we'd like to find another function F such that $F'(x) = f(x)$. We call such a function F an **antiderivative** of f.

EXAMPLE 1.1 Finding Several Antiderivatives of a Given Function

Find an antiderivative of $f(x) = x^2$.

Solution Notice that $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x)$, since

$$F'(x) = \frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2.$$

Further, observe that
$$\frac{d}{dx}\left(\frac{1}{3}x^3 + 5\right) = x^2,$$

so that $G(x) = \frac{1}{3}x^3 + 5$ is also an antiderivative of f. In fact, for any constant c, we have

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + c\right) = x^2.$$

Thus, $H(x) = \frac{1}{3}x^3 + c$ is also an antiderivative of $f(x)$, for any choice of the constant c. Graphically, this gives us a family of antiderivative curves, as illustrated in Figure 1.1. Note that each curve is a vertical translation of every other curve in the family. ■

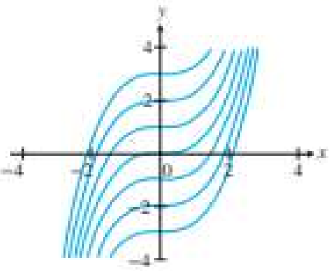


FIGURE 1.1
A family of antiderivative curves

In general, observe that if F is any antiderivative of f and c is any constant, then

$$\frac{d}{dx}[F(x) + c] = F'(x) + 0 = f(x).$$

Thus, $F(x) + c$ is also an antiderivative of $f(x)$, for any constant c. On the other hand, are there any other antiderivatives of $f(x)$ besides $F(x) + c$? The answer, as provided in Theorem 1.1, is no.

THEOREM 1.1

Suppose that F and G are both antiderivatives of f on an interval I. Then,

$$G(x) = F(x) + c,$$

for some constant c.

PROOF

Since F and G are both antiderivatives for f, we have that $G'(x) = F'(x)$. It now follows, from Corollary 5.1 in Grade 11, section 8.5, that $G(x) = F(x) + c$, for some constant, c, as desired. ■

NOTES

Theorem 1.1 says that given *any* antiderivative F of f, *every* possible antiderivative of f can be written in the form $F(x) + c$, for some constant, c. We give this most general antiderivative a name in Definition 1.1.

DEFINITION 1.1

Let F be any antiderivative of f on an interval I. The **indefinite integral** of $f(x)$ (with respect to x) on I, is defined by

$$\int f(x)\, dx = F(x) + c,$$

where c is an arbitrary constant (the **constant of integration**).

The process of computing an integral is called **integration.** Here, $f(x)$ is called the **integrand** and the term dx identifies x as the **variable of integration**.

EXAMPLE 1.2 An Indefinite Integral

Evaluate $\int 3x^2 dx$.

Solution You should recognize $3x^2$ as the derivative of x^3 and so,

$$\int 3x^2 dx = x^3 + c.$$

EXAMPLE 1.3 Evaluating an Indefinite Integral

Evaluate $\int t^5 dt$.

Solution We know that $\frac{d}{dt}t^6 = 6t^5$ and so, $\frac{d}{dt}\left(\frac{1}{6}t^6\right) = t^5$. Therefore,

$$\int t^5 dt = \frac{1}{6}t^6 + c.$$

We should point out that every differentiation rule gives rise to a corresponding integration rule. For instance, recall that for every rational power, r, $\frac{d}{dx}x^r = rx^{r-1}$. Likewise, we have

$$\frac{d}{dx}x^{r+1} = (r+1)x^r.$$

This proves the following result.

REMARK 1.1

Theorem 1.2 says that to integrate a power of x (other than x^{-1}), you simply raise the power by 1 and divide by the new power. Notice that this rule obviously doesn't work for $r = -1$, since this would produce a division by 0. Later in this section, we develop a rule to cover this case.

THEOREM 1.2 (Power Rule)

For any rational power $r \neq -1$,

$$\int x^r dx = \frac{x^{r+1}}{r+1} + c.$$

Here, if $r < -1$, the interval I on which this is defined can be any interval that does not include $x = 0$.

EXAMPLE 1.4 Using the Power Rule

Evaluate $\int x^{17} dx$.

Solution From the power rule, we have

$$\int x^{17} dx = \frac{x^{17+1}}{17+1} + c = \frac{x^{18}}{18} + c.$$

EXAMPLE 1.5 The Power Rule with a Negative Exponent

Evaluate $\int \frac{1}{x^3} dx$.

Solution We can use the power rule if we first rewrite the integrand. In any interval not containing 0, we have

$$\int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{-3+1}}{-3+1} + c = -\frac{1}{2}x^{-2} + c.$$

EXAMPLE 1.6 The Power Rule with a Fractional Exponent

Evaluate (a) $\int \sqrt{x}\, dx$ and (b) $\int \frac{1}{\sqrt[3]{x}}\, dx$.

Solution (a) As in example 1.5, we first rewrite the integrand and then apply the power rule. We have

$$\int \sqrt{x}\, dx = \int x^{1/2} dx = \frac{x^{1/2+1}}{1/2+1} + c = \frac{x^{3/2}}{3/2} + c = \frac{2}{3}x^{3/2} + c.$$

Notice that the fraction $\frac{2}{3}$ in the last expression is exactly what it takes to cancel the new exponent 3/2. (This is what happens if you differentiate.)

(b) Similarly, in any interval not containing 0,

$$\int \frac{1}{\sqrt[3]{x}}\, dx = \int x^{-1/3} dx = \frac{x^{-1/3+1}}{-1/3+1} + c$$
$$= \frac{x^{2/3}}{2/3} + c = \frac{3}{2}x^{2/3} + c. \blacksquare$$

Notice that since $\frac{d}{dx}\sin x = \cos x$, we have

$$\boxed{\int \cos x\, dx = \sin x + c.}$$

Again, by reversing any derivative formula, we get a corresponding integration formula. If we take the latter example a step forward, we end up having:

$$\frac{d}{dx}\left(\frac{1}{2}\sin(2x)\right) = \frac{1}{2}\cdot 2\cdot \cos(2x) = \cos(2x) \Rightarrow \int \cos(2x)dx = \frac{1}{2}\sin(2x) + c$$

The following table contains a number of important formulas. The proofs of these are easily verified by differentiating the general form for the antiderivative—a straightforward, yet important process to understand the beauty of calculus.

$$\int x^r dx = \frac{x^{r+1}}{r+1} + c, \text{ for } r \neq -1 \text{ (power rule)}$$

$\int \sin kx\, dx = -\frac{1}{k}\cos kx + c$	$\int \sec kx \tan kx\, dx = \frac{1}{k}\sec kx + c$
$\int \cos kx\, dx = \frac{1}{k}\sin kx + c$	$\int \csc kx \cot kx\, dx = -\frac{1}{k}\csc kx + c$
$\int \sec^2 kx\, dx = \frac{1}{k}\tan kx + c$	$\int e^{kx}\, dx = \frac{1}{k}e^{kx} + c$
$\int \csc^2 kx\, dx = -\frac{1}{k}\cot kx + c$	$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + c$
$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + c$	$\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\, dx = \sec^{-1} x + c$

At this point, we are simply reversing the most basic derivative rules we know. We will develop more sophisticated techniques later. For now, we need a general rule to allow us to combine our basic integration formulas.

THEOREM 1.3

Suppose that $f(x)$ and $g(x)$ have antiderivatives. Then, for any constants, a and b,

$$\int [af(x) + bg(x)]dx = a\int f(x)dx + b\int g(x)dx.$$

PROOF

We have that $\frac{d}{dx}\int f(x)\,dx = f(x)$ and $\frac{d}{dx}\int g(x)\,dx = g(x)$. It then follows that

$$\frac{d}{dx}\left[a\int f(x)\,dx + b\int g(x)\,dx\right] = af(x) + bg(x),$$

as desired. ■

Note that Theorem 1.3 says that we can easily compute integrals of sums, differences and constant multiples of functions. However, it turns out that the integral of a product (or a quotient) is not generally the product (or quotient) of the integrals.

EXAMPLE 1.7 An Indefinite Integral of a Sum

Evaluate $\int(3\cos 5x + 4x^8)\,dx$.

Solution

$$\begin{aligned}\int(3\cos 5x + 4x^8)\,dx &= 3\int \cos 5x\,dx + 4\int x^8\,dx \quad \text{From Theorem 1.3.}\\ &= 3\cdot\frac{1}{5}\sin 5x + 4\frac{x^9}{9} + c\\ &= \frac{3}{5}\sin x + \frac{4}{9}x^9 + c.\end{aligned}$$ ■

EXAMPLE 1.8 An Indefinite Integral of a Difference

Evaluate $\int\left(4e^{2x} - \frac{2}{1+x^2}\right)dx$.

Solution

$$\begin{aligned}\int\left(4e^{2x} - \frac{2}{1+x^2}\right)dx &= 4\int e^{2x}\,dx - 2\int\frac{1}{1+x^2}\,dx = 4\cdot\frac{1}{2}e^{2x} - 2\tan^{-1}x + c\\ &= 2e^{2x} - 2\tan^{-1} + c\end{aligned}$$ ■

From the power rule, we know how to evaluate $\int x^r\,dx$ for any rational exponent *except* $r = -1$. We can deal with this exceptional case if we make the following observation. First, recall that for $x > 0$,

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Now, note that $\ln|x|$ is defined for $x \neq 0$. For $x > 0$, we have $\ln|x| = \ln x$ and hence,

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}.$$

Similarly, for $x < 0$, $\ln|x| = \ln(-x)$, and hence,

$$\begin{aligned}\frac{d}{dx}\ln|x| &= \frac{d}{dx}\ln(-x)\\ &= \frac{1}{-x}\frac{d}{dx}(-x) \quad \text{By the chain rule.}\\ &= \frac{1}{-x}(-1) = \frac{1}{x}.\end{aligned}$$

Notice that we got the same derivative in either case. This proves the following result:

THEOREM 1.4

For $x \neq 0$, $\frac{d}{dx} \ln|x| = \frac{1}{x}$.

EXAMPLE 1.9 The Derivative of the Log of an Absolute Value

For any x for which $\tan x \neq 0$, evaluate $\frac{d}{dx} \ln|\tan x|$.

Solution From Theorem 1.4 and the chain rule, we have

$$\frac{d}{dx} \ln|\tan x| = \frac{1}{\tan x} \frac{d}{dx} \tan x$$
$$= \frac{1}{\tan x} \sec^2 x = \frac{1}{\sin x \cos x}.$$

With the new differentiation rule in Theorem 1.4, we get a new integration rule.

COROLLARY 1.1

In any interval not containing 0,

$$\int \frac{1}{x}\, dx = \ln|x| + c.$$

More generally, notice that if $f(x) \neq 0$ and f is differentiable, we have by the chain rule that

$$\frac{d}{dx} \ln|f(x)| = \frac{1}{f(x)} f'(x) = \frac{f'(x)}{f(x)}.$$

This proves the following integration rule:

COROLLARY 1.2

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c,$$

in any interval in which $f(x) \neq 0$.

EXAMPLE 1.10 The Indefinite Integral of a Fraction of the Form $\frac{f'(x)}{f(x)}$

Evaluate $\int \frac{\sec^2 x}{\tan x}\, dx$.

Solution Notice that the numerator ($\sec^2 x$) is the derivative of the denominator ($\tan x$). From Corollary 1.2, we then have

$$\int \frac{\sec^2 x}{\tan x}\, dx = \ln|\tan x| + c.$$

Before concluding the section by examining another falling object, we should emphasize that we have developed only a small number of integration rules. Further, unlike with derivatives, we will never have rules to cover all of the functions with which we are familiar. Thus, it is important to recognize when you *cannot* find an antiderivative.

EXAMPLE 1.11 Identifying Integrals That We Cannot Yet Evaluate

Which of the following integrals can you evaluate given the rules developed in this section? (a) $\int \frac{1}{\sqrt[3]{x^2}}\,dx$, (b) $\int \sec x\,dx$, (c) $\int \frac{2x}{x^2+1}\,dx$, (d) $\int \frac{x^3+1}{x}\,dx$, (e) $\int (x+1)(x-1)\,dx$ and (f) $\int x \sin 2x\,dx$.

Solution First, notice that we can rewrite problems (a), (c), (d) and (e) into forms where we can recognize an antiderivative, as follows. For (a),

$$\int \frac{1}{\sqrt[3]{x^2}}\,dx = \int x^{-2/3}\,dx = \frac{x^{-2/3+1}}{-\frac{2}{3}+1} + c = 3x^{1/3} + c.$$

In part (c), notice that $\frac{d}{dx}(x^2+1) = 2x$ (the numerator). From Corollary 1.2, we then have

$$\int \frac{2x}{x^2+1}\,dx = \ln|x^2+1| + c = \ln(x^2+1) + c,$$

where we can remove the absolute value signs since $x^2 + 1 > 0$ for all x.

In part (d), if we divide out the integrand, we find

$$\int \frac{x^3+1}{x}\,dx = \int (x^2 + x^{-1})\,dx = \frac{1}{3}x^3 + \ln|x| + c.$$

Finally, in part (e), if we multiply out the integrand, we get

$$\int (x+1)(x-1)\,dx = \int (x^2-1)\,dx = \frac{1}{3}x^3 - x + c.$$

Parts (b) and (f) require us to find functions whose derivatives equal $\sec x$ and $x \sin 2x$. As yet, we do not know how to evaluate these integrals. ■

Now that we know how to find antiderivatives for a number of functions, we return to the problem of the falling object that opened the section.

EXAMPLE 1.12 Finding the Position of a Falling Object Given its Acceleration

If an object's downward acceleration is given by $y''(t) = -9.8$ m/s^2, find the position function $y(t)$. Assume that the initial velocity is $y'(0) = -100$ m/s and the initial position is $y(0) = 100{,}000$ m.

Solution We have to undo two derivatives, so we compute two antiderivatives. First, we have

$$y'(t) = \int y''(t)\,dt = \int (-9.8)\,dt = -9.8t + c.$$

Since $y'(t)$ is the velocity of the object (given in units of m per second), we can determine the constant c from the given initial velocity. We have

$$v(t) = y'(t) = -9.8t + c$$

and $v(0) = y'(0) = -100$ and so,

$$-100 = v(0) = -9.8(0) + c = c,$$

so that $c = -100$. Thus, the velocity is $y'(t) = -9.8t - 100$. Next, we have

$$y(t) = \int y'(t)\,dt = \int (-9.8t - 100)\,dt = -4.9t^2 - 100t + c.$$

Now, $y(t)$ gives the height of the object (measured in m) and so, from the initial position, we have

$$100{,}000 = y(0) = -4.9(0) - 100(0) + c = c.$$

Thus, $c = 100{,}000$ and

$$y(t) = -4.9t^2 - 100t + 100{,}000.$$

Keep in mind that this models the object's height assuming that the only force acting on the object is gravity (i.e., there is no air drag or lift). ■

EXERCISES 1.1

WRITING EXERCISES

1. In the text, we emphasized that the indefinite integral represents *all* antiderivatives of a given function. To understand why this is important, consider a situation where you know the net force, $F(t)$, acting on an object. By Newton's second law, $F = ma$. For the velocity function $v(t)$, this translates to $a(t) = v'(t) = F(t)/m$. To compute $v(t)$, you need to compute an antiderivative of the force function $F(t)/m$. However, suppose you were unable to find *all* antiderivatives. How would you know whether you had computed the antiderivative that corresponds to the velocity function? In physical terms, explain why it is reasonable to expect that there is only one antiderivative corresponding to a given set of initial conditions.

2. In the text, we presented a one-dimensional model of the motion of a falling object. We ignored some of the forces on the object so that the resulting mathematical equation would be one that we could solve. Weigh the relative worth of having an unsolvable but realistic model versus having a solution of a model that is only partially accurate. Keep in mind that when you toss trash into a wastebasket you do not take the curvature of the earth into account.

3. Verify that $\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + c$ and $\int xe^x\,dx = xe^x - e^x + c$ by computing derivatives of the proposed antiderivatives. Which derivative rules did you use? Why does this make it unlikely that we will find a general product (antiderivative) rule for $\int f(x)g(x)\,dx$?

4. We stated in the text that we do not yet have a formula for the antiderivative of several elementary functions, including $\ln x$, $\sec x$ and $\csc x$. Given a function $f(x)$, explain what determines whether or not we have a simple formula for $\int f(x)\,dx$. For example, why is there a simple formula for $\int \sec x \tan x\,dx$ but not $\int \sec x\,dx$?

In exercises 1–4, sketch several members of the family of functions defined by the antiderivative.

1. $\int x^3\,dx$

2. $\int (x^3 - x)\,dx$

3. $\int e^x\,dx$

4. $\int \cos x\,dx$

In exercises 5–32, find the general antiderivative.

5. $\int (3x^4 - 3x)\,dx$

6. $\int (x^3 - 2)\,dx$

7. $\int \left(3\sqrt{x} - \frac{1}{x^4}\right)dx$

8. $\int \left(2x^{-2} + \frac{1}{\sqrt{x}}\right)dx$

9. $\int \frac{x^{1/3} - 3}{x^{2/3}}\,dx$

10. $\int \frac{x + 2x^{3/4}}{x^{5/4}}\,dx$

11. $\int (2\sin x + \cos x)\,dx$

12. $\int \left(3\cos 2x - \sin\frac{x}{5}\right)dx$

13. $\int 2\sec x \tan x\,dx$

14. $\int \frac{4}{\sqrt{1 - x^2}}\,dx$

15. $\int 5\sec^2 4x\,dx$

16. $\int 4\frac{\cos x}{\sin^2 x}\,dx$

17. $\int (3e^x - 2)\,dx$

18. $\int (4x - 2e^{2x})\,dx$

19. $\int (3\cos x - 1/x)\,dx$

20. $\int (2x^{-1} + \sin(7x))\,dx$

21. $\int \frac{4x}{x^2 + 4}\,dx$

22. $\int \frac{3}{4x^2 + 4}\,dx$

23. $\int \frac{\cos x}{\sin x}\,dx$

24. $\int \left(2\cos x - \sqrt{e^{2x}}\right)dx$

25. $\int \frac{e^x}{e^x + 3}\,dx$

26. $\int \frac{e^{4x} + 3}{e^{2x}}\,dx$

27. $\int x^{1/4}(x^{5/4} - 4)\,dx$

28. $\int x^{2/3}(x^{-4/3} - 3)\,dx$

29. $\int \pi\sec\left(\frac{\pi}{2}\theta\right)\tan\left(\frac{\pi}{2}\theta\right)d\theta$

30. $\int \sin\theta(\cos\theta - 3\csc\theta)\,d\theta$

31. $\int \frac{\sqrt[3]{t^2} - t}{\sqrt{t}}\,dt$

32. $\int \frac{1 - \sin 2y}{3}\,dy$

In exercises 33–36, one of the two antiderivatives can be determined using basic algebra and the antiderivative formulas we have presented. Name the method by finding the antiderivative of this one and label the other "N/A."

33. (a) $\int \sqrt{x^3 + 4}\,dx$ (b) $\int \left(\sqrt{x^3} + 4\right)dx$

34. (a) $\int \frac{3x^2 - 4}{x^2}\,dx$ (b) $\int \frac{x^2}{3x^2 - 4}\,dx$

35. (a) $\int 2 \sec x \, dx$ (b) $\sec^2 x \, dx$

36. (a) $\int \left(\frac{1}{x^2} - 1\right) dx$ (b) $\int \frac{1}{x^2 - 1} dx$

In exercises 37–42, find the function $f(x)$ satisfying the given conditions.

37. $f'(x) = 3e^x + x, f(0) = 4$

38. $f'(x) = 4 \cos x, f(0) = 3$

39. $f''(x) = 12x^2 + 2e^x, f'(0) = 2, f(0) = 3$

40. $f''(x) = 20x^3 + 2e^{2x}, f'(0) = -3, f(0) = 2$

41. $f''(t) = 2 + 2t, f(0) = 2, f(3) = 2$

42. $f''(t) = 4 + 6t, f(1) = 3, f(-1) = -2$

In exercises 43–46, find all functions satisfying the given conditions.

43. $f''(x) = 3 \sin x + 4x^2$

44. $f''(x) = \sqrt{x} - 2 \cos x$

45. $f'''(x) = 4 - 2/x^3$

46. $f'''(x) = \sin x - e^x$

47. Determine the position function if the velocity function is $v(t) = 3 - 12t$ and the initial position is $s(0) = 3$.

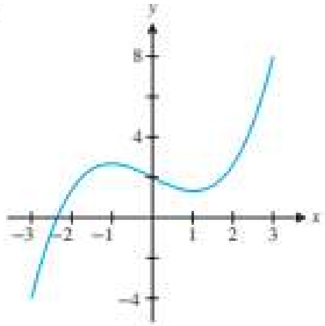
48. Determine the position function if the velocity function is $v(t) = 3e^{-t} - 2$ and the initial position is $s(0) = 0$.

49. Determine the position function if the acceleration function is $a(t) = 3 \sin t + 1$, the initial velocity is $v(0) = 0$ and the initial position is $s(0) = 4$.

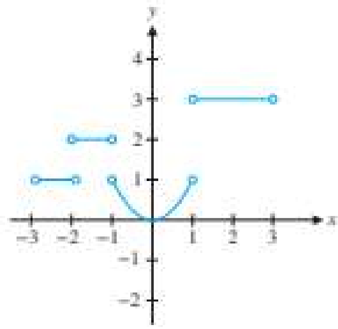
50. Determine the position function if the acceleration function is $a(t) = t^2 + 1$, the initial velocity is $v(0) = 4$ and the initial position is $s(0) = 0$.

51. Sketch the graph of two functions $f(x)$ corresponding to the given graph of $y = f'(x)$.

(a)



(b)



52. Repeat exercise 51 if the given graph is of $f''(x)$.

53. Find a function $f(x)$ such that the point (1, 2) is on the graph of $y = f(x)$, the slope of the tangent line at (1, 2) is 3 and $f''(x) = x - 1$.

54. Find a function $f(x)$ such that the point (−1, 1) is on the graph of $y = f(x)$, the slope of the tangent line at (−1, 1) is 2 and $f''(x) = 6x + 4$.

In exercises 55–60, find an antiderivative by reversing the chain rule, product rule or quotient rule.

55. $\int 2x \cos x^2 \, dx$

56. $\int x^2 \sqrt{x^3 + 2} \, dx$

57. $\int (x \sin 2x + x^2 \cos 2x) \, dx$

58. $\int \frac{2xe^{3x} - 3x^2e^{3x}}{e^{6x}} dx$

59. $\int \frac{x \cos x^2}{\sqrt{\sin x^2}} dx$

60. $\int \left(2\sqrt{x} \cos x + \frac{1}{\sqrt{x}} \sin x\right) dx$

61. In example 1.11, use your CAS to evaluate the antiderivatives in parts (b) and (f). Verify that these are correct by computing the derivatives.

62. For each of the problems in exercises 33–36 that you labeled N/A, try to find an antiderivative on your CAS. Where possible, verify that the antiderivative is correct by computing the derivatives.

63. Use a CAS to find an antiderivative, then verify the answer by computing a derivative.

(a) $\int x^2 e^{-x^3} dx$ (b) $\int \frac{1}{x^2 - x} dx$ (c) $\int \csc x \, dx$

64. Use a CAS to find an antiderivative, then verify the answer by computing a derivative.

(a) $\int \frac{x}{x^4 + 1} dx$ (b) $\int 3x \sin 2x \, dx$ (c) $\int \ln x \, dx$

65. Show that $\int \frac{-1}{\sqrt{1 - x^2}} dx = \cos^{-1} x + c$ and $\int \frac{-1}{\sqrt{1 - x^2}} dx = -\sin^{-1} x + c$. Explain why this does not imply that $\cos^{-1} x = -\sin^{-1} x$. Find an equation relating $\cos^{-1} x$ and $\sin^{-1} x$.

66. Derive the formulas $\int \sec^2 x\,dx = \tan x + c$ and $\int \sec x \tan x\,dx = \sec x + c$.

67. Derive the formulas $\int e^x\,dx = e^x + c$ and $\int e^{-x}\,dx = -e^{-x} + c$.

68. For the antiderivative $\int \frac{1}{kx}\,dx$, (a) factor out the k and then use a basic formula and (b) rewrite the problem as $\frac{1}{k}\int \frac{k}{kx}\,dx$ and use formula (1.4). Discuss the difference between the antiderivatives (a) and (b) and explain why they are both correct.

APPLICATIONS

1. Suppose that a car can accelerate from 36 km/h to 54 km/h in 4 seconds. Assuming a constant acceleration, find the acceleration of the car and find the distance traveled by the car during the 4 seconds.

2. Suppose that a car can come to rest from 72 km/h in 3 seconds. Assuming a constant (negative) acceleration, find the acceleration (in meters per second squared) of the car and find the distance traveled by the car during the 3 seconds (i.e., the stopping distance).

3. The following table shows the velocity of a falling object at different times. For each time interval, estimate the distance fallen and the acceleration.

t (s)	0	0.5	1.0	1.5	2.0
$v(t)$ (m/s)	−4.0	−19.8	−31.9	−37.7	−39.5

4. The following table shows the velocity of a falling object at different times. For each time interval, estimate the distance fallen and the acceleration.

t (s)	0	1.0	2.0	3.0	4.0
$v(t)$ (m/s)	0.0	−9.8	−18.6	−24.9	−28.5

5. The following table shows the acceleration of a car moving in a straight line. If the car is traveling 70 m/s at time $t = 0$, estimate the speed and distance traveled at each time.

t (s)	0	0.5	1.0	1.5	2.0
$a(t)$ (m/s^2)	−4.2	2.4	0.6	−0.4	1.6

6. The following table shows the acceleration of a car moving in a straight line. If the car is traveling 20 m/s at time $t = 0$, estimate the speed and distance traveled at each time.

t (s)	0	0.5	1.0	1.5	2.0
$a(t)$ (m/s^2)	0.6	−2.2	−4.5	−1.2	−0.3

EXPLORATORY EXERCISE

1. Compute the derivatives of $e^{\sin x}$ and e^{x^2}. Given these derivatives, evaluate the indefinite integrals $\int \cos x\, e^{\sin x}\,dx$ and $\int 2x\,e^{x^2}\,dx$. Next, evaluate $\int x\,e^{x^2}\,dx$. (Hint: $\int x\,e^{x^2}dx = \frac{1}{2}\int 2x\,e^{x^2}dx$.) Similarly, evaluate $\int x^2\,e^{x^3}\,dx$. In general, evaluate

$$\int f'(x)e^{f(x)}dx.$$

Next, evaluate $\int e^x \cos(e^x)\,dx$, $\int 2x\cos(x^2)\,dx$ and the more general

$$\int f'(x)\cos(f(x))dx.$$

As we have stated, there is no general rule for the antiderivative of a product, $\int f(x)\,g(x)\,dx$. Instead, there are many special cases that you evaluate case by case.

1.2 SUMS AND SIGMA NOTATION

In section 1.1, we discussed how to calculate backward from the velocity function for an object to arrive at the position function for the object. It's no surprise that driving at a constant 60 km/h, you travel 120 km in 2 hours, or 240 km in 4 hours. Viewing this graphically, note that the area under the graph of the (constant) velocity function $v(t) = 60$ from $t = 0$ to $t = 2$ is 120, the distance traveled in this time interval. (See the shaded area in Figure 1.2a.) Likewise, in Figure 1.2b, the shaded region from $t = 0$ to $t = 4$ has an area equal to the distance of 240 km.

It turns out that, in general, the distance traveled over a particular time interval equals the area of the region bounded by $y = v(t)$ and the t-axis on that interval. For the case of constant velocity, this is no surprise, as we have that

$$d = r \times t = \text{velocity} \times \text{time}.$$

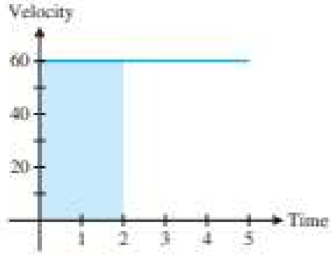


FIGURE 1.2a
$y = v(t)$ on $[0, 2]$

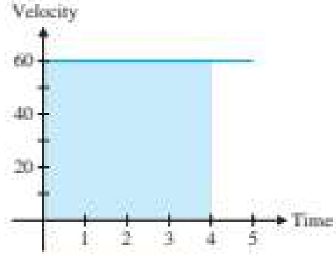


FIGURE 1.2b
$y = v(t)$ on $[0, 4]$

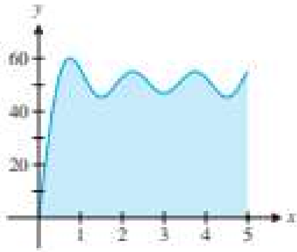


FIGURE 1.3
Area under a curve

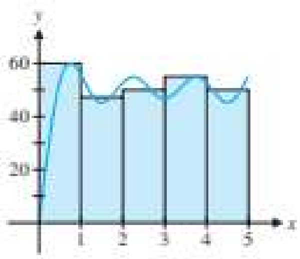


FIGURE 1.4
Approximate area

Our aim over the next several sections is to compute the area under the curve for a nonconstant function, such as the one shown in Figure 1.3. Our work in this section provides the first step toward a powerful technique for computing such areas. To indicate the direction we will take, note that we can approximate the area in Figure 1.3 by the sum of the areas of the five rectangles indicated in Figure 1.4:

$$A \approx 60 + 45 + 50 + 55 + 50 = 260 \text{ km.}$$

Of course, this is a crude estimate of the area, but you should observe that we could get a better estimate by approximating the area using more (and smaller) rectangles. Certainly, we had no problem adding up the areas of five rectangles, but for 5000 rectangles, you will want some means for simplifying and automating the process. Dealing with such sums is the topic of this section.

We begin by introducing some notation. Suppose that you want to sum the squares of the first 20 positive integers. Notice that

$$1 + 4 + 9 + \cdots + 400 = 1^2 + 2^2 + 3^2 + \cdots + 20^2,$$

where each term in the sum has the form i^2, for $i = 1, 2, 3, \ldots, 20$. To reduce the amount of writing, we use the Greek capital letter sigma, Σ, as a symbol for *sum* and write the sum in **summation notation** as

$$\sum_{i=1}^{20} i^2 = 1^2 + 2^2 + 3^2 + \cdots + 20^2,$$

to indicate that we add together terms of the form i^2, starting with $i = 1$ and ending with $i = 20$. The variable i is called the **index of summation**.

In general, for any real numbers $a_1, a_2, \ldots, a_n$, we have

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

EXAMPLE 2.1 Using Summation Notation

Write in summation notation: (a) $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{10}$ and (b) $3^3 + 4^3 + 5^3 + \cdots + 45^3$.

Solution (a) We have the sum of the square roots of the integers from 1 to 10:

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{10} = \sum_{i=1}^{10} \sqrt{i}.$$

(b) Similarly, the sum of the cubes of the integers from 3 to 45:

$$3^3 + 4^3 + 5^3 + \cdots + 45^3 = \sum_{i=3}^{45} i^3.$$ ■

REMARK 2.1

The index of summation is a **dummy variable**, since it is used only as a counter to keep track of terms. The value of the summation does not depend on the letter used as the index. For this reason, you may use any letter you like as an index. By tradition, we most frequently use i, j, k, m and n, but any index will do. For instance,

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k.$$

EXAMPLE 2.2 Summation Notation for a Sum Involving Odd Integers

Write in summation notation: the sum of the first 200 odd positive integers.

Solution First, notice that $(2i)$ is even for every integer i and hence, both $(2i - 1)$ and $(2i + 1)$ are odd. So, we have

$$1 + 3 + 5 + \cdots + 399 = \sum_{i=1}^{200}(2i - 1).$$

Alternatively, we can write this as the equivalent expression $\sum_{i=0}^{199}(2i + 1)$. (Write out the terms to see why these are equivalent.) ■

EXAMPLE 2.3 Computing Sums Given in Summation Notation

Write out all terms and compute the sums (a) $\sum_{i=1}^{8}(2i + 1)$, (b) $\sum_{i=2}^{6}\sin(2\pi i)$ and (c) $\sum_{i=4}^{10} 5$.

Solution We have

(a) $$\sum_{i=1}^{8}(2i + 1) = 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 80.$$

(b) $$\sum_{i=2}^{6}\sin(2\pi\, i) = \sin 4\pi + \sin 6\pi + \sin 8\pi + \sin 10\pi + \sin 12\pi = 0.$$

(Note that the sum started at $i = 2$.) Finally, we have

(c) $$\sum_{i=4}^{10} 5 = 5 + 5 + 5 + 5 + 5 + 5 + 5 = 35.$$ ■

We give several shortcuts for computing sums in the following result.

THEOREM 2.1

If n is any positive integer and c is any constant, then

(i) $\sum_{i=1}^{n} c = cn$ **(sum of constants)**,

(ii) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ **(sum of the first n positive integers)** and

(iii) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ **(sum of the squares of the first n positive integers)**.

HISTORICAL NOTES

Karl Friedrich Gauss (1777–1855)
A German mathematician widely considered to be the greatest mathematician of all time. A prodigy who had proved important theorems by age 14, Gauss was the acknowledged master of almost all areas of mathematics. He proved the Fundamental Theorem of Algebra and numerous results in number theory and mathematical physics. Gauss was instrumental in starting new fields of research, including the analysis of complex variables, statistics, vector calculus and non-Euclidean geometry. Gauss was truly the "Prince of Mathematicians."

PROOF

(i) $\sum_{i=1}^{n} c$ indicates to add the same constant c to itself n times and hence, the sum is simply c times n.

(ii) The following clever proof has been credited to then 10-year-old Karl Friedrich Gauss. (For more on Gauss, see the historical note.) First notice that

$$\sum_{i=1}^{n} i = \underbrace{1 + 2 + 3 + \cdots + (n-2) + (n-1) + n}_{n \text{ terms}}. \tag{2.1}$$

Since the order in which we add the terms does not matter, we add the terms in (2.1) in reverse order, to get

$$\sum_{i=1}^{n} i = \underbrace{n + (n-1) + (n-2) + \cdots + 3 + 2 + 1}_{\text{same } n \text{ terms (backward)}}. \tag{2.2}$$

Adding equations (2.1) and (2.2) term by term, we get

$$\begin{aligned} 2\sum_{i=1}^{n} i &= (1+n) + (2+n-1) + (3+n-2) + \cdots + (n-1+2) + (n+1) \\ &= \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)}_{n \text{ terms}} \\ &= n(n+1), \quad \text{Adding each term in parentheses} \end{aligned}$$

since $(n+1)$ appears n times in the sum. As desired, dividing both sides by 2 gives us

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

The proof of (iii) requires a more sophisticated proof using mathematical induction and we defer it to the end of this section. ■

We also have the following general rule for expanding sums. The proof is straightforward and is left as an exercise.

THEOREM 2.2

For any constants c and d,

$$\sum_{i=1}^{n} (ca_i + db_i) = c\sum_{i=1}^{n} a_i + d\sum_{i=1}^{n} b_i.$$

Using Theorems 2.1 and 2.2, we can now compute several simple sums with ease. Note that we have no more difficulty summing 800 terms than we do summing 8.

EXAMPLE 2.4 Computing Sums Using Theorems 2.1 and 2.2

Compute (a) $\sum_{i=1}^{8} (2i+1)$ and (b) $\sum_{i=1}^{800} (2i+1)$.

Solution (a) From Theorems 2.1 and 2.2, we have

$$\sum_{i=1}^{8} (2i+1) = 2\sum_{i=1}^{8} i + \sum_{i=1}^{8} 1 = 2\frac{8(9)}{2} + (1)(8) = 72 + 8 = 80.$$

(b) Similarly,

$$\begin{aligned} \sum_{i=1}^{800} (2i+1) &= 2\sum_{i=1}^{800} i + \sum_{i=1}^{800} 1 = 2\frac{800(801)}{2} + (1)(800) \\ &= 640{,}800 + 800 = 641{,}600. \end{aligned}$$ ■

EXAMPLE 2.5 Computing Sums Using Theorems 2.1 and 2.2

Compute (a) $\sum_{i=1}^{20} i^2$ and (b) $\sum_{i=1}^{20}\left(\frac{i}{20}\right)^2$.

Solution (a) From Theorems 2.1 and 2.2, we have

$$\sum_{i=1}^{20} i^2 = \frac{20(21)(41)}{6} = 2870.$$

(b)

$$\sum_{i=1}^{20}\left(\frac{i}{20}\right)^2 = \frac{1}{20^2}\sum_{i=1}^{20} i^2 = \frac{1}{400}\frac{20(21)(41)}{6} = \frac{1}{400}2870 = 7.175.$$

In the beginning of this section, we approximated distance by summing several values of the velocity function. In section 1.3, we will further develop these sums to allow us to compute areas exactly. However, our immediate interest in sums is to use these to sum a number of values of a function, as we illustrate in examples 2.6 and 2.7.

EXAMPLE 2.6 Computing a Sum of Function Values

Sum the values of $f(x) = x^2 + 3$ evaluated at $x = 0.1, x = 0.2, \ldots, x = 1.0$.

Solution We first formulate this in summation notation, so that we can use the rules we have developed in this section. The terms to be summed are $a_1 = f(0.1) = 0.1^2 + 3$, $a_2 = f(0.2) = 0.2^2 + 3$ and so on. Note that since each of the x-values is a multiple of 0.1, we can write the x's in the form $0.1i$, for $i = 1, 2, \ldots, 10$. In general, we have

$$a_i = f(0.1i) = (0.1i)^2 + 3, \quad \text{for } i = 1, 2, \ldots, 10.$$

From Theorem 2.1 (i) and (iii), we then have

$$\sum_{i=1}^{10} a_i = \sum_{i=1}^{10} f(0.1i) = \sum_{i=1}^{10}[(0.1i)^2 + 3] = 0.1^2\sum_{i=1}^{10} i^2 + \sum_{i=1}^{10} 3$$
$$= 0.01\frac{10(11)(21)}{6} + (3)(10) = 3.85 + 30 = 33.85.$$

EXAMPLE 2.7 A Sum of Function Values at Equally Spaced x's

Sum the values of $f(x) = 3x^2 - 4x + 2$ evaluated at $x = 1.05, x = 1.15, x = 1.25, \ldots, x = 2.95$.

Solution You will need to think carefully about the x's. The distance between successive x-values is 0.1, and there are 20 such values. (Be sure to count these for yourself.) Notice that we can write the x's in the form $0.95 + 0.1i$, for $i = 1, 2, \ldots, 20$.

We now have

$$\sum_{i=1}^{20} f(0.95 + 0.1i) = \sum_{i=1}^{20}\left[3(0.95 + 0.1i)^2 - 4(0.95 + 0.1i) + 2\right]$$
$$= \sum_{i=1}^{20}(0.03i^2 + 0.17i + 0.9075) \quad \text{Multiply out terms.}$$
$$= 0.03\sum_{i=1}^{20} i^2 + 0.17\sum_{i=1}^{20} i + \sum_{i=1}^{20} 0.9075 \quad \text{From Theorem 2.2.}$$
$$= 0.03\frac{20(21)(41)}{6} + 0.17\frac{20(21)}{2} + 0.9075(20) \quad \text{From Theorem 2.1 (i), (ii) and (iii).}$$
$$= 139.95.$$

Over the next several sections, we will see how sums such as those found in examples 2.6 and 2.7 play a very significant role. We end this section by looking at a powerful mathematical principle.

○ Principle of Mathematical Induction

For any proposition that depends on a positive integer, n, we first show that the result is true for a specific value $n = n_0$. We then *assume* that the result is true for an *unspecified* $n = k \geq n_0$. (This is called the **induction assumption**.) If we can show that it follows that the proposition is true for $n = k + 1$, then we have proved that the result is true for any positive integer $n \geq n_0$. Think about why this must be true. (Hint: If P_1 is true and P_k true implies P_{k+1} is true, then P_1 true implies P_2 is true, which in turn implies P_3 is true and so on.)

We can now use mathematical induction to prove the last part of Theorem 2.1, which states that for any positive integer n, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

PROOF of Theorem 2.1 (iii)

For $n = 1$, we have

$$1 = \sum_{i=1}^{1} i^2 = \frac{1(2)(3)}{6},$$

as desired. So, the proposition is true for $n = 1$. Next, **assume** that

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}, \quad \text{Induction assumption.} \qquad (2.3)$$

for some integer $k \geq 1$.

In this case, as desired, we have by the induction assumption that for $n = k + 1$,

$$\begin{aligned}
\sum_{i=1}^{n} i^2 &= \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + \sum_{i=k+1}^{k+1} i^2 && \text{Split off the last term.} \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{From (2.3).} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} && \text{Add the fractions.} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} && \text{Factor out } (k+1). \\
&= \frac{(k+1)\left[2k^2 + 7k + 6\right]}{6} && \text{Combine terms.} \\
&= \frac{(k+1)(k+2)(2k+3]}{6} && \text{Factor the quadratic.} \\
&= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} && \text{Rewrite the terms.} \\
&= \frac{n(n+1)(2n+1)}{6}. && \text{Since } n = k+1. \blacksquare
\end{aligned}$$

EXERCISES 1.2

WRITING EXERCISES

1. In the text, we mentioned that one of the benefits of using the summation notation is the simplification of calculations. To help understand this, write out in words what is meant by $\sum_{i=1}^{40}(2i^2 - 4i + 11)$.

2. Following up on exercise 1, calculate the sum $\sum_{i=1}^{40}(2i^2 - 4i + 11)$ and then describe in words how you did so. Be sure to describe any formulas and your use of them in words.

In exercises 1 and 2, translate into summation notation.

1. $2(1)^2 + 2(2)^2 + 2(3)^2 + \cdots + 2(14)^2$
2. $\sqrt{2-1} + \sqrt{3-1} + \sqrt{4-1} + \cdots + \sqrt{15-1}$

In exercises 3 and 4, calculations are described in words. Translate each into summation notation and then compute the sum.

3. (a) The sum of the squares of the first 50 positive integers.
   (b) The square of the sum of the first 50 positive integers.
4. (a) The sum of the square roots of the first 10 positive integers.
   (b) The square root of the sum of the first 10 positive integers.

In exercises 5–8, write out all terms and compute the sums.

5. $\sum_{i=1}^{6} 3i^2$
6. $\sum_{i=3}^{7}(i^2 + i)$
7. $\sum_{i=6}^{10}(4i + 2)$
8. $\sum_{i=6}^{8}(i^2 + 2)$

In exercises 9–18, use summation rules to compute the sum.

9. $\sum_{i=1}^{70}(3i - 1)$
10. $\sum_{i=1}^{45}(3i - 4)$
11. $\sum_{i=1}^{40}(4 - i^2)$
12. $\sum_{i=1}^{50}(8 - i)$
13. $\sum_{n=1}^{100}(n^2 - 3n + 2)$
14. $\sum_{n=1}^{140}(n^2 + 2n - 4)$
15. $\sum_{i=3}^{30}[(i-3)^2 + i - 3]$
16. $\sum_{i=4}^{20}(i-3)(i+3)$
17. $\sum_{k=3}^{n}(k^2 - 3)$
18. $\sum_{k=0}^{n}(k^2 + 5)$

In exercises 19–22, compute sums of the form $\sum_{i=1}^{n} f(x_i)\Delta x$ for the given values of x_i.

19. $f(x) = x^2 + 4x$; $x = 0.2, 0.4, 0.6, 0.8, 1.0$; $\Delta x = 0.2$; $n = 5$
20. $f(x) = 3x + 5$; $x = 0.4, 0.8, 1.2, 1.6, 2.0$; $\Delta x = 0.4$; $n = 5$
21. $f(x) = 4x^2 - 2$; $x = 2.1, 2.2, 2.3, 2.4, \ldots, 3.0$; $\Delta x = 0.1$; $n = 10$
22. $f(x) = x^3 + 4$; $x = 2.05, 2.15, 2.25, 2.35, \ldots, 2.95$; $\Delta x = 0.1$; $n = 10$

In exercises 23–26, compute the sum and the limit of the sum as $n \to \infty$.

23. $\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 + 2\left(\frac{i}{n}\right)\right]$
24. $\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 - 5\left(\frac{i}{n}\right)\right]$
25. $\sum_{i=1}^{n} \frac{1}{n}\left[4\left(\frac{2i}{n}\right)^2 - \left(\frac{2i}{n}\right)\right]$
26. $\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{2i}{n}\right)^2 + 4\left(\frac{i}{n}\right)\right]$

27. Use mathematical induction to prove that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for all integers $n \geq 1$.
28. Use mathematical induction to prove that $\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2 + 2n - 1)}{12}$ for all integers $n \geq 1$.

In exercises 29–32, use the formulas in exercises 27 and 28 to compute the sums.

29. $\sum_{i=1}^{10}(i^3 - 3i + 1)$
30. $\sum_{i=1}^{20}(i^3 + 2i)$
31. $\sum_{i=1}^{100}(i^5 - 2i^2)$
32. $\sum_{i=1}^{100}(2i^5 + 2i + 1)$

33. Prove Theorem 2.2.
34. Use induction to derive the geometric series formula $a + ar + ar^2 + \cdots + ar^n = \frac{a - ar^{n+1}}{1 - r}$ for constants a and $r \neq 1$.

In exercises 35 and 36, use the result of exercise 34 to evaluate the sum and the limit of the sum as $n \to \infty$.

35. $\sum_{i=1}^{n} e^{(6i/n)} \frac{6}{n}$
36. $\sum_{i=1}^{n} e^{(2i/n)} \frac{2}{n}$

APPLICATIONS

1. Suppose that a car has velocity 50 km/h for 2 hours, velocity 60 km/h for 1 hour, velocity 70 km/h for 30 minutes and velocity 60 km/h for 3 hours. Find the distance traveled.
2. Suppose that a car has velocity 50 km/h for 1 hour, velocity 40 km/h for 1 hour, velocity 60 km/h for 30 minutes and velocity 55 km/h for 3 hours. Find the distance traveled.

3. The table shows the velocity of a projectile at various times. Estimate the distance traveled.

time (s)	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0
velocity (m/s)	120	116	113	110	108	106	104	103	102

4. The table shows the (downward) velocity of a falling object. Estimate the distance fallen.

time (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
velocity (m/s)	10	14.9	19.8	24.7	29.6	34.5	39.4	44.3	49.2

EXPLORATORY EXERCISES

1. Suppose that the velocity of a car is given by $v(t) = 3\sqrt{t} + 30$ km/h at time t hours ($0 \le t \le 4$). We will try to determine the distance traveled in the 4 hours. The velocity at $t = 0$ is $v(0) = 3\sqrt{0} + 30 = 30$ km/h and the velocity at time $t = 1$ is $v(1) = 3\sqrt{1} + 30 = 33$ km/h. Since the average of these velocities is 31.5 km/h, we could estimate that the car traveled 31.5 km in the first hour. Carefully explain why this is not necessarily correct. Since $v(1) = 33$ km/h and $v(2) = 3\sqrt{2} + 30 \approx 34$ km/h, we estimate that the car traveled 33.5 km in the second hour. Using $v(3) \approx 35$ km/h and $v(4) = 36$ km/h, find similar estimates for the distance traveled in the third and fourth hours and then estimate the total distance. To improve this estimate, we can find an estimate for the distance covered each half hour. The first estimate would take $v(0) = 30$ km/h and $v(0.5) \approx 32.1$ km/h and estimate a distance of 15.525 km. Estimate the average velocity and then the distance for the remaining 7 half hours and estimate the total distance. By estimating the average velocity every quarter hour, find a third estimate of the total distance. Based on these three estimates, conjecture the limit of these approximations as the time interval considered goes to zero.

2. In this exercise, we investigate a generalization of a finite sum called an *infinite series*. Suppose a bouncing ball has **coefficient of restitution** equal to 0.6. This means that if the ball hits the ground with velocity v m/s, it rebounds with velocity $0.6v$. Ignoring air resistance, a ball launched with velocity v m/s will stay in the air $v/16$ seconds before hitting the ground. Suppose a ball with coefficient of restitution 0.6 is launched with initial velocity 60 m/s. Explain why the total time in the air is given by $60/16 + (0.6)(60)/16 + (0.6)(0.6)(60)/16 + \cdots$. It might seem like the ball would continue to bounce forever. To see otherwise, use the result of exercise 40 to find the limit that these sums approach. The limit is the number of seconds that the ball continues to bounce.

3. The following statement is obviously false: Given any set of n numbers, the numbers are all equal. Find the flaw in the attempted use of mathematical induction. Let $n = 1$. One number is equal to itself. Assume that for $n = k$, any k numbers are equal. Let S be any set of $k + 1$ numbers $a_1, a_2, \ldots, a_{k+1}$. By the induction hypothesis, the first k numbers are equal: $a_1 = a_2 = \cdots = a_k$ and the last k numbers are equal: $a_2 = a_3 = \cdots = a_{k+1}$. Combining these results, all $k + 1$ numbers are equal: $a_1 = a_2 = \cdots = a_k = a_{k+1}$, as desired.

1.3 AREA UNDER A CURVE AND INTEGRATION

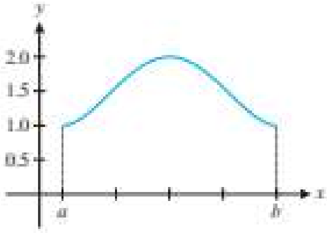


FIGURE 1.5
Area under $y = f(x)$

In this section, we develop a method for computing the area beneath the graph of $y = f(x)$ and above the x-axis on an interval $a \le x \le b$. You are familiar with the formulas for computing the area of a rectangle, a circle and a triangle. However, how would you compute the area of a region that's not a rectangle, circle or triangle?

We need a more general description of area, one that can be used to find the area of almost any two-dimensional region imaginable. It turns out that this process (which we generalize to the notion of the *definite integral* in section 1.4) is one of the central ideas of calculus, with applications in a wide variety of fields.

First, assume that $f(x) \ge 0$ and f is continuous on the interval $[a, b]$, as in Figure 1.5. We start by dividing the interval $[a, b]$ into n equal pieces. This is called a **regular partition** of $[a, b]$. The width of each subinterval in the partition is then $\frac{b-a}{n}$, which we denote by Δx (meaning a small change in x). The points in the partition are denoted by $x_0 = a$, $x_1 = x_0 + \Delta x$, $x_2 = x_1 + \Delta x$ and so on. In general,

$$x_i = x_0 + i\Delta x, \qquad \text{for } i = 1, 2, \ldots, n.$$

See Figure 1.6 for an illustration of a regular partition for the case where $n = 6$. On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, \ldots, n$) construct a rectangle of height $f(x_i)$ (the value of the function at the right endpoint of the subinterval), as illustrated in Figure 1.7 for the case where $n = 4$. It should be clear from Figure 1.7 that the area under the curve A is roughly the same as the sum of the areas of the four rectangles,

$$A \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x = A_4.$$

FIGURE 1.6
Regular partition of $[a, b]$

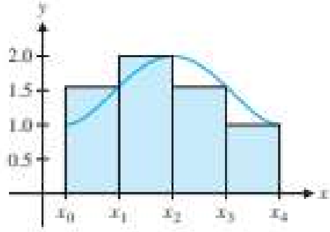


FIGURE 1.7
$A \approx A_4$

In particular, notice that although two of these rectangles enclose more area than that under the curve and two enclose less area, on the whole, the sum of the areas of the four rectangles provides an approximation to the total area under the curve. More generally, if we construct n rectangles of equal width on the interval $[a, b]$, we have

$$\begin{aligned} A &\approx f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x \\ &= \sum_{i=1}^{n} f(x_i)\,\Delta x = A_n. \end{aligned} \tag{3.1}$$

EXAMPLE 3.1 Approximating an Area with Rectangles

Approximate the area under the curve $y = f(x) = 2x - 2x^2$ on the interval $[0, 1]$, using (a) 10 rectangles and (b) 20 rectangles.

Solution (a) The partition divides the interval into 10 subintervals, each of length $\Delta x = 0.1$, namely, $[0, 0.1], [0.1, 0.2], \ldots, [0.9, 1.0]$. In Figure 1.8, we have drawn in rectangles of height $f(x_i)$ on each subinterval $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, 10$. Notice that the sum of the areas of the 10 rectangles indicated provides an approximation to the area under the curve. That is,

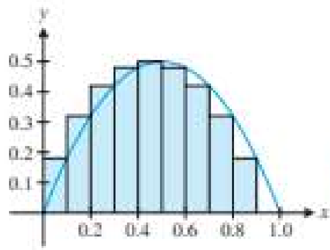


FIGURE 1.8
$A \approx A10$

$$\begin{aligned} A \approx A_{10} &= \sum_{i=1}^{10} f(x_i)\Delta x \\ &= [f(0.1) + f(0.2) + \cdots + f(1.0)](0.1) \\ &= (0.18 + 0.32 + 0.42 + 0.48 + 0.5 + 0.48 + 0.42 + 0.32 + 0.18 + 0)(0.1) \\ &= 0.33. \end{aligned}$$

(b) Here, we partition the interval $[0, 1]$ into 20 subintervals, each of width

$$\Delta x = \frac{1-0}{20} = \frac{1}{20} = 0.05.$$

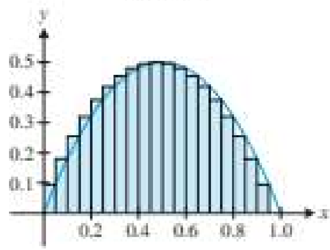


FIGURE 1.9
$A \approx A_{20}$

We then have $x_0 = 0$, $x_1 = 0 + \Delta x = 0.05$, $x_2 = x_1 + \Delta x = 2(0.05)$ and so on, so that $x_i = (0.05)i$, for $i = 0, 1, 2, \ldots, 20$. From (3.1), the area is then approximately

$$\begin{aligned} A \approx A_{20} &= \sum_{i=1}^{20} f(x_i)\,\Delta x = \sum_{i=1}^{20} (2x_i - 2x_i^2)\,\Delta x \\ &= \sum_{i=1}^{20} 2[0.05i - (0.05i)^2](0.05) = 0.3325, \end{aligned}$$

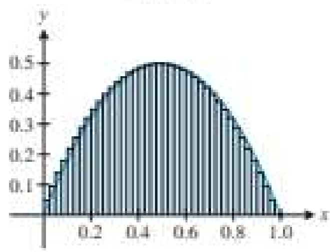


FIGURE 1.10
$A \approx A_{40}$

where the details of the calculation are left for the reader. Figure 1.9 shows an approximation using 20 rectangles and in Figure 1.10, we see 40 rectangles.

Based on Figures 1.8–1.10, you should expect that the larger we make n, the better A_n will approximate the actual area, A. The obvious drawback to this idea is the length of time it would take to compute A_n, for n large. However, your CAS or programmable calculator can compute these sums for you, with ease. The table shown in the margin indicates approximate values of A_n for various values of n.

n	A_n
10	0.33
20	0.3325
30	0.332963
40	0.333125
50	0.3332
60	0.333241
70	0.333265
80	0.333281
90	0.333292
100	0.3333

Notice that as n gets larger and larger, A_n seems to be approaching $\frac{1}{3}$. ■

Example 3.1 gives strong evidence that the larger the number of rectangles we use, the better our approximation of the area becomes. Thinking this through, we arrive at the following definition of the area under a curve.

DEFINITION 3.1

For a function f defined on the interval $[a, b]$, if f is continuous on $[a, b]$ and $f(x) \geq 0$ on $[a, b]$, the **area A under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x. \tag{3.2}$$

In example 3.2, we use the limit defined in (3.2) to find the exact area under the curve from Example 3.1.

EXAMPLE 3.2 Computing the Area Exactly

Find the area under the curve $y = f(x) = 2x - 2x^2$ on the interval $[0, 1]$.

Solution Here, using n subintervals, we have

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

and so, $x_0 = 0$, $x_1 = \frac{1}{n}$, $x_2 = x_1 + \Delta x = \frac{2}{n}$ and so on. Then, $x_i = \frac{i}{n}$, for $i = 0, 1, 2, \ldots, n$. From (3.1), the area is approximately

$$\begin{aligned}
A \approx A_n &= \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\frac{i}{n} - 2\left(\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right) \\
&= \sum_{i=1}^{n}\left[2\left(\frac{i}{n}\right)\left(\frac{1}{n}\right)\right] - \sum_{i=1}^{n}\left[2\left(\frac{i^2}{n^2}\right)\left(\frac{1}{n}\right)\right] \\
&= \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{2}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{2}{n^2}\frac{n(n+1)}{2} - \frac{2}{n^3}\frac{n(n+1)(2n+1)}{6} \quad \text{From Theorem 2.1 (ii) and (iii).} \\
&= \frac{n+1}{n} - \frac{(n+1)(2n+1)}{3n^2} \\
&= \frac{(n+1)(n-1)}{3n^2}.
\end{aligned}$$

Since we have a formula for A_n, for any n, we can compute various values with ease. We have

$$A_{200} = \frac{(201)(199)}{3(40{,}000)} = 0.333325,$$
$$A_{500} = \frac{(501)(499)}{3(250{,}000)} = 0.333332$$

and so on. Finally, we can compute the limiting value of A_n explicitly. We have

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{n^2-1}{3n^2} = \lim_{n\to\infty} \frac{1 - 1/n^2}{3} = \frac{1}{3}.$$

Therefore, the exact area in Figure 1.8 is 1/3, as we had suspected. ■

EXAMPLE 3.3 Estimating the Area Under a Curve

Estimate the area under the curve $y = f(x) = \sqrt{x+1}$ on the interval $[1, 3]$.

Solution Here, we have

$$\Delta x = \frac{3-1}{n} = \frac{2}{n}$$

and $x_0 = 1$, so that

$$x_1 = x_0 + \Delta x = 1 + \frac{2}{n},$$

$$x_2 = 1 + 2\left(\frac{2}{n}\right)$$

and so on, so that

$$x_i = 1 + \frac{2i}{n}, \quad \text{for } i = 0, 1, 2, \ldots, n.$$

Thus, we have from (3.1) that

n	A_n
10	3.50595
50	3.45942
100	3.45357
500	3.44889
1000	3.44830
5000	3.44783

$$\begin{aligned} A \approx A_n &= \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \sqrt{x_i + 1}\,\Delta x \\ &= \sum_{i=1}^{n} \sqrt{\left(1 + \frac{2i}{n}\right) + 1}\left(\frac{2}{n}\right) \\ &= \frac{2}{n}\sum_{i=1}^{n}\sqrt{2 + \frac{2i}{n}}. \end{aligned}$$

We have no formulas like those in Theorem 2.1 for simplifying this last sum (unlike the sum in example 3.2). Our only choice, then, is to compute A_n for a number of values of n using a CAS or programmable calculator. The table shown in the margin lists approximate values of A_n. Although we can't compute the area exactly (as yet), you should get the sense that the area is approximately 3.4478. ■

We pause now to define some of the mathematical objects we have been examining.

HISTORICAL NOTES

Bernhard Riemann (1826–1866)

A German mathematician who made important generalizations to the definition of the integral. Riemann died at a young age without publishing many papers, but each of his papers was highly influential. His work on integration was a small portion of a paper on Fourier series. Pressured by Gauss to deliver a talk on geometry, Riemann developed his own geometry, which provided a generalization of both Euclidean and non-Euclidean geometry. Riemann's work often formed unexpected and insightful connections between analysis and geometry.

DEFINITION 3.2

Let $\{x_0, x_1, \ldots, x_n\}$ be a regular partition of the interval $[a, b]$, with $x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$, for all i. Pick points $c_1, c_2, \ldots, c_n$, where c_i is any point in the subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$. (These are called **evaluation points.**) The **Riemann sum** for this partition and set of evaluation points is

$$\sum_{i=1}^{n} f(c_i)\,\Delta x.$$

So far, we have seen that for a continuous, non-negative function f, the area under the curve $y = f(x)$ is the limit of the Riemann sums:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x, \tag{3.3}$$

where $c_i = x_i$, for $i = 1, 2, \ldots, n$. Surprisingly, for any continuous function f, the limit in (3.3) is the same for *any* choice of the evaluation points $c_i \in [x_{i-1}, x_i]$ (although the proof is beyond the level of this course). In examples 3.2 and 3.3, we used the evaluation points $c_i = x_i$, for each i (the right endpoint of each subinterval). This is usually the most convenient choice when working by hand, but does *not* generally produce the most accurate approximation for a given value of n.

REMARK 3.1

Most often, we *cannot* compute the limit of Riemann sums indicated in (3.3) exactly (at least not directly). However, we can *always* obtain an approximation to the area by calculating Riemann sums for some large values of n. The most common (and obvious) choices for the evaluation points c_i are x_i (the right endpoint), x_{i-1} (the left endpoint) and $\frac{1}{2}(x_{i-1} + x_i)$ (the midpoint). See Figures 1.11a, 1.11b and 4.11c for the right endpoint, left endpoint and midpoint approximations, respectively, for $f(x) = 9x^2 + 2$, on the interval [0, 1], using $n = 10$. You should note that in this case (as with any increasing function), the rectangles corresponding to the right endpoint evaluation (Figure 1.11a) give too much area on each subinterval, while the rectangles corresponding to left endpoint evaluation (Figure 1.11b) give too little area. We leave it to you to observe that the reverse is true for a decreasing function.

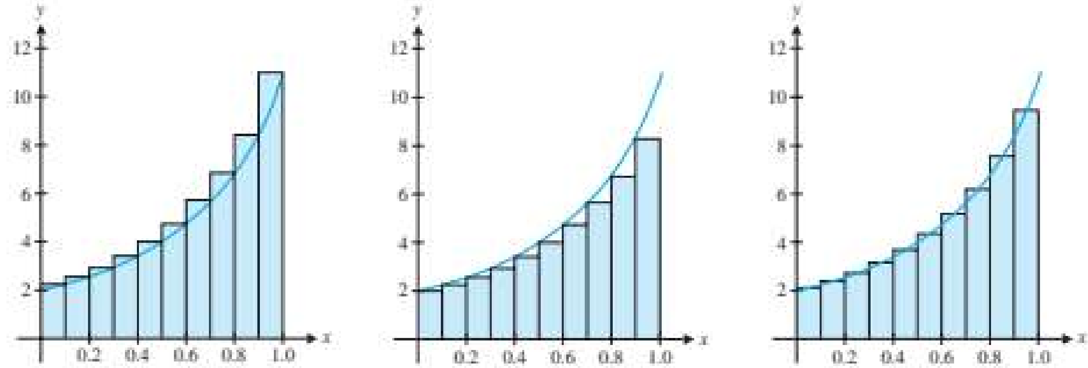


FIGURE 1.11a $c_i = x_i$

FIGURE 1.11b $c_i = x_{i-1}$

FIGURE 1.11c $c_i = \frac{1}{2}(x_{i-1} + x_i)$

EXAMPLE 3.4 Computing Riemann Sums with Different Evaluation Points

Compute Riemann sums for $f(x) = \sqrt{x+1}$ on the interval [1, 3], for $n = 10, 50, 100, 500, 1000$ and 5000, using the left endpoint, right endpoint and midpoint of each subinterval as the evaluation points.

Solution The numbers given in the following table are from a program written for a programmable calculator. We suggest that you test your own program or one built into your CAS against these values (rounded off to six digits).

n	*Left Endpoint*	*Midpoint*	*Right Endpoint*
10	3.38879	3.44789	3.50595
50	3.43599	3.44772	3.45942
100	3.44185	3.44772	3.45357
500	3.44654	3.44772	3.44889
1000	3.44713	3.44772	3.44830
5000	3.44760	3.44772	3.44783

There are several conclusions to be drawn from these numbers. First, there is good evidence that all three sets of numbers are converging to a common limit of approximately 3.4477. Second, even though the limits are the same, the different rules approach the limit at different rates. You should try computing left and right endpoint sums for larger values of n, to see that these eventually approach 3.44772, also. ■

TODAY IN MATHEMATICS

Louis de Branges (1932–Present)
A French-American mathematician who proved the Bieberbach conjecture in 1985. To solve this famous 70-year-old problem, de Branges actually proved a related but much stronger result. In 2004, de Branges posted on the Internet what he believed is a proof of another famous problem, the Riemann hypothesis. To qualify for the $1 million prize offered for the first proof of the Riemann hypothesis, the result will have to be verified by expert mathematicians.

Riemann sums using midpoint evaluation are usually more accurate than left or right endpoint rules for a given n. If you think about the corresponding rectangles, you may be able to explain why. Finally, notice that the left and right endpoint sums in example 3.4 approach the limit from opposite directions and at about the same rate.

BEYOND FORMULAS

We have now developed a technique for using limits to compute certain areas exactly. This parallels the derivation of the slope of the tangent line as the limit of the slopes of secant lines. Recall that this limit became known as the derivative and turned out to have applications far beyond the slope of a tangent line. Similarly, Riemann sums lead us to a second major area of calculus, called integration. Based on your experience with the derivative, do you expect this new limit to solve problems beyond the area of a region? Do you expect that there will be rules developed to simplify the calculations?

EXERCISES 1.3

WRITING EXERCISES

1. For many functions, the limit of the Riemann sums is independent of the choice of evaluation points. As the number of partition points gets larger, the distance between the endpoints gets smaller. For a continuous function $f(x)$, explain why the difference between the function values at any two points in a given subinterval will have to get smaller.

2. Rectangles are not the only basic geometric shapes for which we have an area formula. Discuss how you might approximate the area under a parabola using circles or triangles. Which geometric shape do you think is the easiest to use?

In exercises 1–4, list the evaluation points corresponding to the midpoint of each subinterval, sketch the function and approximating rectangles and evaluate the Riemann sum.

1. $f(x) = x^2 + 1$, (a) $[0, 1]$, $n = 4$; (b) $[0, 2]$, $n = 4$
2. $f(x) = x^3 - 1$, (a) $[1, 2]$, $n = 4$; (b) $[1, 3]$, $n = 4$
3. $f(x) = \sin x$, (a) $[0, \pi]$, $n = 4$; (b) $[0, \pi]$, $n = 8$
4. $f(x) = 4 - x^2$, (a) $[-1, 1]$, $n = 4$; (b) $[-3, -1]$, $n = 4$

In exercises 5–10, approximate the area under the curve on the given interval using n rectangles and the evaluation rules (a) left endpoint, (b) midpoint, (c) right endpoint.

5. $y = x^2 + 1$ on $[0, 1]$, $n = 16$
6. $y = x^2 + 1$ on $[0, 2]$, $n = 16$
7. $y = \sqrt{x + 2}$ on $[1, 4]$, $n = 16$
8. $y = e^{-2x}$ on $[-1, 1]$, $n = 16$
9. $y = \cos x$ on $[0, \pi/2]$, $n = 50$
10. $y = x^3 - 1$ on $[-1, 1]$, $n = 100$

In exercises 11–14, use Riemann sums and a limit to compute the exact area under the curve.

11. $y = x^2 + 1$ on (a) $[0, 1]$; (b) $[0, 2]$; (c) $[1, 3]$
12. $y = x^2 + 3x$ on (a) $[0, 1]$; (b) $[0, 2]$; (c) $[1, 3]$
13. $y = 2x^2 + 1$ on (a) $[0, 1]$; (b) $[-1, 1]$; (c) $[1, 3]$
14. $y = 4x^2 - x$ on (a) $[0, 1]$; (b) $[-1, 1]$; (c) $[1, 3]$

In exercises 15–18, construct a table of Riemann sums as in example 3.4 to show that sums with right-endpoint, midpoint and left-endpoint evaluation all converge to the same value as $n \to \infty$.

15. $f(x) = 4 - x^2$, $[-2, 2]$
16. $f(x) = \sin x$, $[0, \pi/2]$
17. $f(x) = x^3 - 1$, $[1, 3]$
18. $f(x) = x^3 - 1$, $[-1, 1]$

In exercises 19–22, graphically determine whether a Riemann sum with (a) left-endpoint, (b) midpoint and (c) right-endpoint evaluation points will be greater than or less than the area under the curve $y = f(x)$ on $[a, b]$.

19. $f(x)$ is increasing and concave up on $[a, b]$.
20. $f(x)$ is increasing and concave down on $[a, b]$.
21. $f(x)$ is decreasing and concave up on $[a, b]$.
22. $f(x)$ is decreasing and concave down on $[a, b]$.

23. For the function $f(x) = x^2$ on the interval $[0, 1]$, by trial and error find evaluation points for $n = 2$ such that the Riemann sum equals the exact area of 1/3.

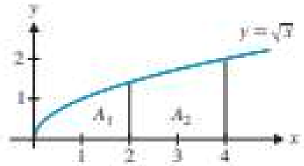
24. For the function $f(x) = \sqrt{x}$ on the interval $[0, 1]$, by trial and error find evaluation points for $n = 2$ such that the Riemann sum equals the exact area of 2/3.

25. (a) Show that for right-endpoint evaluation on the interval $[a, b]$ with each subinterval of length $\Delta x = (b - a)/n$, the

evaluation points are $c_i = a + i\Delta x$, for $i = 1, 2, \dots, n$. (b) Find a formula for the evaluation points for midpoint evaluation.

26. (a) Show that for left-endpoint evaluation on the interval $[a, b]$ with each subinterval of length $\Delta x = (b - a)/n$, the evaluation points are $c_i = a + (i - 1)\Delta x$, for $i = 1, 2, \dots, n$. (b) Find a formula for evaluation points that are one-third of the way from the left endpoint to the right endpoint.

27. In the figure, which area equals $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{2}\,\sqrt{1 + i/n}\,\frac{2}{n}$?



28. Which area equals $\lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{\frac{1}{n}}\,\sqrt{1 + 2i}\,\frac{2}{n}$?

In exercises 29–32, use the following definitions. The upper sum of f on P is given by $U(P, f) = \sum_{i=1}^{n} f(c_i)\,\Delta x$, where $f(c_i)$ is the maximum of f on the subinterval $[x_{i-1}, x_i]$. Similarly, the lower sum of f on P is given by $L(P, f) = \sum_{i=1}^{n} f(d_i)\,\Delta x$, where $f(d_i)$ is the minimum of f on the subinterval $[x_{i-1}, x_i]$.

29. Compute the upper sum and lower sum of $f(x) = x^2$ on $[0, 2]$ for the regular partition with $n = 4$.

30. Compute the upper sum and lower sum of $f(x) = x^2$ on $[-2, 2]$ for the regular partition with $n = 8$.

31. Find (a) the general upper sum and (b) the general lower sum for $f(x)^2$ on $[0, 2]$ and show that both sums approach the same number as $n \to \infty$.

32. Repeat exercise 31 $f(x) = x^3 + 1$ on the interval $[0, 2]$.

33. The following result has been credited to Archimedes. (See the historical note in section 5.2.) For the general parabola $y = a^2 - x^2$ with $-a \le x \le a$, show that the area under the parabola is $\frac{2}{3}$ of the base times the height [that is, $\frac{2}{3}(2a)(a^2)$].

34. Show that the area under $y = ax^2$ for $0 \le x \le b$ is $\frac{1}{3}$ of the base times the height.

In exercises 35–38, use the given function values to estimate the area under the curve using left-endpoint and right-endpoint evaluation.

35.

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$f(x)$	2.0	2.4	2.6	2.7	2.6	2.4	2.0	1.4	0.6

36.

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$f(x)$	2.0	2.2	1.6	1.4	1.6	2.0	2.2	2.4	2.0

37.

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
$f(x)$	1.8	1.4	1.1	0.7	1.2	1.4	1.8	2.4	2.6

38.

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6
$f(x)$	0.0	0.4	0.6	0.8	1.2	1.4	1.2	1.4	1.0

APPLICATIONS

1. Economists use a graph called the **Lorentz curve** to describe how equally a given quantity is distributed in a given population. For example, the **gross domestic product** (GDP) varies considerably from country to country. The accompanying data from the Energy Information Administration show percentages for the 100 top GDP countries in the world in 2001, arranged in order of increasing GDP. The data indicate that the first 10 (lowest 10%) countries account for only 0.2% of the world's total GDP; the first 20 countries account for 0.4% and so on. The first 99 countries account for 73.6% of the total GDP. What percentage does country #100 (the United States) produce? The Lorentz curve is a plot of y versus x. Graph the Lorentz curve for these data. Estimate the area between the curve and the x-axis. (Hint: Notice that the x-values are not equally spaced. You will need to decide how to handle this.)

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7
y	0.002	0.004	0.008	0.014	0.026	0.048	0.085

x	0.8	0.9	0.95	0.98	0.99	1.0
y	0.144	0.265	0.398	0.568	0.736	1.0

2. The Lorentz curve (see exercise 1) can be used to compute the **Gini index,** a numerical measure of how inequitable a given distribution is. Let A_1 equal the area between the Lorentz curve and the x-axis. Construct the Lorentz curve for the situation of all countries being exactly equal in GDP and let A_2 be the area between this new Lorentz curve and the x-axis. The Gini index G equals A_1 divided by A_2. Explain why $0 \le G \le 1$ and show that $G = 2A_1$. Estimate G for the data in exercise 1.

EXPLORATORY EXERCISES

1. Riemann sums can also be defined on **irregular partitions,** for which subintervals are not of equal size. An example of an irregular partition of the interval $[0, 1]$ is $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.6$, $x_3 = 0.9$, $x_4 = 1$. Explain why the corresponding Riemann sum would be

$$f(c_1)(0.2) + f(c_2)(0.4) + f(c_3)(0.3) + f(c_4)(0.1),$$

for evaluation points c_1, c_2, c_3 and c_4. Identify the interval from which each c_i must be chosen and give examples of evaluation points. To see why irregular partitions might be useful, consider the function $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2 + 1 & \text{if } x \ge 1 \end{cases}$ on the interval

[0, 2]. One way to approximate the area under the graph of this function is to compute Riemann sums using midpoint evaluation for $n = 10$, $n = 50$, $n = 100$ and so on. Show graphically and numerically that with midpoint evaluation, the Riemann sum with $n = 2$ gives the correct area on the subinterval [0, 1]. Then explain why it would be wasteful to compute Riemann sums on this subinterval for larger and larger values of n. A more efficient strategy would be to compute the areas on [0, 1] and [1, 2] separately and add them together. The area on [0, 1] can be computed exactly using a small value of n, while the area on [1, 2] must be approximated using larger and larger values of n. Use this technique to estimate the area for $f(x)$ on the interval [0, 2]. Try to determine the area to within an error of 0.01 and discuss why you believe your answer is this accurate.

2. Graph the function $f(x) = e^{-x^2}$. You may recognize this curve as the so-called "bell curve," which is of fundamental importance in statistics. We define the **area function** $g(t)$ to be the area between this graph and the x-axis between $x = 0$ and $x = t$ (for now, assume that $t > 0$). Sketch the area that defines $g(1)$ and $g(2)$ and argue that $g(2) > g(1)$. Explain why the function $g(x)$ is increasing and hence $g'(x) > 0$ for $x > 0$. Further, argue that $g'(2) < g'(1)$. Explain why $g'(x)$ is a decreasing function. Thus, $g'(x)$ has the same general properties (positive, decreasing) that $f(x)$ does. In fact, we will discover in section 1.5 that $g'(x) = f(x)$. To collect some evidence for this result, use Riemann sums to estimate $g(2)$, $g(1.1)$, $g(1.01)$ and $g(1)$. Use these values to estimate $g'(1)$ and compare to $f(1)$.

1.4 THE DEFINITE INTEGRAL

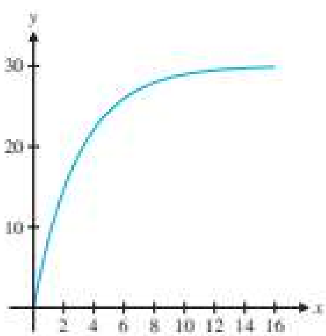


FIGURE 1.12
$y = f(x)$

A sky diver who steps out of an airplane (starting with zero downward velocity) gradually picks up speed until reaching *terminal velocity*, the speed at which the force due to air resistance cancels out the force due to gravity. A function that models the velocity x seconds into the jump is $f(x) = 30(1 - e^{-x/3})$. (See Figure 1.12.)

We saw in section 1.2 that the area A under this curve on the interval $0 \leq x \leq t$ corresponds to the distance fallen in the first t seconds. For any given value of t, the area is given by the limit of the Riemann sums,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x, \tag{4.1}$$

where for each i, c_i is taken to be any point in the subinterval $[x_{i-1}, x_i]$. Notice that the sum in (4.1) still makes sense even when some (or all) of the function values $f(c_i)$ are negative. The general definition follows.

REMARK 4.1

Definition 4.1 is adequate for most functions (those that are continuous except for at most a finite number of discontinuities). For more general functions, we broaden the definition to include partitions with subintervals of different lengths.

DEFINITION 4.1

For any function f defined on $[a, b]$, the **definite integral** of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x,$$

whenever the limit exists and is the same for every choice of evaluation points, $c_1, c_2, \ldots, c_n$. When the limit exists, we say that f is **integrable** on $[a, b]$.

We should observe that in the Riemann sum, the Greek letter Σ indicates a sum; so does the elongated "S", with $\int$ used as the integral sign. The **lower** and **upper limits of integration**, a and b, respectively, indicate the endpoints of the interval over which you are integrating. The dx in the integral corresponds to the increment Δx in the Riemann sum and also indicates the variable of integration. The letter used for the variable of integration (called a dummy variable) is irrelevant since the value of the integral is a constant and not a function of x. Here, $f(x)$ is called the **integrand**.

So, when will the limit defining a definite integral exist? Theorem 4.1 indicates that many familiar functions are integrable.

NOTES

If f is continuous on $[a, b]$ and $f(x) \geq 0$ on $[a, b]$, then

$$\int_a^b f(x)dx = \text{Area under the curve} \geq 0.$$

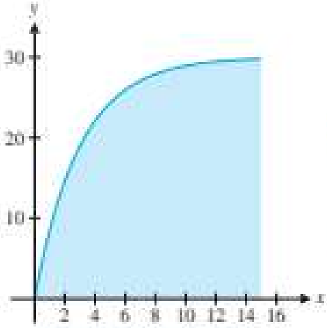


FIGURE 1.13
$y = 30(1 - e^{-x/3})$

n	R_n
10	361.5
20	360.8
50	360.6
100	360.6

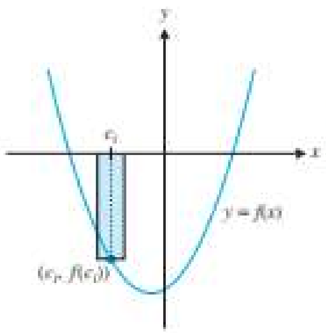


FIGURE 1.14
$f(c_i) < 0$

THEOREM 4.1

If f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$.

The proof of Theorem 4.1 is too technical to include here. However, if you think about the area interpretation of the definite integral, the result should seem plausible.

To calculate a definite integral of an integrable function, we have two options: if the function is simple enough (say, a polynomial of degree 2 or less) we can symbolically compute the limit of the Riemann sums. Otherwise, we can numerically compute a number of Riemann sums and approximate the value of the limit. We frequently use the **Midpoint Rule**, which uses the midpoints as the evaluation points for the Riemann sum.

EXAMPLE 4.1 A Midpoint Rule Approximation of a Definite Integral

Use the Midpoint Rule to estimate $\int_0^{15} 30(1 - e^{-x/3})\, dx$.

Solution The integral gives the area under the curve indicated in Figure 1.13. (Note that this corresponds to the distance fallen by the sky diver in this section's introduction.) From the Midpoint Rule we have

$$\int_0^{15} 30(1 - e^{-x/3})\, dx \approx \sum_{i=1}^{n} f(c_i)\, \Delta x = 30 \sum_{i=1}^{n} (1 - e^{-c_i/3}) \left(\frac{15 - 0}{n}\right),$$

where $c_i = \dfrac{x_i + x_{i-1}}{2}$. Using a CAS or a calculator program, you can get the sequence of approximations found in the accompanying table.

One remaining question is when to stop increasing n. In this case, we continued to increase n until it seemed clear that 361 meters was a reasonable approximation. ■

Now, think carefully about the limit in Definition 4.1. How can we interpret this limit when f is both positive and negative on the interval $[a, b]$? Notice that if $f(c_i) < 0$, for some i, then the height of the rectangle shown in Figure 1.14 is $-f(c_i)$ and so,

$$f(c_i)\, \Delta x = -\text{Area of the } i\text{th rectangle}.$$

To see the effect this has on the sum, consider example 4.2.

EXAMPLE 4.2 A Riemann Sum for a Function with Positive and Negative Values

For $f(x) = \sin x$ on $[0, 2\pi]$, give an area interpretation of $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(c_i)\, \Delta x$.

Solution For this illustration, we take c_i to be the midpoint of $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$. In Figure 1.15a, we see 10 rectangles constructed between the x-axis and the curve $y = f(x)$.

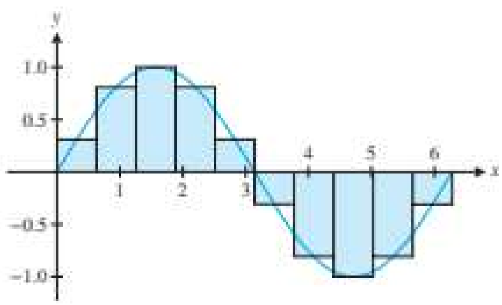


FIGURE 1.15a
Ten rectangles

The first five rectangles [where $f(c_i) > 0$] lie above the x-axis and have height $f(c_i)$. The remaining five rectangles [where $f(c_i) < 0$] lie below the x-axis and have height $-f(c_i)$. So, here

$$\sum_{i=1}^{10} f(c_i)\,\Delta x = (\text{Area of rectangles above the } x\text{-axis}) \\ -(\text{Area of rectangles below the } x\text{-axis}).$$

In Figures 1.15b and 1.15c, we show 20 and 40 rectangles, respectively, constructed in the same way. From this, observe that

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x = (\text{Area above the } x\text{-axis}) - (\text{Area below the } x\text{-axis}),$$

which turns out to be zero, in this case.

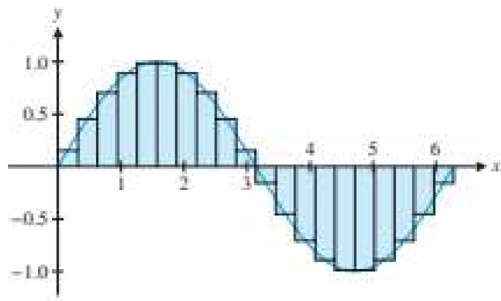


FIGURE 1.15b
Twenty rectangles

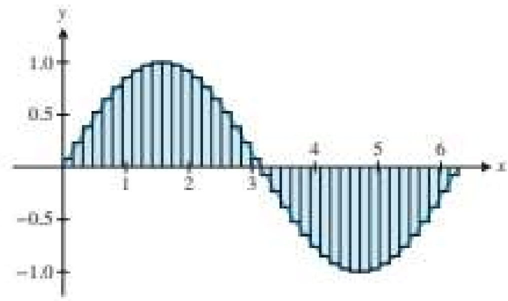


FIGURE 1.15c
Forty rectangles

■

More generally, we have the notion of **signed area**, which we now define.

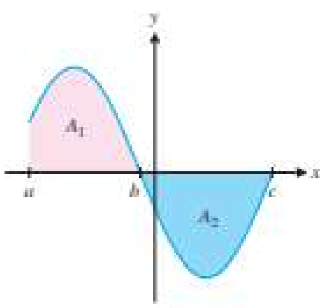


FIGURE 1.16
Signed area

DEFINITION 4.2

Suppose that $f(x) \geq 0$ on the interval $[a, b]$ and A_1 is the area bounded between the curve $y = f(x)$ and the x-axis for $a \leq x \leq b$. Further, suppose that $f(x) \leq 0$ on the interval $[b, c]$ and A_2 is the area bounded between the curve $y = f(x)$ and the x-axis for $b \leq x \leq c$. The **signed area** between $y = f(x)$ and the x-axis for $a \leq x \leq c$ is $A_1 - A_2$, and the **total area** between $y = f(x)$ and the x-axis for $a \leq x \leq c$ is $A_1 + A_2$. (See Figure 1.16.)

Definition 4.2 says that the signed area is the difference between any areas lying above the x-axis and any areas lying below the x-axis, while the total area is the sum total of the area bounded between the curve $y = f(x)$ and the x-axis.

Example 4.3 examines the general case where the integrand may be both positive and negative on the interval of integration.

EXAMPLE 4.3 Relating Definite Integrals to Signed Area

Compute the integrals: (a) $\int_0^2 (x^2 - 2x)\,dx$ and (b) $\int_0^3 (x^2 - 2x)\,dx$, and interpret each in terms of area.

Solution First, note that the integrand is continuous everywhere and so, it is also integrable on any interval. (a) The definite integral is the limit of a sequence of

Riemann sums, where we can choose any evaluation points we wish. It is usually easiest to write out the formula using right endpoints, as we do here. In this case,

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}.$$

We then have $x_0 = 0$, $x_1 = x_0 + \Delta x = \frac{2}{n}$,

$$x_2 = x_1 + \Delta x = \frac{2}{n} + \frac{2}{n} = \frac{2(2)}{n}$$

and so on. We then have $c_i = x_i = \frac{2i}{n}$. The nth Riemann sum R_n is then

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}(x_i^2 - 2x_i)\Delta x \\ &= \sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2 - 2\left(\frac{2i}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left(\frac{4i^2}{n^2} - \frac{4i}{n}\right)\left(\frac{2}{n}\right) \\ &= \frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{8}{n^2}\sum_{i=1}^{n} i \\ &= \left(\frac{8}{n^3}\right)\frac{n(n+1)(2n+1)}{6} - \left(\frac{8}{n^2}\right)\frac{n(n+1)}{2} \qquad \text{From Theorem 2.1 (ii) and (iii).} \\ &= \frac{4(n+1)(2n+1)}{3n^2} - \frac{4(n+1)}{n} = \frac{8n^2+12n+4}{3n^2} - \frac{4n+4}{n}. \end{aligned}$$

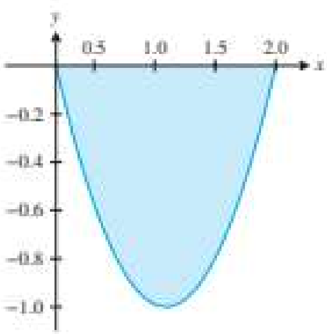


FIGURE 1.17
$y = x^2 - 2x$ on $[0, 2]$

Taking the limit of R_n as $n \to \infty$ gives us the exact value of the integral:

$$\int_0^2 (x^2 - 2x)\,dx = \lim_{n\to\infty}\left(\frac{8n^2+12n+4}{3n^2} - \frac{4n+4}{n}\right) = \frac{8}{3} - 4 = -\frac{4}{3}.$$

A graph of $y = x^2 - 2x$ on the interval [0, 2] is shown in Figure 1.17. Notice that since the function is always negative on the interval [0, 2], the integral is negative and equals −A, where A is the area lying between the x-axis and the curve.

(b) On the interval [0, 3], we have $\Delta x = \frac{3}{n}$ and $x_0 = 0$, $x_1 = x_0 + \Delta x = \frac{3}{n}$,

$$x_2 = x_1 + \Delta x = \frac{3}{n} + \frac{3}{n} = \frac{3(2)}{n}$$

and so on. Using right-endpoint evaluation, we have $c_i = x_i = \frac{3i}{n}$. This gives us the Riemann sum

$$\begin{aligned} R_n &= \sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^2 - 2\left(\frac{3i}{n}\right)\right]\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left(\frac{9i^2}{n^2} - \frac{6i}{n}\right)\left(\frac{3}{n}\right) \\ &= \frac{27}{n^3}\sum_{i=1}^{n} i^2 - \frac{18}{n^2}\sum_{i=1}^{n} i \\ &= \left(\frac{27}{n^3}\right)\frac{n(n+1)(2n+1)}{6} - \left(\frac{18}{n^2}\right)\frac{n(n+1)}{2} \qquad \text{From Theorem 2.1(ii) and (iii)} \\ &= \frac{9(n+1)(2n+1)}{2n^2} - \frac{9(n+1)}{n}. \end{aligned}$$

Taking the limit as $n \to \infty$ gives us

$$\int_0^3 (x^2 - 2x)\,dx = \lim_{n\to\infty}\left[\frac{9(n+1)(2n+1)}{2n^2} - \frac{9(n+1)}{n}\right] = \frac{18}{2} - 9 = 0.$$

On the interval [0, 2], notice that the curve $y = x^2 - 2x$ lies below the x-axis and the area bounded between the curve and the x-axis is $\frac{4}{3}$. On the interval [2, 3], the curve lies above the x-axis and so, the integral of 0 on the interval [0, 3] indicates that the signed areas have canceled out one another. (See Figure 1.18 for a graph of $y = x^2 - 2x$ on the

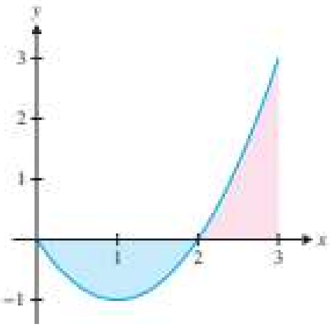


FIGURE 1.18
$y = x^2 - 2x$ on $[0, 3]$

interval [0, 3].) Note that this also says that the area under the curve on the interval [2, 3] must be $\frac{4}{3}$. You should also observe that the total area A bounded between $y = x^2 - 2x$ and the x-axis is the sum of the two areas indicated in Figure 1.18, $A = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$. ■

We can also interpret signed area in terms of velocity and position. Suppose that $v(t)$ is the velocity function for an object moving back and forth along a straight line. Notice that the velocity may be both positive and negative. If the velocity is positive on the interval $[t_1, t_2]$, then $\int_{t_1}^{t_2} v(t)\,dt$ gives the distance traveled (here, in the positive direction). If the velocity is negative on the interval $[t_3, t_4]$, then the object is moving in the negative direction and the distance traveled (here, in the negative direction) is given by $-\int_{t_3}^{t_4} v(t)\,dt$. Notice that if the object starts moving at time 0 and stops at time T, then $\int_0^T v(t)\,dt$ gives the distance traveled in the positive direction minus the distance traveled in the negative direction. That is, $\int_0^T v(t)\,dt$ corresponds to the *overall change* in position from start to finish.

EXAMPLE 4.4 Estimating Overall Change in Position

An object moving along a straight line has velocity function $v(t) = \sin t$. If the object starts at position 0, determine the total distance traveled and the object's position at time $t = 3\pi/2$.

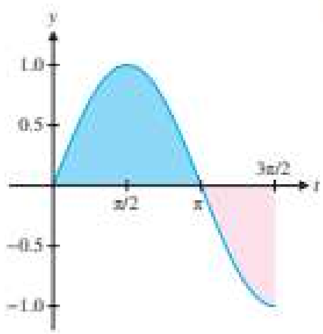


FIGURE 1.19
$y = \sin t$ on $\left[0, \frac{3\pi}{2}\right]$

Solution From the graph (see Figure 1.19), notice that $\sin t \geq 0$ for $0 \leq t \leq \pi$ and $\sin t \leq 0$ for $\pi \leq t \leq 3\pi/2$. The total distance traveled corresponds to the area of the shaded regions in Figure 1.19, given by

$$A = \int_0^{\pi} \sin t\,dt - \int_{\pi}^{3\pi/2} \sin t\,dt.$$

You can use the Midpoint Rule to get the following Riemann sums:

n	$R_n \approx \int_0^{\pi} \sin t\,dt$
10	2.0082
20	2.0020
50	2.0003
100	2.0001

n	$R_n \approx \int_{\pi}^{3\pi/2} \sin t\,dt$
10	−1.0010
20	−1.0003
50	−1.0000
100	−1.0000

Observe that the sums appear to be converging to 2 and −1, respectively, which we will soon be able to show are indeed correct. The total area bounded between $y = \sin t$ and the t-axis on $\left[0, \frac{3\pi}{2}\right]$ is then

$$\int_0^{\pi} \sin t\,dt - \int_{\pi}^{3\pi/2} \sin t\,dt = 2 + 1 = 3,$$

so that the total distance traveled is 3 units. The overall change in position of the object is given by

$$\int_0^{3\pi/2} \sin t\,dt = \int_0^{\pi} \sin t\,dt + \int_{\pi}^{3\pi/2} \sin t\,dt = 2 + (-1) = 1.$$

So, if the object starts at position 0, it ends up at position $0 + 1 = 1$. ■

Next, we give some general rules for integrals.

THEOREM 4.2

If f and g are integrable on $[a, b]$, then the following are true.

(i) For any constants c and d, $\int_a^b [cf(x) + dg(x)]\,dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx$ and

(ii) For any c in $[a, b]$, $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

PROOF

By definition, for any constants c and d, we have

$$\begin{aligned}\int_a^b [cf(x) + dg(x)]\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} [cf(c_i) + dg(c_i)]\,\Delta x \\ &= \lim_{n\to\infty} \left[c\sum_{i=1}^{n} f(c_i)\,\Delta x + d\sum_{i=1}^{n} g(c_i)\,\Delta x\right] \quad \text{From Theorem 2.2.} \\ &= c\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x + d\lim_{n\to\infty} \sum_{i=1}^{n} g(c_i)\,\Delta x \\ &= c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx,\end{aligned}$$

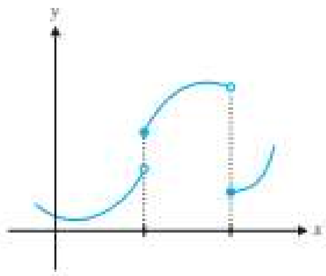


FIGURE 1.20
Piecewise continuous function

where we have used our usual rules for summations plus the fact that f and g are integrable. We leave the proof of part (ii) to the exercises, but note that we have already illustrated the idea in example 4.4 ■

We now make a pair of reasonable definitions. First, for any integrable function f, if $a < b$, we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx. \tag{4.2}$$

This should appear reasonable in that if we integrate "backward" along an interval, the width of the rectangles corresponding to a Riemann sum (Δx) would seem to be negative. Second, if $f(a)$ is defined, we define

$$\int_a^a f(x)\,dx = 0.$$

If you think of the definite integral as area, this says that the area from a up to a is zero.

It turns out that a function is integrable even when it has a finite number of jump discontinuities, but is otherwise continuous. (Such a function is called **piecewise continuous**; see Figure 1.20 for the graph of such a function.)

In example 4.5, we evaluate the integral of a discontinuous function.

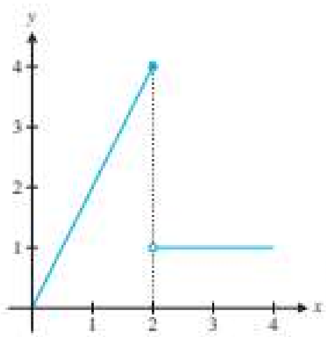


FIGURE 1.21a
$y = f(x)$

EXAMPLE 4.5 An Integral with a Discontinuous Integrand

Evaluate $\int_0^3 f(x)\,dx$, where $f(x)$ is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x \le 2 \\ 1, & \text{if } x > 2 \end{cases}.$$

Solution We start by looking at a graph of $y = f(x)$ in Figure 1.21a. Notice that although f is discontinuous at $x = 2$, it has only a single jump discontinuity and, so, is piecewise continuous on [0, 3]. By Theorem 4.2 (ii), we have that

$$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx.$$

Referring to Figure 1.21b, observe that $\int_0^2 f(x)\,dx$ corresponds to the area of the triangle of base 2 and altitude 4 shaded in the figure, so that

$$\int_0^2 f(x)\,dx = \frac{1}{2}(\text{base})(\text{height}) + \frac{1}{2}(2)(4) = 4.$$

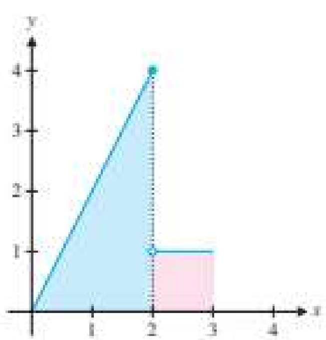


FIGURE 1.21b
The area under the curve $y = f(x)$ on [0, 3]

Next, also notice from Figure 1.21b that $\int_2^3 f(x)\,dx$ corresponds to the area of the square of side 1, so that

$$\int_2^3 f(x)\,dx = 1.$$

We now have that

$$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = 4 + 1 = 5.$$

Notice that in this case, the areas corresponding to the two integrals could be computed using simple geometric formulas and so, there was no need to compute Riemann sums here. ■

Another simple property of definite integrals is the following:

THEOREM 4.3

Suppose that $g(x) \le f(x)$ for all $x \in [a, b]$ and that f and g are integrable on $[a, b]$. Then,

$$\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx.$$

PROOF

Since $g(x) \le f(x)$, we must also have that $0 \le [f(x) - g(x)]$ on $[a, b]$ and in view of this, $\int_a^b [f(x) - g(x)]\,dx$ represents the area under the curve $y = f(x) - g(x)$, which can't be negative. Using Theorem 4.2 (i), we now have

$$0 \le \int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx,$$

from which the result follows. ■

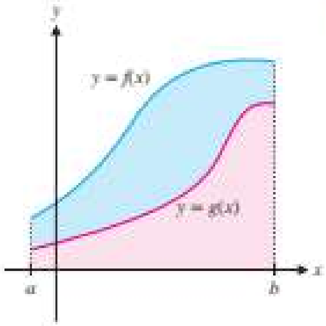


FIGURE 1.22
Larger functions have larger integrals

Notice that Theorem 4.3 simply says that larger functions have larger integrals. We illustrate this for the case of two positive functions in Figure 1.22.

Average Value of a Function

To compute the average age of the students in your calculus class, note that you need only add up each student's age and divide the total by the number of students in your class. By contrast, how would you find the average depth of a cross-section of a lake? You would get a reasonable idea of the average depth by sampling the depth of the lake at a number of points spread out along the length of the lake and then averaging these depths, as indicated in Figure 1.23.

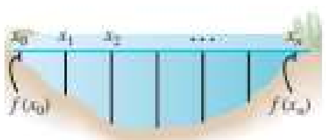


FIGURE 1.23
Average depth of a cross-section of a lake

More generally, we often want to calculate the average value of a function f on some interval $[a, b]$. To do this, we form a partition of $[a, b]$:

$$a = x_0 < x_1 < \cdots < x_n = b,$$

where the difference between successive points is $\Delta x = \dfrac{b-a}{n}$. The **average value**, f_{ave}, is then given approximately by the average of the function values at $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} f_{ave} &\approx \frac{1}{n}[f(x_1) + f(x_2) + \cdots + f(x_n)] \\ &= \frac{1}{n}\sum_{i=1}^{n} f(x_i) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right) && \text{Multiply and divide by } (b-a). \\ &= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\,\Delta x. && \text{Since } \Delta x = \frac{b-a}{n}. \end{aligned}$$

Notice that the last summation is a Riemann sum. Further, observe that the more points we sample, the better our approximation should be. So, letting $n \to \infty$, we arrive at an integral representing average value:

$$f_{ave} = \lim_{n\to\infty}\left[\frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\,\Delta x\right] = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{4.3}$$

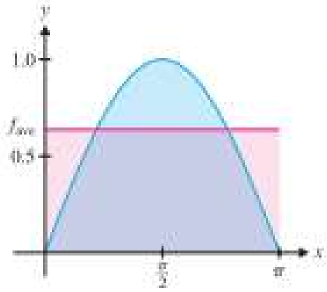


FIGURE 1.24
$y = \sin x$ and its average

EXAMPLE 4.6 Computing the Average Value of a Function

Compute the average value of $f(x) = \sin x$ on the interval $[0, \pi]$.

Solution From (4.3), we have

$$f_{ave} = \frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx.$$

We can approximate the value of this integral by calculating some Riemann sums, to obtain the approximate average, $f_{ave} \approx 0.6366198$. (See example 4.4.) In Figure 1.24, we show a graph of $y = \sin x$ and its average value on the interval $[0, \pi]$. You should note that the two shaded regions have the same area. ■

Notice in Figure 1.24 that there are two points at which the function equals its average value. We give a precise statement of this (unsurprising) result in Theorem 4.4. First, observe that for any constant, c,

$$\int_a^b c\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} c\,\Delta x = c\lim_{n\to\infty}\sum_{i=1}^{n}\Delta x = c(b-a),$$

since $\sum_{i=1}^{n}\Delta x$ is simply the sum of the lengths of the subintervals in the partition.

Let f be any continuous function defined on $[a, b]$. Recall that by the Extreme Value Theorem, since f is continuous, it has a minimum, m, and a maximum, M, on $[a, b]$, so that

$$m \le f(x) \le M, \qquad \text{for all } x \in [a, b]$$

and consequently, from Theorem 4.3,

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx.$$

Since m and M are constants, we get

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a). \tag{4.4}$$

Finally, dividing by $(b - a) > 0$, we obtain

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

That is, $\frac{1}{b-a}\int_a^b f(x)\,dx$ (the average value of f on $[a, b]$) lies between the minimum and the maximum values of f on $[a, b]$. Since f is a continuous function, we have by the Intermediate Value Theorem (Theorem 4.4 in section 1.4) that there must be some $c \in (a, b)$ for which

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

We have just proved a theorem:

THEOREM 4.4 (Integral Mean Value Theorem)

If f is continuous on $[a, b]$, then there is a number $c \in (a, b)$ for which

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The Integral Mean Value Theorem is a fairly simple idea (that a continuous function will take on its average value at some point), but it has some significant applications. The first of these will be found in section 1.5, in the proof of one of the most significant results in the calculus, the Fundamental Theorem of Calculus.

Referring back to our derivation of the Integral Mean Value Theorem, observe that along the way we proved that for any integrable function f, if $m \le f(x) \le M$, for all $x \in [a, b]$, then inequality (4.4) holds:

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

This enables us to estimate the value of a definite integral. Although the estimate is generally only a rough one, it still has importance in that it gives us an interval in which the value must lie. We illustrate this in example 4.7.

EXAMPLE 4.7 Estimating the Value of an Integral

Use inequality (4.4) to estimate the value of $\int_0^1 \sqrt{x^2+1}\,dx$.

Solution First, notice that it's beyond your present abilities to compute the value of this integral exactly. However, notice that

$$1 \le \sqrt{x^2+1} \le \sqrt{2}, \quad \text{for all } x \in [0, 1].$$

From inequality (4.4), we now have

$$1 \le \int_0^1 \sqrt{x^2+1}\,dx \le \sqrt{2} \approx 1.414214.$$

In other words, although we still do not know the exact value of the integral, we know that it must be between 1 and $\sqrt{2} \approx 1.414214$. ■

EXERCISES 1.4

WRITING EXERCISES

1. Sketch a graph of a function f that has both positive and negative values on an interval $[a, b]$. Explain in terms of area what it means to have $\int_a^b f(x)\,dx = 0$. Also, explain what it means to have $\int_a^b f(x)\,dx > 0$ and $\int_a^b f(x)\,dx < 0$.

2. To get a physical interpretation of the result in Theorem 4.3, suppose that $f(x)$ and $g(x)$ are velocity functions for two different objects starting at the same position. If $f(x) \ge g(x) \ge 0$, explain why it follows that $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

3. The Integral Mean Value Theorem says that if $f(x)$ is continuous on the interval $[a, b]$, then there exists a number c between a and b such that $f(c)(b-a) = \int_a^b f(x)\,dx$. By thinking of the left-hand side of this equation as the area of a rectangle, sketch a picture that illustrates this result, and explain why the result follows.

4. Write out the Integral Mean Value Theorem as applied to the derivative $f'(x)$. Then write out the Mean Value Theorem for derivatives (see section 2.10). If the c-values identified by each theorem are the same,

what does $\int_a^b f'(x)\,dx$ have to equal? Explain why, at this point, we don't know whether the c-values are the same.

In exercises 1–4, use the Midpoint Rule with $n = 6$ to estimate the value of the integral.

1. $\int_0^3 (x^3 + x)\,dx$
2. $\int_0^3 \sqrt{x^2 + 1}\,dx$
3. $\int_0^{\pi} \sin x^2\,dx$
4. $\int_{-2}^{2} e^{-x^2}\,dx$

In exercises 5–8, give an area interpretation of the integral.

5. $\int_1^3 x^2\,dx$
6. $\int_0^1 e^x\,dx$
7. $\int_0^2 (x^2 - 2)\,dx$
8. $\int_0^2 (x^3 - 3x^2 + 2x)\,dx$

In exercises 9–14, evaluate the integral by computing the limit of Riemann sums.

9. $\int_0^1 2x\,dx$
10. $\int_1^2 2x\,dx$
11. $\int_0^2 x^2\,dx$
12. $\int_0^3 (x^2 + 1)\,dx$
13. $\int_1^3 (x^2 - 3)\,dx$
14. $\int_{-2}^{2} (x^2 - 1)\,dx$

In exercises 15–20, write the given (total) area as an integral or sum of integrals

15. The area above the x-axis and below $y = 4 - x^2$
16. The area above the x-axis and below $y = 4x - x^2$
17. The area below the x-axis and above $y = x^2 - 4$
18. The area below the x-axis and above $y = x^2 - 4x$
19. The area between $y = \sin x$ and the x-axis for $0 \le x \le \pi$
20. The area between $y = \sin x$ and the x-axis for $-\frac{\pi}{2} \le x \le \frac{\pi}{4}$.

In exercises 21 and 22, use the given velocity function and initial position to estimate the final position $s(b)$.

21. $v(t) = 40(1 - e^{-2t})$, $s(0) = 0$, $b = 4$
22. $v(t) = 30e^{-t/4}$, $s(0) = -1$, $b = 4$

In exercises 23 and 24, compute $\int_0^4 f(x)\,dx$.

23. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 4 & \text{if } x \ge 1 \end{cases}$
24. $f(x) = \begin{cases} 2 & \text{if } x \le 2 \\ 3x & \text{if } x > 2 \end{cases}$

In exercises 25–28, compute the average value of the function on the given interval.

25. $f(x) = 2x + 1$, $[0, 4]$
26. $f(x) = x^2 + 2x$, $[0, 1]$
27. $f(x) = x^2 - 1$, $[1, 3]$
28. $f(x) = 2x - 2x^2$, $[0, 1]$

In exercises 29–32, use the Integral Mean Value Theorem to estimate the value of the integral.

29. $\int_{\pi/3}^{\pi/2} 3\cos x^2\,dx$
30. $\int_0^{1/2} e^{-x^2}\,dx$
31. $\int_0^2 \sqrt{2x^2 + 1}\,dx$
32. $\int_{-1}^{1} \frac{3}{x^3 + 2}\,dx$

In exercises 33 and 34, find a value of c that satisfies the conclusion of the Integral Mean Value Theorem.

33. $\int_0^2 3x^2\,dx \,(= 8)$
34. $\int_{-1}^{1} (x^2 - 2x)\,dx \,(= \frac{2}{3})$

In exercises 35 and 36, use Theorem 4.2 to write the expression as a single integral.

35. (a) $\int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx$ (b) $\int_0^3 f(x)\,dx - \int_2^3 f(x)\,dx$
36. (a) $\int_0^2 f(x)\,dx + \int_2^1 f(x)\,dx$ (b) $\int_{-1}^{2} f(x)\,dx + \int_2^3 f(x)\,dx$

In exercises 37 and 38, assume that $\int_1^3 f(x)\,dx = 3$ and $\int_1^3 g(x)\,dx = -2$ and find

37. (a) $\int_1^3 [f(x) + g(x)]\,dx$ (b) $\int_1^3 [2f(x) - g(x)]\,dx$
38. (a) $\int_1^3 [f(x) - g(x)]\,dx$ (b) $\int_1^3 [4g(x) - 3f(x)]\,dx$

In exercises 39 and 40, sketch the area corresponding to the integral.

39. (a) $\int_1^2 (x^2 - x)\,dx$ (b) $\int_2^4 (x^2 - x)\,dx$
40. (a) $\int_0^{\pi/2} \cos x\,dx$ (b) $\int_{-2}^{2} e^{-x^2}\,dx$

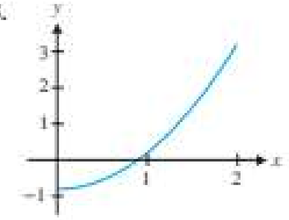
41. (a) Use Theorem 4.3 to show that $\sin(1) \le \int_1^2 x^2 \sin x\,dx \le 4$.
   (b) Use Theorem 4.3 to show that $\frac{7}{3}\sin(1) \le \int_1^2 x^2 \sin x\,dx \le \frac{7}{3}$.
   (c) Is the result of part (a) or part (b) more useful? Briefly explain.
42. Use Theorem 4.3 to find bounds for $\int_1^2 x^2 e^{-\sqrt{x}}\,dx$.

43. Prove that if f is continuous on the interval $[a, b]$, then there exists a number c in (a, b) such that $f(c)$ equals the average value of f on the interval $[a, b]$.

44. Prove part (ii) of Theorem 4.2 for the special case where $c = \frac{1}{2}(a + b)$.

In exercises 45–48, use the graph to determine whether $\int_0^2 f(x)\,dx$ is positive or negative.

45.



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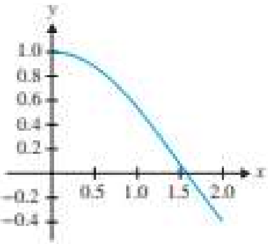

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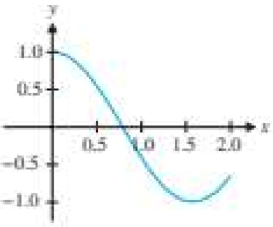

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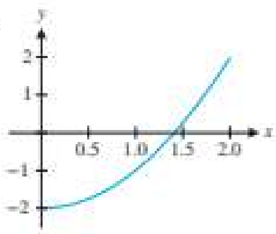


In exercises 49–52, use a geometric formula to compute the integral.

49. $\int_0^2 3x\,dx$

50. $\int_1^4 2x\,dx$

51. $\int_0^2 \sqrt{4 - x^2}\,dx$

52. $\int_{-3}^0 \sqrt{9 - x^2}\,dx$

53. Express each limit as an integral

(a) $\lim_{n\to\infty} \frac{1}{n}\left[\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{n\pi}{n}\right]$

(b) $\lim_{n\to\infty} \left(\frac{n+1}{n^2} + \frac{n+2}{n^2} + \cdots + \frac{2n}{n^2}\right)$

(c) $\lim_{n\to\infty} \frac{f(1/n) + f(2/n) + \cdots + f(n/n)}{n}$

54. Suppose that the average value of a function $f(x)$ over an interval $[a, b]$ is v and the average value of $f(x)$ over the interval $[b, c]$ is w. Find the average value of $f(x)$ over the interval $[a, c]$.

APPLICATIONS

1. Suppose that, for a particular population of organisms, the birth rate is given by $b(t) = 410 - 0.3t$ organisms per month and the death rate is given by $a(t) = 390 + 0.2t$ organisms per month. Explain why $\int_0^{12} [b(t) - a(t)]\,dt$ represents the net change in population in the first 12 months. Determine for which values of t it is true that $b(t) > a(t)$. At which times is the population increasing? Decreasing? Determine the time at which the population reaches a maximum.

2. Suppose that, for a particular population of organisms, the birth rate is given by $b(t) = 400 - 3\sin t$ organisms per month and the death rate is given by $a(t) = 390 + t$ organisms per month. Explain why $\int_0^{12} [b(t) - a(t)]\,dt$ represents the net change in population in the first 12 months. Graphically determine for which values of t it is true that $b(t) > a(t)$. At which times is the population increasing? Decreasing? Estimate the time at which the population reaches a maximum.

3. For a particular ideal gas at constant temperature, pressure P and volume V are related by $PV = 10$. The work required to increase the volume from $V = 2$ to $V = 4$ is given by the integral $\int_2^4 P(V)\,dV$. Estimate the value of this integral.

4. Suppose that the temperature t months into the year is given by $T(t) = 25 - 24\cos\frac{\pi}{6}t$ (degrees Celsius). Estimate the average temperature over an entire year. Explain why this answer is obvious from the graph of $T(t)$.

Exercises 5–8 involve the just-in-time inventory discussed in the chapter introduction.

5. For a business using just-in-time inventory, a delivery of Q items arrives just as the last item is shipped out. Suppose that items are shipped out at the constant rate of r items per day. If a delivery arrives at time 0, show that $f(t) = Q - rt$ gives the number of items in inventory for $0 \le t \le \frac{Q}{r}$. Find the average value of f on the interval $\left[0, \frac{Q}{r}\right]$.

6. The **Economic Order Quantity (EOQ)** model uses the assumptions in exercise 5 to determine the optimal quantity Q to order at any given time. Assume that D items are ordered annually, so that the number of shipments equals $\frac{D}{Q}$. If C_o is the cost of placing an order and C_c is the annual cost for storing an item in inventory, then the total annual cost is given by $f(Q) = C_o\frac{D}{Q} + C_c\frac{Q}{2}$. Find the value of Q that minimizes the total cost. For the optimal order size, show that the total ordering cost $C_o\frac{D}{Q}$ equals the total carrying cost (for storage) $C_c\frac{Q}{2}$.

7. The EOQ model of exercise 6 can be modified to take into account noninstantaneous receipt. In this case, instead of a full delivery arriving at one instant, the delivery arrives at a rate of p items per day. Assume that a delivery of size Q starts at time 0, with shipments out continuing at the rate of r items per day (assume that $p > r$). Show that when the delivery is completed, the inventory equals $Q(1 - r/p)$. From there, inventory drops at a steady rate of r items per day until no items are left. Show that the average inventory equals $\frac{1}{2}Q(1 - r/p)$ and find the order size Q that minimizes the total cost.

8. A further refinement we can make to the EOQ model of exercises 6–7 is to allow discounts for ordering large quantities. To make the calculations easier, take specific values of $D = 4000$, $C_0 = \$50{,}000$ and $C_c = \$3800$. If 1–99 items are ordered, the price is \$2800 per item. If 100–179 items are ordered, the price is \$2200 per item. If 180 or more items are ordered, the price is \$1800 per item. The total cost is now $C_o\frac{D}{Q} + C_c\frac{Q}{2} + PD$, where P is the price per item. Find the order size Q that minimizes the total cost.

9. The **impulse-momentum equation** states the relationship between a force $F(t)$ applied to an object of mass m and the resulting change in velocity Δv of the object. The equation is $m\Delta v = \int_a^b F(t)\,dt$, where $\Delta v = v(b) - v(a)$. Suppose that the force of a baseball bat on a ball is approximately $F(t) = 9 - 10^8(t - 0.0003)^2$ thousand newtons, for t between 0 and 0.0006 second. What is the maximum force on the ball? Using $m = 0.01$ for the mass of a baseball, estimate the change in velocity Δv (in m/s).

10. Measurements taken of the feet of badminton players lunging for a shot indicate a vertical force of approximately $F(t) = 1000 - 25{,}000(t - 0.2)^2$ newtons, for t between 0 and 0.4 second (see *The Science of Racquet Sports*).[2] For a player of mass $m = 5$, use the impulse-momentum equation in exercise 9 to estimate the change in vertical velocity of the player.

[2] Reilly, T., Hughes, M. and Lees, A. (2013). *Science and Racket Sports I* (London: Taylor & Francis).

EXPLORATORY EXERCISES

1. Many of the basic quantities used by epidemiologists to study the spread of disease are described by integrals. In the case of AIDS, a person becomes infected with the HIV virus and, after an incubation period, develops AIDS. Our goal is to derive a formula for the number of AIDS cases given the HIV infection rate $g(t)$ and the incubation distribution $F(t)$. To take a simple case, suppose that the infection rate the first month is 20 people per month, the infection rate the second month is 30 people per month and the infection rate the third month is 25 people per month. Then $g(1) = 20$, $g(2) = 30$ and $g(3) = 25$. Also, suppose that 20% of those infected develop AIDS after 1 month, 50% develop AIDS after 2 months and 30% develop AIDS after 3 months (fortunately, these figures are not at all realistic). Then $F(1) = 0.2$, $F(2) = 0.5$ and $F(3) = 0.3$. Explain why the number of people developing AIDS in the fourth month would be $g(1)F(3) + g(2)F(2) + g(3)F(1)$. Compute this number. Next, suppose that $g(0.5) = 16$, $g(1) = 20$, $g(1.5) = 26$, $g(2) = 30$, $g(2.5) = 28$, $g(3) = 25$ and $g(3.5) = 22$. Further, suppose that $F(0.5) = 0.1$, $F(1) = 0.1$, $F(1.5) = 0.2$, $F(2) = 0.3$, $F(2.5) = 0.1$, $F(3) = 0.1$ and $F(3.5) = 0.1$. Compute the number of people developing AIDS in the fourth month. If we have $g(t)$ and $F(t)$ defined at all real numbers t, explain why the number of people developing AIDS in the fourth month equals $\int_0^4 g(t)\,F(4 - t)\,dt$.

2. **Riemann's condition** states that $\int_a^b f(x)\,dx$ exists if and only if for every $\epsilon > 0$ there exists a partition P such that the upper sum U and lower sum L (see exercises 29–32 in section 4.3 satisfy $|U - L| < \epsilon$. Use this condition to prove that $f(x) = \begin{cases} -1 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$ is not integrable on the interval $[0, 1]$. A function f is called a **Lipschitz** function on the interval $[a, b]$ if $|f(x) - f(y)| \le |x - y|$ for all x and y in $[a, b]$. Use Riemann's condition to prove that every Lipschitz function on $[a, b]$ is integrable on $[a, b]$.

1.5 THE FUNDAMENTAL THEOREM OF CALCULUS

In this section, we present a pair of results known collectively as the Fundamental Theorem of Calculus. On a practical level, the Fundamental Theorem provides us with a much-needed shortcut for computing definite integrals without struggling to find limits of Riemann sums. On a conceptual level, the Fundamental Theorem unifies the seemingly disconnected studies of derivatives and definite integrals, showing us that differentiation and integration are, in fact, inverse processes. In this sense, the theorem is truly *fundamental* to calculus as a coherent discipline.

One hint as to the nature of the first part of the Fundamental Theorem is that we used suspiciously similar notations for indefinite and definite integrals. However, the Fundamental Theorem makes much stronger statements about the relationship between differentiation and integration.

THEOREM 5.1 (The Fundamental Theorem of Calculus, Part I)

If f is continuous on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{5.1}$$

NOTES

The Fundamental Theorem, Part 1, says that to compute a definite integral, we need only find an antiderivative and then evaluate it at the two limits of integration. Observe that this is a vast improvement over computing limits of Riemann sums, which we could compute exactly for only a few simple cases.

PROOF

First, we partition $[a, b]$:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

where $x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$, for $i = 1, 2, \ldots, n$. Working backward, note that by virtue of all the cancellations, we can write

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= [F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + \cdots + [F(x_n) - F(x_{n-1})] \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]. \end{aligned} \tag{5.2}$$

Since F is an antiderivative of f, F is differentiable on (a, b) and continuous on $[a, b]$. By the Mean Value Theorem, we then have for each $i = 1, 2, \ldots, n$, that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\,\Delta x, \tag{5.3}$$

for some $c_i \in (x_{i-1}, x_i)$. Thus, from (5.2) and (5.3), we have

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} f(c_i)\,\Delta x. \tag{5.4}$$

You should recognize this last expression as a Riemann sum for f on $[a, b]$. Taking the limit of both sides of (5.4) as $n \to \infty$, we find that

$$\begin{aligned} \int_a^b f(x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x = \lim_{n\to\infty} [F(b) - F(a)] \\ &= F(b) - F(a), \end{aligned}$$

as desired, since this last quantity is a constant. ■

HISTORICAL NOTES

The Fundamental Theorem of Calculus marks the beginning of calculus as a unified discipline and is credited to both Isaac Newton and Gottfried Leibniz. Newton developed his calculus in the late 1660s but did not publish his results until 1687. Leibniz rediscovered the same results in the mid-1670s but published before Newton in 1684 and 1686. Leibniz' original notation and terminology, much of which is in use today, is superior to Newton's (Newton referred to derivatives and integrals as *fluxions* and *fluents*), but Newton developed the central ideas earlier than Leibniz. A bitter controversy, centering on some letters from Newton to Leibniz in the 1670s, developed over which man would receive credit for inventing calculus. The dispute evolved into a battle between England and the rest of the European mathematical community. Communication between the two groups ceased for over 100 years and greatly influenced the development of mathematics in the 1700s.

REMARK 5.1

We will often use the notation

$$F(x)\Big|_a^b = F(b) - F(a).$$

This enables us to write down the antiderivative before evaluating it at the endpoints.

EXAMPLE 5.1 Using the Fundamental Theorem

Compute $\int_0^2 (x^2 - 2x)\,dx$

Solution Notice that $f(x) = x^2 - 2x$ is continuous on the interval $[0, 2]$ and so, we can apply the Fundamental Theorem. We find an antiderivative from the power rule and simply evaluate:

$$\int_0^2 (x^2 - 2x)\,dx = \left(\frac{1}{3}x^3 - x^2\right)\Big|_0^2 = \left(\frac{8}{3} - 4\right) - (0) = -\frac{4}{3}.$$ ■

Recall that we had already evaluated the integral in example 5.1 by computing the limit of Riemann sums. (See example 4.3.) Given a choice, which method would you prefer?

While you had a choice in example 5.1, you *cannot* evaluate the integrals in examples 5.2–5.5 by computing the limit of a Riemann sum directly, as we have no formulas for the summations involved.

EXAMPLE 5.2 Computing a Definite Integral Exactly

Compute $\int_1^4 \left(\sqrt{x} - \frac{1}{x^2}\right) dx$.

Solution Observe that since $f(x) = x^{1/2} - x^{-2}$ is continuous on [1, 4], we can apply the Fundamental Theorem. Since an antiderivative of $f(x)$ is $F(x) = \frac{2}{3}x^{3/2} + x^{-1}$, we have

$$\int_1^4 \left(\sqrt{x} - \frac{1}{x^2}\right) dx = \left(\frac{2}{3}x^{3/2} + x^{-1}\right)\Big|_1^4 = \left[\frac{2}{3}(4)^{3/2} + 4^{-1}\right] - \left(\frac{2}{3} + 1\right) = \frac{47}{12}.$$

EXAMPLE 5.3 Using the Fundamental Theorem to Compute Areas

Find the area under the curve $f(x) = \sin x$ on the interval $[0, \pi]$.

Solution Since $\sin x \geq 0$ and $\sin x$ is continuous on $[0, \pi]$, we have that

$$\text{Area} = \int_0^\pi \sin x\, dx.$$

Notice that an antiderivative of $\sin x$ is $F(x) = -\cos x$. By the Fundamental Theorem, then, we have

$$\int_0^\pi \sin x\, dx = F(\pi) - F(0) = (-\cos \pi) - (-\cos 0) = -(-1) - (-1) = 2.$$

TODAY IN MATHEMATICS

Benoit Mandelbrot (1924–2010)
A Polish-born French-American mathematician who invented and developed fractal geometry (see the Mandelbrot set in the exercises for section 9.1). Mandelbrot has always been guided by a strong geometric intuition. He explains, "Faced with some complicated integral, I instantly related it to a familiar shape. . . . I knew an army of shapes I'd encountered once in some book and remembered forever, with their properties and their peculiarities."* The fractal geometry that Mandelbrot developed has greatly extended our ability to accurately describe the peculiarities of such phenomena as the structure of the lungs and heart, or mountains and clouds, as well as the stock market and weather.

*Barcellos A. (1984) Interview of B. B. Mandelbrot, Mathematical People, Birkhaüser, Boston. Available at: http://users.math.yale.edu/~bbm3/web_pdfs/inHisOwnWords.pdf

EXAMPLE 5.4 A Definite Integral Involving an Exponential Function

Compute $\int_0^4 e^{-2x} dx$.

Solution Since $f(x) = e^{-2x}$ is continuous, we can apply the Fundamental Theorem. Notice that an antiderivative of e^{-2x} is $-\frac{1}{2}e^{-2x}$, so that

$$\int_0^4 e^{-2x} dx = -\frac{1}{2}e^{-2x}\Big|_0^4 = -\frac{1}{2}e^{-8} - \left(-\frac{1}{2}e^{0}\right) \approx 0.49983.$$

EXAMPLE 5.5 A Definite Integral Involving a Logarithm

Evaluate $\int_{-3}^{-1} \frac{2}{x} dx$.

Solution Since $f(x) = \frac{2}{x}$ is continuous on $[-3, -1]$, we can apply the Fundamental Theorem. First, recall that an antiderivative for $f(x)$ is $2 \ln |x|$. (It's a common error to leave off the absolute values. In this case, the error is fatal! Look carefully at the following to see why.)

$$\int_{-3}^{-1} \frac{2}{x}\,dx = 2\ln|x|\Big|_{-3}^{-1} = 2(\ln|-1| - \ln|-3|)$$
$$= 2(\ln 1 - \ln 3) = -2\ln 3.$$

EXAMPLE 5.6 A Definite Integral with a Variable Upper Limit

Evaluate $\int_1^x 12t^5\,dt$.

Solution Even though the upper limit of integration is a variable, we can use the Fundamental Theorem to evaluate this, since $f(t) = 12t^5$ is continuous on any interval. We have

$$\int_1^x 12t^5\,dt = 12\frac{t^6}{6}\Big|_1^x = 2(x^6 - 1).$$

It's not surprising that the definite integral in example 5.6 is a function of x, since one of the limits of integration involves x. The following observation may be surprising, though. Note that

$$\frac{d}{dx}[2(x^6 - 1)] = 12x^5,$$

which is the same as the original integrand, except that the (dummy) variable of integration, t, has been replaced by the variable in the upper limit of integration, x.

The seemingly odd coincidence observed here is, in fact, not an isolated occurrence, as we see in Theorem 5.2. First, you need to be clear about what a function such as $F(x) = \int_1^x 12t^5\,dt$ means. Notice that the function value at $x = 2$ is found by replacing x by 2:

$$F(2) = \int_1^2 12t^5\,dt,$$

which corresponds to the area under the curve $y = 12t^5$ from $t = 1$ to $t = 2$. (See Figure 1.25a.) Similarly, the function value at $x = 3$ is

$$F(3) = \int_1^3 12t^5\,dt,$$

which is the area under the curve $y = 12t^5$ from $t = 1$ to $t = 3$. (See Figure 1.25b.) More generally, for any $x > 1$, $F(x)$ gives the area under the curve $y = 12t^5$ from $t = 1$ up to $t = x$. (See Figure 1.25c.) For this reason, the function F is sometimes called an **area function.** Notice that for $x > 1$, as x increases, $F(x)$ gives more and more of the area under the curve to the right of $t = 1$.

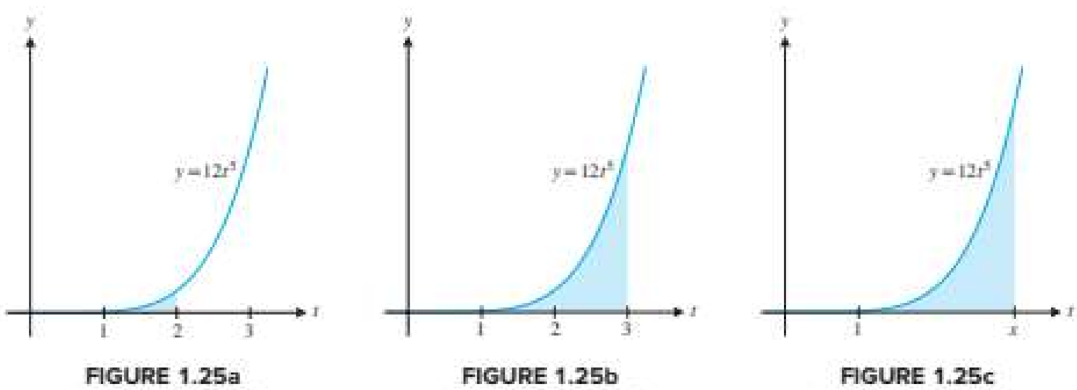


FIGURE 1.25a Area from $t = 1$ to $t = 2$

FIGURE 1.25b Area from $t = 1$ to $t = 3$

FIGURE 1.25c Area from $t = 1$ to $t = x$

THEOREM 5.2 (The Fundamental Theorem of Calculus, Part II)

If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$, on $[a, b]$.

PROOF

Using the definition of derivative, we have

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right] \tag{5.5}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right] = \lim_{h \to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt,$$

where we switched the limits of integration according to equation (4.2) and combined the integrals according to Theorem 4.2 (ii).

Look very carefully at the last term in (5.5). You may recognize it as the limit of the average value of $f(t)$ on the interval $[x, x+h]$ (if $h > 0$). By the Integral Mean Value Theorem (Theorem 4.4), we have

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c), \tag{5.6}$$

REMARK 5.2

Part II of the Fundamental Theorem says that *every* continuous function f has an antiderivative, namely, $\int_a^x f(t)\,dt$.

for some number c between x and $x + h$. Finally, since c is between x and $x + h$, we have that $c \to x$, as $h \to 0$. Since f is continuous, it follows from (5.5) and (5.6) that

$$F'(x) = \lim_{h \to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h \to 0} f(c) = f(x),$$

as desired. ■

EXAMPLE 5.7 Using the Fundamental Theorem, Part II

For $F(x) = \int_1^x (t^2 - 2t + 3)\,dt$, compute $F'(x)$.

Solution Here, the integrand is $f(t) = t^2 - 2t + 3$. By Theorem 5.2, the derivative is

$$F'(x) = f(x) = x^2 - 2x + 3.$$

That is, $F'(x)$ is the function in the integrand with t replaced by x. ■

Before moving on to more complicated examples, let's look at example 5.7 in more detail, just to get more comfortable with the meaning of Part II of the Fundamental Theorem. First, we can use Part I of the Fundamental Theorem to find

$$F(x) = \int_1^x (t^2 - 2t + 3)\,dt = \frac{1}{3}t^3 - t^2 + 3t\Big|_1^x = \left(\frac{1}{3}x^3 - x^2 + 3x\right) - \left(\frac{1}{3} - 1 + 3\right).$$

It's easy to differentiate this directly, to get

$$F'(x) = \frac{1}{3} \cdot 3x^2 - 2x + 3 - 0 = x^2 - 2x + 3.$$

Notice that the lower limit of integration (in this case, 1) has no effect on the value of $F'(x)$. In the definition of $F(x)$, the lower limit of integration merely determines the value of the constant that is subtracted at the end of the calculation of $F(x)$. Since the derivative of any constant is 0, this value does not affect $F'(x)$.

REMARK 5.3

The general form of the chain rule used in example 5.8 is: if $g(x) = \int_a^{u(x)} f(t)\,dt$, then $g'(x) = f(u(x))u'(x)$ or

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f(u(x))u'(x).$$

EXAMPLE 5.8 Using the Chain Rule and the Fundamental Theorem, Part II

If $F(x) = \int_2^{x^2} \cos t\,dt$, compute $F'(x)$.

Solution Let $u(x) = x^2$, so that

$$F(x) = \int_2^{u(x)} \cos t\,dt.$$

From the chain rule,

$$F'(x) = \cos u(x)\frac{du}{dx} = \cos u(x)(2x) = 2x\cos x^2. \blacksquare$$

EXAMPLE 5.9 An Integral with Variable Upper and Lower Limits

If $F(x) = \int_{2x}^{x^2} \sqrt{t^2+1}\,dt$, compute $F'(x)$.

Solution The Fundamental Theorem applies only to definite integrals with variables in the upper limit, so we will first rewrite the integral by Theorem 4.2 (ii) as

$$F(x) = \int_{2x}^{0} \sqrt{t^2+1}\,dt + \int_0^{x^2} \sqrt{t^2+1}\,dt = -\int_0^{2x} \sqrt{t^2+1}\,dt + \int_0^{x^2} \sqrt{t^2+1}\,dt,$$

where we have also switched the limits of integration in the first integral. Using the chain rule as in example 5.8, we get

$$\begin{aligned} F'(x) &= -\sqrt{(2x)^2+1}\,\frac{d}{dx}(2x) + \sqrt{(x^2)^2+1}\,\frac{d}{dx}(x^2) \\ &= -2\sqrt{4x^2+1} + 2x\sqrt{x^4+1}. \blacksquare \end{aligned}$$

Before discussing the theoretical significance of the two parts of the Fundamental Theorem, we present two examples that remind you of why you might want to compute integrals and derivatives.

EXAMPLE 5.10 Computing the Distance Fallen by an Object

Suppose the (downward) velocity of a sky diver is given by $v(t) = 30(1 - e^{-t})$ m/s for the first 5 seconds of a jump. Compute the distance fallen.

Solution Recall that the distance d is given by the definite integral

$$\begin{aligned} d &= \int_0^5 (30 - 30e^{-t})\,dt = (30t + 30e^{-t})\Big|_0^5 \\ &= (150 + 30e^{-5}) - (0 + 30e^0) = 120 + 30e^{-5} \approx 120.2 \text{ m}. \blacksquare \end{aligned}$$

Recall that velocity is the *instantaneous rate of change* of the distance function with respect to time. We see in example 5.10 that the definite integral of velocity gives the *total change* of the distance function over the given time interval. A similar interpretation of derivative and the definite integral holds for many quantities of interest. In example 5.11, we look at the rate of change and total change of water in a tank.

EXAMPLE 5.11 Rate of Change and Total Change of Volume of a Tank

Suppose that water flows in and out of a storage tank. The net rate of change (that is, the rate in minus the rate out) of water is $f(t) = 20(t^2 - 1)$ liters per minute. (a) For $0 \le t \le 3$, determine when the water level is increasing and when the water level is decreasing.

(b) If the tank has 200 liters of water at time $t = 0$, determine how many liters are in the tank at time $t = 3$ minutes.

Solution Let $w(t)$ be the number of liters in the tank at time t. (a) Notice that the water level decreases if $w'(t) = f(t) < 0$. We have

$$f(t) = 20(t^2 - 1) < 0, \quad \text{if } 0 \le t < 1.$$

Alternatively, the water level increases if $w'(t) = f(t) > 0$. In this case, we have

$$f(t) = 20(t^2 - 1) > 0, \quad \text{if } 1 < t \le 3.$$

(b) We start with $w'(t) = 20(t^2 - 1)$. Integrating from $t = 0$ to $t = 3$, we have

$$\int_0^3 w'(t)\,dt = \int_0^3 20(t^2 - 1)\,dt$$

Evaluating the integrals on both sides yields

$$w(3) - w(0) = 20\left(\frac{t^3}{3} - t\right)\Big|_{t=0}^{t=3}.$$

Since $w(0) = 200$, we have

$$w(3) - 200 = 20(9 - 3) = 120$$

and hence,
$$w(3) = 200 + 120 = 320,$$

so that the tank will have 320 liters at time 3 minutes. ■

In example 5.12, we use Part II of the Fundamental Theorem to determine information about a seemingly complicated function. Notice that although we don't know how to evaluate the integral, we can use the Fundamental Theorem to obtain some important information about the function.

EXAMPLE 5.12 Finding a Tangent Line for a Function Defined as an Integral

For the function $F(x) = \int_4^{x^2} \ln(t^3 + 4)\,dt$, find an equation of the tangent line at $x = 2$.

Solution Notice that there are almost no function values that we can compute exactly, yet we can easily find an equation of a tangent line! From Part II of the Fundamental Theorem and the chain rule, we get the derivative

$$F'(x) = \ln[(x^2)^3 + 4]\frac{d}{dx}(x^2) = \ln[(x^2)^3 + 4](2x) = 2x\ln(x^6 + 4).$$

So, the slope at $x = 2$ is $F'(2) = 4\ln(68) \approx 16.878$. The tangent passes through the point with $x = 2$ and $y = F(2) = \int_4^4 \ln(t^3 + 4)\,dt = 0$ (since the upper limit equals the lower limit). An equation of the tangent line is then

$$y = (4\ln 68)(x - 2).$$ ■

BEYOND FORMULAS

The two parts of the Fundamental Theorem are different sides of the same theoretical coin. Recall the conclusions of Parts I and II of the Fundamental Theorem:

$$\int_a^b F'(x)\,dx = F(b) - F(a) \quad \text{and} \quad \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

In both cases, we are saying that differentiation and integration are in some sense *inverse operations:* their effects (with appropriate hypotheses) cancel each other out. This fundamental connection is what unifies seemingly unrelated calculation techniques into *the calculus*.

EXERCISES 1.5

WRITING EXERCISES

1. To explore Part I of the Fundamental Theorem graphically, first suppose that $F(x)$ is increasing on the interval $[a, b]$. Explain why both of the expressions $F(b) - F(a)$ and $\int_a^b F'(x)\,dx$ will be positive. Further, explain why the faster $F(x)$ increases, the larger each expression will be. Similarly, explain why if $F(x)$ is decreasing, both expressions will be negative.

2. You can think of Part I of the Fundamental Theorem in terms of position $s(t)$ and velocity $v(t) = s'(t)$. Start by assuming that $v(t) \geq 0$. Explain why $\int_a^b v(t)\,dt$ gives the total distance traveled and explain why this equals $s(b) - s(a)$. Discuss what changes if $v(t) < 0$.

3. To explore Part II of the Fundamental Theorem graphically, consider the function $g(x) = \int_a^x f(t)\,dt$. If $f(t)$ is positive on the interval $[a, b]$, explain why $g'(x)$ will also be positive. Further, the larger $f(t)$ is, the larger $g'(x)$ will be. Similarly, explain why if $f(t)$ is negative then $g'(x)$ will also be negative.

4. In Part I of the Fundamental Theorem, F can be *any* antiderivative of f. Recall that any two antiderivatives of f differ by a constant. Explain why $F(b) - F(a)$ is **well defined;** that is, if F_1 and F_2 are different antiderivatives, explain why $F_1(b) - F_1(a) = F_2(b) - F_2(a)$. When evaluating a definite integral, explain why you do not need to include "$+ c$" with the antiderivative.

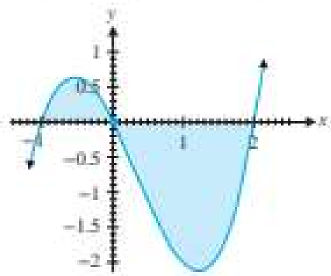
In exercises 1–18, use Part I of the Fundamental Theorem to compute each integral exactly.

1. $\int_0^2 (2x - 3)\,dx$
2. $\int_0^3 (x^2 - 2)\,dx$
3. $\int_{-1}^1 (x^3 + 2x)\,dx$
4. $\int_0^2 (x^3 + 3x - 1)\,dx$
5. $\int_1^4 \left(x\sqrt{x} + \frac{3}{x}\right)dx$
6. $\int_1^2 \left(4x - \frac{2}{x^2}\right)dx$
7. $\int_0^1 (6e^{-3x} + 4)\,dx$
8. $\int_0^2 \left(\frac{e^{2x} - 2e^{3x}}{e^{3x}}\right)dx$
9. $\int_{\pi/2}^{\pi} (2\sin x - \cos x)\,dx$
10. $\int_{\pi/4}^{\pi/2} 3\csc x \cot x\,dx$
11. $\int_0^{\pi/4} \sec t \tan t\,dt$
12. $\int_0^{\pi/4} \sec^2 t\,dt$
13. $\int_0^{1/2} \frac{3}{\sqrt{1 - x^2}}\,dx$
14. $\int_{-1}^1 \frac{4}{1 + x^2}\,dx$
15. $\int_1^4 \frac{t - 3}{t}\,dt$
16. $\int_0^3 t(t - 2)\,dt$
17. $\int_0^1 (e^{x/2})^2\,dx$
18. $\int_0^{\pi} (\sin^2 x + \cos^2 x)\,dx$

In exercises 19–24, find the given area.

19. The area above the x-axis and below $y = 4 - x^2$
20. The area below the x-axis and above $y = x^2 - 4x$
21. The area of the region bounded by $y = x^2$, $x = 2$ and the x-axis
22. The area of the region bounded by $y = x^3$, $x = 3$ and the x-axis
23. The area between $y = \sin x$ and the x-axis for $0 \leq x \leq \pi$
24. The area between $y = \sin x$ and the x-axis for $-\pi/2 \leq x \leq \pi/4$

In exercises 25–28, find the total area of the shaded region.

25.


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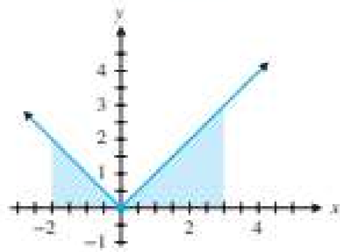

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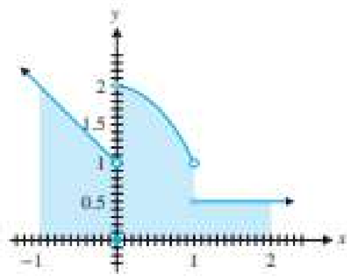
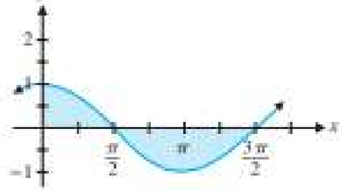


28.


In exercises 29–36, find the derivative $f'(x)$.

29. $f(x) = \int_0^x (t^2 - 3t + 2)\,dt$
30. $f(x) = \int_2^x (t^2 - 3t - 4)\,dt$
31. $f(x) = \int_0^{x^2} (e^{-t^2} + 1)\,dt$
32. $f(x) = \int_x^2 \sec t\,dt$

33. $f(x)=\int_{e^x}^{2-x}\sin t^2\,dt$ **34.** $f(x)=\int_{2-x}^{x^2} e^{2t}\,dt$

35. $f(x)=\int_{x^3}^{x^2}\sin(3t)\,dt$ **36.** $f(x)=\int_{3x}^{\sin x}(t^2+4)\,dt$

In exercises 37–40, find the position function $s(t)$ from the given velocity or acceleration function and initial value(s). Assume that units are meters and seconds.

37. $v(t)=40-\sin t,\ s(0)=2$

38. $v(t)=10\,e^{-t},\ s(0)=2$

39. $a(t)=4-t,\ v(0)=8,\ s(0)=0$

40. $a(t)=16-t^2,\ v(0)=0,\ s(0)=0$

41. Suppose that the rate of change of water in a storage tank is $f(t)=10\sin(t)$ liters per minute. (a) For $0\le t\le 2\pi$, determine when the water level is increasing and when the water level is decreasing. (b) If the tank has 100 liters of water at time $t=0$, determine how many liters are in the tank at $t=\pi$.

42. Suppose that the rate of change of water in a pond is $f(t)=4t-t^2$ thousand liters per minute. (a) For $0\le t\le 6$, determine when the water level is rising and when it is falling. (b) If the pond has 40 thousand liters at time $t=0$, determine how many liters are in the pond at $t=6$.

In exercises 43–46, find an equation of the tangent line at the given value of x.

43. $y=\int_0^x \sin\sqrt{t^2+\pi^2}\,dt,\ x=0$

44. $y=\int_{-1}^x \ln(t^2+2t+2)\,dt,\ x=-1$

45. $y=\int_2^x \cos(\pi t^3)\,dt,\ x=2$

46. $y=\int_0^x e^{-t^2+1}\,dt,\ x=0$

In exercises 47–52, name the method by using the Fundamental Theorem if possible or estimating the integral using Riemann sums. (Hint: Three problems can be worked using antiderivative formulas we have covered so far.)

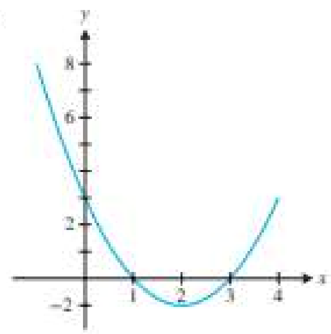
47. $\int_0^2 \sqrt{x^2+1}\,dx$ **48.** $\int_0^2 (\sqrt{x}+1)^2\,dx$

49. $\int_1^4 \frac{x^2}{x^2+4}\,dx$ **50.** $\int_1^4 \frac{x^2+4}{x^2}\,dx$

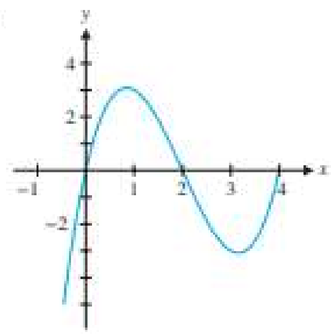
51. $\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx$ **52.** $\int_0^{\pi/4} \frac{\tan x}{\sec^2 x}\,dx$

In exercises 53 and 54, use the graph to list $\int_0^1 f(x)\,dx$, $\int_0^2 f(x)\,dx$ and $\int_0^3 f(x)\,dx$ in order, from smallest to largest. For $g(x)=\int_0^x f(t)\,dt$, determine intervals where g is increasing and identify critical points for g.

53.



54.



In exercises 55 and 56, (a) explain how you know the proposed integral value is wrong and (b) find all mistakes.

55. $\int_{-1}^1 \frac{1}{x^2}\,dx=-\frac{1}{x}\Big|_{x=-1}^{x=1}=-1-(1)=-2$

56. $\int_0^\pi \sec^2 x\,dx=\tan x\Big|_{x=0}^{x=\pi}=\tan\pi-\tan 0=0$

In exercises 57 and 58, identify the integrals to which the Fundamental Theorem of Calculus applies; the other integrals are called improper integrals.

57. (a) $\int_0^4 \frac{1}{x-4}\,dx$ (b) $\int_0^1 \sqrt{x}\,dx$ (c) $\int_0^1 \ln x\,dx$

58. (a) $\int_0^1 \frac{1}{\sqrt{x+2}}\,dx$ (b) $\int_0^4 \frac{1}{(x-3)^2}\,dx$ (c) $\int_0^2 \sec x\,dx$

In exercises 59–62, find the average value of the function on the given interval.

59. $f(x)=x^2-1$, $[1,3]$ **60.** $f(x)=2x-2x^2$, $[0,1]$

61. $f(x)=\cos x$, $[0,\pi/2]$ **62.** $f(x)=e^x$, $[0,2]$

63. Use the Fundamental Theorem of Calculus to find an antiderivative of

(a) e^{-x^2}; (b) $\sin\sqrt{x^2+1}$.

64. For $f(x) = \begin{cases} x^2+1, & 0 \le x \le 4 \\ x^3 - x, & 4 < x \end{cases}$ find $g(x) = \int_0^x f(x)\,dt$ for $x > 0$. Is $g'(x) = f(x)$ for all $x > 0$?

65. Identify all local extrema of $f(x) = \int_0^x (t^2 - 3t + 2)\,dt$.

66. Find the first and second derivatives of $g(x) = \int_0^x \left(\int_0^u f(t)\,dt\right)du$, where f is a continuous function. Identify the graphical feature of $y = g(x)$ that corresponds to a zero of $f(x)$.

67. Let $f(x) = \begin{cases} x & \text{if } x < 2 \\ x+1 & \text{if } x \ge 2 \end{cases}$ and define $F(x) = \int_0^x f(t)\,dt$. Show that $F(x)$ is continuous but that it is not true that $F'(x) = f(x)$ for all x. Explain why this does not contradict the Fundamental Theorem of Calculus.

68. Let f be a continuous function on the interval $[0, 1]$, and define $g_n(x) = f(x^n)$ for $n = 1, 2$ and so on. For a given x with $0 \le x \le 1$, find $\lim_{n\to\infty} g_n(x)$. Then, find $\lim_{n\to\infty} \int_0^1 g_n(x)\,dx$.

APPLICATIONS

1. Katie drives a car at speed $f(t) = 55 + 10\cos t$ km/h, and Michael drives a car at speed $g(t) = 50 + 2t$ km/h at time t minutes. Suppose that Katie and Michael are at the same location at time $t = 0$. Compute $\int_0^{\pi} [f(t) - g(t)]\,dt$, and interpret the integral in terms of a race between Katie and Michael.

2. The number of items that consumers are willing to buy depends on the price of the item. Let $p = D(q)$ represent the price (in dollars) at which q items can be sold. The integral $\int_0^Q D(q)\,dq$ is interpreted as the total number of dollars that consumers would be willing to spend on Q items. If the price is fixed at $P = D(Q)$ dollars, then the actual amount of money spent is PQ. The **consumer surplus** is defined by $CS = \int_0^Q D(q)\,dq - PQ$. (a) Compute the consumer surplus for $D(q) = 150 - 2q - 3q^2$ at $Q = 4$ and at $Q = 6$. What does the difference in CS values tell you about how many items to produce? (b) Repeat for $D(q) = 40e^{-0.05q}$ at $Q = 10$ and $Q = 20$.

3. For a business using just-in-time inventory, a delivery of Q items arrives just as the last item is shipped out. Suppose that items are shipped out at a nonconstant rate such that $f(t) = Q - r\sqrt{t}$ gives the number of items in inventory. Find the time T at which the next shipment must arrive. Find the average value of f on the interval $[0, T]$.

4. The Economic Order Quantity (EOQ) model uses the assumptions in exercise 3 to determine the optimal quantity Q to order at any given time. If C_o is the cost of placing an order, C_c is the annual cost for storing an item in inventory and A is the average value from exercise 3, then the total annual cost is given by $f(Q) = C_o\frac{D}{Q} + C_cA$. Find the value of Q that minimizes the total cost. Show that for this order size, the total ordering cost $C_o\frac{D}{Q}$ equals the total carrying cost (for storage) C_cA.

EXPLORATORY EXERCISES

1. When solving differential equations of the form $\frac{dy}{dt} = f(y)$ for the unknown function $y(t)$, it is often convenient to make use of a **potential function** $V(y)$. This is a function such that $-\frac{dV}{dy} = f(y)$. For the function $f(y) = y - y^3$, find a potential function $V(y)$. Find the locations of the local minima of $V(y)$ and use a graph of $V(y)$ to explain why this is called a "double-well" potential. Explain each step in the calculation

$$\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = -f(y)f(y) \le 0.$$

Since $\frac{dV}{dt} \le 0$, does the function V increase or decrease as time goes on? Use your graph of V to predict the possible values of $\lim_{t\to\infty} y(t)$. Thus, you can predict the limiting value of the solution of the differential equation without ever solving the equation itself. Use this technique to predict $\lim_{t\to\infty} y(t)$ if $y' = 2 - 2y$.

2. Let $f_n(x) = \begin{cases} 2n + 4n^2x & -\frac{1}{2n} \le x \le 0 \\ 2n - 4n^2x & 0 \le x \le \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$

for $n = 1, 2, 3, \ldots$. For an arbitrary n, sketch $y = f_n(x)$ and show that $\int_{-1}^1 f_n(x)\,dx = 1$. Compute $\lim_{n\to\infty} \int_{-1}^1 f_n(x)\,dx$. For an arbitrary $x \ne 0$ in $[-2, 2]$, compute $\lim_{n\to\infty} f_n(x)$ and compute $\int_{-1}^1 \lim_{n\to\infty} f_n(x)\,dx$. Is it always true that $\lim_{n\to\infty} \int_{-1}^1 f_n(x)\,dx = \int_{-1}^1 \lim_{n\to\infty} f_n(x)\,dx$?

1.6 INTEGRATION BY SUBSTITUTION

In this section, we significantly expand our ability to compute antiderivatives by developing a useful technique called **integration by substitution.**

EXAMPLE 6.1 Finding an Antiderivative by Trial and Error

Evaluate $\int 2xe^{x^2}\,dx$.

Solution We need to find a function $F(x)$ for which $F'(x) = 2xe^{x^2}$. You might be tempted to guess that

$$F(x) = x^2e^{x^2}$$

is an antiderivative of $2xe^{x^2}$. However, from the product rule,

$$\frac{d}{dx}(x^2e^{x^2}) = 2xe^{x^2} + x^2e^{x^2}(2x) \neq 2xe^{x^2}.$$

Now, look closely at the integrand and notice that $2x$ is the derivative of x^2 and x^2 appears as the exponent of e^{x^2}. Further, by the chain rule, for $F(x) = e^{x^2}$,

$$F'(x) = e^{x^2}\frac{d}{dx}(x^2) = 2xe^{x^2},$$

which is the integrand. To finish this example, recall that we need to add an arbitrary constant, to get

$$\int 2xe^{x^2}\,dx = e^{x^2} + c. \blacksquare$$

More generally, recognize that when one factor in an integrand is the derivative of another part of the integrand, you may be looking at a chain rule derivative.

Note that, in general, if F is any antiderivative of f, then from the chain rule, we have

$$\frac{d}{dx}[F(u)] = F'(u)\frac{du}{dx} = f(u)\frac{du}{dx}.$$

From this, we have that

$$\int f(u)\frac{du}{dx}\,dx = \int \frac{d}{dx}[F(u)]\,dx = F(u) + c = \int f(u)\,du, \tag{6.1}$$

since F is an antiderivative of f. If you read the expressions on the far left and the far right sides of (6.1), this suggests that

$$du = \frac{du}{dx}dx.$$

So, if we cannot compute the integral $\int h(x)\,dx$ directly, we often look for a new variable u and function $f(u)$ for which

$$\int h(x)\,dx = \int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du,$$

where the second integral is easier to evaluate than the first.

NOTES

In deciding how to choose a new variable, there are several things to look for:

- think in terms of a main function and its derivative.
- terms that are derivatives of other terms (or pieces thereof).
- terms that are particularly complicated/ troublesome. (You can often substitute your troubles away.)

EXAMPLE 6.2 Using Substitution to Evaluate an Integral

Evaluate $\int (x^3 + 5)^{100}(3x^2)\,dx$.

Solution You probably cannot evaluate this as it stands. However, observe that

$$\frac{d}{dx}(x^3 + 5) = 3x^2,$$

which is a factor in the integrand. This leads us to make the substitution $u = x^3 + 5$, so that $du = \frac{d}{dx}(x^3 + 5)\,dx = 3x^2\,dx$. This gives us

$$\int \underbrace{(x^3 + 5)^{100}}_{u^{100}}\underbrace{(3x^2)\,dx}_{du} = \int u^{100}\,du = \frac{u^{101}}{101} + c.$$

We are not done quite yet. Since we invented the new variable u, we need to convert back to the original variable x, to obtain

$$\int (x^3 + 5)^{100}(3x^2)\,dx = \frac{u^{101}}{101} + c = \frac{(x^3 + 5)^{101}}{101} + c.$$

It's always a good idea to perform a quick check on the antiderivative. (Remember that integration and differentiation are inverse processes!) Here, we compute

$$\frac{d}{dx}\left[\frac{(x^3+5)^{101}}{101}\right] = \frac{101(x^3+5)^{100}(3x^2)}{101} = (x^3+5)^{100}(3x^2),$$

which is the original integrand. This confirms that we have indeed found an antiderivative. ■

INTEGRATION BY SUBSTITUTION

Integration by substitution consists of the following general steps, as illustrated in example 6.2.

- **Choose a new variable *u*:** a common choice is the innermost expression or "inside" term of a composition of functions. (In example 6.2, note that $x^3 + 5$ is the inside term of $(x^3+5)^{100}$.)
- **Compute** $du = \frac{du}{dx}\,dx$.
- **Replace all terms** in the original integrand with expressions involving u and du.
- **Evaluate** the resulting (u) integral. If you still can't evaluate the integral, you may need to try a different choice of u.
- **Replace each occurrence of *u*** in the antiderivative with the corresponding expression in x.

Always keep in mind that finding antiderivatives is the reverse process of finding derivatives. In example 6.3, we are not so fortunate as to have the exact derivative we want in the integrand.

EXAMPLE 6.3 Using Substitution: A Power Function Inside a Cosine

Evaluate $\int x \cos x^2\,dx$.

Solution Notice that

$$\frac{d}{dx}x^2 = 2x.$$

While we don't quite have a factor of $2x$ in the integrand, we can always push constants back and forth past an integral sign and rewrite the integral as

$$\int x\cos x^2\,dx = \frac{1}{2}\int 2x\cos x^2\,dx.$$

We now substitute $u = x^2$, so that $du = 2x\,dx$ and we have

$$\int x\cos x^2\,dx = \frac{1}{2}\int \underbrace{\cos x^2}_{\cos u}\ \underbrace{(2x)\,dx}_{du}$$

$$= \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + c = \frac{1}{2}\sin x^2 + c.$$

Again, as a check, observe that

$$\frac{d}{dx}\left(\frac{1}{2}\sin x^2\right) = \frac{1}{2}\cos x^2(2x) = x\cos x^2,$$

which is the original integrand. ■

EXAMPLE 6.4 Using Substitution: A Trigonometric Function Inside a Power

Evaluate $\int (3\tan x + 4)^5 \sec^2 x\,dx$.

Solution As with most integrals, you probably can't evaluate this one as it stands. However, observe that there's a $\tan x$ term and a factor of $\sec^2 x$ in the integrand and that $\frac{d}{dx}\tan x = \sec^2 x$. Thus, we let $u = 3\tan x + 4$, so that $du = 3\sec^2 x\,dx$. We then have

$$\begin{aligned}\int (3\tan x + 4)^5 \sec^2 x\,dx &= \frac{1}{3}\int \underbrace{(3\tan x + 4)^5}_{u^5}\underbrace{(3\sec^2 x)\,dx}_{du} \\ &= \frac{1}{3}\int u^5 du = \left(\frac{1}{3}\right)\frac{u^6}{6} + c \\ &= \frac{1}{18}(3\tan x + 4)^6 + c.\end{aligned}$$

Sometimes you will need to look a bit deeper into an integral to see terms that are derivatives of other terms, as in example 6.5.

EXAMPLE 6.5 Using Substitution: A Root Function Inside a Sine

Evaluate $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$.

Solution This integral is not especially obvious. It never hurts to try something, though. If you had to substitute for something, what would you choose? You might notice that $\sin\sqrt{x} = \sin x^{1/2}$ and letting $u = \sqrt{x} = x^{1/2}$ (the "inside"), we get $du = \frac{1}{2}x^{-1/2}dx = \frac{1}{2\sqrt{x}}dx$. Since there is a factor of $\frac{1}{\sqrt{x}}$ in the integrand, we can proceed. We have

$$\begin{aligned}\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx &= 2\int \underbrace{\sin\sqrt{x}}_{\sin u}\underbrace{\left(\frac{1}{2\sqrt{x}}\right)dx}_{du} \\ &= 2\int \sin u\,du = -2\cos u + c = -2\cos\sqrt{x} + c.\end{aligned}$$

EXAMPLE 6.6 Substitution: Where the Numerator Is the Derivative of the Denominator

Evaluate $\int \frac{x^2}{x^3+5}\,dx$.

Solution Since $\frac{d}{dx}(x^3+5) = 3x^2$, we let $u = x^3 + 5$, so that $du = 3x^2dx$. We now have

$$\begin{aligned}\int \frac{x^2}{x^3+5}\,dx &= \frac{1}{3}\int \underbrace{\frac{1}{x^3+5}}_{u}\underbrace{(3x^2)\,dx}_{du} = \frac{1}{3}\int \frac{1}{u}\,du \\ &= \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln|x^3+5| + c.\end{aligned}$$

Example 6.6 is an illustration of a very common type of integral, one where the numerator is the derivative of the denominator. More generally, we have the result in Theorem 6.1.

THEOREM 6.1

For any continuous function, f

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c,$$

on any interval in which $f(x) \neq 0$.

PROOF

Let $u = f(x)$. Then $du = f'(x)\,dx$ and

$$\int \frac{f'(x)}{f(x)}\,dx = \int \underbrace{\frac{1}{f(x)}}_{u}\,\underbrace{f'(x)\,dx}_{du}$$
$$= \int \frac{1}{u}\,du = \ln|u| + c = \ln|f(x)| + c,$$

as desired. As an alternative to this proof, you might simply compute $\frac{d}{dx}\ln|f(x)|$ directly, to obtain the integrand. ■

You should recall that we already stated this result in section 1.1 (as Corollary 1.2). It is important enough to repeat here in the context of substitution.

EXAMPLE 6.7 An Antiderivative for the Tangent Function

Evaluate $\int \tan x\,dx$.

Solution Note that this is *not* one of our basic integration formulas. However, you might notice that with $u = \cos x$,

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \underbrace{\frac{1}{\cos x}}_{u}\,\underbrace{(-\sin x)\,dx}_{du}$$
$$= -\int \frac{1}{u}\,du = -\ln|u| + c = -\ln|\cos x| + c,$$

where we have used the fact that $\frac{d}{dx}(\cos x) = -\sin x$. ■

EXAMPLE 6.8 A Substitution for an Inverse Tangent

Evaluate $\int \frac{(\tan^{-1} x)^2}{1 + x^2}\,dx$.

Solution Again, the key is to look for a substitution. Since

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1 + x^2},$$

we let $u = \tan^{-1} x$, so that $du = \frac{1}{1 + x^2}\,dx$. We now have

$$\int \frac{(\tan^{-1} x)^2}{1 + x^2}\,dx = \int \underbrace{(\tan^{-1} x)^2}_{u^2}\,\underbrace{\tfrac{1}{1+x^2}\,dx}_{du}$$
$$= \int u^2\,du = \frac{1}{3}u^3 + c = \frac{1}{3}(\tan^{-1} x)^3 + c. \quad ■$$

So far, every one of our examples has been solved by spotting a term in the integrand that was the derivative of another term. We present an integral now where this is not the case, but where a substitution is made to deal with a particularly troublesome term in the integrand.

EXAMPLE 6.9 A Substitution That Lets You Expand the Integrand

Evaluate $\int x\sqrt{2-x}\,dx$.

Solution You certainly cannot evaluate this as it stands. If you look for terms that are derivatives of other terms, you will come up empty-handed. The real problem here is that there is a square root of a sum (or difference) in the integrand. A reasonable step would be to substitute for the expression under the square root. We let $u = 2 - x$, so that $du = -dx$. That doesn't seem so bad, but what are we to do with the extra x in the integrand? Well, since $u = 2 - x$, it follows that $x = 2 - u$. Making these substitutions in the integral, we get

$$\int x\sqrt{2-x}\,dx = (-1)\int \underbrace{x}_{2-u}\,\underbrace{\sqrt{2-x}}_{\sqrt{u}}\,\underbrace{(-1)\,dx}_{du}$$
$$= -\int (2-u)\sqrt{u}\,du.$$

While we can't evaluate this integral directly, if we multiply out the terms, we get

$$\begin{aligned}\int x\sqrt{2-x}\,dx &= -\int (2-u)\sqrt{u}\,du \\ &= -\int (2u^{1/2} - u^{3/2})\,du \\ &= -2\frac{u^{3/2}}{\left(\frac{3}{2}\right)} + \frac{u^{5/2}}{\left(\frac{5}{2}\right)} + c \\ &= -\frac{4}{3}u^{3/2} + \frac{2}{5}u^{5/2} + c \\ &= -\frac{4}{3}(2-x)^{3/2} + \frac{2}{5}(2-x)^{5/2} + c.\end{aligned}$$

You should check the validity of this antiderivative via differentiation. ■

○ Substitution in Definite Integrals

There is only one slight difference in using substitution for evaluating a definite integral: you must also change the limits of integration to correspond to the new variable. The procedure here is then precisely the same as that used for examples 6.2 through 6.9, except that when you introduce the new variable u, the limits of integration change from $x = a$ and $x = b$ to the corresponding limits for u: $u = u(a)$ and $u = u(b)$. We have

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

EXAMPLE 6.10 Using Substitution in a Definite Integral

Evaluate $\int_1^2 x^3\sqrt{x^4+5}\,dx$.

Solution Of course, you probably can't evaluate this as it stands. However, since $\frac{d}{dx}(x^4+5) = 4x^3$, we make the substitution $u = x^4 + 5$, so that $du = 4x^3dx$. For the limits of integration, note that when $x = 1$,

$$u = x^4 + 5 = 1^4 + 5 = 6$$

and when $x = 2$,
$$u = x^4 + 5 = 2^4 + 5 = 21.$$

CAUTION

You must change the limits of integration as soon as you change variables!

We now have

$$\int_1^2 x^3\sqrt{x^4+5}\,dx = \frac{1}{4}\int_1^2 \underbrace{\sqrt{x^4+5}}_{\sqrt{u}}\,\underbrace{(4x^3)\,dx}_{du} = \frac{1}{4}\int_6^{21}\sqrt{u}\,du$$

$$= \frac{1}{4}\frac{u^{3/2}}{\left(\frac{3}{2}\right)}\Bigg|_6^{21} = \left(\frac{1}{4}\right)\left(\frac{2}{3}\right)(21^{3/2}-6^{3/2}).$$

Notice that because we changed the limits of integration to match the new variable, we did *not* need to convert back to the original variable, as we do when we make a substitution in an indefinite integral. (Note that, if we had switched the variables back, we would also have needed to switch the limits of integration back to their original values before evaluating!) ■

It may have occurred to you that you could use a substitution in a definite integral only to find an antiderivative and then switch back to the original variable to do the evaluation. Although this method will work for many problems, we recommend that you avoid it, for several reasons. First, changing the limits of integration is not very difficult and results in a much more readable mathematical expression. Second, in many applications requiring substitution, you will *need* to change the limits of integration, so you might as well get used to doing so now.

EXAMPLE 6.11 Substitution in a Definite Integral Involving an Exponential

Compute $\int_0^{15} te^{-t^2/2}dt$.

Solution As always, we look for terms that are derivatives of other terms. Here, you should notice that $\frac{d}{dt}\left(\frac{-t^2}{2}\right) = -t$. So, we set $u = -\frac{t^2}{2}$ and compute $du = -t\,dt$. For the upper limit of integration, we have that $t = 15$ corresponds to $u = -\frac{(15)^2}{2} = -\frac{225}{2}$. For the lower limit, we have that $t = 0$ corresponds to $u = 0$. This gives us

$$\int_0^{15} te^{-t^2/2}dt = -\int_0^{15}\underbrace{e^{-t^2/2}}_{e^u}\,\underbrace{(-t)dt}_{du}$$

$$= -\int_0^{-225/2} e^u du = -e^u\Big|_0^{-112.5} = -e^{-112.5}+1.$$ ■

EXERCISES 1.6

WRITING EXERCISES

1. It is never *wrong* to make a substitution in an integral, but sometimes it is not very helpful. For example, using the substitution $u = x^2$, you can correctly conclude that

$$\int x^3\sqrt{x^2+1}\,dx = \int \frac{1}{2}u\sqrt{u+1}\,du,$$

but the new integral is no easier than the original integral. Find a better substitution and evaluate this integral. Give guidelines on when to give up on a substitution.

2. It is not uncommon for students learning substitution to use incorrect notation in the intermediate steps. Be aware of this—it can be harmful to your grade! Carefully examine the following string of equalities and find each mistake. Using $u = x^2$,

$$\int_0^2 x\sin x^2\,dx = \int_0^2 (\sin u)x\,dx = \int_0^2 (\sin u)\frac{1}{2}du$$

$$= -\frac{1}{2}\cos u\Big|_0^2 = -\frac{1}{2}\cos x^2\Big|_0^2$$

$$= -\frac{1}{2}\cos 4 + \frac{1}{2}.$$

The final answer is correct, but because of several errors, this work would not earn full credit. Discuss each error and write this in a way that would earn full credit.

3. Suppose that an integrand has a term of the form $e^{f(x)}$. For example, suppose you are trying to evaluate $\int x^2e^{x^3}dx$. Discuss why you should immediately try the substitution $u = f(x)$.

4. Suppose that an integrand has a composite function of the form $f(g(x))$. Explain why you should look to see if the integrand also has the term $g'(x)$. Discuss possible substitutions.

In exercises 1–4, use the given substitution to evaluate the indicated integral.

1. $\int x^2\sqrt{x^3+2}\,dx,\ u = x^3+2$

2. $\int x^3(x^4+1)^{-2/3}dx,\ u = x^4+1$

3. $\int \frac{(\sqrt{x}+2)^3}{\sqrt{x}}dx,\ u = \sqrt{x}+2$

4. $\int \sin x \cos x\,dx,\ u = \sin x$

In exercises 5–30, evaluate the indicated integral.

5. $\int x^3\sqrt{x^4+3}\,dx$

6. $\int \sqrt{1+10x}\,dx$

7. $\int \frac{\sin x}{\sqrt{\cos x}}dx$

8. $\int \sin^3 x \cos x\,dx$

9. $\int t^2 \cos t^3\,dt$

10. $\int \sin t(\cos t+3)^{3/4}dt$

11. $\int xe^{x^2+1}dx$

12. $\int e^x\sqrt{e^x+4}\,dx$

13. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

14. $\int \frac{\cos(1/x)}{x^2}dx$

15. $\int \frac{\sqrt{\ln x}}{x}dx$

16. $\int \sec^2 x\sqrt{\tan x}\,dx$

17. $\int \frac{1}{\sqrt{u}(\sqrt{u}+1)}du$

18. $\int \frac{v}{v^2+4}dv$

19. $\int \frac{4}{x(\ln x+1)^2}dx$

20. $\int \tan 2x\,dx$

21. $\int \frac{(\sin^{-1}x)^3}{\sqrt{1-x^2}}dx$

22. $\int x^2\sec^2 x^3\,dx$

23. (a) $\int \frac{x}{\sqrt{1-x^4}}dx$ (b) $\int \frac{x^3}{\sqrt{1-x^4}}dx$

24. (a) $\int \frac{x^2}{\sqrt{1+x^6}}dx$ (b) $\int \frac{x^5}{\sqrt{1+x^6}}dx$

25. (a) $\int \frac{1+x}{1+x^2}dx$ (b) $\int \frac{1+x}{1-x^2}dx$

26. (a) $\int \frac{3\sqrt{x}}{1+x^3}dx$ (b) $\int \frac{x\sqrt{x}}{1+x^5}dx$

27. $\int \frac{2t+3}{t+7}dt$

28. $\int \frac{t^2}{\sqrt[3]{t+3}}dt$

29. $\int \frac{1}{\sqrt{1+\sqrt{x}}}dx$

30. $\int \frac{1}{x\sqrt{x^4-1}}dx$

In exercises 31–40, evaluate the definite integral.

31. $\int_0^2 x\sqrt{x^2+1}\,dx$

32. $\int_1^2 x\sin(\pi x^2)dx$

33. $\int_{-1}^1 \frac{t}{(t^2+1)^2}dt$

34. $\int_0^2 t^2e^{t^3}dt$

35. $\int_0^2 \frac{e^x}{1+e^{2x}}dx$

36. $\int_0^2 \frac{e^x}{1+e^x}dx$

37. $\int_{\pi/4}^{\pi/2} \cot x\,dx$

38. $\int_1^e \frac{\ln x}{x}dx$

39. $\int_1^4 \frac{x-1}{\sqrt{x}}dx$

40. $\int_0^1 \frac{x}{\sqrt{x^2+1}}dx$

In exercises 41–44, name the method by evaluating the integral exactly, if possible. Otherwise, estimate it numerically.

41. (a) $\int_0^\pi \sin x^2\,dx$ (b) $\int_0^\pi x\sin x^2\,dx$

42. (a) $\int_{-1}^1 xe^{-x^2}dx$ (b) $\int_{-1}^1 e^{-x^2}dx$

43. (a) $\int_0^2 \frac{4x^2}{(x^2+1)^2}dx$ (b) $\int_0^2 \frac{4x^3}{(x^2+1)^2}dx$

44. (a) $\int_0^{\pi/4} \sec x\,dx$ (b) $\int_0^{\pi/4} \sec^2 x\,dx$

In exercises 45–48, make the indicated substitution for an unspecified function $f(x)$.

45. $u = x^2$ for $\int_0^2 xf(x^2)dx$

46. $u = x^3$ for $\int_1^2 x^2f(x^3)dx$

47. $u = \sin x$ for $\int_0^{\pi/2} (\cos x)f(\sin x)dx$

48. $u = \sqrt{x}$ for $\int_0^4 \frac{f(\sqrt{x})}{\sqrt{x}}dx$

49. A function f is said to be **even** if $f(-x) = f(x)$ for all x. A function f is said to be **odd** if $f(-x) = -f(x)$. Suppose that f is continuous for all x. Show that if f is even, then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)\,dx$. Also, if f is odd, show that $\int_{-a}^a f(x)\,dx = 0$.

50. Assume that f is periodic with period T; that is, $f(x+T) = f(x)$ for all x. Show that $\int_0^T f(x)dx = \int_a^{a+T} f(x)dx$ for any real number a. (Hint: First, work with $0 \le a \le T$.)

51. (a) For the integral $I = \int_0^{10} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}dx$, use a substitution to show that $I = \int_0^{10} \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx$. Use these two representations of I to evaluate I.

(b) Generalize to $I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}dx$ for any positive, continuous function f and then quickly evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}dx$.

52. (a) For $I = \int_2^4 \frac{\sin^2(9-x)}{\sin^2(9-x)+\sin^2(x+3)}\,dx$, use the substitution $u = 6 - x$ to show that $I = \int_2^4 \frac{\sin^2(x+3)}{\sin^2(9-x)+\sin^2(x+3)}\,dx$ and evaluate I.

(b) Generalize to $\int_2^4 \frac{f(9-x)}{f(9-x)+f(x+3)}\,dx$, for any positive, continuous function f on [2, 4].

53. Evaluate $\int_0^2 \frac{f(x+4)}{f(x+4)+f(6-x)}\,dx$ for any positive, continuous function f on [0, 2].

54. (a) Use the substitution $u = x^{1/6}$ to evaluate $\int \frac{1}{x^{5/6}+x^{2/3}}\,dx$.

(b) Evaluate $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx$.

(c) Generalize to $\int \frac{1}{x^{(p+1)/q}+x^{p/q}}\,dx$ for positive integers p and q.

55. There are often multiple ways of computing an antiderivative. For $\int \frac{1}{x \ln \sqrt{x}}\,dx$, first use the substitution $u = \ln\sqrt{x}$ to find the indefinite integral $2\ln|\ln\sqrt{x}| + c$. Then rewrite $\ln\sqrt{x}$ and use the substitution $u = \ln x$ to find the indefinite integral $2\ln|\ln x| + c$. Show that these two answers are equivalent.

56. Sketch the area between $y = \pi x - x^2$ and the x-axis for $0 \le x \le 1$ and the area between $y = (\pi\cos x - \cos^2 x)\sin x$ and the x-axis for $0 \le x \le \frac{\pi}{2}$. Show that the areas are equal.

57. Find each mistake in the following calculations and then show how to correctly do the substitution. Start with $\int_{-2}^{1} 4x^4\,dx = \int_{-2}^{1} x(4x^3)\,dx$ and then use the substitution $u = x^4$ with $du = 4x^3 dx$. Then

$$\int_{-2}^{1} x(4x^3)\,dx = \int_{16}^{1} u^{1/4}\,du = \frac{4}{5}u^{5/4}\Big|_{u=16}^{u=1} = \frac{4}{5} - \frac{32}{5} = -\frac{18}{5}$$

58. Find each mistake in the following calculations and then show how to correctly do the substitution. Start with $\int_0^{\pi} \cos^2 x\,dx = \int_0^{\pi} \cos x(\cos x)dx$ and then use the substitution $u = \sin x$ with $du = \cos x\,dx$. Then

$$\int_0^{\pi} \cos x(\cos x)dx = \int_0^0 \sqrt{1-u^2}\,du = 0$$

59. For $a > 0$, show that $\int_a^1 \frac{1}{x^2+1}\,dx = \int_1^{1/a} \frac{1}{x^2+1}\,dx$. Use this equality to derive an identity involving $\tan^{-1} x$.

60. Evaluate $\int \frac{1}{|x|\sqrt{x^2-1}}\,dx$ by rewriting the integrand as $\frac{1}{x^2\sqrt{1-1/x^2}}$ and then making the substitution $u = 1/x$. Use your answer to derive an identity involving $\sin^{-1}(1/x)$ and $\sec^{-1} x$.

APPLICATIONS

1. The location $(\bar{x}, \bar{y})$ of the center of gravity (balance point) of a flat plate bounded by $y = f(x) > 0$, $a \le x \le b$ and the x-axis is given by $\bar{x} = \frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx}$ and $\bar{y} = \frac{\int_a^b [f(x)]^2\,dx}{2\int_a^b f(x)\,dx}$. For the semicircle $y = f(x) = \sqrt{4-x^2}$, use symmetry to argue that $\bar{x} = 0$ and $\bar{y} = \frac{1}{2\pi}\int_0^2 (4-x^2)\,dx$. Compute $\bar{y}$.

2. Suppose that the population density of a group of animals can be described by $f(x) = xe^{-x^2}$ thousand animals per km for $0 \le x \le 2$, where x is the distance from a pond. Graph $y = f(x)$ and briefly describe where these animals are likely to be found. Find the total population $\int_0^2 f(x)\,dx$.

3. The voltage in an AC (alternating current) circuit is given by $V(t) = V_p \sin(2\pi f t)$, where f is the frequency. A voltmeter does not indicate the amplitude V_p. Instead, the voltmeter reads the **root-mean-square** (rms), the square root of the average value of the square of the voltage over one cycle. That is, rms $= \sqrt{f\int_0^{1/f} V^2(t)\,dt}$. Use the trigonometric identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ to show that rms $= V_p/\sqrt{2}$.

4. Graph $y = f(t)$ and find the root-mean-square of

$$f(t) = \begin{cases} -1 & \text{if } -2 \le t < -1 \\ t & \text{if } -1 \le t \le 1, \\ 1 & \text{if } 1 < t \le 2 \end{cases}$$

where rms $= \sqrt{\frac{1}{4}\int_{-2}^{2} f^2(t)\,dt}$

EXPLORATORY EXERCISES

1. A **predator-prey system** is a set of differential equations modeling the change in population of interacting species of organisms. A simple model of this type is

$$\begin{cases} x'(t) = x(t)[a - by(t)] \\ y'(t) = y(t)[dx(t) - c] \end{cases}$$

for positive constants a, b, c and d. Both equations include a term of the form $x(t)y(t)$, which is intended to represent the result of confrontations between the species. Noting that the contribution of this term is negative to $x'(t)$ but positive to $y'(t)$, explain why it must be that $x(t)$ represents the population of the prey and $y(t)$ the population of the predator. If $x(t) = y(t) = 0$, compute $x'(t)$ and $y'(t)$. In this case, will x and y increase, decrease or stay constant? Explain why this makes sense physically. Determine $x'(t)$ and $y'(t)$ and the subsequent change in x and y at the so-called **equilibrium point** $x = c/d$, $y = a/b$. If the population is periodic, we can show that the equilibrium point gives the average population (even if the population does not remain constant). To do so, note that $\frac{x'(t)}{x(t)} = a - by(t)$. Integrating both sides of this equation from $t = 0$ to $t = T$ [the period of $x(t)$ and $y(t)$], we get $\int_0^T \frac{x'(t)}{x(t)}dt = \int_0^T a\,dt - \int_0^T by(t)dt$. Evaluate each integral to show that $\ln x(T) - \ln x(0) = aT - \int_0^T by(t)\,dt$. Assuming that $x(t)$ has period T, we have $x(T) = x(0)$ and so, $0 = aT - \int_0^T by(t)dt$. Finally, rearrange terms to show that $1/T\int_0^T y(t)\,dt = a/b$; that is, the average value of the population $y(t)$ is the equilibrium value $y = a/b$. Similarly, show that the average value of the population $x(t)$ is the equilibrium value $x = c/d$.

2. Define the **Dirac delta** $\delta(x)$, to have the defining property $\int_{-a}^{b} \delta(x)\,dx = 1$ for any $a, b > 0$. Assuming that $\delta(x)$ acts like a continuous function (this is a significant issue!), use this property to evaluate (a) $\int_0^1 \delta(x-2)\,dx$, (b) $\int_0^1 \delta(2x-1)\,dx$ and (c) $\int_{-1}^{1} \delta(2x)\,dx$. Assuming that it applies, use the Fundamental Theorem of Calculus to prove that $\delta(x) = 0$ for all $x \neq 0$ and to prove that $\delta(x)$ is unbounded in $[-1, 1]$. What do you find troublesome about this? Do you think that $\delta(x)$ is really a continuous function, or even a function at all?

3. Suppose that f is a continuous function such that for all x, $f(2x) = 3f(x)$ and $f(x + \frac{1}{2}) = \frac{1}{3} + f(x)$. Compute $\int_0^1 f(x)\,dx$.

1.7 NUMERICAL INTEGRATION

Thus far, our development of the integral has paralleled our development of the derivative. In both cases, we began with a limit definition that was difficult to use for calculation and then proceeded to develop simplified rules for calculation. At this point, you should be able to find the derivative of nearly any function you can write down. You might expect that with a few more rules you will be able to do the same for integrals. Unfortunately, this is not the case. There are many functions for which *no* elementary antiderivative is available. (By elementary antiderivative, we mean an antiderivative expressible in terms of the elementary functions with which you are familiar: the algebraic, trigonometric, exponential and logarithmic functions.) For instance,

$$\int_0^2 \cos(x^2)\,dx$$

cannot be calculated exactly, since $\cos(x^2)$ does not have an elementary antiderivative. (Try to find one, but don't spend much time on it.)

In fact, most definite integrals cannot be calculated exactly. When we can't compute the value of an integral exactly, we do the next best thing: we approximate its value numerically. In this section, we develop three methods of approximating definite integrals. None will replace the built-in integration routine on your calculator or computer. However, by exploring these methods, you will gain a basic understanding of some of the ideas behind more sophisticated numerical integration routines.

Since a definite integral is the limit of a sequence of Riemann sums, any Riemann sum serves as an approximation of the integral,

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(c_i)\,\Delta x,$$

where c_i is any point chosen from the subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$. Further, the larger n is, the better the approximation tends to be. The most common choice of the evaluation points $c_1, c_2, \ldots, c_n$ leads to a method called the **Midpoint Rule**:

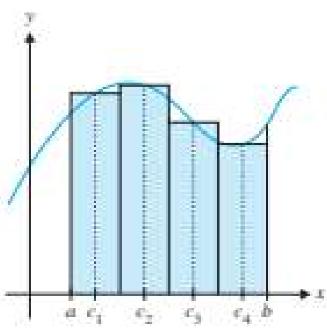


FIGURE 1.26
Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(c_i)\,\Delta x,$$

where c_i is the midpoint of the subinterval $[x_{i-1}, x_i]$,

$$c_i = \frac{1}{2}(x_{i-1} + x_i), \quad \text{for } i = 1, 2, \ldots, n.$$

We illustrate this approximation for the case where $f(x) \geq 0$ on $[a, b]$, in Figure 1.26.

EXAMPLE 7.1 Using the Midpoint Rule

Write out the Midpoint Rule approximation of $\int_0^1 3x^2\,dx$ with $n = 4$.

Solution For $n = 4$, the regular partition of the interval [0, 1] is $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$ and $x_4 = 1$. The midpoints are then $c_1 = \frac{1}{8}$, $c_2 = \frac{3}{8}$, $c_3 = \frac{5}{8}$ and $c_4 = \frac{7}{8}$. With $\Delta x = \frac{1}{4}$, the Riemann sum is then

$$\left[f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+f\left(\frac{5}{8}\right)+f\left(\frac{7}{8}\right)\right]\left(\frac{1}{4}\right) = \left(\frac{3}{64}+\frac{27}{64}+\frac{75}{64}+\frac{147}{64}\right)\left(\frac{1}{4}\right)$$

$$= \frac{252}{256} = 0.984375.$$

Of course, from the Fundamental Theorem, the exact value of the integral in example 7.1 is

$$\int_0^1 3x^2\,dx = \frac{3x^3}{3}\bigg|_0^1 = 1.$$

So, our approximation in example 7.1 is not especially accurate. To obtain greater accuracy, notice that you could always compute an approximation using more rectangles. You can simplify this process by writing a simple program for your calculator or computer to implement the Midpoint Rule. A suggested outline for such a program follows.

MIDPOINT RULE

1. Store $f(x)$, a, b and n.
2. Compute $\Delta x = \dfrac{b-a}{n}$.
3. Compute $c_1 = a + \dfrac{\Delta x}{2}$ and start the sum with $f(c_1)$.
4. Compute the next $c_i = c_{i-1} + \Delta x$ and add $f(c_i)$ to the sum.
5. Repeat step 4 until $i = n$ [i.e., perform step 4 a total of $(n - 1)$ times].
6. Multiply the sum by Δx.

EXAMPLE 7.2 Using a Program for the Midpoint Rule

Repeat example 7.1 using a program to compute the Midpoint Rule approximations for $n = 8$, 16, 32, 64 and 128.

Solution You should confirm the values in the following table. We include a column displaying the error in the approximation for each n (i.e., the difference between the exact value of 1 and the approximate values).

n	*Midpoint Rule*	*Error*
4	0.984375	0.015625
8	0.99609375	0.00390625
16	0.99902344	0.00097656
32	0.99975586	0.00024414
64	0.99993896	0.00006104
128	0.99998474	0.00001526

You should note that each time the number of steps is doubled, the error is reduced approximately by a factor of 4. Although this precise reduction in error will not occur with all integrals, this rate of improvement in the accuracy of the approximation is typical of the Midpoint Rule. ■

Of course, we won't know the error in a Midpoint Rule approximation, except where we know the value of the integral exactly. We started with a simple integral, whose value we knew exactly, so that you could get a sense of how accurate the Midpoint Rule approximation is.

Note that in example 7.3, we can't compute an exact value of the integral, since we do not know an antiderivative for the integrand.

EXAMPLE 7.3 Finding an Approximation with a Given Accuracy

Use the Midpoint Rule to approximate $\int_0^2 \sqrt{x^2+1}\, dx$ accurate to three decimal places.

n	*Midpoint Rule*
10	2.95639
20	2.95751
30	2.95772
40	2.95779

Solution To obtain the desired accuracy, we continue increasing n until it appears unlikely the third decimal will change further. (The size of n will vary substantially from integral to integral.) You should confirm the numbers in the accompanying table.

From the table, we can make the reasonable approximation

$$\int_0^2 \sqrt{x^2+1}\, dx \approx 2.958.$$

While this is reasonable, note that there is no guarantee that the digits shown are correct. To get a guarantee, we will need the error bounds derived later in this section. ■

REMARK 7.1

Computer and calculator programs that estimate the values of integrals face the same challenge we did in example 7.3—that is, knowing when a given approximation is good enough. Such software generally includes sophisticated algorithms for estimating the accuracy of its approximations. You can find an introduction to such algorithms in most texts on numerical analysis.

Another important reason for pursuing numerical methods is for the case where we don't know the function that we're trying to integrate. That's right: we often know only some *values* of a function at a collection of points, while a symbolic representation of a function is unavailable. This is often the case in the physical and biological sciences and engineering, where the only information available about a function comes from measurements made at a finite number of points.

EXAMPLE 7.4 Estimating an Integral from a Table of Function Values

Estimate $\int_0^1 f(x)\, dx$, where we have values of the unknown function $f(x)$ as given in the table shown in the margin.

x	$f(x)$
0.0	1.0
0.25	0.8
0.5	1.3
0.75	1.1
1.0	1.6

Solution Approaching the problem graphically, we have five data points. (See Figure 1.27a.) How can we estimate the area under the curve from five points? Conceptually, we have two tasks. First, we need a reasonable way to connect the given points. Second, we need to compute the area of the resulting region. The most obvious way to connect the dots is with straight-line segments as in Figure 1.27b.

Notice that the region bounded by the graph and the x-axis on the interval [0, 1] consists of four trapezoids. (See Figure 1.27c.)

It's an easy exercise to show that the area of a trapezoid with sides h_1 and h_2 and base b is given by $\left(\frac{h_1+h_2}{2}\right)b$. (Think of this as the average of the areas of the rectangle whose height is the value of the function at the left endpoint and the rectangle whose height is the value of the function at the right endpoint.)

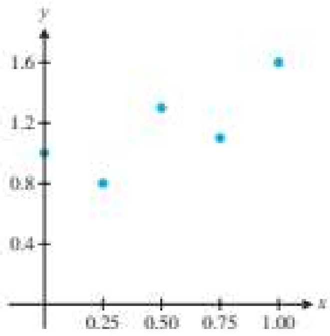


FIGURE 1.27a
Data from an unknown function

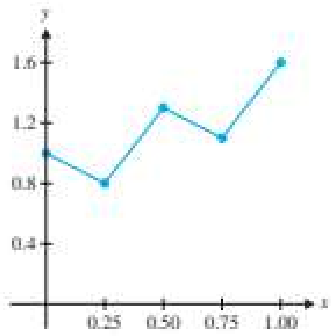


FIGURE 1.27b
Connecting the dots

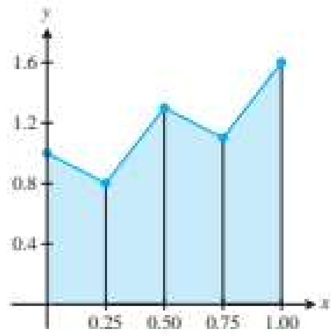


FIGURE 1.27c
Four trapezoids

The total area of the four trapezoids is then

$$\frac{f(0)+f(0.25)}{2}0.25+\frac{f(0.25)+f(0.5)}{2}0.25+\frac{f(0.5)+f(0.75)}{2}0.25$$
$$+\frac{f(0.75)+f(1)}{2}0.25$$
$$=[f(0)+2f(0.25)+2f(0.5)+2f(0.75)+f(1)]\frac{0.25}{2}=1.125. \blacksquare$$

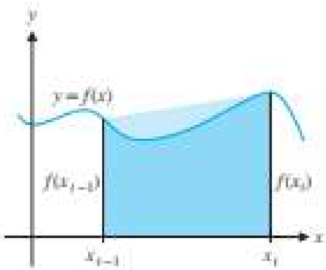


FIGURE 1.28
Trapezoidal Rule

More generally, for any continuous function f defined on the interval $[a, b]$, we partition $[a, b]$ as follows:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

where the points in the partition are equally spaced, with spacing $\Delta x = \frac{b-a}{n}$. On each subinterval $[x_{i-1}, x_i]$, approximate the area under the curve by the area of the trapezoid whose sides have length $f(x_{i-1})$ and $f(x_i)$, as indicated in Figure 1.28. The area under the curve on the interval $[x_{i-1}, x_i]$ is then approximately

$$A_i \approx \frac{1}{2}[f(x_{i-1}) + f(x_i)]\,\Delta x,$$

for each $i = 1, 2, \ldots, n$. Adding together the approximations for the area under the curve on each subinterval, we get that

$$\int_a^b f(x)\,dx \approx \left[\frac{f(x_0)+f(x_1)}{2}+\frac{f(x_1)+f(x_2)}{2}+\cdots+\frac{f(x_{n-1})+f(x_n)}{2}\right]\Delta x$$
$$=\frac{b-a}{2n}[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)].$$

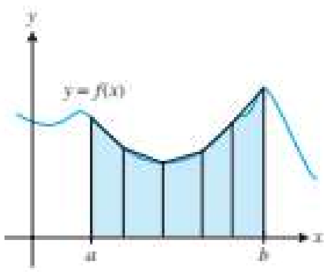


FIGURE 1.29
The $(n + 1)$-point Trapezoidal Rule

We illustrate this in Figure 1.29. Notice that each of the middle terms is multiplied by 2, since each one is used in two trapezoids, once as the height of the trapezoid at the right endpoint and once as the height of the trapezoid at the left endpoint. We refer to this as the $(n + 1)$-point **Trapezoidal Rule**, $T_n(f)$,

Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx T_n(f) = \frac{b-a}{2n}[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)].$$

One way to write a program for the Trapezoidal Rule is to add together $[f(x_{i-1}) + f(x_i)]$ for $i = 1, 2, \ldots, n$ and then multiply by $\Delta x/2$. As discussed in the exercises, an alternative is to add together the Riemann sums using left- and right-endpoint evaluations, and then divide by 2.

EXAMPLE 7.5 Using the Trapezoidal Rule

Compute the Trapezoidal Rule approximations with $n = 4$ (by hand) and $n = 8, 16, 32,$ 64 and 128 (using a program) for $\int_0^1 3x^2\,dx$.

Solution As we saw in examples 7.1 and 7.2, the exact value of this integral is 1. For the Trapezoidal Rule with $n = 4$, we have

$$T_4(f) = \frac{1-0}{(2)(4)}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right]$$
$$= \frac{1}{8}\left(0 + \frac{3}{8} + \frac{12}{8} + \frac{27}{8} + 3\right) = \frac{66}{64} = 1.03125.$$

Using a program, you can easily get the values in the accompanying table.

n	$T_n(f)$	*Error*
4	1.03125	0.03125
8	1.0078125	0.0078125
16	1.00195313	0.00195313
32	1.00048828	0.00048828
64	1.00012207	0.00012207
128	1.00003052	0.00003052

We have included a column showing the error (the absolute value of the difference between the exact value of 1 and the approximate value). Notice that (as with the Midpoint Rule) as the number of steps doubles, the error is reduced by approximately a factor of 4. ■

NOTES

Since the Trapezoidal Rule formula is an average of two Riemann sums, we have

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} T_n(f).$$

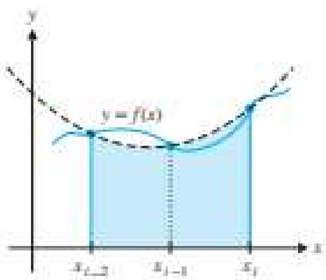


FIGURE 1.30
Simpson's Rule

○ Simpson's Rule

Consider the following alternative to the Trapezoidal Rule. First, construct a regular partition of the interval $[a, b]$:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

where
$$x_i - x_{i-1} = \frac{b-a}{n} = \Delta x,$$

for each $i = 1, 2, \ldots, n$ and where n is an *even* number. Instead of connecting each pair of points with a straight line segment (as we did with the Trapezoidal Rule), we connect each set of *three* consecutive points, $(x_{i-2}, f(x_{i-2}))$, $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$ for $i = 2, 4, \ldots, n$, with a parabola. (See Figure 1.30.) That is, we look for the quadratic function $p(x)$ whose graph passes through these three points, so that

$$p(x_{i-2}) = f(x_{i-2}), \quad p(x_{i-1}) = f(x_{i-1}) \quad \text{and} \quad p(x_i) = f(x_i).$$

Using this to approximate the value of the integral of f on the interval $[x_{i-2}, x_i]$, we have

$$\int_{x_{i-2}}^{x_i} f(x)\,dx \approx \int_{x_{i-2}}^{x_i} p(x)\,dx.$$

Notice why we want to approximate f by a polynomial: polynomials are easy to integrate. A straightforward though tedious computation (try this; your CAS may help) gives

$$\int_{x_{i-2}}^{x_i} f(x)\,dx \approx \int_{x_{i-2}}^{x_i} p(x)\,dx = \frac{x_i - x_{i-2}}{6}[f(x_{i-2}) + 4f(x_{i-1}) + f(x_i)]$$
$$= \frac{b-a}{3n}[f(x_{i-2}) + 4f(x_{i-1}) + f(x_i)].$$

HISTORICAL NOTES

Thomas Simpson (1710–1761)
An English mathematician who popularized the numerical method now known as Simpson's Rule. Trained as a weaver, Simpson also earned a living as a fortune-teller, as the editor of the *Ladies' Diary* and as a textbook author. Simpson's calculus textbook (titled *A New Treatise on Fluxions*, using Newton's calculus terminology) introduced many mathematicians to Simpson's Rule, although the method had been developed years earlier.

Adding together the integrals over each subinterval $[x_{i-2}, x_i]$, for $i = 2, 4, 6, \ldots, n$, we get

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + f(x_2)] + \frac{b-a}{3n}[f(x_2) + 4f(x_3) + f(x_4)] + \cdots \\ &\qquad + \frac{b-a}{3n}[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \\ &= \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)].\end{aligned}$$

Be sure to notice the pattern that the coefficients follow. We refer to this as the $(n+1)$-point **Simpson's Rule,** $S_n(f)$.

SIMPSON'S RULE

$$\int_a^b f(x)\,dx \approx S_n(f) = \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)].$$

Next, we illustrate the use of Simpson's Rule for a simple integral.

EXAMPLE 7.6 Using Simpson's Rule

Approximate the value of $\int_0^1 3x^2\,dx$ using Simpson's Rule with $n = 4$.

Solution We have

$$S_4(f) = \frac{1-0}{(3)(4)}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right] = 1,$$

which is in fact, the *exact* value. Notice that this is far more accurate than the Midpoint and Trapezoidal Rules and yet requires no more effort. ■

Recall that Simpson's Rule computes the area beneath approximating parabolas. Given this, it shouldn't surprise you that Simpson's Rule gives the exact area in example 7.6. As you will discover in the exercises, Simpson's Rule gives exact values of integrals for *any* polynomial of degree 3 or less.

In example 7.7, we illustrate Simpson's Rule for an integral that you do not know how to compute exactly.

EXAMPLE 7.7 Using a Program for Simpson's Rule

Compute Simpson's Rule approximations with $n = 4$ (by hand), $n = 8$, 16, 32, 64 and 128 (using a program) for $\int_0^2 \sqrt{x^2+1}\,dx$.

Solution For $n = 4$, we have

$$\begin{aligned}S_4(f) &= \frac{2-0}{(3)(4)}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + f(2)\right] \\ &= \left(\frac{1}{6}\right)\left[1 + 4\sqrt{\frac{5}{4}} + 2\sqrt{2} + 4\sqrt{\frac{13}{4}} + \sqrt{5}\right] \approx 2.95795560.\end{aligned}$$

n	$S_n(f)$
4	2.9579556
8	2.9578835
16	2.95788557
32	2.95788571
64	2.95788571
128	2.95788572

Using a program, you can easily obtain the values in the accompanying table. Based on these calculations, we would expect 2.9578857 to be a very good approximation of $\int_0^2 \sqrt{x^2+1}\,dx$. ■

Since most graphs curve somewhat, you might expect the parabolas of Simpson's Rule to better track the curve than the line segments of the Trapezoidal Rule. As example 7.8 shows, Simpson's Rule can be much more accurate than either the Midpoint Rule or the Trapezoidal Rule.

EXAMPLE 7.8 Comparing the Midpoint, Trapezoidal and Simpson's Rules

Compute the Midpoint, Trapezoidal and Simpson's Rule approximations of $\int_0^1 \frac{4}{x^2+1}\,dx$ with $n = 10$, $n = 20$, $n = 50$ and $n = 100$. Compare to the exact value of π.

Solution

n	Midpoint Rule	Trapezoidal Rule	Simpson's Rule
10	3.142425985	3.139925989	3.141592614
20	3.141800987	3.141175987	3.141592653
50	3.141625987	3.141525987	3.141592654
100	3.141600987	3.141575987	3.141592654

Compare these values to the exact value of $\pi \approx 3.141592654$. Note that the Midpoint Rule tends to be slightly closer to π than the Trapezoidal Rule, but neither is as close with $n = 100$ as Simpson's Rule is with $n = 10$. ■

REMARK 7.2

Notice that for a given value of n, the number of computations (and hence the effort) required to produce the Midpoint, Trapezoidal and Simpson's Rule approximations are all roughly the same. So, example 7.8 gives an indication of how much more efficient Simpson's Rule is than the other two methods. This is particularly significant when the function $f(x)$ is difficult to evaluate. For instance, in the case of experimental data, each function value $f(x)$ could be the result of an expensive and time-consuming experiment.

In example 7.9, we revise our estimate of the area in Figure 1.27a, first examined in example 7.4.

EXAMPLE 7.9 Using Simpson's Rule with Data

Use Simpson's Rule to estimate $\int_0^1 f(x)\,dx$, where the only information known about f is given in the table of values shown in the margin.

x	$f(x)$
0.0	1.0
0.25	0.8
0.5	1.3
0.75	1.1
1.0	1.6

Solution From Simpson's Rule with $n = 4$, we have

$$\int_0^1 f(x)\,dx \approx \frac{1-0}{(3)(4)}[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)]$$
$$= \left(\frac{1}{12}\right)[1 + 4(0.8) + 2(1.3) + 4(1.1) + 1.6] \approx 1.066667.$$

Since Simpson's Rule is generally much more accurate than the Trapezoidal Rule (for the same number of points), we expect that this approximation is more accurate than the approximation of 1.125 found in example 7.4 via the Trapezoidal Rule. ■

REMARK 7.3

Most graphing calculators and computer algebra systems have very fast and accurate programs for numerical approximation of definite integrals. Some ask you to specify an error tolerance and then calculate a value accurate to within that tolerance. Most calculators and CAS's use *adaptive quadrature* routines, which automatically calculate how many points are needed to obtain a desired accuracy. You should feel comfortable using these programs. If the integral you are approximating is a critical part of an important project, you can check your result by using Simpson's Rule, $S_n(f)$, for a sequence of values of n. Of course, if all you know about a function is its value at a fixed number of points, most calculator and CAS programs will not help you, but the three methods discussed here will, as we saw in examples 7.4 and examples 7.9. We will pursue this idea further in the exercises.

○ Error Bounds for Numerical Integration

We have used examples where we know the value of an integral exactly to compare the accuracy of our three numerical integration methods. However, in practice, where the value of an integral is not known exactly, how do we determine how accurate a given numerical estimate is? In Theorems 7.1 and 7.2, we give bounds on the error in our three numerical integration methods. First, we introduce some notation. Let ET_n represent the error in using the $(n + 1)$-point Trapezoidal Rule to approximate $\int_a^b f(x)\,dx$. That is,

$$ET_n = \text{exact} - \text{approximate} = \int_a^b f(x)\,dx - T_n(f).$$

Similarly, we denote the error in the Midpoint Rule and Simpson's Rule by EM_n and ES_n, respectively. We now have

THEOREM 7.1

Suppose that f'' is continuous on $[a, b]$ and that $|f''(x)| \le K$, for all x in $[a, b]$. Then,

$$|ET_n| \le K\frac{(b-a)^3}{12n^2}$$

and

$$|EM_n| \le K\frac{(b-a)^3}{24n^2}.$$

Notice that both of the estimates in Theorem 7.1 say that the error in using the indicated numerical method is no larger (in absolute value) than the given bound. This says that if the bound is small, so too will be the error. In particular, observe that the error bound for the Midpoint Rule is *half* that for the Trapezoidal Rule. This doesn't say that the actual error in the Midpoint Rule will be half that of the Trapezoidal Rule, but it does explain why the Midpoint Rule tends to be somewhat more accurate than the Trapezoidal Rule for the same value of n. Also notice that the constant K depends on $|f''(x)|$. The larger $|f''(x)|$ is, the more the graph curves and consequently, the less accurate are the straight-line approximations of the Midpoint Rule and the Trapezoidal Rule. An error bound for Simpson's Rule follows.

THEOREM 7.2

Suppose that $f^{(4)}$ is continuous on $[a, b]$ and that $|f^{(4)}(x)| \le L$, for all x in $[a, b]$. Then,

$$|ES_n| \le L\frac{(b-a)^5}{180n^4}.$$

The proofs of Theorems 7.1 and 7.2 are beyond the level of this course and we refer the interested reader to a text on numerical analysis. In comparing Theorems 7.1 and 7.2, notice that the denominators of the error bounds for both the Trapezoidal Rule and the Midpoint Rule contain a factor of n^2, while the error bound for Simpson's Rule contains a factor of n^4. For $n = 10$, observe that $n^2 = 100$, while $n^4 = 10{,}000$. Since these powers of n are in the denominators of the error bounds, this says that the error bound for Simpson's Rule tends to be *much* smaller than that of either the Trapezoidal Rule or the Midpoint Rule for the same value of n. This accounts for the far greater accuracy we have seen with using Simpson's Rule over the other two methods. We illustrate the use of the error bounds in example 7.10.

EXAMPLE 7.10 Finding a Bound on the Error in Numerical Integration

Find bounds on the error in using each of the Midpoint Rule, the Trapezoidal Rule and Simpson's Rule to approximate the value of the integral $\int_1^3 \frac{1}{x}\,dx$, using $n = 10$.

Solution Your first inclination might be to observe that you already know the value of this integral exactly, since by the Fundamental Theorem of Calculus,

$$\int_1^3 \frac{1}{x}\,dx = \ln|x|\Big|_1^3 = \ln 3 - \ln 1 = \ln 3.$$

However, you don't really know the value of ln 3, but must use your calculator to compute an approximate value of this. On the other hand, you can approximate this integral using Trapezoidal, Midpoint or Simpson's Rules. Here, $f(x) = 1/x = x^{-1}$, so that $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$ and $f^{(4)}(x) = 24x^{-5}$. This says that for $x \in [1, 3]$,

$$|f''(x)| = |2x^{-3}| = \frac{2}{x^3} \le 2.$$

From Theorem 7.1, we now have

$$|EM_{10}| \le K\frac{(b-a)^3}{24n^2} = 2\frac{(3-1)^3}{24(10^2)} \approx 0.006667.$$

Similarly, we have

$$|ET_{10}| \le K\frac{(b-a)^3}{12n^2} = 2\frac{(3-1)^3}{12(10^2)} \approx 0.013333.$$

Turning to Simpson's Rule, for $x \in [1, 3]$, we have $S_{10}(f) \approx 1.09866$ and

$$|f^{(4)}(x)| = |24x^{-5}| = \frac{24}{x^5} \le 24,$$

so that Theorem 7.2 now gives us

$$|ES_{10}| \le L\frac{(b-a)^5}{180n^4} = 24\frac{(3-1)^5}{180(10^4)} \approx 0.000427.$$ ■

From example 7.10, we now know that the Simpson's Rule approximation $S_{10}(f) \approx 1.09866$ is off by no more than about 0.000427. However, a more interesting question is to determine the number of points needed to obtain a given accuracy. We explore this in example 7.11.

EXAMPLE 7.11 Determining the Number of Steps That Guarantee a Given Accuracy

Determine the number of steps that will guarantee an accuracy of at least 10^{-7} for using each of Trapezoidal Rule and Simpson's Rule to approximate $\int_1^3 \frac{1}{x}\,dx$.

Solution From example 7.10, we know that $|f''(x)| \le 2$ and $|f^{(4)}(x)| \le 24$, for all $x \in [1, 3]$. So, from Theorem 7.1, we now have that

$$|ET_n| \le K\frac{(b-a)^3}{12n^2} = 2\frac{(3-1)^3}{12n^2} = \frac{4}{3n^2}.$$

If we require the above bound on the error to be no larger than the required accuracy of 10^{-7}, we have

$$|ET_n| \le \frac{4}{3n^2} \le 10^{-7}.$$

Solving this inequality for n^2 gives us

$$\frac{4}{3}10^7 \le n^2$$

and taking the square root of both sides yields

$$n \ge \sqrt{\frac{4}{3}10^7} \approx 3651.48.$$

So, any value of $n \ge 3652$ will give the required accuracy. Similarly, for Simpson's Rule, we have

$$|ES_n| \le L\frac{(b-a)^5}{180n^4} = 24\frac{(3-1)^5}{180n^4}.$$

Again, requiring that the error bound be no larger than 10^{-7} gives us

$$|ES_n| \le 24\frac{(3-1)^5}{180n^4} \le 10^{-7}$$

and solving for n^4, we have $$n^4 \ge 24\frac{(3-1)^5}{180}10^7.$$

Upon taking fourth roots, we get

$$n \ge \sqrt[4]{24\frac{(3-1)^5}{180}10^7} \approx 80.8,$$

so that taking any value of $n \ge 82$ will guarantee the required accuracy. (If you expected us to say that $n \ge 81$, keep in mind that Simpson's Rule requires n to be even.) ■

In example 7.11, compare the number of steps required to guarantee 10^{-7} accuracy in Simpson's Rule (82) to the number required to guarantee the same accuracy in the Trapezoidal Rule (3652). Simpson's Rule typically requires far fewer steps than either the Trapezoidal Rule or the Midpoint Rule to get the same accuracy. Finally, from example 7.11, observe that we now know that

$$\ln 3 = \int_1^3 \frac{1}{x}dx \approx S_{82} \approx 1.0986123,$$

which is guaranteed (by Theorem 7.2) to be correct to within 10^{-7}. Compare this with the approximate value of ln 3 generated by your calculator.

EXERCISES 1.7

WRITING EXERCISES

1. Ideally, approximation techniques should be both simple and accurate. How do the numerical integration methods presented in this section compare in terms of simplicity and accuracy? Which criterion would be more important if you were working entirely by hand? Which method would you use? Which criterion would be more important if you were using a very fast computer? Which method would you use?

2. Suppose you were going to construct your own rule for approximate integration. (Name it after yourself!) In the text, new methods were obtained both by choosing evaluation points for Riemann sums (Midpoint Rule) and by geometric construction (Trapezoidal Rule and Simpson's Rule). Without working out the details, explain how you would develop a very accurate but simple rule.

3. Test your calculator or computer on $\int_0^1 \sin(1/x)dx$. Discuss what your options are when your technology does not immediately return an accurate approximation. Based on a quick sketch of $y = \sin(1/x)$, describe why a numerical integration routine would have difficulty with this integral.

4. Explain why we did not use the Midpoint Rule in example 7.4.

In exercises 1–4, compute Midpoint, Trapezoidal and Simpson's Rule approximations by hand (leave your answer as a fraction) for $n = 4$.

1. $\int_0^1 (x^2 + 1)\,dx$
2. $\int_0^2 (x^2 + 1)\,dx$
3. $\int_1^3 \frac{1}{x}\,dx$
4. $\int_{-1}^1 (2x - x^2)\,dx$

In exercises 5–8, approximate the given value using (a) Midpoint Rule, (b) Trapezoidal Rule and (c) Simpson's Rule with $n = 4$. Determine if each approximation is too small or too large.

5. $\ln 4 = \int_1^4 \frac{1}{x}\,dx$
6. $\ln 8 = \int_1^8 \frac{1}{x}\,dx$
7. $\sin 1 = \int_0^1 \cos x\,dx$
8. $e^2 = \int_0^1 (2e^{2x} + 1)\,dx$

In exercises 9–14, use a computer or calculator to compute the Midpoint, Trapezoidal and Simpson's Rule approximations with $n = 10$, $n = 20$ and $n = 50$. Compare these values to the approximation given by your calculator or computer.

9. $\int_0^{\pi} \cos x^2\,dx$
10. $\int_0^{\pi/4} \sin \pi x^2\,dx$
11. $\int_0^2 e^{-x^2}\,dx$
12. $\int_0^3 e^{-x^2}\,dx$
13. $\int_0^{\pi} e^{\cos x}\,dx$
14. $\int_0^1 \sqrt[3]{x^2 + 1}\,dx$

In exercises 15–18, compute the exact value and compute the error (the difference between the approximation and the exact value) in each of the Midpoint, Trapezoidal and Simpson's Rule approximations using $n = 10$, $n = 20$, $n = 40$ and $n = 80$.

15. $\int_0^1 5x^4\,dx$
16. $\int_1^2 \frac{1}{x}\,dx$
17. $\int_0^{\pi} \cos x\,dx$
18. $\int_0^{\pi/4} \cos x\,dx$

19. Fill in the blanks with the most appropriate power of 2 (2, 4, 8, etc.). If you double n, the error in the Midpoint Rule is divided by ______. If you double n, the error in the Trapezoidal Rule is divided by ______. If you double n, the error in Simpson's Rule is divided by ______.

20. Fill in the blanks with the most appropriate power of 2 (2, 4, 8, etc.). If you halve the interval length $b - a$, the error in the Midpoint Rule is divided by ______, the error in the Trapezoidal Rule is divided by ______, and the error in Simpson's Rule is divided by ______.

In exercises 21 and 22, use (a) Trapezoidal Rule and (b) Simpson's Rule to estimate $\int_0^2 f(x)\,dx$ from the given data.

21.

x	0.0	0.25	0.5	0.75	1.0
$f(x)$	4.0	4.6	5.2	4.8	5.0

x	1.25	1.5	1.75	2.0
$f(x)$	4.6	4.4	3.8	4.0

22.

x	0.0	0.25	0.5	0.75	1.0
$f(x)$	1.0	0.6	0.2	−0.2	−0.4

x	1.25	1.5	1.75	2.0
$f(x)$	0.4	0.8	1.2	2.0

23. For exercise 5, (a) find bounds on the errors made by each method. (b) Find the number of steps needed to guarantee an accuracy of 10^{-7}.

24. For exercise 7, (a) find bounds on the errors made by each method. (b) Find the number of steps needed to guarantee an accuracy of 10^{-7}.

In exercises 25–28, determine the number of steps to guarantee an accuracy of 10^{-6} using (a) the Trapezoidal Rule; (b) the Midpoint Rule; (c) Simpson's Rule.

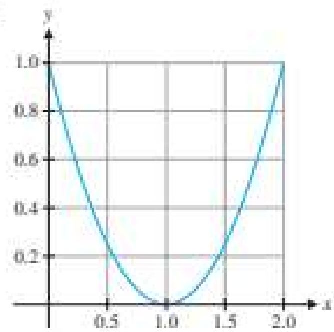
25. $\int_1^2 \ln x\,dx$
26. $\int_1^4 x \ln x\,dx$
27. $\int_0^1 e^{-x^2}\,dx$
28. $\int_1^2 xe^x\,dx$

29. For each rule in exercise 15, compute the error bound and compare it to the actual error.

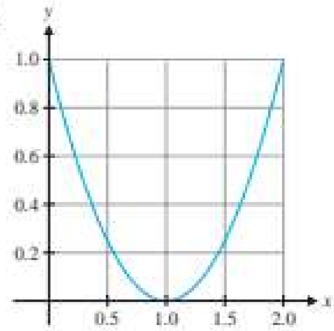
30. For each rule in exercise 17, compute the error bound and compare it to the actual error.

In exercises 31 and 32, use the graph to estimate (a) Riemann sum with left-endpoint evaluation, (b) Midpoint Rule, (c) Trapezoidal Rule, and (d) Simpson's Rule approximations with $n = 4$ of $\int_0^2 f(x)\,dx$.

31.



32.



In exercises 33–38, use the given information about $f(x)$ and its derivatives to determine whether (a) the Midpoint Rule would be exact, underestimate or overestimate the integral (or there's not enough information to tell). Repeat for (b) Trapezoidal Rule and (c) Simpson's Rule.

33. $f''(x) > 0, f'(x) > 0$

34. $f''(x) > 0, f'(x) < 0$

35. $f''(x) < 0, f'(x) > 0$

36. $f''(x) < 0, f'(x) < 0$

37. $f''(x) = 4, f'(x) > 0$

38. $f''(x) = 0, f'(x) > 0$

39. Suppose that R_L and R_R are the Riemann sum approximations of $\int_a^b f(x)\,dx$ using left- and right-endpoint evaluation rules, respectively, for some $n > 0$. Show that the trapezoidal approximation T_n is equal to $(R_L + R_R)/2$.

40. For the data in Figure 1.27a, sketch in the two approximating parabolas for Simpson's Rule. Compare the Simpson's Rule approximation to the Trapezoidal Rule approximation. Explain graphically why the Simpson's Rule approximation is smaller.

41. Show that both $\int_0^1 \sqrt{1-x^2}\,dx$ and $\int_0^1 \frac{1}{1+x^2}\,dx$ equal $\frac{\pi}{4}$. Use Simpson's Rule on each integral with $n = 4$ and $n = 8$ and compare to the exact value. Which integral provides a better algorithm for estimating π?

42. Prove the following formula, which is basic to Simpson's Rule. If $f(x) = Ax^2 + Bx + C$, then $\int_{-h}^h f(x)\,dx = \frac{h}{3}[f(-h) + 4f(0) + f(h)]$.

43. A commonly used type of numerical integration algorithm is called **Gaussian quadrature.** For an integral on the interval $[-1, 1]$, a simple Gaussian quadrature approximation is $\int_{-1}^1 f(x)\,dx \approx f(\frac{-1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$. Show that, like Simpson's Rule, this Gaussian quadrature gives the exact value of the integrals of the power functions x, x^2 and x^3.

44. Referring to exercise 43, compare the Simpson's Rule ($n = 2$) and Gaussian quadrature approximations of $\int_{-1}^1 \pi\cos(\frac{\pi x}{2})\,dx$ to the exact value.

45. Explain why Simpson's Rule can't be used to approximate $\int_0^\pi \frac{\sin x}{x}\,dx$. Find $L = \lim_{x\to 0}\frac{\sin x}{x}$ and argue that if $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ L & \text{if } x = 0 \end{cases}$ then $\int_0^\pi f(x)\,dx = \int_0^\pi \frac{\sin x}{x}\,dx$. Use an appropriate numerical method to conjecture that $\int_0^\pi \frac{\sin x}{x}\,dx \approx 1.18\left(\frac{\pi}{2}\right)$.

46. As in exercise 45, approximate $\int_{-\pi/2}^{\pi/2} \frac{\sin x}{x}\,dx$.

47. In most of the calculations that you have done, it is true that the Trapezoidal Rule and Midpoint Rule are on opposite sides of the exact integral (i.e., one is too large, the other too small). Also, you may have noticed that the Trapezoidal Rule tends to be about twice as far from the exact value as the Midpoint Rule. Given this, explain why the linear combination $\frac{1}{3}T_n + \frac{2}{3}M_n$ should give a good estimate of the integral. (Here, T_n represents the Trapezoidal Rule approximation using n partitions and M_n the corresponding Midpoint Rule approximation.)

48. Show that the approximation rule $\frac{1}{3}T_n + \frac{2}{3}M_n$ in exercise 47 is identical to Simpson's Rule.

49. For $f(x) = \frac{x^2}{2x^2 - 2x + 1}$, show that $f(x) + f(1 - x) = 1$ for $0 \le x \le 1$. Show that this implies that the Trapezoidal Rule approximation of $\int_0^1 f(x)\,dx = \frac{1}{2}$ for any n. This is, in fact, the exact value of the integral. (For more information, see M. A. Khan's article in the January 2008 issue of *College Mathematics Journal.*)

50. Show that the Trapezoidal Rule approximation of $\int_0^n x^n\,dx$ is too large (if $n > 1$). Conclude that

$$1^n + 2^n + 3^n + \cdots + n^n > n^n\frac{3n+1}{2n+2}.$$

APPLICATIONS

In exercises 1 and 2, the velocity of an object at various times is given. Use the data to estimate the distance traveled.

1.

$t(s)$	0	1	2	3	4	5	6
$v(t)$ (m/s)	40	42	40	44	48	50	46

$t(s)$	7	8	9	10	11	12
$v(t)$ (m/s)	46	42	44	40	42	42

2.

$t(s)$	0	2	4	6	8	10	12
$v(t)$ (m/s)	26	30	28	30	28	32	30

$t(s)$	14	16	18	20	22	24
$v(t)$ (m/s)	33	31	28	30	32	32

In exercises 3 and 4, the data come from a pneumotachograph, which measures air flow through the throat (in liters per second). The integral of the air flow equals the volume of air exhaled. Estimate this volume.

3.

$t(s)$	0	0.2	0.4	0.6	0.8	1.0	1.2
$f(t)$ (l/s)	0	0.2	0.4	1.0	1.6	2.0	2.2

$t(s)$	1.4	1.6	1.8	2.0	2.2	2.4
$f(t)$ (l/s)	2.0	1.6	1.2	0.6	0.2	0

4.

$t(s)$	0	0.2	0.4	0.6	0.8	1.0	1.2
$f(t)$ (l/s)	0	0.1	0.4	0.8	1.4	1.8	2.0

$t(s)$	1.4	1.6	1.8	2.0	2.2	2.4
$f(t)$ (l/s)	2.0	1.6	1.0	0.6	0.2	0

EXPLORATORY EXERCISES

1. Compute the Trapezoidal Rule approximations T_4, T_8 and T_{16} of $\int_0^1 3x^2\,dx$, and compute the error for each. Verify that when the step size is cut in half, the error is divided by four. When such patterns emerge, they can be taken advantage of using **extrapolation.** Given that $(T_4 - I) = 4(T_8 - I)$, where $I = 1$ is the exact integral, show that $I = T_8 + \frac{T_8 - T_4}{3}$. Also, show that

$I = T_{16} + \frac{T_{16} - T_8}{3}$. In general, we have the *approximations* $(T_4 - I) \approx 4(T_8 - I)$ and $I \approx T_8 + \frac{T_8 - T_4}{3}$. Then the extrapolation $E_{2n} = T_{2n} + \frac{T_{2n} - T_n}{3}$ is closer to the exact integral than either of the individual Trapezoidal Rule approximations T_{2n} and T_n. Show that, in fact, E_{2n} equals the Simpson's Rule approximation for $2n$.

2. The geometric construction of Simpson's Rule makes it clear that Simpson's Rule will compute integrals such as $\int_0^1 3x^2\,dx$ exactly. Briefly explain why. Now, compute Simpson's Rule with $n = 2$ for $\int_0^1 4x^3\,dx$. Simpson's Rule also computes integrals of cubics exactly. In this exercise, we see why a method that uses parabolas can compute integrals of cubics exactly. To see how Simpson's Rule works on $\int_0^1 4x^3\,dx$, we need to determine the actual parabola being used. The parabola must pass through the points $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(1, 4)$. Find the quadratic function $y = ax^2 + bx + c$ that accomplishes this. (Hint: Explain why $0 = 0 + 0 + c$, $\frac{1}{2} = \frac{a}{4} + \frac{b}{2} + c$ and $4 = a + b + c$, and then solve for a, b and c.) Graph this parabola and $y = 4x^3$ on the same axes, carefully choosing the graphing window so that you can see what is happening on the interval $[0, 1]$. Where is the vertex of the parabola? How do the integrals of the parabola and cubic compare on the subinterval $[0, \frac{1}{2}]$? $[\frac{1}{2}, 1]$? Why does Simpson's Rule compute the integral exactly?

1.8 THE NATURAL LOGARITHM AS AN INTEGRAL

In most instances, we define the natural logarithm as the ordinary logarithm with base e. That is,

$$\ln x = \log_e x,$$

where e is a (so far) mysterious transcendental number, $e \approx 2.718\ldots$ So, what is *natural* or even interesting about such a seemingly unusual function? We will resolve both of these issues in this section.

First, recall the power rule for integrals,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad \text{for } n \neq -1.$$

Of course, this rule doesn't hold for $n = -1$, since this would result in division by zero. Assume, for the moment, that we have not yet defined $\ln x$. Then, what can we say about

$$\int \frac{1}{x}\,dx?$$

(Although we found this integral in section 1.1, our observation hinged on the conjecture that $\frac{d}{dx}\ln x = \frac{1}{x}$, whose proof is not yet complete.) From Theorem 4.1, we know that since $f(x) = \frac{1}{x}$ is continuous for $x \neq 0$, it must be integrable on any interval not including $x = 0$. Notice that by Part II of the Fundamental Theorem of Calculus,

$$\int_1^x \frac{1}{t}\,dt$$

is an antiderivative of $\frac{1}{x}$ for $x > 0$. We give this new (and naturally arising) function a name in Definition 8.1.

DEFINITION 8.1

For $x > 0$, we define the **natural logarithm** function, written $\ln x$, by

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

We'll see later that this definition is, in fact, consistent with how we define the function in most instances. First, let's interpret this function graphically. Notice that for $x > 1$,

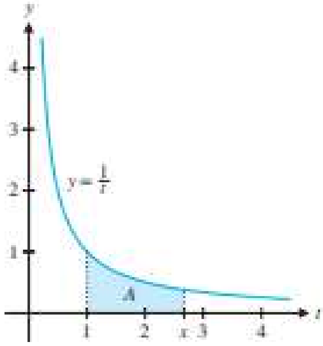


FIGURE 1.31a
$\ln x\ (x > 1)$

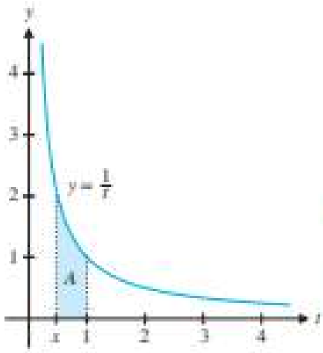


FIGURE 1.31b
$\ln x\ (0 < x < 1)$

this definite integral corresponds to the area A under the curve $y = \frac{1}{t}$ from 1 to x, as indicated in Figure 1.31a. That is,

$$\ln x = \int_1^x \frac{1}{t}\,dt = A > 0.$$

Similarly, for $0 < x < 1$, notice from Figure 1.31b that for the area A under the curve $y = \frac{1}{t}$ from x to 1, we have

$$\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt = -A < 0.$$

Using Definition 8.1, we get by Part II of the Fundamental Theorem of Calculus that

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}, \quad \text{for } x > 0, \tag{8.1}$$

which is the same derivative formula we had obtained in section 2.7.

Recall that we had shown in section 1.1 that for any $x \neq 0$, we can extend (8.1) to obtain $\frac{d}{dx}\ln|x| = \frac{1}{x}$. This, in turn, gives us the familiar integration rule

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + c.}$$

EXAMPLE 8.1 Approximating Several Values of the Natural Logarithm

Approximate the value of ln 2 and ln 3.

Solution Notice that since $\ln x$ is defined by a definite integral, we can use any convenient numerical integration method to compute approximate values of the function. For instance, using Simpson's Rule, we obtain

$$\ln 2 = \int_1^2 \frac{1}{t}\,dt \approx 0.693147$$

and

$$\ln 3 = \int_1^3 \frac{1}{t}\,dt \approx 1.09861.$$

We leave the details of these approximations as exercises. (You should also check the values with the 'ln' key on your calculator.) ■

We now briefly sketch a graph of $y = \ln x$. As we've already observed, the domain of $f(x) = \ln x$ is $(0, \infty)$ and

$$\ln x \begin{cases} < 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x > 1. \end{cases}$$

Further, we've shown that

$$f'(x) = \frac{1}{x} > 0, \quad \text{for } x > 0,$$

so that f is increasing throughout its domain. Next,

$$f''(x) = -\frac{1}{x^2} < 0, \quad \text{for } x > 0$$

and hence, the graph is concave down everywhere. You can easily use Simpson's Rule or the Trapezoidal Rule (this is left as an exercise) to make the conjectures

$$\lim_{x\to\infty} \ln x = \infty \tag{8.2}$$

and

$$\lim_{x\to 0^+} \ln x = -\infty. \tag{8.3}$$

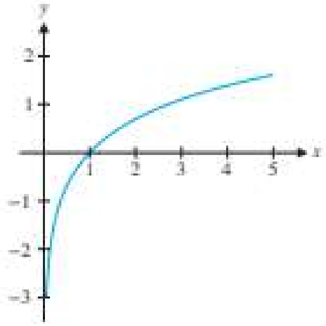


FIGURE 1.32
$y = (\ln x)$

We postpone the proof of (8.2) until after Theorem 8.1. The proof of (8.3) is left as an exercise. We now obtain the graph pictured in Figure 1.32.

Now, it remains for us to explain why this function should be called a *logarithm*. The answer is simple: it satisfies all of the properties satisfied by other logarithms. Since $\ln x$ behaves like any other logarithm, we call it (what else?) a logarithm. We summarize this in Theorem 8.1.

THEOREM 8.1

For any real numbers $a, b > 0$ and any rational number r,

(i) $\ln 1 = 0$
(ii) $\ln (ab) = \ln a + \ln b$
(iii) $\ln \left(\frac{a}{b}\right) = \ln a - \ln b$ and
(iv) $\ln (a^r) = r \ln a$.

PROOF

(i) From Definition 8.1,

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0.$$

(ii) Also from the definition, we have

$$\ln (ab) = \int_1^{ab} \frac{1}{t}\,dt = \int_1^{a} \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt,$$

from part (ii) of Theorem 4.2 in section 1.4. Make the substitution $u = \frac{t}{a}$ in the last integral only. This gives us $du = \frac{1}{a}dt$. Finally, the limits of integration must be changed to reflect the new variable (when $t = a$, we have $u = \frac{a}{a} = 1$, and when $t = ab$, we have $u = \frac{ab}{a} = b$) to yield

$$\ln (ab) = \int_1^{a} \frac{1}{t}\,dt + \int_a^{ab} \underbrace{\frac{a}{t}}_{\frac{1}{u}} \underbrace{\left(\frac{1}{a}\right)dt}_{du}$$

$$= \int_1^{a} \frac{1}{t}\,dt + \int_1^{b} \frac{1}{u}\,du = \ln a + \ln b.$$ From Definition 8.1

(iv) Note that

$$\frac{d}{dx}\ln (x^r) = \frac{1}{x^r}\frac{d}{dx}x^r$$ From (8.1) and the chain rule.

$$= \frac{1}{x^r} r x^{r-1} = \frac{r}{x}.$$ From the power rule.

Likewise,

$$\frac{d}{dx}[r \ln x] = r\frac{d}{dx}(\ln x) = \frac{r}{x}.$$

Now, since $\ln (x^r)$ and $r \ln x$ have the same derivative, it follows from Corollary 10.1 in section 2.10 that for all $x > 0$,

$$\ln (x^r) = r \ln x + k,$$

for some constant, k. In particular, taking $x = 1$, we find that

$$\ln(1^r) = r \ln 1 + k,$$

where since $1^r = 1$ and $\ln 1 = 0$, we have

$$0 = r(0) + k.$$

So, $k = 0$ and $\ln(x^r) = r \ln x$, for all $x > 0$.

Part (iii) follows from (ii) and (iv) and is left as an exercise. ■

Using the properties of logarithms will often simplify the calculation of derivatives. We illustrate this in example 8.2.

EXAMPLE 8.2 Using Properties of Logarithms to Simplify Differentiation

Find the derivative of $\ln \sqrt{\dfrac{(x-2)^3}{x^2+5}}$.

Solution Rather than directly differentiating this expression by applying the chain rule and the quotient rule, notice that we can considerably simplify our work by first using the properties of logarithms. We have

$$\begin{aligned}
\frac{d}{dx}\ln\sqrt{\frac{(x-2)^3}{x^2+5}} &= \frac{d}{dx}\ln\left[\frac{(x-2)^3}{x^2+5}\right]^{1/2} \\
&= \frac{1}{2}\frac{d}{dx}\ln\left[\frac{(x-2)^3}{x^2+5}\right] && \text{From Theorem 8.1 (iv).} \\
&= \frac{1}{2}\frac{d}{dx}[\ln(x-2)^3 - \ln(x^2+5)] && \text{From Theorem 8.1 (iii).} \\
&= \frac{1}{2}\frac{d}{dx}[3\ln(x-2) - \ln(x^2+5)] && \text{From Theorem 8.1 (iv).} \\
&= \frac{1}{2}\left[3\left(\frac{1}{x-2}\right)\frac{d}{dx}(x-2) - \left(\frac{1}{x^2+5}\right)\frac{d}{dx}(x^2+5)\right] && \text{From (8.1) and the chain rule.} \\
&= \frac{1}{2}\left(\frac{3}{x-2} - \frac{2x}{x^2+5}\right).
\end{aligned}$$

Compare our work here to computing the derivative directly using the original expression to see how the rules of logarithms have simplified our work. ■

EXAMPLE 8.3 Examining the Limiting Behavior of ln x

Use the properties of logarithms in Theorem 8.1 to prove that

$$\lim_{x\to\infty} \ln x = \infty.$$

Solution We can verify this as follows. First, recall that $\ln 3 \approx 1.0986 > 1$. Taking $x = 3^n$, we have by the rules of logarithms that for any integer n,

$$\ln 3^n = n \ln 3.$$

Since $x = 3^n \to \infty$, as $n \to \infty$, it now follows that

$$\lim_{x\to\infty} \ln x = \lim_{n\to\infty} \ln 3^n = \lim_{n\to\infty}(n \ln 3) = +\infty,$$

where the first equality depends on the fact that $\ln x$ is a strictly increasing function. ■

The Exponential Function as the Inverse of the Natural Logarithm

Next, we revisit the natural exponential function, e^x. As we did with the natural logarithm, we now carefully define this function and develop its properties. First, recall that we gave the (usual) mysterious description of e as an irrational number $e = 2.71828\ldots$, without attempting to explain why this number is significant. We then proceeded to define $\ln x$ as $\log_e x$, the logarithm with base e. Now that we have carefully defined $\ln x$ (independent of the definition of e), we can clearly define e, as well as calculate its approximate value.

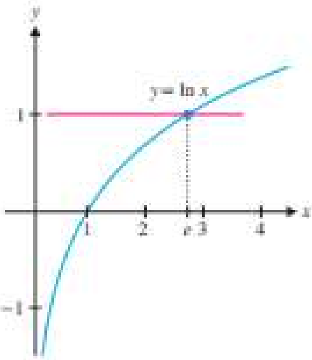


FIGURE 1.33
Definition of e

DEFINITION 8.2

We define e to be that number for which

$$\ln e = 1.$$

That is, e is the x-coordinate of the point of intersection of the graphs of $y = \ln x$ and $y = 1$. (See Figure 1.33.) In other words, e is the solution of the equation

$$\ln x - 1 = 0.$$

You can solve this equation approximately (e.g., using Newton's method) to obtain

$$e \approx 2.71828182846.$$

Previously, we had defined e by $e = \lim_{n\to\infty} (1 + 1/n)^n$. We leave it as an exercise to show that these two definitions of e define the same number. So, having defined the irrational number e, you might wonder what the big deal is with defining the function e^x? Of course, there's no problem at all, when x is rational. For instance, we have

$$\begin{aligned} e^2 &= e \cdot e \\ e^3 &= e \cdot e \cdot e \\ e^{1/2} &= \sqrt{e} \\ e^{5/7} &= \sqrt[7]{e^5} \end{aligned}$$

and so on. In fact, for any rational power, $x = p/q$ (where p and q are integers), we have

$$e^x = e^{p/q} = \sqrt[q]{e^p}.$$

On the other hand, what does it mean to raise a number to an *irrational* power? For instance, what is e^π? Previously, we gave a necessarily vague answer to this important question. We are now in a position to give a complete answer.

First, observe that for $f(x) = \ln x$ $(x > 0)$, $f'(x) = 1/x > 0$. So, f is a strictly increasing and hence one-to-one function, which therefore has an inverse, f^{-1}. As is often the case, there is no algebraic method of solving for this inverse function. However, from Theorem 8.1 (iv), we have that for any rational power x,

$$\ln(e^x) = x \ln e = x,$$

since we had defined e so that $\ln e = 1$. Observe that this says that

$$f^{-1}(x) = e^x, \quad \text{for } x \text{ rational.}$$

That is, the (otherwise unknown) inverse function, $f^{-1}(x)$, agrees with e^x at every rational number x. Since e^x so far has no meaning when x is irrational, we now *define* it to be the value of $f^{-1}(x)$, as follows.

DEFINITION 8.3

For x irrational, we define $y = e^x$ to be that number for which

$$\ln y = \ln(e^x) = x.$$

This says that for any irrational exponent x, we define e^x to be that real number for which $\ln(e^x) = x$. According to this definition, notice that for any $x > 0$, $e^{\ln x}$ is that real number for which

$$\ln(e^{\ln x}) = \ln x. \tag{8.4}$$

Since $\ln x$ is a one-to-one function, (8.4) says that

$$\boxed{e^{\ln x} = x, \quad \text{for } x > 0.} \tag{8.5}$$

Notice that (8.5) says that

$$\ln x = \log_e x.$$

That is, the integral definition of $\ln x$ is consistent with our earlier definition of $\ln x$ as $\log_e x$. Observe also that with this definition of the exponential function, we have

$$\boxed{\ln(e^x) = x, \quad \text{for all } x \in (-\infty, \infty).}$$

Together with (8.5), this says that e^x and $\ln x$ are inverse functions. Keep in mind that for x irrational, e^x is defined *only* through the inverse function relationship given in Definition 8.3. We now state some familiar laws of exponents and prove that they hold even for the case of irrational exponents.

THEOREM 8.2

For r, s any real numbers and t any rational number,

(i) $e^r e^s = e^{r+s}$

(ii) $\dfrac{e^r}{e^s} = e^{r-s}$ and

(iii) $(e^r)^t = e^{rt}$.

PROOF

These laws are already known when the exponents are rational. If the exponent is irrational though, we only know the value of these exponentials indirectly, through the inverse function relationship with $\ln x$, given in Definition 8.3.

(i) Note that using the rules of logarithms, we have

$$\ln(e^r e^s) = \ln(e^r) + \ln(e^s) = r + s = \ln(e^{r+s}).$$

Since $\ln x$ is one-to-one, it must follow that

$$e^r e^s = e^{r+s}.$$

The proofs of (ii) and (iii) are similar and are left as exercises. ■

Previously, we found the derivative of e^x using the limit definition of derivative. You may recall that the derivation was complete, except for the evaluation of the limit

$$\lim_{h \to 0} \frac{e^h - 1}{h}.$$

At the time, we conjectured that the value of this limit is 1, but we were unable to prove it, with the tools at our disposal. We revisit this limit in exercises 37 and 38 at the end of this section. We now present an alternative derivation, based on our new, refined definition of the exponential function. Again, from Definition 8.3, we have that

$$y = e^x \quad \text{if and only if} \quad \ln y = x.$$

Differentiating this last equation with respect to x gives us

$$\frac{d}{dx} \ln y = \frac{d}{dx} x = 1.$$

From the chain rule, we now have

$$1 = \frac{d}{dx} \ln y = \frac{1}{y} \frac{dy}{dx}. \tag{8.6}$$

Multiplying both sides of (8.6) by y, we have

$$\frac{dy}{dx} = y = e^x$$

or

$$\frac{d}{dx}(e^x) = e^x. \tag{8.7}$$

$$\int e^x \, dx = e^x + c.$$

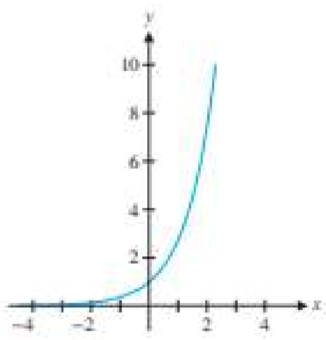


FIGURE 1.34
$y = e^x$

We now have the tools to revisit the graph of $f(x) = e^x$. Since $e = 2.718\ldots > 1$, we have

$$\lim_{x \to \infty} e^x = \infty \quad \text{and} \quad \lim_{x \to -\infty} e^x = 0.$$

We also have that $\quad f'(x) = e^x > 0,$

so that f is increasing for all x and $\quad f''(x) = e^x > 0,$

so that the graph is concave up everywhere. You should now readily obtain the graph in Figure 1.34.

Similarly, for $f(x) = e^{-x}$, we have

$$\lim_{x \to \infty} e^{-x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} e^{-x} = \infty.$$

Further, from the chain rule, $\quad f'(x) = -e^{-x} < 0,$

so that f is decreasing for all x. We also have

$$f''(x) = -e^{-x} < 0,$$

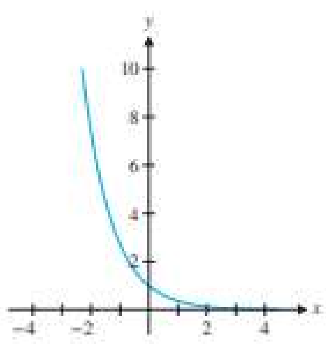


FIGURE 1.35
$y = e^{-x}$

so that the graph is concave up everywhere. You should easily obtain the graph in Figure 1.35.

More general exponential functions, such as $f(x) = b^x$, for any base $b > 0$, are easy to express in terms of the natural exponential function, as follows. Notice that by the usual rules of logs and exponentials, we have

$$\boxed{b^x = e^{\ln(b^x)} = e^{x \ln b}.}$$

It now follows that

$$\frac{d}{dx} b^x = \frac{d}{dx} e^{x \ln b} = e^{x \ln b} \frac{d}{dx}(x \ln b)$$
$$= e^{x \ln b}(\ln b) = b^x(\ln b),$$

as we had conjectured in section 2.7. Similarly, for $b > 0$ $(b \neq 1)$, we have

$$\int b^x\,dx = \int e^{x\ln b}dx = \frac{1}{\ln b}\int e^{\overbrace{x\ln b}^{u}}\underbrace{(\ln b)\,dx}_{du}$$
$$= \frac{1}{\ln b}e^{x\ln b} + c = \frac{1}{\ln b}b^x + c.$$

You can now see that the general exponential functions are easily dealt with in terms of the natural exponential. In fact, you should not bother to memorize the formulas for the derivatives and integrals of general exponentials. Rather, each time you encounter the exponential function $f(x) = b^x$, simply rewrite it as $f(x) = e^{x\ln b}$ and then use the familiar rules for the derivative and integral of the natural exponential and the chain rule.

NOTES

We will occasionally write $e^x = \exp(x)$. This is particularly helpful when the exponent is a complicated expression. For example,

$$\exp(x^3 - 5x^2 + 2x + 7) = e^{x^3-5x^2+2x+7},$$

where the former is more easily read than the latter.

EXAMPLE 8.4 Differentiating an Exponential Function

Find the derivative of $f(x) = 2^{x^2}$.

Solution We first rewrite the function as

$$f(x) = e^{\ln 2^{x^2}} = e^{x^2\ln 2}.$$

From the chain rule, we now have

$$f'(x) = e^{x^2\ln 2}(2x\ln 2) = (2\ln 2)x2^{x^2}. \blacksquare$$

In a similar way, we can use our knowledge of the natural logarithm to discuss more general logarithms. First, recall that for any base $a > 0$ $(a \neq 1)$ and any $x > 0$, $y = \log_a x$ if and only if $x = a^y$. Taking the natural logarithm of both sides of this equation, we have

$$\ln x = \ln(a^y) = y\ln a.$$

Solving for y gives us
$$y = \frac{\ln x}{\ln a},$$

which proves Theorem 8.3.

THEOREM 8.3

For any base $a > 0$ $(a \neq 1)$ and any $x > 0$, $\log_a x = \dfrac{\ln x}{\ln a}$.

Calculators typically have built-in programs for evaluating $\ln x$ and $\log_{10} x$, but not for general logarithms. Theorem 8.3 enables us to easily evaluate logarithms with any base. For instance, we have

$$\log_7 3 = \frac{\ln 3}{\ln 7} \approx 0.564575.$$

More importantly, observe that we can use Theorem 8.3 to find derivatives of general logarithms in terms of the derivative of the natural logarithm. In particular, for any base $a > 0$ $(a \neq 1)$, we have

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln x)$$
$$= \frac{1}{\ln a}\left(\frac{1}{x}\right) = \frac{1}{x\ln a}.$$

As with the derivative formula for general exponentials, there is little point in learning this as a new differentiation rule. Rather, simply use Theorem 8.3.

BEYOND FORMULAS

You might wonder why we have returned to the natural logarithm and exponential functions to carefully define them. Part of the answer is versatility. The integral definition of the logarithm gives you a convenient formula for working with properties of the natural logarithm. A more basic reason for this section is to be sure that the logarithm function is not just a button on your calculator. Instead, $\ln x$ can be thought of in terms of area. You can visualize this easily and use this image to help understand properties of the function. What are some examples from your life (such as behavior rules or sports techniques) where you used a rule before understanding the reason for the rule? Did understanding the rule help?

EXERCISES 1.8

WRITING EXERCISES

1. Explain why it is mathematically legal to define $\ln x$ as $\int_1^x \frac{1}{t}\,dt$. For some, this type of definition is not very satisfying. Try the following comparison. Clearly, it is easier to compute function values for x^2 than for $\ln x$ and therefore x^2 is easier to understand. However, compare how you would compute (without a calculator) function values for $\sin x$ versus function values for $\ln x$. Describe which is more "natural" and easy to understand.
2. In this section, we discussed two different "definitions" of $\ln x$. Explain why it is logically invalid to give different definitions unless you can show that they define the same thing. If they define the same object, both definitions are equally valid and you should use whichever definition is clearer for the task at hand. Explain why, in this section, the integral definition is more convenient than the base e logarithm.
3. Use the integral definition of $\ln x$ (interpreted as area) to explain why it is reasonable that $\lim_{x\to 0^+} \ln x = -\infty$ and $\lim_{x\to\infty} \ln x = \infty$.
4. Use the integral definition of $\ln x$ (interpreted as area) to explain why the graph of $\ln x$ is increasing and concave down for $x > 0$.

In exercises 1–4, express the number as an integral and sketch the corresponding area.

1. $\ln 4$
2. $\ln 5$
3. $\ln 8.2$
4. $\ln 24$

5. Use Simpson's Rule with $n = 4$ to estimate $\ln 4$.
6. Use Simpson's Rule with $n = 4$ to estimate $\ln 5$.
7. Use Simpson's Rule with (a) $n = 32$ and (b) $n = 64$ to estimate $\ln 4$.
8. Use Simpson's Rule with (a) $n = 32$ and (b) $n = 64$ to estimate $\ln 5$.

In exercises 9–12, use the properties of logarithms to rewrite the expression as a single term.

9. $\ln\sqrt{2} + 3\ln 2$
10. $\ln 8 - 2\ln 2$
11. $2\ln 3 - \ln 9 + \ln\sqrt{3}$
12. $2\ln\left(\frac{1}{3}\right) - \ln 3 + \ln\left(\frac{1}{9}\right)$

In exercises 13–20, evaluate the derivative using properties of logarithms where needed.

13. $\frac{d}{dx}(\ln\sqrt{x^2+1})$
14. $\frac{d}{dx}[\ln(x^3 \sin x \cos x)]$
15. $\frac{d}{dx}\left(\ln\frac{x^4}{x^3+1}\right)$
16. $\frac{d}{dx}\left(\ln\sqrt{\frac{x^5}{x^3+1}}\right)$
17. $\frac{d}{dx}\log_7\sqrt{x^2+1}$
18. $\frac{d}{dx}\log_{10}(2^x)$
19. $\frac{d}{dx}(3^{\sin x})$
20. $\frac{d}{dx}(4^{\sqrt{x}})$

In exercises 21–30, evaluate the integral.

21. $\int \frac{1}{x\ln x}\,dx$
22. $\int \frac{1}{\sqrt{1-x^2}\sin^{-1}x}\,dx$
23. $\int x3^{x^2}\,dx$
24. $\int 2^x \sin(2^x)\,dx$
25. $\int \frac{e^{2/x}}{x^2}\,dx$
26. $\int \frac{\sin(\ln x^2)}{x}\,dx$
27. $\int_0^1 \frac{x^2}{x^3-4}\,dx$
28. $\int_0^1 \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$
29. $\int_0^1 \tan x\,dx$
30. $\int_1^2 \frac{\ln x}{x}\,dx$

31. Use property (ii) of Theorem 8.1 to prove property (iii) that $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

32. Use the properties of logarithms in Theorem 8.1 to prove that $\lim_{x\to 0^+} \ln x = -\infty$

33. Show graphically that, for any integer $n > 1$, $\ln(n) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$. Conclude that $\lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) = \infty$

34. Show graphically that, for any integer $n > 1$, $\ln(n) > \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$.

35. Use Definition 8.1 to show that the graph of $y = \ln x$ is concave down for $x > 0$.

36. Prove parts (ii) and (iii) of Theorem 8.2.

37. In the text, we deferred the proof of $\lim_{h\to 0}\frac{e^h - 1}{h} = 1$ to the exercises. In this exercise, we guide you through one possible proof. (Another proof is given in exercise 38.) Starting with $h > 0$, write $h = \ln e^h = \int_1^{e^h}\frac{1}{x}dx$. Use the Integral Mean Value Theorem to write $\int_1^{e^h}\frac{1}{x}dx = \frac{e^h - 1}{\bar{x}}$ for some number $\bar{x}$ between 1 and e^h. This gives you $\frac{e^h - 1}{h} = \bar{x}$. Now, take the limit as $h \to 0^+$. For $h < 0$, repeat this argument, with h replaced with $-h$.

38. In this exercise, we guide you through a different proof of $\lim_{h\to 0}\frac{e^h - 1}{h} = 1$. Start with $f(x) = \ln x$ and the fact that $f'(1) = 1$. Using the alternative definition of derivative, we write this as $f'(1) = \lim_{x\to 1}\frac{\ln x - \ln 1}{x - 1} = 1$. Explain why this implies that $\lim_{x\to 1}\frac{x - 1}{\ln x} = 1$. Finally, substitute $x = e^h$.

39. Starting with $e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$, show that $\ln x = \lim_{n\to\infty}[n(x^{1/n} - 1)]$. Assume that if $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty}[n(x_n^{1/n} - 1)] = \lim_{n\to\infty}[n(x^{1/n} - 1)]$.

40. In this exercise, we show that if $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$, then $\ln e = 1$. Define $x_n = \left(1 + \frac{1}{n}\right)^n$. By the continuity of $\ln x$, we have $\ln e = \ln\left[\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n\right] = \lim_{n\to\infty}\left[\ln\left(1 + \frac{1}{n}\right)^n\right]$. Use l'Hôpital's Rule on $\lim_{n\to\infty}\left[\frac{\ln\left(1 + \frac{1}{n}\right)}{1/n}\right]$ to evaluate this limit.

41. Apply Newton's method to the function $f(x) = \ln x - 1$ to find an iterative scheme for approximating e. Discover how many steps are needed to start at $x_0 = 3$ and obtain five digits of accuracy.

42. Show that $\ln(1 + x) \approx x$ for small x by (a) using a linear approximation and (b) considering the area under the graph of $y = \frac{1}{x}$ between 1 and $1 + x$.

APPLICATIONS

1. Suppose you have a 1-in-10 chance of winning a prize with some purchase (like a lottery). If you make 10 purchases (i.e., you get 10 tries), the probability of winning at least one prize is $1 - (9/10)^{10}$. If the prize had probability 1-in-20 and you tried 20 times, would the probability of winning at least once be higher or lower? Compare $1 - (9/10)^{10}$ and $1 - (19/20)^{20}$ to find out. To see what happens for larger and larger odds, compute $\lim_{n\to\infty}[1 - ((n-1)/n)^n]$.

2. The **sigmoid function** $f(x) = \frac{1}{1 + e^{-x}}$ is used to model situations with a threshold. For example, in the brain, each neuron receives inputs from numerous other neurons and fires only after its total input crosses some threshold. Graph $y = f(x)$ and find $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$. Define the function $g(x)$ to be the value of $f(x)$ rounded off to the nearest integer. What value of x is the threshold for this function to switch from "off" (0) to "on" (1)? How could you modify the function to move the threshold to $x = 4$ instead?

3. A telegraph cable is made of an outer winding around an inner core. If x is defined as the core radius divided by the outer radius, the transmission speed is proportional to $s(x) = x^2\ln(1/x)$. Find an x that maximizes the transmission speed.

EXPLORATORY EXERCISES

1. A special function used in many applications is the **error function,** defined by $\text{erf}(x) = \frac{2}{\sqrt{\pi}}f(x)$, where $f(x) = \int_0^x e^{-u^2}du$. Explore the function $f(x)$. For which x's is $f(x)$ positive? Negative? Increasing? Decreasing? Concave up? Concave down? Estimate some function values for large x. Conjecture $\lim_{x\to\infty} f(x)$. Sketch a graph of $f(x)$.

2. Verify that $\int \sec x\,dx = \ln|\sec x + \tan x| + c$. (Hint: Differentiate the suspected antiderivative and show that you get the integrand.) This integral appears in the construction of a special type of map called a Mercator map. On this map, the latitude lines are not equally spaced. Instead, they are placed so that straight lines on a Mercator map correspond to paths of constant heading. (If you travel due northeast, your path on a map with equally spaced latitude will appear to curve due to the curvature of the earth.) Let R be the (average) radius of the earth. Assuming the earth is a sphere, the actual distance from the equator to a place at latitude $b°$ is $\frac{\pi}{180}Rb$. On a Mercator map, this distance is scaled to $\frac{\pi}{180}R\int_0^b \sec x\,dx$. Tampa, Florida, has latitude 28° north. Moscow, Russia, is twice as far from the equator at 56° north. What is the spacing for Tampa and Moscow on a Mercator map?

3. Define the **log integral function** $Li(x) = \int_0^x \frac{1}{\ln t}dt$ for $x > 1$. For $x = 4$ and $n = 4$, explain why Simpson's Rule does not give an estimate of $Li(4)$. Sketch a picture of the area represented by $Li(4)$. It turns out that $Li(x) = 0$ for $x \approx 1.45$. Explain why $Li(4) \approx \int_{1.45}^{4} \frac{1}{\ln t}\,dt$ and estimate this with Simpson's Rule using $n = 4$. This function is used to estimate $\pi(N)$, the number of prime numbers less than N. Another common estimate of $\pi(N)$ is $\frac{N}{\ln N}$. Estimate $\frac{N}{\ln N}$, $\pi(N)$ and $Li(N)$ for (a) $N = 20$; (b) $N = 40$ and (c) $N = 100{,}000{,}000$, where we'll give you $\pi(N) = 5{,}761{,}455$. Discuss any patterns that you find. (See "Prime Obsession" by John Derbyshire for more about this area of number theory.)[3]

[3] Derbyshire, J. (2004). Prime Obsession. *The Mathematical Intelligencer*, 26(1), 55–59.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Area	Average value	Definite integral
Signed area	Integration by substitution	Indefinite integral
Midpoint Rule	Trapezoidal Rule	Simpson's Rule
Integral Mean Value Theorem	Fundamental Theorem of Calculus	Natural logarithm
Riemann sum		

TRUE OR FALSE

State whether each statement is true or false, and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to make a new statement that is true.

1. The Midpoint Rule always gives better approximations than left-endpoint evaluation.
2. The larger n is, the better is the Riemann sum approximation.
3. All piecewise continuous functions are integrable.
4. The definite integral of velocity gives the total distance traveled.
5. There are some elementary functions that do not have an antiderivative.
6. To evaluate a definite integral, you can use *any* antiderivative.
7. A substitution is not correct unless the derivative term du is present in the original integrand.
8. With Simpson's Rule, if n is doubled, the error is reduced by a factor of 16.

In exercises 1–20, find the antiderivative.

1. $\int (4x^2 - 3)\,dx$
2. $\int (x - 3x^5)\,dx$
3. $\int \frac{4}{x}\,dx$
4. $\int \frac{4}{x^2}\,dx$
5. $\int 2 \sin 4x\,dx$
6. $\int 3 \sec^2 x\,dx$
7. $\int (x - e^{4x})\,dx$
8. $\int 3\sqrt{x}\,dx$
9. $\int \frac{x^2 + 4}{x}\,dx$
10. $\int \frac{x}{x^2 + 4}\,dx$
11. $\int e^x(1 - e^{-x})\,dx$
12. $\int e^x(1 + e^x)^2\,dx$
13. $\int x\sqrt{x^2 + 4}\,dx$
14. $\int x(x^2 + 4)\,dx$
15. $\int 6x^2 \cos x^3\,dx$
16. $\int 4x \sec x^2 \tan x^2\,dx$
17. $\int \frac{e^{1/x}}{x^2}\,dx$
18. $\int \frac{\ln x}{x}\,dx$
19. $\int \tan x\,dx$
20. $\int \sqrt{3x + 1}\,dx$

21. Find a function $f(x)$ satisfying $f'(x) = 3x^2 + 1$ and $f(0) = 2$.
22. Find a function $f(x)$ satisfying $f'(x) = e^{-2x}$ and $f(0) = 3$.
23. Determine the position function if the velocity is $v(t) = -32t + 10$ and the initial position is $s(0) = 2$.
24. Determine the position function if the acceleration is $a(t) = 6$ with initial velocity $v(0) = 10$ and initial position $s(0) = 0$.
25. Write out all terms and compute $\sum_{i=1}^{6}(i^2 + 3i)$.
26. Translate into summation notation and compute: the sum of the squares of the first 12 positive integers.

In exercises 27 and 28, use summation rules to compute the sum.

27. $\sum_{i=1}^{100}(i^2 - 1)$
28. $\sum_{i=1}^{100}(i^2 + 2i)$

29. Compute the sum $\frac{1}{n^3}\sum_{i=1}^{n}(i^2 - i)$ and the limit of the sum as n approaches ∞.
30. For $f(x) = x^2 - 2x$ on the interval $[0, 2]$, list the evaluation points for the Midpoint Rule with $n = 4$, sketch the function and approximating rectangles and evaluate the Riemann sum.

Review Exercises

In exercises 31–34, approximate the area under the curve using n rectangles and the given evaluation rule.

31. $y = x^2$ on $[0, 2]$, $n = 8$, midpoint evaluation

32. $y = x^2$ on $[-1, 1]$, $n = 8$, right-endpoint evaluation

33. $y = \sqrt{x+1}$ on $[0, 3]$, $n = 8$, midpoint evaluation

34. $y = e^{-x}$ on $[0, 1]$, $n = 8$, left-endpoint evaluation

In exercises 35 and 36, use the given function values to estimate the area under the curve using (a) left-endpoint evaluation, (b) right-endpoint evaluation, (c) Trapezoidal Rule and (d) Simpson's Rule.

35.

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$f(x)$	1.0	1.4	1.6	2.0	2.2	2.4	2.0	1.6	1.4

36.

x	1.0	1.4	1.8	2.2	2.6	3.0	3.4	3.8	4.2
$f(x)$	4.0	3.4	3.6	3.0	2.6	2.4	3.0	3.6	3.4

37. In exercises 35 and 36, which of the four area estimates would you expect to be the most accurate? Briefly explain.

38. If $f(x)$ is positive and concave up, will the Midpoint Rule give an overestimate or underestimate of the actual area? Will the Trapezoidal Rule give an overestimate or underestimate of the actual area?

In exercises 39 and 40, evaluate the integral by computing the limit of Riemann sums.

39. $\int_0^1 2x^2\,dx$

40. $\int_0^2 (x^2+1)\,dx$

In exercises 41 and 42, write the total area as an integral or sum of integrals and then evaluate it.

41. The area above the x-axis and below $y = 3x - x^2$

42. The area between the x-axis and $y = x^3 - 3x^2 + 2x$, $0 \le x \le 2$

In exercises 43 and 44, use the velocity function to compute the distance traveled in the given time interval.

43. $v(t) = 40 - 10t$, $[1, 2]$

44. $v(t) = 20e^{-t/2}$, $[0, 2]$

In exercises 45 and 46, compute the average value of the function on the interval.

45. $f(x) = e^x$, $[0, 2]$

46. $f(x) = 4x - x^2$, $[0, 4]$

In exercises 47–58, evaluate the integral.

47. $\int_0^2 (x^2 - 2)\,dx$

48. $\int_{-1}^1 (x^3 - 2x)\,dx$

49. $\int_0^{\pi/2} \sin 2x\,dx$

50. $\int_0^{\pi/4} \sec^2 x\,dx$

51. $\int_0^{10} (1 - e^{-t/4})\,dt$

52. $\int_0^1 te^{-t^2}\,dt$

53. $\int_0^2 \frac{x}{x^2+1}\,dx$

54. $\int_1^2 \frac{\ln x}{x}\,dx$

55. $\int_0^2 x\sqrt{x^2+4}\,dx$

56. $\int_0^2 x(x^2+1)\,dx$

57. $\int_0^1 (e^x - 2)^2\,dx$

58. $\int_{-\pi}^{\pi} \cos(x/2)\,dx$

In exercises 59 and 60, find the derivative.

59. $f(x) = \int_2^x (\sin t^2 - 2)\,dt$

60. $f(x) = \int_0^{x^2} \sqrt{t^2+1}\,dt$

In exercises 61 and 62, compute the (a) Midpoint Rule, (b) Trapezoidal Rule and (c) Simpson's Rule approximations with $n = 4$ by hand.

61. $\int_0^1 \sqrt{x^2+4}\,dx$

62. $\int_0^2 e^{-x^2/4}\,dx$

63. Repeat exercise 61 using a computer or calculator and $n = 20$; $n = 40$.

64. Repeat exercise 62 using a computer or calculator and $n = 20$; $n = 40$.

65. Show that if $u = \tanh(t/2)$, then $\cosh t = \frac{1+u^2}{1-u^2}$ and $\sinh t = \frac{2u}{1-u^2}$. Use the substitution $u = \tanh(t/2)$ to evaluate (a) $\int \frac{1}{\sinh t + \cosh t}\,dt$ and (b) $\int \frac{\sinh t + \cosh t}{1 + \cosh t}\,dt$.

APPLICATIONS

1. The linear density of a rod is given by $\rho(x) = 12 - x$ in grams per centimeter, where x is measured from one end of the rod. Find the total mass of a rod of length 10 cm.

2. An underground storage tank is leaking at a rate of $r(t) = 25e^{-0.2t}$ liters per minute. Suppose the leaking starts at $t = 0$, find how much water leaks out during the first half an hour.

3. Miriam skates at speed $M(t) = \sqrt{t}$ meters per second, and Lyanne drives her scooter at speed $L(t) = \frac{1}{2}t$ m/s. Suppose that Miriam and Lyanne are at the same location at time $t = 0$.

Review Exercises

Compute and interpret the following integrals in terms of a race between Miriam and Lyanne $\int_0^4 [M(t) - L(t)]dt$ and $\int_4^8 [M(t) - L(t)]dt$

4. Much energy needed for walking and running is saved by the arch of a human foot. It stores energy as the foot stretches and returns energy as the foot recoils. Suppose that the data provided shows x as the vertical displacement of the arch, $f_s(x)$ is the force during stretching and $f_r(x)$ is the force during recoil. Use Simpson's Rule to estimate the proportion of energy returned by the arch.

x (mm)	0	2.0	4.0	6.0	8.0
$f_s(x)$ (N)	0	250	700	1100	2400
$f_r(x)$ (N)	0	120	500	900	2400

EXPLORATORY EXERCISES

1. Suppose that $f(t)$ is the rate of occurrence of some event (e.g., the birth of an animal or the lighting of a firefly). Then the average rate of occurrence R over a time interval $[0, T]$ is $R = \frac{1}{T}\int_0^T f(t)\,dt$. We will assume that the function $f(t)$ is periodic with period T. [That is, $f(t + T) = f(t)$ for all t.] **Perfect asynchrony** means that the event is equally likely to occur at all times. Argue that this corresponds to a constant rate function $f(t) = c$ and find the value of c (in terms of R and T). **Perfect synchrony** means that the event occurs only once every period (e.g., the fireflies all light at the same time, or all babies are born simultaneously). We will see what the rate function $f(t)$ looks like in this case. First, define the **degree of synchrony** to be $\frac{\text{area under } f \text{ and above } R}{RT}$. Show that if $f(t)$ is constant, then the degree of synchrony is 0. Then graph and find the degree of synchrony for the following functions (assuming $T > 2$):

$$f_1(t) = \begin{cases} (RT)(t - \frac{T}{2}) + RT & \text{if } \frac{T}{2} - 1 \le t \le \frac{T}{2} \\ (-RT)(t - \frac{T}{2}) + RT & \text{if } \frac{T}{2} \le t \le \frac{T}{2} + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(t) = \begin{cases} (4RT)(t - \frac{T}{2}) + 2RT & \text{if } \frac{T}{2} - \frac{1}{2} \le t \le \frac{T}{2} \\ (-4RT)(t - \frac{T}{2}) + 2RT & \text{if } \frac{T}{2} \le t \le \frac{T}{2} + \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

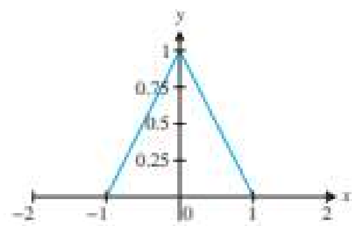
$$f_3(t) = \begin{cases} (9RT)(t - \frac{T}{2}) + 3RT & \text{if } \frac{T}{2} - \frac{1}{3} \le t \le \frac{T}{2} \\ (-9RT)(t - \frac{T}{2}) + 3RT & \text{if } \frac{T}{2} \le t \le \frac{T}{2} + \frac{1}{3} \\ 0 & \text{otherwise} \end{cases}$$

What would you conjecture as the limit of the degrees of synchrony of $f_n(t)$ as $n \to \infty$? The "function" that $f_n(t)$ approaches as $n \to \infty$ is called an **impulse function** of strength RT. Discuss the appropriateness of this name.

2. The **Omega function** is used for risk/reward analysis of financial investments. Suppose that $f(x)$ is a function defined on the interval (A, B) that gives the distribution of returns on an investment. (This means that $\int_a^b f(x)dx$ is the probability that the investment returns between \$$a$ and \$$b$.) Let $F(x) = \int_A^x f(t)\,dt$ be the **cumulative distribution function** for returns.

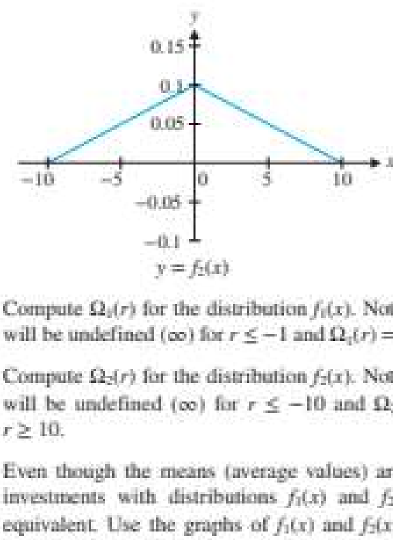
Then $\Omega(r) = \dfrac{\int_r^B [1 - F(x)]\,dx}{\int_A^r F(x)\,dx}$ is the Omega function for the investment.

(a) For the distribution $f_1(x)$ shown, compute the cumulative distribution function $F_1(x)$.



$y = f_1(x)$

(b) Repeat part (a) for the distribution $f_2(x)$ shown.



$y = f_2(x)$

(c) Compute $\Omega_1(r)$ for the distribution $f_1(x)$. Note that $\Omega_1(r)$ will be undefined (∞) for $r \le -1$ and $\Omega_1(r) = 0$ for $r \ge 1$.

(d) Compute $\Omega_2(r)$ for the distribution $f_2(x)$. Note that $\Omega_2(r)$ will be undefined (∞) for $r \le -10$ and $\Omega_2(r) = 0$ for $r \ge 10$.

(e) Even though the means (average values) are the same, investments with distributions $f_1(x)$ and $f_2(x)$ are not equivalent. Use the graphs of $f_1(x)$ and $f_2(x)$ to explain why $f_2(x)$ corresponds to a riskier investment than $f_1(x)$.

(f) Show that $\Omega_2(r) > \Omega_1(r)$ for $r > 0$ and $\Omega_2(r) < \Omega_1(r)$ for $r < 0$. In general, the larger $\Omega(r)$ is, the better the investment is. Explain this in terms of this example.

Applications of the Definite Integral

CHAPTER

2

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Chapter Topics

2.1 Area Between Curves

2.2 Volume: Slicing, Disks and Washers

2.3 Volumes by Cylindrical Shells

2.4 Arc Length and Surface Area

2.5 Projectile Motion

2.6 Applications of Integration to Physics and Engineering

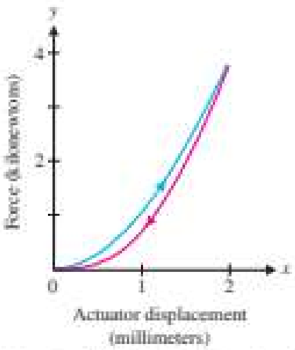
2.7 Probability

Athletes who can jump high are often said to have "springs in their legs." It turns out that tendons and the arches in your feet act very much like springs, storing and releasing energy. For example, your Achilles tendon stretches as you stride when walking and contracts as your foot hits the ground. Much like a spring that is stretched and then released, the tendon stores energy during the stretching phase and releases it when contracting.

Physiologists measure the efficiency of the springlike action of tendons by computing the percentage of energy released during contraction relative to the energy stored during the stretch. The stress-strain curve presented here shows force as a function of stretch during stretch (top curve) and recoil (bottom curve) for a human arch. (Figure reprinted with permission from *Exploring Biomechanics* by R. McNeill Alexander.)[1] If no energy is lost, the two curves are identical. The area between the curves is a measure of the energy lost.

aiokaji © 123RF.com

The corresponding curve for a kangaroo (see Alexander) shows almost no area between the curves. The efficiency of the kangaroo's legs means that very little energy is required to hop. In fact, biologist Terry Dawson found in treadmill tests that the faster kangaroos run, the less energy they burn (up to the test limit of 32 km/h). The same principle applies to human athletes, in that the more the Achilles tendons stretch, the more efficient the running process becomes. For this reason, athletes spend considerable time stretching and strengthening their Achilles tendons.



This chapter demonstrates the versatility of the integral by exploring numerous applications. We start with calculations of the area between two curves. The integral can be viewed from a variety of perspectives: graphical (areas), numerical (Riemann sum approximations) and symbolic (the Fundamental Theorem of Calculus). As you study each new application, pay close attention to how we develop the integral(s) measuring the quantity of interest.

[1] Alexander, R.M. (1992). *Exploring Biomechanics* (New York: W.H. Freeman).

2.1 AREA BETWEEN CURVES

We initially developed the definite integral (in Chapter 1) to compute the area under a curve. In particular, let f be a continuous function defined on $[a, b]$, where $f(x) \geq 0$ on $[a, b]$. To find the area under the curve $y = f(x)$ on the interval $[a, b]$, we begin by partitioning $[a, b]$ into n subintervals of equal size, $\Delta x = \frac{b-a}{n}$. The points in the partition are then $x_0 = a$, $x_1 = x_0 + \Delta x$, $x_2 = x_1 + \Delta x$ and so on. That is,

$$x_i = a + i\Delta x, \quad \text{for } i = 0, 1, 2, \ldots, n.$$

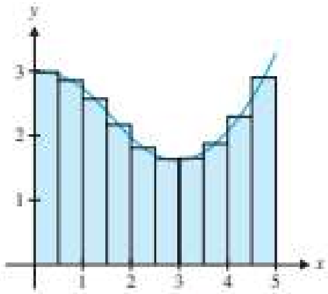


FIGURE 2.1
Approximation of area

On each subinterval $[x_{i-1}, x_i]$, we construct a rectangle of height $f(c_i)$, for some $c_i \in [x_{i-1}, x_i]$, as indicated in Figure 2.1 and take the sum of the areas of the n rectangles as an approximation of the area A under the curve:

$$A \approx \sum_{i=1}^{n} f(c_i)\,\Delta x.$$

As we take more and more rectangles, this sum approaches the exact area, which is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)dx.$$

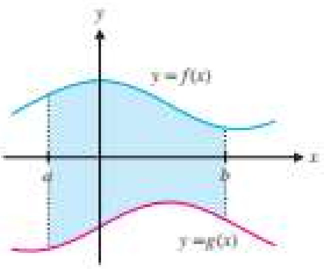


FIGURE 2.2
Area between two curves

We now extend this notion to find the area bounded between the two curves $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ (see Figure 2.2), where f and g are continuous and $f(x) \geq g(x)$ on $[a, b]$. We first use rectangles to approximate the area. In this case, on each subinterval $[x_{i-1}, x_i]$, construct a rectangle, stretching from the lower curve $y = g(x)$ to the upper curve $y = f(x)$, as shown in Figure 2.3a. Referring to Figure 2.3b, the ith rectangle has height $h_i = f(c_i) - g(c_i)$, for some $c_i \in [x_{i-1}, x_i]$.

So, the area of the ith rectangle is

$$\text{Area} = \text{length} \times \text{width} = h_i\,\Delta x = [\,f(c_i) - g(c_i)]\,\Delta x.$$

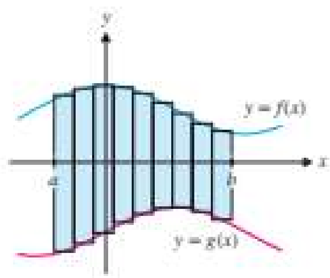


FIGURE 2.3a
Approximate area

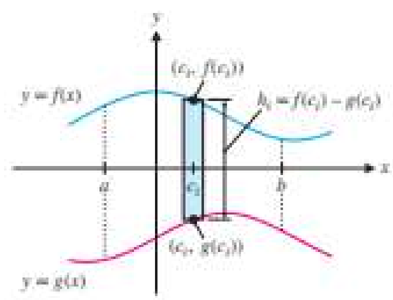


FIGURE 2.3b
Area of ith rectangle

The total area is then approximately equal to the sum of the areas of the n indicated rectangles,

$$A \approx \sum_{i=1}^{n} [\,f(c_i) - g(c_i)]\,\Delta x.$$

Finally, observe that if the limit as $n \to \infty$ exists, we will get the exact area, which we recognize as a definite integral:

AREA BETWEEN TWO CURVES

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [\,f(c_i) - g(c_i)]\,\Delta x = \int_a^b [\,f(x) - g(x)]\,dx. \quad (1.1)$$

REMARK 1.1

Formula (1.1) is valid only when $f(x) \geq g(x)$ on the interval $[a, b]$. In general, the area between $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$ is given by $\int_a^b |f(x) - g(x)|\,dx$. Notice that to evaluate this integral, you must evaluate $\int_c^d [f(x) - g(x)]\,dx$ on all subintervals where $f(x) \geq g(x)$, then evaluate $\int_c^d [g(x) - f(x)]\,dx$ on all subintervals where $g(x) \geq f(x)$ and finally, add the integrals together.

EXAMPLE 1.1 Finding the Area between Two Curves

Find the area bounded by the graphs of $y = 3 - x$ and $y = x^2 - 9$. (See Figure 2.4.)

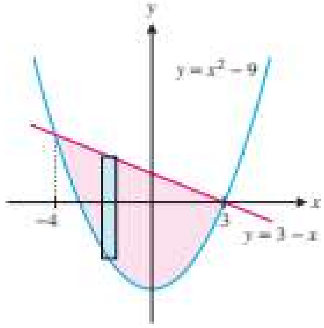


FIGURE 2.4
$y = 3 - x$ and $y = x^2 - 9$

Solution Notice that the limits of integration correspond to the x-coordinates of the points of intersection of the two curves. Setting the two functions equal, we have

$$3 - x = x^2 - 9 \quad \text{or} \quad 0 = x^2 + x - 12 = (x - 3)(x + 4).$$

Thus, the curves intersect at $x = -4$ and $x = 3$. Take note that the upper boundary of the region is formed by $y = 3 - x$ and the lower boundary is formed by $y = x^2 - 9$. So, for each fixed value of x, the height of a rectangle (such as the one indicated in Figure 2.4) is

$$h(x) = (3 - x) - (x^2 - 9).$$

From (1.1), the area between the curves is then

$$\begin{aligned} A &= \int_{-4}^{3} [(3 - x) - (x^2 - 9)]\,dx \\ &= \int_{-4}^{3} (-x^2 - x + 12) = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 12x\right]_{-4}^{3} \\ &= \left[-\frac{3^3}{3} - \frac{3^2}{2} + 12(3)\right] - \left[-\frac{(-4)^3}{3} - \frac{(-4)^2}{2} + 12(-4)\right] = \frac{343}{6}. \end{aligned}$$

Sometimes, the upper or lower boundary is not defined by a single function, as in the following case of intersecting graphs.

EXAMPLE 1.2 Finding the Area between Two Curves That Cross

Find the area bounded by the graphs of $y = x^2$ and $y = 2 - x^2$ for $0 \leq x \leq 2$.

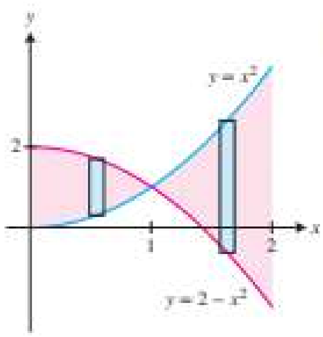


FIGURE 2.5
$y = x^2$ and $y = 2 - x^2$

Solution Notice from Figure 2.5 that since the two curves intersect in the middle of the interval, we will need to compute two integrals, one on the interval where $2 - x^2 \geq x^2$ and one on the interval where $x^2 \geq 2 - x^2$. To find the point of intersection, we solve $x^2 = 2 - x^2$, so that $2x^2 = 2$ or $x^2 = 1$ or $x = \pm 1$. Since $x = -1$ is outside the interval of interest, the only intersection of note is at $x = 1$. From (1.1), the area is

$$\begin{aligned} A &= \int_0^1 [(2 - x^2) - x^2]\,dx + \int_1^2 [x^2 - (2 - x^2)]\,dx \\ &= \int_0^1 (2 - 2x^2)\,dx + \int_1^2 (2x^2 - 2)\,dx = \left[2x - \frac{2x^3}{3}\right]_0^1 + \left[\frac{2x^3}{3} - 2x\right]_1^2 \\ &= \left(2 - \frac{2}{3}\right) - (0 - 0) + \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 2\right) = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4. \end{aligned}$$

In example 1.3, the intersection points must be approximated numerically.

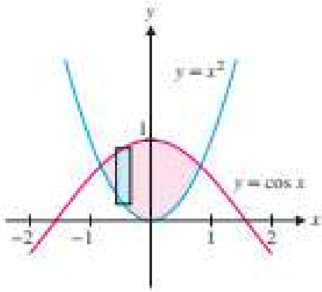


FIGURE 2.6
$y = \cos x$ and $y = x^2$

EXAMPLE 1.3 A Case Where the Intersection Points Are Known Only Approximately

Find the area bounded by the graphs of $y = \cos x$ and $y = x^2$.

Solution The graph of $y = \cos x$ and $y = x^2$ in Figure 2.6 indicates intersections at about $x = -1$ and $x = 1$, where $\cos x = x^2$. However, this equation cannot be solved exactly. Instead, we use a rootfinding method to find the approximate solutions $x = \pm 0.824132$. [For instance, you can use Newton's method to find values of x for which $f(x) = \cos x - x^2 = 0$.] From the graph, we can see that between these two x-values, $\cos x \geq x^2$ and so, the desired area is given by

$$A \approx \int_{-0.824132}^{0.824132} (\cos x - x^2)\,dx = \left[\sin x - \frac{1}{3}x^3\right]_{-0.824132}^{0.824132}$$
$$= \sin 0.824132 - \frac{1}{3}(0.824132)^3 - \left[\sin(-0.824132) - \frac{1}{3}(-0.824132)^3\right]$$
$$\approx 1.09475.$$

Note that we have approximated both the limits of integration and the final calculations. ■

Finding the area of some regions may require breaking the region up into several pieces, each having different upper and/or lower boundaries.

Although it was certainly not difficult to break up the region in example 1.4 into several pieces, we want to suggest an alternative that will prove to be surprisingly useful. Notice that if you turn the page sideways, Figure 2.7a will look like a region with a single curve determining each of the upper and lower boundaries. Of course, by turning the page sideways, you are essentially reversing the roles of x and y.

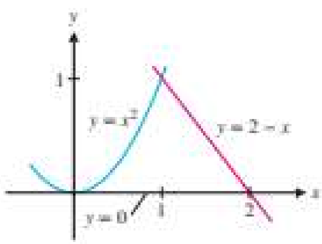


FIGURE 2.7a
$y = x^2$ and $y = 2 - x$

EXAMPLE 1.4 The Area of a Region Determined by Three Curves

Find the area bounded by the graphs of $y = x^2$, $y = 2 - x$ and $y = 0$.

Solution A sketch of the three defining curves is shown in Figure 2.7a. Notice that the top boundary of the region is the curve $y = x^2$ on the first portion of the interval and the line $y = 2 - x$ on the second portion. To determine the point of intersection, we solve

$$2 - x = x^2 \quad \text{or} \quad 0 = x^2 + x - 2 = (x + 2)(x - 1).$$

Since $x = -2$ is to the left of the y-axis, the intersection we seek occurs at $x = 1$. We then break the region into two pieces, as shown in Figure 2.7b and find the area of each separately. The total area is then

$$A = A_1 + A_2 = \int_0^1 (x^2 - 0)\,dx + \int_1^2 [(2 - x) - 0]\,dx$$
$$= \frac{x^3}{3}\bigg|_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 = \frac{5}{6}.$$ ■

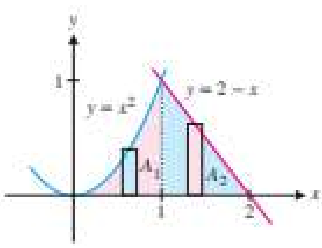


FIGURE 2.7b
$y = x^2$ and $y = 2 - x$

More generally, for two continuous functions, f and g, where $f(y) \geq g(y)$ for all y on the interval $c \leq y \leq d$, to find the area bounded between the two curves $x = f(y)$ and $x = g(y)$, we first partition the interval $[c, d]$ into n equal subintervals, each of width $\Delta y = \frac{d - c}{n}$. (See Figure 2.8a.) We denote the points in the partition by $y_0 = c$, $y_1 = y_0 + \Delta y$, $y_2 = y_1 + \Delta y$ and so on. That is,

$$y_i = c + i\,\Delta y, \quad \text{for } i = 0, 1, 2, \ldots, n.$$

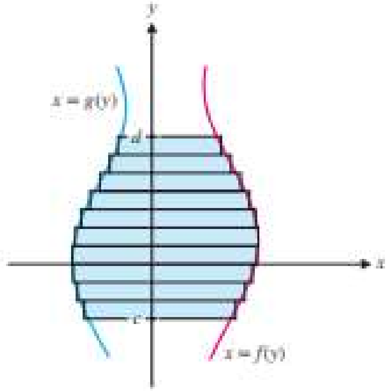


FIGURE 2.8a
Area between $x = g(y)$ and $x = f(y)$

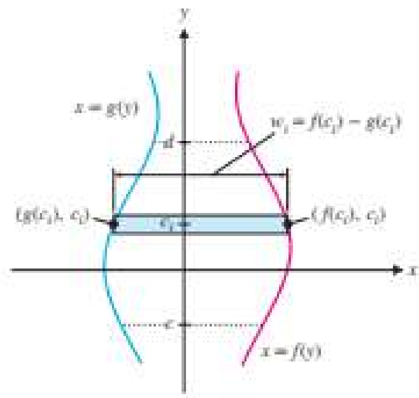


FIGURE 2.8b
Area of ith rectangle

On each subinterval $[y_{i-1}, y_i]$ (for $i = 1, 2, \ldots, n$), we then construct a rectangle of width $w_i = [f(c_i) - g(c_i)]$, for some $c_i \in [y_{i-1}, y_i]$, as shown in Figure 2.8b. The area of the ith rectangle is given by

$$\text{Area} = \text{length} \times \text{width} = [f(c_i) - g(c_i)]\,\Delta y.$$

The total area between the two curves is then given approximately by

$$A \approx \sum_{i=1}^{n} [f(c_i) - g(c_i)]\,\Delta y.$$

We get the exact area by taking the limit as $n \to \infty$ and recognizing the limit as a definite integral. We have

Area between two curves

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(c_i) - g(c_i)]\,\Delta y = \int_c^d [f(y) - g(y)]\,dy. \tag{1.2}$$

EXAMPLE 1.5 An Area Computed by Integrating with Respect to *y*

Repeat example 1.4, but integrate with respect to y instead.

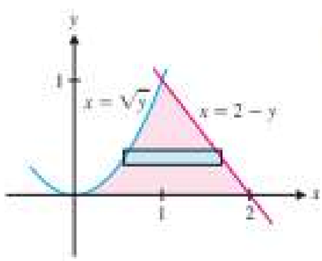


FIGURE 2.9
$y = x^2$ and $y = 2 - x$

Solution From Figure 2.9, notice that the left-hand boundary of the region is formed by the graph of $y = x^2$ or $x = \sqrt{y}$ (since only the right half of the parabola forms the left boundary). The right-hand boundary of the region is formed by the line $y = 2 - x$ or $x = 2 - y$. These boundary curves intersect where $\sqrt{y} = 2 - y$; squaring both sides gives us

$$y = (2 - y)^2 = 4 - 4y + y^2$$

or

$$0 = y^2 - 5y + 4 = (y - 1)(y - 4).$$

So, the curves intersect at $y = 1$ and $y = 4$. From Figure 2.9, it is clear that $y = 1$ is the solution we need. (What does the solution $y = 4$ correspond to?) From (1.2), the area is given by

$$A = \int_0^1 [(2 - y) - \sqrt{y}]\,dy = \left[2y - \frac{1}{2}y^2 - \frac{2}{3}y^{3/2}\right]_0^1 = 2 - \frac{1}{2} - \frac{2}{3} = \frac{5}{6}.$$ ■

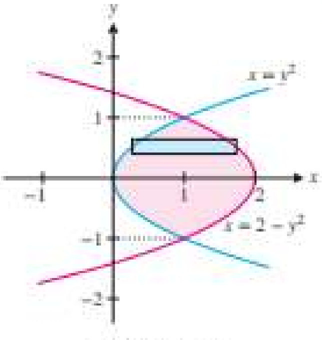


FIGURE 2.10
$x = y^2$ and $x = 2 - y^2$

EXAMPLE 1.6 The Area of a Region Bounded by Functions of *y*

Find the area bounded by the graphs of $x = y^2$ and $x = 2 - y^2$.

Solution From Figure 2.10, observe that it's easiest to compute this area by integrating with respect to y, since integrating with respect to x would require us to break the region into two pieces. The two intersections of the curves occur where $y^2 = 2 - y^2$, or $y^2 = 1$, so that $y = \pm 1$. On the interval $[-1, 1]$, notice that $2 - y^2 \geq y^2$ (since the curve $x = 2 - y^2$ stays to the right of the curve $x = y^2$). So, from (1.2), the area is given by

$$A = \int_{-1}^{1} [(2 - y^2) - y^2]\,dy = \int_{-1}^{1} (2 - 2y^2)\,dy$$
$$= \left[2y - \frac{2}{3}y^3\right]_{-1}^{1} = \left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right) = \frac{8}{3}.$$

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In collisions between a tennis racket and ball, the ball changes shape, first compressing and then expanding. Let x represent how far the ball is compressed, where $0 \leq x \leq m$, and let $f(x)$ represent the force exerted on the ball by the racket. Then, the energy transferred is proportional to the area under the curve $y = f(x)$. Suppose that $f_c(x)$ is the force during compression of the ball and $f_e(x)$ is the force during expansion of the ball. Energy is transferred to the ball during compression and transferred away from the ball during expansion, so that the energy lost by the ball in the collision (due to friction) is proportional to $\int_0^m [f_c(x) - f_e(x)]\,dx$. The percentage of energy lost in the collision is then given by

$$100\,\frac{\int_0^m [f_c(x) - f_e(x)]\,dx}{\int_0^m f_c(x)\,dx}.$$

EXAMPLE 1.7 Estimating the Energy Lost by a Tennis Ball

Suppose that test measurements provide the following data on the collision of a tennis ball with a racket. Estimate the percentage of energy lost in the collision.

x (cm)	0.0	0.1	0.2	0.3	0.4
$f_c(x)$ (N)	0	25	50	90	160
$f_e(x)$ (N)	0	23	46	78	160

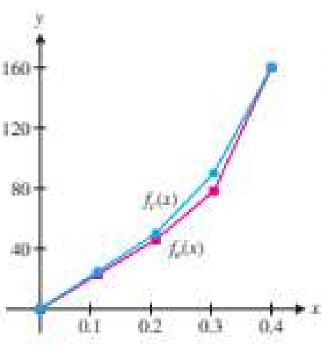

FIGURE 2.11
Force exerted on a tennis ball

Solution The data are plotted in Figure 2.11, connected by line segments.

We need to estimate the area between the curves and the area under the top curve. Since we don't have a formula for either function, we must use a numerical method such as Simpson's Rule. For $\int_0^{0.4} f_c(x)\,dx$, we get

$$\int_0^{0.4} f_c(x)\,dx \approx \frac{0.1}{3}[0 + 4(25) + 2(50) + 4(90) + 160] = 24.$$

To use Simpson's Rule to approximate $\int_0^{0.4} [f_c(x) - f_e(x)]\,dx$, we need a table of function values for $f_c(x) - f_e(x)$. Subtraction gives us

x	0.0	0.1	0.2	0.3	0.4
$f_c(x) - f_e(x)$	0	2	4	12	0

from which Simpson's Rule gives us

$$\int_0^{0.4} [f_c(x) - f_e(x)]\,dx \approx \frac{0.1}{3}[0 + 4(2) + 2(4) + 4(12) + 0] = \frac{6.4}{3}.$$

The percentage of energy lost is then $\frac{100(6.4/3)}{24} \approx 8.9\%$. With over 90% of its energy retained in the collision, this is a lively tennis ball.

BEYOND FORMULAS

In example 1.5, we viewed the given graphs as functions of y and set up the area as an integral of y. This idea indicates the direction that much of the rest of the course takes. The derivative and integral remain the two most important tools, but we diversify our options for working with them, often by changing variables. The flexible thinking that this promotes is key in calculus, as well as in other areas of mathematics and science. We develop some general techniques and often the first task in solving an application problem is to make the technique fit the problem at hand.

EXERCISES 2.1

WRITING EXERCISES

1. Suppose the functions f and g satisfy $f(x) \geq g(x) \geq 0$ for all x in the interval $[a, b]$. Explain in terms of the areas $A_1 = \int_a^b f(x)\,dx$ and $A_2 = \int_a^b g(x)\,dx$ why the area between the curves $y = f(x)$ and $y = g(x)$ is given by $\int_a^b [f(x) - g(x)]\,dx$.
2. Suppose the functions f and g satisfy $f(x) \leq g(x) \leq 0$ for all x in the interval $[a, b]$. Explain in terms of the areas $A_1 = \int_a^b f(x)dx$ and $A_2 = \int_a^b g(x)\,dx$ why the area between the curves $y = f(x)$ and $y = g(x)$ is given by $\int_a^b [f(x) - g(x)]\,dx$.
3. Suppose that the speeds of racing cars A and B are $v_A(t)$ and $v_B(t)$ km/h, respectively. If $v_A(t) \geq v_B(t)$ for all t, $v_A(0) = v_B(0)$ and the race lasts from $t = 0$ to $t = 2$ hours, explain why car A will win the race by $\int_0^2 [v_A(t) - v_B(t)]\,dt$ km.
4. Suppose that the speeds of racing cars A and B are $v_A(t)$ and $v_B(t)$ km/h, respectively. If $v_A(t) \geq v_B(t)$ for $0 \leq t \leq 0.5$ and $1.1 \leq t \leq 1.6$ and $v_B(t) \geq v_A(t)$ for $0.5 \leq t \leq 1.1$ and $1.6 \leq t \leq 2$, describe the difference between $\int_0^2 |v_A(t) - v_B(t)|\,dt$ and $\int_0^2 [v_A(t) - v_B(t)]\,dt$. Which integral will tell you which car wins the race?

In exercises 1–4, find the area between the curves on the given interval.

1. $y = x^3$, $y = x^2 - 1$, $1 \leq x \leq 3$
2. $y = \cos x$, $y = x^2 + 2$, $0 \leq x \leq 2$
3. $y = e^x$, $y = x - 1$, $-2 \leq x \leq 0$
4. $y = e^{-x}$, $y = x^2$, $1 \leq x \leq 4$

In exercises 5–12, sketch and find the area of the region determined by the intersections of the curves.

5. $y = x^2 - 1$, $y = 7 - x^2$
6. $y = x^2 - 1$, $y = \frac{1}{2}x^2$
7. $y = x^3$, $y = 3x + 2$
8. $y = \sqrt{x}$, $y = x^2$
9. $y = 4xe^{-x^2}$, $y = |x|$
10. $y = \frac{2}{x^2 + 1}$, $y = |x|$
11. $y = \frac{5x}{x^2 + 1}$, $y = x$
12. $y = \sin x$ $(0 \leq x \leq 2\pi)$, $y = \cos x$

In exercises 13–20, find the area of the shaded region.

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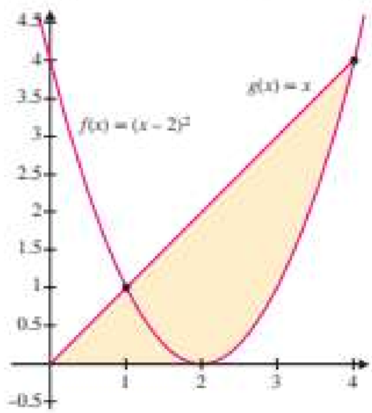

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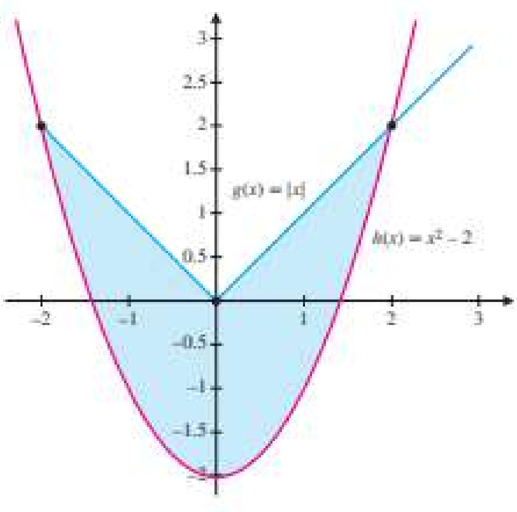

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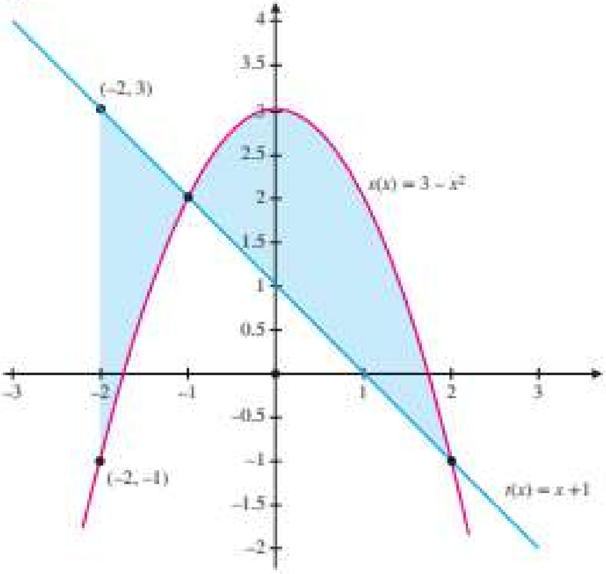

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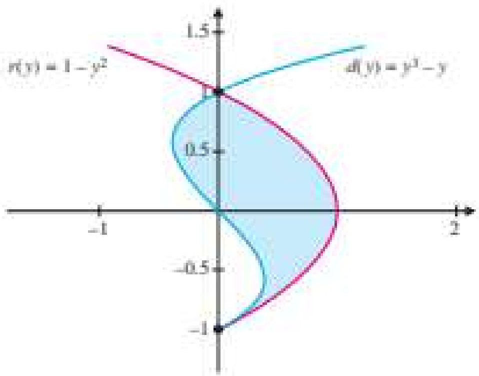
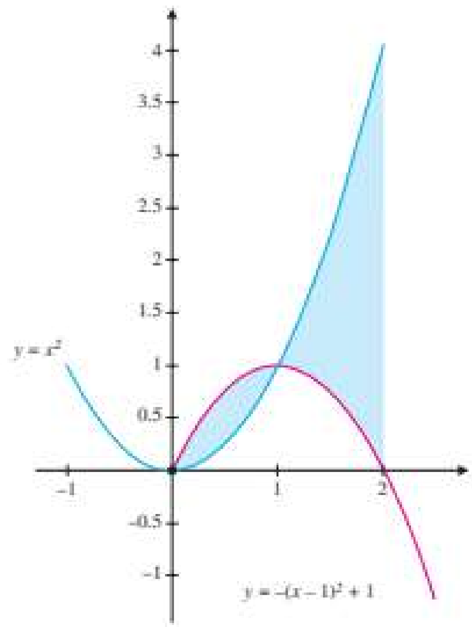


17.



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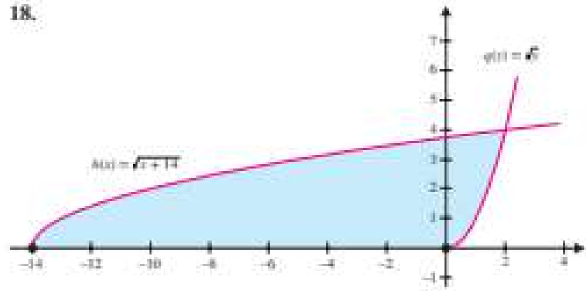

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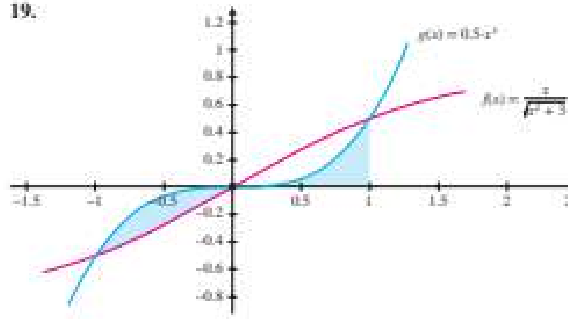

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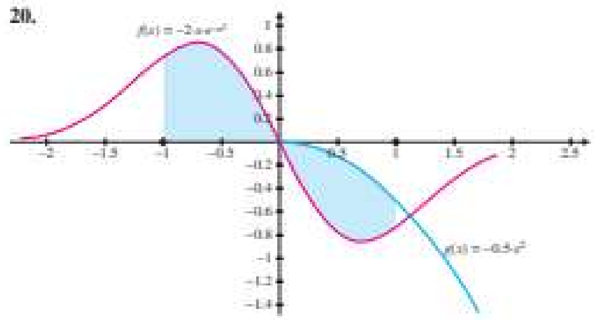


In exercises 21–26, sketch and estimate the area determined by the intersections of the curves.

21. $y = e^x$, $y = 1 - x^2$

22. $y = x^4$, $y = 1 - x$

23. $y = \sin x$, $y = x^2$

24. $y = \cos x$, $y = x^4$

25. $y = x^4$, $y = 2 + x$

26. $y = \ln x$, $y = x^2 - 2$

In exercises 27–34, sketch and find the area of the region bounded by the given curves. Choose the variable of integration so that the area is written as a single integral. Verify your answers to exercises 27–29 with a basic geometric area formula.

27. $y = x$, $y = 2 - x$, $y = 0$

28. $y = x$, $y = 2$, $y = 6 - x$, $y = 0$

29. $x = y$, $x = -y$, $x = 1$

30. $x = 3y$, $x = 2 + y^2$

31. $y = 2x\ (x > 0)$, $y = 3 - x^2$, $x = 0$

32. $x = y^2$, $x = 4$

33. $y = e^x$, $y = 4e^{-x}$, $x = 0$

34. $y = \dfrac{\ln x}{x}$, $y = \dfrac{1-x}{x^2+1}$, $1 \le x \le 4$

---

35. The force exerted by a tendon as a function of its extension determines the loss of energy (see the chapter introduction). Thus, the proportion of energy lost is $\int_0^m [f_s(x) - f_r(x)]\,dx / \int_0^m f_s(x)\,dx$. Suppose that x is the extension of the tendon, $f_s(x)$ is the force during stretching of the tendon and $f_r(x)$ is the force during recoil of the tendon. The data given are for a hind leg tendon of a wallaby (see Alexander's book *Exploring Biomechanics*)[2]:

x (mm)	0	0.75	1.5	2.25	3.0
$f_s(x)$ (N)	0	110	250	450	700
$f_r(x)$ (N)	0	100	230	410	700

Use Simpson's Rule to estimate the proportion of energy returned by the tendon.

36. The arch of a human foot acts like a spring during walking and jumping, storing energy as the foot stretches (i.e., the arch flattens) and returning energy as the foot recoils. In the data, x is the vertical displacement of the arch, $f_s(x)$ is the force on the foot during stretching and $f_r(x)$ is the force during recoil (see Alexander's book *Exploring Biomechanics*):

x (mm)	0	2.0	4.0	6.0	8.0
$f_s(x)$ (N)	0	300	1000	1800	3500
$f_r(x)$ (N)	0	150	700	1300	3500

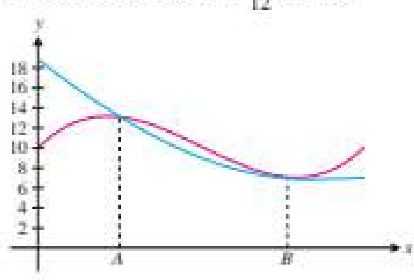
Use Simpson's Rule to estimate the proportion of energy returned by the arch.

37. The average value of a function $f(x)$ on the interval $[a, b]$ is $A = \dfrac{1}{b-a}\int_a^b f(x)\,dx$. Compute the average value of $f(x) = x^2$ on $[0, 3]$ and show that the area above $y = A$ and below $y = f(x)$ equals the area below $y = A$ and above $y = f(x)$.

38. Find t such that the area between $y = \dfrac{2}{x+1}$ and $y = \dfrac{2x}{x^2+1}$ for $0 \le x \le t$ equals $\ln(3/2)$.

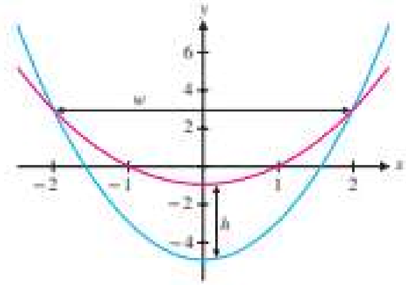
39. Suppose that the parabola $y = ax^2 + bx + c$ and the line $y = mx + n$ intersect at $x = A$ and $x = B$ with $A < B$. Show that the area between the curves equals $\dfrac{|a|}{6}(B - A)^3$. (Hint: Use A and B to rewrite the integrand and then integrate.)

40. Suppose that the cubic $y = ax^3 + bx^2 + cx + d$ and the parabola $y = kx^2 + mx + n$ intersect at $x = A$ and $x = B$ with B repeated (that is, the curves are tangent at B; see the figure). Show that the area between the curves equals $\dfrac{|a|}{12}(B - A)^4$.



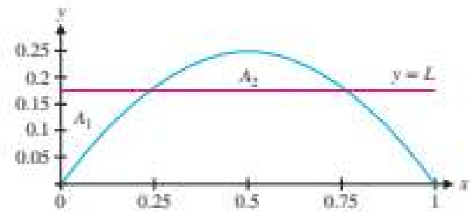
[2]Alexander, R.M. (1992). *Exploring Biomechanics* (New York: W.H. Freeman).

41. Consider two parabolas, each of which has its vertex at $x = 0$, but with different concavities. Let h be the difference in y-coordinates of the vertices, and let w be the difference in the x-coordinates of the intersection points. Show that the area between the curves is $\frac{2}{3}hw$.

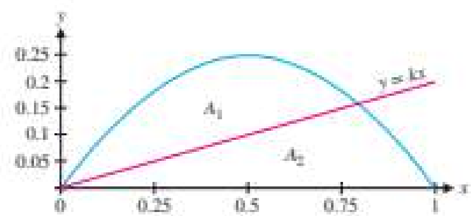


42. Show that for any constant m, the area between $y = 2 - x^2$ and $y = mx$ is $\frac{1}{6}(m^2 + 8)^{3/2}$. Find the minimum such area.

43. For $y = x - x^2$ as shown, find the value of L such that $A_1 = A_2$.



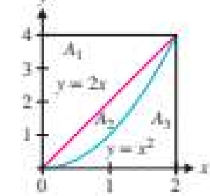
44. For $y = x - x^2$ and $y = kx$ as shown, find k such that $A_1 = A_2$.



45. In terms of A_1, A_2 and A_3, identify the area given by each integral.

(a) $\int_0^2 (2x - x^2)\,dx$ (b) $\int_0^2 (4 - x^2)\,dx$

(c) $\int_0^4 (2 - \sqrt{y})\,dy$ (d) $\int_0^4 \left(\sqrt{y} - \dfrac{y}{2}\right)dy$



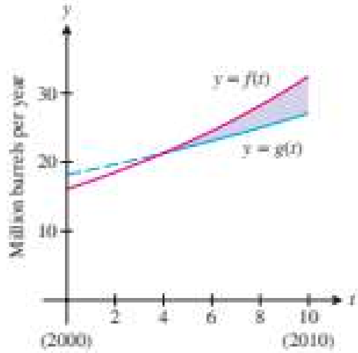
46. Give an integral equal to each area.
(a) $A_2 + A_3$ (b) $A_1 + A_2$ (c) A_1 (c) A_3

47. Let $f(t)$ be the area between $y = \sin^2 x$ and $y = 1$ for $0 \le x \le t$. Find all critical points, local extrema and inflection points for $f(t)$, $t \ge 0$.

48. Let g be a continuous function defined for $x \geq 0$ with $|g(x)| \leq 1$ for $x \geq 0$. Let $f(t)$ be the area between $y = g(x)$ and $y = 1$ for $0 \leq x \leq t$. If g has a local max at $x = a$, does f have a critical point at a? An inflection point at a? What if there is local min at $x = a$?

APPLICATIONS

1. Suppose that a country's oil consumption for the years 2000–2004 was approximately equal to $f(t) = 16.1e^{0.07t}$ million barrels per year, where $t = 0$ corresponds to 2000. Due to the reliance on alternative resources in 2004, the country's consumption changed and was better modeled by $g(t) = 21.3e^{0.04(t-4)}$ million barrels per year, for $t \geq 4$. Show that $f(4) \approx g(4)$ and explain what this number represents. Compute the area between $f(t)$ and $g(t)$ for $4 \leq t \leq 10$. Use this number to estimate the number of barrels of oil saved from 2000 to 2010.



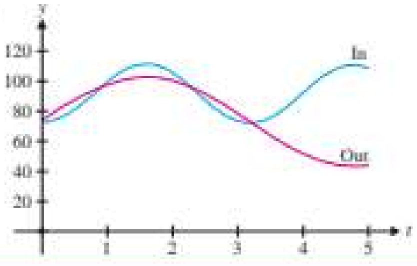
2. Suppose that a nation's fuelwood consumption is given by $76e^{0.03t}$ m^3/yr and new tree growth is $50 - 6e^{0.09t}$ m^3/yr. Compute and interpret the area between the curves for $0 \leq t \leq 10$.

3. Suppose that the birth rate for a certain population is $b(t) = 2e^{0.04t}$ million people per year and the death rate for the same population is $d(t) = 2e^{0.02t}$ million people per year. Show that $b(t) \geq d(t)$ for $t \geq 0$ and explain why the area between the curves represents the increase in population. Compute the increase in population for $0 \leq t \leq 10$.

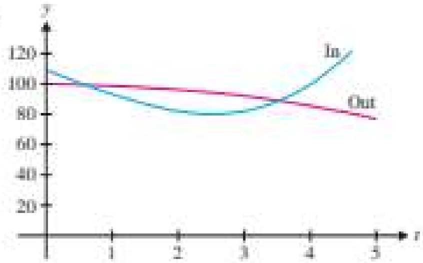
4. Suppose that the birth rate for a population is $b(t) = 2e^{0.04t}$ million people per year and the death rate for the same population is $d(t) = 3e^{0.02t}$ million people per year. Find the intersection T of the curves ($T > 0$). Interpret the area between the curves for $0 \leq t \leq T$ and the area between the curves for $T \leq t \leq 30$. Compute the net change in population for $0 \leq t \leq 30$.

In exercises 5 and 6, the graph shows the rate of flow of water in liters per hour into and out of a tank. Assuming that the tank starts with 400 liters, estimate the amount of water in the tank at hours 1, 2, 3, 4 and 5 and sketch a graph of the amount of water in the tank.

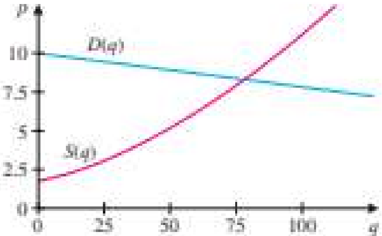
5.



6.

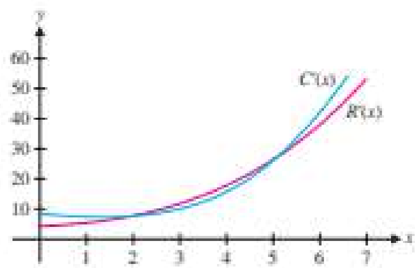


7. The graph shows the supply and demand curves for a product. The point of intersection (q^*, p^*) gives the equilibrium quantity and equilibrium price for the product. The **consumer surplus** is defined to be $\text{CS} = \int_0^{q^*} D(q)\,dq - p^*q^*$. Shade in the area of the graph that represents the consumer surplus, and compute this in the case where $D(q) = 10 - \frac{1}{40}q$ and $S(q) = 2 + \frac{1}{120}q + \frac{1}{1200}q^2$.



8. Repeat exercise 7 for the **producer surplus** defined by $\text{PS} = p^*q^* - \int_0^{q^*} S(q)\,dq$.

9. Let $C'(x)$ be the marginal cost of producing x thousand copies of an item and let $R'(x)$ be the marginal revenue from the sale of that item, with graphs as shown. Assume that $R'(x) = C'(x)$ at $x = 2$ and $x = 5$. Interpret the area between the curves for each interval: (a) $0 \leq x \leq 2$, (b) $2 \leq x \leq 5$, (c) $0 \leq x \leq 5$ and (d) $5 \leq x \leq 6$.



10. A basic principle of economics is that profit is maximized when marginal cost equals marginal revenue. At which intersection is profit maximized in exercise 9? Explain your answer. In terms of profit, what does the other intersection point represent?

EXPLORATORY EXERCISES

1. Find the area between $y = x^2$ and $y = mx$ for any constant $m > 0$. Without doing further calculations, use this area to find the area between $y = \sqrt{x}$ and $y = mx$.

2. For $x > 0$, let $f(x)$ be the area between $y = 1$ and $y = \sin^2 t$ for $0 \leq t \leq x$. Without calculating $f(x)$, find as many relationships as possible between the graphical features (zeros, extrema, inflection points) of $y = f(x)$ and the graphical features of $y = \sin^2 x$.

2.2 VOLUME: SLICING, DISKS AND WASHERS

HISTORICAL NOTES

Archimedes (ca. 287–212 BCE) A Greek mathematician and scientist who was among the first to derive formulas for volumes and areas. Archimedes is known for discovering the basic laws of hydrostatics (he reportedly leapt from his bathtub, shouting "Eureka!" and ran into the streets to share his discovery) and levers ("Give me a place to stand on and I can move the earth."). An ingenious engineer, his catapults, grappling cranes and reflecting mirrors terrorized a massive Roman army that eventually conquered his hometown of Syracuse. Archimedes was especially proud of his proof that the volume of a sphere inscribed in a cylinder is 2/3 of the volume of the cylinder (see exercises 31–34), and requested that this be inscribed on his tombstone. Many of his techniques were very similar to those that we use in calculus today, but many of his writings were lost in the Middle Ages. The amazing story of the recent discovery of his book *The Method* is told in *The Archimedes Codex*, by Netz and Noel.[3]

As we shall see throughout this chapter, the integral is an amazingly versatile tool. In this section, we use integrals to compute the volume of a three-dimensional solid. We begin with a simple problem.

When designing a building, architects must perform numerous detailed calculations. For instance, in order to analyze a building's heating and cooling systems, engineers must calculate the volume of air being processed.

FIGURE 2.12a
© McGraw-Hill Education

FIGURE 2.12b
© McGraw-Hill Education

There are probably only a few solids whose volume you know how to compute. For instance, the building shown in Figure 2.12a is essentially a rectangular box, whose volume is given by lwh, where l is the length, w is the width and h is the height. The right circular cylinders seen in the buildings in Figure 2.12b have volume given by $\pi r^2 h$, where h is the height and r is the radius of the circular cross-section. Notice in each case that the building has a familiar cross-section (a rectangle in Figure 2.12a and a circle in Figure 2.12b) that is extended vertically. We call any such solid a **cylinder** (*any* solid whose cross-sections perpendicular to some axis running through the solid are all the same). There is a connection between the volume formulas for these two cylinders. The volume of a right circular cylinder is

$$V = \underbrace{(\pi r^2)}_{\text{cross-sectional area}} \times \underbrace{h}_{\text{height}},$$

while the volume of a box is

$$V = \underbrace{(\text{length} \times \text{width})}_{\text{cross-sectional area}} \times \text{height}.$$

In general, the volume of any cylinder is found by

$$V = (\text{cross-sectional area}) \times (\text{height}).$$

Volumes by Slicing

Even relatively simple solids, such as pyramids and domes, do not have constant cross-sectional area, as seen in Figures 2.13a and 2.13b. To find the volume in such a case, we take the approach we've used a number of times now: first approximate the volume and then improve the approximation.

[3]Netz, R. and Noel, W. (2011). *The Archimedes Codex: Revealing the Secrets of the World's Greatest Palimpsest* (London: Orion Books).

FIGURE 2.13a
Sheikh Zayed Mosque in Abu Dhabi

FIGURE 2.13b
The Pyramid at the Louvre, Paris

More generally, for a solid that extends from $x = a$ to $x = b$, we start by partitioning the interval $[a, b]$ on the x-axis into n subintervals, each of width $\Delta x = \frac{b-a}{n}$. As usual, we denote $x_0 = a$, $x_1 = a + \Delta x$ and so on, so that

$$x_i = a + i\Delta x, \quad \text{for } i = 0, 1, 2, \ldots, n.$$

We then slice the solid perpendicular to the x-axis at each of the $(n - 1)$ points $x_1, x_2, \ldots, x_{n-1}$. (See Figure 2.14a.) Notice that if n is large, then each slice of the solid will be thin, with nearly constant cross-sectional area. Suppose that the area of the cross-section corresponding to any particular value of x is given by $A(x)$. Observe that the slice between $x = x_{i-1}$ and $x = x_i$ is nearly a cylinder. (See Figure 2.14b.) So, for any point c_i in the interval $[x_{i-1}, x_i]$, the area of the cross-sections on that interval are all approximately $A(c_i)$. The volume V_i of the ith slice is then approximately the volume of the cylinder lying along the interval $[x_{i-1}, x_i]$, with constant cross-sectional area $A(c_i)$ (see Figure 2.14c), so that

$$V_i \approx \underbrace{A(c_i)}_{\text{cross-sectional area}} \underbrace{\Delta x}_{\text{width}}.$$

FIGURE 2.14a
Sliced solid

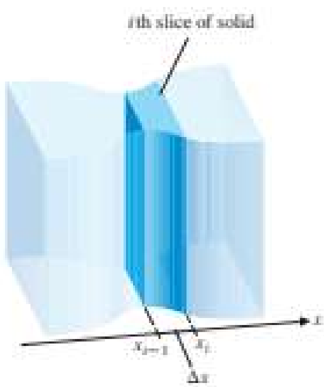


FIGURE 2.14b
ith slice of solid

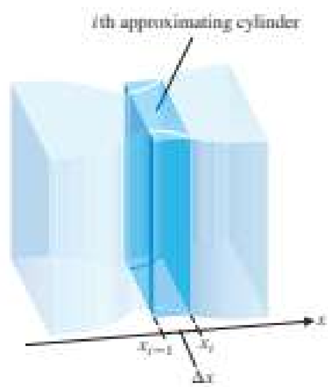


FIGURE 2.14c
ith approximating cylinder

Repeating this process for each of the n slices, we find that the total volume V of the solid is approximately

$$V \approx \sum_{i=1}^{n} A(c_i)\Delta x.$$

Notice that as the number of slices increases, the volume approximation should improve and we get the exact volume by computing the limit

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(c_i)\Delta x,$$

assuming the limit exists. You should recognize this limit as the definite integral

Volume of a solid with cross-sectional area $A(x)$

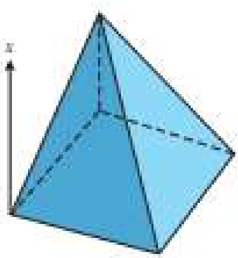
$$V = \int_a^b A(x)\,dx. \tag{2.1}$$

REMARK 2.1

We use the same process followed here to derive many important formulas. In each case, we divide an object into n smaller pieces, approximate the quantity of interest for each of the small pieces, sum the approximations and then take a limit, ultimately recognizing that we have derived a definite integral. For this reason, it is essential that you understand the concept behind formula (2.1). Memorization will not do this for you. However, if you understand how the various pieces of this puzzle fit together, then the rest of this chapter should fall into place for you nicely.

EXAMPLE 2.1 Computing Volume from Cross-Sectional Areas

The Pyramid Arena in Memphis has a square base of side approximately 180 m and a height of approximately 98 m. Find the volume of the pyramid with these measurements.



A Pyramid

Solution Since the pyramid has square horizontal cross-sections, we need only find a formula for the size of the square at each height. Let x represent the height above the ground. At $x = 0$, the cross-section is a square of side 180 m. At $x = 98$, the cross-section can be thought of as a square of side 0 m. If $f(x)$ represents the side length of the square cross-section at height x, we know that $f(0) = 180$, $f(98) = 0$ and $f(x)$ must be a linear function. (Think about this; the sides of the pyramid do not curve.) The slope of the line is $m = \frac{180-0}{0-98} = -\frac{90}{49}$ and we use the y-intercept of 180 to get

$$f(x) = -\frac{90}{49}x + 180.$$

The cross-sectional area is simply the square of $f(x)$, so that from (2.1), we have

$$V = \int_0^{98} A(x)\,dx = \int_0^{98}\left(-\frac{90}{49}x + 180\right)^2 dx.$$

Observe that we can evaluate this integral by substitution, by taking $u = -\frac{90}{49}x + 180$, so that $du = -\frac{90}{49}\,dx$. This gives us

$$V = \int_0^{98}\left(-\frac{90}{49}x + 180\right)^2 dx = -\frac{49}{90}\int_{180}^{0} u^2\,du$$
$$= \frac{49}{90}\int_0^{180} u^2\,du = \frac{49}{90}\frac{u^3}{3}\bigg|_0^{180} = 1{,}058{,}400\,\text{m}^3.$$

In many important applications, the cross-sectional area is not known exactly, but must be approximated using measurements. In such cases, we can approximate the volume using numerical integration.

EXAMPLE 2.2 Estimating Volume from Cross-Sectional Data

In medical imaging, such as CT (computerized tomography) and MRI (magnetic resonance imaging) processes, numerous measurements are taken and processed by a computer to construct a three-dimensional image of the tissue the physician wishes to study. The process is similar to the slicing process we have used to find the volume of a solid. In this case, however, mathematical representations of various slices of the tissue are combined to produce a three-dimensional image that physicians view to determine the health of the tissue. Suppose that an MRI scan indicates that the cross-sectional areas of adjacent slices of a tumor are given by the values in the table.

x (cm)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$A(x)$ (cm^2)	0.0	0.1	0.4	0.3	0.6	0.9	1.2	0.8	0.6	0.2	0.1

Estimate the volume of the tumor.

Solution To find the volume of the tumor, we would compute [following (2.1)]

$$V = \int_0^1 A(x)\,dx,$$

except that we only know $A(x)$ at a finite number of points. Although we can't compute this exactly, we can use Simpson's Rule with $\Delta x = 0.1$ to estimate the value of this integral:

$$\begin{aligned} V &= \int_0^1 A(x)\,dx \\ &\approx \frac{b-a}{3n}\begin{bmatrix} A(0) + 4A(0.1) + 2A(0.2) + 4A(0.3) + 2A(0.4) + 4A(0.5) \\ + 2A(0.6) + 4A(0.7) + 2A(0.8) + 4A(0.9) + A(1) \end{bmatrix} \\ &= \frac{0.1}{3}(0 + 0.4 + 0.8 + 1.2 + 1.2 + 3.6 + 2.4 + 3.2 + 1.2 + 0.8 + 0.1) \\ &\approx 0.49667 \text{ cm}^3. \end{aligned}$$

We now turn to the problem of finding the volume of the dome in Figure 2.13a (The Sheikh Zayed Mosque). Since the horizontal cross-sections are circles, we need only to determine the radius of each circle.

Observe that an alternative way of stating the problem in example 2.3 is to say: Find the volume formed by revolving the region bounded by the curve $x = \sqrt{\frac{45}{2}(90 - y)}$ and the y-axis, for $0 \le y \le 90$, about the y-axis.

EXAMPLE 2.3 Computing the Volume of a Dome

Suppose that a dome has circular cross-sections, with outline $y = -\frac{2}{45}x^2 + 90$, for $-45 \le x \le 45$. A graph is shown in Figure 2.15. Find the volume of the dome.

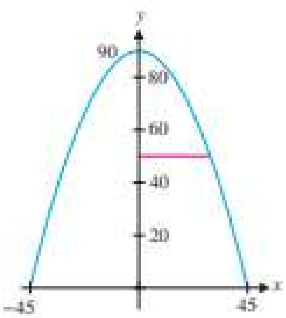


FIGURE 2.15
$y = -\frac{2}{45}x^2 + 90$

Solution As seen in Figure 2.15, the circular cross-sections occur at each value of y, with $0 \le y \le 90$. For a given y, the radius extends from $x = 0$ to $x = \sqrt{\frac{45}{2}(90 - y)}$. The radius for this value of y is given by $r(y) = \sqrt{\frac{45}{2}(90 - y)}$, so that the cross-sectional areas are given by

$$A(y) = \pi\left(\sqrt{\frac{45}{2}(90 - y)}\right)^2,$$

for $0 \le y \le 90$. The volume is then given by

$$\begin{aligned} V &= \int_0^{90} A(y)\,dy = \int_0^{90} \pi\left(\sqrt{\frac{45}{2}(90 - y)}\right)^2 dy = \int_0^{90} \pi\left(2025 - \frac{45}{2}y\right)dy \\ &= \pi\left[2025y - \frac{45}{4}y^2\right]_0^{90} = 91{,}125\pi \approx 286{,}278 \text{ m}^3. \end{aligned}$$

Example 2.3 can be generalized to the *method of disks* used to compute the volume of a solid formed by revolving a two-dimensional region about a vertical or horizontal line. We consider this general method next.

The Method of Disks

Suppose that $f(x) \geq 0$ and f is continuous on the interval $[a, b]$. Take the region bounded by the curve $y = f(x)$ and the x-axis, for $a \leq x \leq b$, and revolve it about the x-axis, generating a solid. (See Figures 2.16a and 2.16b.) We can find the volume of this solid by slicing it perpendicular to the x-axis and recognizing that each cross-section is a circular disk of radius $r = f(x)$. (See Figure 2.16b.) From (2.1), we then have that the volume of the solid is

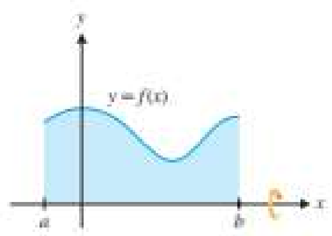


FIGURE 2.16a
$y = f(x) \geq 0$

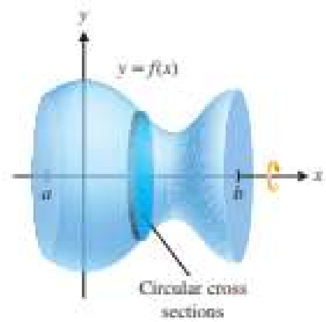


FIGURE 2.16b
Solid of revolution

Volume of a solid of revolution (Method of disks)

$$V = \int_a^b \underbrace{\pi[f(x)]^2}_{\text{cross-sectional area} = \pi r^2} dx. \tag{2.2}$$

Since the cross-sections of such a solid of revolution are all disks, we refer to this method of finding volume as the **method of disks**.

EXAMPLE 2.4 Using the Method of Disks to Compute Volume

Revolve the region under the curve $y = \sqrt{x}$ on the interval $[0, 4]$ about the x-axis and find the volume of the resulting solid of revolution.

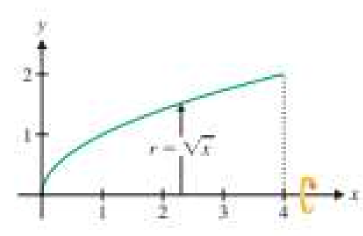


FIGURE 2.17a
$y = \sqrt{x}$

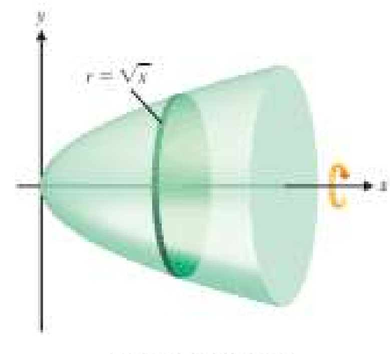


FIGURE 2.17b
Solid of revolution

Solution It's critical to draw a picture of the region and the solid of revolution, so that you get a clear idea of the radii of the circular cross-sections. You can see from Figures 2.17a and 2.17b that the radius of each cross-section is given by $r = \sqrt{x}$. From (2.2), we then get the volume:

$$V = \int_0^4 \underbrace{\pi[\sqrt{x}]^2}_{\text{cross-sectional area} = \pi r^2} dx = \pi \int_0^4 x \, dx = \pi \frac{x^2}{2}\Bigg|_0^4 = 8\pi.$$

In a similar way, suppose that $g(y) \geq 0$ and g is continuous on the interval $[c, d]$. Then, revolving the region bounded by the curve $x = g(y)$ and the y-axis, for $c \leq y \leq d$, about the y-axis generates a solid. (See Figures 2.18a and 2.18b.) Once again, notice from Figure 2.18b that the cross-sections of the resulting solid of revolution are circular disks of radius $r = g(y)$. All that has changed here is that we have interchanged the roles of the variables x and y. The volume of the solid is then given by

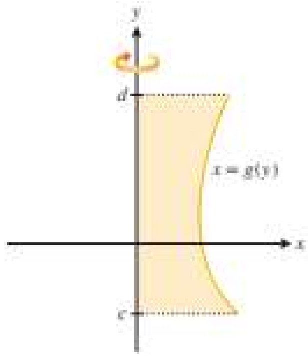


FIGURE 2.18a
Revolve about the y-axis

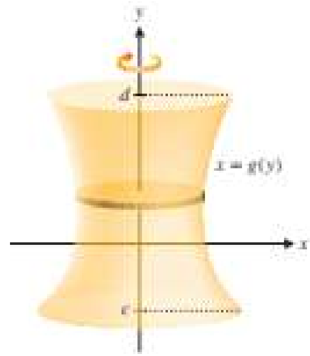


FIGURE 2.18b
Solid of revolution

Volume of a solid of revolution (Method of disks)

$$V = \int_c^d \underbrace{\pi [g(y)]^2}_{\text{cross-sectional area} = \pi r^2} dy. \tag{2.3}$$

REMARK 2.2

When using the method of disks, the variable of integration depends solely on the axis about which you revolve the two-dimensional region: revolving about the x-axis requires integration with respect to x, while revolving about the y-axis requires integration with respect to y. This is easily determined by looking at a sketch of the solid. Don't make the mistake of simply looking for what you can plug in where. This is a recipe for disaster, for the rest of this chapter will require you to make similar choices, each based on distinctive requirements of the problem at hand.

EXAMPLE 2.5 Using the Method of Disks with y as the Independent Variable

Find the volume of the solid resulting from revolving the region bounded by the curves $y = 4 - x^2$ and $y = 1$ from $x = 0$ to $x = \sqrt{3}$ about the y-axis.

Solution You will find a graph of the curve in Figure 2.19a and of the solid in Figure 2.19b.

Notice from Figures 2.19a and 2.19b that the radius of any of the circular cross-sections is given by x. So, we must solve the equation $y = 4 - x^2$ for x, to get $x = \sqrt{4 - y}$. Since the surface extends from $y = 1$ to $y = 4$, the volume is given by (2.3) to be

$$V = \int_1^4 \underbrace{\pi\left(\sqrt{4-y}\right)^2}_{\pi r^2} dy = \int_1^4 \pi(4-y)\,dy$$

$$= \pi\left[4y - \frac{y^2}{2}\right]_1^4 = \pi\left[(16-8) - \left(4 - \frac{1}{2}\right)\right] = \frac{9\pi}{2}.$$

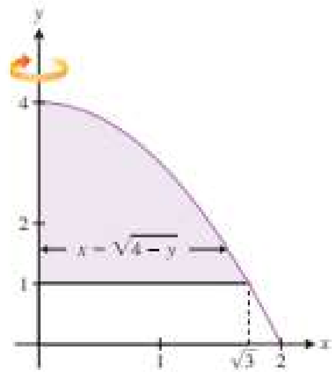


FIGURE 2.19a
$y = 4 - x^2$

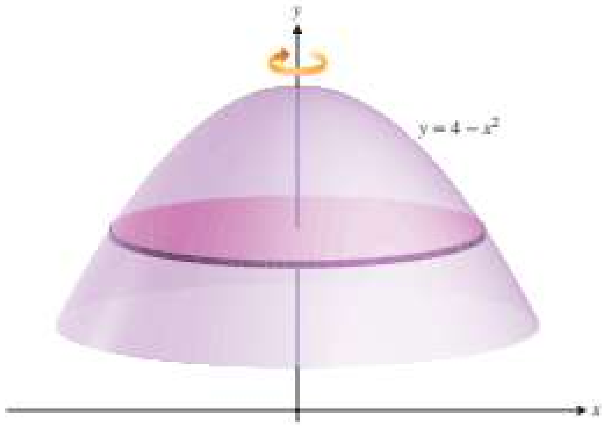


FIGURE 2.19b
Solid of revolution

○ The Method of Washers

One complication that occurs in computing volumes is that the solid may contain a cavity or "hole." Another occurs when a region is revolved about a line other than the x-axis or the y-axis. Neither case will present you with any significant difficulties, if you look carefully at the figures. We illustrate these ideas in examples 2.6 and 2.7.

EXAMPLE 2.6 Computing Volumes of Solids with and without Cavities

Let R be the region bounded by the graphs of $y = \frac{1}{4}x^2$, $x = 0$ and $y = 1$. Compute the volume of the solid formed by revolving R about (a) the y-axis, (b) the x-axis and (c) the line $y = 2$.

Solution (a) The region R is shown in Figure 2.20a and the solid formed by revolving it about the y-axis is shown in Figure 2.20b. Notice that this part of the problem is similar to example 2.5.

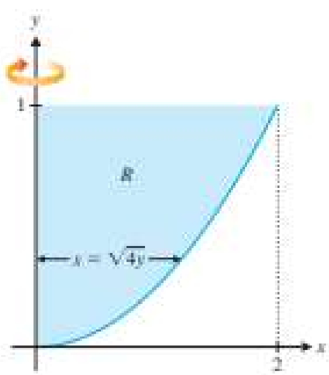


FIGURE 2.20a
$x = \sqrt{4y}$

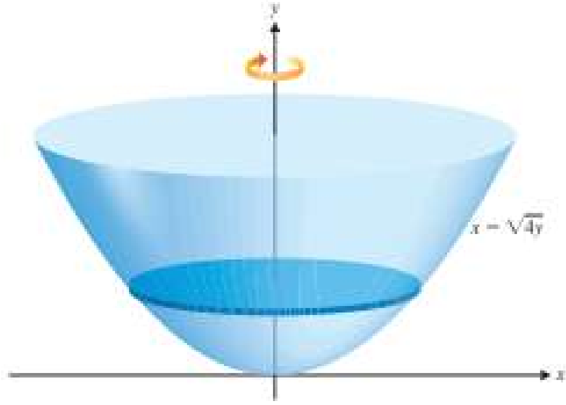


FIGURE 2.20b
Solid of revolution

From (2.3), the volume is given by

$$V = \int_0^1 \underbrace{\pi\left(\sqrt{4y}\right)^2}_{\pi r^2} dy = \pi \frac{4}{2} y^2 \Big|_0^1 = 2\pi.$$

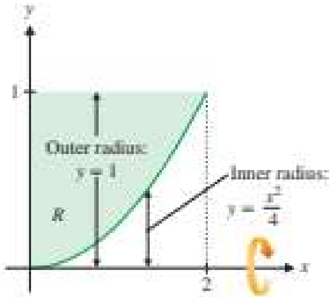


FIGURE 2.21a
$y = \frac{1}{4}x^2$

(b) Revolving the region R about the x-axis produces a cavity in the middle of the solid. See Figure 2.21a for a graph of the region R and Figure 2.21b for a picture of the solid. Our strategy is to compute the volume of the outside of the object (as if it were solid) and then subtract the volume of the cavity. Before diving into a computation, be sure to visualize the geometry behind this. Here, the outside surface of the solid is formed by revolving the line $y = 1$ about the x-axis. The cavity is formed by revolving the curve $y = \frac{1}{4}x^2$ about the x-axis. Look carefully at Figures 2.21a and 2.21b and make certain that you see this. The outer radius, r_O, is the distance from the x-axis to the line $y = 1$, or $r_O = 1$. The inner radius, r_I, is the distance from the x-axis to the curve $y = \frac{1}{4}x^2$, or $r_I = \frac{1}{4}x^2$. Applying (2.2) twice, we see that the volume is given by

$$V = \int_0^2 \underbrace{\pi(1)^2}_{\pi(\text{outer radius})^2} dx - \int_0^2 \underbrace{\pi\left(\frac{1}{4}x^2\right)^2}_{\pi(\text{inner radius})^2} dx$$

$$= \pi \int_0^2 \left(1 - \frac{x^4}{16}\right) dx = \pi\left(x - \frac{1}{80}x^5\right)\Big|_0^2 = \pi\left(2 - \frac{32}{80}\right) = \frac{8}{5}\pi.$$

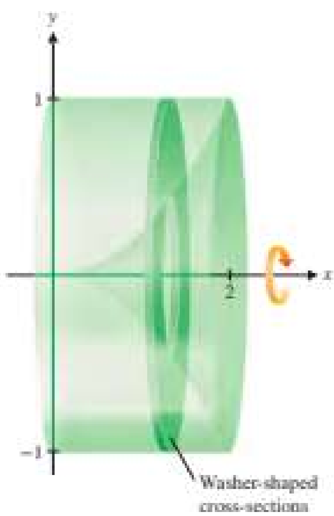


FIGURE 2.21b
Solid with cavity

(c) Revolving the region R about the line $y = 2$ produces a washer-like solid with a cylindrical hole in the middle. The region R is shown in Figure 2.22a and the solid is shown in Figure 2.22b.

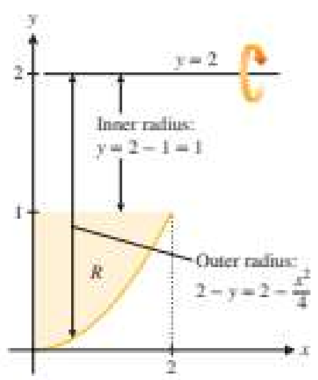


FIGURE 2.22a
Revolve about $y = 2$

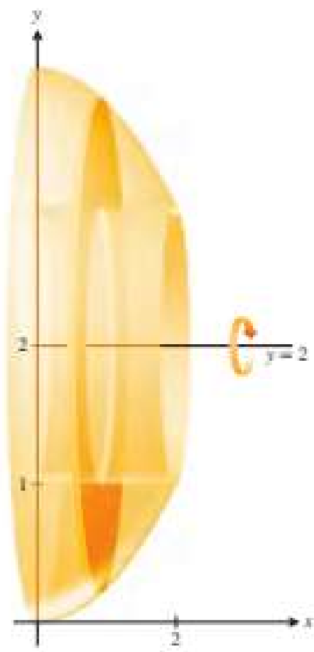


FIGURE 2.22b
Solid of revolution

The volume is computed in the same way as in part (b), by subtracting the volume of the cavity from the volume of the outside solid. From Figures 2.22a and 2.22b, notice that the radius of the outer surface is the distance from the line $y = 2$ to the curve

$y = \frac{1}{4}x^2$ or $r_O = 2 - \frac{1}{4}x^2$. The radius of the inner hole is the distance from the line $y = 2$ to the line $y = 1$ or $r_I = 2 - 1 = 1$. From (2.2), the volume is given by

$$V = \int_0^2 \underbrace{\pi\left(2 - \frac{1}{4}x^2\right)^2}_{\pi(\text{outer radius})^2} dx - \int_0^2 \underbrace{\pi(2-1)^2}_{\pi(\text{inner radius})^2} dx$$
$$= \pi\int_0^2 \left[\left(4 - x^2 + \frac{x^4}{16}\right) - 1\right] dx = \pi\left[3x - \frac{1}{3}x^3 + \frac{1}{80}x^5\right]_0^2$$
$$= \pi\left(6 - \frac{8}{3} + \frac{32}{80}\right) = \frac{56}{15}\pi. \quad \blacksquare$$

In parts (b) and (c) of example 2.6, the volume was computed by subtracting an inner volume from an outer volume in order to compensate for a cavity inside the solid. This technique is a slight generalization of the method of disks and is referred to as the **method of washers**, since the cross-sections of the solids look like washers.

EXAMPLE 2.7 Revolving a Region about Different Lines

Let R be the region bounded by $y = 4 - x^2$ and $y = 0$. Find the volume of the solids obtained by revolving R about each of the following: (a) the y-axis, (b) the line $y = -3$, (c) the line $y = 7$ and (d) the line $x = 3$.

Solution For part (a), we draw the region R in Figure 2.23a and the solid of revolution in Figure 2.23b.

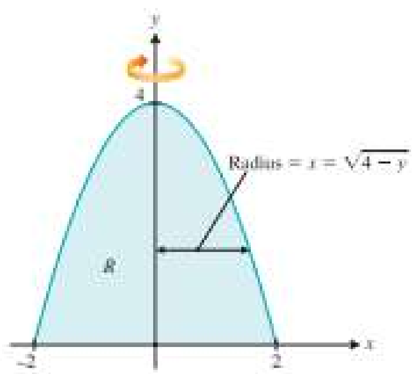


FIGURE 2.23a
Revolve about y-axis

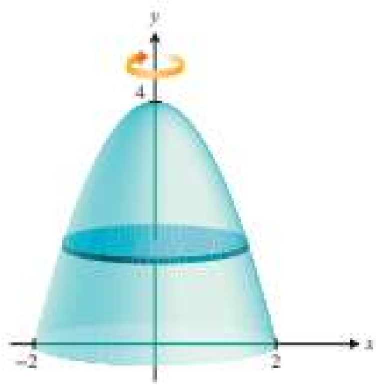


FIGURE 2.23b
Solid of revolution

From Figure 2.23b, notice that each cross-section of the solid is a circular disk, whose radius is simply x. Solving for x, we get $x = \sqrt{4 - y}$, where we have selected x to be positive, since in this context, x represents a distance. From (2.3), the volume of the solid of revolution is given by

$$V = \int_0^4 \underbrace{\pi\left(\sqrt{4-y}\right)^2}_{\pi(\text{radius})^2} dy = \pi\int_0^4 (4 - y)\,dy = \pi\left[4y - \frac{y^2}{2}\right]_0^4 = 8\pi.$$

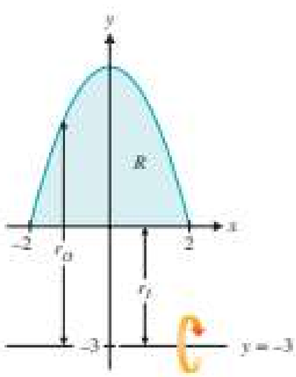


FIGURE 2.24a
Revolve about $y = -3$

For part (b), we have sketched the region R in Figure 2.24a and the solid of revolution in Figure 2.24b. Notice from Figure 2.24b that the cross-sections of the solid are shaped like washers and the outer radius r_O is the distance from the axis of revolution $y = -3$ to the curve $y = 4 - x^2$. That is,

$$r_O = y - (-3) = (4 - x^2) - (-3) = 7 - x^2,$$

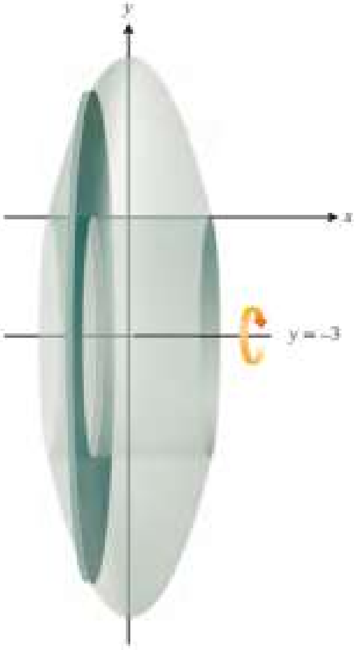


FIGURE 2.24b
Solid of revolution

while the inner radius is the distance from the x-axis to the line $y = -3$. That is,

$$r_I = 0 - (-3) = 3.$$

From (2.2), the volume is

$$V = \int_{-2}^{2} \underbrace{\pi(7 - x^2)^2}_{\pi\,(\text{outer radius})^2} dx - \int_{-2}^{2} \underbrace{\pi(3)^2}_{\pi(\text{inner radius})^2} dx = \frac{1472}{15}\pi,$$

where we have left the details of the computation as an exercise.

Part (c) (revolving about the line $y = 7$) is very similar to part (b). You can see the region R in Figure 2.25a and the solid in Figure 2.25b.

The cross-sections of the solid are again shaped like washers, but this time, the outer radius is the distance from the line $y = 7$ to the x-axis, that is, $r_O = 7$. The inner radius is the distance from the line $y = 7$ to the curve $y = 4 - x^2$,

$$r_I = 7 - (4 - x^2) = 3 + x^2.$$

From (2.2), the volume of the solid is then

$$V = \int_{-2}^{2} \underbrace{\pi(7)^2}_{\pi\,(\text{outer radius})^2} dx - \int_{-2}^{2} \underbrace{\pi(3 + x^2)^2}_{\pi(\text{inner radius})^2} dx = \frac{576}{5}\pi,$$

where we again leave the details of the calculation as an exercise.

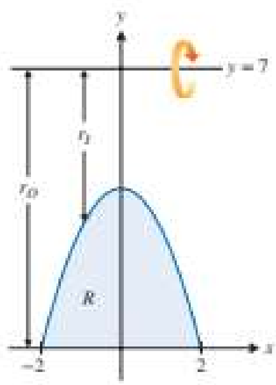


FIGURE 2.25a
Revolve about $y = 7$

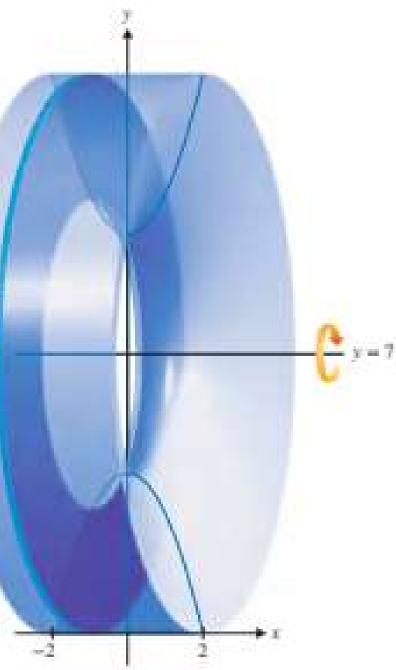


FIGURE 2.25b
Solid of revolution

Finally, for part (d) (revolving about the line $x = 3$), we show the region R in Figure 2.26a and the solid of revolution in Figure 2.26b. In this case, the cross-sections of the solid are washers, but the inner and outer radii are a bit trickier to determine than in the previous parts. The outer radius is the distance between the line $x = 3$ and the *left* half of the parabola, while the inner radius is the distance between the line $x = 3$ and the *right* half of the parabola. The parabola is given by $y = 4 - x^2$, so that $x = \pm\sqrt{4-y}$.

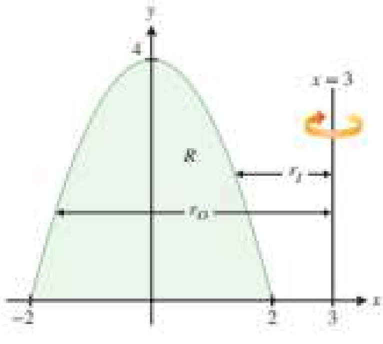


FIGURE 2.26a
Revolve about $x = 3$

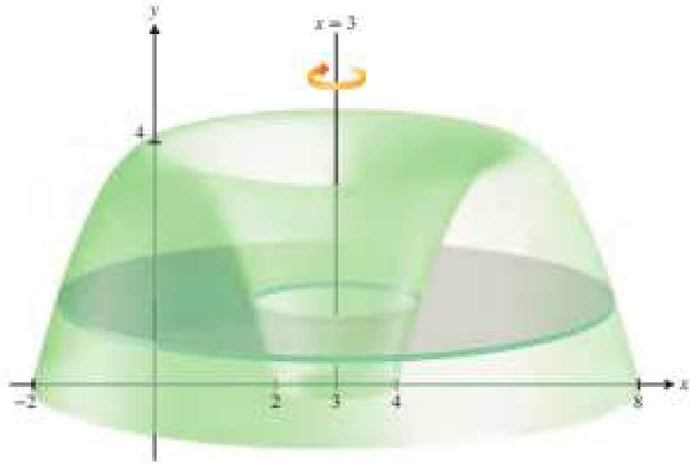


FIGURE 2.26b
Solid of revolution

Notice that $x = \sqrt{4-y}$ corresponds to the right half of the parabola, while $x = -\sqrt{4-y}$ describes the left half of the parabola. This gives us

$$r_O = 3 - (-\sqrt{4-y}) = 3 + \sqrt{4-y} \quad \text{and} \quad r_I = 3 - \sqrt{4-y}.$$

Consequently, we get the volume

$$V = \int_0^4 \underbrace{\pi(3+\sqrt{4-y})^2}_{\pi(\text{outer radius})^2} dy - \int_0^4 \underbrace{\pi(3-\sqrt{4-y})^2}_{\pi(\text{inner radius})^2} dy = 64\pi,$$

where we leave the details of this rather messy calculation to you. In section 5.3, we present an alternative method for finding the volume of a solid of revolution that, for the present problem, will produce much simpler integrals. ■

REMARK 2.3

You will be most successful in finding volumes of solids of revolution if you draw reasonable figures and label them carefully. Don't simply look for what to plug in where. You only need to keep in mind how to find the area of a cross-section of the solid. Integration does the rest.

EXERCISES 2.2

WRITING EXERCISES

1. Discuss the relationships (e.g., perpendicular or parallel) to the x-axis and y-axis of the disks in examples 2.4 and 2.5. Explain how this relationship enables you to correctly determine the variable of integration.

2. The methods of disks and washers were developed separately in the text, but each is a special case of the general volume formula. Discuss the advantages of learning separate formulas versus deriving each example separately from the general formula. For example, would you prefer to learn the extra formulas or have to work each problem from basic principles? How many formulas is too many to learn?

3. To find the area of a triangle of the form Δ in section 5.1, explain why you would use y-integration. In this section, would it be easier to compute the volume of the solid formed by revolving this triangle about the x-axis or y-axis? Explain your preference.

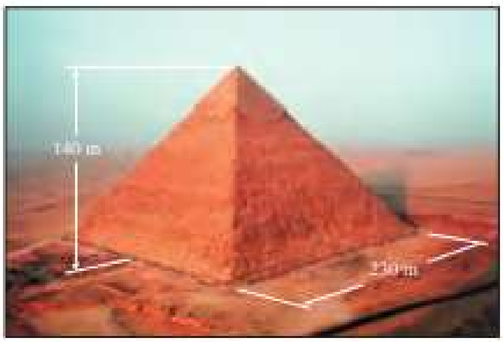
4. In part (a) of example 2.7, Figure 2.23a extends from $x = -\sqrt{4-y}$ to $x = \sqrt{4-y}$, but we used $\sqrt{4-y}$ as the radius. Explain why this is the correct radius and not $2\sqrt{4-y}$.

In exercises 1–4, find the volume of the solid with cross-sectional area $A(x)$.

1. $A(x) = x + 2$, $-1 \le x \le 3$
2. $A(x) = 10e^{0.01x}$, $0 \le x \le 10$
3. $A(x) = \pi(4 - x)^2$, $0 \le x \le 2$
4. $A(x) = 2(x + 1)^2$, $1 \le x \le 4$

In exercises 5–12, set up an integral and compute the volume.

5. (a) The great pyramid at Giza is approximately 140 m high, rising from a square base of side 230 m. Compute its volume using integration.



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(b) Suppose that instead of completing a pyramid, the builders at Giza had stopped at height 70 m (with a square plateau top of side 115 m). Compute the volume of this structure. Explain why the volume is greater than half the volume of the pyramid in part (a).

6. Find the volume of a pyramid of height 49 m that has a square base of side 90 m. These dimensions are half those of the pyramid in example 2.1. How does the volume compare?

7. A church steeple is 30 m tall with square cross-sections. The square at the base has side 3 m, the square at the top has side 0.5 m and the side varies linearly in between. Compute the volume.

8. A house attic has rectangular cross-sections parallel to the ground and triangular cross-sections perpendicular to the ground. The rectangle is 30 m by 60 m at the bottom of the attic and the triangles have base 30 m and height 10 m. Compute the volume of the attic.

9. The outline of a dome is given by $y = 60 - \frac{x^2}{60}$ for $-60 \le x \le 60$ (units of m), with circular cross-sections perpendicular to the y-axis. Find its volume.

10. A dome "twice as big" as that of exercise 9 has outline $y = 120 - \frac{x^2}{120}$ for $-120 \le x \le 120$ (units of m). Find its volume.

11. A pottery jar has circular cross-sections of radius $4 + \sin\frac{x}{2}$ cm for $0 \le x \le 2\pi$. Sketch a picture of the jar and compute its volume.

12. A pottery jar has circular cross-sections of radius $4 - \sin\frac{x}{2}$ cm for $0 \le x \le 2\pi$. Sketch a picture of the jar and compute its volume.

13. Suppose an MRI scan indicates that cross-sectional areas of adjacent slices of a tumor are as given in the table. Use Simpson's Rule to estimate the volume.

x (cm)	0.0	0.1	0.2	0.3	0.4	0.5
$A(x)$ (cm^2)	0.0	0.1	0.2	0.4	0.6	0.4

x (cm)	0.6	0.7	0.8	0.9	1.0
$A(x)$ (cm^2)	0.3	0.2	0.2	0.1	0.0

14. Suppose an MRI scan indicates that cross-sectional areas of adjacent slices of a tumor are as given in the table. Use Simpson's Rule to estimate the volume.

x (cm)	0.0	0.2	0.4	0.6	0.8	1.0	1.2
$A(x)$ (cm^2)	0.0	0.2	0.3	0.2	0.4	0.2	0.0

15. Estimate the volume from the cross-sectional areas.

x (m)	0.0	0.5	1.0	1.5	2.0
$A(x)$ (m^2)	1.0	1.2	1.4	1.3	1.2

16. Estimate the volume from the cross-sectional areas.

x (m)	0.0	0.1	0.2	0.3	0.4
$A(x)$ (m^2)	2.0	1.8	1.7	1.6	1.8

x (m)	0.5	0.6	0.7	0.8
$A(x)$ (m^2)	2.0	2.1	2.2	2.4

In exercises 17–20, compute the volume of the solid formed by revolving the given region about the given line.

17. Region bounded by $y = 2 - x$, $y = 0$ and $x = 0$ about (a) the x-axis; (b) $y = 3$

18. Region bounded by $y = x^2$, $y = 4 - x^2$ about (a) the x-axis; (b) $y = 4$

19. Region bounded by $y = \sqrt{x}$, $y = 2$ and $x = 0$ about (a) the y-axis; (b) $x = 4$

20. Region bounded by $y = x^2$ and $x = y^2$ about (a) the y-axis; (b) $x = 1$

In exercises 21–24, a solid is formed by revolving the given region about the given line. Compute the volume exactly if possible and estimate if necessary.

21. Region bounded by $y = e^x$, $x = 0$, $x = 2$ and $y = 0$ about (a) the y-axis; (b) $y = -2$

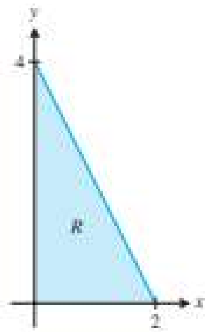
22. Region bounded by $y = \sec x$, $y = 0$, $x = -\pi/4$ and $x = \pi/4$ about (a) $y = 2$; (b) the x-axis

23. Region bounded by $y = \sqrt{\frac{x}{x^2+2}}$, the x-axis and $x = 1$ about (a) the x-axis; (b) $y = 3$

24. Region bounded by $y = e^{-x^2}$ and $y = x^2$ about (a) the x-axis; (b) $y = -1$

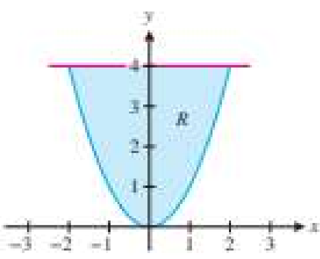
25. Let R be the region bounded by $y = 4 - 2x$, the x-axis and the y-axis. Compute the volume of the solid formed by revolving R about the given line.

(a) the y-axis	(b) the x-axis	(c) $y = 4$
(d) $y = -4$	(e) $x = 2$	(f) $x = -2$



26. Let R be the region bounded by $y = x^2$ and $y = 4$. Compute the volume of the solid formed by revolving R about the given line.

(a) $y = 4$	(b) the y-axis	(c) $y = 6$
(d) $y = -2$	(e) $x = 2$	(f) $x = -4$



27. Let R be the region bounded by $y = x^2$, $y = 0$ and $x = 1$. Compute the volume of the solid formed by revolving R about the given line.

(a) the y-axis	(b) the x-axis	(c) $x = 1$
(d) $y = 1$	(e) $x = -1$	(f) $y = -1$

28. Let R be the region bounded by $y = x$, $y = -x$ and $x = 1$. Compute the volume of the solid formed by revolving R about the given line.

(a) the x-axis	(b) the y-axis
(c) $y = 1$	(d) $y = -1$

29. Let R be the region bounded by $y = ax^2$, $y = h$ and the y-axis (where a and h are positive constants). Compute the volume of the solid formed by revolving this region about the y-axis. Show that your answer equals half the volume of a cylinder of height h and radius $\sqrt{h/a}$. Sketch a picture to illustrate this.

30. Use the result of exercise 29 to immediately write down the volume of the solid formed by revolving the region bounded by $y = ax^2$, $x = \sqrt{h/a}$ and the x-axis about the y-axis.

31. Suppose that the square consisting of all points (x, y) with $-1 \le x \le 1$ and $-1 \le y \le 1$ is revolved about the y-axis. Show that the volume of the resulting solid is 2π.

32. Suppose that the circle $x^2 + y^2 = 1$ is revolved about the y-axis. Show that the volume of the resulting solid is $\frac{4}{3}\pi$.

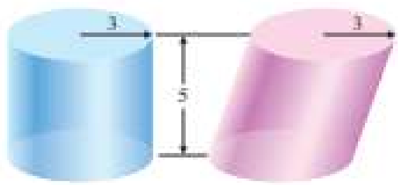
33. Suppose that the triangle with vertices $(-1, -1)$, $(0, 1)$ and $(1, -1)$ is revolved about the y-axis. Show that the volume of the resulting solid is $\frac{2}{3}\pi$.

34. Sketch the square, circle and triangle of exercises 31–33 on the same axes. Show that the relative volumes of the revolved regions (cylinder, sphere and cone, respectively) are 3:2:1.

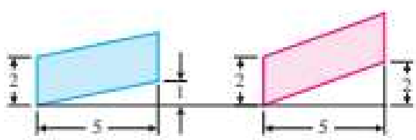
35. Verify the formula for the volume of a sphere by revolving the circle $x^2 + y^2 = r^2$ about the y-axis.

36. Verify the formula for the volume of a cone by revolving the line segment $y = -\frac{h}{r}x + h$, $0 \le x \le r$, about the y-axis.

37. Let A be a right circular cylinder with radius 3 and height 5. Let B be the tilted circular cylinder with radius 3 and height 5. Determine whether A and B enclose the same volume.



38. Determine whether the two indicated parallelograms have the same area. (Exercises 37 and 38 illustrate Cavalieri's Theorem.)



39. The base of a solid V is the circle $x^2 + y^2 = 1$. Find the volume if V has (a) square cross-sections and (b) semicircular cross-sections perpendicular to the x-axis.

40. The base of a solid V is the triangle with vertices $(-1, 0)$, $(0, 1)$ and $(1, 0)$. Find the volume if V has (a) square cross-sections and (b) semicircular cross-sections perpendicular to the x-axis.

41. The base of a solid V is the region bounded by $y = x^2$ and $y = 2 - x^2$. Find the volume if V has (a) square cross-sections, (b) semicircular cross-sections and (c) equilateral triangle cross-sections perpendicular to the x-axis.

42. The base of a solid V is the region bounded by $y = \ln x$, $x = 2$ and $y = 0$. Find the volume if V has (a) square cross-sections, (b) semicircular cross-sections and (c) equilateral triangle cross-sections perpendicular to the x-axis.

43. The base of a solid V is the region bounded by $y = e^{-2x}$, $y = 0$, $x = 0$ and $x = \ln 5$. Find the volume if V has (a) square cross-sections and (b) semicircular cross-sections perpendicular to the x-axis.

44. The base of a solid V is the region bounded by $y = x^2$ and $y = \sqrt{x}$. Find the volume if V has (a) square cross-sections and (b) semicircular cross-sections perpendicular to the x-axis.

45. Find the volume of the intersection of two spheres, one formed by revolving $x^2 + y^2 = 1$ about the y-axis, the other formed by revolving $(x-1)^2 + y^2 = 1$ about $x = 1$.

46. Let S be the sphere formed by revolving $x^2 + y^2 = 4$ about the y-axis, and C the cylinder formed by revolving $x = 1$, $-4 \le y \le 4$, about the y-axis. Find the volume of the intersection of S and C.

APPLICATIONS

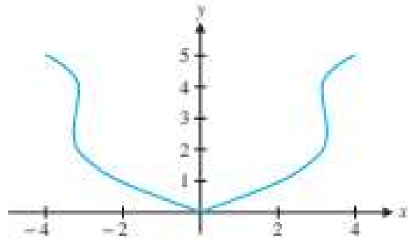
1. Use the given table of values to estimate the volume of the solid formed by revolving $y = f(x)$, $0 \le x \le 3$, about the x-axis.

x	0	0.5	1.0	1.5	2.0	2.5	3.0
$f(x)$	2.0	1.2	0.9	0.4	1.0	1.4	1.6

2. Use the given table of values to estimate the volume of the solid formed by revolving $y = f(x)$, $0 \le x \le 2$, about the x-axis.

x	0	0.25	0.50	0.75	1.0	1.25	1.50	1.75	2.0
$f(x)$	4.0	3.6	3.4	3.5	3.2	3.8	4.2	4.6	5.0

3. Water is poured at a constant rate into the vase with outline as shown and circular cross-sections. Sketch a graph showing the height of the water in the vase as a function of time.



4. Sketch a graph of the rate of flow versus time if you poured water into the vase of exercise 3 in such a way that the height of the water in the vase increased at a constant rate.

EXPLORATORY EXERCISES

1. Generalize the result of exercise 34 to any rectangle. That is, sketch the rectangle with $-a \le x \le a$ and $-b \le y \le b$, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the triangle with vertices $(-a, -b)$, $(0, b)$ and $(a, -b)$. Show that the relative volumes of the solid formed by revolving these regions about the y-axis are 3:2:1.

2. Take the circle $(x-2)^2 + y^2 = 1$ and revolve it about the y-axis. The resulting donut-shaped solid is called a **torus**. Compute its volume. Show that the volume equals the area of the circle times the distance traveled by the center of the circle. This is an example of **Pappus' Theorem**, dating from the fourth century BC. Verify that the result also holds for the triangle in exercise 25, parts (c) and (d).

2.3 VOLUMES BY CYLINDRICAL SHELLS

In this section, we present an alternative to the method of washers discussed in section 5.2. Let R denote the region bounded by the graph of $y = f(x)$ and the x-axis on the interval $[a, b]$, where $0 < a < b$ and $f(x) \ge 0$ on $[a, b]$. (See Figure 2.27a.) If we revolve this region about the y-axis, we get the solid shown in Figure 2.27b. Finding the volume of this solid by the method of washers is awkward, since we would need to break up the region into several pieces.

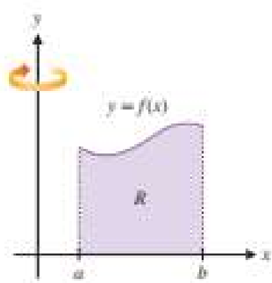


FIGURE 2.27a
Revolve about y-axis

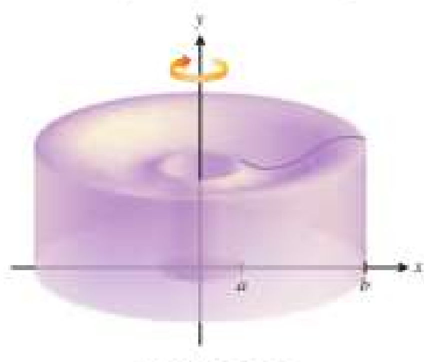


FIGURE 2.27b
Solid of revolution

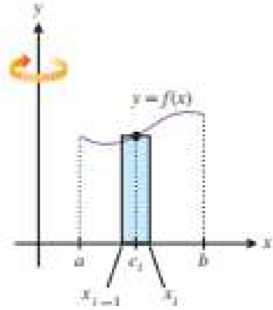


FIGURE 2.28
*i*th rectangle

Alternatively, we partition the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. On each subinterval $[x_{i-1}, x_i]$, pick a point c_i and construct the rectangle of height $f(c_i)$ as indicated in Figure 2.28. Revolving this rectangle about the y-axis forms a thin cylindrical shell (i.e., a hollow cylinder, like a pipe), as in Figure 2.29a.

To find the volume of this thin cylindrical shell, imagine cutting the cylinder from top to bottom and then flattening out the shell. After doing this, you should have essentially a thin rectangular sheet, as seen in Figure 2.29b.

Notice that the length of such a thin sheet corresponds to the circumference of the cylindrical shell, which is $2\pi \cdot \text{radius} \approx 2\pi c_i$. So, the volume V_i of the ith cylindrical shell is approximately

$$\begin{aligned} V_i &\approx \text{length} \times \text{width} \times \text{height} \\ &= (2\pi \times \text{radius}) \times \text{thickness} \times \text{height} \\ &\approx (2\pi c_i)\Delta x\, f(c_i). \end{aligned}$$

The total volume V of the solid can then be approximated by the sum of the volumes of the n cylindrical shells:

$$V \approx \sum_{i=1}^{n} 2\pi \underbrace{c_i}_{\text{radius}} \underbrace{f(c_i)}_{\text{height}} \underbrace{\Delta x}_{\text{thickness}}.$$

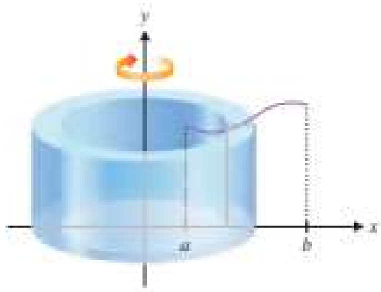


FIGURE 2.29a
Cylindrical shell

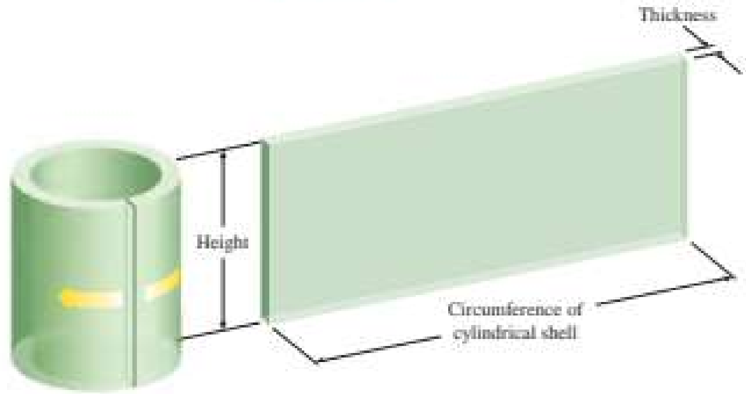


FIGURE 2.29b
Flattened cylindrical shell

As we have done many times now, we can get the exact volume of the solid by taking the limit as $n \to \infty$ and recognizing the resulting definite integral. We have

Volume of a solid of revolution (cylindrical shells)

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi c_i f(c_i)\Delta x = \int_a^b 2\pi \underbrace{x}_{\text{radius}} \underbrace{f(x)}_{\text{height}} \underbrace{dx}_{\text{thickness}}. \tag{3.1}$$

For obvious reasons, we call this the *method of cylindrical shells*.

EXAMPLE 3.1 Using the Method of Cylindrical Shells

Use the method of cylindrical shells to find the volume of the solid formed by revolving the region bounded by the graphs of $y = x$ and $y = x^2$ in the first quadrant about the y-axis.

Solution From Figure 2.30a, notice that the region has an upper boundary of $y = x$ and a lower boundary of $y = x^2$ and runs from $x = 0$ to $x = 1$. Here, we have drawn a sample rectangle that generates a cylindrical shell. The resulting solid of revolution can be seen in Figure 2.30b. We can write down an integral for the volume by analyzing the various components of the solid in Figures 2.30a and 2.30b. From (3.1), we have

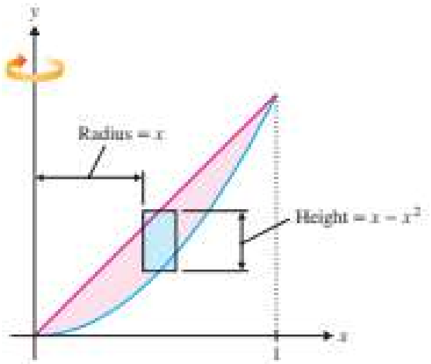


FIGURE 2.30a
Sample rectangle generating a cylindrical shell

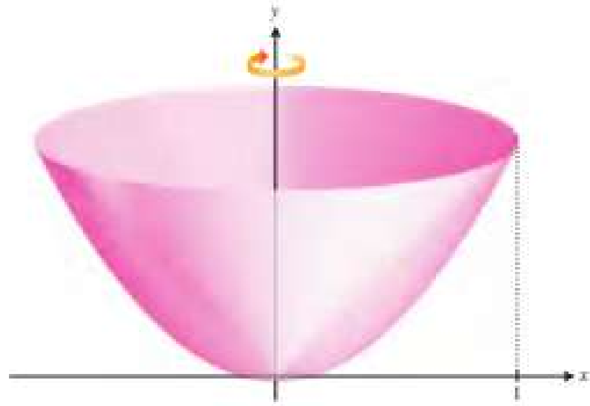


FIGURE 2.30b
Solid of revolution

$$V = \int_0^1 2\pi \underbrace{x}_{\text{radius}} \underbrace{(x - x^2)}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$

$$= 2\pi \int_0^1 (x^2 - x^3)\,dx = 2\pi\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{\pi}{6}.$$

REMARK 3.1

Do not rely on simply memorizing formula (3.1). You must strive to understand the meaning of the components. It's simple to do if you just think of how they correspond to the volume of a cylindrical shell:

2π(radius) (height) (thickness).

If you think of volumes in this way, you will have no difficulty with the method of cylindrical shells.

We can now generalize this method to solve the problem in example 2.7 part (d) in a much simpler fashion.

EXAMPLE 3.2 A Volume Where Shells Are Simpler Than Washers

Find the volume of the solid formed by revolving the region bounded by the graph of $y = 4 - x^2$ and the x-axis about the line $x = 3$.

Solution Look carefully at Figure 2.31a, where we have drawn a sample rectangle that generates a cylindrical shell, and at the solid shown in Figure 2.31b. Notice that the radius of a cylindrical shell is the distance from the line $x = 3$ to the shell:

$$r = 3 - x.$$

This gives us the volume

$$V = \int_{-2}^{2} 2\pi \underbrace{(3 - x)}_{\text{radius}} \underbrace{(4 - x^2)}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$

$$= 2\pi \int_{-2}^{2} (x^3 - 3x^2 - 4x + 12)\,dx = 64\pi,$$

where the routine details of the calculation of the integral are left to the reader.

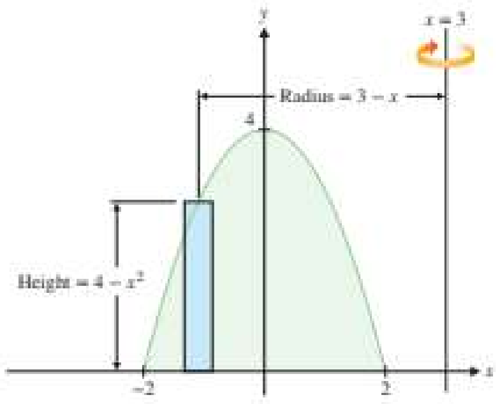


FIGURE 2.31a
Typical rectangle generating a cylindrical shell

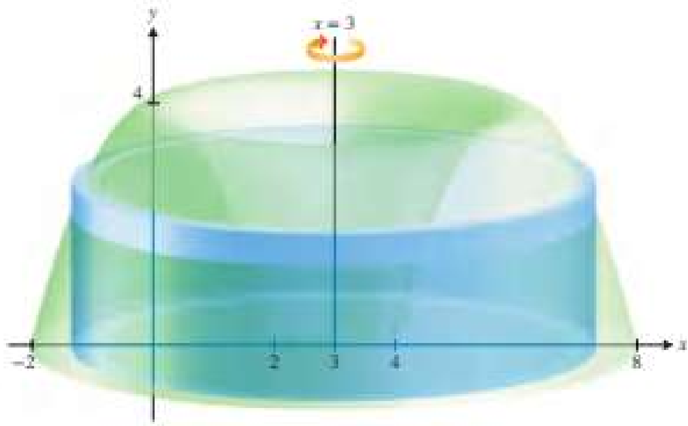


FIGURE 2.31b
Solid of revolution

Your first step in a volume calculation should be to analyze the geometry of the solid to decide whether it's easier to integrate with respect to x or y. Note that for a given solid, the variable of integration in the method of cylindrical shells is exactly *opposite* that of the method of washers. So, your choice of integration variable will determine which method you use.

EXAMPLE 3.3 Computing Volumes Using Shells and Washers

Let R be the region bounded by the graphs of $y = x$, $y = 2 - x$ and $y = 0$. Compute the volume of the solid formed by revolving R about the lines (a) $y = 2$, (b) $y = -1$ and (c) $x = 3$.

Solution The region R is shown in Figure 2.32a. The geometry of the region suggests that we should consider y as the variable of integration. Look carefully at the differences among the following three volumes.

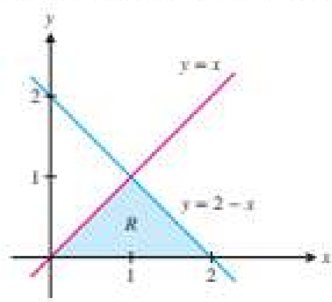


FIGURE 2.32a
$y = x$ and $y = 2 - x$

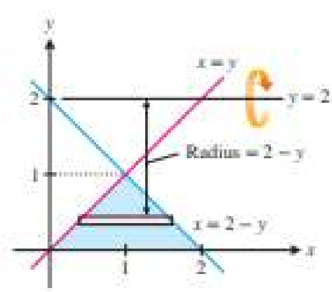


FIGURE 2.32b
Revolve about $y = 2$

(a) Revolving R about the line $y = 2$, observe that the radius of a cylindrical shell is the distance from the line $y = 2$ to the shell: $r = 2 - y$, for $0 \le y \le 1$. (See Figure 2.32b.) The height is the difference in the x-values on the two curves: solving for x, we have $x = y$ and $x = 2 - y$. Following (3.1), we get the volume

$$V = \int_0^1 2\pi \underbrace{(2-y)}_{\text{radius}} \underbrace{[(2-y)-y]}_{\text{height}} \underbrace{dy}_{\text{thickness}} = \frac{10}{3}\pi,$$

where we leave the routine details of the calculation to you.

(b) Revolving R about the line $y = -1$, notice that the height of the cylindrical shells is the same as in part (a), but the radius r is the distance from the line $y = -1$ to the shell: $r = y - (-1) = y + 1$. (See Figure 2.32c.) This gives us the volume

(c) Finally, revolving R about the line $x = 3$, notice that to find the volume using cylindrical shells, we would need to break the calculation into two pieces, since the height of the cylindrical shells would be different for $x \in [0, 1]$ than for $x \in [1, 2]$. (Think about this some.) On the other hand, this is done easily by the method of washers. Observe that the outer radius is the distance from the line $x = 3$ to the line $x = y$: $r_o = 3 - y$, while the inner radius is the distance from the line $x = 3$ to the line $x = 2 - y$: $r_I = 3 - (2 - y)$. (See Figure 2.32d.) This gives us the volume

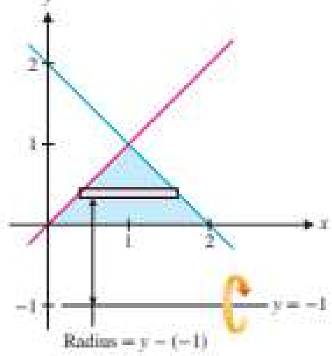


FIGURE 2.32c
Revolve about $y = -1$

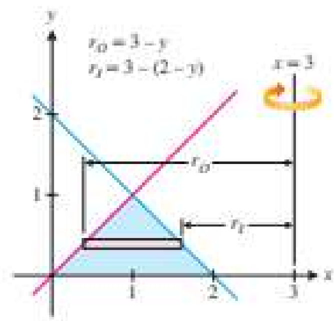


FIGURE 2.32d
Revolve about $x = 3$

$$V = \int_0^1 \pi\left\{ \underbrace{(3-y)^2}_{\text{outer radius}^2} - \underbrace{[3-(2-y)]^2}_{\text{inner radius}^2} \right\} dy = 4\pi.$$

Once again, you should note the importance of sketching and carefully labeling the region. Doing so will make it much easier to correctly set up the integral. Finally, do whatever it takes to evaluate the integral. If you don't know how to evaluate it, you can try your CAS or approximate it numerically (e.g., by Simpson's Rule).

EXAMPLE 3.4 Approximating Volumes Using Shells and Washers

Let R be the region bounded by the graphs of $y = \cos x$ and $y = x^2$. Compute the volume of the solid formed by revolving R about the lines (a) $x = 2$ and (b) $y = 2$.

Solution First, we sketch the region R. (See Figure 2.33a.) Since the top and bottom of R are each defined by a curve of the form $y = f(x)$, we will want to integrate with respect to x. We next look for the points of intersection of the two curves, by solving the equation $\cos x = x^2$. Since we can't solve this exactly, we must use an approximate method (e.g., Newton's method) to obtain the approximate intersections at $x = \pm 0.824132$.

(a) If we revolve the region about the line $x = 2$, we should use cylindrical shells. (See Figure 2.33b.) In this case, observe that the radius r of a cylindrical shell is the distance from the line $x = 2$ to the shell: $r = 2 - x$, while the height of a shell is $\cos x - x^2$. We get the volume

$$V \approx \int_{-0.824132}^{0.824132} 2\pi \underbrace{(2-x)}_{\text{radius}} \underbrace{(\cos x - x^2)}_{\text{height}} dx \approx 13.757,$$

where we have approximated the value of the integral numerically. (We will see how to find an antiderivative for this integrand in Chapter 3.)

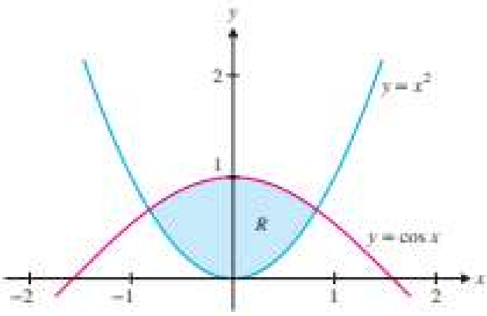


FIGURE 2.33a
$y = \cos x$, $y = x^2$

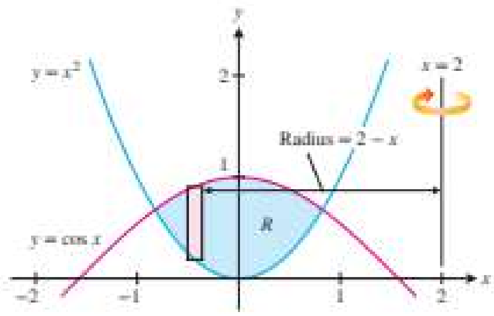


FIGURE 2.33b
Revolve about $x = 2$

(b) If we revolve the region about the line $y = 2$ (see Figure 2.33c), we use the method of washers. In this case, observe that the outer radius of a washer is the distance

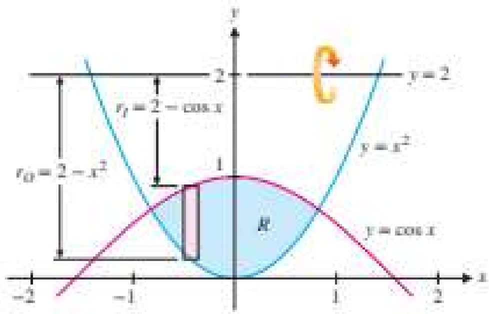


FIGURE 2.33c
Revolve about $y = 2$

from the line $y = 2$ to the curve $y = x^2$: $r_o = 2 - x^2$, while the inner radius is the distance from the line $y = 2$ to the curve $y = \cos x$: $r_I = 2 - \cos x$. (Again, see Figure 2.33c.) This gives us the volume

$$V \approx \int_{-0.824132}^{0.824132} \pi \left[\underbrace{(2 - x^2)^2}_{\text{outer radius}^2} - \underbrace{(2 - \cos x)^2}_{\text{inner radius}^2} \right] dx \approx 10.08,$$

where we have approximated the value of the integral numerically. ■

We close this section with a summary of strategies for computing volumes of solids of revolution.

VOLUME OF A SOLID OF REVOLUTION

- Sketch the region to be revolved and the axis of revolution.
- Determine the variable of integration (x if the region has a well-defined top and bottom, y if the region has well-defined left and right boundaries).
- Based on the axis of revolution and the variable of integration, determine the method (disks or washers for x-integration about a horizontal axis or y-integration about a vertical axis, shells for x-integration about a vertical axis or y-integration about a horizontal axis).
- Label your picture with the inner and outer radii for disks or washers; label the radius and height for cylindrical shells.
- Set up the integral(s) and evaluate.

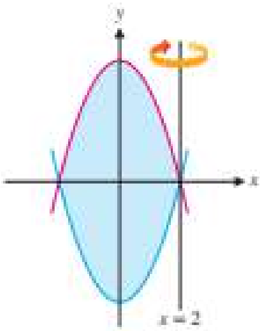
EXERCISES 2.3

WRITING EXERCISES

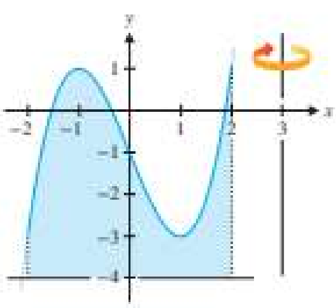
1. Explain why the method of cylindrical shells produces an integral with x as the variable of integration when revolving about a vertical axis. (Describe where the shells are and which direction to move in to go from shell to shell.)

2. Explain why the method of cylindrical shells has the same form whether or not the solid has a hole or cavity. That is, there is no need for separate methods analogous to disks and washers.

3. Suppose that the region bounded by $y = x^2 - 4$ and $y = 4 - x^2$ is revolved about the line $x = 2$. Carefully explain which method (disks, washers or shells) would be easiest to use to compute the volume.



4. Suppose that the region bounded by $y = x^3 - 3x - 1$ and $y = -4$, $-2 \le x \le 2$, is revolved about $x = 3$. Explain what would be necessary to compute the volume using the method of washers and what would be necessary to use the method of cylindrical shells. Which method would you prefer and why?



In exercises 1–8, sketch the region, draw in a typical shell, identify the radius and height of each shell and compute the volume.

1. The region bounded by $y = x^2$ and the x-axis, $-1 \le x \le 1$, revolved about $x = 2$

2. The region bounded by $y = x^2$ and the x-axis, $-1 \le x \le 1$, revolved about $x = -2$

3. The region bounded by $y = x$, $y = -x$ and $x = 1$ revolved about the y-axis

4. The region bounded by $y = x$, $y = -x$ and $x = 1$ revolved about $x = 1$

5. The region bounded by $y = \sqrt{x^2 + 1}$ and $y = 0$, $0 \le x \le 4$ revolved about $x = 0$

6. The region bounded by $y = x^2$ and $y = 0$, $-1 \le x \le 1$, revolved about $x = 2$

7. The region bounded by $x^2 + y^2 = 1$ revolved about $y = 2$

8. The region bounded by $x^2 + y^2 = 2y$ revolved about $y = 4$

In exercises 9–16, use cylindrical shells to compute the volume.

9. The region bounded by $y = x^2$ and $y = 2 - x^2$, revolved about $x = -2$

10. The region bounded by $y = x^2$ and $y = 2 - x^2$, revolved about $x = 2$

11. The region bounded by $x = y^2$ and $x = 4$, revolved about $y = -2$

12. The region bounded by $x = y^2$ and $x = 4$, revolved about $y = 2$

13. The region bounded by $y = x^2 + 2$, $y = x + 1$ and $x = 2$, revolved about $x = 3$

14. The region bounded by $y = x$ and $y = x^2 - 2$, revolved about $x = 3$

15. The region bounded by $x = (y - 1)^2$ and $x = 9$, revolved about $y = 5$

16. The region bounded by $x = (y - 1)^2$ and $x = 9$, revolved about $y = -3$

In exercises 17–26, use the best method available to find each volume.

17. The region bounded by $y = 4 - x$, $y = 4$ and $y = x$ revolved about
(a) the x-axis (b) the y-axis (c) $x = 4$ (d) $y = 4$

18. The region bounded by $y = x + 2$, $y = -x - 2$ and $x = 0$ revolved about
(a) $y = -2$ (b) $x = -2$ (c) the y-axis (d) the x-axis

19. The region bounded by $y = x$ and $y = x^2 - 6$ revolved about
(a) $x = 3$ (b) $y = 3$ (c) $x = -3$ (d) $y = -6$

20. The region bounded by $x = y^2$ and $x = 2 + y$ revolved about
(a) $x = -1$ (b) $y = -1$ (c) $x = -2$ (d) $y = -2$

21. The region bounded by $y = x^2$ $(x \ge 0)$, $y = 2 - x$ and $x = 0$ revolved about
(a) the x-axis (b) the y-axis (c) $x = 1$ (d) $y = 2$

22. The region bounded by $y = 2 - x^2$, $y = x (x > 0)$ and the y-axis revolved about
(a) the x-axis (b) the y-axis (c) $x = -1$ (d) $y = -1$

23. The region to the right of $x = y^2$ and to the left of $y = 2 - x$ and $y = x - 2$ revolved about

(a) the x-axis (b) the y-axis

24. The region bounded by $y = e^x - 1$, $y = 2 - x$ and the x-axis revolved about

(a) x-axis (b) y-axis

25. The region bounded by $y = \cos x$ and $y = x^2$ revolved about

(a) $x = 2$ (b) $y = 2$ (c) the x-axis (d) the y-axis

26. The region bounded by $y = \sin x$ and $y = x^2$ revolved about

(a) $y = 1$ (b) $x = 1$ (c) the y-axis (d) the x-axis

In exercises 27–30, the integral represents the volume of a solid. Sketch the region and axis of revolution that produce the solid.

27. $\int_0^1 \pi[(\sqrt{y})^2 - y^2]\,dy$ **28.** $\int_0^2 \pi(4 - y^2)^2\,dy$

29. $\int_0^1 2\pi x(x - x^2)\,dx$ **30.** $\int_0^2 2\pi(4 - y)(y + y)\,dy$

31. Use a method similar to our derivation of equation (3.1) to derive the following fact about a circle of radius R. Area $= \pi R^2 = \int_0^R c(r)\,dr$, where $c(r) = 2\pi r$ is the circumference of a circle of radius r.

32. You have probably noticed that the circumference of a circle $(2\pi r)$ equals the derivative with respect to r of the area of the circle (πr^2). Use exercise 31 to explain why this is not a coincidence.

APPLICATIONS

1. A jewelry bead is formed by drilling a $\frac{1}{2}$-cm radius hole from the center of a 1-cm radius sphere. Explain why the volume is given by $\int_{1/2}^1 4\pi x\sqrt{1 - x^2}\,dx$. Evaluate this integral or compute the volume in some easier way.

2. Find the size of the hole in exercise 1 such that exactly half the volume is removed.

3. An anthill is in the shape formed by revolving the region bounded by $y = 1 - x^2$ and the x-axis about the y-axis. A researcher removes a cylindrical core from the center of the hill. What should the radius be to give the researcher 10% of the dirt?

4. The outline of a rugby ball has the shape of $\frac{x^2}{30} + \frac{y^2}{16} = 1$. The ball itself is the revolution of this ellipse about the x-axis. Find the volume of the ball.

EXPLORATORY EXERCISES

1. From a sphere of radius R, a hole of radius r is drilled out of the center. Compute the volume removed in terms of R and r. Compute the length L of the hole in terms of R and r. Rewrite the volume in terms of L. Is it reasonable to say that the volume removed depends on L and not on R?

2. In each case, sketch the solid and find the volume formed by revolving the region about (i) the x-axis and (ii) the y-axis. Compute the volume exactly if possible and estimate numerically if necessary. (a) Region bounded by $y = \sec x\sqrt{\tan x + 1}$, $y = 0$, $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$. (b) Region bounded by $x = \sqrt{y^2 + 1}$, $x = 0$, $y = -1$ and $y = 1$. (c) Region bounded by $y = \frac{\sin x}{x}$, $y = 0$, $x = \pi$ and $x = 0$. (d) Region bounded by $y = x^3 - 3x^2 + 2x$ and $y = 0$. (e) Region bounded by $y = e^{-x^2}$ and $y = (x - 1)^2$.

2.4 ARC LENGTH AND SURFACE AREA

In this section, we compute the length of a curve in two dimensions and the area of a surface in three dimensions. As always, pay particular attention to the derivations.

Arc Length

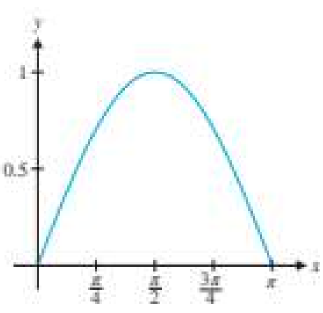


FIGURE 2.34a
$y = \sin x$

How could we find the *length* of the portion of the sine curve shown in Figure 2.34a? (We call the length of a curve its **arc length**.) If the curve were actually a piece of string, you could straighten out the string and then measure its length with a ruler. With this in mind, we begin with an approximation.

We first approximate the curve with several line segments joined together. In Figure 2.34b, the line segments connect the points $(0, 0)$, $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{\pi}{2}, 1\right)$, $\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $(\pi, 0)$ on the curve $y = \sin x$. An approximation of the arc length s of the curve is given by the sum of the lengths of these line segments:

$$s \approx \sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} + \sqrt{\left(\frac{\pi}{4}\right)^2 + \left(1 - \frac{1}{\sqrt{2}}\right)^2} + \sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{1}{\sqrt{2}} - 1\right)^2} + \sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \approx 3.79.$$

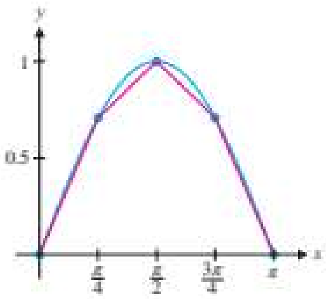


FIGURE 2.34b
Four line segments approximating $y = \sin x$

n	*Length*
8	3.8125
16	3.8183
32	3.8197
64	3.8201
128	3.8202

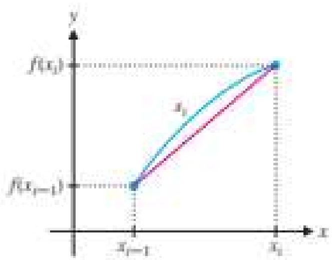


FIGURE 2.35
Straight-line approximation of arc length

You might notice that this estimate is too small. (Why is that?) We will improve our approximation by using more than four line segments. In the following table, we show estimates of the length of the curve using n line segments for larger values of n. As you would expect, the approximation of length will get closer to the actual length of the curve, as the number of line segments increases. This general idea should sound familiar.

We develop this notion further now for the more general problem of finding the arc length of the curve $y = f(x)$ on the interval $[a, b]$. Here, we'll assume that f is continuous on $[a, b]$ and differentiable on (a, b). (Where have you seen hypotheses like these before?) As usual, we begin by partitioning the interval $[a, b]$ into n equal pieces:

$$a = x_0 < x_1 < \cdots < x_n = b, \text{ where } x_i - x_{i-1} = \Delta x = \frac{b-a}{n}, \text{ for each } i = 1, 2, \ldots, n.$$

Between each pair of adjacent points on the curve $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$, we approximate the arc length s_i by the straight-line distance between the two points. (See Figure 2.35.) From the usual distance formula, we have

$$s_i \approx d\left\{(x_{i-1}, f(x_{i-1})), (x_i, f(x_i))\right\} = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2}.$$

Since f is continuous on all of $[a, b]$ and differentiable on (a, b), f is also continuous on the subinterval $[x_{i-1}, x_i]$ and is differentiable on (x_{i-1}, x_i). By the Mean Value Theorem, we then have

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}),$$

for some number $c_i \in (x_{i-1}, x_i)$. This gives us the approximation

$$\begin{aligned} s_i &\approx \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2} \\ &= \sqrt{(x_i - x_{i-1})^2 + [f'(c_i)(x_i - x_{i-1})]^2} \\ &= \sqrt{1 + [f'(c_i)]^2}\,\underbrace{(x_i - x_{i-1})}_{\Delta x} = \sqrt{1 + [f'(c_i)]^2}\,\Delta x. \end{aligned}$$

Adding together the lengths of these n line segments, we get an approximation of the total arc length,

$$s \approx \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\,\Delta x.$$

Notice that as n gets larger, this approximation should approach the exact arc length, that is,

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\,\Delta x.$$

You should recognize this as the limit of a Riemann sum for $\sqrt{1 + [f'(x)]^2}$, so that the arc length is given exactly by the definite integral:

Arc length of $y = f(x)$ on the interval $[a, b]$

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx, \tag{4.1}$$

whenever the limit exists.

REMARK 4.1

The formula for arc length is very simple. Unfortunately, very few functions produce arc length integrals that can be evaluated exactly. You should expect to use a numerical integration method on your calculator or computer to compute most arc lengths.

EXAMPLE 4.1 Using the Arc Length Formula

Find the arc length of the portion of the curve $y = \sin x$ with $0 \le x \le \pi$. (We estimated this as 3.79 in our introductory example.)

Solution From (4.1), the arc length is

$$s = \int_0^{\pi} \sqrt{1 + (\cos x)^2}\,dx.$$

Try to find an antiderivative of $\sqrt{1+\cos^2 x}$, but don't try for too long. (The best our CAS can do is $\sqrt{2}$ EllipticE[x, $\frac{1}{2}$], which doesn't seem especially helpful.) Using a numerical integration method, the arc length is

$$s = \int_0^{\pi} \sqrt{1+(\cos x)^2}\,dx \approx 3.8202.$$

Even for very simple curves, evaluating the arc length integral exactly can be quite challenging.

EXAMPLE 4.2 Estimating an Arc Length

Find the arc length of the portion of the curve $y = x^2$ with $0 \le x \le 1$.

Solution Using the arc length formula (4.1), we get

$$s = \int_0^1 \sqrt{1+(2x)^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx \approx 1.4789,$$

where we have again evaluated the integral numerically. (In this case, you can find an antiderivative using a technique developed in section 6.3 and evaluate the integral exactly as $\frac{\ln(\sqrt{5}+2)}{4} + \frac{\sqrt{5}}{2}$.)

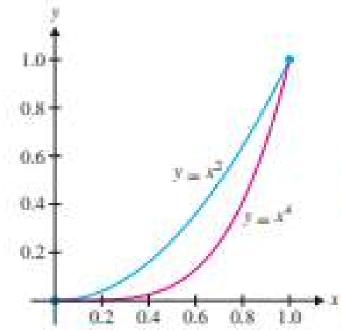


FIGURE 2.36
$y = x^2$ and $y = x^4$

The graphs of $y = x^2$ and $y = x^4$ look surprisingly similar on the interval [0, 1]. (See Figure 2.36.) They both connect the points (0, 0) and (1, 1), are increasing and are concave up. If you graph them simultaneously, you will note that $y = x^4$ starts out flatter and then becomes steeper from about $x = 0.7$ on. (Try *proving* that this is true!) Arc length gives us one way to quantify the difference between the two graphs.

EXAMPLE 4.3 A Comparison of Arc Lengths of Power Functions

Find the arc length of the portion of the curve $y = x^4$ with $0 \le x \le 1$ and compare to the arc length of the portion of the curve $y = x^2$ on the same interval.

Solution From (4.1), the arc length for $y = x^4$ is given by

$$\int_0^1 \sqrt{1+(4x^3)^2}\,dx = \int_0^1 \sqrt{1+16x^6}\,dx \approx 1.6002.$$

Notice that this arc length is about 8% larger than that of $y = x^2$, as found in example 4.2.

In the exercises, you will be asked to explore the trend in the lengths of the portion of the curves $y = x^6$, $y = x^8$ and so on, on the interval [0, 1]. Can you guess now what happens to the arc length of the portion of $y = x^n$, on the interval [0, 1], as $n \to \infty$?

Everyday usage of words such as *length* can be ambiguous and misleading. For instance, the *length* of a frisbee throw usually refers to the horizontal distance covered, not to the arc length of the frisbee's flight path. On the other hand, suppose you need to hang a banner between two poles that are 20 m apart. In this case, you'll need more than 20 m of rope since the length of rope required is determined by the arc length, rather than the horizontal distance.

EXAMPLE 4.4 Computing the Length of Cable Hanging between Two Poles

A cable is to be hung between two poles of equal height that are 20 m apart. It can be shown that such a hanging cable assumes the shape of a *catenary*, the general form of which is $y = a \cosh x/a = \frac{a}{2}(e^{x/a} + e^{-x/a})$. In this case, suppose that the cable takes the shape of $y = 5(e^{x/10} + e^{-x/10})$, for $-10 \le x \le 10$, as seen in Figure 2.37. How long is the cable?

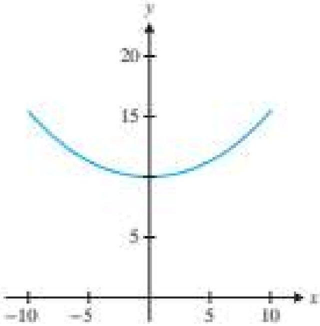


FIGURE 2.37
$y = 5(e^{x/10} + e^{-x/10})$

Solution From (4.1), the arc length of the curve is given by

$$\begin{aligned} s &= \int_{-10}^{10} \sqrt{1 + \left(\frac{e^{x/10}}{2} - \frac{e^{-x/10}}{2}\right)^2}\, dx \\ &= \int_{-10}^{10} \sqrt{1 + \frac{1}{4}(e^{x/5} - 2 + e^{-x/5})}\, dx \\ &= \int_{-10}^{10} \sqrt{\frac{1}{4}(e^{x/5} + 2 + e^{-x/5})}\, dx \\ &= \int_{-10}^{10} \sqrt{\frac{1}{4}(e^{x/10} + e^{-x/10})^2}\, dx \\ &= \int_{-10}^{10} \frac{1}{2}(e^{x/10} + e^{-x/10})\, dx \\ &= 5(e^{x/10} - e^{-x/10})\Big|_{x=-10}^{x=10} \\ &= 10(e - e^{-1}) \\ &\approx 23.504 \text{ m}, \end{aligned}$$

which corresponds to the horizontal distance of 20 m plus about $3\frac{1}{2}$ feet of slack. ■

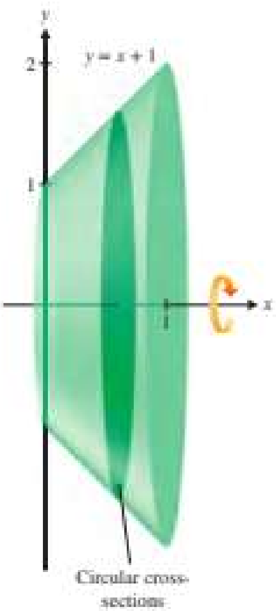


FIGURE 2.38
Surface of revolution

Surface Area

In sections 2.2 and 2.3, we saw how to compute the volume of a solid formed by revolving a two-dimensional region about a fixed axis. In addition, we often want to determine the area of the *surface* that is generated by the revolution. For instance, when revolving the line $y = x + 1$, for $0 \le x \le 1$, about the x-axis, the surface generated is the bottom portion of a right circular cone whose top is cut off by a plane parallel to the base, as shown in Figure 2.38.

FIGURE 2.39a
Right circular cone

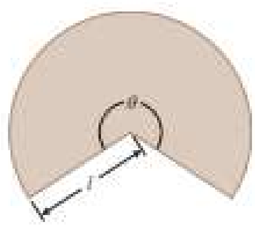


FIGURE 2.39b
Flattened cone

We first pause to find the curved surface area of a right circular cone. In Figure 2.39a, we show a right circular cone of base radius r and **slant height** l. (As you'll see later, it is more convenient in this context to specify the slant height than the altitude.) If we cut the cone along a seam and flatten it out, we get the circular sector shown in Figure 2.39b. Notice that the curved surface area of the cone is the same as the area A of the circular sector. This is the area of a circle of radius l multiplied by the fraction of the circle included: θ out of a possible 2π radians, or

$$A = \pi\,(\text{radius})^2 \frac{\theta}{2\pi} = \pi l^2 \frac{\theta}{2\pi} = \frac{\theta}{2} l^2. \tag{4.2}$$

The only problem with this is that we don't know θ. However, notice that by the way we constructed the sector (i.e. by flattening the cone), the circumference of the sector is the same as the circumference of the base of the cone. That is,

$$2\pi r = 2\pi l \frac{\theta}{2\pi} = l\theta.$$

FIGURE 2.40
Frustum of a cone

Dividing by l gives us $$\theta = \frac{2\pi r}{l}.$$

From (4.2), the curved surface area of the cone is then

$$A = \frac{\theta}{2} l^2 = \frac{\pi r}{l} l^2 = \pi r l.$$

Recall that we were originally interested in finding the surface area of only a portion of a right circular cone. (Look back at Figure 2.38.) For the **frustum** of a cone shown in Figure 2.40, the curved surface area is given by

$$A = \pi(r_1 + r_2)L.$$

You can verify this by subtracting the curved surface area of two cones, where you must use similar triangles to find the height of the larger cone from which the frustum is cut. We leave the details of this as an exercise.

Returning to the original problem of revolving the line $y = x + 1$ on the interval $[0, 1]$ about the x-axis (seen in Figure 2.38), we have $r_1 = 1$, $r_2 = 2$ and $L = \sqrt{2}$ (from the Pythagorean Theorem). The curved surface area is then

$$A = \pi(1 + 2)\sqrt{2} = 3\pi\sqrt{2} \approx 13.329.$$

For the general problem of finding the curved surface area of a surface of revolution, consider the case where $f(x) \geq 0$ and where f is continuous on the interval $[a, b]$ and differentiable on (a, b). If we revolve the graph of $y = f(x)$ about the x-axis on the interval $[a, b]$ (see Figure 2.41a), we get the surface of revolution seen in Figure 2.41b.

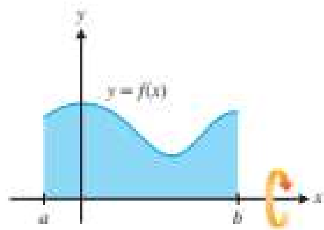


FIGURE 2.41a
Revolve about x-axis

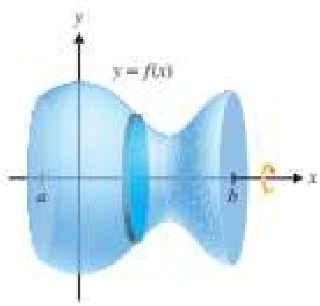


FIGURE 2.41b
Surface of revolution

As we have done many times now, we first partition the interval $[a, b]$ into n pieces of equal size: $a = x_0 < x_1 < \cdots < x_n = b$, where $x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$, for each $i = 1, 2, \ldots, n$. On each subinterval $[x_{i-1}, x_i]$, we can approximate the curve by the straight line segment joining the points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$, as in Figure 2.42. Notice that revolving this line segment around the x-axis generates the frustum of a cone. The surface area of this frustum will give us an approximation to the actual surface area on the interval $[x_{i-1}, x_i]$. First, observe that the slant height of this frustum is

$$L_i = d\left\{(x_{i-1}, f(x_{i-1})), (x_i, f(x_i))\right\} = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2},$$

from the usual distance formula. Because of our assumptions on f, we can apply the Mean Value Theorem, to obtain

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}),$$

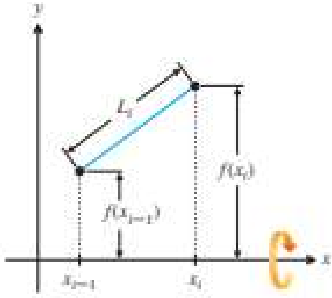


FIGURE 2.42
Revolve about x-axis

for some number $c_i \in (x_{i-1}, x_i)$. This gives us

$$L_i = \sqrt{(x_i - x_{i-1})^2 + [f(x_i) - f(x_{i-1})]^2} = \sqrt{1 + [f'(c_i)]^2}\,\underbrace{(x_i - x_{i-1})}_{\Delta x}.$$

The surface area S_i of that portion of the surface on the interval $[x_{i-1}, x_i]$ is approximately the surface area of the frustum of the cone,

$$\begin{aligned} S_i &\approx \pi[f(x_i) + f(x_{i-1})]\sqrt{1 + [f'(c_i)]^2}\,\Delta x \\ &\approx 2\pi f(c_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x, \end{aligned}$$

since if Δx is small, $$f(x_i) + f(x_{i-1}) \approx 2f(c_i).$$

Repeating this argument for each subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, gives us an approximation to the total surface area S,

$$S \approx \sum_{i=1}^{n} 2\pi f(c_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x.$$

As n gets larger, this approximation approaches the actual surface area,

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(c_i)\sqrt{1 + [f'(c_i)]^2}\,\Delta x.$$

Recognizing this as the limit of a Riemann sum gives us the integral

Surface area of a solid of revolution

$$\boxed{S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx,} \tag{4.3}$$

whenever the integral exists.

You should notice that the factor of $\sqrt{1 + [f'(x)]^2}\,dx$ in the integrand in (4.3) corresponds to the arc length of a small section of the curve $y = f(x)$, while the factor $2\pi f(x)$ corresponds to the circumference of the solid of revolution. This should make sense to you, as follows. For any small segment of the curve, if we approximate the surface area by revolving a small segment of the curve of radius $f(x)$ around the x-axis, the surface area generated is simply the surface area of a cylinder,

$$S = 2\pi r h = 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx,$$

since the radius of such a small cylindrical segment is $f(x)$ and the height of the cylinder is $h = \sqrt{1 + [f'(x)]^2}\,dx$. It is far better to think about the surface area formula in this way than to simply memorize the formula.

REMARK 4.1

There are exceptionally few functions f for which the integral in (4.3) can be computed exactly. Don't worry; we have numerical integration for just such occasions.

EXAMPLE 4.5 Computing Surface Area

Find the surface area of the surface generated by revolving $y = x^4$, for $0 \le x \le 1$, about the x-axis.

Solution Using the surface area formula (4.3), we have

$$S = \int_0^1 2\pi x^4\sqrt{1 + (4x^3)^2}\,dx = \int_0^1 2\pi x^4\sqrt{1 + 16x^6}\,dx \approx 3.4365,$$

where we have used a numerical method to approximate the value of the integral. ■

EXERCISES 2.4

WRITING EXERCISES

1. Explain in words how the arc length integral is derived from the lengths of the approximating secant line segments.

2. Explain why the sum of the lengths of the line segments in Figure 2.34b is less than the arc length of the curve in Figure 2.34a.

3. Discuss whether the arc length integral is more accurately called a formula or a definition (i.e., can you precisely define the length of a curve without using the integral?).

4. Suppose you graph the trapezoid bounded by $y = x + 1$, $y = -x - 1$, $x = 0$ and $x = 1$, cut it out and roll it up. Explain why you would not get Figure 2.38. (Hint: Compare areas and carefully consider Figures 2.39a and 2.39b.)

In exercises 1–4, approximate the length of the curve using n secant lines for $n = 2$; $n = 4$.

1. $y = x^2, 0 \le x \le 1$
2. $y = x^4, 0 \le x \le 1$
3. $y = \cos x, 0 \le x \le \pi$
4. $y = \ln x, 1 \le x \le 3$

In exercises 5–14, compute the arc length exactly.

5. $y = 2x + 1, 0 \le x \le 2$
6. $y = \sqrt{1 - x^2}, -1 \le x \le 1$
7. $y = 4x^{3/2} + 1, 1 \le x \le 2$
8. $y = \frac{1}{4}(e^{2x} + e^{-2x}), 0 \le x \le 1$
9. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x, 1 \le x \le 2$
10. $y = \frac{1}{6}x^3 + \frac{1}{2x}, 1 \le x \le 3$
11. $x = \frac{1}{8}y^4 + \frac{1}{4y^2}, -2 \le y \le -1$
12. $x = e^{y/2} + e^{-y/2}, -1 \le y \le 1$
13. $y = \frac{1}{3}x^{3/2} - x^{1/2}, 1 \le x \le 4$
14. $y = 2\ln(4 - x^2), 0 \le x \le 1$

In exercises 15–22, set up the integral for arc length and then approximate the integral with a numerical method.

15. $y = x^3, -1 \le x \le 1$
16. $y = x^3, -2 \le x \le 2$
17. $y = 2x - x^2, 0 \le x \le 2$
18. $y = \tan x, 0 \le x \le \pi/4$
19. $y = \cos x, 0 \le x \le \pi$
20. $y = \ln x, 1 \le x \le 3$
21. $y = \int_0^x u \sin u\, du, 0 \le x \le \pi$
22. $y = \int_0^x e^{-u} \sin u\, du, 0 \le x \le \pi$

23. A rope is to be hung between two poles 40 m apart. If the rope assumes the shape of the catenary $y = 10(e^{x/20} + e^{-x/20})$, $-20 \le x \le 20$, compute the length of the rope.

24. A rope is to be hung between two poles 60 m apart. If the rope assumes the shape of the catenary $y = 15(e^{x/30} + e^{-x/30})$, $-30 \le x \le 30$, compute the length of the rope.

25. In example 4.4, compute the "sag" in the cable—that is, the difference between the y-values in the middle ($x = 0$) and at the poles ($x = 10$). Given this, is the arc length calculation surprising?

26. Sketch and compute the length of the **astroid** defined by $x^{2/3} + y^{2/3} = 1$.

27. For $y = x^6$, $y = x^8$ and $y = x^{10}$, compute the arc length for $0 \le x \le 1$. Using results from examples 4.2 and 4.3, identify the pattern for the length of $y = x^n$, $0 \le x \le 1$, as n increases. Conjecture the limit as $n \to \infty$.

28. (a) To help understand the result of exercise 27, determine $\lim_{n\to\infty} x^n$ for each x such that $0 \le x < 1$. Compute the length of this limiting curve. Connecting this curve to the endpoint (1, 1), what is the total length?
    (b) Prove that $y = x^4$ is flatter than $y = x^2$ for $0 < x < \sqrt{1/2}$ and steeper for $x > \sqrt{1/2}$. Compare the flatness and steepness of $y = x^6$ and $y = x^4$.

In exercises 29–36, set up the integral for the surface area of the surface of revolution and approximate the integral with a numerical method.

29. $y = x^2, 0 \le x \le 1$, revolved about the x-axis
30. $y = \sin x, 0 \le x \le \pi$, revolved about the x-axis
31. $y = 2x - x^2, 0 \le x \le 2$, revolved about the x-axis
32. $y = x^3 - 4x, -2 \le x \le 0$, revolved about the x-axis
33. $y = e^x, 0 \le x \le 1$, revolved about the x-axis
34. $y = \ln x, 1 \le x \le 2$, revolved about the x-axis
35. $y = \cos x, 0 \le x \le \pi/2$, revolved about the x-axis
36. $y = \sqrt{x}, 1 \le x \le 2$, revolved about the x-axis

In exercises 37 and 38, compute the arc length L_1 of the curve and the length L_2 of the secant line connecting the endpoints of the curve. Compute the ratio L_2/L_1; the closer this number is to 1, the straighter the curve is.

37. (a) $y = \sin x, -\frac{\pi}{6} \le x \le \frac{\pi}{6}$ (b) $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
38. (a) $y = e^x, 3 \le x \le 5$ (b) $-5 \le x \le -3$

39. (a) Suppose that the square consisting of all (x, y) with $-1 \le x \le 1$ and $-1 \le y \le 1$ is revolved about the y-axis. Compute the surface area.

(b) Suppose that the circle $x^2 + y^2 = 1$ is revolved about the y-axis. Compute the surface area.

(c) Suppose that the triangle with vertices $(-1, -1)$, $(0, 1)$ and $(1, -1)$ is revolved about the y-axis. Compute the surface area.

(d) Sketch the square, circle and triangle of parts (a)–(c) on the same axes. Show that the relative surface areas of the solids of revolution (cylinder, sphere and cone, respectively) are $3{:}2{:}\tau$, where τ is the **golden mean** defined by $\tau = \dfrac{1+\sqrt{5}}{2}$.

40. Derive the general formulas for the surface area of (a) a right circular cylinder of radius r and height h, (b) a sphere of radius r and (c) a cone of radius r and height h.

APPLICATIONS

1. Two people walk along different paths starting at the origin. They have the same positive x-coordinate at each time. One follows the positive x-axis and the other follows $y = \frac{2}{3}x^{3/2}$. (a) Find the point at which one person has walked twice as far as the other. (b) Let $f(t)$ be the distance walked along $y = \frac{2}{3}x^{3/2}$ for $0 \le x \le t$. Compute $f'(t)$ and use it to determine at which point the ratio of the speeds of the walkers equals 2. (Suggested by Tim Pennings.)

2. (a) The **elliptic integral of the second kind** is defined by $\text{EllipticE}(\phi, m) = \int_0^\phi \sqrt{1 - m\sin^2 u}\, du$. Referring to example 4.1, many CASs report $\sqrt{2}\ \text{EllipticE}(x, \frac{1}{2})$ as an antiderivative of $\sqrt{1+\cos^2 x}$. Verify that this is an antiderivative.

(b) Many CASs report the antiderivative

$$\int \sqrt{1+16x^6}\, dx = \frac{1}{4}x\sqrt{1+16x^6} + \int \frac{3/4}{\sqrt{1+16x^6}}\, dx.$$

Verify that this is an antiderivative.

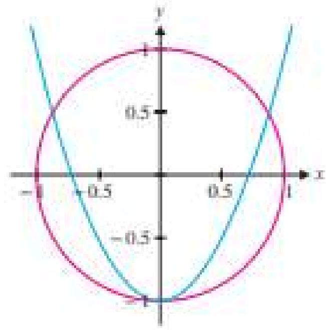
3. A football punt follows the path $y = \frac{1}{15}x(60 - x)$ m. Sketch a graph. How far did the punt go horizontally? How high did it go? Compute the arc length. If the ball was in the air for 4 seconds, what was the ball's average velocity?

4. A baseball outfielder's throw follows the path $y = \dfrac{1}{300}x(100 - x)$ m. Sketch a graph. How far did the ball go horizontally? How high did it go? Compute the arc length. Explain why the baseball player would want a small arc length, while the football player in exercise 3 would want a large arc length.

EXPLORATORY EXERCISES

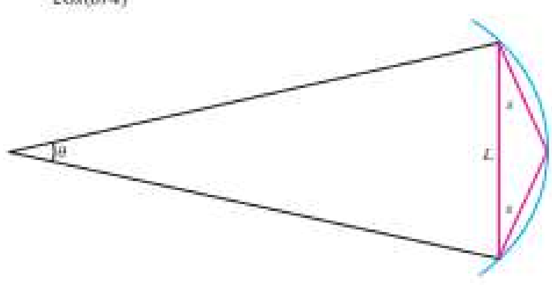
1. In this exercise, you will explore a famous paradox (often called Gabriel's horn). Suppose that the curve $y = 1/x$, for $1 \le x \le R$ (where R is a large positive constant), is revolved about the x-axis. Compute the enclosed volume and the surface area of the resulting surface. (In both cases, antiderivatives can be found, although you may need help from your CAS to get the surface area.) Determine the limit of the volume and surface area as $R \to \infty$. Now for the paradox. Based on your answers, you should have a solid with finite volume, but infinite surface area. Thus, the three-dimensional solid could be completely filled with a finite amount of paint but the outside surface could never be completely painted.

2. Let C be the portion of the parabola $y = ax^2 - 1$ inside the circle $x^2 + y^2 = 1$.



Find the value of $a > 0$ that maximizes the arc length of C.

3. The figure shows an arc of a circle subtended by an angle θ, with a chord of length L and two chords of length s. Show that $2s = \dfrac{L}{\cos(\theta/4)}$.



Start with a quarter-circle and use this formula repeatedly to derive the infinite product

$$\cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}\cos\frac{\pi}{32}\cdots = \frac{2}{\pi}$$

where the left-hand side represents $\lim_{n\to\infty}\left(\cos\frac{\pi}{2^n}\cos\frac{\pi}{2^{n-1}}\cdots\cos\frac{\pi}{4}\right)$.

2.5 PROJECTILE MOTION

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In previous sections, we discussed aspects of the motion of an object moving in a straight line path (rectilinear motion). We saw that if we know a function describing the position of an object at any time t, then we can determine its velocity and acceleration by differentiation. A much more important problem is to go backward; that is, to find the position and velocity of an object, given its acceleration. Mathematically, this means that, starting with the derivative of a function, we must find the original function. Now that we have integration at our disposal, we can accomplish this with ease.

You may already be familiar with **Newton's second law of motion**, which says that

$$F = ma,$$

where F is the sum of the forces acting on an object, m is the mass of the object and a is the acceleration of the object.

Start by imagining that you are diving. The primary force acting on you throughout the dive is gravity. The force due to gravity is your own weight, which is related to mass by $W = mg$, where g is the gravitational constant. (Common approximations of g, accurate near sea level, are 32 ft/s^2 and 9.8 m/s^2.) To keep the problem simple mathematically, we will ignore any other forces, such as air resistance.

Let $h(t)$ represent your height above the water t seconds after starting your dive. Then the force due to gravity is $F = -mg$, where the minus sign indicates that the force is acting downward, in the negative direction. From our earlier work, we know that the acceleration is $a(t) = h''(t)$. Newton's second law then gives us $-mg = mh''(t)$ or

$$h''(t) = -g.$$

Notice that the position function of *any* object (regardless of its mass) subject to gravity and no other forces will satisfy the same equation. The only differences from situation to situation are the initial conditions (the initial velocity and initial position) and the questions being asked.

In example 5.1, the negative sign of the velocity indicated that the diver was coming down. In many situations, both upward and downward motions are important.

EXAMPLE 5.1 Finding the Velocity of a Diver at Impact

If a diving board is 5 m above the surface of the water and a diver starts with initial velocity 2 m/s (in the upward direction), what is the diver's velocity at impact (assuming no air resistance)?

Solution If the height at time t is given by $h(t)$, Newton's second law gives us $h(t) = -9.8$. Since the diver starts 5 m above the water with initial velocity of 2 m/s, we have the initial conditions $h(0) = 5$ and $h'(0) = 2$. Finding $h(t)$ now takes little more than elementary integration. We have

$$\int h''(t)\,dt = \int -9.8\,dt$$

or

$$h'(t) = -9.8t + c.$$

From the initial velocity, we have

$$2 = h'(0) = -9.8(0) + c = c,$$

so that $c = 8$ and the velocity at any time t is given by

$$h'(t) = -9.8t + 2.$$

To find the velocity at impact, you first need to find the *time* of impact. Notice that the diver will hit the water when $h(t) = 0$ (i.e., when the height above the water is 0). Integrating the velocity function gives us the height function:

$$\int h'(t)\,dt = \int (-9.8t + 2)\,dt$$

or

$$h(t) = -4.9t^2 + 2t + c.$$

From the initial height, we have

$$5 = h(0) = -4.9(0)^2 + 8(0) + c = c,$$

so that $c = 15$ and the height above the water at any time t is given by

$$h(t) = -4.9t^2 + 2t + 5.$$

Impact then occurs when

$$0 = h(t) = -4.9t^2 + 2t + 5$$

$$t = 1.23s \text{ and } t = -0.82s \text{ (ignore)}$$

$$h'(t) = -10 \text{ m/s}$$

EXAMPLE 5.2 An Equation for the Vertical Motion of a Ball

A ball is propelled straight upward from the ground with initial velocity 20 m/s. Ignoring air resistance, find an equation for the height of the ball at any time t. Also, determine the maximum height and the amount of time the ball spends in the air.

Solution With gravity as the only force, the height $h(t)$ satisfies $h''(t) = -9.8$. The initial conditions are $h'(0) = 20$ and $h(0) = 0$. We then have

$$\int h''(t)\,dt = \int -9.8\,dt$$

or

$$h'(t) = -9.8t + c.$$

From the initial velocity, we have

$$20 = h'(0) = -9.8(0) + c = c$$

and so,

$$h'(t) = 20 - 9.8t$$

Integrating one more time gives us

$$\int h'(t)\,dt = \int (20 - 9.8t)\,dt$$

or

$$h(t) = 20t - 4.9t^2 + c.$$

From the initial height we have

$$0 = h(0) = 20(0) - 4.9(0)^2 + c = c,$$

and so,

$$h(t) = 20t - 4.9t^2.$$

Since the height function is quadratic, its maximum occurs at the one time when $h'(t) = 0$. [You should also consider the physics of the situation: what happens physically when $h'(t) = 0$?] Solving $20 - 9.8t = 0$ gives t^2 (the time at the maximum height) and the corresponding height is $h(2) = -4.9(2)^2 + 64(2) \approx 20.4$ m. Again, the ball lands when $h(t) = 0$. Solving

$$0 = h(t) = -4.9t^2 + 20t$$

gives $t = 0$ (launch time) and $t = 4.08$ (landing time). The time of flight is approximately 4 seconds.

TODAY IN MATHEMATICS

Vladimir Arnold (1937–2010)
A Russian mathematician with important contributions to numerous areas of mathematics, both in research and popular exposition. The esteem in which he was held by his colleagues can be measured by the international conference known as "Arnold-fest" held in Toronto in honor of his 60th birthday. Many of his books are widely used today, including a collection of challenges titled *Arnold's Problems*.[4] A review of this book states that "Arnold did not consider mathematics a game with deductive reasoning and symbols, but a part of natural science (especially of physics), i.e., an experimental science."

[4]Arnold, V. I. (ed.) (2004). *Arnold's Problems* (Berlin: Springer Science & Business Media).

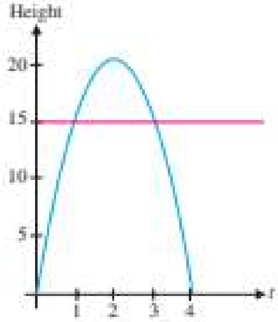


FIGURE 2.43
Height of the ball at time t

You can observe an interesting property of projectile motion by graphing the height function from example 5.2 along with the line $y = 15$. (See Figure 2.43.) Notice that the graphs intersect at $t \approx 1$ and $t \approx 3$. Further, the time interval corresponds to half the time spent in the air. Notice that this says that the ball stays in the top one-fourth of its height for almost half of its time in the air. You may have marveled at how some athletes jump so high that they seem to "hang in the air." As this calculation suggests, all objects tend to hang in the air.

EXAMPLE 5.3 Finding the Initial Velocity Required to Reach a Certain Height

It has been reported that former basketball star Michael Jordan had a vertical leap of 1.4 m. Ignoring air resistance, what is the initial velocity required to jump this high?

Solution Once again, Newton's second law leads us to the equation $h''(t) = -9.8$ for the height $h(t)$. We call the initial velocity v_0, so that $h'(0) = v_0$ and look for the value of v_0 that will give a maximum altitude of 1.4 m. As before, we integrate to get

$$h'(t) = -9.8t + c.$$

Using the initial velocity, we get

$$v_0 = h'(0) = -9.8(0) + c = c.$$

This gives us the velocity function

$$h'(t) = v_0 - 9.8t.$$

Integrating once again and using the initial position $h(0) = 0$, we get

$$h(t) = v_0 t - 4.9t^2.$$

The maximum height occurs when $h'(t) = 0$. (Why?) Setting

$$0 = h'(t) = v_0 - 9.8t,$$

gives us $t = \frac{v_0}{9.8}$. The height at this time (i.e., the maximum altitude) is then

$$h\left(\frac{v_0}{9.8}\right) = v_0\left(\frac{v_0}{9.8}\right) - 4.9\left(\frac{v_0}{9.8}\right)^2 = \frac{v_0^2}{9.8} - \frac{5v_0^2}{98} = \frac{5v_0^2}{98}.$$

$\frac{5v_0^2}{98} = 1.4$, so that $v_0 = 5.24$ m/s ■

FIGURE 2.44a
Path of projectile

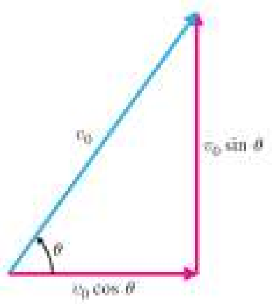


FIGURE 2.44b
Vertical and horizontal components of velocity

So far, we have only considered projectiles moving vertically. In practice, we must also consider movement in the horizontal direction. Ignoring air resistance, these calculations are also relatively straightforward. The idea is to apply Newton's second law separately to the horizontal and vertical components of the motion. If $y(t)$ represents the vertical position, then we have $y''(t) = -g$, as before. Ignoring air resistance, there are no forces acting horizontally on the projectile. So, if $x(t)$ represents the horizontal position, Newton's second law gives us $x''(t) = 0$.

The initial conditions are slightly more complicated here. In general, we want to consider projectiles that are launched with an initial speed v_0 at an angle θ from the horizontal. Figure 2.44a shows a projectile fired with $\theta > 0$. Notice that an initial angle of $\theta < 0$ would mean a downward initial velocity.

As shown in Figure 2.44b, the initial velocity can be separated into horizontal and vertical components. From elementary trigonometry, the horizontal component of the initial velocity is $v_x = v_0 \cos\theta$ and the vertical component is $v_y = v_0 \sin\theta$.

EXAMPLE 5.4 The Motion of a Projectile in Two Dimensions

An object is launched at angle $\theta = \pi/6$ from the horizontal with initial speed $v_0 = 98$ m/s. Determine the time of flight and the (horizontal) range of the projectile.

Solution Starting with the vertical component of the motion (and again ignoring air resistance), we have $y''(t) = -9.8$ (since the initial speed is given in terms of meters per second). Referring to Figure 2.44b, notice that the vertical component of the initial velocity is $y'(0) = 98 \sin \pi/6 = 49$ and the initial altitude is $y(0) = 0$. A pair of simple integrations gives us the velocity function $y'(t) = -9.8t + 49$ and the position function $y(t) = -4.9t^2 + 49t$. The object hits the ground when $y(t) = 0$ (i.e., when its height above the ground is 0). Solving

$$0 = y(t) = -4.9t^2 + 49t = 49t(1 - 0.1t)$$

gives $t = 0$ (launch time) and $t = 10$ (landing time). The time of flight is then 10 seconds. The horizontal component of motion is determined from the equation $x''(t) = 0$ with initial velocity $x'(0) = 98 \cos \pi/6 = 49\sqrt{3}$ and initial position $x(0) = 0$. Integration gives us $x'(t) = 49\sqrt{3}$ and $x(t) = (49\sqrt{3})t$. In Figure 2.45, we plot the path of the ball. (You can do this using the parametric plot mode on your graphing calculator or CAS, by entering equations for $x(t)$ and $y(t)$ and setting the range of t-values to be $0 \le t \le 10$. Alternatively, you can easily solve for t, to get $t = \frac{1}{49\sqrt{3}}x$, to see that the curve is simply a parabola.) The horizontal range is then the value of $x(t)$ at $t = 10$ (the landing time),

$$x(10) = (49\sqrt{3})(10) = 490\sqrt{3} \approx 849 \text{ meters.} \quad ■$$

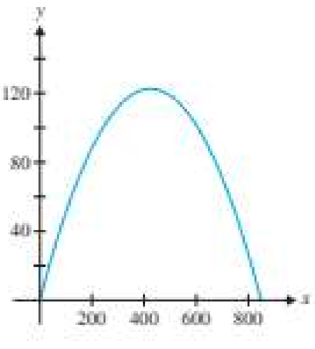


FIGURE 2.45
Path of ball

REMARK 5.1

You should resist the temptation to reduce this section to a few memorized formulas. It is true that if you ignore air resistance, the vertical component of position will always turn out to be $y(t) = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + y(0)$. However, your understanding of the process and your chances of finding the correct answer will improve dramatically if you start each problem with Newton's second law and work through the integrations (which are not difficult).

EXAMPLE 5.5 The Motion of a Tennis Serve

Venus Williams has one of the fastest serves in women's tennis. Suppose that she hits a serve from a height of 3.3 m at an initial speed of 108 km/h and at an angle of 7° below the horizontal. The serve is "in" if the ball clears a 0.91 m high net that is 11.9 m away and hits the ground in front of the service line 18.3 m away. (We illustrate this situation in Figure 2.46.) Determine whether the serve is in or out.

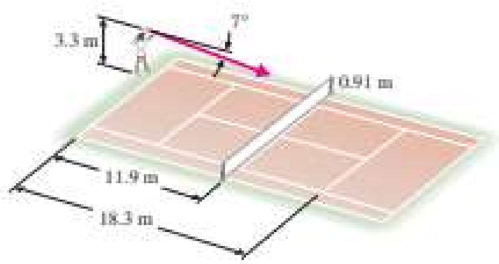


FIGURE 2.46
Height of tennis serve

Solution As in example 5.4, we start with the vertical motion of the ball. Since distance is given in meters, the equation of motion is $y''(t) = -9.8$. The initial speed must be converted to meters per second: 108 km/h = 30 m/s. The vertical component of the initial velocity is then $y(0) = 30 \sin(-7°) \approx -3.66$ m/s.

$$y'(t) = -9.8t - 3.65$$

The initial height is $y(0) = 3$ m, so another integration gives us

$$y(t) = -4.9t^2 - 3.65t + 3$$

The horizontal component of motion is determined from $x(t) = 0$, with initial velocity $x(0) = 30 \cos(-7°) \approx 29.78$ m/s and initial position $x(0) = 0$. Integrations give us $x'(t) = 29.78$ m/s and $x(t) = 29.78t$ m. Summarizing, we have

$$x(t) = 29.78t,$$
$$y(t) = -4.9t^2 - 3.65t + 3.$$

For the ball to clear the net, y must be at least 0.91 when $x = 11.9$. We have $x(t) = 11.9$ when $29.78t = 11.9$ or $t \approx 0.399$. At this time, $y(0.399) \approx 0.76$, showing that the ball is not high enough to clear the net. The serve is not in. ■

One reason you should start each problem with Newton's second law is so that you pause to consider the forces that are (and are not) being considered. For example, we have thus far ignored air resistance, as a simplification of reality. Some calculations using such simplified equations are reasonably accurate. Others, such as in example 5.6, are not.

EXAMPLE 5.6 An Example Where Air Resistance Can't Be Ignored

Suppose a raindrop falls from a cloud 900 meters above the ground. Ignoring air resistance, how fast would the raindrop be falling when it hits the ground?

Solution If the height of the raindrop at time t is given by $y(t)$, Newton's second law of motion tells us that $y''(t) = -9.8$. Further, we have the initial velocity $y'(0) = 0$ (since the drop falls—as opposed to being thrown down) and the initial altitude $y(0) = 900$. Integrating and using the initial conditions gives us $y'(t) = -9.8t$ and $y(t) = 900 - 4.9t^2$. The raindrop hits the ground when $y(t) = 0$. Setting

$$0 = y(t) = 900 - 4.9t^2$$

gives us $t = 13.55$ sec. The velocity at this time is then

$$y'(t) = -9.8t = -132.79 \text{ m/s}$$

This corresponds to nearly 478 km/h! Fortunately, air resistance does play a significant role in the fall of a raindrop, which has an actual landing speed of about 16 km/h. ■

The obvious lesson from example 5.6 is that it is not always reasonable to ignore air resistance.

The air resistance (more precisely, *air drag*) that slows the raindrop down is only one of the ways in which air can affect the motion of an object. The **Magnus force**, produced by the spinning of an object or lack of symmetry in the shape of an object, can cause the object to change directions and curve. Perhaps the most common example of a Magnus force occurs on an airplane. One side of an airplane wing is curved and the other side is comparatively flat. (See Figure 2.47.) The lack of symmetry causes the air to move over the top of the wing faster than it moves over the bottom. This produces a Magnus force in the upward direction (lift), lifting the airplane into the air.

FIGURE 2.47
Cross-section of a wing

EXERCISES 2.5

WRITING EXERCISES

1. In example 5.6, the assumption that air resistance can be ignored is obviously invalid. Discuss the validity of this assumption in examples 5.1 and 5.3.

2. In the discussion preceding example 5.3, we showed that Michael Jordan (and any other human) spends half of his air-time in the top one-fourth of the height. Compare his velocities at various points in the jump to explain why relatively more time is spent at the top than at the bottom.

3. In example 5.4, we derived separate equations for the horizontal and vertical components of position. To discover one consequence of this separation, consider the following situation. Two people are standing next to each other with arms raised to the same height. One person fires a bullet horizontally from a gun. At the same time, the other person drops a bullet. Explain why (ignoring air resistance) the bullets will hit the ground at the same time.

4. For the falling raindrop in example 5.6, a more accurate model would be $y''(t) = -9.8 + f(t)$, where $f(t)$ represents the force due to air resistance (divided by the mass). If $v(t)$ is the downward velocity of the raindrop, explain why this equation is equivalent to $v'(t) = 9.8 - f(t)$. Explain in physical terms why the larger $v(t)$ is, the larger $f(t)$ is. Thus, a model such as $f(t) = v(t)$ or $f(t) = [v(t)]^2$ would be reasonable. (In most situations, it turns out that $[v(t)]^2$ matches the experimental data better.)

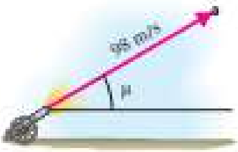
In exercises 1–4, identify the initial conditions $y(0)$ and $y'(0)$.

1. An object is dropped from a height of 80 m.
2. An object is dropped from a height of 100 m.
3. An object is released from a height of 60 m with an upward velocity of 10 m/s.
4. An object is released from a height of 20 m with a downward velocity of 4 m/s.

In exercises 5–20 and 1–6 in the Applications section, ignore air resistance.

5. A diver drops from 9 m above the water (about the height of an Olympic platform dive). What is the diver's velocity at impact?
6. A diver drops from 36 m above the water (about the height of divers at the Acapulco Cliff Diving competition). What is the diver's velocity at impact?
7. Compare the impact velocities of objects falling from 9 m (exercise 5), 36 m (exercise 6) and 900 m (example 5.6). If height is increased by a factor of h, by what factor does the impact velocity increase?
8. Suppose that a football was dropped from the top of the Aspire towers in Doha, Qatar. If the tower is 300 meters high, how fast would the ball be going at ground level?
9. A certain not-so-wily coyote discovers that he just stepped off the edge of a cliff. Four seconds later, he hits the ground in a puff of dust. How high in meters was the cliff?
10. A large boulder dislodged by the falling coyote in exercise 9 falls for 3 seconds before landing on the coyote. How far in meters did the boulder fall? What was its velocity in m/s when it flattened the coyote?
11. The coyote's next scheme involves launching himself into the air with an Acme catapult. If the coyote is propelled vertically from the ground with initial velocity 19.6 m/s, find an equation for the height of the coyote at any time t. Find his maximum height, the amount of time spent in the air and his velocity when he smacks back into the catapult.
12. On the rebound, the coyote in exercise 11 is propelled to a height of 78.4 m. What is the initial velocity required to reach this height?
13. One of the authors has a vertical "jump" of 50 cm. What is the initial velocity required to jump this high? How does this compare to Michael Jordan's velocity, found in example 5.3?
14. If the author underwent an exercise program and increased his initial velocity by 10%, by what percentage would he increase his vertical jump?
15. (a) Show that an object dropped from a height of H m will hit the ground at time $T = \frac{1}{7}\sqrt{10H}$ seconds with impact velocity $V = -\frac{7}{5}\sqrt{10H}$ m/s.

    (b) Show that an object propelled from the ground with initial velocity v_0 m/s will reach a maximum height of $\frac{5}{98}v_0^2$ m.
16. An object is launched at angle $\theta = \pi/3$ radians from the horizontal with an initial speed of 98 m/s. Determine the time of flight and the horizontal range. Compare to example 5.4.

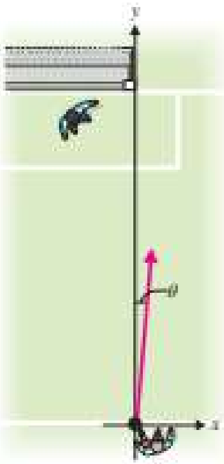


17. Find the time of flight and horizontal range of an object launched at angle 30° with initial speed 40 m/s. Repeat with an angle of 60°.
18. Repeat example 5.5 with an initial angle of 6°.
19. A daredevil plans to jump over 25 cars. If the cars are all compact cars with a width of 1.5 feet and the ramp angle is 30°, determine the initial velocity required to complete the jump successfully. Repeat with a takeoff angle of 45°. In spite of the reduced initial velocity requirement, why might the daredevil prefer an angle of 30° to 45°?
20. A plane at an altitude of 80 m wants to drop supplies to a specific location on the ground. If the plane has a horizontal velocity of 30 m/s, how far away from the target should the plane release the supplies in order to hit the target location? (Hint: Use the y-equation to determine the time of flight, then use the x-equation to determine how far the supplies will drift.)

APPLICATIONS

1. You can measure your reaction time using a ruler. Hold your thumb and forefinger on either side of a meter stick. Have a friend drop the meter stick and grab it as fast as you can. Take the distance d that the meter stick falls and compute how long the ruler fell. Show that if d is measured in cm, your reaction time is approximately $t \approx 0.45\sqrt{d}$. For comparison purposes, a top athlete has a reaction time of about 0.15 s.
2. The coefficient of restitution of a ball measures how "lively" the bounce is. By definition, the coefficient equals $\frac{v_2}{v_1}$, where v_1 is the (downward) speed of the ball when it hits the ground and v_2 is the (upward) launch speed after it hits the ground. If a ball is dropped from a height of H m and rebounds to a height of cH for some constant c with $0 < c < 1$, compute its coefficient of restitution.
3. For the Olympic diver in exercise 5, what would be the average angular velocity (measured in radians per second) necessary to complete $2\frac{1}{2}$ somersaults?
4. In the Flying Zucchini Circus' human cannonball act, a performer is shot out of a cannon from a height of 3 m at an angle of 45° with an initial speed of 48 m/s. If the safety net stands 1.5 m above the ground, how far should the safety net be placed from the cannon? If the safety net can withstand an impact velocity of only 48 m/s, will the Flying Zucchini land safely or come down squash?
5. Soccer player Roberto Carlos of Brazil is known for his curving kicks. Suppose that he has a free kick from 28 m out. Orienting the x- and y-axes as shown in the figure, suppose the kick has initial speed 30 m/s at an angle of 5° from the positive

y-axis. Assume that the only force on the ball is a Magnus force to the left caused by the spinning of the ball. (a) With $x''(t) = -6$ and $y''(t) = 0$, determine whether the ball goes in the goal at $y = 28$ and $-7.32 \le x \le 0$.



(b) A wall of players is lined up 9 m away, extending from $x = -4$ to $x = 1$. Determine whether the kick goes around the wall.

6. To train astronauts to operate in a weightless environment, NASA sends them up in a special plane (nicknamed the *Vomit Comet*). To allow the passengers to experience weightlessness, the vertical acceleration of the plane must exactly match the acceleration due to gravity. If $y''(t)$ is the vertical acceleration of the plane, then $y''(t) = -g$. Show that, for a constant horizontal velocity, the plane follows a parabolic path. NASA's plane flies parabolic paths of approximately 762 m in height (762 m up and 762 m down). The time to complete such a path is the amount of weightless time for the passengers. Compute this time.

In exercises 7–12, we explore two aspects of juggling. More information can be found in "The Mathematics of Juggling" by Polster and Behrends.[5]

7. Professional jugglers generally agree that 10 is the maximum number of balls that a human being can successfully maintain. To get an idea why, suppose that it takes $\frac{1}{2}$ second to catch and toss a ball. (In other words, using both hands, the juggler can process 4 balls per second.) To juggle 10 balls, each ball would need to be in the air for 2.5 seconds. Neglecting air resistance, how high would the ball have to be tossed to stay in the air this long? How much higher would the balls need to be tossed to juggle 11 balls?

8. Another aspect of juggling balls is accuracy. A ball juggled from the right hand to the left hand must travel the correct horizontal distance to be catchable. Suppose that a ball is tossed with initial horizontal velocity v_{0x} and initial vertical velocity v_{0y}. Assume that the ball is caught at the height from which it is thrown. Show that the horizontal distance traveled is $w = \frac{10\, v_{0x} v_{0y}}{49}$ m. (Hint: This is a basic projectile problem, like example 5.4.)

[5]Polster, B. and Behrends, E. (2006). The Mathematics of Juggling. *The Mathematical Intelligencer*, 28(2), 88–89.

9. Referring to exercise 8, suppose that a ball is tossed at an angle of α from the vertical. Show that $\tan \alpha = \frac{v_{0x}}{v_{0y}}$. Combining this result with exercises 15(b) and 8, show that $w = 4h \tan \alpha$, where h is the maximum height of the toss.

10. Find a linear approximation for $\tan^{-1} x$ at $x = 0$. Use this approximation and exercise 9 to show that $\alpha \approx \frac{w}{4h}$. If an angle of α produces a distance of w and an angle of $\alpha + \Delta\alpha$ produces a distance of $w + \Delta w$, show that $\Delta\alpha \approx \frac{\Delta w}{4h}$.

11. Suppose that Δw is the difference between the ideal horizontal distance for a toss and the actual horizontal distance of a toss. For the average juggler, an error of $\Delta w = 1$ m is manageable. Let $\Delta\alpha$ be the corresponding error in the angle of toss. If h is the height needed to juggle 10 balls (see exercise 7), find the maximum error in tossing angle.

12. Repeat exercise 11 using the height needed to juggle 11 balls. How much more accurate does the juggler need to be to juggle 11 balls?

13. Astronaut Alan Shepard modified some of his lunar equipment and became the only person to hit a golf ball on the Moon. Assume that the ball was hit with speed 20 m/s at an angle of 25° above the horizontal. Assuming no air resistance, find the distance the ball would have traveled on Earth. Then find how far it would travel on the Moon, where there really is no air resistance (use $g = 1.6$ m/s^2). The gravitational force of the Moon is about one-sixth that of Earth. A simple guess might be that a golf ball would travel six times as high and six times as far on the Moon compared to on Earth. Determine whether this is correct.

14. Suppose that a firefighter holds a water hose at slope m and the water exits the hose with speed v m/s. Show that the water follows the path $y = -4.9\left(\frac{1+m^2}{v^2}\right)x^2 + mx$. If the firefighter stands 20 m from a wall for a given speed v, what is the maximum height on the wall that the water can reach?

15. Suppose a target is dropped vertically at a horizontal distance of 20 m from you. If you fire a paint ball horizontally directly at the target, show that you will hit it (assuming no air resistance and assuming that the paint ball reaches the target before either hits the ground).

16. An object is dropped from a height of m feet. Another object directly below the first is launched vertically from the ground with initial velocity 40 m/s. Determine when and how high up the objects collide.

17. How fast is a vert skateboarder like Tony Hawk going at the bottom of a ramp? Ignoring friction and air resistance, the answer comes from conservation of energy, which states that the kinetic energy $\frac{1}{2}mv^2$ plus the potential energy mgy remains constant. Assume that the energy at the top of a trick at height H is all potential energy and the energy at the bottom of the ramp is all kinetic energy. (a) Find the speed at the bottom as a function of H. (b) Compute the speed if $H = 16$ m. (c) Find the speed halfway down ($y = 8$). (d) If the ramp has the shape $y = x^2$ for $-4 \le x \le 4$, find the horizontal and vertical components of speed halfway at $y = 8$.

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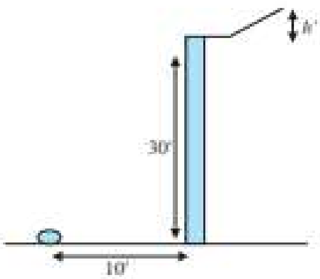
18. A science class builds a ramp to roll a bowling ball out of a window that is 30 m above the ground. Their goal is for the ball to land on a watermelon that is 10 m from the building. Assuming no friction or air resistance, determine how high the ramp should be to smash the watermelon.



EXPLORATORY EXERCISES

1. Although we have commented on some inadequacies of the gravity-only model of projectile motion, we have not presented any alternatives. Such models tend to be somewhat more mathematically complex. The model explored in this exercise takes into account air resistance in a way that is mathematically tractable but still not very realistic. Assume that the force of air resistance is proportional to the speed and acts in the opposite direction of velocity. For a horizontal motion (no gravity), we have $a(t) = F(t)/m = -cv(t)$ for some constant c. Explain what the minus sign indicates. Since $a(t) = v'(t)$, the model is $v'(t) = -cv(t)$. Show that the function $v(t) = v_0e^{-ct}$ satisfies the equation $v'(t) = -cv(t)$ and the initial condition $v(0) = v_0$. If an object starts at $x(0) = a$, integrate $v(t) = v_0e^{-ct}$ to find its position at any time t. Show that the amount of time needed to reach $x = b$ (for $a < b$) is given by $T = -\frac{1}{c}\ln\left(1 - c\frac{b-a}{v_0}\right)$.

2. The goal in the old computer game called "Gorillas" is to enter a speed and angle to launch an explosive banana to try to hit a gorilla at some other location. Suppose that you are located at the origin and the gorilla is at (40, 20). (a) Find two speed/angle combinations that will hit the gorilla. (b) Estimate the smallest speed that can be used to hit the target. (c) Repeat parts (a) and (b) if there is a building in the way that occupies $20 \leq x \leq 30$ and $0 \leq y \leq 30$.

2.6 APPLICATIONS OF INTEGRATION TO PHYSICS AND ENGINEERING

In this section, we explore several applications of integration in physics. In each case, we will define a basic concept and then use the definite integral to generalize the concept to solve a broader range of problems.

Imagine that you are at the bottom of a snow-covered hill with a sled. To get a good ride, you want to push the sled as far up the hill as you can. A physicist would say that the higher up you are, the more *potential energy* you have. Sliding down the hill converts the potential energy into *kinetic energy*. (This is the fun part!) But pushing the sled up the hill requires you to do some work: you must exert a force over a long distance.

Our first task is to quantify work. Certainly, if you push twice the weight (i.e., exert twice the force), you're doing twice the work. Further, if you push the sled twice as far, you've done twice the work. In view of these observations, for any constant force F applied over a distance d, we define the **work** W done as

$$W = Fd.$$

We extend this notion of work to the case of a nonconstant force $F(x)$ applied on the interval $[a, b]$ as follows. First, we partition the interval $[a, b]$ into n equal subintervals, each of width $\Delta x = \frac{b-a}{n}$ and consider the work done on each subinterval. If Δx is small, then the force $F(x)$ applied on the subinterval $[x_{i-1}, x_i]$ can be approximated by the constant force $F(c_i)$ for some point $c_i \in [x_{i-1}, x_i]$. The work done moving the object along the subinterval is then approximately $F(c_i)\,\Delta x$. The total work W is then approximately

$$W \approx \sum_{i=1}^{n} F(c_i)\,\Delta x.$$

You should recognize this as a Riemann sum, which, as n gets larger, approaches the actual work,

Work

$$\boxed{W = \lim_{n\to\infty} \sum_{i=1}^{n} F(c_i)\,\Delta x = \int_a^b F(x)\,dx.} \tag{6.1}$$

We take (6.1) as our definition of work.

You've probably noticed that the farther a spring is compressed (or stretched) from its natural length, the more force is required to further compress (or stretch) the spring. According to **Hooke's Law**, the force required to maintain a spring in a given position is proportional to the distance it's compressed (or stretched). That is, if x is the distance a spring is compressed (or stretched) from its natural length, the force $F(x)$ exerted by the spring is given by

$$F(x) = kx, \tag{6.2}$$

for some constant k (the **spring constant**).

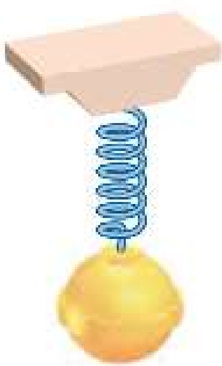


FIGURE 2.48
Stretched spring

EXAMPLE 6.1 Computing the Work Done Stretching a Spring

A force of 30 N stretches a spring from 20 cm to 40 cm. (See Figure 2.48.) Find the work done in stretching the spring.

Solution First, we determine the value of the spring constant. From Hooke's Law (6.2), we have that

$$30 = k(0.40 - 0.20)$$
$$kk = 150 \text{ N/m}$$

so that $k = 150$ and $F(x) = 150x$. From (6.1), the work done in stretching the spring 20 cm is then

$$W = \int_0^{0.2} F(x)\,dx = \int_0^{0.2} 150x\,dx = 3 \text{ J}$$

In this case, notice that stretching the spring transfers potential energy to the spring. (If the spring is later released, it springs back toward its natural length, converting the potential energy to kinetic energy.) ■

EXAMPLE 6.2 Computing the Work Done by a Weightlifter

A weightlifter lifts a 200 kg barbell a distance of 0.7 m. How much work was done? Also, determine the work done by the weightlifter if the weight is raised 0.4 m above the ground and then lowered back into place.

Solution Since the force (the weight) is constant here, we simply have

$$W = Fd = 200 \times 0.7 = 140 \text{ J}.$$

It may seem strange, but if the weightlifter lifts the same weight 0.4 m from the ground and then lowers it back into place, then since the barbell ends up in the same place as it started, the net distance covered is zero and the work done is zero. Of course, it would feel like work to the weightlifter, but as we have defined it, work accounts for the energy change in the object. Since the barbell has the same kinetic and potential energy that it started with, the total work done on it is zero. ■

In example 6.3, both the force and the distance are nonconstant. This presents some unique challenges and we'll need to first approximate the work and then recognize the definite integral that this approximation process generates.

EXAMPLE 6.3 Computing the Work Required to Pump Water Out of a Tank

A spherical tank of radius 10 m is filled with water. Find the work done in pumping all of the water out through the top of the tank.

Solution The basic formula $W = Fd$ does not directly apply here, for several reasons. The most obvious of these is that the distance traveled by the water in each part of the tank is different, as the water toward the bottom of the tank must be pumped all the way to the top, while the water near the top of the tank must be pumped only a short distance. Let x represent distance as measured from the bottom of the tank, as in Figure 2.49a. The entire tank corresponds to the interval $0 \le x \le 20$, which we partition into

$$0 = x_0 < x_1 < \cdots < x_n = 20,$$

where $x_i - x_{i-1} = \Delta x = \dfrac{20}{n}$, for each $i = 1, 2, \ldots, n$.

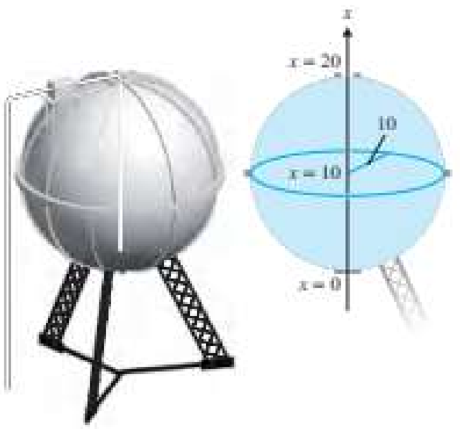


FIGURE 2.49a
Spherical tank

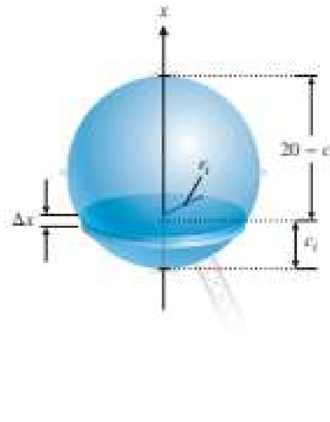


FIGURE 2.49b
The ith slice of water

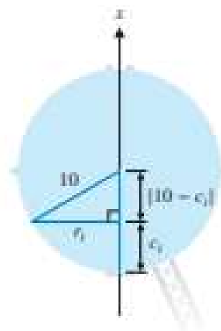


FIGURE 2.49c
Cross section of tank

This partitions the tank into n thin layers, each corresponding to an interval $[x_{i-1}, x_i]$. (See Figure 2.49b.) You can think of the water in the layer corresponding to $[x_{i-1}, x_i]$ as being approximately cylindrical, of height Δx. This layer must be pumped a distance of approximately $20 - c_i$, for some $c_i \in [x_{i-1}, x_i]$. Notice from Figure 2.49b that the radius of the ith layer depends on the value of x. From Figure 2.49c (where we show a cross-section of the tank), the radius r_i corresponding to a depth of $x = c_i$ is the base of a right triangle with hypotenuse 10 and height $|10 - c_i|$. From the Pythagorean Theorem, we now have

$$(10 - c_i)^2 + r_i^2 = 10^2.$$

Solving this for r_i^2, we have

$$\begin{aligned} r_i^2 &= 10^2 - (10 - c_i)^2 = 100 - (100 - 20c_i + c_i^2) \\ &= 20c_i - c_i^2 \end{aligned}$$

The force F_i required to move the ith layer is then simply the force exerted on the water by gravity (i.e., its weight). Since the weight density of water is 9809 kg/m^3, we now have

$$\begin{aligned} F_i &\approx \text{(Volume of cylindrical slice) (weight of water per unit volume)} \\ &= (\pi r_i^2 h)(9809 \text{ kg/m}^3) \\ &= 9809\pi(20c_i - c_i^2)\Delta x. \end{aligned}$$

The work required to pump out the ith slice is then given approximately by

$$\begin{aligned} W_i &\approx \text{(Force) (distance)} \\ &= 9809\pi(20c_i - c_i^2)\Delta x(20 - c_i) \\ &= 9809\pi c_i(20 - c_i)^2 \Delta x. \end{aligned}$$

The work required to pump out all of the water is then the sum of the work required for each of the n slices:

$$W \approx \sum_{i=1}^{n} 9809\pi c_i(20 - c_i)^2 \Delta x.$$

Finally, taking the limit as $n \to \infty$ gives the exact work, which you should recognize as a definite integral:

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} 9809\pi c_i(20 - c_i)^2 \Delta x = \int_0^{20} 9809\pi x(20 - x)^2\, dx \\ &= 9809\pi \int_0^{20} (400x - 40x^2 + x^3)\, dx \\ &= 9809\pi \left[400\frac{x^2}{2} - 40\frac{x^3}{3} + \frac{x^4}{4}\right]_0^{20} \\ &= 9809\pi\left(\frac{40,000}{3}\right) \approx 4.1 \times 10^8 \text{ J.} \end{aligned}$$

Impulse is a physical quantity closely related to work. Instead of relating force and distance to account for changes in energy, impulse relates force and time to account for changes in velocity. First, suppose that a constant force F is applied to an object from time $t = 0$ to time $t = T$. If the position of the object at time t is given by $x(t)$, then Newton's second law says that $F = ma = mx''(t)$. Integrating this equation once with respect to t gives us

$$\int_0^T F dt = m\int_0^T x''(t)\, dt,$$

or

$$F(T - 0) = m[x'(T) - x'(0)].$$

Recall that $x'(t)$ is the velocity $v(t)$, so that

$$FT = m[v(T) - v(0)]$$

or $FT = m\Delta v$, where $\Delta v = v(T) - v(0)$ is the change in velocity. The quantity FT is called the **impulse**, $mv(t)$ is the **momentum** at time t and the equation relating the impulse to the change in velocity is called the **impulse-momentum equation**.

Just as we extended the concept of work to include nonconstant forces, we must generalize the notion of impulse. Think about this and try to guess what the definition should be.

We define the impulse J of a force $F(t)$ applied over the time interval $[a, b]$ to be

Impulse

$$J = \int_a^b F(t)\,dt.$$

We leave the derivation of this as an exercise. The impulse-momentum equation likewise generalizes to:

Impulse-momentum equation

$$J = m[v(b) - v(a)].$$

EXAMPLE 6.4 Estimating the Impulse for a Cricket ball

Suppose that a cricket ball traveling at 25 m/s collides with a cricket bat. The following data shows the force exerted by the bat on the ball at 0.0001-second intervals.

t (s)	0	0.0001	0.0002	0.0003	0.0004	0.0005	0.0006	0.0007
$F(t)$ (N)	0	5560	7125	12325	18430	7250	5560	0

Estimate the impulse of the bat on the ball and (using $m = 0.15$ kg) the speed of the ball after impact.

Solution In this case, the impulse J is given by $\int_0^{0.0007} F(t)dt$. Since we're given only a fixed number of measurements of $F(t)$, the best we can do is approximate the integral numerically (e.g., using Simpson's Rule). Recall that Simpson's Rule requires an even number n of subintervals, which means that you need an odd number $n + 1$ of points in the partition. Using $n = 8$ and adding a 0 function value at $t = 0.0008$ (why is it fair to do this?), Simpson's Rule gives us

$$\begin{aligned} J &\approx [0 + 4(5560) + 2(7125) + 4(12325) + 2(18430) + 4(7250) \\ &\quad + 2(5560) + 4(0) + 0]\frac{0.0001}{3} \\ &\approx 5.42. \end{aligned}$$

In this case, the impulse-momentum equation $J = m\,\Delta v$ becomes $5.42 = 0.15\,\Delta v$ or $\Delta v = 36.2$ m/s. Since the ball started out with a speed of 25 m/s in one direction and it ended up moving in the opposite direction, the speed after impact is 11.2 m/s. ■

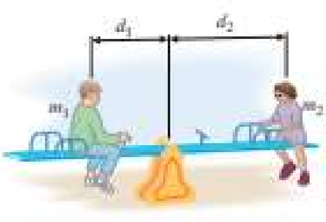


FIGURE 2.50a
Balancing two masses

Consider two children on a playground seesaw (or teeter-totter). Suppose that the child on the left in Figure 2.50a is heavier (i.e., has larger mass) than the child on the right. If the children sit an equal distance from the pivot point, you know what will happen: the left side will be pulled down. However, the children can balance each other if the heavier child moves closer to the pivot point. That is, the balance is determined both by weight (force) and distance from the pivot point. If the children have masses m_1 and m_2 and are sitting at distances d_1 and d_2, respectively, from the pivot point, then they balance each other if and only if

$$m_1 d_1 = m_2 d_2. \tag{6.3}$$

Turning the problem around slightly, suppose there are two objects, of mass m_1 and m_2, located at x_1 and x_2, respectively, with $x_1 < x_2$. We consider the objects to be **point-masses**. That is, each is treated as a single point, with all of the mass concentrated at that point. (See Figure 2.50b.)

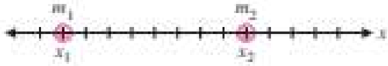


FIGURE 2.50b
Two point-masses

We want to find the **center of mass** $\bar{x}$, that is, the location at which we could place the pivot of a seesaw and have the objects balance. From the balance equation (6.3), we'll need $m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$. Solving this equation for $\bar{x}$ gives us

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

Notice that the denominator in this equation is the total mass of the "system" (i.e., the total mass of the two objects). The numerator of this expression is called the **first moment** of the system.

More generally, for a system of n masses $m_1, m_2, \ldots, m_n$, located at $x = x_1, x_2, \ldots, x_n$, respectively, the center of mass $\bar{x}$ is given by the first moment divided by the total mass, that is,

Center of mass

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n}.$$

Now, suppose that we wish to find the mass and center of mass of an object of variable density $\rho(x)$ (measured in units of mass per unit length) that extends from $x = a$ to $x = b$. Note that if the density is a constant ρ, the mass of the object is simply given by $m = \rho L$, where $L = b - a$ is the length of the object. On the other hand, if the density varies throughout the object, we can approximate the mass by partitioning the interval $[a, b]$ into n pieces of equal width $\Delta x = \frac{b-a}{n}$. On each subinterval $[x_{i-1}, x_i]$, the mass is approximately $\rho(c_i)\Delta x$, where c_i is a point in the subinterval. The total mass is then approximately

$$m \approx \sum_{i=1}^{n} \rho(c_i)\,\Delta x.$$

You should recognize this as a Riemann sum, which approaches the total mass as $n \to \infty$,

Mass

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(c_i)\,\Delta x = \int_a^b \rho(x)\,dx. \tag{6.4}$$

EXAMPLE 6.5 Computing the Mass of an Object

The density function of an object extending from $x = 0$ to $x = 3$, with density $\rho(x) = \frac{2x}{3} + 4$ kg/m. Find the mass of the object.

Solution From (6.4), the mass is given by

$$m = \int_0^3 \left(\frac{2x}{3} + 4\right) dx$$
$$= \left(\frac{x^2}{3} + 4x\right)\Big|_0^3 = 15 \text{ kg}$$

To compute the first moment for an object of nonconstant density $\rho(x)$ extending from $x = a$ to $x = b$, we again partition the interval into n equal pieces. From our earlier argument, for each $i = 1, 2, \ldots, n$, the mass of the ith slice of the object is approximately $\rho(c_i)\,\Delta x$, for any choice of $c_i \in [x_{i-1}, x_i]$. We then represent the ith slice of the object with a particle of mass $m_i = \rho(c_i)\,\Delta x$ located at $x = c_i$. We can now think of the original object as having been approximated by n distinct point-masses, as indicated in Figure 2.51.

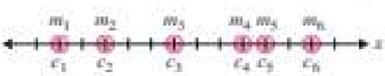


FIGURE 2.51
Six point-masses

Notice that the first moment M_n of this approximate system is

$$M_n = [\rho(c_1)\Delta x]c_1 + [\rho(c_2)\Delta x]c_2 + \cdots + [\rho(c_n)\Delta x]c_n$$
$$= [c_1\rho(c_1) + c_2\rho(c_2) + \cdots + c_n\rho(c_n)]\,\Delta x = \sum_{i=1}^{n} c_i\,\rho(c_i)\Delta x.$$

Taking the limit as $n \to \infty$, the sum approaches the first moment

First moment

$$\boxed{M = \lim_{n\to\infty} \sum_{i=1}^{n} c_i\,\rho(c_i)\,\Delta x = \int_a^b x\rho(x)\,dx.} \qquad (6.5)$$

The center of mass of the object is then given by

Center of mass

$$\boxed{\bar{x} = \frac{M}{m} = \frac{\displaystyle\int_a^b x\rho(x)\,dx}{\displaystyle\int_a^b \rho(x)\,dx}.} \qquad (6.6)$$

Aswan Dam, Egypt
Adwo/Shutterstock

EXAMPLE 6.6 Finding the Center of Mass of an Object

Find the center of mass of the object from example 6.5.

Solution From (6.5), the first moment is given by

$$M = \int_0^3 \left(\frac{2x}{3} + 4\right)dx = \left[\frac{2x^3}{9} + 2x^2\right]_0^3 = 24.$$

Recall that we had already found the mass to be $m = 15$ kg and so, from (6.6), the center of mass of the object is

$$\bar{x} = \frac{M}{m} = \frac{24}{15} = 1.6 \text{ meters.} \quad ■$$

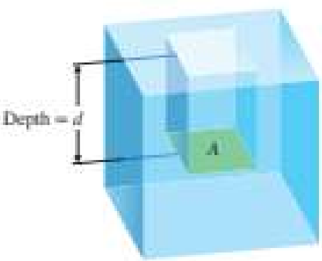


FIGURE 2.52
A plate of area A submerged to depth d

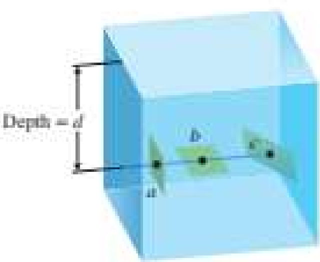


FIGURE 2.53
Pressure at a given depth is the same, regardless of the orientation

For our final application of integration in this section, we consider hydrostatic force. Imagine a dam holding back a lake full of water. What force must the dam withstand?

As usual, we solve a simpler problem first. If you have a flat rectangular plate oriented horizontally underwater, notice that the force exerted on the plate by the water (the **hydrostatic force**) is simply the weight of the water lying above the plate. This is the product of the volume of the water lying above the plate and the weight density of water (9809 N/m^3). If the area of the plate is A m^2 and it lies d m below the surface (see Figure 2.52), then the force on the plate is

$$F = 62.4\,Ad.$$

According to Pascal's Principle, the pressure at a given depth d in a fluid is the same in all directions. This says that if a flat plate is submerged in a fluid, then the pressure on one side of the plate at any given point is $\rho \cdot d$, where ρ is the weight density of the fluid and d is the depth. In particular, this says that it's irrelevant whether the plate is submerged vertically, horizontally or otherwise. (See Figure 2.53.)

Consider now a vertically oriented wall (a dam) holding back a lake. It is convenient to orient the x-axis vertically with $x = 0$ located at the surface of the water and the bottom of the wall at $x = a > 0$. (See Figure 2.54.) In this way, x measures the depth of a section of the dam. Suppose $w(x)$ is the width of the wall at depth x (where all distances are measured in meters).

Partition the interval $[0, a]$ into n subintervals of equal width $\Delta x = \frac{a}{n}$. This has the effect of slicing the dam into n slices, each of width Δx. For each $i = 1, 2, \ldots, n$, observe that the area of the ith slice is approximately $w(c_i)\,\Delta x$, where c_i is some point in the

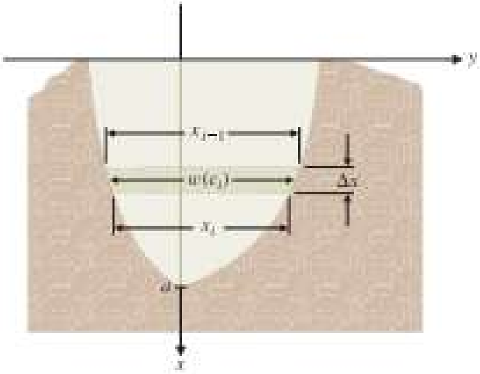


FIGURE 2.54
Force acting on a dam

subinterval $[x_{i-1}, x_i]$. Further, the depth at every point on this slice is approximately c_i. We can then approximate the force F_i acting on this slice of the dam by the weight of the water lying above a plate the size of this portion but which is oriented horizontally:

$$F_i \approx \underbrace{9809}_{\text{weight density}} \underbrace{w(c_i)}_{\text{length}} \underbrace{\Delta x}_{\text{width}} \underbrace{c_i}_{\text{depth}} = 9809\, c_i w(c_i) \Delta x.$$

Adding together the forces acting on each slice, the total force F on the dam is approximately

$$F \approx 9809\, c_i w(c_i) \Delta x. \sum_{i=1}^{n}$$

Recognizing this as a Riemann sum and taking the limit as $n \to \infty$, we obtain the total hydrostatic force on the dam,

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 9809\, c_i w(c_i) \Delta x = \int_0^a 9809 x w(x)\, dx. \qquad (6.7)$$

EXAMPLE 6.7 Finding the Hydrostatic Force on a Dam

A dam is shaped like a trapezoid with height 60 m. The width at the top is 100 m and the width at the bottom is 40 m. (See Figure 2.55.) Find the maximum hydrostatic force that the dam will need to withstand. Find the hydrostatic force if a drought lowers the water level by 10 m.

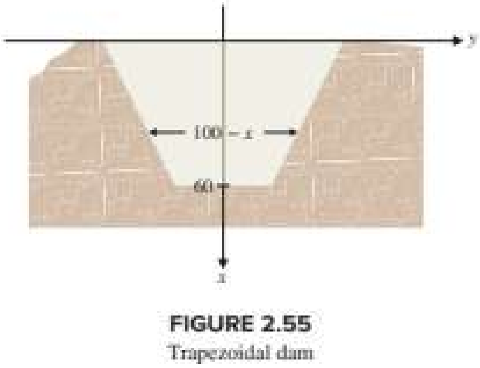


FIGURE 2.55
Trapezoidal dam

Solution Notice that the width function is a linear function of depth with $w(0) = 100$ and $w(60) = 40$. The slope is then $\frac{60}{-60} = -1$ and so, $w(x) = 100 - x$. From (6.7), the hydrostatic force is then

$$F = \int_0^{60} \underbrace{9809}_{\text{weight density}} \underbrace{x}_{\text{depth}} \underbrace{(100 - x)}_{\text{width}} \, dx$$
$$= 490450x^2 - 9809\frac{x^3}{3}\bigg|_0^{60} = 1059372000 \text{ N}.$$

If the water level dropped 10 m, the width of the dam at the water level would be 90 m. Lowering the origin by 10 m, the new width function satisfies $w(0) = 90$ and $w(50) = 40$. The slope is still -1 and so the width is given by $w(x) = 90 - x$. From (6.7), the hydrostatic force is now

$$F = \int_0^{50} \underbrace{9809}_{\text{weight density}} \underbrace{x}_{\text{depth}} \underbrace{(90 - x)}_{\text{width}} \, dx$$
$$= 441405x^2 - 9809\frac{x^3}{3}\bigg|_0^{50} = 694804166.7 \text{ N}$$

EXERCISES 2.6

WRITING EXERCISES

1. For each of work, impulse and the first moment: identify the quantities in the definition (e.g., force and distance) and the calculations for which it is used (e.g., change in velocity).

2. People who play catch have a seemingly instinctive method of pulling their hand back as they catch the ball. To catch a ball, you must apply an impulse equal to the mass times velocity of the ball. By pulling your hand back, you increase the amount of time in which you decelerate the ball. Use the impulse-momentum equation to explain why this reduces the average force on your hand.

3. A tennis ball comes toward you at 50 km/h. After you hit the ball, it is moving away from you at 50 km/h. Work measures changes in energy. Explain why work has been done by the tennis racket on the ball even though the ball has the same speed before and after the hit.

---

1. A force of 8 N stretches a spring 0.4 m. Find the work done in stretching this spring 0.6 m beyond its natural length.

2. A force of 100 N stretches a spring 1/6 m. Find the work done in stretching this spring 0.25 m beyond its natural length.

3. A weightlifter lifts 1112 N a distance of 20 cm. Find the work done (as measured in joules).

4. A wrestler lifts his 1500 N opponent overhead, a height of 2 m. Find the work done (as measured in joules).

5. A rocket full of fuel weighs 10,000 N at launch. After launch, the rocket gains altitude and loses weight as the fuel burns. Assume that the rocket loses 1 N of fuel for every 15 m of altitude gained. Explain why the work done raising the rocket to an altitude of 3000 m is $\int_0^{3000} (10{,}000 - x/15)\, dx$ and compute the integral.

6. Referring to exercise 5, suppose that a rocket weighs 8000 N at launch and loses 1 N of fuel for every 10 N of altitude gained. Find the work needed to raise the rocket to a height of 1000 m.

7. A 40 m chain weighs 1000 N and is hauled up to the deck of a boat. The chain is oriented vertically and the top of the chain starts in the water 30 m below the deck. Compute the work done.

8. A bucket is lifted a distance of 80 m at the rate of 4 m/s. The bucket initially contains 100 kg of sand but leaks at a rate of 2 kg/s. Compute the work done.

9. (a) Suppose that a car engine exerts a force of $800x(1 - x)$ N when the car is at position x km, $0 \le x \le 1$. Compute the work done. (b) Horsepower measures the rate of work done as a function of time. Explain why this is not equal to $800x(1 - x)$. If the car takes 80 seconds to travel the km, compute the average horsepower (1 hp = 550 J/s).

10. (a) A water tower is spherical in shape with radius 5 m, extending from 20 m to 30 m above ground. Compute the work done in filling the tank from the ground. (b) Compute the work done in filling the tank halfway. The weight density of water is 9809 N/m^3.

11. A right circular cylinder of radius 1 m and height 3 m is filled with water. Compute the work done pumping all of the water out of the top of the cylinder if (a) the cylinder stands upright (the circular cross-sections are parallel to the ground) and (b) the cylinder is on its side (the circular cross-sections are perpendicular to the ground).

12. A water tank is in the shape of a right circular cone of altitude 10 m and base radius 5 m, with its vertex at the ground. (Think of an ice cream cone with its point facing down.) If the tank is full, find the work done in pumping all of the water out the top of the tank.

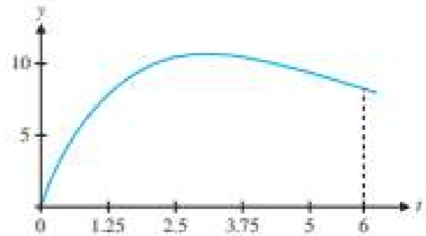
13. Two laborers share the job of digging a rectangular hole 10 m deep. The dirt from the hole is cleared away by other laborers. Assuming a constant density of dirt, how deep should the first worker dig to do half the work? Explain why 5 m is not the answer.

14. A trough is to be dug 6 m deep. Cross-sections have the shape ⋃ and are 2 m wide at the bottom and 5 feet wide at the top. Find the depth at which half the work has been done.

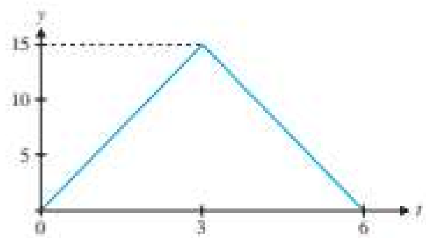
15. A crash test is performed on a vehicle. The force of the wall on the front bumper is shown in the table. Estimate the impulse and the speed of the vehicle (use $m = 2000$ kg).

t (s)	0	0.1	0.2	0.3	0.4	0.5	0.6
F (N)	0	30,000	60,000	90,000	50,000	40,000	0

16. A **thrust-time curve** $f(t) = 10t\,e^{-t/3}$ for a model rocket is shown. Compute the maximum thrust. Compute the impulse.



17. A thrust-time curve for a model rocket is shown. Compute the impulse. Based on your answers to exercises 16 and 17, which rocket would reach a higher altitude?



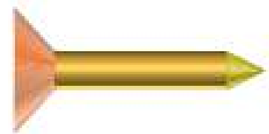
18. Compute the mass and center of mass of an object with density $\rho(x) = \frac{x}{6} + 2$ kg/m, $0 \le x \le 6$. Briefly explain in terms of the density function why the center of mass is not at $x = 3$.

19. Compute the mass and center of mass of an object with density $\rho(x) = 3 - \frac{x}{6}$ kg/m, $0 \le x \le 6$. Briefly explain in terms of the density function why the center of mass is not at $x = 3$.

20. Compute the weight in kg of an object extending from $x = -3$ to $x = 27$ with density $\rho(x) = \left(\frac{1}{46} + \frac{x+3}{690}\right)^2$ kg/m.

21. Compute the weight in kg of an object extending from $x = 0$ to $x = 32$ with density $\rho(x) = \left(\frac{1}{46} + \frac{x+3}{690}\right)^2$ kg/m.

22. The accompanying figure shows the outline of a model rocket. Assume that the vertical scale is 3 units high and the horizontal scale is 6 units wide. Use basic geometry to compute the area of each of the three regions of the rocket outline. Assuming a constant density ρ, locate the x-coordinate of the center of mass of each region. (Hint: The first region can be thought of as extending from $x = 0$ to $x = 1$ with density $\rho(3 - 2x)$. The third region extends from $x = 5$ to $x = 6$ with density $\rho(6 - x)$.)



23. For the model rocket in exercise 22, replace the rocket with 3 particles, one for each region. Assume that the mass of each particle equals the area of the region and the location of the particle on the x-axis equals the center of mass of the region. Find the center of mass of the 3-particle system. [Rockets are designed with bottom fins large enough that the center of mass is shifted near the bottom (or, in the figure here, left) of the rocket. This improves the flight stability of the rocket.]

In exercises 24–27, find the centroid of each region. The centroid is the center of mass of a region with constant density. (Hint: Modify (6.6) to find the y-coordinate $\bar{y}$.)

24. The triangle with vertices (0, 0), (4, 0) and (4, 6)

25. The rhombus with vertices (0, 0), (3, 4), (8, 4) and (5, 0)

26. The region bounded by $y = 4 - x^2$ and $y = 0$

27. The region bounded by $y = x$, $y = -x$ and $x = 4$

APPLICATIONS

1. A dam is in the shape of a trapezoid with height 60 m. The width at the top is 40 m and the width at the bottom is 100 m. Find the maximum hydrostatic force the wall would need to withstand. Explain why the force is so much greater than the force in example 6.7.

2. Find the maximum hydrostatic force in exercise 1 if a drought lowers the water level by 10 m.

3. An underwater viewing window is installed at an aquarium. The window is circular with radius 2 m. The center of the window is 10 m below the surface of the water. Find the hydrostatic force on the window.

4. An underwater viewing window is rectangular with width 40 m. The window extends from the surface of the water to a depth of 10 m. Find the hydrostatic force on the window.

5. The camera's window on a robotic submarine is circular with radius 10 cm. How much hydrostatic force would the window need to withstand to descend to a depth of 100 m?

6. A diver wears a watch to a depth of 30 m. The face of the watch is circular with a radius of 2 cm. How much hydrostatic force will the face need to withstand if the watch is to keep on ticking?

7. Given that power is the product of force and velocity, compute the horsepower needed to lift a 100-ton object such as a blue whale at 20 km/h (1 hp = 745.7 Nm/s). (Note that blue whales swim so efficiently that they can maintain this speed with an output of 60–70 hp.)

8. For a constant force F exerted over a length of time t, impulse is defined by $F \cdot t$. For a variable force $F(t)$, derive the impulse formula $J = \int_a^b F(t)dt$.

EXPLORATORY EXERCISES

1. As equipment has improved, heights cleared in the pole vault have increased. A crude estimate of the maximum pole vault possible can be derived from conservation of energy principles. Assume that the maximum speed a pole-vaulter could run carrying a long pole is 40.23 km/h. Convert this speed to m/s. The kinetic energy of this vaulter would be $1/2mv^2$. (Leave m as an unknown for the time being.) This initial kinetic energy would equal the potential energy at the top of the vault minus whatever energy is absorbed by the pole (which we will ignore). Set the potential energy, $9.8mh$, equal to the kinetic energy and solve for h. This represents the maximum amount the vaulter's center of mass could be raised. Add 1 meter for the height of the vaulter's center of mass and you have an estimate of the maximum vault possible. Compare this to Sergei Bubka's 1994 world record vault of 6.14 meters.

2. An object will remain on a table as long as the center of mass of the object lies over the table. For example, a board of length 1 will balance if half the board hangs over the edge of the table. Show that two homogeneous boards of length 1 will balance if $\frac{1}{4}$ of the first board hangs over the edge of the table and $\frac{1}{2}$ of the second board hangs over the edge of the first board. Show that three boards of length 1 will balance if $\frac{1}{6}$ of the first board hangs over the edge of the table, $\frac{1}{4}$ of the second board hangs over the edge of the first board and $\frac{1}{2}$ of the third board hangs over the edge of the second board. Generalize this to a procedure for balancing n boards. How many boards are needed so that the last board hangs completely over the edge of the table?

2.7 PROBABILITY

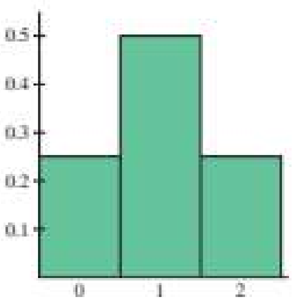


FIGURE 2.56
Histogram for two-coin toss

The mathematical fields of probability and statistics focus on the analysis of random processes. In this section, we give a brief introduction to the use of calculus in probability theory.

We begin with a simple example involving coin-tossing. Suppose that you toss two coins, each of which has a 50% chance of coming up heads. Because of the randomness involved, you cannot calculate exactly how many heads you will get on a given number of tosses. But you *can* calculate the *likelihood* of each of the possible outcomes. If we denote heads by H and tails by T, then the four possible outcomes from tossing two coins are HH, HT, TH and TT. Each of these four outcomes is equally likely, so we can say that each has probability $\frac{1}{4}$. This means that, *on average*, each of these events will occur in one-fourth of your tries. Said a different way, the **relative frequency** with which each event occurs in a large number of trials will be approximately $\frac{1}{4}$.

Note that based on our calculations above, the probability of getting two heads is $\frac{1}{4}$, the probability of getting one head is $\frac{2}{4}$ (since there are two ways for this to happen: HT and TH) and the probability of getting zero heads is $\frac{1}{4}$. We often summarize such information by displaying it in a **histogram**, a bar graph where the outcomes are listed on the horizontal axis. (See Figure 2.56.)

If we instead toss eight coins, the probabilities for getting a given number of heads are given in the accompanying table and the corresponding histogram is shown in Figure 5.57. You should notice that the sum of all the probabilities is 1 (or 100%, since it's certain that *one* of the possible outcomes will occur on a given try). This is one of the defining properties of probability theory. Another basic property is called the **addition principle**: to compute the probability of getting 6, 7 or 8 heads (or any other mutually exclusive outcomes), simply add together the individual probabilities:

Number of Heads	*Probability*
0	1/256
1	8/256
2	28/256
3	56/256
4	70/256
5	56/256
6	28/256
7	8/256
8	1/256

$$P(6, 7 \text{ or } 8 \text{ heads}) = \frac{28}{256} + \frac{8}{256} + \frac{1}{256} = \frac{37}{256} \approx 0.145.$$

A graphical interpretation of this calculation is very revealing. In the histogram in Figure 2.58, notice that each bar is a rectangle of width 1. Then the probability associated with each bar equals the area of the rectangle. In graphical terms,

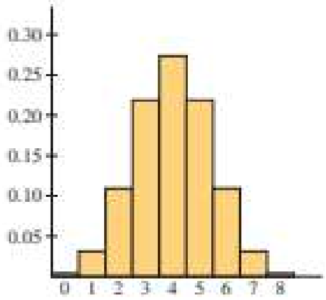


FIGURE 2.57
Histogram for eight-coin toss

- The total area in such a histogram is 1.
- The probability of getting between 6 and 8 heads (inclusive) equals the sum of the areas of the rectangles located between 6 and 8 (inclusive).

A more complicated question is to ask the probability that a randomly chosen child (aged 2 to 4 years) will have a height of 69 cm or 70 cm. There is no easy theory we can use here to compute the probabilities (since not all heights are equally likely). In this case, we use the correspondence between probability and relative frequency. If we collect information about the heights of a large number of children of the same age interval, we might find the following.

Height (cm)	<64	64	65	66	67	68	69	70	71	72	73	>73
Number of children	23	32	61	94	133	153	155	134	96	62	31	26

Since the total number of children in the survey is 1000, the relative frequency of the height 69 cm is $\frac{155}{1000} = 0.155$ and the relative frequency of the height 70 cm is $\frac{134}{1000} = 0.134$. An estimate of the probability of being 69 cm or 70 cm is then $0.155 + 0.134 = 0.289$. A histogram is shown in Figure 2.58.

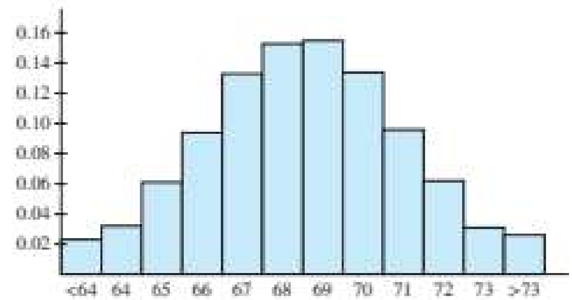


FIGURE 2.58
Histogram for relative frequency of heights

To answer a more specific question, such as what the probability is that a randomly chosen child is 68.5 cm or 69 cm, we would need to have our data broken down further, as in the following partial table.

Height (cm)	$66\frac{1}{2}$	67	$67\frac{1}{2}$	68	$68\frac{1}{2}$	69	$69\frac{1}{2}$	70	$70\frac{1}{2}$	71
Number of children	52	61	72	71	82	81	74	69	65	58

The probability that a child is 69 cm can be estimated by the relative frequency of 69 cm children in our survey, which is $\frac{81}{1000} = 0.081$. Similarly, the probability that a child is 68.5 cm is approximately $\frac{82}{1000} = 0.082$. The probability of being 58.5 cm or 69 cm is then approximately $0.081 + 0.082 = 0.163$. A histogram for this portion of the data is shown in Figure 2.59a.

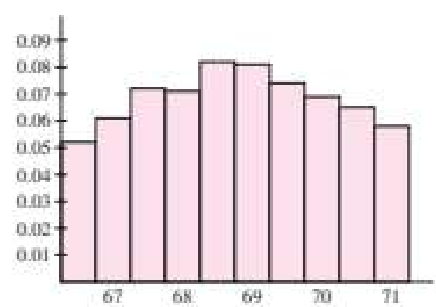


FIGURE 2.59a
Histogram for relative frequency of heights

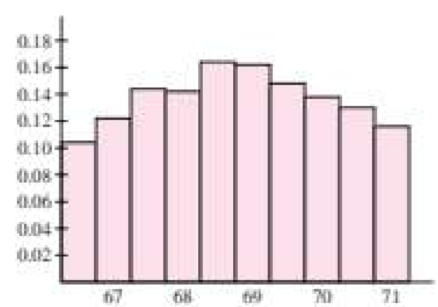


FIGURE 2.59b
Histogram showing double the relative frequency

Notice that since each bar of the histogram now represents a half-inch range of height, we can no longer interpret area in the histogram as the probability. We will modify the histogram to make the area connection clearer. In Figure 2.59b, we have labeled the horizontal axis with the height in centimeters, while the vertical axis shows twice the relative frequency. The bar at 69 cm has height 0.162 and width $\frac{1}{2}$. Its area, $\frac{1}{2}(0.162) = 0.081$, corresponds to the relative frequency (or probability) of the height 69 cm.

Of course, we could continue subdividing the height intervals into smaller and smaller pieces. Think of doing this while modifying the scale on the vertical axis so that the area of each rectangle (length times width of interval) always gives the relative frequency (probability) of that height interval. For example, suppose that there are n height intervals between 68 cm and 69 cm. Let x represent height in centimeters and $f(x)$ equal the height of the histogram bar for the interval containing x. Let $x_1 = 68 + \frac{1}{n}$, $x_2 = 68 + \frac{2}{n}$ and so on, so that $x_i = 68 + \frac{i}{n}$, for $1 \leq i \leq n$ and let $\Delta x = \frac{1}{n}$. For a randomly selected person, the probability that their height is between 68 cm and 69 cm is estimated by the sum of the areas of the corresponding histogram rectangles, given by

$$P(68 \leq x \leq 69) \approx f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x. \qquad (7.1)$$

Observe that as n increases, the histogram of Figure 2.60 will "smooth out," approaching a curve like the one shown in Figure 2.61.

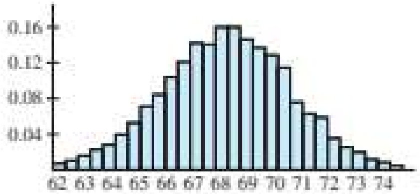


FIGURE 2.60
Histogram for heights

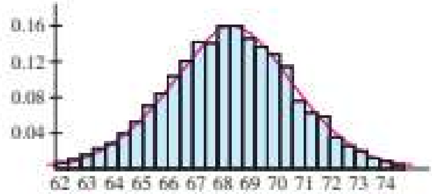


FIGURE 2.61
Probability density function and histogram for heights

We call this limiting function $f(x)$, the **probability density function (PDF)** for heights. Notice that for any given $i = 1, 2, \ldots, n$, $f(x_i)$ does not give the probability that a child's height equals x_i. Instead, for small values of Δx, the quantity $f(x_i)\,\Delta x$ is an approximation of the probability that a randomly selected height is in the range $[x_{i-1}, x_i]$.

Observe that as $n \to \infty$, the Riemann sum in (7.1) should approach an integral $\int_a^b f(x)\,dx$. Here, the limits of integration are 68 cm and 69 cm. We have

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_{68}^{69} f(x)\,dx.$$

Notice that by adjusting the function values so that probability corresponds to area, we have found a familiar and direct technique for computing probabilities. We now summarize our discussion with some definitions. The preceding examples are of **discrete probability distributions** (*discrete* since the quantity being measured can only assume values from a certain finite set). For instance, in coin-tossing, the number of heads must be an integer. By contrast, many distributions are **continuous**. That is, the quantity of interest (the random variable) assumes values from a continuous range of numbers (an interval). For instance, although height is normally rounded off to the nearest integer number of centimeters, a child's actual height can be any number.

HISTORICAL NOTES

Blaise Pascal (1623–1662)
A French mathematician and physicist who teamed with Pierre Fermat to begin the systematic study of probability. (See *The Unfinished Game*, by Keith Devlin for an account of this.[5]) Pascal is credited with numerous inventions, including a wrist watch, barometer, hydraulic press, syringe and a variety of calculating machines. He also discovered what is now known as Pascal's Principle in hydrostatics. (See section 5.6.) Pascal may well have become one of the founders of calculus, but poor health and large periods of time devoted to religious and philosophical contemplation reduced his mathematical output.

[5]Devlin, K. (2010). *The Unfinished Game: Pascal, Fermat, and the Seventeenth-century Letter that Made the World Modern* (New York: Basic Books).

For continuous distributions, the graph corresponding to a histogram is the graph of a probability density function (PDF). We now give a precise definition of a PDF.

DEFINITION 7.1

Suppose that X is a random variable that may assume any value x with $a \le x \le b$. A probability density function for X is a function $f(x)$ satisfying

(i) $f(x) \ge 0$ for $a \le x \le b$. Probability density functions are never negative.

and

(ii) $\int_a^b f(x)\,dx = 1$. The total probability is 1.

The probability that the (observed) value of X falls between c and d is given by the area under the graph of the PDF on that interval. That is,

$$P(c \le X \le d) = \int_c^d f(x)\,dx.$$

Probability corresponds to area under the curve.

To verify that a function defines a PDF for *some* (unknown) random variable, we must show that it satisfies properties (i) and (ii) of Definition 7.1.

EXAMPLE 7.1 Verifying That a Function is a PDF on an Interval

Show that $f(x) = 3x^2$ defines a pdf on the interval [0, 1] by verifying properties (i) and (ii) of Definition 7.1.

Solution Clearly, $f(x) \ge 0$. For property (ii), we integrate the pdf over its domain. We have

$$\int_0^1 3x^2\,dx = x^3\Big|_0^1 = 1.$$

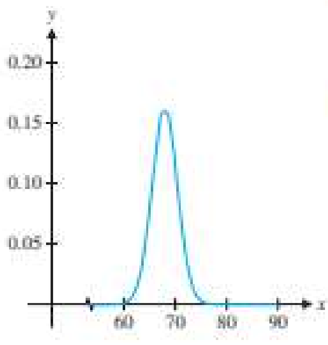


FIGURE 2.62
Heights of adult males

The PDF for the heights of children in a certain country looks like the graph of $f(x) = \dfrac{0.4}{\sqrt{2\pi}}e^{-0.08(x-68)^2}$ shown in Figure 2.62 and used in example 7.2. You probably have seen *bell-shaped curves* like this before. This distribution is referred to as a **normal distribution**. Besides the normal distribution, there are many other probability distributions that are important in applications.

EXAMPLE 7.2 Using a PDF to Estimate Probabilities

Suppose that $f(x) = \dfrac{0.4}{\sqrt{2\pi}}e^{-0.08(x-68)^2}$ is the probability density function for the heights in centimeters of children between the ages of 2 and 4 in a certain country. Find the probability that a randomly selected child will be between 68 cm and 69 cm. Also, find the probability that a randomly selected child will be between 74 and 76 centimeters.

Solution The probability of being between 68 and 69 centimeters tall is

$$P(68 \le X \le 69) = \int_{68}^{69} \frac{0.4}{\sqrt{2\pi}}e^{-0.08(x-68)^2}\,dx \approx 0.15542.$$

Here, we approximated the value of the integral numerically. (You can use Simpson's Rule or the numerical integration method built into your calculator or CAS.) Similarly, the probability of being between 74 and 76 centimeters is

$$P(74 \le X \le 76) = \int_{74}^{76} \frac{0.4}{\sqrt{2\pi}}e^{-0.08(x-68)^2}\,dx \approx 0.00751,$$

where we have again approximated the value of the integral numerically.

EXAMPLE 7.3 Computing Probability with an Exponential PDF

Suppose that the lifetime in years of a certain brand of lightbulb is exponentially distributed with pdf $f(x) = 4e^{-4x}$. Find the probability that a given lightbulb lasts 3 months or less.

Solution First, since the random variable measures lifetime in years, convert 3 months to $\frac{1}{4}$ year. The probability is then

$$P\left(0 \le X \le \frac{1}{4}\right) = \int_0^{1/4} 4e^{-4x}\,dx = 4\left(-\frac{1}{4}\right)e^{-4x}\Big|_0^{1/4}$$
$$= -e^{-1} + e^0 = 1 - e^{-1} \approx 0.63212.$$

In some cases, there may be theoretical reasons for assuming that a PDF has a certain form. In this event, the first task is to determine the values of any constants to achieve the properties of a PDF.

EXAMPLE 7.4 Determining the Coefficient of a PDF

Suppose that the pdf for a random variable has the form $f(x) = ce^{-3x}$ for some constant c, with $0 \le x \le 1$. Find the value of c that makes this a PDF.

Solution To be a PDF, we first need that $f(x) = ce^{-3x} \ge 0$, for all $x \in [0, 1]$. (This will be the case as long as $c \ge 0$.) Also, the integral over the domain must equal 1. So, we set

$$1 = \int_0^1 c\,e^{-3x}\,dx = c\left(-\frac{1}{3}\right)e^{-3x}\Big|_0^1 = \frac{c}{3}(1 - e^{-3}).$$

It now follows that $c = \dfrac{3}{1 - e^{-3}} \approx 3.1572.$

TODAY IN MATHEMATICS

Persi Diaconis (1945–Present) An American statistician who was one of the first recipients of a lucrative MacArthur Foundation Fellowship, often called a "genius grant." Diaconis trained on the violin at Juilliard until age 14, when he left home to become a professional magician for 10 years. His varied interests find expression in his work, where he uses all areas of mathematics and statistics to solve problems from throughout science and engineering. He says, "What makes somebody a good applied mathematician is a balance between finding an interesting real-world problem and finding an interesting real-world problem which relates to beautiful mathematics."[7]

[7]Albers, D. and Alexanderson, G. L. (2008). *Mathematical People: Profiles and Interviews* (Florida: CRC Press).

Given a PDF, it is possible to compute various statistics to summarize the properties of the random variable. The most common statistic is the **mean**, the best-known measure of average value. If you wanted to average test scores of 85, 89, 93 and 93, you would probably compute the mean, given by $\frac{85+89+93+93}{4} = 90$.

Notice here that there were three different test scores recorded: 85, which has a relative frequency of $\frac{1}{4}$, 89, also with a relative frequency of $\frac{1}{4}$ and 93, with a relative frequency of $\frac{2}{4}$. We can also compute the mean by multiplying each value by its relative frequency and then summing: $(85)\frac{1}{4} + (89)\frac{1}{4} + (93)\frac{2}{4} = 90$.

Now, suppose we wanted to compute the mean height of children aged 2–4 in the following table.

Height (cm)	63	64	65	66	67	68	69	70	71	72	73	74
Number of children	23	32	61	94	133	153	155	134	96	62	31	26

It would be silly to write out the heights of all 1000 children, add and divide by 1000. It is much easier to multiply each height by its relative frequency and add the results. Following this route, the mean m is given by

$$m = (63)\frac{23}{1000} + (64)\frac{32}{1000} + (65)\frac{61}{1000} + (66)\frac{94}{1000} + (67)\frac{133}{1000} + \cdots + (74)\frac{26}{1000}$$
$$= 68.523.$$

If we denote the heights by $x_1, x_2, \ldots, x_n$ and let $f(x_i)$ be the relative frequency or probability corresponding to $x = x_i$, the mean then has the form

$$m = x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + \cdots + x_{12} f(x_{12}).$$

If the heights in our data set were given for every half-centimeter, we would compute the mean by multiplying each x_i by the corresponding probability $f(x_i)\Delta x$, where Δx is the fraction of a centimeter between data points. The mean now has the form

$$m = [x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + \cdots + x_n f(x_n)]\,\Delta x = \sum_{i=1}^{n} x_i f(x_i)\Delta x,$$

where n is the number of data points. Notice that, as n increases and Δx approaches 0, the Riemann sum approaches the integral $\int_a^b xf(x)dx$. This gives us the following definition.

DEFINITION 7.2

The **mean** μ of a random variable with pdf $f(x)$ on the interval $[a, b]$ is given by

$$\mu = \int_a^b xf(x)dx. \tag{7.2}$$

Although the mean is commonly used to report the average value of a random variable, it is important to realize that it is not the only measure of *average* used by statisticians. An alternative measurement of average is the **median**, the x-value that divides the probability in half. (That is, half of all values of the random variable lie at or below the median and half lie at or above the median.) In example 7.5 and in the exercises, you will explore situations in which each measure provides a different indication about the average of a random variable.

EXAMPLE 7.5 Finding the Mean Age and Median Age of a Group of Cells

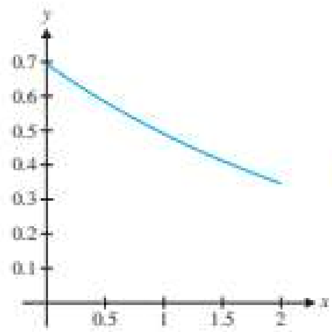


FIGURE 2.63
$y = (\ln 2)e^{-(\ln 2)x/2}$

Suppose that the age in days of a type of single-celled organism has PDF $f(x) = (\ln 2)\,e^{-kx}$, where $k = \frac{1}{2}\ln 2$. The domain is $0 \le x \le 2$. (The assumption here is that upon reaching an age of 2 days, each cell divides into two daughter cells.) Find (a) the mean age of the cells, (b) the proportion of cells that are younger than the mean and (c) the median age of the cells.

Solution For part (a), we have from (7.2) that the mean is given by

$$\mu = \int_0^2 x(\ln 2)e^{-(\ln 2)x/2}\,dx \approx 0.88539 \text{ day},$$

where we have approximated the value of the integral numerically. Notice that even though the cells range in age from 0 to 2 days, the mean is not 1. The graph of the PDF in Figure 2.63 shows that younger ages are more likely than older ages, and this causes the mean to be less than 1.

For part (b), notice that the proportion of cells younger than the mean is the same as the probability that a randomly selected cell is younger than the mean. This probability is given by

$$P(0 \le X \le \mu) \approx \int_0^{0.88539} (\ln 2)e^{-(\ln 2)x/2}\,dx \approx 0.52848,$$

where we have again approximated the value of the integral numerically. Therefore, the proportion of cells younger than the mean is about 53%. Notice that in this case the mean does not represent the 50% mark for probabilities. In other words, the mean is not the same as the median.

To find the median in part (c), we must solve for the constant c such that

$$0.5 = \int_0^c (\ln 2)e^{-(\ln 2)x/2}\,dx.$$

Since an antiderivative of $e^{-(\ln 2)x/2}$ is $-\frac{2}{\ln 2}e^{-(\ln 2)x/2}$, we have

$$\begin{aligned} 0.5 &= 2e^{-(\ln 2)x/2}\,dx \int_0^c \ln \\ &= \ln 2\left[-\frac{2}{\ln 2}e^{-(\ln 2)x/2}\right]_0^c \\ &= -2e^{-(\ln 2)c/2} + 2. \end{aligned}$$

Subtracting 2 from both sides, we have

$$-1.5 = -2e^{-(\ln 2)c/2},$$

so that dividing by -2 yields $$0.75 = e^{-(\ln 2)c/2}.$$

Taking the natural log of both sides gives us

$$\ln 0.75 = -(\ln 2)c/2.$$

Finally, solving for c gives us $$c = \frac{-2\ln 0.75}{\ln 2},$$

so that the median is $-2\ln 0.75/\ln 2 \approx 0.83$. We can now conclude that half of the cells are younger than 0.83 day and half the cells are older than 0.83 day. ■

EXERCISES 2.7

WRITING EXERCISES

1. In the text, we stated that the probability of tossing two fair coins and getting two heads is $\frac{1}{4}$. If you try this experiment four times, explain why you will not always get two heads exactly one out of four times. If probability doesn't give precise predictions, what is its usefulness? To answer this question, discuss the information conveyed by knowing that in the above experiment the probability of getting one head and one tail is $\frac{1}{2}$ (twice as big as $\frac{1}{4}$).
2. Suppose you toss two coins numerous times (or simulate this on your calculator or computer). Theoretically, the probability of getting two heads is $\frac{1}{4}$. In the long run (as the coins are tossed more and more often), what proportion of the time should two heads occur? Try this and discuss how your results compare to the theoretical calculation.
3. Based on Figures 2.56 and 2.57, describe what you expect the histogram to look like for larger numbers of coins. Compare to Figure 2.62.
4. The height of a person is determined by numerous factors, both hereditary and environmental (e.g., diet). Explain why this might produce a histogram similar to that produced by tossing a large number of coins.

In exercises 1–6, show that the given function is a PDF on the indicated interval.

1. $f(x) = 4x^3$, $[0, 1]$
2. $f(x) = \frac{3}{8}x^2$, $[0, 2]$
3. $f(x) = x + 2x^3$, $[0, 1]$
4. $f(x) = \cos x$, $[0, \pi/2]$
5. $f(x) = \frac{1}{2}\sin x$, $[0, \pi]$
6. $f(x) = e^{-x/2}$, $[0, \ln 4]$

In exercises 7–12, find a value of c for which $f(x)$ is a PDF on the indicated interval.

7. $f(x) = cx^3$, $[0, 1]$
8. $f(x) = cx + x^2$, $[0, 1]$
9. $f(x) = ce^{-4x}$, $[0, 1]$
10. $f(x) = 2ce^{-cx}$, $[0, 2]$
11. $f(x) = \frac{c}{1+x^2}$, $[0, 1]$
12. $f(x) = \frac{c}{\sqrt{1-x^2}}$, $[0, 1]$

In exercises 13–16, use the pdf in example 7.2 to find the probability that a randomly selected child has height in the indicated range.

13. Between 70 cm and 72 cm.
14. Between 76 cm and 80 cm.
15. Between 84 cm and 120 cm.
16. Between 70 cm and 120 cm.

In exercises 17–20, find the indicated probabilities, given that the lifetime of a lightbulb is exponentially distributed with PDF $f(x) = 6e^{-6x}$ (with x measured in years).

17. The lightbulb lasts less than 3 months.
18. The lightbulb lasts less than 6 months.
19. The lightbulb lasts between 1 and 2 years.
20. The lightbulb lasts between 3 and 10 years.

In exercises 21–24, suppose the lifetime of an organism has PDF $f(x)\ 4xe^{-2x}$ (with x measured in years).

21. Find the probability that the organism lives less than 1 year.
22. Find the probability that the organism lives between 1 and 2 years.
23. Find the mean lifetime ($0 \le x \le 10$).
24. Graph the pdf and compare the maximum value of the PDF to the mean.

In exercises 25–30, find (a) the mean and (b) the median of the random variable with the given PDF.

25. $f(x) = 3x^2$, $[0, 1]$
26. $f(x) = 4x^3$, $[0, 1]$
27. $f(x) = \frac{4/\pi}{1+x^2}$, $[0, 1]$
28. $f(x) = \frac{2/\pi}{\sqrt{1-x^2}}$, $[0, 1]$
29. $f(x) = \frac{1}{2}\sin x$, $[0, \pi]$
30. $f(x) = \cos x$, $[0, \pi/2]$

31. For $f(x) = ce^{-4x}$, find c so that $f(x)$ is a pdf on the interval $[0, b]$ for $b > 0$. What happens to c as $b \to \infty$?

32. For the pdf of exercise 31 find the mean exactly (use a CAS for the antiderivative). As b increases, what happens to the mean?

33. Repeat exercises 31 and 32 for $f(x) = ce^{-6x}$.

34. Based on the results of exercises 31–33, conjecture the values for c and the mean as $a \to \infty$, for $f(x) = ce^{-ax}$, $a > 0$.

APPLICATIONS

1. In one version of the game of keno, you choose 10 numbers between 1 and 80. A random drawing selects 20 numbers between 1 and 80. Your payoff depends on how many of your numbers are selected. Use the given probabilities (rounded to 4 digits) to find the probability of each event indicated below. (To win, at least 5 of your numbers must be selected. On a \$2 bet, you win \$40 or more if 6 or more of your numbers are selected.)

Number selected	0	1	2	3	4
Probability	0.0458	0.1796	0.2953	0.2674	0.1473

Number selected	5	6	7	8	9	10
Probability	0.0514	0.0115	0.0016	0.0001	0.0	0.0

(a) winning (at least 5 selected)
(b) losing (4 or fewer selected)
(c) winning big (6 or more)
(d) 3 or 4 numbers selected

2. Suppose a basketball player makes 70% of her free throws. If she shoots three free throws and the probability of making each one is 0.7, the probabilities for the total number made are as shown. Find the probability of each event indicated below.

Number made	0	1	2	3
Probability	0.027	0.189	0.441	0.343

(a) She makes 2 or 3
(b) She makes at least 1

3. (a) Suppose that a game player has won m games out of n, with a winning percentage of $100\frac{m}{n} < 75$. The player then wins several games in a row, so that the winning percentage exceeds 75%. Show that at some point in this process the player's winning percentage is *exactly* 75%.

(b) Generalize to any winning percentage that can be written as $100\dfrac{k}{k+1}$, for some integer k.

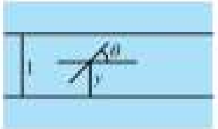
4. In example 7.5, we found the median (also called the second quartile). Now find the first and third quartiles, the ages such that the probability of being younger are 0.25 and 0.75, respectively.

5. The pdf in example 7.2 is the pdf for a normally distributed random variable. The mean is easily read off from $f(x)$; in example 7.2, the mean is 68. The mean and a number called the **standard deviation** characterize normal distributions. As Figure 2.62 indicates, the graph of the pdf has a maximum at the mean and has two inflection points located on opposite sides of the mean. The standard deviation equals the distance from the mean to an inflection point. Find the standard deviation in example 7.2.

6. In exercise 5, you found the standard deviation for the PDF in example 7.2. Denoting the mean as μ and the standard deviation as σ, find the probability that a given height is between $\mu - \sigma$ and $\mu + \sigma$ (that is, within one standard deviation of the mean). Find the probability that a given height is within two standard deviations of the mean ($\mu - 2\sigma$ to $\mu + 2\sigma$) and within three standard deviations of the mean. These probabilities are the same for any normal distribution. So, if you know the mean and standard deviation of a normally distributed random variable, you automatically know these probabilities.

7. If the probability of an event is p, the probability that it will happen m times in n tries is $f(p) = \dfrac{n!}{m!(n-m)!}p^m(1-p)^{n-m}$. Find the value of p that maximizes $f(p)$. This is called the maximum likelihood estimator of p. Briefly explain why your answer makes sense.

8. The **Buffon needle problem** is one of the oldest and most famous of probability problems. Suppose that a series of horizontal lines are spaced one unit apart and a needle of length one is placed randomly. What is the probability that the needle intersects one of the horizontal lines?



In the figure, y is the distance from the center of the needle to the nearest line and θ is the positive angle that the needle makes with the horizontal. Show that the needle intersects the line if and only if $0 \le y \le \frac{1}{2}\sin\theta$. Since $0 \le \theta \le \pi$ and $0 \le y \le \frac{1}{2}$, the desired probability is $\dfrac{\int_0^\pi \frac{1}{2}\sin\theta\, d\theta}{\int_0^\pi \frac{1}{2}\, d\theta}$. Compute this.

9. The Maxwell-Boltzmann pdf for molecular speeds in a gas at equilibrium is $f(x) = ax^2e^{-b^2x^2}$, for positive parameters a and b. Find the most common speed [i.e., find x to maximize $f(x)$].

10. The PDF for inter–spike intervals of neurons firing in the cochlear nucleus of a cat is $f(t) = kt^{-3/2}e^{bt-a/t}$, where $a = 100$, $b = 0.38$ and t is measured in microseconds (see Glass and Mackey, *From Clocks to Chaos*).[7] Use your CAS to find the value of k that makes f a PDF on the interval $[0, 40]$. Then find the probability that neurons fire between 20 and 30 microseconds apart.

11. Suppose that a soccer team has a probability p of scoring the next goal in a game. The probability of a 2-goal game ending in a 1–1 tie is $2p(1-p)$, the probability of a 4-goal game ending in a 2–2 tie is $\dfrac{4\cdot3}{2\cdot1}p^2(1-p)^2$, the probability of a 6-goal game ending in a 3–3 tie is $\dfrac{6\cdot5\cdot4}{3\cdot2\cdot1}p^3(1-p)^3$ and so

[7]Glass, L. and Mackey, M.C. (1988). *From Clocks to Chaos: The Rhythms of Life* (Princeton, NJ: Princeton University Press).

on. Assume that an even number of goals is scored. Show that the probability of a tie is a decreasing function of the number of goals scored.

12. Two players flip a fair coin until either the sequence HTT or HHT occurs. Player A wins if HTT occurs first, and player B wins if HHT occurs first. Show that player B is twice as likely to win.

13. Let f be a function such that both f and g are PDF's on $[0, 1]$, where $g(x) = f(x^2)$.

(a) Find such a function of the form $f(x) = a + bx + cx^2$.

(b) Find the mean of any random variable with PDF g.

EXPLORATORY EXERCISES

1. The mathematical theory of chaos indicates that numbers generated by very simple algorithms can look random. Chaos researchers look at a variety of graphs to try to distinguish randomness from deterministic chaos. For example, iterate the function $f(x) = 4x(1 - x)$ starting at $x = 0.1$. That is, compute $f(0.1) = 0.36$, $f(0.36) = 0.9216$, $f(0.9216) \approx 0.289$ and so on. Iterate 50 times and record how many times each first digit occurs (so far, we've got a 1, a 3, a 9 and a 2). If the process were truly random, the digits would occur about the same number of times. Does this seem to be happening? To unmask this process as nonrandom, you can draw a **phase portrait**. To do this, take consecutive iterates as coordinates of a point (x, y) and plot the points. The first three points are $(0.1, 0.36)$, $(0.36, 0.9216)$ and $(0.9216, 0.289)$. Describe the (nonrandom) pattern that appears, identifying it as precisely as possible.

2. Suppose that a spring is oscillating up and down with vertical position given by $u(t) = \sin t$. If you pick a random time and look at the position of the spring, would you be more likely to find the spring near an extreme ($u = 1$ or $u = -1$) or near the middle ($u = 0$)? The PDF is inversely proportional to speed. (Why is this reasonable?) Show that speed is given by $|\cos t| = \sqrt{1 - u^2}$, so the pdf is $f(u) = c/\sqrt{1 - u^2}$, for some constant c. Show that $c = 1/\pi$, then graph $f(x)$ and describe the positions in which the spring is likely to be found. Use this result to explain the following. If you are driving in a residential neighborhood, you are more likely to meet a car coming the other way at an intersection than in the middle of a block.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Volume by slicing	Volume by disks	Volume by washers
Volume by shells	Arc length	Surface area
Newton's second law	Work	Impulse
Center of mass	Probability density function	Mean

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. The area between f and g is given by $\int_a^b [f(x) - g(x)]\,dx$.

2. The method of disks is a special case of volume by slicing.

3. For a given region, the methods of disks and shells will always use different variables of integration.

4. A Riemann sum for arc length always gives an approximation that is too large.

5. For most functions, the integral for arc length can be evaluated exactly.

6. The only force on a projectile is gravity.

7. For two-dimensional projectile motion, you can always solve for $x(t)$ and $y(t)$ independently.

8. The more you move an object, the more work you have done.

9. The mean of a random variable is always larger than the median.

In exercises 1–8, find the indicated area exactly if possible (estimate if necessary).

1. The area between $y = x^2 + 2$ and $y = \sin x$ for $0 \le x \le \pi$

2. The area between $y = e^x$ and $y = e^{-x}$ for $0 \le x \le 1$

3. The area between $y = x^3$ and $y = 2x^2 - x$

4. The area between $y = x^2 - 3$ and $y = -x^2 + 5$

5. The area between $y = e^{-x}$ and $y = 2 - x^2$

6. The area between $x = y^2$ and $y = 1 - x$

7. The area of the region bounded by $y = x^2$, $y = 2 - x$ and $y = 0$

8. The area of the region bounded by $y = x^2$, $y = 0$ and $x = 2$

9. A town has a population of 10,000 with a birth rate of $10 + 2t$ people per year and a death rate of $4 + t$ people per year. Compute the town's population after 6 years.

Review Exercises

10. From the given data, estimate the area between the curves for $0 \le x \le 2$.

x	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	3.2	3.6	3.8	3.7	3.2	3.4	3.0	2.8	2.4	2.9	3.4
$g(x)$	1.2	1.5	1.6	2.2	2.0	2.4	2.2	2.1	2.3	2.8	2.4

11. Find the volume of the solid with cross-sectional area $A(x) = \pi(3 + x)^2$ for $0 \le x \le 2$.

12. A swimming pool viewed from above has an outline given by $y = \pm(5 + x)$ for $0 \le x \le 2$. The depth is given by $4 + x$ (all measurements in meters). Compute the volume.

13. The cross-sectional areas of an underwater object are given. Estimate the volume.

x	0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2
$A(x)$	0.4	1.4	1.8	2.0	2.1	1.8	1.1	0.4	0

In exercises 14–18, find the volume of the indicated solid of revolution.

14. The region bounded by $y = x^2$, $y = 0$ and $x = 1$ revolved about (a) the x-axis; (b) the y-axis; (c) $x = 2$; (d) $y = -2$

15. The region bounded by $y = x^2$ and $y = 4$ revolved about (a) the x-axis; (b) the y-axis; (c) $x = 2$; (d) $y = -2$

16. The region bounded by $y = x$, $y = 2x$ and $x = 2$ revolved about (a) the x-axis; (b) the y-axis; (c) $x = -1$; (d) $y = 4$

17. The region bounded by $y = x$, $y = 2 - x$ and $y = 0$ revolved about (a) the x-axis; (b) the y-axis; (c) $x = -1$; (d) $y = 4$

18. The region bounded by $x = 4 - y^2$ and $x = y^2 - 4$ revolved about (a) the x-axis; (b) the y-axis; (c) $x = 4$; (d) $y = 4$

In exercises 19–22, set up an integral for the arc length and numerically estimate the integral.

19. The portion of $y = x^4$ for $-1 \le x \le 1$

20. The portion of $y = x^2 + x$ for $-1 \le x \le 0$

21. The portion of $y = e^{x/2}$ for $-2 \le x \le 2$

22. The portion of $y = \sin 2x$ for $0 \le x \le \pi$

In exercises 23 and 24, set up an integral for the surface area and numerically estimate the integral.

23. The surface generated by revolving $y = 1 - x^2$, $0 \le x \le 1$, about the x-axis

24. The surface generated by revolving $y = x^3$, $0 \le x \le 1$, about the x-axis

In exercises 25–32, ignore air resistance.

25. A diver drops from a height of 20 meters. What is the velocity at impact?

26. If the diver in exercise 25 has an initial upward velocity of 4 m/s, what will be the impact velocity?

27. An object is launched from the ground at an angle of 20° with an initial speed of 48 m/s. Find the time of flight and the horizontal range.

28. Repeat exercise 27 for an object launched from a height of 6 meters.

29. A football is thrown from a height of 1.83 m with initial speed 24 m/s at an angle of 8°. A person stands 36 m downfield in the direction of the throw. Is it possible to catch the ball?

30. Repeat exercise 29 with a launch angle of 24°. By trial and error, find the range of angles (rounded to the nearest degree) that produce a catchable throw.

31. Find the initial velocity needed to propel an object to a height of 128 m. Find the object's velocity at impact.

32. A plane at an altitude of 120 m drops supplies to a location on the ground. If the plane has a horizontal velocity of 100 m/s, how far from the target should the supplies be released?

33. A force of 60 N stretches a spring 10 cm. Find the work done to stretch the spring 8 cm beyond its natural length.

34. A car engine exerts a force of $800 + 2x$ N when the car is at position x kilometers. Find the work done as the car moves from $x = 0$ to $x = 8$.

35. Compute the mass and center of mass of an object with density $\rho(x) = x^2 - 2x + 8$ for $0 \le x \le 4$. Explain why the center of mass is not at $x = 2$.

36. Compute the mass and center of mass of an object with density $\rho(x) = x^2 - 2x + 8$ for $0 \le x \le 2$. Explain why the center of mass is at $x = 1$.

37. A dam has the shape of a trapezoid with height 80 m. The width at the top of the dam is 60 m and the width at the bottom of the dam is 140 m. Find the maximum hydrostatic force that the dam will need to withstand.

38. An underwater viewing window is a rectangle with width 20 m extending from 5 m below the surface to 10 m below the surface. Find the maximum hydrostatic force that the window will need to withstand.

39. The force exerted by a bat on a ball over time is shown in the table. Use the data to estimate the impulse. If the ball (mass $m = 0.15$ kg) had speed 8 m/s before the collision, estimate its speed after the collision.

t (s)	0	0.0001	0.0002	0.0003	0.0004
$F(t)$ (N)	0	800	1600	2400	3000

t (s)	0.0005	0.0006	0.0007	0.0008
$F(t)$ (N)	3600	2200	1200	0

Review Exercises

40. If a wall applies a force of $f(t) = 3000t(2 - t)$ N to a car for $0 \le t \le 2$, find the impulse. If the car (mass $m = 1000$ kg) is motionless after the collision, compute its speed before the collision.

41. Show that $f(x) = x + 2x^3$ is a pdf on the interval $[0, 1]$.

42. Show that $f(x) = \frac{8}{3}e^{-2x}$ is a pdf on the interval $[0, \ln 2]$.

43. Find the value of c such that $f(x) = \frac{c}{x^2}$ is a PDF on the interval $[1, 2]$.

44. Find the value of c such that $f(x) = ce^{-2x}$ is a PDF on the interval $[0, 4]$.

45. The lifetime of a lightbulb has PDF $f(x) = 4e^{-4x}$ (x in years). Find the probability that the lightbulb lasts

    (a) less than 6 months;

    (b) between 6 months and 1 year.

46. The lifetime of an organism has PDF $f(x) = 9xe^{-3x}$ (x in years). Find the probability that the organism lasts (a) less than 2 months; (b) between 3 months and 1 year.

47. Find the

    (a) mean and

    (b) median of a random variable with pdf $f(x) = x + 2x^3$ on the interval $[0, 1]$.

48. Find the (a) mean and (b) median of a random variable with pdf $f(x) = \frac{8}{3}e^{-2x}$ on the interval $[0, \ln 2]$.

49. Suppose that a 100-m high voltage power cable weighs 60 N/m. Find the work done in winding 40 m of the cable.

50. An oil tank is in the shape of a right circular cone of height 4 m and radius 2 m, with its vertex on the ground. The tank is filled with motor oil weighing 8950 N/m³. Find the work done in pumping all of the motor oil out the top of the tank.

51. Compute the mass and the center of mass of a 5 m wire with density $\rho(x) = e^{-x}$ *kg/m*.

EXPLORATORY EXERCISES

1. As indicated in section 5.5, general formulas can be derived for many important quantities in projectile motion. For an object launched from the ground at angle θ_0 with initial speed v_0 m/s, find the horizontal range R m and use the trig identity $\sin(2\theta_0) = 2\sin\theta_0\cos\theta_0$ to show that $R = \frac{v_0^2 \sin(2\theta_0)}{32}$. Conclude that the maximum range is achieved with angle $\theta_0 = \pi/4$ (45°).

2. To follow up on exploratory exercise 1, suppose that the ground makes an angle of $A°$ with the horizontal. If $A > 0$ (i.e., the projectile is being launched uphill), explain why the maximum range would be achieved with an angle larger than 45°. If $A < 0$ (launching downhill), explain why the maximum range would be achieved with an angle less than 45°. To determine the exact value of the optimal angle, first argue that the ground can be represented by the line $y = (\tan A)x$. Show that the projectile reaches the ground at time $t = v_0 \frac{\sin\theta_0 - \tan A\cos\theta_0}{16}$. Compute $x(t)$ for this value of t and use a trig identity to replace the quantity $\sin\theta_0\cos A - \sin A\cos\theta_0$ with $\sin(\theta_0 - A)$. Then use another trig identity to replace $\cos\theta_0\sin(\theta_0 - A)$ with $\sin(2\theta_0 - A) - \sin A$. At this stage, the only term involving θ_0 will be $\sin(2\theta_0 - A)$. To maximize the range, maximize this term by taking $\theta_0 = \frac{\pi}{4} + \frac{1}{2}A$.

Chapter

3

Integration Techniques

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Chapter Topics

3.1 Review of Formulas and Techniques

3.2 Integration by Parts

3.3 Trigonometric Techniques of Integration

3.4 Integration of Rational Functions using Partial Fractions

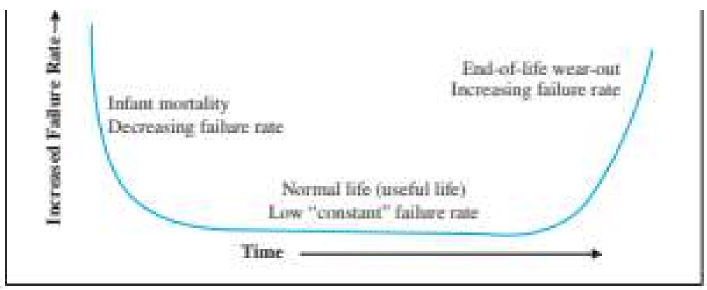
3.5 Integration Tables and Computer Algebra Systems

3.6 Improper Integrals

Electronics companies constantly test their products for reliability. The lifetime of an electronics component is often viewed as having three stages, as illustrated by the so-called *bathtub curve* shown in the figure.

This curve indicates the average failure rate of a product as a function of age. In the first stage (called the **infant mortality phase**), the failure rate drops rapidly as faulty components quickly fail. Components that survive this initial phase enter a lengthy second phase (the **useful life phase**) of constant failure rate. The third phase shows an increase in failure rate as the components reach the physical limit of their lifespan.

The constant failure rate of the useful life phase has several interesting consequences. First, the failures are "memoryless," in the sense that the probability that the component lasts another hour is independent of the age of the component. A component that is 40 hours old may be as likely to last another hour as a component that is only 10 hours old. This unusual property holds for electronics components such as lightbulbs, during the useful life phase.



A constant failure rate also implies that component failures follow what is called an exponential distribution. (See exercises 80 and 81 in the Review Exercises at the end of this chapter.) The computation of statistics for the exponential distribution requires more sophisticated integration techniques than those discussed so far. For instance, the mean (average) lifetime of certain electronics components is given by an integral of the form $\int_0^\infty cxe^{-cx}dx$, for some constant $c > 0$. To evaluate this, we will first need to extend our notion of integral to include *improper integrals* such as this, where one or both of the limits of integration are infinite. We do this in section 6.6. Another challenge is that we do not presently know an antiderivative for $f(x) = xe^{-cx}$. In section 6.2, we introduce a powerful technique called *integration by parts* that can be used to find antiderivatives of many such functions.

The new techniques of integration introduced in this chapter provide us with a broad range of tools used to solve countless problems of interest to engineers, mathematicians and scientists.

3.1 REVIEW OF FORMULAS AND TECHNIQUES

In this brief section, we draw together all of the integration formulas and the one integration technique (integration by substitution) that we have developed so far. We use these to develop some more general formulas, as well as to solve more complicated integration problems. First, look over the following table of the basic integration formulas developed in Chapter 1.

$\int x^r dx = \frac{x^{r+1}}{r+1} + c,$ for $r \neq -1$ (power rule)	$\int \frac{1}{x} dx = \ln\lvert x\rvert + c,$ for $x \neq 0$
$\int \sin kx\, dx = -\frac{1}{k}\cos kx + c$	$\int \cos kx = \frac{1}{k}\sin kx + c$
$\int \sec^2 kx\, dx = \frac{1}{k}\tan kx + c$	$\int \sec kx \cdot \tan kx\, dx = \frac{1}{k}\sec kx + c$
$\int \csc^2 kx\, dx = -\frac{1}{k}\cot kx + c$	$\int \csc kx \cdot \cot kx\, dx = -\frac{1}{k}\csc kx + c$
$\int e^{kx}\, dx = \frac{1}{k}e^{kx} + c$	$\int \cot kx\, dx = \frac{1}{k}\ln\lvert\sin kx\rvert + c$
$\int \tan kx\, dx = -\frac{1}{k}\ln\lvert\cos kx\rvert + c$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}x + c$
$\int \frac{1}{1+x^2} dx = \tan^{-1}x + c$	$\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}} dx = \sec^{-1}x + c$

Recall that each of these follows from a corresponding differentiation rule. So far, we have expanded this list slightly by using the method of substitution, as in example 1.1.

EXAMPLE 1.1 A Simple Substitution

Evaluate $\int \sin(ax)\, dx$, for $a \neq 0$.

Solution The obvious choice here is to let $u = ax$, so that $du = a\, dx$. This gives us

$$\int \sin(ax)\, dx = \frac{1}{a}\int \underbrace{\sin(ax)}_{\sin u}\underbrace{a\, dx}_{du} = \frac{1}{a}\int \sin u\, du$$

$$= -\frac{1}{a}\cos u + c = -\frac{1}{a}\cos(ax) + c.$$

There is no need to memorize general rules like the ones given in examples 1.1 and 1.2, although it is often convenient to do so. You can reproduce such general rules any time you need them using substitution.

EXAMPLE 1.2 Generalizing a Basic Integration Rule

Evaluate $\int \frac{1}{a^2+x^2}\,dx$, for $a \neq 0$.

Solution Notice that this is nearly the same as $\int \frac{1}{1+x^2}\,dx$ and we can write

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx.$$

Now, letting $u = \frac{x}{a}$, we have $du = \frac{1}{a}\,dx$ and so,

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx = \frac{1}{a}\int \underbrace{\frac{1}{1+\left(\frac{x}{a}\right)^2}}_{1+u^2}\underbrace{\left(\frac{1}{a}\right)dx}_{du}$$

$$= \frac{1}{a}\int \frac{1}{1+u^2}\,du = \frac{1}{a}\tan^{-1}u + c = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c.$$

Substitution will not resolve all of your integration difficulties, as we see in example 1.3.

EXAMPLE 1.3 An Integrand That Must Be Expanded

Evaluate $\int (x^2-5)^2\,dx$.

Solution Your first impulse might be to substitute $u = x^2 - 5$. However, this fails, as we don't have $du = 2x\,dx$ in the integral. (We can force the constant 2 into the integral, but we can't get the x in there.) On the other hand, you can always multiply out the binomial to obtain

$$\int (x^2-5)^2\,dx = \int (x^4 - 10x^2 + 25)\,dx = \frac{x^5}{5} - 10\frac{x^3}{3} + 25x + c.$$

The moral of example 1.3 is to make certain you don't overlook simpler methods. The most general rule in integration is to *keep trying*. Sometimes, you will need to do some algebra before you can recognize the form of the integrand.

EXAMPLE 1.4 An Integral Where We Must Complete the Square

Evaluate $\int \frac{1}{\sqrt{-5+6x-x^2}}\,dx$.

Solution Not much may come to mind here. Substitution for either the entire denominator or the quantity under the square root does not work. (Why not?) So, what's left? Recall that there are essentially only two things you can do to a quadratic polynomial: either factor it or complete the square. Here, doing the latter sheds some light on the integral. We have

$$\int \frac{1}{\sqrt{-5+6x-x^2}}\,dx = \int \frac{1}{\sqrt{-5-(x^2-6x+9)+9}}\,dx = \int \frac{1}{\sqrt{4-(x-3)^2}}\,dx.$$

Notice how much this looks like $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$. If we factor out the 4 in the square root, we get

$$\int \frac{1}{\sqrt{-5+6x-x^2}}\,dx = \int \frac{1}{\sqrt{4-(x-3)^2}}\,dx = \int \frac{1}{\sqrt{1-\left(\frac{x-3}{2}\right)^2}}\frac{1}{2}\,dx.$$

Taking $u = \frac{x-3}{2}$, we have $du = \frac{1}{2}dx$ and so,

$$\int \frac{1}{\sqrt{-5+6x-x^2}}dx = \int \underbrace{\frac{1}{\sqrt{1-\left(\frac{x-3}{2}\right)^2}}}_{\sqrt{1-u^2}} \underbrace{\frac{1}{2}dx}_{du} = \int \frac{1}{\sqrt{1-u^2}}du$$

$$= \sin^{-1} u + c = \sin^{-1}\left(\frac{x-3}{2}\right) + c.$$

Example 1.5 illustrates the value of perseverance.

EXAMPLE 1.5 An Integral Requiring Some Imagination

Evaluate $\int \frac{4x+1}{2x^2+4x+10}dx$.

Solution As with most integrals, you cannot evaluate this as it stands. However, the numerator is very nearly the derivative of the denominator (but not quite). Recognize that you can complete the square in the denominator, to obtain

$$\int \frac{4x+1}{2x^2+4x+10}dx = \int \frac{4x+1}{2(x^2+2x+1)-2+10}dx = \int \frac{4x+1}{2(x+1)^2+8}dx.$$

Now, the denominator nearly looks like the denominator in $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$. Factoring out an 8 from the denominator, we have

$$\int \frac{4x+1}{2x^2+4x+10}dx = \int \frac{4x+1}{2(x+1)^2+8}dx$$

$$= \frac{1}{8}\int \frac{4x+1}{\frac{1}{4}(x+1)^2+1}dx$$

$$= \frac{1}{8}\int \frac{4x+1}{\left(\frac{x+1}{2}\right)^2+1}dx.$$

Now, taking $u = \frac{x+1}{2}$, we have $du = \frac{1}{2}dx$ and $x = 2u - 1$ and so,

$$\int \frac{4x+1}{2x^2+4x+10}dx = \frac{1}{8}\int \frac{4x+1}{\left(\frac{x+1}{2}\right)^2+1}dx = \frac{1}{4}\int \frac{\overbrace{4x+1}^{4(2u-1)+1}}{\underbrace{\left(\frac{x+1}{2}\right)^2+1}_{u^2+1}} \underbrace{\frac{1}{2}dx}_{du}$$

$$= \frac{1}{4}\int \frac{4(2u-1)+1}{u^2+1}du = \frac{1}{4}\int \frac{8u-3}{u^2+1}du$$

$$= \frac{4}{4}\int \frac{2u}{u^2+1}du - \frac{3}{4}\int \frac{1}{u^2+1}du$$

$$= \ln(u^2+1) - \frac{3}{4}\tan^{-1}u + c$$

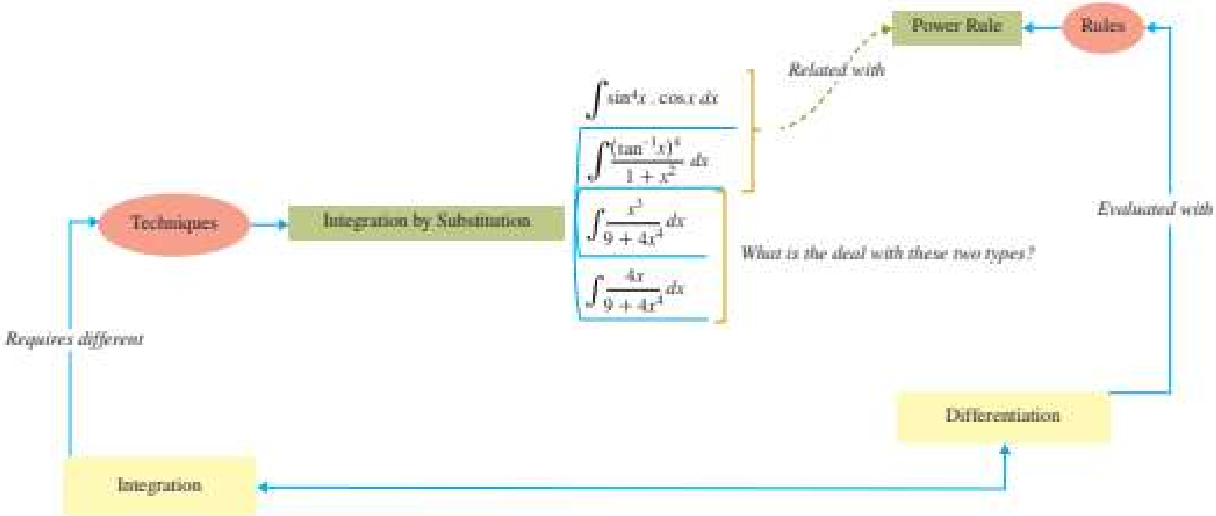
$$= \ln\left[\left(\frac{x+1}{2}\right)^2+1\right] - \frac{3}{4}\tan^{-1}\left(\frac{x+1}{2}\right) + c.$$

Example 1.5 was tedious, but reasonably straightforward. The issue in integration is to recognize what pieces are present in a given integral and to see how you might rewrite the integral in a more familiar form.

Concept Mapping

The concept map that you started earlier in differentiation should provide a solid baseline that you can easily build on as you learn the different integration techniques in this chapter. Your success in subsequent courses that require the use of the latter concepts rests on your ability to comprehend, appreciate and extend the links that you have established so far. By the time you finish this useful exercise, you will have a good overview of these two powerful concepts.

In this map, there are two areas that we would like you to work on. The first is to try to establish a connection between the different integration techniques that you will be learning and the differentiation rules (whenever possible). In the second, you will provide a summary of the various integration techniques and reveal the connections in between. Give examples to support your judgment and include those integrals that you find most challenging. By the time you finish these two exercises, you will have acquired a skill that you can utilize beyond your course of study.



EXERCISES 3.1

WRITING EXERCISES

1. In example 1.2, explain how you should know to write the denominator as $a^2\left[1+\left(\frac{x}{a}\right)^2\right]$. Would this still be a good first step if the numerator were x instead of 1? What would you do if the denominator were $\sqrt{a^2-x^2}$?

2. In both examples 1.4 and 1.5, we completed the square and found antiderivatives involving $\sin^{-1} x$, $\tan^{-1} x$ and $\ln(x^2+1)$. Briefly describe how the presence of an x in the numerator or a square root in the denominator affects which of these functions will be in the antiderivative.

In exercises 1–40, evaluate the integral.

1. $\int e^{ax}\, dx,\ a \neq 0$

2. $\int \cos(ax)\, dx,\ a \neq 0$

3. $\int \frac{1}{\sqrt{a^2-x^2}}\, dx,\ a > 0$

4. $\int \frac{1}{|x|\sqrt{x^2-a^2}}\, dx,\ a > 0$

5. $\int \sin 6t\, dt$

6. $\int \sec 2t \tan 2t\, dt$

7. $\int (x^2+4)^2\, dx$

8. $\int x(x^2+4)^2\, dx$

9. $\int \frac{3}{16+x^2}\, dx$

10. $\int \frac{2}{4+4x^2}\, dx$

11. $\int \frac{1}{\sqrt{3-2x-x^2}}\, dx$

12. $\int \frac{x+1}{\sqrt{3-2x-x^2}}\, dx$

13. $\int \frac{4}{5+2x+x^2}\, dx$

14. $\int \frac{4x+4}{5+2x+x^2}\, dx$

15. $\int \frac{4t}{5+2t+t^2}\, dt$

16. $\int \frac{t+1}{t^2+2t+4}\, dt$

17. $\int e^{3-2x}\, dx$

18. $\int \frac{3}{e^{6x}}\, dx$

19. $\int \frac{4}{x^{1/3}(1+x^{2/3})}\, dx$

20. $\int \frac{2}{x^{1/4}+x}\, dx$

21. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx$

22. $\int \frac{\cos(1/x)}{x^2}\, dx$

23. $\int_0^{\pi} \cos x\, e^{\sin x}\, dx$

24. $\int_0^{\pi/4} \sec^2 x\, e^{\tan x}\, dx$

25. $\int_{-\pi/4}^{0} \frac{\sin t}{\cos^2 t}\,dt$

26. $\int_{\pi/4}^{\pi/2} \frac{1}{\sin^2 t}\,dt$

27. $\int \frac{x^2}{1+x^6}\,dx$

28. $\int \frac{x^5}{1+x^6}\,dx$

29. $\int \frac{1}{\sqrt{4-x^2}}\,dx$

30. $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

31. $\int \frac{x}{\sqrt{1-x^4}}\,dx$

32. $\int \frac{2x^3}{\sqrt{1-x^4}}\,dx$

33. $\int \frac{1+x}{1+x^2}\,dx$

34. $\int \frac{1}{\sqrt{x}+x}\,dx$

35. $\int_{-2}^{-1} \frac{\ln x^2}{x}\,dx$

36. $\int_{1}^{3} e^{2\ln x}\,dx$

37. $\int_{3}^{4} x\sqrt{x-3}\,dx$

38. $\int_{0}^{1} x(x-3)^2\,dx$

39. $\int_{1}^{4} \frac{x^2+1}{\sqrt{x}}\,dx$

40. $\int_{-2}^{0} xe^{-x^2}\,dx$

In exercises 41–46, you are given a pair of integrals. Evaluate the integral that can be worked using the techniques covered so far (the other cannot).

41. $\int \frac{5}{3+x^2}\,dx$ and $\int \frac{5}{3+x^3}\,dx$

42. $\int \sin 3x\,dx$ and $\int \sin^3 x\,dx$

43. $\int \ln x\,dx$ and $\int \frac{\ln x}{2x}\,dx$

44. $\int \frac{x^3}{1+x^8}\,dx$ and $\int \frac{x^4}{1+x^8}\,dx$

45. $\int e^{-x^2}\,dx$ and $\int xe^{-x^2}\,dx$

46. $\int \sec x\,dx$ and $\int \sec^2 x\,dx$

47. Find $\int_0^2 f(x)\,dx$, where $f(x) = \begin{cases} x/(x^2+1) & \text{if } x \le 1 \\ x^2/(x^2+1) & \text{if } x > 1 \end{cases}$

48. Rework example 1.5 by rewriting the integral as $\int \frac{4x+4}{2x^2+4x+10}\,dx - \int \frac{3}{2x^2+4x+10}\,dx$ and completing the square in the second integral.

49. Find $\int \frac{1}{1+x^2}\,dx$, $\int \frac{x}{1+x^2}\,dx$, $\int \frac{x^2}{1+x^2}\,dx$ and $\int \frac{x^3}{1+x^2}\,dx$. Generalize to give the form of $\int \frac{x^n}{1+x^2}\,dx$ for any positive integer n, as completely as you can.

50. Find $\int \frac{x}{1+x^4}\,dx$, $\int \frac{x^3}{1+x^4}\,dx$ and $\int \frac{x^5}{1+x^4}\,dx$. Generalize to give the form of $\int \frac{x^n}{1+x^4}\,dx$ for any odd positive integer n, as completely as you can.

EXPLORATORY EXERCISES

1. Find $\int xe^{-x^2}\,dx$, $\int x^3e^{-x^2}\,dx$ and $\int x^5e^{-x^2}\,dx$. Generalize to give the form of $\int x^ne^{-x^2}\,dx$ for any odd positive integer n.

2. In many situations, the integral as we've defined it must be extended to the **Riemann–Stieltjes integral** considered in this exercise. For functions f and g, let P be a regular partition of $[a, b]$ with evaluation points $c_i \in [x_{i-1}, x_i]$ and define the sums $R(f, g, P) = \sum_{i=1}^{n} f(c_i)[g(x_i) - g(x_{i-1})]$. The integral $\int_a^b f(x)dg(x)$ equals the limit of the sums $R(f, g, P)$ as $n \to \infty$, if the limit exists and equals the same number for all evaluation points c_i. (a) Show that if g' exists, then $\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)\,dx$. (b) If $g(x) = \begin{cases} 1 & a \le x \le d \\ 2 & d < x \le b \end{cases}$ for some constant d with $a < d < b$, evaluate $\int_a^b f(x)dg(x)$. (c) Find a function $g(x)$ such that $\int_0^1 \frac{1}{x}\,dg(x)$ exists.

3.2 INTEGRATION BY PARTS

At this point, you will have recognized that there are many integrals that cannot be evaluated using our basic formulas or integration by substitution. For instance,

$$\int x \sin x\,dx$$

cannot be evaluated with what you presently know.

We have observed that every differentiation rule gives rise to a corresponding integration rule. So, for the product rule:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x),$$

integrating both sides of this equation gives us

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int f'(x)g(x)dx + \int f(x)g'(x)\,dx.$$

HISTORICAL NOTES

Brook Taylor (1685–1731) An English mathematician who is credited with devising integration by parts. Taylor made important contributions to probability, the theory of magnetism and the use of vanishing lines in linear perspective. However, he is best known for Taylor's Theorem (see section 8.7), in which he generalized results of Newton, Halley, the Bernoullis and others. Personal tragedy (both his wives died during childbirth) and poor health limited the mathematical output of this brilliant mathematician.

Ignoring the constant of integration, the integral on the left-hand side is simply $f(x)\,g(x)$. Solving for the second integral on the right-hand side then yields

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

This rule is called **integration by parts**. In short, this new rule lets us replace a given integral with an easier one. We'll let the examples convince you of the power of this technique. First, it's usually convenient to write this using the notation $u = f(x)$ and $v = g(x)$. Then,

$$du = f'(x)\,dx \quad \text{and} \quad dv = g'(x)dx,$$

so that the integration by parts algorithm becomes

INTEGRATION BY PARTS

$$\int u\,dv = uv - \int v\,du. \tag{2.1}$$

To apply integration by parts, you need to make a judicious choice of u and dv so that the integral on the right-hand side of (2.1) is one that you know how to evaluate.

EXAMPLE 2.1 Integration by Parts

Evaluate $\int x \sin x\,dx$.

Solution First, observe that this is not one of our basic integrals and there's no obvious substitution that will help. To use integration by parts, you will need to choose u (something to differentiate) and dv (something to integrate). If we let

$$u = x \quad \text{and} \quad dv = \sin x\,dx,$$

then $du = dx$ and integrating dv, we have

$$v = \int \sin x\,dx = -\cos x + k.$$

In performing integration by parts, we drop this constant of integration. (Think about why it makes sense to do this.) Also, we usually write this information as the block

$$\begin{array}{ll} u = x & dv = \sin x\,dx \\ du = dx & v = -\cos x \end{array}$$

This gives us

$$\begin{aligned} \int \underbrace{x}_{u}\ \underbrace{\sin x\,dx}_{dv} &= \int u\,dv = uv - \int v\,du \\ &= -x\cos x - \int (-\cos x)\,dx \\ &= -x\cos x + \sin x + c. \end{aligned} \tag{2.2}$$

It's a simple matter to differentiate the expression on the right-hand side of (2.2) and verify directly that you have indeed found an antiderivative of $x \sin x$. ■

You should quickly realize that the choice of u and dv is critical. Observe what happens if we switch the choice of u and dv made in example 2.1.

EXAMPLE 2.2 A Poor Choice of u and dv

Consider $\int x \sin x\,dx$ as in example 2.1, but this time, reverse the choice of u and dv.

Solution Here, we let

$$\begin{array}{ll} u = \sin x & dv = x\,dx \\ du = \cos x\,dx & v = \frac{1}{2}x^2 \end{array}$$

This gives us $\int \underbrace{(\sin x)}_{u} \underbrace{x\,dx}_{dv} = uv - \int v\,du = \frac{1}{2}x^2 \sin x - \frac{1}{2}\int x^2 \cos x\,dx.$

Notice that the last integral is one that we do *not* know how to calculate any better than the original one. In fact, we have made the situation worse in that the power of x in the new integral is higher than in the original integral. ■

REMARK 2.1

When using integration by parts, keep in mind that you are splitting up the integrand into two pieces. One of these pieces, corresponding to u, will be differentiated and the other, corresponding to dv, will be integrated. Since you can differentiate virtually every function you run across, you should choose a dv for which you know an antiderivative and make a choice of both that will result in an easier integral. If possible, a choice of $u = x$ results in the simple $du = dx$. You will learn what works best by working through lots of problems. Even if you don't see how the problem is going to end up, *try something!*

EXAMPLE 2.3 An Integrand with a Single Term

Evaluate $\int \ln x\,dx$.

Solution This may look like it should be simple, but it's not one of our basic integrals and there's no obvious substitution that will simplify it. That leaves us with integration by parts. Remember that you must pick u (to be differentiated) and dv (to be integrated). You obviously can't pick $dv = \ln x\,dx$, since the problem here is to find a way to integrate this very term. So, try

$$u = \ln x \qquad dv = dx$$
$$du = \frac{1}{x}dx \qquad v = x$$

Integration by parts now gives us

$$\int \underbrace{\ln x}_{u} \underbrace{dx}_{dv} = uv - \int v\,du = x \ln x - \int x\left(\frac{1}{x}\right)dx$$
$$= x \ln x - \int 1\,dx = x \ln x - x + c. \ ■$$

Frequently, an integration by parts results in an integral that we cannot evaluate directly, but instead, one that we can evaluate only by repeating integration by parts one or more times.

EXAMPLE 2.4 Repeated Integration by Parts

Evaluate $\int x^2 \sin x\,dx$.

Solution Certainly, you cannot evaluate this as it stands and there is no simplification or obvious substitution that will help. We choose

$$u = x^2 \qquad dv = \sin x\,dx$$
$$du = 2x\,dx \qquad v = -\cos x$$

With this choice, integration by parts yields

$$\int \underbrace{x^2}_{u} \underbrace{\sin x\,dx}_{dv} = -x^2 \cos x + 2\int x \cos x\,dx.$$

Of course, this last integral cannot be evaluated as it stands, but we could do it using a further integration by parts. We now choose

$$u = x \qquad dv = \cos x\,dx$$
$$du = dx \qquad v = \sin x$$

Applying integration by parts to the last integral, we now have

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2\int \underbrace{x}_{u} \underbrace{\cos x\,dx}_{dv}$$
$$= -x^2 \cos x + 2\left(x \sin x - \int \sin x\,dx\right)$$
$$= -x^2 \cos x + 2x \sin x + 2 \cos x + c. \ ■$$

REMARK 2.2

In the second integration by parts in example 2.4, if you choose $u = \cos x$ and $dv = x\,dx$, then integration by parts will leave you with the less than astounding conclusion that the integral that you started with equals itself. (Try this as an exercise.)

Based on our work in example 2.4, try to figure out how many integrations by parts would be required to evaluate $\int x^n \sin x\,dx$, for a positive integer n. (There will be more on this, including a shortcut, in the exercises.)

Repeated integration by parts sometimes takes you back to the integral you started with. This can be bad news (see Remark 2.2), or this can give us a clever way of evaluating an integral, as in example 2.5.

EXAMPLE 2.5 Repeated Integration by Parts with a Twist

Evaluate $\int e^{2x} \sin x\,dx$.

Solution None of our elementary methods works on this integral. For integration by parts, there are two viable choices for u and dv. We take

$$\begin{aligned} u &= e^{2x} & dv &= \sin x\,dx \\ du &= 2e^{2x}\,dx & v &= -\cos x \end{aligned}$$

(The opposite choice also works. Try this as an exercise.) Integration by parts yields

$$\int \underbrace{e^{2x}}_{u} \underbrace{\sin x\,dx}_{dv} = -e^{2x} \cos x + 2\int e^{2x} \cos x\,dx.$$

The remaining integral again requires integration by parts. We choose

$$\begin{aligned} u &= e^{2x} & dv &= \cos x\,dx \\ du &= 2e^{2x}\,dx & v &= \sin x \end{aligned}$$

It now follows that

$$\begin{aligned} \int e^{2x} \sin x\,dx &= -e^{2x} \cos x + 2\int \underbrace{e^{2x}}_{u} \underbrace{\cos x\,dx}_{dv} \\ &= -e^{2x} \cos x + 2\left(e^{2x} \sin x - 2\int e^{2x} \sin x\,dx\right) \\ &= -e^{2x} \cos x + 2e^{2x} \sin x - 4\int e^{2x} \sin x\,dx. \end{aligned} \tag{2.3}$$

Observe that the last line includes the integral that we started with. Treating the integral $\int e^{2x} \sin x\,dx$ as the unknown, we can add $4\int e^{2x} \sin x\,dx$ to both sides of equation (2.3), leaving

$$5\int e^{2x} \sin x\,dx = -e^{2x} \cos x + 2e^{2x} \sin x + K,$$

where we have added the constant of integration K on the right side. Dividing both sides by 5 then gives us

$$\int e^{2x} \sin x\,dx = -\frac{1}{5}e^{2x} \cos x + \frac{2}{5}e^{2x} \sin x + c,$$

where we have replaced the arbitrary constant of integration $\frac{K}{5}$ by c. ■

REMARK 2.3

For integrals like $\int e^{2x} \sin x\,dx$ (or related integrals like $\int e^{-3x} \cos 2x\,dx$), repeated integration by parts as in example 2.5 will produce an antiderivative. The first choice of u and dv is up to you (either choice will work) but your choice of u and dv in the second integration by parts must be consistent with your first choice. For instance, in example 2.5, our initial choice of $u = e^{2x}$ commits us to using $u = e^{2x}$ for the second integration by parts, as well. To see why, rework the second integral taking $u = \cos x$ and observe what happens!

Observe that for any positive integer n, the integral $\int x^n e^x\,dx$ will require integration by parts. At this point, it should be no surprise that we take

$$\begin{aligned} u &= x^n & dv &= e^x\,dx \\ du &= nx^{n-1}\,dx & v &= e^x \end{aligned}$$

Applying integration by parts gives us

$$\int \underbrace{x^n}_{u} \underbrace{e^x dx}_{dv} = x^n e^x - n \int x^{n-1} e^x \, dx. \tag{2.4}$$

Notice that if $n - 1 > 0$, we will need to perform another integration by parts. In fact, we'll need to perform a total of n integrations by parts to complete the process. An alternative is to apply formula (2.4) (called a **Tabular Integration**).

The movie *Stand and Deliver* tells the story of mathematics teacher Jaime Escalante, who developed a remarkable AP calculus program in inner-city Los Angeles. In one scene, Escalante shows a student how to evaluate the integral $\int x^3 e^{2x} \, dx$ by forming a table like the one we see in the examples below. Of course, the latter integral requires repeated integration by parts (twice in this case) to be evaluated, however tedious that process might seem. If you have a product of a polynomial that can be differentiated repeatedly to become zero and another function that is easy to integrate repeatedly. In this case, a tabular integration is a faster method to evaluate an integral as illustrated in the following examples.

EXAMPLE 2.6 Using a Tabular Integration

Evaluate the integral $\int x^3 e^{2x} \, dx$.

Solution *Let* $u = x^3$ *and* $dv = e^{2x}$

Differentiate repeatedly	*Integrate repeatedly*	$\int x^3 e^2 dx$
$+ \; x^3$	e^{2x}	
$- \; 3x^2$	$\frac{1}{2}e^{2x}$	$+\frac{1}{2}x^3 e^{2x}$
$+ \; 6x$	$\frac{1}{4}e^{2x}$	$-\frac{3}{4}x^2 e^{2x}$
$- \; 6$	$\frac{1}{8}e^{2x}$	$+\frac{3}{4}xe^{2x}$
0	$\frac{1}{16}e^{2x}$	$-\frac{3}{8}e^{2x}$

$$\int x^3 e^{2x} dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}x e^{2x} - \frac{3}{8}e^{2x} + c$$

where we leave the details of the remaining calculations to you.

Note that to evaluate a definite integral, it is always possible to apply integration by parts to the corresponding indefinite integral and then simply evaluate the resulting antiderivative between the limits of integration. Whenever possible, however (i.e., when the integration is not too involved), you should apply integration by parts directly to the definite integral. Observe that the integration by parts algorithm for definite integrals is simply

Integration by parts for a definite integral

$$\boxed{\int_{x=a}^{x=b} u \, dv = uv \Big|_{x=a}^{x=b} - du, \int_{x=a}^{x=b} v}$$

where we have written the limits of integration as we have to remind you that these refer to the values of x. (Recall that we derived the integration by parts formula by taking u and v both to be functions of x.) ■

EXAMPLE 2.7 Using a Tabular Integration

Evaluate $\int x^2 \sin x \, dx$.

Solution

Differentiate repeatedly	*Integrate repeatedly*	$\int x^2 \sin x \, dx$
$+\ x^2$	$\sin x$	
$-\ 2x$	$-\cos x$	$-x^2 \cos x$
$+\ 2$	$-\sin x$	$+2x \sin x$
0	$\cos x$	$+2 \cos x$

Integration by parts is the most powerful tool in our integration arsenal. In order to master its use, you will need to work through many problems. We provide a wide assortment of these in the exercise set that follows. ■

EXERCISES 3.2

WRITING EXERCISES

1. Discuss your best strategy for determining which part of the integrand should be u and which part should be dv.

2. Integration by parts comes from the product rule for derivatives. Derive an integration technique that comes from the quotient rule. Briefly discuss why it would not be a very useful rule.

In exercises 1–28, evaluate the integrals.

1. $\int x \cos x \, dx$
2. $\int x \sin 4x \, dx$
3. $\int xe^{2x} \, dx$
4. $\int x \ln x \, dx$
5. $\int x^2 \ln x \, dx$
6. $\int \frac{\ln x}{x} \, dx$
7. $\int x^2 e^{-3x} dx$
8. $\int x^2 e^{x^3} \, dx$
9. $\int e^x \sin 4x \, dx$
10. $\int e^{2x} \cos x \, dx$
11. $\int \cos x \cos 2x \, dx$
12. $\int \sin x \sin 2x \, dx$
13. $\int x \sec^2 x \, dx$
14. $\int (\ln x)^2 \, dx$
15. $\int x^3 e^{x^2} \, dx$
16. $\int \frac{x^3}{(4+x^2)^{3/2}} dx$
17. $\int \cos x \ln(\sin x) \, dx$
18. $\int x \sin x^2 \, dx$
19. $\int_0^1 x \sin 2x \, dx$
20. $\int_0^\pi 2x \cos x \, dx$
21. $\int_0^1 x^2 \cos \pi x \, dx$
22. $\int_0^1 x^2 e^{3x} \, dx$
23. $\int_1^{10} \ln 2x \, dx$
24. $\int_1^2 x \ln x \, dx$
25. $\int e^{ax} x^2 \, dx, a \neq 0$
26. $\int x \sin(ax) \, dx, a \neq 0$
27. $\int x^n \ln x \, dx, n \neq -1$
28. $\int \sin(ax) \cos(bx) \, dx, a \neq 0, 0, b \neq 0$

29. Several useful integration formulas (called *reduction formulas*) are used to automate the process of performing multiple integrations by parts. Prove that for any positive integer n,

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

(Use integration by parts with $u = \cos^{n-1} x$ and $dv = \cos x \, dx$.)

30. Use integration by parts to prove that for any positive integer n,

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

In exercises 31–38, evaluate the integral using the reduction formulas from exercises 29 and 30 and (2.4).

31. $\int x^3 e^x \, dx$
32. $\int \cos^5 x \, dx$
33. $\int \cos^3 x \, dx$
34. $\int \sin^4 x \, dx$
35. $\int_0^1 x^4 e^x \, dx$
36. $\int_0^{\pi/2} \sin^4 x \, dx$
37. $\int_0^{\pi/2} \sin^5 x \, dx$
38. $\int_0^{\pi/2} \sin^6 x \, dx$

39. Based on exercises 36–38 and similar integrals, conjecture a formula for $\int_0^{\pi/2} \sin^m x \, dx$. (Note: You will need different formulas for m odd and for m even.)

40. Conjecture a formula for $\int_0^{\pi/2} \cos^m x \, dx$.

In exercises 41–50, evaluate the integral using integration by parts and substitution. (As we recommended in the text, "Try something!")

41. $\int \cos^{-1} x \, dx$
42. $\int \tan^{-1} x \, dx$
43. $\int \sin \sqrt{x} \, dx$
44. $\int e^{\sqrt{x}} \, dx$
45. $\int \sin(\ln x) dx$
46. $\int x \ln(4 + x^2) dx$

47. $\int e^{6x}\sin(e^{2x})\,dx$

48. $\int \cos\sqrt[3]{x}\,dx$

49. $\int_0^8 e^{\sqrt[3]{x}}\,dx$

50. $\int_0^1 x\tan^{-1}x\,dx$

51. How many times would integration by parts need to be performed to evaluate $\int x^n \sin x\,dx$ (where n is a positive integer)?

52. How many times would integration by parts need to be performed to evaluate $\int x^n \ln x\,dx$ (where n is a positive integer)?

In exercises 53 and 54, name the method by identifying whether substitution or integration by parts can be used to evaluate the integral.

53. (a) $\int x\sin x^2\,dx$ (b) $\int x^2\sin x\,dx$

(c) $\int x\ln x\,dx$ (d) $\int \frac{\ln x}{x}\,dx$

54. (a) $\int x^3e^{4x}\,dx$ (b) $\int x^3e^{x^2}\,dx$

(c) $\int x^{-2}e^{4/x}\,dx$ (d) $\int x^2e^{-4x}\,dx$

In exercises 55–60, use the tabular method to evaluate the integral.

55. $\int x^4\sin 2x\,dx$

56. $\int (5x^2-1)\cos \pi x\,dx$

57. $\int x^2e^{\frac{1}{2}x}\,dx$

58. $\int x^4e^{2x}\,dx$

59. $\int x^5\cos 2x\,tdx$

60. $\int x^3e^{-3x}\,dx$

61. Show that $\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx = 0$ and $\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx = 0$ for positive integers $m \neq n$.

62. Show that $\int_{-\pi}^{\pi}\cos(mx)\sin(nx)\,dx = 0$ for positive integers m and n and $\int_{-\pi}^{\pi}\cos^2(nx)\,dx = \int_{-\pi}^{\pi}\sin^2(nx)\,dx = \pi$, for any positive integer n.

63. Find all mistakes in the following (invalid) attempted proof that $0 = -1$. Start with $\int e^xe^{-x}\,dx$ and apply integration by parts with $u = e^x$ and $dv = e^{-x}dx$. This gives $\int e^xe^{-x}\,dx = -1 + \int e^xe^{-x}\,dx$. Then subtract $\int e^xe^{-x}\,dx$ to get $0 = -1$.

64. Find the volume of the solid formed by revolving the region bounded by $y = x\sqrt{\sin x}$ and $y = 0$ $(0 \le x \le \pi)$ about the x-axis.

65. Evaluate $\int e^x\left(\ln x + \frac{1}{x}\right)dx$ by using integration by parts on $\int e^x\ln x\,dx$.

66. Generalize the technique of exercise 65 to any integral of the form $\int e^x[f(x) + f'(x)]dx$. Prove your result without using integration by parts.

67. Suppose that f and g are functions with $f(0) = g(0) = 0$, $f(1) = g(1) = 0$ and with continuous second derivatives f'' and g''. Use integration by parts twice to show that

$$\int_0^1 f''(x)g(x)dx = \int_0^1 f(x)g''(x)dx.$$

68. Assume that f is a function with a continuous second derivative. Show that $f(b) = f(a) + f'(a)(b-a) + \int_a^b f''(x)(b-x)dx$. Use this result to show that $|\sin b - b| = |\int_0^b (b-x)\sin x\,dx|$ and conclude that the error in the approximation $\sin x \approx x$ is at most $\frac{1}{2}x^2$.

EXPLORATORY EXERCISES

1. Integration by parts can be used to compute coefficients for important functions called **Fourier series**. Here, you will discover what some of the fuss is about. Start by computing $a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx$ for an unspecified positive integer n. Write out the specific values for a_1, a_2, a_3 and a_4 and then form the function

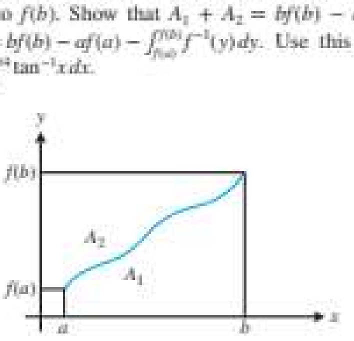
$$f(x) = a_1\sin x + a_2\sin 2x + a_3\sin 3x + a_4\sin 4x.$$

Compare the graphs of $y = x$ and $y = f(x)$ on the interval $[-\pi, \pi]$. From writing out a_1 through a_4, you should notice a nice pattern. Use it to form the function

$$g(x) = f(x) + a_5\sin 5x + a_6\sin 6x + a_7\sin 7x + a_8\sin 8x.$$

Compare the graphs of $y = x$ and $y = g(x)$ on the interval $[-\pi, \pi]$. Is it surprising that you can add sine functions together and get something close to a straight line? It turns out that Fourier series can be used to find cosine and sine approximations to any continuous function on a closed interval.

2. Assume that f is an increasing continuous function on $[a, b]$ with $0 \le a < b$ and $f(x) \ge 0$. Let A_1 be the area under $y = f(x)$ from $x = a$ to $x = b$ and let A_2 be the area to the left of $y = f(x)$ from $f(a)$ to $f(b)$. Show that $A_1 + A_2 = bf(b) - af(a)$ and $\int_a^b f(x)dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y)dy$. Use this result to evaluate $\int_0^{\pi/4}\tan^{-1}x\,dx$.



3.3 TRIGONOMETRIC TECHNIQUES OF INTEGRATION

○ Integrals Involving Powers of Trigonometric Functions

Evaluating an integral whose integrand contains powers of one or more trigonometric functions often involves making a clever substitution. These integrals are sufficiently common that we present them here as a group.

We first consider integrals of the form

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are positive integers.

Case 1: *m* or *n* is an Odd Positive Integer

If m is odd, first isolate one factor of $\sin x$. (You'll need this for du.) Then, replace any factors of $\sin^2 x$ with $1 - \cos^2 x$ and make the substitution $u = \cos x$. Likewise, if n is odd, first isolate one factor of $\cos x$. (You'll need this for du.) Then, replace any factors of $\cos^2 x$ with $1 - \sin^2 x$ and make the substitution $u = \sin x$.

We illustrate this for the case where m is odd in example 3.1.

EXAMPLE 3.1 A Typical Substitution

Evaluate $\int \cos^4 x \sin x \, dx$.

Solution Since you cannot evaluate this integral as it stands, you should consider substitution. (Hint: Look for terms that are derivatives of other terms.) Here, letting $u = \cos x$, so that $du = -\sin x \, dx$, gives us

$$\int \cos^4 x \sin x \, dx = -\int \underbrace{\cos^4 x}_{u^4} \underbrace{(-\sin x)\, dx}_{du} = -\int u^4 du$$

$$= -\frac{u^5}{5} + c = -\frac{\cos^5 x}{5} + c. \quad \text{Since } u = \cos x.$$

While this first example was not particularly challenging, it should give you an idea of what to do with example 3.2.

EXAMPLE 3.2 An Integrand with an Odd Power of Sine

Evaluate $\int \cos^4 x \sin^3 x \, dx$.

Solution Here, with $u = \cos x$, we have $du = -\sin x \, dx$, so that

$$\int \cos^4 x \sin^3 x \, dx = \int \cos^4 x \sin^2 x \sin x \, dx = -\int \cos^4 x \sin^2 x (-\sin x) dx$$

$$= -\int \underbrace{\cos^4 x(1 - \cos^2 x)}_{u^4(1-u^2)} \underbrace{(-\sin x) dx}_{du} = -\int u^4(1 - u^2) du \quad \text{Since } \sin^2 x + \cos^2 x = 1, \ \sin^2 x = 1 - \cos^2 x$$

$$= -\int (u^4 - u^6) du = -\left(\frac{u^5}{5} - \frac{u^7}{7}\right) + c$$

$$= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + c. \quad \text{Since } u = \cos x.$$

The ideas used in example 3.2 can be applied to any integral of the specified form.

EXAMPLE 3.3 An Integrand with an Odd Power of Cosine

Evaluate $\int \sqrt{\sin x}\cos^5 x\,dx$.

Solution Observe that we can rewrite this as

$$\int \sqrt{\sin x}\cos^5 x\,dx = \int \sqrt{\sin x}\cos^4 x\cos x\,dx = \int \sqrt{\sin x}(1-\sin^2 x)^2\cos x\,dx.$$

Substituting $u = \sin x$, so that $du = \cos x\,dx$, we have

$$\begin{aligned}\int \sqrt{\sin x}\cos^5 x\,dx &= \int \underbrace{\sqrt{\sin x}(1-\sin^2 x)^2}_{\sqrt{u}(1-u^2)^2}\underbrace{\cos x\,dx}_{du}\\ &= \int \sqrt{u}(1-u^2)^2\,du = \int u^{1/2}(1-2u^2+u^4)du\\ &= \int (u^{1/2}-2u^{5/2}+u^{9/2})du\\ &= \frac{2}{3}u^{3/2} - 2\left(\frac{2}{7}\right)u^{7/2} + \frac{2}{11}u^{11/2}+c\\ &= \frac{2}{3}\sin^{3/2}x - \frac{4}{7}\sin^{7/2}x + \frac{2}{11}\sin^{11/2}x + c. \quad \text{Since } u = \sin x.\end{aligned}$$

Looking beyond the details of calculation here, you should see the main point: that all integrals of this form are calculated in essentially the same way. ■

NOTES

Half-angle formulas

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x)$$
$$\cos^2 x = \tfrac{1}{2}(1+\cos 2x)$$

Case 2: *m* and *n* Are Both Even Positive Integers

In this case, we can use the half-angle formulas for sine and cosine (shown in the Notes box) to reduce the powers in the integrand.

We illustrate this case in example 3.4.

EXAMPLE 3.4 An Integrand with an Even Power of Sine

Evaluate $\int \sin^2 x\,dx$.

Solution Using the half-angle formula, we can rewrite the integral as

$$\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)dx.$$

We can evaluate this last integral by using the substitution $u = 2x$, so that $du = 2\,dx$. This gives us

$$\begin{aligned}\int \sin^2 x\,dx &= \frac{1}{2}\left(\frac{1}{2}\right)\int \underbrace{(1-\cos 2x)}_{1-\cos u}\underbrace{2dx}_{du} = \frac{1}{4}\int (1-\cos u)du\\ &= \frac{1}{4}(u-\sin u)+c = \frac{1}{4}(2x-\sin 2x)+c. \quad \text{Since } u = 2x.\end{aligned}$$ ■

With some integrals, you may need to apply the half-angle formulas several times, as in example 3.5.

EXAMPLE 3.5 An Integrand with an Even Power of Cosine

Evaluate $\int \cos^4 x\, dx$.

Solution Using the half-angle formula for cosine, we have

$$\int \cos^4 x\, dx = \int (\cos^2 x)^2\, dx = \frac{1}{4}\int (1 + \cos 2x)^2 dx$$
$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)dx.$$

Using the half-angle formula again, on the last term in the integrand, we get

$$\int \cos^4 x\, dx = \frac{1}{4}\int \left[1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right]dx$$
$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c,$$

where we leave the details of the final integration as an exercise. ■

Our next aim is to devise a strategy for evaluating integrals of the form

$$\int \tan^m x \sec^n x\, dx,$$

where m and n are integers.

Case 3: *m* is an Odd Positive Integer

First, isolate one factor of $\sec x \tan x$. (You'll need this for du.) Then, replace any factors of $\tan^2 x$ with $\sec^2 x - 1$ and make the substitution $u = \sec x$.

We illustrate this in example 3.6.

EXAMPLE 3.6 An Integrand with an Odd Power of Tangent

Evaluate $\int \tan^3 x \sec^3 x\, dx$.

Solution Looking for terms that are derivatives of other terms, we rewrite the integral as

$$\int \tan^3 x \sec^3 x\, dx = \int \tan^2 x \sec^2 x(\sec x \tan x)dx$$
$$= \int (\sec^2 x - 1)\sec^2 x\,(\sec x \tan x)dx,$$

where we have used the Pythagorean identity

$$\tan^2 x = \sec^2 x - 1.$$

You should see the substitution now. We let $u = \sec x$, so that $du = \sec x \tan x\, dx$ and hence,

$$\int \tan^3 x \sec^3 x\, dx = \int \underbrace{(\sec^2 x - 1)\sec^2 x}_{(u^2-1)u^2}\, \underbrace{(\sec x \tan x)\,dx}_{du}$$
$$= \int (u^2 - 1)u^2 du = du\int (u^4 - u^2)$$
$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + c = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + c. \quad \text{Since } u = \sec x.$$

■

Case 4: *n* is an Even Positive Integer

First, isolate one factor of $\sec^2 x$. (You'll need this for du.) Then, replace any remaining factors of $\sec^2 x$ with $1 + \tan^2 x$ and make the substitution $u = \tan x$.

We illustrate this in example 3.7.

EXAMPLE 3.7 An Integrand with an Even Power of Secant

Evaluate $\int \tan^2 x \sec^4 x\, dx$.

Solution Since $\frac{d}{dx}\tan x = \sec^2 x$, we rewrite the integral as

$$\int \tan^2 x \sec^4 x\, dx = \int \tan^2 x \sec^2 x \sec^2 x\, dx = \int \tan^2 x(1 + \tan^2 x)\sec^2 x\, dx.$$

Now, we let $u = \tan x$, so that $du = \sec^2 x\, dx$ and

$$\begin{aligned}\int \tan^2 x \sec^4 x\, dx &= \int \underbrace{\tan^2 x(1+\tan^2 x)}_{u^2(1+u^2)}\underbrace{\sec^2 x\, dx}_{du} \\ &= \int u^2(1+u^2)\,du = du\int(u^2+u^4) \\ &= \frac{1}{3}u^3 + \frac{1}{5}u^5 + c \\ &= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + c. \quad \text{Since } u = \tan x.\end{aligned}$$

■

Case 5: *m* is an Even Positive Integer and *n* is an Odd Positive Integer

Replace any factors of $\tan^2 x$ with $\sec^2 x - 1$ and then use a special *reduction formula* (given in the exercises) to evaluate integrals of the form $\int \sec^n x\, dx$. This complicated case will be covered briefly in the exercises. Much of this depends on example 3.8.

EXAMPLE 3.8 An Unusual Integral

Evaluate the integral $\int \sec x\, dx$.

Solution Finding an antiderivative here depends on an unusual observation. Notice that if we multiply the integrand by the fraction $\frac{\sec x + \tan x}{\sec x + \tan x}$ (which is of course equal to 1), we get

$$\begin{aligned}\int \sec x\, dx &= \int \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx.\end{aligned}$$

Now, observe that the numerator is exactly the derivative of the denominator. That is,

$$\frac{d}{dx}(\sec x + \tan x) = \sec x \tan x + \sec^2 x,$$

so that taking $u = \sec x + \tan x$ gives us

$$\begin{aligned}\int \sec x\, dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx \\ &= \int \frac{1}{u}\,du = \ln|u| + c \\ &= \ln|\sec x + \tan x| + c. \quad \text{Since } u = \sec x + \tan x.\end{aligned}$$

■

○ Trigonometric Substitution

If an integral contains a term of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$, for some $a > 0$, you can often evaluate the integral by making a substitution involving a trig function (hence, the name *trigonometric substitution*).

First, suppose that an integrand contains a term of the form $\sqrt{a^2 - x^2}$, for some $a > 0$. Letting $x = a \sin \theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, we can eliminate the square root, as follows:

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - (a \sin \theta)^2} = \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= a\sqrt{1 - \sin^2 \theta} = a\sqrt{\cos^2 \theta} = a \cos \theta,\end{aligned}$$

since for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\cos \theta \ge 0$. Example 3.9 is typical of how these substitutions are used.

NOTES

Terms of the form $\sqrt{a^2 - x^2}$ can also be simplified using the substitution $x = a \cos \theta$, using a different restriction for θ.

EXAMPLE 3.9 An Integral Involving $\sqrt{a^2 - x^2}$

Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{4 - x^2}}\, dx$.

Solution You should always first consider whether an integral can be done directly, by substitution or by parts. Since none of these methods help here, we consider trigonometric substitution. Keep in mind that the immediate objective here is to eliminate the square root. A substitution that will accomplish this is

$$x = 2 \sin \theta, \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

(Why do we need *strict* inequalities here?) This gives us

$$dx = 2 \cos \theta \, d\theta$$

and hence,

$$\begin{aligned}\int \frac{1}{x^2\sqrt{4 - x^2}}\, dx &= \int \frac{1}{(2 \sin \theta)^2 \sqrt{4 - (2 \sin \theta)^2}} 2 \cos \theta \, d\theta \\ &= \int \frac{2 \cos \theta}{4 \sin^2 \theta \sqrt{4 - 4 \sin^2 \theta}}\, d\theta \\ &= \int \frac{\cos \theta}{(2 \sin^2 \theta)\, 2\sqrt{1 - \sin^2 \theta}}\, d\theta \\ &= \int \frac{\cos \theta}{4 \sin^2 \theta \cos \theta}\, d\theta && \text{Since } 1 - \sin^2 \theta = \cos^2 \theta \\ &= \frac{1}{4}\int \csc^2 \theta \, d\theta = -\frac{1}{4}\cot \theta + c. && \text{Since } \frac{1}{\sin^2 \theta} = \csc^2 \theta\end{aligned}$$

The only remaining problem is that the antiderivative is presently written in terms of the variable θ. When converting back to the original variable $x = 2 \sin \theta$, we urge you to draw a diagram, as in Figure 3.1. Since the substitution was $x = 2 \sin \theta$, we have $\sin \theta = \frac{x}{2} = \frac{\text{opposite}}{\text{hypotenuse}}$ and so we label the hypotenuse as 2. The side opposite the angle θ is then $2 \sin \theta$. By the Pythagorean Theorem, we get that the adjacent side is $\sqrt{4 - x^2}$, as indicated. So, we have

$$\cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{\sqrt{4 - x^2}}{x}.$$

It now follows that

$$\int \frac{1}{x^2\sqrt{4 - x^2}}\, dx = -\frac{1}{4}\cot \theta + c = -\frac{1}{4}\frac{\sqrt{4 - x^2}}{x} + c.$$

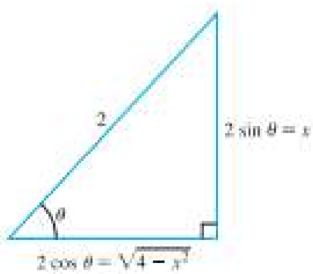


FIGURE 3.1

Next, suppose that an integrand contains a term of the form $\sqrt{a^2+x^2}$, for some $a>0$. Taking $x = a\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, we eliminate the square root, as follows:

$$\begin{aligned}\sqrt{a^2+x^2} &= \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2+a^2\tan^2\theta}\\ &= a\sqrt{1+\tan^2\theta} = a\sqrt{\sec^2\theta} = a\sec\theta,\end{aligned}$$

since for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, $\sec\theta>0$. Example 3.10 is typical of how these substitutions are used.

EXAMPLE 3.10 An Integral Involving $\sqrt{a^2+x^2}$

Evaluate the integral $\int \frac{1}{\sqrt{9+x^2}}\,dx$.

Solution You can eliminate the square root by letting $x = 3\tan\theta$, for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. This gives us $dx = 3\sec^2\theta d\theta$, so that

$$\begin{aligned}\int \frac{1}{\sqrt{9+x^2}}\,dx &= \int \frac{1}{\sqrt{9+(3\tan\theta)^2}}\,3\sec^2\theta\,d\theta\\ &= \int \frac{3\sec^2\theta}{\sqrt{9+9\tan^2\theta}}\,d\theta\\ &= \int \frac{3\sec^2\theta}{3\sqrt{1+\tan^2\theta}}\,d\theta\\ &= \int \frac{\sec^2\theta}{\sec\theta}\,d\theta \qquad \text{Since } 1+\tan^2\theta=\sec^2\theta\\ &= \int \sec\theta\,d\theta\\ &= \ln|\sec\theta+\tan\theta|+c,\end{aligned}$$

from example 3.8. We're not done here, though, since we must still express the integral in terms of the original variable x. Observe that we had $x = 3\tan\theta$, so that $\tan\theta=\frac{x}{3}$. It remains only to solve for $\sec\theta$. Although you can do this with a triangle, as in example 3.9, the simplest way to do this is to recognize that for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$,

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{x}{3}\right)^2}.$$

This leaves us with

$$\begin{aligned}\int \frac{1}{\sqrt{9+x^2}}\,dx &= \ln|\sec\theta+\tan\theta|+c\\ &= \ln\left|\sqrt{1+\left(\frac{x}{3}\right)^2}+\frac{x}{3}\right|+c.\end{aligned}$$

■

Finally, suppose that an integrand contains a term of the form $\sqrt{x^2-a^2}$, for some $a>0$. Taking $x = a\sec\theta$, where $\theta\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$, we eliminate the square root, as follows:

$$\begin{aligned}\sqrt{x^2-a^2} &= \sqrt{(a\sec\theta)^2-a^2} = \sqrt{a^2\sec^2\theta-a^2}\\ &= a\sqrt{\sec^2\theta-1} = a\sqrt{\tan^2\theta} = a|\tan\theta|.\end{aligned}$$

Notice that the absolute values are needed, as $\tan\theta$ can be both positive and negative on $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$. Example 3.11 is typical of how these substitutions are used.

EXAMPLE 3.11 An Integral Involving $\sqrt{x^2 - a^2}$

Evaluate the integral $\int \frac{\sqrt{x^2-25}}{x}\,dx$, for $x \geq 5$.

Solution Here, we let $x = 5\sec\theta$, for $\theta \in [0, \frac{\pi}{2})$, where we chose the first half of the domain $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, so that $x = 5\sec\theta > 5$. (If we had $x < -5$, we would have chosen $\theta \in (\frac{\pi}{2}, \pi]$.) This gives us $dx = 5\sec\theta\tan\theta\,d\theta$ and the integral then becomes:

$$\begin{aligned}\int \frac{\sqrt{x^2-25}}{x}\,dx &= \int \frac{\sqrt{(5\sec\theta)^2-25}}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta\\ &= \int \sqrt{25\sec^2\theta-25}\tan\theta\,d\theta\\ &= \int 5\sqrt{\sec^2\theta-1}\ \text{tran}\,\theta\,d\theta\\ &= 5\int \tan^2\theta\,d\theta \quad \text{Since } \sec^2\theta-1=\tan^2\theta.\\ &= 5\int (\sec^2\theta-1)\,d\theta\\ &= 5(\tan\theta-\theta)+c.\end{aligned}$$

Finally, observe that since $x = 5\sec\theta$, for $\theta \in [0, \frac{\pi}{2})$, we have that

$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{x}{5}\right)^2 - 1} = \frac{1}{5}\sqrt{x^2-25}$$

and $\theta = \sec^{-1}(\frac{x}{5})$. We now have

$$\begin{aligned}\int \frac{\sqrt{x^2-25}}{x}\,dx &= 5(\tan\theta-\theta)+c\\ &= \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right)+c.\end{aligned}$$ ■

You will find a number of additional integrals requiring trigonometric substitution in the exercises. The principal idea here is to see that you can eliminate certain square root terms in an integrand by making use of a carefully chosen trigonometric substitution.

We summarize the three trigonometric substitutions presented here in the following table.

Expression	*Trigonometric Substitution*	*Interval*	*Identity*
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$	$\sec^2\theta - 1 = \tan^2\theta$

EXERCISES 3.3

WRITING EXERCISES

1. Suppose a friend in your calculus class tells you that this section just has too many rules to memorize. Help your friend out by making it clear that each rule indicates when certain substitutions will work. In turn, a substitution $u(x)$ works if the expression $u'(x)$ appears in the integrand and the resulting integral is easier to integrate. For each of the rules covered in the text, identify $u'(x)$ and point out why n has to be odd (or whatever the rule says) for the remaining integrand to be workable. Without memorizing rules, you can remember a small number of potential substitutions and see which one works for a given problem.

2. In the text, we suggested that when the integrand contains a term of the form $\sqrt{4-x^2}$, you might try the trigonometric

substitution $x = 2\sin\theta$. We should admit now that this does not always work. After a substitution, how can you tell whether the substitution worked?

In exercises 1–44, evaluate the integrals.

1. $\int \cos x \sin^4 x\,dx$
2. $\int \cos^3 x \sin^4 x\,dx$
3. $\int_0^{\pi/4} \cos 2x \sin^3 2x\,dx$
4. $\int_{\pi/4}^{\pi/3} \cos^3 3x \sin^3 3x\,dx$
5. $\int_0^{\pi/2} \cos^2 x \sin x\,dx$
6. $\int_{-\pi/2}^{0} \cos^3 x \sin x\,dx$
7. $\int \cos^2 (x+1)\,dx$
8. $\int \sin^4 (x-3)\,dx$
9. $\int \tan x \sec^3 x\,dx$
10. $\int \cot x \csc^4 x\,dx$
11. $\int x \tan^3 (x^2+1) \sec (x^2+1)\,dx$
12. $\int \tan(2x+1) \sec^3 (2x+1)\,dx$
13. $\int \cot^2 x \csc^4 x\,dx$
14. $\int \cot^2 x \csc^2 x\,dx$
15. $\int_0^{\pi/4} \tan^4 x \sec^4 x\,dx$
16. $\int_{-\pi/4}^{\pi/4} \tan^4 x \sec^2 x\,dx$
17. $\int \cos^2 x \sin^2 x\,dx$
18. $\int (\cos^2 x + \sin^2 x)\,dx$
19. $\int_{-\pi/3}^{0} \sqrt{\cos x}\, \sin^3 x\,dx$
20. $\int_{\pi/4}^{\pi/2} \cot^2 x \csc^4 x\,dx$
21. $\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$
22. $\int \frac{1}{x^2\sqrt{16-x^2}}\,dx$
23. $\int \frac{x^2}{\sqrt{16-x^2}}\,dx$
24. $\int \frac{x^3}{\sqrt{9-x^2}}\,dx$
25. $\int_0^2 \sqrt{4-x^2}\,dx$
26. $\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx$
27. $\int \frac{x^2}{\sqrt{x^2-9}}\,dx$
28. $\int x^3\sqrt{x^2-1}\,dx$
29. $\int \frac{2}{\sqrt{x^2-4}}\,dx$
30. $\int \frac{x}{\sqrt{x^2-4}}\,dx$
31. $\int \frac{\sqrt{4x^2-9}}{x}\,dx$
32. $\int \frac{\sqrt{x^2-4}}{x^2}\,dx$
33. $\int \frac{x^2}{\sqrt{9+x^2}}\,dx$
34. $\int x^3\sqrt{8+x^2}\,dx$
35. $\int \sqrt{16+x^2}\,dx$
36. $\int \frac{1}{\sqrt{4+x^2}}\,dx$
37. $\int_0^1 x\sqrt{x^2+8}\,dx$
38. $\int_0^2 x^2\sqrt{x^2+9}\,dx$
39. $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$
40. $\int \frac{x+1}{\sqrt{4+x^2}}\,dx$
41. $\int \frac{x}{\sqrt{x^2+4x}}\,dx$
42. $\int \frac{2}{\sqrt{x^2-6x}}\,dx$
43. $\int \frac{x}{\sqrt{10+2x+x^2}}\,dx$
44. $\int \frac{2}{\sqrt{4x-x^2}}\,dx$

In exercises 45 and 46, evaluate the integral using both substitutions $u = \tan x$ and $u = \sec x$ and compare the results.

45. $\int \tan x \sec^4 x\,dx$
46. $\int \tan^3 x \sec^4 x\,dx$

47. (a) Show that for any integer $n > 1$, we have the reduction formula

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx.$$

Evaluate (b) $\int \sec^3 x\,dx$, (c) $\int \sec^4 x\,dx$ and (d) $\int \sec^5 x\,dx$.

48. The area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by $\frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$. Compute this integral.

49. Show that $\int \csc x\,dx = \ln|\csc x - \cot x| + c$ and evaluate $\int \csc^3 x\,dx$.

50. Show that $\int \frac{1}{\cos x - 1}\,dx = \csc x + \cot x + c$ and $\int \frac{1}{\cos x + 1}\,dx = \csc x - \cot x + c$.

51. Evaluate the antiderivatives in examples 3.2, 3.3, 3.5, 3.6 and 3.7 using your CAS. Based on these examples, speculate whether or not your CAS uses the same techniques that we do. In the cases where your CAS gives a different antiderivative than we do, comment on which antiderivative looks simpler.

52. (a) One CAS produces $-\frac{1}{7}\sin^2 x \cos^5 x - \frac{2}{35}\cos^5 x$ as an antiderivative in example 3.2. Find c such that this equals our antiderivative of $-\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + c$.

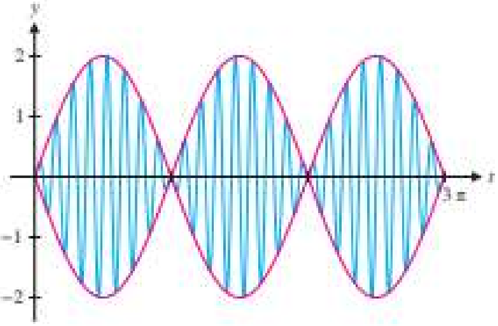
(b) One CAS produces $-\frac{2}{15}\tan x - \frac{1}{15}\sec^2 x \tan x + \frac{1}{5}\sec^4 x \tan x$ as an antiderivative in example 3.7. Find c such that this equals our antiderivative of $\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + c$.

53. In an AC circuit, the current has the form $i(t) = I\cos(\omega t)$ for constants I and ω. The power is defined as Ri^2 for a constant R. Find the average value of the power by integrating over the interval $[0, 2\pi/\omega]$.

EXPLORATORY EXERCISES

1. In section 3.2, you were asked to show that for positive integers m and n with $m \neq n$, $\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx = 0$ and $\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx = 0$. Also, $\int_{-\pi}^{\pi}\cos^2(nx)\,dx = \int_{-\pi}^{\pi}\sin^2(nx)\,dx = \pi$. Finally, $\int_{-\pi}^{\pi}\cos(mx)\sin(nx)\,dx = 0$, for any positive integers m and n. We will use these formulas to explain how a radio can tune in an AM station.

Amplitude modulation (or AM) radio sends a signal (e.g., music) that modulates the carrier frequency. For example, if the signal is $2\sin t$ and the carrier frequency is 16, then the radio sends out the modulated signal $2\sin t\sin 16t$. The graphs of $y = 2\sin t$, $y = -2\sin t$ and $y = 2\sin t\sin 16t$ are shown in the figure.



The graph of $y = 2\sin t \sin 16t$ oscillates as rapidly as the carrier $\sin 16t$, but the amplitude varies between $2 \sin t$ and $-2 \sin t$ (hence the term amplitude modulation). The radio's problem is to tune in the frequency 16 and recover the signal $2 \sin t$. The difficulty is that other radio stations are broadcasting simultaneously. A radio receives all the signals mixed together. To see how this works, suppose a second station broadcasts the signal $3 \sin t$ at frequency 32. The combined signal that the radio receives is $2\sin t \sin 16t + 3\sin t \sin 32t$. We will decompose this signal. The first step is to rewrite the signal using the identity

$$\sin A \sin B = \frac{1}{2}\cos(B - A) - \frac{1}{2}\cos(B + A).$$

The signal then equals

$$f(t) = \cos 15t - \cos 17t + \frac{3}{2}\cos 31t - \frac{3}{2}\cos 33t.$$

If the radio "knows" that the signal has the form $c \sin t$, for some constant c, it can determine the constant c at frequency 16 by computing the integral $\int_{-\pi}^{\pi} f(t)\cos 15t\,dt$ and multiplying by $2/\pi$. Show that $\int_{-\pi}^{\pi} f(t)\cos 15t\,dt = \pi$, so that the correct constant is $c = \pi(2/\pi) = 2$. The signal is then $2 \sin t$. To recover the signal sent out by the second station, compute $\int_{-\pi}^{\pi} f(t)\cos 31t\,dt$ and multiply by $2/\pi$. Show that you correctly recover the signal $3 \sin t$.

2. In this exercise, we derive an important result called **Wallis' product**. Define the integral $I_n = \int_0^{\pi/2} \sin^n x\,dx$ for a positive integer n. (a) Show that $I_n = \frac{n}{n-1} I_{n-2}$. (b) Show that $\frac{I_{2n+1}}{I_{2n}} = \frac{2^2 4^2 \cdots (2n)^2 2}{3^2 5^2 \cdots (2n-1)^2 (2n+1)\pi}$. (c) Given that $\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$, conclude that $\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^2 4^2 \cdots (2n)^2}{3^2 5^2 \cdots (2n-1)^2 (2n+1)}$.

3.4 INTEGRATION OF RATIONAL FUNCTIONS USING PARTIAL FRACTIONS

In this section, we introduce a method for rewriting certain rational functions that is very useful in integration as well as in other applications. We begin with a simple observation. Note that

$$\frac{3}{x+2} - \frac{2}{x-5} = \frac{3(x-5) - 2(x+2)}{(x+2)(x-5)} = \frac{x-19}{x^2 - 3x - 10}. \tag{4.1}$$

So, suppose that you wanted to evaluate the integral of the function on the right-hand side of (4.1). While it's not clear how to evaluate this integral, the integral of the (equivalent) function on the left-hand side of (4.1) is easy to evaluate. From (4.1), we have

$$\int \frac{x-19}{x^2 - 3x - 10}\,dx = \int \left(\frac{3}{x+2} - \frac{2}{x-5}\right) dx = 3\ln|x+2| - 2\ln|x-5| + c.$$

The second integrand,

$$\frac{3}{x+2} - \frac{2}{x-5}$$

is called a **partial fractions decomposition** of the first integrand. More generally, if the three factors $a_1x + b_1$, $a_2x + b_2$ and $a_3x + b_3$ are all distinct (i.e., none is a constant multiple of another), then we can write

$$\boxed{\frac{a_1x + b_1}{(a_2x + b_2)(a_3x + b_3)} = \frac{A}{a_2x + b_2} + \frac{B}{a_3x + b_3},}$$

for some choice of constants A and B to be determined. Notice that if you wanted to integrate this expression, the partial fractions on the right-hand side are very easy to integrate.

EXAMPLE 4.1 Partial Fractions: Distinct Linear Factors

Evaluate $\int \frac{1}{x^2+x-2}\,dx$.

Solution First, note that you can't evaluate this as it stands and all of our earlier methods fail to help. (Consider each of these for this problem.) However, we can make a partial fractions decomposition, as follows.

$$\frac{1}{x^2+x-2} = \frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

Multiplying both sides of this equation by the common denominator $(x-1)(x+2)$, we get

$$1 = A(x+2) + B(x-1). \tag{4.2}$$

We would like to solve this equation for A and B. The key is to realize that this equation must hold for all x. In particular, for $x = 1$, notice that from (4.2), we have

$$1 = A(1+2) + B(1-1) = 3A,$$

so that $A = \frac{1}{3}$. Likewise, taking $x = -2$, we have

$$1 = A(-2+2) + B(-2-1) = -3B,$$

so that $B = -\frac{1}{3}$. Thus, we have

$$\int \frac{1}{x^2+x-2}\,dx = \int \left[\frac{1}{3}\left(\frac{1}{x-1}\right) - \frac{1}{3}\left(\frac{1}{x+2}\right)\right] dx$$
$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + c.$$

We can do the same as we did in example 4.1 whenever a rational expression has a denominator that factors into n distinct linear factors, as follows. If the degree of $P(x) < n$ and the factors $(a_i x + b_i)$, for $i = 1, 2, \ldots, n$ are all distinct, then we can write

Partial fractions: distinct linear factors

$$\frac{P(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)} = \frac{c_1}{a_1x+b_1} + \frac{c_2}{a_2x+b_2} + \cdots + \frac{c_n}{a_nx+b_n},$$

for some constants $c_1, c_2, \ldots, c_n$. ■

EXAMPLE 4.2 Partial Fractions: Three Distinct Linear Factors

Evaluate $\int \frac{3x^2-7x-2}{x^3-x}\,dx$.

Solution Once again, our earlier methods fail us, but we can rewrite the integrand using partial fractions. We have

$$\frac{3x^2-7x-2}{x^3-x} = \frac{3x^2-7x-2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

Multiplying by the common denominator $x(x-1)(x+1)$, we get

$$3x^2 - 7x - 2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1). \tag{4.3}$$

In this case, note that taking $x = 0$, we get

$$-2 = A(-1)(1) = -A,$$

so that $A = 2$. Likewise, taking $x = 1$, we find $B = -3$ and taking $x = -1$, we find $C = 4$. Thus, we have

$$\int \frac{3x^2 - 7x - 2}{x^3 - x}\,dx = \int \left(\frac{2}{x} - \frac{3}{x-1} + \frac{4}{x+1}\right) dx$$
$$= 2\ln|x| - 3\ln|x-1| + 4\ln|x+1| + c. \blacksquare$$

REMARK 4.1

If the numerator of a rational expression has the same or higher degree than the denominator, you must first perform a long division and follow this with a partial fractions decomposition of the remaining (proper) fraction.

EXAMPLE 4.3 Partial Fractions Where Long Division Is Required

Find the indefinite integral of $f(x) = \dfrac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8}$ using a partial fractions decomposition.

Solution Since the degree of the numerator exceeds that of the denominator, first divide

$$\begin{array}{r} 2x \\ x^2 - 2x - 8 \overline{\big)\, 2x^3 - 4x^2 - 15x + 5} \\ \underline{2x^3 - 4x^2 - 16x \quad} \\ x + 5 \end{array}$$

Thus, we have $\quad f(x) = \dfrac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} = 2x + \dfrac{x+5}{x^2 - 2x - 8}.$

The remaining proper fraction can be expanded as

$$\frac{x+5}{x^2 - 2x - 8} = \frac{x+5}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}.$$

It is a simple matter to solve for the constants: $A = \frac{3}{2}$ and $B = -\frac{1}{2}$. (This is left as an exercise.) We now have

$$\int \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8}\,dx = \int \left[2x + \frac{3}{2}\left(\frac{1}{x-4}\right) - \frac{1}{2}\left(\frac{1}{x+2}\right)\right] dx$$
$$= x^2 + \frac{3}{2}\ln|x-4| - \frac{1}{2}\ln|x+2| + c. \blacksquare$$

If the denominator of a rational expression contains repeated linear factors, the decomposition looks like the following. If the degree of $P(x)$ is less than n, then we can write

Partial fractions: repeated linear factors

$$\frac{P(x)}{(ax+b)^n} = \frac{c_1}{ax+b} + \frac{c_2}{(ax+b)^2} + \cdots + \frac{c_n}{(ax+b)^n},$$
for constants $c_1, c_2, \ldots, c_n$ to be determined.

Example 4.4 is typical.

EXAMPLE 4.4 Partial Fractions with a Repeated Linear Factor

Use a partial fractions decomposition to find an antiderivative of

$$f(x) = \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}.$$

Solution First, note that there is a repeated linear factor in the denominator. We have

$$\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \frac{5x^2 + 20x + 6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Multiplying by the common denominator $x(x+1)^2$, we have

$$5x^2 + 20x + 6 = A(x+1)^2 + Bx(x+1) + Cx.$$

Taking $x = 0$, we find $A = 6$. Likewise, taking $x = -1$, we find that $C = 9$. To determine B, substitute any convenient value for x, say $x = 1$. (Unfortunately, notice that there is no choice of x that will make the two terms containing A and C both zero, without also making the term containing B zero.) You should find that $B = -1$. So, we have

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\,dx = \int \left[\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right] dx$$
$$= 6 \ln|x| - \ln|x+1| - 9(x+1)^{-1} + c. \blacksquare$$

We can extend the notion of partial fractions decomposition to rational expressions with denominators containing irreducible quadratic factors (i.e., quadratic factors that have no real factorization). If the degree of $P(x)$ is less than $2n$ (the degree of the denominator) and all of the factors in the denominator are distinct, then we can write

Partial fractions: irreducible quadratic factors

$$\frac{P(x)}{(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_nx^2 + b_nx + c_n)} = \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{a_2x^2 + b_2x + c_2} + \cdots + \frac{A_nx + B_n}{a_nx^2 + b_nx + c_n}. \tag{4.4}$$

Think of this in terms of irreducible quadratic denominators in a partial fractions decomposition getting linear numerators, while linear denominators get constant numerators. If you think this looks messy, you're right, but only the algebra is messy (and you can always use a CAS to do the algebra for you). You should note that the partial fractions on the right-hand side of (4.4) are integrated comparatively easily using substitution together with possibly completing the square.

EXAMPLE 4.5 Partial Fractions with a Quadratic Factor

Use a partial fractions decomposition to find an antiderivative of $f(x) = \dfrac{2x^2 - 5x + 2}{x^3 + x}$.

Solution First, note that

$$\frac{2x^2 - 5x + 2}{x^3 + x} = \frac{2x^2 - 5x + 2}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}.$$

Multiplying through by the common denominator $x(x^2 + 1)$ gives us

$$2x^2 - 5x + 2 = A(x^2 + 1) + (Bx + C)x$$
$$= (A + B)x^2 + Cx + A.$$

Rather than substitute numbers for x (notice that there are no convenient values to plug in, except for $x = 0$), we instead match up the coefficients of like powers of x:

$$2 = A + B$$
$$-5 = C$$
$$2 = A.$$

This leaves us with $B = 0$ and so,

$$\int \frac{2x^2 - 5x + 2}{x^3 + x}\,dx = \int \left(\frac{2}{x} - \frac{5}{x^2 + 1}\right) dx = 2 \ln|x| - 5 \tan^{-1} x + c. \blacksquare$$

Partial fractions decompositions involving irreducible quadratic terms often lead to expressions that require further massaging (such as completing the square) before we can find an antiderivative. We illustrate this in example 4.6.

EXAMPLE 4.6 Partial Fractions with a Quadratic Factor

Use a partial fractions decomposition to find an antiderivative for

$$f(x) = \frac{5x^2 + 6x + 2}{(x+2)(x^2+2x+5)}.$$

Solution First, notice that the quadratic factor in the denominator does not factor and so, the correct decomposition is

$$\frac{5x^2+6x+2}{(x+2)(x^2+2x+5)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+2x+5}.$$

Multiplying through by $(x+2)(x^2+2x+5)$, we get

$$5x^2 + 6x + 2 = A(x^2+2x+5) + (Bx+C)(x+2).$$

Matching up the coefficients of like powers of x, we get

$$\begin{aligned} 5 &= A + B \\ 6 &= 2A + 2B + C \\ 2 &= 5A + 2C. \end{aligned}$$

You'll need to solve this by elimination. We leave it as an exercise to show that $A = 2$, $B = 3$ and $C = -4$. Integrating, we have

$$\int \frac{5x^2+6x+2}{(x+2)(x^2+2x+5)}\,dx = \int \left(\frac{2}{x+2} + \frac{3x-4}{x^2+2x+5} \right) dx. \tag{4.5}$$

The integral of the first term is easy, but what about the second term? Since the denominator doesn't factor, you have very few choices. Try substituting for the denominator: let $u = x^2 + 2x + 5$, so that $du = (2x+2)\,dx$. Notice that we can write the integral of the second term as

$$\begin{aligned} \int \frac{3x-4}{x^2+2x+5}\,dx &= \int \frac{3(x+1)-7}{x^2+2x+5}\,dx = \int \left[\left(\frac{3}{2}\right) \frac{2(x+1)}{x^2+2x+5} - \frac{7}{x^2+2x+5} \right] dx \\ &= \frac{3}{2}\int \frac{2(x+1)}{x^2+2x+5}\,dx - \int \frac{7}{x^2+2x+5}\,dx \\ &= \frac{3}{2}\ln(x^2+2x+5) - \int \frac{7}{x^2+2x+5}\,dx. \end{aligned} \tag{4.6}$$

Completing the square in the denominator of the remaining integral, we get

$$\int \frac{7}{x^2+2x+5}\,dx = \int \frac{7}{(x+1)^2+4}\,dx = \frac{7}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c.$$

(We leave the details of this last integration as an exercise.) Putting this together with (4.5) and (4.6), we now have

$$\int \frac{5x^2+6x+2}{(x+2)(x^2+2x+5)}\,dx = 2\ln|x+2| + \frac{3}{2}\ln(x^2+2x+5) - \frac{7}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c.$$ ■

REMARK 4.2

Most CASs include commands for performing partial fractions decomposition. Even so, we urge you to work through the exercises in this section by hand. Once you have the idea of how these decompositions work, by all means, use your CAS to do the drudge work for you. Until that time, be patient and work carefully by hand.

Rational expressions with repeated irreducible quadratic factors in the denominator are explored in the exercises. The idea of these is the same as the preceding decompositions, but the algebra is even messier.

Using the techniques covered in this section, you should be able to find the partial fractions decomposition of *any* rational function, since polynomials can always be factored into linear and quadratic factors, some of which may be repeated.

Brief Summary of Integration Techniques

At this point, we pause to briefly summarize what we have learned about techniques of integration. While you can differentiate virtually any function that you can write down, we are not nearly so fortunate with integrals. Many cannot be evaluated at all exactly, while others can be evaluated, but only by recognizing which technique might lead to a solution. With these things in mind, we present now a few hints for evaluating integrals.

Integration by Substitution: $\int f(u(x))u'(x)\,dx = \int f(u)\,du$

What to look for:

1. Compositions of the form $f(u(x))$, where the integrand also contains $u'(x)$; for example,

$$\int 2x\cos(x^2)dx = \int \underbrace{\cos(x^2)}_{\cos u}\,\underbrace{2x\,dx}_{du} = \int \cos u\,du.$$

2. Compositions of the form $f(ax+b)$; for example,

$$\int \frac{\overbrace{x}^{u-1}}{\underbrace{\sqrt{x+1}}_{\sqrt{u}}}\,\underbrace{dx}_{du} = \int \frac{u-1}{\sqrt{u}}\,du.$$

Integration by Parts: $\int u\,dv = uv - \int v\,du$

What to look for: products of different types of functions: x^n, $\cos x$, e^x ; for example,

$$\int 2x\cos x\,dx \qquad \begin{cases} u = x & dv = \cos x\,dx \\ du = dx & v = \sin x \end{cases}$$

$$= x\sin x - \int \sin x\,dx.$$

Trigonometric Substitution:

What to look for:

1. Terms like $\sqrt{a^2 - x^2}$: Let $x = a\sin\theta\,(-\pi/2 \le \theta \le \pi/2)$, so that $dx = a\cos\theta\,d\theta$ and $\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta$; for example,

$$\int \frac{\overbrace{x^2}^{\sin^2\theta}}{\underbrace{\sqrt{1-x^2}}_{\cos\theta}}\,\underbrace{dx}_{\cos\theta\,d\theta} = \int \sin^2\theta\,d\theta.$$

2. Terms like $\sqrt{x^2 + a^2}$: Let $x = a\tan\theta\,(-\pi/2 < \theta < \pi/2)$, so that $dx = a\sec^2\theta\,d\theta$ and $\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2} = a\sec\theta$; for example,

$$\int \frac{\overbrace{x^3}^{27\tan^3\theta}}{\underbrace{\sqrt{x^2+9}}_{3\sec\theta}}\,\underbrace{dx}_{3\sec^2\theta\,d\theta} = 27\int \tan^3\theta\sec\theta\,d\theta.$$

3. Terms like $\sqrt{x^2 - a^2}$: Let $x = a\sec\theta$, for $\theta \in [0, \pi/2) \cup (\pi/2, \pi]$, so that $dx = a\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\tan\theta$; for example,

$$\int \underbrace{x^3}_{8\sec^3\theta}\,\underbrace{\sqrt{x^2-4}}_{2\tan\theta}\,\underbrace{dx}_{2\sec\theta\tan\theta\,d\theta} = 32\int \sec^4\theta\tan^2\theta\,d\theta.$$

Partial Fractions:

What to look for: rational functions; for example,

$$\int \frac{x+2}{x^2-4x+3}\,dx = \int \frac{x+2}{(x-1)(x-3)}\,dx = \int \left(\frac{A}{x-1} + \frac{B}{x-3}\right)dx.$$

EXERCISES 3.4

WRITING EXERCISES

1. There is a shortcut for determining the constants for linear terms in a partial fractions decomposition. For example, take

$$\frac{x-1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}.$$

To compute A, take the original fraction on the left, cover up the $x+1$ in the denominator and replace x with -1: $A = \frac{-1-1}{-1-2} = \frac{2}{3}$. Similarly, to solve for B, cover up the $x-2$ and replace x with 2: $B = \frac{2-1}{2+1} = \frac{1}{3}$. Explain why this works on this decomposition but does not work on the decomposition of $\frac{x-1}{(x+1)^2(x-2)}$.

2. For partial fractions, there is a big distinction between quadratic functions that factor into linear terms and quadratic functions that are irreducible. Recall that a quadratic function factors as $(x-a)(x-b)$ if and only if a and b are zeros of the function. Explain how you can use the quadratic formula to determine whether a given quadratic function is irreducible.

In exercises 1–20, find the partial fractions decomposition and an antiderivative. If you have a CAS available, use it to check your answer.

1. $\frac{x-5}{x^2-1}$
2. $\frac{5x-2}{x^2-4}$
3. $\frac{6x}{x^2-x-2}$
4. $\frac{3x}{x^2-3x-4}$
5. $\frac{-x+5}{x^3-x^2-2x}$
6. $\frac{3x+8}{x^3+5x^2+6x}$
7. $\frac{5x-23}{6x^2-11x-7}$
8. $\frac{3x+5}{5x^2-4x-1}$
9. $\frac{x-1}{x^3+4x^2+4x}$
10. $\frac{4x-5}{x^3-3x^2}$
11. $\frac{x+2}{x^3+x}$
12. $\frac{1}{x^3+4x}$
13. $\frac{4x^2-7x-17}{6x^2-11x-10}$
14. $\frac{x^3+x}{x^2-1}$
15. $\frac{2x+3}{x^2+2x+1}$
16. $\frac{2x}{x^2-6x+9}$
17. $\frac{x^3-4}{x^3+2x^2+2x}$
18. $\frac{4}{x^3-2x^2+4x}$
19. $\frac{3x^2+1}{x^3-x^2+x-1}$
20. $\frac{2x^4+9x^2+x-4}{x^3+4x}$

In exercises 21–36, evaluate the integral.

21. $\int \frac{x^3+x+2}{x^2+2x-8}\,dx$
22. $\int \frac{x^2+1}{x^2-5x-6}\,dx$
23. $\int \frac{x+4}{x^3+3x^2+2x}\,dx$
24. $\int \frac{1}{x^3-1}\,dx$
25. $\int \frac{4x^3-1}{x^4-x}\,dx$
26. $\int \frac{x}{x^4+1}\,dx$
27. $\int \frac{4x-2}{16x^4-1}\,dx$
28. $\int \frac{3x+7}{x^4-16}\,dx$
29. $\int \frac{x^3+x}{3x^2+2x+1}\,dx$
30. $\int \frac{x^3-2x}{2x^2-3x+2}\,dx$
31. $\int \frac{4x^2+3}{x^3+x^2+x}\,dx$
32. $\int \frac{4x+4}{x^4+x^3+2x^2}\,dx$
33. $\int x^2 \sin x\,dx$
34. $\int xe^{2x}\,dx$
35. $\int \frac{\sin x \cos x}{\sin^2 x-4}\,dx$
36. $\int \frac{2e^x}{e^{3x}+e^x}\,dx$

37. In this exercise, we find the partial fractions decomposition of $\frac{4x^2+2}{(x^2+1)^2}$. The form for the decomposition is

$$\frac{4x^2+2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

Multiplying through by $(x^2+1)^2$, we get

$$\begin{aligned} 4x^2+2 &= (Ax+B)(x^2+1) + Cx + D \\ &= Ax^3 + Bx^2 + Ax + B + Cx + D \end{aligned}$$

As in example 4.5, we match up coefficients of like powers of x. For x^3, we have $0 = A$. For x^2, we have $4 = B$. Match the coefficients of x and the constants to finish the decomposition.

In exercises 38–40, find the partial fractions decomposition. (Refer to exercise 37.)

38. $\frac{x^3+2}{(x^2+1)^2}$
39. $\frac{4x^2+3}{(x^2+x+1)^2}$
40. $\frac{x^4+x^3}{(x^2+4)^2}$

41. Often, more than one integration technique can be applied. Evaluate $\int \frac{3}{x^4+x}\,dx$ in each of the following ways. First, use the substitution $u = x^3+1$ and partial fractions. Second, use the substitution $u = \frac{1}{x}$ and evaluate the resulting integral. Show that the two answers are equivalent.

42. Evaluate $\int \frac{2}{x^3+x}\,dx$ in each of the following ways. First, use the substitution $u = x^2 + 1$ and partial fractions. Second, use the substitution $u = \frac{1}{x}$ and evaluate the resulting integral. Show that the two answers are equivalent.

In exercises 43 and 44, name the method by identifying whether the integral can be evaluated using substitution, integration by parts, or partial fractions.

43. (a) $\int \frac{2}{x^2-1}\,dx$ (b) $\int \frac{x}{x^2-1}\,dx$

(c) $\int \frac{x+1}{x^2-1}\,dx$ (d) $\int \frac{2}{x^2+1}\,dx$

44. (a) $\int \frac{2}{(x+1)^2}\,dx$ (b) $\int \frac{2x+2}{(x+1)^2}\,dx$

(c) $\int \frac{x-1}{(x+1)^2}\,dx$ (d) $\int \frac{x-1}{(x^2+1)^2}\,dx$

45. Evaluate $\int \sec^3 x\,dx$ by rewriting the integrand as $\frac{\cos x}{\cos^4 x}$, making the substitution $u = \sin x$ and using partial fractions.

EXPLORATORY EXERCISES

1. In developing the definite integral, we looked at sums such as $\sum_{i=1}^{n} \frac{2}{i^2+i}$. For sums like this, we are especially interested in the limit as $n \to \infty$. Write out several terms of the sum and try to guess what the limit is. It turns out that this is one of the few sums for which a precise formula exists, because this is a **telescoping sum**. To find out what this means, write out the partial fractions decomposition for $\frac{2}{i^2+i}$. Using the partial fractions form, write out several terms of the sum and notice how much cancellation there is. Briefly describe why the term *telescoping* is appropriate, and determine $\sum_{i=1}^{n} \frac{2}{i^2+i}$. Then find the limit as $n \to \infty$. Repeat this process for the telescoping sum $\sum_{i=2}^{n} \frac{4}{i^2-1}$.

2. Use the substitution $u = x^{1/4}$ to evaluate $\int \frac{1}{x^{5/4}+x}\,dx$. Use similar substitutions to evaluate $\int \frac{1}{x^{1/4}+x^{1/3}}\,dx$, $\int \frac{1}{x^{1/5}+x^{1/7}}\,dx$ and $\int \frac{1}{x^{1/4}+x^{1/6}}\,dx$. Find the form of the substitution for the general integral $\int \frac{1}{x^p+x^q}\,dx$.

3.5 INTEGRATION TABLES AND COMPUTER ALGEBRA SYSTEMS

Ask anyone who has ever needed to evaluate a large number of integrals as part of their work (this includes engineers, mathematicians, physicists and others) and they will tell you that they have made extensive use of integral tables and/or a computer algebra system. These are extremely powerful tools for the professional user of mathematics. However, they do *not* take the place of learning all the basic techniques of integration. To use a table, you often must first rewrite the integral in the form of one of the integrals in the table. This may require you to perform some algebraic manipulation or to make a substitution. While a CAS will report an antiderivative, it will occasionally report it in an inconvenient form. More significantly, a CAS will from time to time report an answer that is (at least technically) incorrect. We will point out some of these shortcomings in the examples that follow.

Using Tables of Integrals

We include a small table of indefinite integrals at the back of the book. A larger table can be found in the *CRC Standard Mathematical Tables*. An amazingly extensive table is found in the book *Table of Integrals, Series and Products*, compiled by Gradshteyn and Ryzhik.[1]

EXAMPLE 5.1 Using an Integral Table

Use a table to evaluate $\int \frac{\sqrt{3+4x^2}}{x}\,dx$.

Solution Certainly, you could evaluate this integral using trigonometric substitution. However, if you look in our integral table, you will find

$$\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + c. \tag{5.1}$$

[1] Gradshteyn, I.S. and Ryzhik, I.M. (2014). *Table of Integrals, Series, and Products* (New York: Academic Press).

Unfortunately, the integral in question is not quite in the form of (5.1). However, we can fix this with the substitution $u = 2x$, so that $du = 2\,dx$. This gives us

$$\int \frac{\sqrt{3+4x^2}}{x}\,dx = \int \frac{\sqrt{3+(2x)^2}}{2x}(2)\,dx = \int \frac{\sqrt{3+u^2}}{u}\,du$$
$$= \sqrt{3+u^2} - \sqrt{3}\ln\left|\frac{\sqrt{3}+\sqrt{3+u^2}}{u}\right| + c$$
$$= \sqrt{3+4x^2} - \sqrt{3}\ln\left|\frac{\sqrt{3}+\sqrt{3+4x^2}}{2x}\right| + c.$$

A number of the formulas in the table are called **reduction formulas**. These are of the form

$$\int f(u)\,du = g(u) + \int h(u)\,du,$$

where the second integral is simpler than the first. These are often applied repeatedly, as in example 5.2.

EXAMPLE 5.2 Using a Reduction Formula

Use a reduction formula to evaluate $\int \sin^6 x\,dx$.

Solution You should recognize that this integral can be evaluated using techniques you already know. (How?) However, for any integer $n \geq 1$, we have the reduction formula

$$\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1}u\cos u + \frac{n-1}{n}\int \sin^{n-2}u\,du. \tag{5.2}$$

(See number 59 in the table of integrals found inside the back cover of the book.) If we apply (5.2) with $n = 6$, we get

$$\int \sin^6 x\,dx = -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\int \sin^4 x\,dx.$$

Applying the same reduction formula (this time with $n = 4$) to evaluate $\int \sin^4 x\,dx$, we get

$$\int \sin^6 x\,dx = -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\int \sin^4 x\,dx$$
$$= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\left(-\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\,dx\right).$$

Finally, for $\int \sin^2 x\,dx$, we can use (5.2) once again (with $n = 2$), or evaluate the integral using a half-angle formula. We choose the former here and obtain

$$\int \sin^6 x\,dx = -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\left(-\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\,dx\right)$$
$$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x + \frac{5}{8}\left(-\frac{1}{2}\sin x\cos x + \frac{1}{2}\int dx\right)$$
$$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x + c.$$

We should remind you at this point that there are many different ways to find an antiderivative. Antiderivatives found through different means may look quite different, even though they are equivalent. For instance, notice that if an antiderivative has the form $\sin^2 x + c$, then an equivalent antiderivative is $-\cos^2 x + c$, since we can write

$$\sin^2 x + c = 1 - \cos^2 x + c = -\cos^2 x + (1 + c).$$

Finally, since c is an arbitrary constant, so is $1 + c$. In example 5.2, observe that the first three terms all have factors of $\sin x \cos x$, which equals $\frac{1}{2}\sin 2x$. Using this and other identities, you can show that our solution in example 5.2 is equivalent to the following solution obtained from a popular CAS:

$$\int \sin^6 x\,dx = \frac{5}{16}x - \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{192}\sin 6x + c.$$

So, do not panic if your answer differs from the one in the back of the book. Both answers may be correct. If you're unsure, find the derivative of your answer. If you get the integrand, you're right.

You will sometimes want to apply different reduction formulas at different points in a given problem.

EXAMPLE 5.3 Making a Substitution Before Using a Reduction Formula

Evaluate $\int x^3 \sin 2x\,dx$.

Solution From our table (see number 63), we have the reduction formula

$$\int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du. \tag{5.3}$$

In order to use (5.3), we must first make the substitution $u = 2x$, so that $du = 2dx$, which gives us

$$\begin{aligned}\int x^3 \sin 2x\,dx &= \frac{1}{2}\int \frac{(2x)^3}{2^3}\sin 2x(2)\,dx = \frac{1}{16}\int u^3 \sin u\,du\\ &= \frac{1}{16}\left(-u^3\cos u + 3\int u^2 \cos u\,du\right),\end{aligned}$$

where we have used the reduction formula (5.3) with $n = 3$. Now, to evaluate this last integral, we use the reduction formula (number 64 in our table)

$$\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du,$$

with $n = 2$ to get

$$\begin{aligned}\int x^3 \sin 2x\,dx &= -\frac{1}{16}u^3\cos u + \frac{3}{16}\int u^2\cos u\,du\\ &= -\frac{1}{16}u^3\cos u + \frac{3}{16}\left(u^2\sin u - 2\int u\sin u\,du\right).\end{aligned}$$

Applying the first reduction formula (5.3) one more time (this time, with $n = 1$), we get

$$\begin{aligned}\int x^3 \sin 2x\,dx &= -\frac{1}{16}u^3\cos u + \frac{3}{16}u^2\sin u - \frac{3}{8}\int u\sin u\,du\\ &= -\frac{1}{16}u^3\cos u + \frac{3}{16}u^2\sin u - \frac{3}{8}\left(-u\cos u + \int u^0\cos u\,du\right)\\ &= -\frac{1}{16}u^3\cos u + \frac{3}{16}u^2\sin u + \frac{3}{8}u\cos u - \frac{3}{8}\sin u + c\\ &= -\frac{1}{16}(2x)^3\cos 2x + \frac{3}{16}(2x)^2\sin 2x + \frac{3}{8}(2x)\cos 2x - \frac{3}{8}\sin 2x + c\\ &= -\frac{1}{2}x^3\cos 2x + \frac{3}{4}x^2\sin 2x + \frac{3}{4}x\cos 2x - \frac{3}{8}\sin 2x + c.\end{aligned}$$

As we'll see in example 5.4, some integrals require insight before using an integral table.

EXAMPLE 5.4 Making a Substitution Before Using an Integral Table

Evaluate $\int \frac{\sin 2x}{\sqrt{4\cos x - 1}}\,dx$.

Solution You won't find this integral or anything particularly close to it in our integral table. However, with a little fiddling, we can rewrite this in a simpler form. First, use the double-angle formula to rewrite the numerator of the integrand. We get

$$\int \frac{\sin 2x}{\sqrt{4\cos x - 1}}\,dx = 2\int \frac{\sin x\cos x}{\sqrt{4\cos x - 1}}\,dx.$$

Remember to always be on the lookout for terms that are derivatives of other terms. Here, taking $u = \cos x$, we have $du = -\sin x\,dx$ and so,

$$\int \frac{\sin 2x}{\sqrt{4\cos x - 1}}\,dx = 2\int \frac{\sin x\cos x}{\sqrt{4\cos x - 1}}\,dx = -2\int \frac{u}{\sqrt{4u - 1}}\,du.$$

From our table (see number 18), notice that

$$\int \frac{u}{\sqrt{a + bu}}\,du = \frac{2}{3b^2}(bu - 2a)\sqrt{a + bu} + c. \tag{5.4}$$

Taking $a = -1$ and $b = 4$ in (5.4), we have

$$\begin{aligned}\int \frac{\sin 2x}{\sqrt{4\cos x - 1}}\,dx &= -2\int \frac{u}{\sqrt{4u - 1}}\,du = (-2)\frac{2}{3(4^2)}(4u + 2)\sqrt{4u - 1} + c\\ &= -\frac{1}{12}(4\cos x + 2)\sqrt{4\cos x - 1} + c.\end{aligned}$$

○ Integration Using a Computer Algebra System

Computer algebra systems are some of the most powerful new tools to arrive on the mathematical scene in the last 25 years. They run the gamut from handheld calculators (like the TI-89 and the HP-48) to powerful software systems (like Mathematica and Maple).

The examples that follow focus on some of the rare problems you may encounter using a CAS. We admit that we intentionally searched for CAS mistakes. The good news is that the mistakes were very uncommon and the CAS you're using won't necessarily make any of them. Be aware that these are software bugs and the next version of your CAS may eliminate these completely. As an intelligent user of technology, you need to be aware of common errors and have the calculus skills to catch mistakes when they occur.

The first thing you notice when using a CAS to evaluate an indefinite integral is that it typically supplies *an* antiderivative, instead of the most general one (the indefinite integral) by leaving off the constant of integration (a minor shortcoming of this very powerful software).

EXAMPLE 5.5 A Shortcoming of Some Computer Algebra Systems

Use a computer algebra system to evaluate $\int \frac{1}{x}\,dx$.

Solution Many CASs evaluate

$$\int \frac{1}{x}\,dx = \ln x.$$

(Actually, one CAS reports the integral as $\log x$, where it is using the notation $\log x$ to denote the natural logarithm.) Not only is this missing the constant of integration, but notice that this antiderivative is valid only for $x > 0$. A popular calculator returns the more general antiderivative

$$\int \frac{1}{x}\,dx = \ln|x|,$$

which, while still missing the constant of integration, at least is valid for all $x \neq 0$. On the other hand, all of the CASs we tested correctly evaluate

$$\int_{-2}^{-1} \frac{1}{x}\,dx = -\ln 2,$$

even though the reported antiderivative $\ln x$ is not defined at the limits of integration. ■

Sometimes the antiderivative reported by a CAS is not valid, as written, for *any* real values of x, as in example 5.6. (In some cases, CASs give an antiderivative that is correct for the more advanced case of a function of a complex variable.)

EXAMPLE 5.6 An Incorrect Antiderivative

Use a computer algebra system to evaluate $\int \frac{\cos x}{\sin x - 2}\,dx$.

Solution One CAS reports the incorrect antiderivative

$$\int \frac{\cos x}{\sin x - 2}\,dx = \ln(\sin x - 2).$$

At first glance, this may not appear to be wrong, especially since the chain rule seems to indicate that it's correct

$$\frac{d}{dx}\ln(\sin x - 2) = \frac{\cos x}{\sin x - 2}.$$ This is incorrect!

The error is more fundamental (and subtle) than a misuse of the chain rule. Notice that the expression $\ln(\sin x - 2)$ is undefined for *all* real values of x, as $\sin x - 2 < 0$ for all x. A general antiderivative rule that applies here is

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c,$$

where the absolute value is important. The correct antiderivative is $\ln|\sin x - 2| + c$, which can also be written as $\ln(2 - \sin x) + c$ since $2 - \sin x > 0$ for all x. ■

Probably the most common errors you will run into are actually your own. If you give your CAS a problem in the wrong form, it may solve a different problem than you intended. One simple, but common, mistake is shown in example 5.7.

EXAMPLE 5.7 A Problem Where the CAS Misinterprets What You Enter

Use a computer algebra system to evaluate $\int 4x8x\,dx$.

Solution After entering the integrand as $4x8x$, one CAS returned the odd answer

$$\int 4x8x\,dx = 4x\,8xx.$$

You can easily evaluate the integral (first, rewrite the integrand as $32x^2$) to show this is incorrect, but what was the error? Because of the odd way in which we wrote the integrand, the CAS interpreted it as four times a variable named $x8x$, which is unrelated to the variable of integration, x. Its answer is of the form $\int 4c\,dx = 4cx$. ■

The form of the antiderivative reported by a CAS will not always be the most convenient.

EXAMPLE 5.8 An Inconvenient Form of an Antiderivative

Use a computer algebra system to evaluate $\int x(x^2+3)^5\,dx$.

Solution Several CASs evaluate

$$\int x(x^2+3)^5\,dx = \frac{1}{12}x^{12} + \frac{3}{2}x^{10} + \frac{45}{4}x^8 + 45x^6 + \frac{405}{4}x^4 + \frac{243}{2}x^2,$$

while others return the much simpler expression

$$\int x(x^2+3)^5\,dx = \frac{(x^2+3)^6}{12}.$$

The two answers are equivalent, although they differ by a constant. ■

Typically, a CAS will perform even lengthy integrations with ease.

EXAMPLE 5.9 Some Good Integrals for Using a CAS

Use a computer algebra system to evaluate $\int x^3 \sin 2x\,dx$ and $\int x^{10} \sin 2x\,dx$.

Solution Using a CAS, you can get in one step

$$\int x^3 \sin 2x\,dx = -\frac{1}{2}x^3\cos 2x + \frac{3}{4}x^2\sin 2x + \frac{3}{4}x\cos 2x - \frac{3}{8}\sin 2x + c.$$

With the same effort, you can obtain

$$\begin{aligned}\int x^{10}\sin 2x\,dx = &-\frac{1}{2}x^{10}\cos 2x + \frac{5}{2}x^9\sin 2x + \frac{45}{4}x^8\cos 2x - 45x^7\sin 2x\\ &-\frac{315}{2}x^6\cos 2x + \frac{945}{2}x^5\sin 2x + \frac{4725}{4}x^4\cos 2x\\ &-\frac{4725}{2}x^3\sin 2x - \frac{14{,}175}{4}x^2\cos 2x + \frac{14{,}175}{4}x\sin 2x\\ &+\frac{14{,}175}{8}\cos 2x + c.\end{aligned}$$

If you wanted to, you could even evaluate

$$\int x^{100}\sin 2x\,dx,$$

although the large number of terms makes displaying the result prohibitive. Think about doing this by hand, using a staggering 100 integrations by parts or by applying a reduction formula 100 times. ■

TODAY IN MATHEMATICS

Jean-Christophe Yoccoz (1957–2016)
A French mathematician who earned a Fields Medal for his contributions to dynamical systems. His citation for the Fields Medal stated, "He combines an extremely acute geometric intuition, an impressive command of analysis, and a penetrating combinatorial sense to play the chess game at which he excels. He occasionally spends half a day on mathematical 'experiments' by hand or by computer. 'When I make such an experiment,' he says, 'it is not just the results that interest me, but the manner in which it unfolds, which sheds light on what is really going on.'"*

*Douady A. (1995) Présentation de Jean-Christophe Yoccoz, *Proceedings of the International Congress of Mathematicians, Zurich*, 1 (Basel), 11–16.

You should get the idea by now: a CAS can perform repetitive calculations (numerical or symbolic) that you could never dream of doing by hand. It is difficult to find a function that has an elementary antiderivative that your CAS cannot find. Consider the following example of a hard integral.

EXAMPLE 5.10 A Very Hard Integral

Evaluate $\int x^7 e^x \sin x\,dx$.

Solution Consider what you would need to do to evaluate this integral by hand and then use a computer algebra system. For instance, one CAS reports the antiderivative

$$\begin{aligned}\int x^7 e^x \sin x\,dx = &\left(-\frac{1}{2}x^7 + \frac{7}{2}x^6 - \frac{21}{2}x^5 + 105x^3 - 315x^2 + 315x\right)e^x\cos x\\ &+\left(\frac{1}{2}x^7 - \frac{21}{2}x^5 + \frac{105}{2}x^4 - 105x^3 + 315x - 315\right)e^x\sin x.\end{aligned}$$

Don't try this by hand unless you have plenty of time and patience. However, based on your experience, observe that the form of the antiderivative is not surprising. (After all, what kind of function could have $x^7 e^x \sin x$ as its derivative?) ■

BEYOND FORMULAS

You may ask why we've spent so much time on integration techniques when you can always let a CAS do the work for you. No, it's not to prepare you in the event that you are shipwrecked on a desert island without a CAS. Your CAS can solve virtually all of the computational problems that arise in this text. On rare occasions, however, a CAS-generated answer may be incorrect or misleading and you need to be prepared for these. More importantly, many important insights in science and engineering require an understanding of basic techniques of integration.

EXERCISES 3.5

WRITING EXERCISES

1. Suppose that you are hired by a company to develop a new CAS. Outline a strategy for symbolic integration. Include provisions for formulas in the Table of Integrals at the back of the book and the various techniques you have studied.

2. In the text, we discussed the importance of knowing general rules for integration. Consider the integral in example 5.4, $\int \frac{\sin 2x}{\sqrt{4\cos x - 1}}\,dx$. Can your CAS evaluate this integral? For many integrals like this that *do* show up in applications (there are harder ones in the exploratory exercises), you have to do some work before the technology can finish the task. For this purpose, discuss the importance of recognizing basic forms and understanding how substitution works.

In exercises 1–28, use the Table of Integrals at the back of the book to find an antiderivative. Note: When checking the back of the book or a CAS for answers, beware of functions that look very different but that are equivalent (through a trig identity, for instance).

1. $\int \frac{x}{(2+4x)^2}\,dx$

2. $\int \frac{x^2}{(2+4x)^2}\,dx$

3. $\int e^{2x}\sqrt{1+e^x}\,dx$

4. $\int e^{3x}\sqrt{1+e^{2x}}\,dx$

5. $\int \frac{x^2}{\sqrt{1+4x^2}}\,dx$

6. $\int \frac{\cos x}{\sin^2 x(3+2\sin x)}\,dx$

7. $\int_0^1 t^8\sqrt{4-t^6}\,dt$

8. $\int_0^{\ln 4} \sqrt{16-e^{2t}}\,dt$

9. $\int_0^{\ln 2} \frac{e^x}{\sqrt{e^{2x}+4}}\,dx$

10. $\int_{\sqrt{3}}^2 \frac{x\sqrt{x^4-9}}{x^2}\,dx$

11. $\int \frac{\sqrt{6x-x^2}}{(x-3)^2}\,dx$

12. $\int \frac{\sec^2 x}{\tan x\sqrt{8\tan x-\tan^2 x}}\,dx$

13. $\int \tan^6 u\,du$

14. $\int \csc^4 u\,du$

15. $\int \frac{\cos x}{\sin x\sqrt{4+\sin x}}\,dx$

16. $\int \frac{x^3}{\sqrt{4+x^2}}\,dx$

17. $\int x^3\cos x^2\,dx$

18. $\int x\sin 3x^2\cos 4x^2\,dx$

19. $\int \frac{\sin 2x}{\sqrt{1+\cos x}}\,dx$

20. $\int \frac{x\sqrt{1+4x^2}}{x^4}\,dx$

21. $\int \frac{\sin^2 t\cos t}{\sqrt{\sin^2 t+4}}\,dt$

22. $\int \frac{\ln\sqrt{t}}{\sqrt{t}}\,dt$

23. $\int \frac{e^{-2/x^2}}{x^3}\,dx$

24. $\int x^3 e^{2x^2}\,dx$

25. $\int \frac{x}{\sqrt{4x-x^2}}\,dx$

26. $\int e^{2x}\cos 3x\,dx$

27. $\int e^x\tan^{-1}(e^x)\,dx$

28. $\int (\ln 4x)^3\,dx$

29. Check your CAS against all examples in this section. Discuss which errors, if any, your CAS makes.

30. Find out how your CAS evaluates $\int x\sin x\,dx$ if you fail to leave a space between x and $\sin x$.

31. Have your CAS evaluate $\int(\sqrt{1-x}+\sqrt{x-1})\,dx$. If you get an answer, explain why it's wrong.

32. To find out if your CAS "knows" integration by parts, try $\int x^2\cos 3x\,dx$ and $\int x^2 e^{2x}\cos 3x\,dx$. To see if it "knows" reduction formulas, try $\int \sec^5 x\,dx$.

33. To find out how many trigonometric techniques your CAS "knows," try $\int \sin^6 x\,dx$, $\int \sin^4 x\cos^3 x\,dx$ and $\int \tan^4 x\sec^3 x\,dx$.

34. Find out if your CAS has a special command (e.g., APART in Mathematica) to do partial fractions decompositions. Also, try $\int \frac{x^2+2x-1}{(x-1)^2(x^2+4)}\,dx$ and $\int \frac{3x}{(x^2+x+2)^2}\,dx$.

35. To find out if your CAS "knows" how to do substitution, try $\int \frac{1}{x^2(3+2x)}dx$ and $\int \frac{\cos x}{\sin^2 x(3+2\sin x)}dx$. Try to find one that your CAS can't do: start with a basic formula like $\int \frac{1}{|x|\sqrt{x^2-1}}dx = \sec^{-1}x + c$ and substitute your favorite function. With $x = e^u$, the preceding integral becomes $\int \frac{e^u}{e^u\sqrt{e^{2u}-1}}du$, which you can use to test your CAS.

36. To compute the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, note that the upper-right quarter of the ellipse is given by

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

for $0 \le x \le a$. Thus, the area of the ellipse is $4b\int_0^a \sqrt{1 - \frac{x^2}{a^2}}\,dx$.

Try this integral on your CAS. The (implicit) assumption we usually make is that $a > 0$, but your CAS should not make this assumption for you. Does your CAS give you πab or $\pi b|a|$?

37. Briefly explain what it means when your CAS returns $\int f(x)\,dx$ when asked to evaluate $\int f(x)\,dx$.

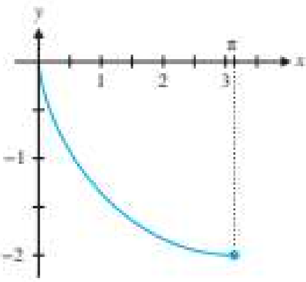
EXPLORATORY EXERCISES

1. This exercise explores two aspects of a very famous problem called the brachistochrone problem. Imagine a bead sliding down a thin wire that extends in some shape from the point (0, 0) to the point $(\pi, -2)$. Assume that gravity pulls the bead down but that there is no friction or other force acting on the bead. This situation is easiest to analyze using **parametric equations** where we have functions $x(u)$ and $y(u)$ giving the horizontal and vertical position of the bead in terms of the variable u. Examples of paths the bead might follow are $\begin{cases} x(u) = \pi u \\ y(u) = -2u \end{cases}$ and $\begin{cases} x(u) = \pi u \\ y(u) = 2(u-1)^2 - 2 \end{cases}$ and $\begin{cases} x(u) = \pi u - \sin \pi u \\ y(u) = \cos \pi u - 1 \end{cases}$. In each case, the bead starts at (0, 0) for $u = 0$ and finishes at $(\pi, -2)$ for $u = 1$. Graph each path on your graphing calculator. The first path is a line, the second is a parabola and the third is a special curve called a cycloid. The time it takes the bead to travel a given path is

$$T = \frac{1}{\sqrt{g}}\int_0^1 \sqrt{\frac{[x'(u)]^2 + [y'(u)]^2}{-2y(u)}}\,du,$$

where g is the gravitational constant. Compute this quantity for the line and the parabola. Explain why the parabola would be a faster path for the bead to slide down, even though the line is shorter in distance. (Think of which would be a faster hill to ski down.) It can be shown that the cycloid is the fastest path possible. Try to get your CAS to compute the optimal time. Comparing the graphs of the parabola and cycloid, what important advantage does the cycloid have at the start of the path?

2. It turns out that the cycloid in exploratory exercise 1 has an amazing property, along with providing the fastest time (which is essentially what the term *brachistochrone* means). The path is shown in the figure.



Suppose that instead of starting the bead at the point (0, 0), you start the bead partway down the path at $x = c$. How would the time to reach the bottom from $x = c$ compare to the total time from $x = 0$? Note that the answer is *not* obvious, since the farther down you start, the less speed the bead will gain. If $x = c$ corresponds to $u = a$, the time to reach the bottom is given by $\frac{\pi}{\sqrt{g}}\int_a^1 \sqrt{\frac{1-\cos \pi u}{\cos a\pi - \cos \pi u}}\,du$. If $a = 0$ (that is, the bead starts at the top), the time is $\pi/\sqrt{g}$ (the integral equals 1). If you have a very good CAS, try to evaluate the integral for various values of a between 0 and 1. If your CAS can't handle it, approximate the integral numerically. You should discover the amazing fact that *the integral always equals 1*. The cycloid also solves the **tautochrone** problem.

3.6 IMPROPER INTEGRALS

Improper Integrals with a Discontinuous Integrand

The phrase "familiarity breeds contempt" has particular relevance for us in this section. You have been using the Fundamental Theorem of Calculus for quite some time now, but do you always check to see that the hypotheses are met? Try to see what is wrong with the following *erroneous* calculation.

$$\int_{-1}^{2} \frac{1}{x^2}dx = \frac{x^{-1}}{-1}\Bigg|_{-1}^{2} = -\frac{3}{2}. \quad \text{This is incorrect!}$$

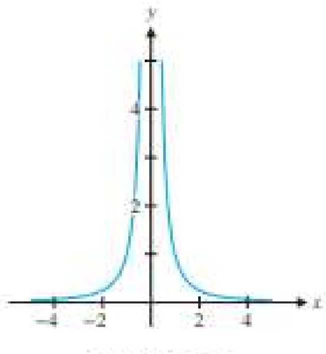


FIGURE 3.2
$y = \frac{1}{x^2}$

There is something fundamentally wrong with this "calculation." Note that $f(x) = 1/x^2$ is not continuous over the interval of integration. (See Figure 3.2.) Since the Fundamental Theorem assumes a continuous integrand, our use of the theorem is invalid and our answer is *incorrect*. Further, note that an answer of $-\frac{3}{2}$ is especially suspicious given that the integrand $\frac{1}{x^2}$ is always positive.

Recall from Chapter 1, that we define the definite integral by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x,$$

where c_i is taken to be any point in the subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$ and where the limit must be the same for any choice of these c_i's. So, if $f(x) \to \infty$ [or $f(x) \to -\infty$] at some point in $[a, b]$, then the limit defining $\int_a^b f(x)\,dx$ is meaningless. [How would we add $f(c_i)$ to the sum, if $f(x) \to \infty$ as $x \to c_i$?] In this case, we call such an integral an **improper integral** and we will need to carefully define what we mean by this. First, we examine a somewhat simpler case.

Consider $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$. Observe that this is *not* a proper definite integral, as the integrand is undefined at $x = 1$. In Figure 3.3a, note that the integrand blows up to ∞ as $x \to 1^-$. Despite this, can we find the area under the curve on the interval [0, 1]? Assuming the area is finite, notice from Figure 3.3b that for $0 < R < 1$, we can approximate it by $\int_0^R \frac{1}{\sqrt{1-x}}\,dx$. This is a proper definite integral, since for $0 \le x \le R < 1$, f is continuous. Further, the closer R is to 1, the better the approximation should be. In the accompanying table, we compute some approximate values of $\int_0^R \frac{1}{\sqrt{1-x}}\,dx$, for a sequence of values of R approaching 1.

R	$\int_0^R \frac{1}{\sqrt{1-x}}\,dx$
0.9	1.367544
0.99	1.8
0.999	1.936754
0.9999	1.98
0.99999	1.993675
0.999999	1.998
0.9999999	1.999368
0.99999999	1.9998

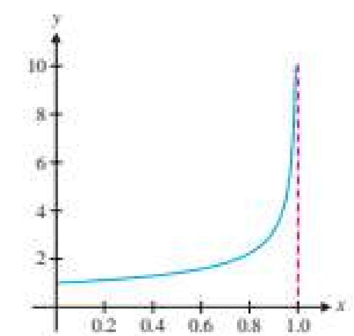


FIGURE 3.3a
$y = \frac{1}{\sqrt{1-x}}$

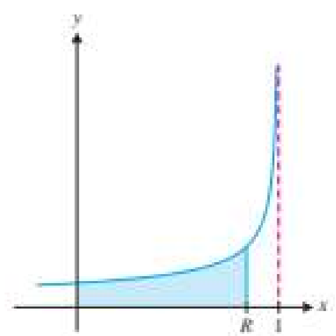


FIGURE 3.3b
$\int_0^R f(x)\,dx$

From the table, the sequence of integrals seems to be approaching 2, as $R \to 1^-$. Notice that since we know how to compute $\int_0^R \frac{1}{\sqrt{1-x}}\,dx$, for any $0 < R < 1$, we can compute this limiting value exactly. We have

$$\lim_{R\to 1^-} \int_0^R \frac{1}{\sqrt{1-x}}\,dx = \lim_{R\to 1^-} \left[-2(1-x)^{1/2}\right]_0^R$$
$$= \lim_{R\to 1^-} \left[-2(1-R)^{1/2} + 2(1-0)^{1/2}\right] = 2.$$

From this computation, we can see that the area under the curve is the limiting value, 2.

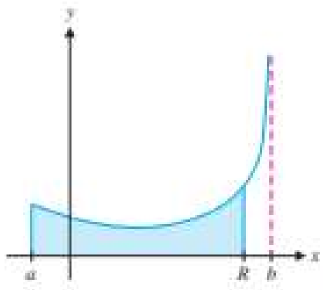


FIGURE 3.4
$\int_a^R f(x)\,dx$

In general, suppose that f is continuous on the interval $[a, b)$ and $|f(x)| \to \infty$, as $x \to b^-$ (i.e., as x approaches b from the left). Then we can approximate $\int_a^b f(x)\,dx$ by $\int_a^R f(x)\,dx$, for some $R < b$, but close to b. [Recall that since f is continuous on $[a, R]$, for any $a < R < b$, $\int_a^R f(x)\,dx$ is defined.] Further, as in our introductory example, the closer R is to b, the better the approximation should be. See Figure 3.4 for a graphical representation of this approximation.

Finally, let $R \to b^-$; if $\int_a^R f(x)\,dx$ approaches some value, L, then we define the **improper integral** $\int_a^b f(x)\,dx$ to be this limiting value. We have the following definition.

DEFINITION 6.1

If f is continuous on the interval $[a, b)$ and $|f(x)| \to \infty$ as $x \to b^-$, we define the improper integral of f on $[a, b]$ by

$$\int_a^b f(x)\,dx = \lim_{R \to b^-} \int_a^R f(x)\,dx.$$

Similarly, if f is continuous on $(a, b]$ and $|f(x)| \to \infty$ as $x \to a^+$, we define the improper integral

$$\int_a^b f(x)\,dx = \lim_{R \to a^+} \int_R^b f(x)\,dx.$$

In either case, if the limit exists (and equals some value L), we say that the improper integral **converges** (to L). If the limit does not exist, we say that the improper integral **diverges.**

EXAMPLE 6.1 An Integrand That Blows Up at the Right Endpoint

Determine whether $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$ converges or diverges.

Solution Based on the work we just completed,

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \lim_{R \to 1^-} \int_0^R \frac{1}{\sqrt{1-x}}\,dx = 2$$

and so, the improper integral converges to 2. ■

In example 6.2, we illustrate a divergent improper integral closely related to this section's introductory example.

EXAMPLE 6.2 A Divergent Improper Integral

Determine whether the improper integral $\int_{-1}^0 \frac{1}{x^2}\,dx$ converges or diverges.

Solution From Definition 6.1, we have

$$\int_{-1}^0 \frac{1}{x^2}\,dx = \lim_{R \to 0^-} \int_{-1}^R \frac{1}{x^2}\,dx = \lim_{R \to 0^-} \left(\frac{x^{-1}}{-1}\right)\Bigg|_{-1}^R$$

$$= \lim_{R \to 0^-} \left(-\frac{1}{R} - \frac{1}{1}\right) = \infty.$$

Since the defining limit does not exist, the improper integral diverges. ■

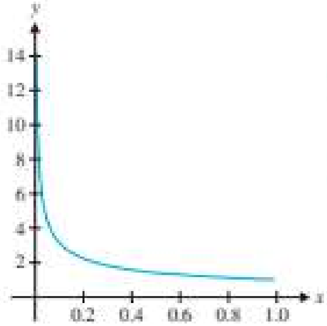


FIGURE 3.5
$y = \frac{1}{\sqrt{x}}$

R	$\int_R^1 \frac{1}{\sqrt{x}}dx$
0.1	1.367544
0.01	1.8
0.001	1.936754
0.0001	1.98
0.00001	1.993675
0.000001	1.998
0.0000001	1.999368
0.00000001	1.9998

In example 6.3, the integrand is discontinuous at the lower limit of integration.

EXAMPLE 6.3 A Convergent Improper Integral

Determine whether the improper integral $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges or diverges.

Solution We show a graph of the integrand on the interval in question in Figure 3.5. Notice that in this case $f(x) = \frac{1}{\sqrt{x}}$ is continuous on $(0, 1]$ and $f(x) \to \infty$ as $x \to 0^+$.

From the computed values shown in the table, it appears that the integrals are approaching 2 as $R \to 0^+$. Since we know an antiderivative for the integrand, we can compute these integrals exactly, for any fixed $0 < R < 1$. We have from Definition 6.1 that

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{R\to 0^+}\int_R^1 \frac{1}{\sqrt{x}}dx = \lim_{R\to 0^+}\frac{x^{1/2}}{\frac{1}{2}}\bigg|_R^1 = \lim_{R\to 0^+} 2(1^{1/2} - R^{1/2}) = 2$$

and so, the improper integral converges to 2. ■

The introductory example in this section represents a third type of improper integral, one where the integrand blows up at a point in the interior of the interval (a, b). We can define such an integral as follows.

DEFINITION 6.2

Suppose that f is continuous on the interval $[a, b]$, except at some $c \in (a, b)$, and $|f(x)| \to \infty$ as $x \to c$. Again, the integral is improper and we write

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

If *both* $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ converge (to L_1 and L_2, respectively), we say that the improper integral $\int_a^b f(x)\,dx$ **converges,** also (to $L_1 + L_2$). If *either* of the improper integrals $\int_a^c f(x)\,dx$ or $\int_c^b f(x)\,dx$ diverges, then we say that the improper integral $\int_a^b f(x)\,dx$ **diverges,** also.

We can now return to our introductory example.

EXAMPLE 6.4 An Integrand That Blows Up in the Middle of an Interval

Determine whether the improper integral $\int_{-1}^2 \frac{1}{x^2}dx$ converges or diverges.

Solution From Definition 6.2, we have

$$\int_{-1}^2 \frac{1}{x^2}dx = \int_{-1}^0 \frac{1}{x^2}dx + \int_0^2 \frac{1}{x^2}dx.$$

In example 6.2, we determined that $\int_{-1}^0 \frac{1}{x^2}dx$ diverges. Thus, $\int_{-1}^2 \frac{1}{x^2}dx$ also diverges. Note that you do *not* need to consider $\int_0^2 \frac{1}{x^2}dx$ (although it's an easy exercise to show that this, too, diverges). Keep in mind that if either of the two improper integrals defining this type of improper integral diverges, then the original integral diverges, too. ■

HISTORICAL NOTES

Pierre Simon Laplace (1749–1827)
A French mathematician who utilized improper integrals to develop the Laplace transform and other important mathematical techniques. Laplace made numerous contributions in probability, celestial mechanics, the theory of heat and a variety of other mathematical topics. Adept at political intrigue, Laplace worked on a new calendar for the French Revolution, served as an advisor to Napoleon and was named a marquis by the Bourbons.

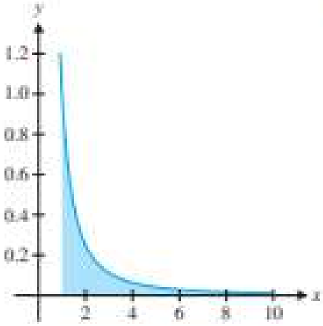


FIGURE 3.6
$y = \frac{1}{x^2}$

R	$\int_1^R \frac{1}{x^2} dx$
10	0.9
100	0.99
1000	0.999
10,000	0.9999
100,000	0.99999
1,000,000	0.999999

Improper Integrals with an Infinite Limit of Integration

Another type of improper integral that is frequently encountered in applications is one where one or both of the limits of integration are infinite. For instance, $\int_0^\infty e^{-x^2} dx$ is of fundamental importance in probability and statistics.

So, given a continuous function f defined on $[a, \infty)$, what could we mean by $\int_a^\infty f(x)\,dx$? Notice that the usual definition of the definite integral:

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^n f(c_i)\,\Delta x,$$

where $\Delta x = \frac{b-a}{n}$, makes no sense when $b = \infty$. We should define $\int_a^\infty f(x)\,dx$ in some way consistent with what we already know about integrals.

Since $f(x) = \frac{1}{x^2}$ is positive and continuous on the interval $[1, \infty)$, $\int_1^\infty \frac{1}{x^2} dx$ should correspond to the area under the curve, assuming this area is, in fact, finite (which is at least plausible, based on the graph in Figure 3.6).

Assuming the area is finite, you could approximate it by $\int_1^R \frac{1}{x^2} dx$, for some large value R. (Notice that this is a proper definite integral, as long as R is finite.) A sequence of values of this integral for increasingly large values of R is displayed in the table.

The sequence of approximating definite integrals seems to be approaching 1, as $R \to \infty$. As it turns out, we can compute this limit exactly. We have

$$\lim_{R \to \infty} \int_1^R x^{-2} dx = \lim_{R \to \infty} \left. \frac{x^{-1}}{-1} \right|_1^R = \lim_{R \to \infty} \left(-\frac{1}{R} + 1 \right) = 1.$$

Thus, the area under the curve on the interval $[1, \infty)$ is seen to be 1, even though the interval has infinite length.

More generally, we have Definition 6.3.

DEFINITION 6.3

If f is continuous on the interval $[a, \infty)$, we define the **improper integral** $\int_a^\infty f(x)\,dx$ to be

$$\int_a^\infty f(x)\,dx = \lim_{R \to \infty} \int_a^R f(x)\,dx.$$

Similarly, if f is continuous on $[-\infty, a)$, we define

$$\int_{-\infty}^a f(x)\,dx = \lim_{R \to -\infty} \int_R^a f(x)\,dx.$$

In either case, if the limit exists (and equals some value L), we say that the improper integral **converges** (to L). If the limit does not exist, we say that the improper integral **diverges**.

You may have already observed that for a decreasing function f, in order for $\int_a^\infty f(x)\,dx$ to converge, it must be the case that $f(x) \to 0$ as $x \to \infty$. (Think about this in terms of area.) However, the reverse need *not* be true. That is, even though $f(x) \to 0$ as $x \to \infty$, the improper integral may diverge, as we see in example 6.5.

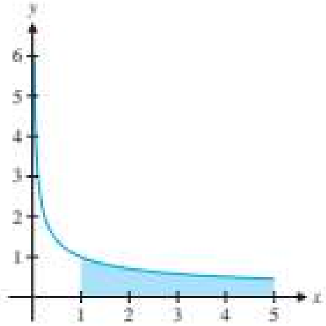


FIGURE 3.7
$y = \frac{1}{\sqrt{x}}$

EXAMPLE 6.5 A Divergent Improper Integral

Determine whether $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ converges or diverges.

Solution Note that $\frac{1}{\sqrt{x}} \to 0$ as $x \to \infty$. Further, from the graph in Figure 3.7, it should seem at least plausible that the area under the curve is finite. However, from Definition 6.3, we have that

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{R\to\infty} \int_1^R x^{-1/2}\,dx = \lim_{R\to\infty} \left.\frac{x^{1/2}}{\frac{1}{2}}\right|_1^R = \lim_{R\to\infty} (2R^{1/2} - 2) = \infty.$$

This says that the improper integral diverges. ■

Note that our introductory example and example 6.5 are special cases of $\int_1^{\infty} \frac{1}{x^p}\,dx$, corresponding to $p = 2$ and $p = 1/2$, respectively. In the exercises, you will show that this integral converges whenever $p > 1$ and diverges for $p \le 1$.

You may need to utilize l'Hôpital's Rule to evaluate the defining limit, as in example 6.6.

EXAMPLE 6.6 A Convergent Improper Integral

Determine whether $\int_{-\infty}^{0} xe^x\,dx$ converges or diverges.

Solution The graph of $y = xe^x$ in Figure 3.8 makes it appear plausible that there could be a finite area under the graph. From Definition 6.3, we have

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{R\to-\infty} \int_R^0 xe^x\,dx.$$

To evaluate the last integral, you will need integration by parts. Let

$$\begin{aligned} u &= x & dv &= e^x\,dx \\ du &= dx & v &= e^x \end{aligned}$$

We then have

$$\begin{aligned} \int_{-\infty}^{0} xe^x\,dx &= \lim_{R\to-\infty} \int_R^0 xe^x\,dx = \lim_{R\to-\infty} \left(xe^x\Big|_R^0 - \int_R^0 e^x\,dx \right) \\ &= \lim_{R\to-\infty} \left[(0 - Re^R) - e^x\Big|_R^0 \right] = \lim_{R\to-\infty} (-Re^R - e^0 + e^R). \end{aligned}$$

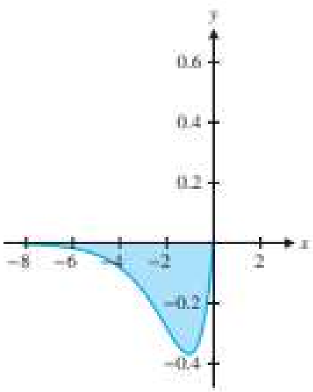


FIGURE 3.8
$y = xe^x$

Note that the limit $\lim_{R\to-\infty} Re^R$ has the indeterminate form $\infty \cdot 0$. We resolve this with l'Hôpital's Rule, as follows:

$$\begin{aligned} \lim_{R\to-\infty} Re^R &= \lim_{R\to-\infty} \frac{R}{e^{-R}} \quad \left(\frac{\infty}{\infty}\right) \\ &= \lim_{R\to-\infty} \frac{\frac{d}{dR}R}{\frac{d}{dR}e^{-R}} = \lim_{R\to-\infty} \frac{1}{-e^{-R}} = 0. \quad \text{By l'Hôpital's Rule} \end{aligned}$$

Returning to the improper integral, we now have

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{R\to-\infty} (-Re^R - e^0 + e^R) = 0 - 1 + 0 = -1.$$

■

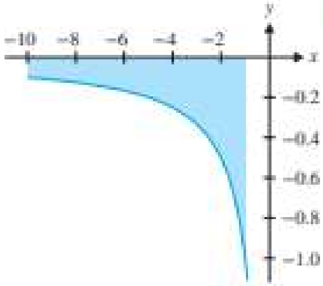


FIGURE 3.9
$y = \frac{1}{x}$

EXAMPLE 6.7 A Divergent Improper Integral

Determine whether $\int_{-\infty}^{-1} \frac{1}{x}\,dx$ converges or diverges.

Solution In Figure 3.9, it appears plausible that there might be a finite area bounded between the graph of $y = \frac{1}{x}$ and the x-axis, on the interval $(-\infty, -1]$. However, from Definition 6.3, we have

$$\int_{-\infty}^{-1} \frac{1}{x}\,dx = \lim_{R\to-\infty} \int_{R}^{-1} \frac{1}{x}\,dx = \lim_{R\to-\infty} \ln|x|\Big|_{R}^{-1} = \lim_{R\to-\infty} [\ln|-1| - \ln|R|] = -\infty$$

and hence, the improper integral diverges. ■

A final type of improper integral is $\int_{-\infty}^{\infty} f(x)\,dx$, defined as follows.

DEFINITION 6.4

If f is continuous on $(-\infty, \infty)$, we write

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx, \quad \text{for any constant } a,$$

where $\int_{-\infty}^{\infty} f(x)\,dx$ converges if and only if *both* $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ converge. If either one diverges, the original improper integral also diverges.

In Definition 6.4, note that you can choose a to be any real number. So, choose it to be something convenient (usually 0).

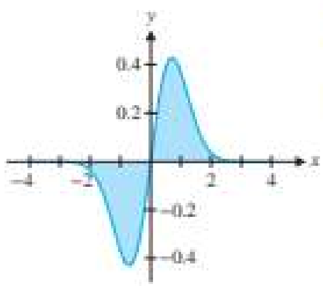


FIGURE 3.10
$y = xe^{-x^2}$

EXAMPLE 6.8 An Integral with Two Infinite Limits of Integration

Determine whether $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$ converges or diverges.

Solution Notice from the graph of the integrand in Figure 3.10 that, since the function tends to 0 relatively quickly (both as $x \to \infty$ and as $x \to -\infty$), it appears plausible that there is a finite area bounded by the graph of the function and the x-axis. From Definition 6.4 we have

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx. \tag{6.1}$$

You must evaluate each of the improper integrals on the right side of (6.1) separately. First, we have

$$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim_{R\to-\infty} \int_{R}^{0} xe^{-x^2}\,dx.$$

Letting $u = -x^2$, we have $du = -2x\,dx$ and so, being careful to change the limits of integration to match the new variable, we have

$$\int_{-\infty}^{0} xe^{-x^2}\,dx = -\frac{1}{2}\lim_{R\to-\infty} \int_{R}^{0} e^{-x^2}(-2x)\,dx = -\frac{1}{2}\lim_{R\to-\infty} \int_{-R^2}^{0} e^{u}\,du$$
$$= -\frac{1}{2}\lim_{R\to-\infty} e^{u}\Big|_{-R^2}^{0} = -\frac{1}{2}\lim_{R\to-\infty} (e^{0} - e^{-R^2}) = -\frac{1}{2}.$$

Similarly, we get (you should fill in the details)

$$\int_{0}^{\infty} xe^{-x^2}\,dx = \lim_{R\to\infty} \int_{0}^{R} xe^{-x^2}\,dx = -\frac{1}{2}\lim_{R\to\infty} e^{u}\Big|_{0}^{-R^2} = -\frac{1}{2}\lim_{R\to\infty} (e^{-R^2} - e^{0}) = \frac{1}{2}.$$

Since both of the preceding improper integrals converge, we get from (6.1) that the original integral also converges, to

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0.$$ ■

EXAMPLE 6.9 An Integral with Two Infinite Limits of Integration

Determine whether $\int_{-\infty}^{\infty} e^{-x}dx$ converges or diverges.

Solution From Definition 6.4, we write

$$\int_{-\infty}^{\infty} e^{-x}dx = \int_{-\infty}^{0} e^{-x}dx + \int_{0}^{\infty} e^{-x}dx.$$

It's easy to show that $\int_{0}^{\infty} e^{-x}dx$ converges. (This is left as an exercise.) However,

$$\int_{-\infty}^{0} e^{-x}dx = \lim_{R\to-\infty}\int_{R}^{0} e^{-x}dx = \lim_{R\to-\infty} -e^{-x}\Big|_{R}^{0} = \lim_{R\to-\infty}(-e^{0} + e^{-R}) = \infty.$$

This says that $\int_{-\infty}^{0} e^{-x}dx$ diverges and hence, $\int_{-\infty}^{\infty} e^{-x}dx$ diverges, also, even though $\int_{0}^{\infty} e^{-x}dx$ converges. ■

CAUTION

Do *not* write

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx.$$

It's certainly tempting to write this, especially since this will often give a correct answer, with about half of the work. Unfortunately, this will often give incorrect answers, too, as the limit on the right-hand side frequently exists for divergent integrals. We explore this issue further in the exercises.

We can't emphasize enough that you should verify the continuity of the integrand for every single integral you evaluate. In example 6.10, we see another reminder of why you must do this.

EXAMPLE 6.10 An Integral That Is Improper for Two Reasons

Determine the convergence or divergence of the improper integral $\int_{0}^{\infty} \frac{1}{(x-1)^2}dx.$

Solution First try to see what is wrong with the following erroneous calculation:

$$\int_{-\infty}^{\infty} \frac{1}{(x-1)^2}dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{1}{(x-1)^2}dx. \quad \text{This is incorrect!}$$

Look carefully at the integrand and observe that it is *not* continuous on $[0, \infty)$. In fact, the integrand blows up at $x = 1$, which is in the interval over which you're trying to integrate. Thus, this integral is improper for several different reasons. In order to deal with the discontinuity at $x = 1$, we must break up the integral into several pieces, as in Definition 6.2. We write

$$\int_{0}^{\infty} \frac{1}{(x-1)^2}dx = \int_{0}^{1} \frac{1}{(x-1)^2}dx + \int_{1}^{\infty} \frac{1}{(x-1)^2}dx. \qquad (6.2)$$

The second integral on the right side of (6.2) must be further broken into two pieces, since it is improper, both at the left endpoint and by virtue of having an infinite limit of integration. You can pick any point on $(1, \infty)$ at which to break up the interval. We'll simply choose $x = 2$. We now have

$$\int_{0}^{\infty} \frac{1}{(x-1)^2}dx = \int_{0}^{1} \frac{1}{(x-1)^2}dx + \int_{1}^{2} \frac{1}{(x-1)^2}dx + \int_{2}^{\infty} \frac{1}{(x-1)^2}dx.$$

Each of these three improper integrals must be evaluated separately, using the appropriate limit definitions. We leave it as an exercise to show that the first two integrals diverge, while the third one converges. This says that the original improper integral diverges (a conclusion you would miss if you did not notice that the integrand blows up at $x = 1$). ■

A Comparison Test

In order to compute the limit(s) defining an improper integral, we first need to find an antiderivative. However, since no antiderivative is available for e^{-x^2}, how would you establish the convergence or divergence of $\int_0^\infty e^{-x^2}\,dx$? An answer lies in the following result.

Given two functions f and g that are continuous on the interval $[a, \infty)$, suppose that

$$0 \le f(x) \le g(x), \text{ for all } x \ge a.$$

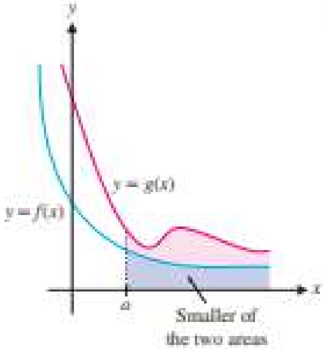


FIGURE 3.11
The Comparison Test

We illustrate this situation in Figure 3.11. In this case, $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ correspond to the areas under the respective curves. Notice that if $\int_a^\infty g(x)\,dx$ (corresponding to the larger area) converges, then this says that there is a finite area under the curve $y = g(x)$ on the interval $[a, \infty)$. Since $y = f(x)$ lies *below* $y = g(x)$, there can be only a finite area under the curve $y = f(x)$, as well. Thus, $\int_a^\infty f(x)\,dx$ converges also.

On the other hand, if $\int_a^\infty f(x)\,dx$ (corresponding to the smaller area) diverges, the area under the curve $y = f(x)$ is infinite. Since $y = g(x)$ lies *above* $y = f(x)$, there must be an infinite area under the curve $y = g(x)$, also, so that $\int_a^\infty g(x)\,dx$ diverges, as well. This comparison of improper integrals based on the relative size of their integrands is called a **Comparison Test** (one of several) and is spelled out in Theorem 6.1.

THEOREM 6.1 (Comparison Test)

Suppose that f and g are continuous on $[a, \infty)$ and $0 \le f(x) \le g(x)$, for all $x \in [a, \infty)$.

(i) If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges, also.

(ii) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges, also.

REMARK 6.1

We can state corresponding comparison tests for improper integrals of the form $\int_{-\infty}^a f(x)\,dx$, where f is continuous on $(-\infty, a]$, as well as for integrals that are improper owing to a discontinuity in the integrand.

We omit the proof of Theorem 6.1, leaving it to stand on the intuitive argument already made.

The idea of the Comparison Test is to compare a given improper integral to another improper integral whose convergence or divergence is already known (or can be more easily determined), as we illustrate in example 6.11.

EXAMPLE 6.11 Using the Comparison Test for an Improper Integral

Determine the convergence or divergence of $\displaystyle\int_0^\infty \frac{1}{x + e^x}\,dx$.

Solution First, note that you do not know an antiderivative for $\dfrac{1}{x + e^x}$ and so, there is no way to compute the improper integral directly. However, notice that for $x \ge 0$,

$$0 \le \frac{1}{x + e^x} \le \frac{1}{e^x}.$$

(See Figure 3.12.) It's an easy exercise to show that $\displaystyle\int_0^\infty \frac{1}{e^x}\,dx$ converges (to 1). From Theorem 6.1, it now follows that $\displaystyle\int_0^\infty \frac{1}{x + e^x}\,dx$ converges, also. While we know that the integral is convergent, the Comparison Test does *not* help to find the value of the integral. We can, however, use numerical integration (e.g., Simpson's Rule) to approximate $\displaystyle\int_0^R \frac{1}{x + e^x}\,dx$, for a sequence of values of R. The accompanying table illustrates some approximate values of $\displaystyle\int_0^R \frac{1}{x + e^x}\,dx$, produced using the numerical integration package built into our CAS. [If you use Simpson's Rule for this, note that you will need to increase the value of n (the number of subintervals in the partition)

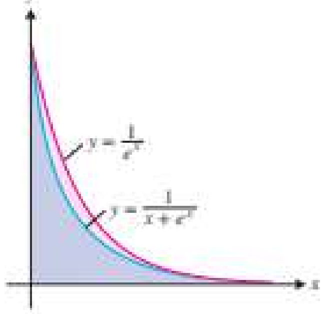


FIGURE 3.12
Comparing $y = \dfrac{1}{e^x}$ and $y = \dfrac{1}{x + e^x}$

R	$\int_0^R \frac{1}{x+e^x}dx$
10	0.8063502
20	0.8063956
30	0.8063956
40	0.8063956

as R increases.] Notice that as R gets larger and larger, the approximate values for the corresponding integrals seem to be approaching 0.8063956, so we take this as an approximate value for the improper integral.

$$\int_0^\infty \frac{1}{x+e^x}dx \approx 0.8063956.$$

You should calculate approximate values for even larger values of R to convince yourself that this estimate is accurate. ■

In example 6.12, we examine an integral that has important applications in probability and statistics.

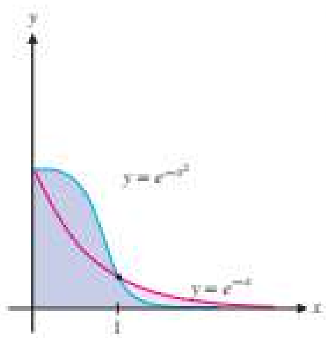


FIGURE 3.13
$y = e^{-x^2}$ and $y = e^{-x}$

EXAMPLE 6.12 Using the Comparison Test for an Improper Integral

Determine the convergence or divergence of $\int_0^\infty e^{-x^2}dx$.

Solution Once again, notice that you do not know an antiderivative for the integrand e^{-x^2}. However, observe that for $x > 1$, $e^{-x^2} < e^{-x}$. (See Figure 3.13.) We can rewrite the integral as

$$\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx.$$

Since the first integral on the right-hand side is a proper definite integral, only the second integral is improper. It's an easy matter to show that $\int_1^\infty e^{-x}\,dx$ converges. By the Comparison Test, it then follows that $\int_1^\infty e^{-x^2}dx$ also converges. We leave it as an exercise to show that

$$\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx \approx 0.8862269.$$

Using more advanced techniques of integration, it is possible to prove the surprising result that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$. ■

The Comparison Test can be used with equal ease to show that an improper integral is divergent.

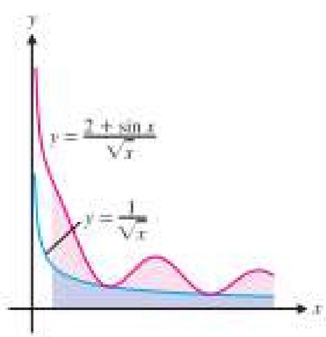


FIGURE 3.14
Comparing $y = \frac{2+\sin x}{\sqrt{x}}$ and $y = \frac{1}{\sqrt{x}}$

EXAMPLE 6.13 Using the Comparison Test: A Divergent Integral

Determine the convergence or divergence of $\int_1^\infty \frac{2+\sin x}{\sqrt{x}}dx$.

Solution As in examples 6.11 and 6.12, you do not know an antiderivative for the integrand and so, your only hope for determining whether or not the integral converges is to use a comparison. First, recall that

$$-1 \le \sin x \le 1, \quad \text{for all } x.$$

We then have that

$$0 < \frac{1}{\sqrt{x}} = \frac{2-1}{\sqrt{x}} \le \frac{2+\sin x}{\sqrt{x}}, \quad \text{for } 1 \le x < \infty.$$

(See Figure 3.14 for a graph of the two functions.) Recall that we showed in example 6.5 that $\int_1^\infty \frac{1}{\sqrt{x}}dx$ diverges. The Comparison Test now tells us that $\int_1^\infty \frac{2+\sin x}{\sqrt{x}}dx$ must diverge, also. ■

The big question, of course, is how to find an improper integral to compare to a given integral. Look carefully at the integrand to see if it resembles any functions whose antiderivative you might know (or at least have a hope of finding using our various techniques of integration). Beyond this, our best answer is that this comes with experience. Comparisons are typically done using only your own experience and trusting your own judgment. We provide ample exercises on this topic to give you some experience with finding appropriate comparisons. Look hard for comparisons and don't give up too easily.

BEYOND FORMULAS

It may seem that this section introduces an overwhelming number of new formulas to memorize. Actually, all of the integrals introduced in this section follow a similar pattern. In each case, we approximate the given integral by integrating over a different interval. The exact value is then found by computing a limit as the approximate interval approaches the desired interval. Answer the following for yourself. How do each of the examples in this section fit this pattern?

EXERCISES 3.6

WRITING EXERCISES

1. For many students, our emphasis on working through the limit process for an improper integral may seem unnecessarily careful. Explain, using examples from this section, why it is important to have and use precise definitions.

2. Identify the following statement as true or false (meaning not always true) and explain why: If the integrand $f(x) \to \infty$ as $x \to a^+$ or as $x \to b^-$, then the area $\int_a^b f(x)\,dx$ is infinite; that is, $\int_a^b f(x)\,dx$ diverges.

In exercises 1 and 2, determine whether or not the integral is improper. If it is improper, explain why

1. (a) $\int_0^2 x^{-2/5}\,dx$ (b) $\int_1^2 x^{-2/5}\,dx$ (c) $\int_0^2 x^{2/5}\,dx$

2. (a) $\int_0^\infty x^{2/5}\,dx$ (b) $\int_{-2}^2 \frac{3}{x}\,dx$ (c) $\int_2^\infty \frac{3}{x}\,dx$

In exercises 3–18, determine whether the integral converges or diverges. Find the value of the integral if it converges.

3. (a) $\int_0^1 x^{-1/3}\,dx$ (b) $\int_0^1 x^{-4/3}\,dx$

4. (a) $\int_1^\infty x^{-4/5}\,dx$ (b) $\int_1^\infty x^{-6/5}\,dx$

5. (a) $\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$ (b) $\int_1^5 \frac{2}{\sqrt{5-x}}\,dx$

6. (a) $\int_0^1 \frac{2}{\sqrt{1-x^2}}\,dx$ (b) $\int_0^1 \frac{2}{x\sqrt{1-x^2}}\,dx$

7. (a) $\int_0^\infty xe^x\,dx$ (b) $\int_1^\infty x^2e^{-2x}\,dx$

8. (a) $\int_{-\infty}^1 x^2e^{3x}\,dx$ (b) $\int_{-\infty}^0 xe^{-4x}\,dx$

9. (a) $\int_{-\infty}^\infty \frac{1}{x^2}\,dx$ (b) $\int_{-\infty}^\infty \frac{1}{\sqrt[3]{x}}\,dx$

10. (a) $\int_0^\infty \cos x\,dx$ (b) $\int_0^\infty \cos x e^{-\sin x}\,dx$

11. (a) $\int_0^1 \ln x\,dx$ (b) $\int_0^\pi \sec^2 x\,dx$

12. (a) $\int_0^\pi \cot x\,dx$ (b) $\int_0^{\pi/2} \tan x\,dx$

13. (a) $\int_0^3 \frac{2}{x^2-1}\,dx$ (b) $\int_{-4}^4 \frac{2x}{x^2-1}\,dx$

14. (a) $\int_0^\pi x\sec^2 x\,dx$ (b) $\int_0^2 \frac{2}{x^3-1}\,dx$

15. (a) $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$ (b) $\int_{-\infty}^\infty \frac{1}{x^2-1}\,dx$

16. (a) $\int_0^2 \frac{x}{x^2-1}\,dx$ (b) $\int_0^\infty \frac{1}{(x-2)^2}\,dx$

17. (a) $\int_0^\infty \frac{1}{\sqrt{x}\,e^{\sqrt{x}}}\,dx$ (b) $\int_0^\infty \tan x\,dx$

18. (a) $\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx$ (b) $\int_0^\infty \frac{x}{\sqrt{x^2+1}}\,dx$

19. (a) Find all values of p for which $\int_0^1 \frac{1}{x^p}\,dx$ converges.

(b) Find all values of p for which $\int_1^\infty \frac{1}{x^p}\,dx$ converges.

(c) Show that $\int_{-\infty}^\infty x^p\,dx$ diverges for every p.

20. (a) Find all values of r for which $\int_0^\infty xe^{rx}\,dx$ converges.

(b) Find all values of r for which $\int_{-\infty}^0 xe^{rx}\,dx$ converges.

In exercises 21–30, use a comparison to determine whether the integral converges or diverges.

21. $\int_1^\infty \frac{x}{1+x^3}\,dx$

22. $\int_1^\infty \frac{x^2-2}{x^4+3}\,dx$

23. $\int_2^\infty \frac{x}{x^{3/2}-1}\,dx$

24. $\int_1^\infty \frac{2+\sec^2 x}{x}\,dx$

25. $\int_0^\infty \frac{3}{x+e^x}\,dx$

26. $\int_1^\infty e^{-x^2}\,dx$

27. $\int_0^\infty \frac{\sin^2 x}{1+e^x}\,dx$

28. $\int_2^\infty \frac{\ln x}{e^x+1}\,dx$

29. $\int_2^\infty \frac{x^2e^x}{\ln x}\,dx$

30. $\int_1^\infty e^{x^2+x+1}\,dx$

In exercises 31 and 32, use integration by parts and l'Hôpital's Rule.

31. $\int_0^1 x\ln 4x\,dx$

32. $\int_0^\infty xe^{-2x}\,dx$

33. In this exercise, you will look at an interesting pair of calculations known as Gabriel's horn. The horn is formed by taking the curve $y = 1/x$ for $x \ge 1$ and revolving it about the x-axis. Show that the volume is finite (i.e., the integral converges), but that the surface area is infinite (i.e., the integral diverges). The paradox is that this would seem to indicate that the horn could be filled with a finite amount of paint but that the outside of the horn could not be covered with any finite amount of paint.

34. Show that $\int_{-\infty}^\infty x^3\,dx$ diverges but $\lim_{R\to\infty}\int_{-R}^R x^3\,dx = 0$.

In exercises 35–38, determine whether the statement is true or false (not always true).

35. If $\lim_{x\to\infty} f(x) = 1$, then $\int_0^\infty f(x)\,dx$ diverges.

36. If $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f(x)\,dx$ converges.

37. If $\lim_{x\to 0} f(x) = \infty$, then $\int_0^1 f(x)\,dx$ diverges.

38. If $f(-x) = -f(x)$ for all x, then $\int_{-\infty}^\infty f(x)\,dx = 0$.

39. (a) Given that $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$, evaluate $\int_{-\infty}^\infty e^{-kx^2}\,dx$ for $k > 0$.

(b) Given that $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$, evaluate $\int_{-\infty}^\infty x^2e^{-kx^2}\,dx$ for $k > 0$.

40. Given that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$, evaluate $\int_0^\infty \frac{\sin kx}{x}\,dx$ for (a) $k > 0$ (b) $k < 0$. Given that $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$, evaluate $\int_0^\infty \frac{\sin^2 kx}{x^2}\,dx$ for (c) $k > 0$, (d) $k < 0$.

41. Noting that $\frac{x}{x^3+1} \approx \frac{1}{x^2}$ for large values of x, explain why you would expect $\int_0^\infty \frac{x}{x^3+1}\,dx$ to converge. Use a comparison test to prove that it does.

42. As in exercise 41, quickly conjecture whether the integral converges or diverges.

(a) $\int_2^\infty \frac{x}{\sqrt{x^3-1}}\,dx$

(b) $\int_2^\infty \frac{x}{\sqrt{x^5-1}}\,dx$

(c) $\int_2^\infty \frac{x}{\sqrt{x^3+x-1}}\,dx$

43. Use the substitution $u = \frac{\pi}{2} - x$ to show that (a) $\int_0^{\pi/2} \ln(\sin x)\,dx = \int_0^{\pi/2} \ln(\cos x)\,dx$. Add $\int_0^{\pi/2} \ln(\sin x)\,dx$ to both sides of this equation and simplify the right-hand side with the identity $\sin 2x = 2\sin x\cos x$. (b) Use this result to show that $2\int_0^{\pi/2} \ln(\sin x)\,dx = -\frac{\pi}{2}\ln 2 + \frac{1}{2}\int_0^\pi \ln(\sin x)\,dx$. (c) Show that $\int_0^\pi \ln(\sin x)\,dx = 2\int_0^{\pi/2} \ln(\sin x)\,dx$. (d) Use parts (b) and (c) to evaluate $\int_0^{\pi/2} \ln(\sin x)\,dx$.

44. Show that for any positive integer n, $\int_0^1 (\ln x)^n\,dx$ equals $n!$ if n is even and $-n!$ if n is odd. [Hint: $\lim_{x\to 0^+} x(\ln x)^n = 0$.]

45. Explain why $\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx$ is an improper integral. Assuming that it converges, explain why it is equal to $\int_0^{\pi/2} f(x)\,dx$, where $f(x) = \begin{cases} \frac{1}{1+\tan x} & \text{if } 0 \le x < \frac{\pi}{2} \\ 0 & \text{if } x = \frac{\pi}{2} \end{cases}$. Similarly, find a function $g(x)$ such that the improper integral $\int_0^{\pi/2} \frac{\tan x}{1+\tan x}\,dx$ equals the proper integral $\int_0^{\pi/2} g(x)\,dx$. Use the substitution $u = x - \frac{\pi}{2}$ to show that $\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx = \int_0^{\pi/2} \frac{\tan x}{1+\tan x}\,dx$. Adding the first integral to both sides of the equation, evaluate $\int_0^{\pi/2} \frac{1}{1+\tan x}\,dx$.

46. Generalize exercise 45 to evaluate $\int_0^{\pi/2} \frac{1}{1+\tan^k x}\,dx$ for any real number k.

47. Assuming that all integrals in this exercise converge, use integration by parts to write $\int_{-\infty}^\infty x^4e^{-x^2}\,dx$ in terms of $\int_{-\infty}^\infty x^2e^{-x^2}\,dx$ and then in terms of $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$. By induction, show that $\int_{-\infty}^\infty x^{2n}e^{-x^2}\,dx = \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2^n}\sqrt{\pi}$, for any positive integer n.

48. Show that $\int_{-\infty}^\infty e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$, for any positive constant $a \ne 0$. Formally (that is, differentiate under the integral sign) compute n derivatives with respect to a of this equation, set $a = 1$ and compare the result to that of exercise 47.

APPLICATIONS

1. A function $f(x) \geq 0$ is a probability density function (PDF) on the interval $[0, \infty)$ if $\int_0^\infty f(x)\,dx = 1$. Find the value of the constant k to make each of the following PDF's on the interval $[0, \infty)$.

   (a) ke^{-2x} (b) ke^{-4x} (c) ke^{-rx}, $r > 0$

2. Find the value of the constant k to make each of the following PDF's on the interval $[0, \infty)$. (See exercise 1.)

   (a) kxe^{-2x} (b) kxe^{-4x} (c) kxe^{-rx}, $r > 0$

3. The **mean** μ (one measure of average) of a random variable with PDF $f(x)$ on the interval $[0, \infty)$ is $\mu = \int_0^\infty xf(x)\,dx$. Find the mean of the exponential distribution $f(x) = re^{-rx}$, $r > 0$.

4. Find the mean of a random variable with PDF $f(x) = r^2xe^{-rx}$.

5. Many probability questions involve **conditional probabilities**. For example, if you know that a lightbulb has already burned for 30 hours, what is the probability that it will last at least 5 more hours? This is the "probability that $x > 35$ given that $x > 30$" and is written as $P(x > 35|x > 30)$. In general, for events A and B, $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$, which in this case reduces to $P(x > 35|x > 30) = \frac{P(x > 35)}{P(x > 30)}$. For the PDF $f(x) = \frac{1}{40}e^{-x/40}$ (in hours), compute $P(x > 35|x > 30)$. Also, compute $P(x > 40|x > 35)$ and $P(x > 45|x > 40)$. (Hint: $P(x > 35) = 1 - P(x \leq 35)$.)

6. Exercise 5 illustrates the "memoryless property" of exponential distributions. The probability that a lightbulb lasts m more hours given that it has already lasted n hours depends only on m and not on n. (a) Prove this for the PDF $f(x) = \frac{1}{40}e^{-x/40}$. (b) Show that any exponential PDF $f(x) = ce^{-cx}$ has the memory-less property, for $c > 0$.

7. The **Omega function** is used for risk/reward analysis of financial investments. Suppose that $f(x)$ is a PDF on $(-\infty, \infty)$ and gives the distribution of returns on an investment. (Then $\int_a^b f(x)\,dx$ is the probability that the investment returns between \$$a$ and \$$b$.) Let $F(x) = \int_{-\infty}^x f(t)\,dt$ be the **cumulative distribution function** for returns. Then $\Omega(r) = \frac{\int_r^\infty [1 - F(x)]dx}{\int_{-\infty}^r F(x)\,dx}$ is the Omega function for the investment.

   (a) Compute $\Omega_1(r)$ for the exponential distribution $f_1(x) = 2e^{-2x}$, $0 \leq x < \infty$. Note that $\Omega_1(r)$ will be undefined (∞) for $r \leq 0$.

   (b) Compute $\Omega_2(r)$ for $f_2(x) = 1$, $0 \leq x \leq 1$.

   (c) Show that the means of $f_1(x)$ and $f_2(x)$ are the same and that $\Omega(r) = 1$ when r equals the mean.

   (d) Even though the means are the same, investments with distributions $f_1(x)$ and $f_2(x)$ are not equivalent. Use the graphs of $f_1(x)$ and $f_2(x)$ to explain why $f_1(x)$ corresponds to a riskier investment than $f_2(x)$.

   (e) Show that for some value c, $\Omega_2(r) > \Omega_1(r)$ for $r < c$ and $\Omega_2(r) < \Omega_1(r)$ for $r > c$. In general, the larger $\Omega(r)$ is, the better the investment is. Explain this in terms of this example.

8. The **reliability function** $R(t)$ gives the probability that $x > t$. For the PDF of a lightbulb, this is the probability that the bulb lasts at least t hours. Compute $R(t)$ for a general exponential PDF $f(x) = ce^{-cx}$.

9. The so-called **Boltzmann integral**

$$I(p) = \int_0^1 p(x) \ln p(x)\,dx$$

is important in the mathematical field of **information theory**. Here, $p(x)$ is a PDF on the interval $[0, 1]$. Graph the PDF's $p_1(x) = 1$ and

$$p_2(x) = \begin{cases} 4x & \text{if } 0 \leq x \leq 1/2 \\ 4 - 4x & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

and compute the integrals $\int_0^1 p_1(x)\,dx$ and $\int_0^1 p_2(x)\,dx$ to verify that they are PDF's. Then compute the Boltzmann integrals $I(p_1)$ and $I(p_2)$. Suppose that you are trying to determine the value of a quantity that you know is between 0 and 1. If the PDF for this quantity is $p_1(x)$, then all values are equally likely. What would a PDF of $p_2(x)$ indicate? Noting that $I(p_2) > I(p_1)$, explain why it is fair to say that the Boltzmann integral measures the amount of *information* available. Given this interpretation, sketch a PDF $p_3(x)$ that would have a larger Boltzmann integral than $p_2(x)$.

EXPLORATORY EXERCISES

1. The **Laplace transform** is an invaluable tool in many engineering disciplines. As the name suggests, the transform turns a function $f(t)$ into a different function $F(s)$. By definition, the Laplace transform of the function $f(t)$ is

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

To find the Laplace transform of $f(t) = 1$, compute

$$\int_0^\infty (1)e^{-st}\,dt = \int_0^\infty e^{-st}dt.$$

Show that the integral equals $1/s$, for $s > 0$. We write $L\{1\} = 1/s$. Show that

$$L\{t\} = \int_0^\infty te^{-st}\,dt = \frac{1}{s^2},$$

for $s > 0$. Compute $L\{t^2\}$ and $L\{t^3\}$ and conjecture the general formula for $L\{t^n\}$. Then, find $L\{e^{at}\}$ for $s > a$.

2. The **gamma function** is defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, if the integral converges. For such a complicated-looking function, the gamma function has some surprising properties. First, show that $\Gamma(1) = 1$. Then use integration by parts and l'Hôpital's Rule to show that $\Gamma(n + 1) = n\Gamma(n)$, for any $n > 0$. Use this property and mathematical induction to show that $\Gamma(n + 1) = n!$, for any positive integer n. (Notice that this includes the value $0! = 1$.) Numerically approximate $\Gamma\left(\frac{3}{2}\right)$ and $\Gamma\left(\frac{5}{2}\right)$. Is it reasonable to define these as $\left(\frac{1}{2}\right)!$ and $\left(\frac{3}{2}\right)!$,

respectively? In this sense, show that $\left(\frac{1}{2}\right)! = \frac{1}{2}\sqrt{\pi}$. Finally, for $x < 1$, the defining integral for $\Gamma(x)$ is improper in two ways. Use a comparison-test to show the convergence of $\int_1^\infty t^{x-1}e^{-t}\,dt$. This leaves $\int_0^1 t^{x-1}e^{-t}\,dt$. Determine the range of p-values for which $\int_0^1 t^p e^{-t}\,dt$ converges and then determine the set of x's for which $\Gamma(x)$ is defined.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Integration by parts	Reduction formula
Partial fractions decomposition	CAS
Improper integral	Integral converges
Integral diverges	Comparison Test

TRUE OR FALSE

State whether each statement is true or false, and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. Integration by parts works only for integrals of the form $\int f(x)g(x)\,dx$.
2. For an integral of the form $\int xf(x)\,dx$, always use integration by parts with $u = x$.
3. The trigonometric techniques in section 6.3 are all versions of substitution.
4. If an integrand contains a factor of $\sqrt{1-x^2}$, you should substitute $x = \sin\theta$.
5. If p and q are polynomials, then any integral of the form $\int \frac{p(x)}{q(x)}\,dx$ can be evaluated.
6. With an extensive integral table, you don't need to know any integration techniques.
7. If $f(x)$ has a vertical asymptote at $x = a$, then $\int_a^b f(x)\,dx$ diverges for any b.
8. If $\lim_{x\to\infty} f(x) = L \neq 0$, then $\int_1^\infty f(x)\,dx$ diverges.

In exercises 1–8, find the missing terms for each of the following cases.

1. $\int \frac{?}{\sec x + \tan x}\,dx = \ln|\sec x + \tan x| + c$

2. $\int \frac{e^x - e^{-x}}{?}\,dx = \ln|?| + c$

3. $\int \frac{e^?}{\sqrt{1-x^2}}\,dx = e^? + c$

4. $\int \frac{?}{?+?}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{3x}{2}\right) + c$

5. $\int \frac{?}{?+9}\,dx = \frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right) + c$

6. $\int \frac{?}{?}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$

7. $\frac{3x^2 - 7x - 2}{x^3 - x} = \frac{A}{x} + \frac{B}{?} + \frac{C}{?}$

8. $\frac{2x^2 - 5x + 2}{x^3 + x} = \frac{?}{x} + \frac{?}{x^2+1}$

In exercises 9–53, evaluate the integral.

9. $\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

10. $\int \frac{\sin(1/x)}{x^2}\,dx$

11. $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

12. $\int \frac{2}{\sqrt{9-x^2}}\,dx$

13. $\int x^2 e^{-3x}\,dx$

14. $\int x^2 e^{-x^3}\,dx$

15. $\int \frac{x}{1+x^4}\,dx$

16. $\int \frac{x^3}{1+x^4}\,dx$

17. $\int \frac{x^3}{4+x^4}\,dx$

18. $\int \frac{x}{4+x^4}\,dx$

19. $\int e^{2\ln x}\,dx$

20. $\int \cos 4x\,dx$

21. $\int_0^1 x\sin 3x\,dx$

22. $\int_0^1 x\sin 4x^2\,dx$

23. $\int_0^{\pi/2} \sin^4 x\,dx$

24. $\int_0^{\pi/2} \cos^3 x\,dx$

25. $\int_{-1}^1 x\sin \pi x\,dx$

26. $\int_0^1 x^2\cos \pi x\,dx$

27. $\int_1^2 x^3 \ln x\,dx$

28. $\int_0^{\pi/4} \sin x\cos x\,dx$

29. $\int \cos x\sin^2 x\,dx$

30. $\int \cos x\sin^3 x\,dx$

31. $\int \cos^3 x\sin^3 x\,dx$

32. $\int \cos^4 x\sin^3 x\,dx$

33. $\int \tan^2 x\sec^4 x\,dx$

34. $\int \tan^3 x\sec^2 x\,dx$

Review Exercises

35. $\int \sqrt{\sin x}\cos^3 x\,dx$

36. $\int \tan^3 x\sec^3 x\,dx$

37. $\int \frac{2}{8+4x+x^2}\,dx$

38. $\int \frac{3}{\sqrt{-2x-x^2}}\,dx$

39. $\int \frac{2}{x^2\sqrt{4-x^2}}\,dx$

40. $\int \frac{x}{\sqrt{9-x^2}}\,dx$

41. $\int \frac{x^3}{\sqrt{9-x^2}}\,dx$

42. $\int \frac{x^3}{\sqrt{x^2-9}}\,dx$

43. $\int \frac{x^3}{\sqrt{x^2+9}}\,dx$

44. $\int \frac{4}{\sqrt{x+9}}\,dx$

45. $\int \frac{x+4}{x^2+3x+2}\,dx$

46. $\int \frac{5x+6}{x^2+x-12}\,dx$

47. $\int \frac{4x^2+6x-12}{x^3-4x}\,dx$

48. $\int \frac{5x^2+2}{x^3+x}\,dx$

49. $\int e^x\cos 2x\,dx$

50. $\int x^3\sin x^2\,dx$

51. $\int x\sqrt{x^2+1}\,dx$

52. $\int \sqrt{1-x^2}\,dx$

In exercises 53–58, find the partial fractions decomposition.

53. $\frac{4}{x^2-3x-4}$

54. $\frac{2x}{x^2+x-6}$

55. $\frac{-6}{x^3+x^2-2x}$

56. $\frac{x^2-2x-2}{x^3+x}$

57. $\frac{x-2}{x^2+4x+4}$

58. $\frac{x^2-2}{(x^2+1)^2}$

In exercises 59–68, use the Table of Integrals to find the integral.

59. $\int e^{3x}\sqrt{4+e^{2x}}\,dx$

60. $\int x\sqrt{x^4-4}\,dx$

61. $\int \sec^4 x\,dx$

62. $\int \tan^3 x\,dx$

63. $\int \frac{4}{x(3-x)^2}\,dx$

64. $\int \frac{\cos x}{\sin^2 x(3+4\sin x)}\,dx$

65. $\int \frac{\sqrt{9+4x^2}}{x^2}\,dx$

66. $\int \frac{x^2}{\sqrt{4-9x^2}}\,dx$

67. $\int \frac{\sqrt{4-x^2}}{x}\,dx$

68. $\int \frac{x^2}{(x^6-4)^{3/2}}\,dx$

In exercises 69–76, determine whether the integral converges or diverges. If it converges, find the limit.

69. $\int_0^1 \frac{x}{x^2-1}\,dx$

70. $\int_4^{10} \frac{2}{\sqrt{x-4}}\,dx$

71. $\int_1^\infty \frac{3}{x^2}\,dx$

72. $\int_1^\infty xe^{-3x}\,dx$

73. $\int_0^\infty \frac{4}{4+x^2}\,dx$

74. $\int_{-\infty}^\infty xe^{-x^2}\,dx$

75. $\int_{-2}^2 \frac{3}{x^2}\,dx$

76. $\int_{-2}^2 \frac{x}{1-x^2}\,dx$

77. Cardiologists test heart efficiency by injecting a dye at a constant rate R into a vein near the heart and measuring the concentration of the dye in the bloodstream over a period of T seconds. If all of the dye is pumped through, the concentration is $c(t)=R$. Compute the total amount of dye $\int_0^T c(t)\,dt$. For a general concentration, the cardiac output is defined by $\frac{RT}{\int_0^T c(t)\,dt}$. Interpret this quantity. Compute the cardiac output if $c(t)=3te^{-2t/T}$.

78. For $\int \ln(x+1)\,dx$, you can use integration by parts with $u=\ln(x+1)$ and $dv=1$. Compare your answers using $v=x$ versus using $v=x+1$.

79. Show that the average value of $\ln x$ on the interval $(0, e^n)$ equals $n-1$ for any positive integer n.

80. Many probability questions involve **conditional probabilities**. For example, if you know that a lightbulb has already burned for 30 hours, what is the probability that it will last at least 5 more hours? This is the "probability that $x>35$ given that $x>30$" and is written as $P(x>35|x>30)$. In general, for events A and B, $P(A|B)=\frac{P(A \text{ and } B)}{P(B)}$. The **failure rate function** is given as the limit of $\frac{P(x<t+\Delta t|x>t)}{\Delta t}$ as $\Delta t \to 0$. For the PDF $f(x)$ of the lifetime of a lightbulb, the numerator is the probability that the bulb burns out between times t and $t+\Delta t$. Use $R(t)=P(x>t)$ to show that the failure rate function can be written as $\frac{f(t)}{R(t)}$.

81. Show that the failure rate function (see exercise 72) of an exponential PDF $f(x)=ce^{-cx}$ is constant.

82. For the gamma distribution $f(x)=xe^{-x}$, (a) use a CAS to show that $P(x>s+t|x>s)=e^{-t}+\frac{t}{1+s}e^{-t}$. (b) Show that this is a decreasing function of s (for a fixed t). (c) If this is the PDF for annual rainfall amounts in a certain city, interpret the result of part (b).

83. Scores on IQ tests are intended to follow the distribution $f(x)=\frac{1}{\sqrt{450\pi}}e^{-(x-100)^2/450}$. Based on this distribution, what percentage of people are supposed to have IQs between 90 and 100? If the top 1% of scores are to be given the title of "genius," how high do you have to score to get this title?

84. Define $I(n)=\int_0^\infty \frac{1}{(1+x^2)^n}\,dx$ for positive integers n and show that $I(1)=\frac{\pi}{2}$. Using integration by parts with $u=\frac{1}{(1+x^2)^n}$, show that $I(n+1)=\frac{2n-1}{2n}I(n)$. Conclude that $I(n)=\frac{\pi}{2}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n-2)}$.

Review Exercises

EXPLORATORY EXERCISES

1. In this exercise, you will try to determine whether or not $\int_0^1 \sin(1/x)\,dx$ converges. Since $|\sin(1/x)| \le 1$, the integral does not diverge to ∞, but that does not necessarily mean it converges. Explain why the integral $\int_0^\infty \sin x\,dx$ diverges (not to ∞, but by oscillating indefinitely). You need to determine whether a similar oscillation occurs for $\int_0^1 \sin(1/x)\,dx$. First, estimate $\int_R^1 \sin(1/x)\,dx$ numerically for $R = 1/\pi$, $1/(2\pi)$, $1/(3\pi)$ and so on. Note that once you have $\int_{1/\pi}^1 \sin(1/x)\,dx$, you can get $\int_{1/(2\pi)}^1 \sin(1/x)\,dx$ by "adding" $\int_{1/(2\pi)}^{1/\pi} \sin(1/x)\,dx$. We put this in quotes because this new integral is negative. Verify that the integrals $\int_{1/(2\pi)}^{1/\pi} \sin(1/x)\,dx$, $\int_{1/(3\pi)}^{1/(2\pi)} \sin(1/x)\,dx$ and so on, are alternately negative and positive, so that the sum $\int_R^1 \sin(1/x)\,dx$ does seem to converge as $R \to 0^+$. It turns out that the limit does converge if the additional integrals $\int_{1/((n+1)\pi)}^{1/(n\pi)} \sin(1/x)\,dx$ tend to 0 as $n \to \infty$. Show that this is true.

2. Suppose that $f(x)$ is a function such that both $\int_{-\infty}^{\infty} f(x)\,dx$ and $\int_{-\infty}^{\infty} f(x - 1/x)\,dx$ converge. Start with $\int_{-\infty}^{\infty} f(x - 1/x)\,dx$ and make the substitution $u = -\frac{1}{x}$. Show that $2\int_{-\infty}^{\infty} f(x - 1/x)\,dx = \int_{-\infty}^{\infty} \frac{1}{u^2} f(u - 1/u)\,du$. Then let $y = u - 1/u$. Show that $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x - 1/x)\,dx$. Use this result to evaluate $\int_{-\infty}^{\infty} \frac{x^2}{x^2 + (x^2 - 1)^2}\,dx$ and $\int_{-\infty}^{\infty} e^{-x^2+2-1/x^2}\,dx$.

3. Evaluate $\int_0^{\pi/2} \frac{ab}{(a\cos x + b\sin x)^2}\,dx$, by dividing all terms by $\cos^2 x$, using the substitution $u = ab\tan x$ and evaluating the improper integral $\int_0^\infty \frac{a^2}{(u + a^2)^2}\,dx$.

First-Order Differential Equations

CHAPTER

4

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Chapter Topics

4.1 Modeling with Differential Equations

4.2 Separable Differential Equations

4.3 First-Order Linear Differential Equations

4.4 Direction Fields and Euler's Method

Well-preserved fossils can provide paleontologists with priceless clues about the early history of life on Earth. In 1993, an amateur fossil hunter found the bones of a massive dinosaur in southern Argentina. The new species of dinosaur, named *Giganotosaurus,* replaced *Tyrannosaurus Rex* as the largest known carnivore. Measuring up to 13.7 m in length and standing 3.7 m high at the hip bone, *Giganotosaurus* is estimated to have weighed roughly 8 tons.

Paleontologists employ several techniques for dating fossils, in order to place them in their correct historical perspective. The most well known of these is radiocarbon dating using carbon-14, an unstable isotope of carbon. In living plants and animals, the ratio of the amount of carbon-14 to the total amount of carbon is constant. When a plant or animal dies, it stops taking in carbon-14 and the existing carbon-14 begins to decay, at a constant (though nearly imperceptible) rate. An accurate measurement of the proportion of carbon-14 remaining can then be converted into an estimate of the time of death. Estimates from carbon-14 dating are considered to be reliable for fossils dating back tens of thousands of years, due to its very slow rate of decay.

Dating using other radioisotopes with slower decay rates than that of carbon-14 works on the same basic principle, but can be used to accurately date very old rock or sediment that surrounds the fossils. Using such techniques, paleontologists estimate that *Giganotosaurus* lived about 100 million years ago. This is critical information to scientists studying life near the end of the Mesozoic era. For example, based on this method of dating, it is apparent that *Giganotosaurus* did not live at the same time and therefore did not compete with the smaller but stronger *Tyrannosaurus Rex.*

The mathematics underlying carbon-14 and other radioisotope dating techniques is developed in this chapter. The study of differential equations provides you with essential tools to analyze many important phenomena such as radiocarbon dating and the population of a bacterial colony. In this chapter, we introduce the basic theory and a few common applications of some elementary differential equations.

4.1 MODELING WITH DIFFERENTIAL EQUATIONS

Growth and Decay Problems

In this age, we are all keenly aware of how infection by micro-organisms such as *Escherichia coli* (*E. coli*) causes disease. Many organisms (such as *E. coli*)

Time (hours)	*Number of Bacteria (millions per ml)*
0	1.2
0.5	2.5
1	5.1
1.5	11.0
2	23.0
2.5	45.0
3	91.0
3.5	180.0
4	350.0

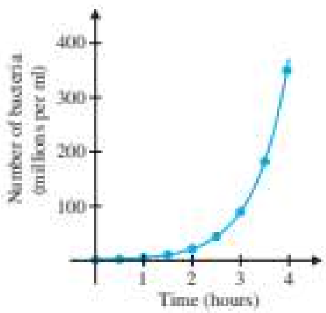


FIGURE 4.1
Growth of bacteria

can reproduce in our bodies at a surprisingly fast rate, overwhelming our bodies' natural defenses with the sheer volume of toxin they are producing. The table shown in the margin indicates the number of *E. coli* bacteria (in millions of bacteria per ml) in a laboratory culture measured at half-hour intervals during the course of an experiment. We have plotted the number of bacteria per milliliter versus time in Figure 4.1.

What would you say the graph most resembles? If you said, "an exponential," you guessed right. Careful analysis of experimental data has shown that many populations grow at a rate proportional to their current level. This is quite easily observed in bacterial cultures, where the bacteria reproduce by binary fission (i.e., each cell reproduces by dividing into two cells). In this case, the rate at which the bacterial culture grows is directly proportional to the current population (until such time as resources become scarce or overcrowding becomes a limiting factor). If we let $y(t)$ represent the number of bacteria in a culture at time t, then the rate of change of the population with respect to time is $y'(t)$. Thus, since $y'(t)$ is proportional to $y(t)$, we have

$$y'(t) = ky(t), \tag{1.1}$$

for some constant of proportionality k (the **growth constant**). Since equation (1.1) involves the derivative of an unknown function, we call it a **differential equation**. Our aim is to *solve* the differential equation, that is, find the *function* $y(t)$. Assuming that $y(t) > 0$ (this is a reasonable assumption, since $y(t)$ represents a population), we have

$$\frac{y'(t)}{y(t)} = k. \tag{1.2}$$

Integrating both sides of equation (1.2) with respect to t, we obtain

$$\int \frac{y'(t)}{y(t)}\, dt = \int k\, dt. \tag{1.3}$$

Substituting $y = y(t)$ in the integral on the left-hand side, we have $dy = y'(t)\, dt$ and so, (1.3) becomes

$$\int \frac{1}{y}\, dy = \int k\, dt.$$

Evaluating these integrals, we obtain

$$\ln |y| + c_1 = kt + c_2,$$

where c_1 and c_2 are constants of integration. Subtracting c_1 from both sides yields

$$\ln |y| = kt + (c_2 - c_1) = kt + c,$$

for some constant c. Since $y(t) > 0$, we have

$$\ln y(t) = kt + c$$

and taking exponentials of both sides, we get

$$y(t) = e^{\ln y(t)} = e^{kt+c} = e^{kt}e^{c}.$$

Since c is an arbitrary constant, we write $A = e^c$ and get

$$y(t) = Ae^{kt}. \tag{1.4}$$

We refer to (1.4) as the **general solution** of the differential equation (1.1). For $k > 0$, equation (1.4) is called an **exponential growth law** and for $k < 0$, it is an **exponential decay law**. (Think about the distinction.)

In example 1.1, we examine how an exponential growth law predicts the number of cells in a bacterial culture.

EXAMPLE 1.1 Exponential Growth of a Bacterial Colony

A freshly inoculated bacterial culture of *Streptococcus A* (a common group of microorganisms that cause strep throat) contains 100 cells. When the culture is checked 60 minutes later, it is determined that there are 450 cells present. Assuming exponential growth, determine the number of cells present at any time t (measured in minutes) and find the doubling time.

Solution Exponential growth means that

$$y'(t) = ky(t)$$

and hence, from (1.4),

$$y(t) = Ae^{kt}, \tag{1.5}$$

where A and k are constants to be determined. If we set the starting time as $t = 0$, we have

$$y(0) = 100. \tag{1.6}$$

Equation (1.6) is called an **initial condition**. Setting $t = 0$ in (1.5), we now have

$$100 = y(0) = Ae^0 = A$$

and hence,

$$y(t) = 100e^{kt}.$$

We can use the second observation to determine the value of the growth constant k. We have

$$450 = y(60) = 100e^{60k}.$$

Dividing both sides by 100 and taking the natural logarithm of both sides, we have

$$\ln 4.5 = \ln e^{60k} = 60k,$$

so that

$$k = \frac{\ln 4.5}{60} \approx 0.02507.$$

We now have a formula representing the number of cells present at any time t:

$$y(t) = 100e^{kt} = 100\exp\left(\frac{\ln 4.5}{60}t\right).$$

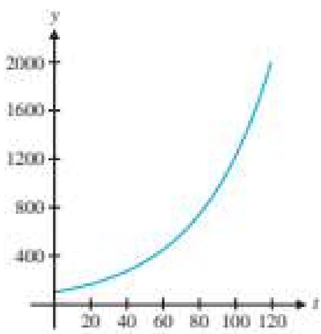


FIGURE 4.2
$y = 100\,e^{\left(\frac{\ln 4.5}{60}t\right)}$

See Figure 4.2 for a graph of the projected bacterial growth over the first 120 minutes. One further question of interest to microbiologists is the **doubling time**, that is, the time it takes for the number of cells to double. We can find this by solving for the time t for which $y(t) = 2y(0) = 200$. We have

$$200 = y(t) = 100\exp\left(\frac{\ln 4.5}{60}t\right).$$

Dividing both sides by 100 and taking logarithms, we obtain

$$\ln 2 = \frac{\ln 4.5}{60}t,$$

so that

$$t = \frac{60\ln 2}{\ln 4.5} \approx 27.65.$$

So, the doubling time for this culture of *Streptococcus A* is about 28 minutes. The doubling time for a bacterium depends on the specific strain of bacteria, as well as the quality and quantity of the food supply, the temperature and other environmental factors. However, it is not dependent on the initial population. Here, you can easily check that the population reaches 400 at time

$$t = \frac{120\ln 2}{\ln 4.5} \approx 55.3$$

(exactly double the time it took to reach 200).

That is, the initial population of 100 doubles to 200 in approximately 28 minutes and it doubles again (to 400) in another 28 minutes and so on. ■

Numerous physical phenomena satisfy exponential growth or decay laws. For instance, experiments have shown that the rate at which a radioactive element decays is directly proportional to the amount present. (Recall that radioactive elements are chemically unstable elements that gradually decay into other, more stable elements.) Let $y(t)$ be the amount (mass) of a radioactive element present at time t. Then, we have that the rate of change (rate of decay) of $y(t)$ satisfies

$$y'(t) = ky(t). \tag{1.7}$$

Note that (1.7) is precisely the same differential equation as (1.1), encountered in example 1.1 for the growth of bacteria and hence, from (1.4), we have that

$$y(t) = Ae^{kt},$$

for some constants A and k (here, the **decay constant**) to be determined.

It is common to discuss the decay rate of a radioactive element in terms of its **half-life**, the time required for half of the initial quantity to decay into other elements. For instance, scientists have calculated that the half-life of carbon-14 (^{14}C) is approximately 5730 years. That is, if you have 2 grams of ^{14}C today and you come back in 5730 years, you will have approximately 1 gram of ^{14}C remaining. It is this long half-life and the fact that living creatures continually take in ^{14}C that make ^{14}C measurements useful for radiocarbon dating. (See the exercise set for more on this important application.)

EXAMPLE 1.2 Radioactive Decay

If you have 50 grams of ^{14}C today, how much will be left in 100 years?

Solution Let $y(t)$ be the mass (in grams) of ^{14}C present at time t. Then, we have

$$y'(t) = ky(t)$$

and as we have already seen, $$y(t) = Ae^{kt}.$$

The initial condition is $y(0) = 50$, so that

$$50 = y(0) = Ae^{0} = A$$

and $$y(t) = 50\,e^{kt}.$$

To find the decay constant k, we use the half-life:

$$25 = y(5730) = 50\,e^{5730k}.$$

Dividing both sides by 50 and taking logarithms gives us

$$\ln\frac{1}{2} = \ln e^{5730k} = 5730k,$$

so that $$k = \frac{\ln\frac{1}{2}}{5730} \approx -1.20968 \times 10^{-4}.$$

A graph of the mass of ^{14}C as a function of time is seen in Figure 4.3. Notice the extremely large timescale shown. This should give you an idea of the incredibly slow rate of decay of ^{14}C. Finally, notice that if we start with 50 grams, then the amount left after 100 years is

$$y(100) = 50\,e^{100k} \approx 49.3988 \text{ grams.}$$

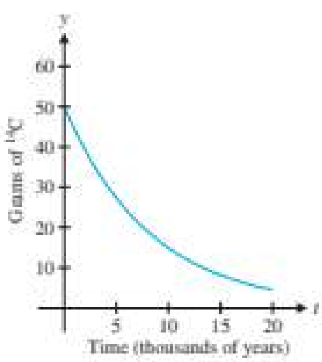


FIGURE 4.3
Decay of ^{14}C

A mathematically similar physical principle is **Newton's Law of Cooling**. If you introduce a hot object into cool surroundings (or equivalently, a cold object into warm surroundings), the rate at which the object cools (or warms) is not proportional to its temperature, but rather, to the difference in temperature between the object and its surroundings. Symbolically, if we let $y(t)$ be the temperature of the object at time t and let T_a be the temperature of the surroundings (the **ambient temperature**, which we assume to be constant), we have the differential equation

$$y'(t) = k[y(t) - T_a]. \tag{1.8}$$

Notice that (1.8) is not the same as the differential equation describing exponential growth or decay. (Compare these; what's the difference?) Even so, we can approach finding a solution in the same way. In the case of cooling, we assume that

$$T_a < y(t).$$

(Why is it fair to assume this?) If we divide both sides of equation (1.8) by $y(t) - T_a$ and then integrate both sides, we obtain

$$\int \frac{y'(t)}{y(t) - T_a}\,dt = \int k\,dt = kt + c_1. \tag{1.9}$$

Notice that we can evaluate the integral on the left-hand side by making the substitution $u = y(t) - T_a$, so that $du = y'(t)\,dt$. Thus, we have

$$\begin{aligned}\int \frac{y'(t)}{y(t) - T_a}\,dt = \int \frac{1}{u}\,du &= \ln|u| + c_2 = \ln|y(t) - T_a| + c_2 \\ &= \ln[y(t) - T_a] + c_2,\end{aligned}$$

since $y(t) - T_a > 0$. From (1.9), we now have

$$\ln[y(t) - T_a] + c_2 = kt + c_1 \quad \text{or} \quad \ln[y(t) - T_a] = kt + c,$$

where we have combined the two constants of integration. Taking exponentials of both sides, we obtain

$$y(t) - T_a = e^{kt+c} = e^{kt}e^c.$$

Finally, for convenience, we write $A = e^c$, to obtain

$$y(t) = Ae^{kt} + T_a,$$

where A and k are constants to be determined.

We illustrate Newton's Law of Cooling in example 1.3.

EXAMPLE 1.3 Newton's Law of Cooling for a Cup of Coffee

A cup of "Caffè macchiato" is 70°C when freshly poured. After 2 minutes in a room at 25°C, the coffee has cooled to 60°C. Find the temperature at any time t and find the time at which the coffee has cooled to 40°C.

Solution Letting $y(t)$ be the temperature of the coffee at time t, we have

$$y'(t) = k[y(t) - 25].$$

Proceeding as above, we obtain

$$y(t) = Ae^{kt} + 25.$$

Observe that the initial condition here is the initial temperature, $y(0) = 70$. This gives us

$$70 = y(0) = Ae^0 + 25 = A + 25,$$

so that $A = 45$ and $$y(t) = 45e^{kt} + 25.$$

We can now use the second measured temperature to solve for the constant k. We have

$$60 = y(2) = 45e^{2k} + 25.$$

Subtracting 25 from both sides and dividing by 45, we have

$$e^{2k} = \frac{60 - 25}{45} = \frac{35}{45} = \frac{7}{9}.$$

Taking logarithms of both sides yields $$2k = \ln\left(\frac{7}{9}\right)$$

and hence, $$k = \frac{1}{2}\ln\left(\frac{7}{9}\right) \approx -0.1256.$$

A graph of the projected temperature against time is shown in Figure 4.4. From Figure 4.4, you might observe that the temperature appears to have fallen to 40°C after about 10 minutes. We can solve this symbolically by finding the time t for which

$$40 = y(t) = 45\,e^{kt} + 25.$$

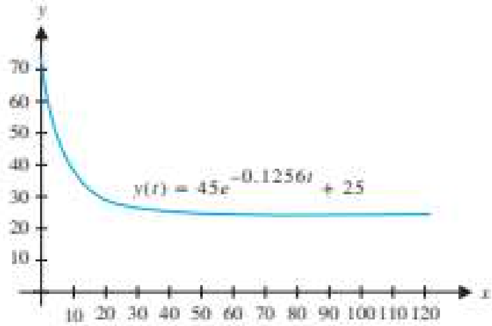


FIGURE 4.4
Temperature of coffee

It is not hard to solve this to obtain

$$t = \frac{1}{k}\ln\frac{1}{3} \approx 8.75 \text{ minutes.}$$

The details are left as an exercise. ■

Compound Interest

If a bank agrees to pay you 8% (annual) interest on your investment of \$10,000, then at the end of a year, you will have

$$\$10{,}000 + (0.08)\$10{,}000 = \$10{,}000(1 + 0.08) = \$10{,}800.$$

On the other hand, if the bank agrees to pay you interest twice a year at the same 8% annual rate, you receive $\frac{8}{2}$% interest twice each year. At the end of the year, you will have

$$\begin{aligned}\$10{,}000\left(1 + \frac{0.08}{2}\right)\left(1 + \frac{0.08}{2}\right) &= \$10{,}000\left(1 + \frac{0.08}{2}\right)^2 \\ &= \$10{,}816.\end{aligned}$$

Continuing in this fashion, notice that paying (compounding) interest monthly would pay $\frac{8}{12}$% each month (period), resulting in a balance of

$$\$10{,}000\left(1 + \frac{0.08}{12}\right)^{12} \approx \$10{,}830.00.$$

Further, if interest is compounded daily, you would end up with

$$\$10{,}000\left(1 + \frac{0.08}{365}\right)^{365} \approx \$10{,}832.78.$$

It should be evident that the more often interest is compounded, the greater the interest will be. A reasonable question to ask is whether there is a limit to how much interest can accrue on a given investment at a given interest rate. If n is the number of times per year that interest is compounded, we wish to calculate the annual percentage yield (APY) under **continuous compounding**.

$$\text{APY} = \lim_{n\to\infty}\left(1 + \frac{0.08}{n}\right)^n - 1.$$

To determine this limit, you must recall that

$$e = \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m.$$

Notice that if we make the change of variable $n = 0.08m$, then we have

$$\begin{aligned}\text{APY} &= \lim_{m\to\infty}\left(1 + \frac{0.08}{0.08m}\right)^{0.08m} - 1 \\ &= \left[\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m\right]^{0.08} - 1 \\ &= e^{0.08} - 1 \approx 0.083287.\end{aligned}$$

Under continuous compounding, you would thus earn approximately 8.3% or

$$\$10{,}000(e^{0.08} - 1) \approx \$832.87$$

in interest, leaving your investment with a total value of \$10,832.87. More generally, suppose that you invest $\$P$ at an annual interest rate r, compounded n times per year. Then the value of your investment after t years is

$$\$P\left(1 + \frac{r}{n}\right)^{nt}.$$

Under continuous compounding (i.e., taking the limit as $n \to \infty$), this becomes

$$\$Pe^{rt}. \tag{1.10}$$

Alternatively, if $y(t)$ is the value of your investment after t years, with continuous compounding, the rate of change of $y(t)$ is proportional to $y(t)$. That is,

$$y'(t) = ry(t),$$

where r is the annual interest rate. From (1.4), we have

$$y(t) = Ae^{rt}.$$

For an initial investment of $\$P$, we have

$$\$P = y(0) = Ae^{0} = A,$$

so that

$$y(t) = \$Pe^{rt},$$

which is the same as (1.10).

EXAMPLE 1.4 Comparing Forms of Compounding Interest

If you invest \$7000 at an annual interest rate of 5.75%, compare the value of your investment after 5 years under various forms of compounding.

Solution With annual compounding, the value is

$$\$7000\left(1 + \frac{0.0575}{1}\right)^{5} \approx \$9257.63.$$

With monthly compounding, this becomes

$$\$7000\left(1 + \frac{0.0575}{12}\right)^{12(5)} \approx \$9325.23.$$

With daily compounding, this yields

$$\$7000\left(1 + \frac{0.0575}{365}\right)^{365(5)} \approx \$9331.42.$$

Finally, with continuous compounding, the value is

$$\$7000\, e^{0.0575(5)} \approx \$9331.63. \blacksquare$$

The mathematics used to describe the compounding of interest also applies to accounts that are decreasing in value.

EXAMPLE 1.5 Depreciation of Assets

(a) Suppose that the value of an \$10,000 asset decreases continuously at a constant rate of 24% per year. Find its worth after 10 years; after 20 years. (b) Compare these values to an \$10,000 asset that is depreciated to no value in 20 years using linear depreciation.

Solution The value $v(t)$ of any quantity that is changing at a constant rate r satisfies $v' = rv$. Here, $r = -0.24$, so that

$$v(t) = Ae^{-0.24t}.$$

Since the value of the asset is initially 10,000, we have

$$10{,}000 = v(0) = Ae^0 = A.$$

We now have

$$v(t) = 10{,}000\, e^{-0.24t}.$$

At time $t = 10$, the value of the asset is then

$$\$10{,}000\, e^{-0.24(10)} \approx \$907.18$$

and at time $t = 20$, the value has decreased to

$$\$10{,}000\, e^{-0.24(20)} \approx \$82.30.$$

For part (b), linear depreciation means we use a linear function $v(t) = mt + b$ for the value of the asset. We start with $v(0) = 10{,}000$ and end at $v(20) = 0$. From $v(0) = 10{,}000$ we get $b = 10{,}000$ and using the points (0, 10,000) and (20, 0), we compute the slope $m = \frac{10{,}000}{-20} = -500$. We then have

$$v(t) = -500t + 10{,}000.$$

At time $t = 10$, $v(10) = \$5000$. Notice that this is considerably more than the approximately \$900 that exponential depreciation gave us. By time $t = 20$, however, the linear depreciation value of \$0 is less than the exponential depreciation value of \$82.30. Figure 4.5 illustrates these comparisons. ■

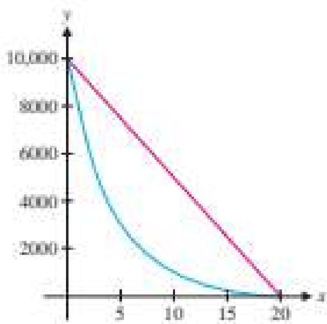


FIGURE 4.5
Linear versus exponential depreciation

BEYOND FORMULAS

With a basic understanding of differential equations, you can model a diverse collection of physical phenomena arising in economics, science and engineering. Understanding the assumptions that go into the model allows you to interpret the meaning of the solution in the context of the original problem. Moreover, part of the power of mathematics lies in its generality. In this case, a given differential equation may model a collection of vastly different phenomena. In this sense, knowing a little bit of mathematics goes a long way. Having solved for doubling time in example 1.1, if you are told that the value of an investment or the size of a tumor is modeled by the same equation, you do not need to re-solve any equations to find the doubling times for the investment or the size of the tumor.

EXERCISES 4.1

WRITING EXERCISES

1. A linear function is characterized by constant slope. If a population showed a constant numerical increase year by year, explain why the population could be represented by a linear function. If the population showed a constant *percentage* increase instead, explain why the population could be represented by an exponential function.
2. If a population has a constant birth rate and a constant death rate (smaller than the birth rate), describe what the population would look like over time. In the United States, is the death rate increasing, decreasing or staying the same? Given this, why is there concern about reducing the birth rate?
3. Explain, in monetary terms, why for a given interest rate the more times the interest is compounded the more money is in the account at the end of a year.
4. In the growth and decay examples, the constant A turned out to be equal to the initial value. In the cooling examples, the constant A did not equal the initial value. Explain why the cooling example worked differently.

In exercises 1–8, find the solution of the given differential equation satisfying the indicated initial condition.

1. $y' = 4y$, $y(0) = 2$
2. $y' = 3y$, $y(0) = -2$
3. $y' = -3y$, $y(0) = 5$
4. $y' = -2y$, $y(0) = -6$
5. $y' = 2y$, $y(1) = 2$
6. $y' = -y$, $y(1) = 2$
7. $y' = y - 50$, $y(0) = 70$
8. $y' = -0.1y - 10$, $y(0) = 80$

Exercises 9–14 involve exponential growth.

9. Suppose a bacterial culture initially has 400 cells. After 1 hour, the population has increased to 800. (a) Quickly determine the population after 3 hours. (b) Find an equation for the population at any time. (c) What will the population be after 3.5 hours?

10. Suppose a bacterial culture initially has 100 cells. After 2 hours, the population has increased to 400. (a) Quickly find the population after 6 hours. (b) Find an equation for the population at any time. (c) What will the population be after 7 hours?

11. Suppose a bacterial culture doubles in population every 4 hours. If the population is initially 100, (a) quickly determine when the population will reach 400. (b) Find an equation for the population at any time. (c) Determine when the population will reach 6000.

12. Suppose a bacterial culture triples in population every 5 hours. If the population is initially 200, (a) quickly determine when the population will reach 5400. (b) Find an equation for the population at any time. (c) Determine when the population will reach 20,000.

13. Suppose that a population of *E. coli* doubles every 20 minutes. A treatment of the infection removes 90% of the *E. coli* present and is timed to accomplish the following. The population starts at size 10^8, grows for T minutes, the treatment is applied and the population returns to size 10^8. Find the time T.

14. Research by Meadows, Meadows, Randers and Behrens indicates that the Earth has 12.95×10^6 km^2 of arable land available. The world population of 1950 required 4.05×10^6 km^2 to sustain it, and the population of 1980 required 8.09×10^6 km^2. If the required acreage grows at a constant percentage rate, in what year will the population reach the maximum sustainable size?

15. Suppose some quantity is increasing exponentially (e.g., the number of cells in a bacterial culture) with growth rate r. Show that the doubling time is $\frac{\ln 2}{r}$.

16. Suppose some quantity is decaying exponentially with decay constant r. Show that the half-life is $-\frac{\ln 2}{r}$. What is the difference between the half-life here and the doubling time in exercise 15?

Exercises 17–22 involve exponential decay.

17. Strontium-90 is a dangerous radioactive isotope. Because of its similarity to calcium, it is easily absorbed into human bones. The half-life of strontium-90 is 28 years. If a certain amount is absorbed into the bones due to exposure to a nuclear explosion, what percentage will remain after (a) 84 years? (b) 100 years?

18. The half-life of uranium ^{235}U is approximately 0.7×10^9 years. If 50 grams are buried at a nuclear waste site, how much will remain after (a) 100 years? (b) 1000 years?

19. The half-life of morphine in the human bloodstream is 3 hours. If initially there is 0.4 mg of morphine in the bloodstream, find an equation for the amount in the bloodstream at any time. When does the amount drop below (a) 0.1 mg? (b) 0.01 mg?

20. Repeat exercise 19 if the half-life is 2.8 hours.

21. Scientists dating a fossil estimate that 20% of the original amount of carbon-14 is present. Recalling that the half-life is 5730 years, approximately how old is the fossil?

22. If a fossil is 1 million years old, what percentage of its original carbon-14 should remain?

Exercises 23–28 involve Newton's Law of Cooling.

23. A bowl of "Umm Ali" dessert at 93°C (too hot) is placed in a 21°C room. One minute later the dessert has cooled to 82°C. When will the temperature be 49°C (just right)?

24. A smaller bowl of porridge served at 93°C cools to 71°C in 1 minute. What temperature (too cold) will this porridge be when the bowl of exercise 23 has reached 49°C (just right)?

25. A cold drink is poured out at 10°C. After 2 minutes of sitting in a 21°C room, its temperature has risen to 13°C. (a) Find the drink's temperature at any time t. (b) What will the temperature be after 10 minutes? (c) When will the drink have warmed to 19°C?

26. Twenty minutes after being served a cup of fast-food coffee, it is still too hot to drink at 71°C. Two minutes later, the temperature has dropped to 70°C. Your friend, whose coffee is also too hot to drink, speculates that since the temperature is dropping an average of 1 degree per minute, it was served at 82°C. (a) Explain what is wrong with this logic. (b) Was the actual serving temperature hotter or cooler than 81°C? (c) Find the actual serving temperature if the room temperature is 20°C.

27. The owner of "Dream of Cream" is a famous cake decorator named Sirine. In celebration of her son's birthday, Sirine is preparing a mix of chocolate ganache to decorate the cake. The mix is poured out at 42°C. After 3 minutes of sitting in a 21°C room, its temperature has fallen to 32°C. (a) Find the mix's temperature at any time t. (b) What will the temperature be after 10 minutes?

28. For the cup of coffee in example 1.3, suppose that the goal is to have the coffee cool to 49°C in 5 minutes. At what temperature should the coffee be served?

Exercises 29–32 involve compound interest.

29. If you invest \$1000 at an annual interest rate of 8%, compare the value of the investment after 1 year under the following forms of compounding: annual, monthly, daily, continuous.

30. Repeat exercise 29 for the value of the investment after 5 years.

31. Person A invests \$10,000 in 1990 and person B invests \$20,000 in 2000. (a) If both receive 12% interest (compounded continuously), what are the values of the investments in 2010? (b) Repeat for an interest rate of 4%. (c) Determine the interest rate such that person A ends up exactly even with person B. (Hint: You want person A to have \$20,000 in 2000.)

32. One of the authors bought a set of basketball trading cards in 1985 for \$34. In 1995, the "book price" for this set was \$9800. (a) Assuming a constant percentage return on this investment, find an equation for the worth of the set at time t years (where $t = 0$ corresponds to 1985). (b) At this rate of return, what would the set have been worth in 2005? (c) The author also bought a set of baseball cards in 1985, costing \$22. In 1995, this set was worth \$32. At this rate of return, what would the set have been worth in 2005?

33. Suppose that the value of a \$40,000 asset decreases at a constant percentage rate of 10%. Find its worth after (a) 10 years and (b) 20 years. Compare these values to a \$40,000 asset that is depreciated to no value in 20 years using linear depreciation.

34. Suppose that the value of a \$400,000 asset decreases (depreciates) at a constant percentage rate of 40%. Find its worth after (a) 5 years and (b) 10 years. Compare these values to a \$40,000 asset that is depreciated to no value in 10 years using linear depreciation.

Exercises 35–38 involve tax rates.

35. In an African country, income between \$16,000 and \$20,000 was taxed at 28% in 2003. In 2016, income between \$16,000 and \$20,000 was taxed at 15%. This makes it seem as if taxes went down considerably between 2003 and 2016. Taking inflation into account, briefly explain why this is not a valid comparison.

36. To make the comparison in exercise 35 a little fairer, note that income above \$30,000 was taxed at 28% in 2016 and assume that inflation averaged 5.5% between 2003 and 2016. Adjust \$16,000 for inflation by computing its value after increasing continuously at 5.5% for 13 years. Based on this calculation, how do the tax rates compare?

37. Suppose the income tax structure is as follows: the first \$30,000 is taxed at 15%, the remainder is taxed at 28%. Compute the tax T_1 on an income of \$40,000. Now, suppose that inflation is 5% and you receive a cost of living (5%) raise to \$42,000. Compute the tax T_2 on this income. To compare the taxes, you should adjust the tax T_1 for inflation (add 5%).

38. In exercise 37, the tax code stayed the same, but (adjusted for inflation) the tax owed did not stay the same. Briefly explain why this happened. What could be done to make the tax owed remain constant?

39. Using the bacterial population data at the beginning of this section, define x to be time and y to be the natural logarithm of the population. Plot the data points (x, y) and comment on how close the data are to being linear. Take two representative points and find an equation of the line through the two points. Then find the population function $p(x) = e^{y(x)}$.

40. (a) As in exercise 39, find an exponential model for the population data (0, 10), (1, 15), (2, 22), (3, 33) and (4, 49). (b) Find an exponential model for the population data (0, 20), (1, 16), (2, 13), (3, 11) and (4, 9).

41. Use the method of exercise 39 to fit an exponential model to the following data representing percentage of the U.S. population classified as living on rural farms (data from the U.S. Census Bureau).

Year	1960	1970	1980	1990
% Pop. Farm	7.5	5.2	2.5	1.6

42. Use the method of exercise 39 to fit an exponential model to the following data representing percentage of the U.S. population classified as living in urban areas (data from the U.S. Census Bureau).

Year	1960	1970	1980	1990
% Pop. Urban	69.9	73.5	73.7	75.2

43. Show that for any constant c, $y = x - \frac{1}{2} + ce^{-2x}$ is a solution of the differential equation $y' + 2y = 2x$.

44. Show that for any constant c, $y = \sqrt{3x^2 + c}$ is a solution of the differential equation $y' = 3x/y$.

APPLICATIONS

1. An Internet site reports that the antidepressant drug amitriptyline has a half-life in humans of 31–46 hours. For a dosage of 150 mg, compare the amounts left in the bloodstream after one day for a person for whom the half-life is 31 hours versus a person for whom the half-life is 46 hours. Is this a large difference?

2. It is reported that Prozac® has a half-life of 2 to 3 days but may be found in your system for several weeks after you stop taking it. What percentage of the original dosage would remain after 2 weeks if the half-life is 2 days? How much would remain if the half-life is 3 days?

3. The antibiotic ertapenem has a half-life of 4 hours in the human bloodstream. The dosage is 1 gm per day. Find and graph the amount in the bloodstream t hours after taking it $(0 \le t \le 24)$.

4. Compare your answer to exercise 3 with a similar drug that is taken with a dosage of 1 gm four times a day and has a half-life of 1 hour. (Note that you will have to do four separate calculations here.)

5. A bank offers to sell a bank note that will reach a maturity value of \$10,000 in 10 years. How much should you pay for it now if you wish to receive an 8% return on your investment? (Note: This is called the **present value** of the bank note.) Show that, in general, the present value of an item worth $\$P$ in t years with constant interest rate r is given by $\$Pe^{-rt}$.

6. Suppose that the value of a piece of land t years from now is $\$40{,}000e^{2\sqrt{t}}$. Given 6% annual inflation, find t that maximizes the present value of your investment: $\$40{,}000e^{2\sqrt{t}-0.06t}$.

7. Suppose that a business has an income stream of $\$P(t)$ per year. The present value at interest rate r of this income for the next T years is $\int_0^T P(t)e^{-rt}\,dt$. Compare the present values at 5% for three people with the following salaries for 3 years: A: $P(t) = 60{,}000$; B: $P(t) = 60{,}000 + 3000t$; and C: $P(t) = 60{,}000e^{0.05t}$.

8. The **future value** of an income stream after T years at interest rate r is given by $\int_0^T P(t)\,e^{r(T-t)}\,dt$. Calculate the future value for cases A, B and C in exercise 7. Briefly describe the difference between what present value and future value measure.

9. If you win a "million dollar" lottery, would you be better off getting your money in four annual installments of $280,000 or in one lump sum of $1 million? (a) Assume 8% interest and payments made at the beginning of the year. Repeat with (b) 6% and (c) 10% interest.

10. The "Rule of 72" is used by many investors to quickly estimate how fast an investment will double in value. For example, at 8% the rule suggests that the doubling time will be about $\frac{72}{8} = 9$ years. Calculate the actual doubling time. Explain why a "Rule of 69" would be more accurate. Give at least one reason why the number 72 is used instead.

EXPLORATORY EXERCISES

1. The amount of carbon-14 in the atmosphere is largely determined by cosmic ray bombardment. Living organisms maintain a constant level of carbon-14 through exchanges with the environment. At death, the organism no longer takes in carbon-14, so the carbon-14 level decreases with the 5730-year half-life. Scientists can measure the rate of disintegration of carbon-14. (You might visualize a Geiger counter.) If $y(t)$ is the amount of carbon-14 remaining at time t, the rate of change is $y'(t) = ky(t)$. We assume that the rate of disintegration at the time of death is the same as it is now for living organisms, and call this $ky(0)$. The ratio of disintegration rates is $y(t)/y(0)$. In particular, suppose that $ky(t) = -2.4$ (disintegrations per minute) and $ky(0) = -6.7$ (disintegrations per minute). Solve for t. Now, suppose the assumption of constant carbon-14 levels is wrong. If $ky(0)$ is decreased by 5%, by what percentage does the estimate of the time t change? If $ky(t)$ is decreased by 5%, by what percentage does the estimate of the time change? Roughly, how do errors in the measurements translate to errors in the estimate of the time?

2. Three confused hunting dogs start at the vertices of an equilateral triangle of side 1. Each dog runs with a constant speed aimed directly at the dog that is positioned clockwise from it. The chase stops when the dogs meet in the middle (having grabbed each other by their tails). How far does each dog run? [Hints: Represent the position of each dog in polar coordinates (r, θ) with the center of the triangle at the origin. By symmetry, each dog has the same r-value, and if one dog has angle θ, then it is aimed at the dog with angle $\theta - \frac{2\pi}{3}$. (a) Set up a differential equation for the motion of one dog and show that there is a solution if $r'(\theta) = \sqrt{3}r$. Use the arc length formula $L = \int_{\theta_1}^{\theta_2} \sqrt{[r'(\theta)]^2 + [r(\theta)]^2}\,d\theta$.] (b) To generalize, suppose that there are n dogs starting at the vertices of a regular n-gon of side s. If α is the interior angle from the center of the n-gon to adjacent vertices, show that the distance run by each dog equals $\frac{s}{1 - \cos\alpha}$. What happens to the distance as n increases without bound? Explain this in terms of the paths of the dogs.

4.2 SEPARABLE DIFFERENTIAL EQUATIONS

In section 4.1, we solved two different differential equations:

$$y'(t) = ky(t) \quad \text{and} \quad y'(t) = k[y(t) - T_a],$$

using essentially the same method. These are both examples of *separable* differential equations. We will examine this type of equation at some length in this section. First, we consider the more general **first-order ordinary differential equation**

$$y' = f(x, y). \tag{2.1}$$

Here, the derivative y' of some unknown function $y(x)$ is given as a function f of both x and y. Our objective is to find some function $y(x)$ (a **solution**) that satisfies equation (2.1) on some interval, I. The equation is **first-order**, since it involves only the first derivative of the unknown function. We will consider the case where the x's and y's can be separated. We call equation (2.1) **separable** if we can separate the variables, i.e., if we can rewrite it in the form

$$g(y)y' = h(x),$$

where all of the x's are on one side of the equation and all of the y's are on the other side.

NOTES

Do not be distracted by the letter used for the independent variable. We frequently use the independent variable x, as in equation (2.1). Whenever the independent variable represents time, we use t as the independent variable, in order to reinforce this connection, as we did in example 1.2. There, the equation describing radioactive decay was given as

$$y'(t) = ky(t).$$

EXAMPLE 2.1 A Separable Differential Equation

Determine whether the differential equation

$$y' = xy^2 - 2xy$$

is separable.

Solution Notice that this equation is separable, since we can rewrite it as

$$y' = x(y^2 - 2y)$$

and then divide by $(y^2 - 2y)$ (assuming this is not zero), to obtain

$$\frac{1}{y^2 - 2y} y' = x. \quad ■$$

EXAMPLE 2.2 An Equation That Is Not Separable

The equation

$$y' = xy^2 - 2x^2y$$

is not separable, as there is no way to separate the x's and the y's. (Try this for yourself!) ■

Essentially, the x's and y's must be separated by multiplication or division in order for a differential equation to be separable. Notice that in example 2.2, you can factor to get $y' = xy(y - 2x)$, but the subtraction $y - 2x$ keeps this equation from being separable.

Separable differential equations are of interest because there is a very simple means of solving them. Notice that if we integrate both sides of

$$g(y)y'(x) = h(x)$$

with respect to x, we get

$$\int g(y)y'(x)\, dx = (x)\, dx. \tag{2.2}$$

Since $dy = y'(x)\, dx$, the integral on the left-hand side of (2.2) becomes

$$\int g(y)\underbrace{y'(x)\, dx}_{dy} = \int g(y)\, dy.$$

Consequently, from (2.2), we have

$$\int g(y)\, dy = \int h(x)\, dx.$$

So, provided we can evaluate both of these integrals, we have an equation relating x and y, which no longer involves y'.

EXAMPLE 2.3 Solving a Separable Equation

Solve the differential equation

$$y' = \frac{x^2 + 7x + 3}{y^2}.$$

Solution Separating variables, observe that we have

$$y^2y' = x^2 + 7x + 3.$$

Integrating both sides with respect to x, we obtain

$$\int y^2 y'(x) dx = \int (x^2 + 7x + 3) dx$$

or

$$\int y^2 dy = \int (x^2 + 7x + 3) dx.$$

Performing the indicated integrations yields

$$\frac{y^3}{3} = \frac{x^3}{3} + 7\frac{x^2}{2} + 3x + c,$$

where we have combined the two constants of integration into one on the right-hand side.

Solving for y, we get

$$y = \sqrt[3]{x^3 + \frac{21}{2}x^2 + 9x + 3c}.$$

Notice that for each value of c, we get a different solution of the differential equation. This is called a **family of solutions** (or the **general solution**) of the differential equation. In Figure 4.6, we have plotted a number of the members of this family of solutions.

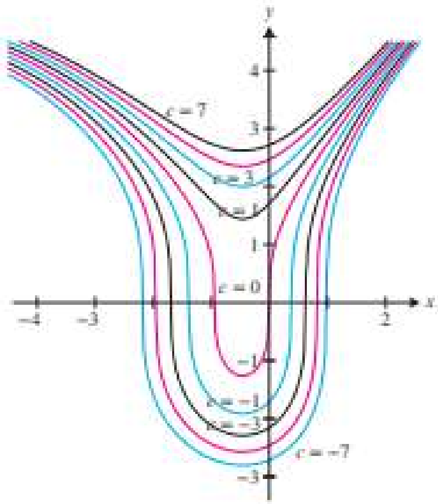


FIGURE 4.6
A family of solutions ■

In general, the solution of a first-order separable equation will include an arbitrary constant (the constant of integration). To select just one of these solution curves, we specify a single point through which the solution curve must pass, say (x_0, y_0). That is, we require that

$$y(x_0) = y_0.$$

This is called an **initial condition** (since this condition often specifies the initial state of a physical system). A first-order differential equation together with an initial condition is referred to as an **initial value problem** (IVP).

EXAMPLE 2.4 Solving an Initial Value Problem

Solve the IVP
$$y' = \frac{x^2 + 7x + 3}{y^2}, \quad y(0) = 3.$$

Solution In example 2.3, we found that the general solution of the differential equation is

$$y = \sqrt[3]{x^3 + \frac{21}{2}x^2 + 9x + 3c}.$$

From the initial condition, we now have

$$3 = y(0) = \sqrt[3]{0 + 3c} = \sqrt[3]{3c}$$

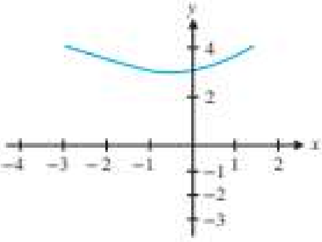


FIGURE 4.7
$y = \sqrt[3]{x^3 + \frac{21}{2}x^2 + 9x + 27}$

and hence, $c = 9$. The solution of the IVP is then

$$y = \sqrt[3]{x^3 + \frac{21}{2}x^2 + 9x + 27}.$$

We show a graph of this solution in Figure 4.7. Notice that this graph would fit above the curves shown in Figure 4.6. We'll explore the effects of other initial conditions in the exercises. ■

We are not always as fortunate as we were in example 2.4. There, we were able to obtain an explicit representation of the solution (i.e., we found a formula for y in terms of x). Most often, we must settle for an *implicit* representation of the solution, that is, an equation relating x and y that cannot be solved for y in terms of x alone.

EXAMPLE 2.5 An Initial Value Problem That Has Only an Implicit Solution

Find the solution of the IVP

$$y' = \frac{9x^2 - \sin x}{\cos y + 5e^y}, \quad y(0) = \pi.$$

Solution First, note that the differential equation is separable, since we can rewrite it as

$$(\cos y + 5e^y)\, y'(x) = 9x^2 - \sin x.$$

Integrating both sides of this equation with respect to x, we find

$$\int (\cos y + 5e^y)\, y'(x)\, dx = \int (9x^2 - \sin x) dx$$

or

$$\int (\cos y + 5e^y) dy = \int (9x^2 - \sin x) dx.$$

Evaluating the integrals, we obtain

$$\sin y + 5e^y = 3x^3 + \cos x + c. \tag{2.3}$$

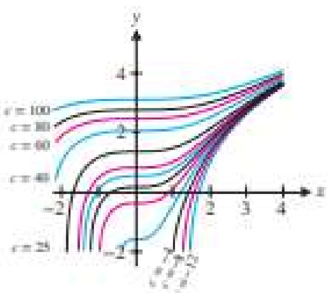


FIGURE 4.8a
A family of solutions

Notice that there is no way to solve this equation explicitly for y in terms of x. However, you can still picture the graphs of some members of this family of solutions by using the implicit plot mode on your graphing utility. Several of these are plotted in Figure 4.8a. Even though we have not solved for y explicitly in terms of x, we can still use the initial condition. Substituting $x = 0$ and $y = \pi$ into equation (2.3), we have

$$\sin \pi + 5e^{\pi} = 0 + \cos 0 + c$$

or

$$5e^{\pi} - 1 = c.$$

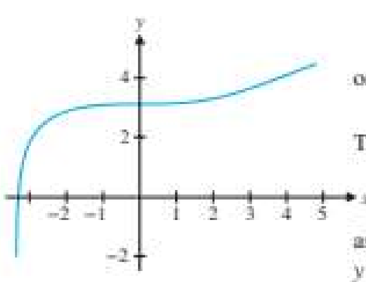


FIGURE 4.8b
The solution of the IVP

This leaves us with

$$\sin y + 5e^y = 3x^3 + \cos x + 5e^{\pi} - 1$$

as an implicit representation of the solution of the IVP. Although we cannot solve for y in terms of x alone, given any particular value for x, we can use Newton's method (or some other numerical method) to approximate the value of the corresponding y. This is essentially what your CAS does (with many, many points) when you use it to plot a graph in implicit mode. We plot the solution of the IVP in Figure 4.8b. ■

Logistic Growth

In section 4.1, we introduced the differential equation

$$y' = ky$$

as a model of bacterial population growth, valid for populations growing with unlimited resources and with unlimited room for growth. Of course, *all* populations have factors that eventually limit their growth. Thus, this particular model generally provides useful information only for relatively short periods of time.

An alternative model of population growth assumes that there is a maximum sustainable population, M (called the **carrying capacity**), determined by the available resources. Further, as the population size approaches M (as available resources become more scarce), the population growth will slow. To reflect this, we assume that the rate of population growth is jointly proportional to the present population level and the difference between the current level and the maximum, M. That is, if $y(t)$ is the population at time t, we assume that

$$y'(t) = ky(M - y).$$

This differential equation is referred to as the **logistic equation**.

> **HISTORICAL NOTES**
>
> **Pierre Verhulst (1804–1849)** A Belgian mathematician who proposed the logistic model for population growth. Verhulst was a professor of mathematics in Brussels and did research on number theory and social statistics. His most important contribution was the logistic equation (also called the Verhulst equation) giving the first realistic model of a population with limited resources. It is worth noting that Verhulst's estimate of Belgium's equilibrium population closely matches the current Belgian population.

Two special solutions of this differential equation are apparent. The constant functions $y = 0$ and $y = M$ are both solutions of this differential equation. These are called **equilibrium solutions** since, under the assumption of logistic growth, once a population hits one of these levels, it remains there for all time. If $y \neq 0$ and $y \neq M$, we can solve the differential equation, since it is separable, as

$$\frac{1}{y(M - y)} y'(t) = k. \tag{2.4}$$

Integrating both sides with respect to t, we obtain

$$\int \frac{1}{y(M - y)} y'(t)\, dt = \int k\, dt$$

or

$$\int \frac{1}{y(M - y)}\, dy = \int k\, dt. \tag{2.5}$$

Using partial fractions, we can write

$$\frac{1}{y(M - y)} = \frac{1}{My} + \frac{1}{M(M - y)}.$$

From (2.5) we now have

$$\int \left[\frac{1}{My} + \frac{1}{M(M - y)}\right] dy = \int k\, dt.$$

Carrying out the integrations gives us

$$\frac{1}{M}\ln|y| - \frac{1}{M}\ln|M - y| = kt + c.$$

Multiplying both sides by M and using the fact that $0 < y < M$, we have

$$\ln y - \ln(M - y) = kMt + Mc.$$

Taking exponentials of both sides, we find

$$\exp[\ln y - \ln(M - y)] = e^{kMt + Mc} = e^{kMt}e^{Mc}.$$

Next, using rules of exponentials and logarithms and replacing the constant term e^{Mc} by a new constant A, we obtain

$$\frac{y}{M - y} = Ae^{kMt}.$$

To solve this for y, we first multiply both sides by $(M - y)$ to obtain

$$\begin{aligned} y &= Ae^{kMt}(M - y) \\ &= AMe^{kMt} - Ae^{kMt}y. \end{aligned}$$

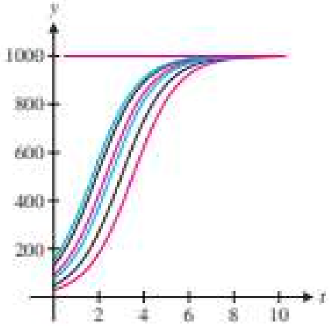


FIGURE 4.9
Several solution curves

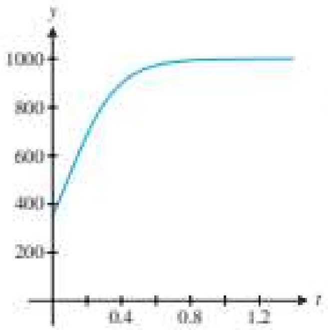


FIGURE 4.10
$y = \dfrac{35{,}000e^{7t}}{65 + 35e^{7t}}$

Combining the two y terms, we find

$$y(1 + Ae^{kMt}) = AMe^{kMt},$$

which gives us the explicit solution of the logistic equation,

$$y = \frac{AMe^{kMt}}{1 + Ae^{kMt}}. \tag{2.6}$$

In Figure 4.9, we plot a number of these solution curves for various values of A (for the case where $M = 1000$ and $k = 0.001$), along with the equilibrium solution $y = 1000$. You can see from Figure 4.9 that logistic growth consists of nearly exponential growth initially, followed by the graph becoming concave down and then asymptotically approaching the maximum, M.

EXAMPLE 2.6 Solving a Logistic Growth Problem

Given a maximum sustainable population of $M = 1000$ (this could be measured in millions or tons, etc.) and growth rate $k = 0.007$, find an expression for the population at any time t, given an initial population of $y(0) = 350$ and assuming logistic growth.

Solution From the solution (2.6) of the logistic equation, we have $kM = 7$ and

$$y = \frac{1000Ae^{7t}}{1 + Ae^{7t}}.$$

From the initial condition, we have

$$350 = y(0) = \frac{1000A}{1 + A}.$$

Solving for A, we obtain $A = \frac{35}{65}$, which gives us the solution of the IVP

$$y = \frac{35{,}000\,e^{7t}}{65 + 35\,e^{7t}}.$$

This solution is plotted in Figure 4.10. ■

We should note that, in practice, the values of M and k are not known and must be estimated from a careful study of the particular population. We explore these issues further in the exercises.

In our final example, we consider growth in an investment plan.

EXAMPLE 2.7 Investment Strategies for Making a Million

Money is invested at 8% interest compounded continuously. If deposits are made continuously at the rate of \$2000 per year, find the size of the initial investment needed to reach \$1 million in 20 years.

Solution Here, interest is earned at the rate of 8% and additional deposits are assumed to be made on a continuous basis. If the deposit rate is \$$d$ per year, the amount $A(t)$ in the account after t years satisfies the differential equation

$$\frac{dA}{dt} = 0.08\,A + d.$$

This equation is separable and can be solved by dividing both sides by $0.08A + d$ and integrating. We have

$$\int \frac{1}{0.08\,A + d}\,dA = \int 1\,dt,$$

so that

$$\frac{1}{0.08}\ln|0.08\,A + d| = t + c.$$

Using $d = 2000$, we have

$$12.5 \ln |0.08\,A + 2000| = t + c.$$

Setting $t = 0$ and taking $A(0) = x$ gives us the constant of integration:

$$12.5 \ln |0.08\,x + 2000| = c,$$

so that

$$12.5 \ln |0.08\,A + 2000| = t + 12.5 \ln |0.08\,x + 2000|. \tag{2.7}$$

To find the value of x such that $A(20) = 1{,}000{,}000$, we substitute $t = 20$ and $A = 1{,}000{,}000$ into equation (2.7) to obtain

$$12.5 \ln |0.08(1{,}000{,}000) + 2000| = 20 + 12.5 \ln |0.08\,x + 2000|$$

or

$$12.5 \ln |82{,}000| = 20 + 12.5 \ln |0.08\,x + 2000|.$$

We can solve this for x, by subtracting 20 from both sides and then dividing by 12.5, to obtain

$$\frac{12.5 \ln 82{,}000 - 20}{12.5} = \ln |0.08\,x + 2000|.$$

Taking the exponential of both sides, we now have

$$e^{(12.5 \ln 82{,}000 - 20)/12.5} = 0.08\,x + 2000.$$

Solving this for x yields

$$x = \frac{e^{\ln 82{,}000 - 1.6} - 2000}{0.08} \approx 181{,}943.93.$$

So, the initial investment needs to be \$181,943.93 (slightly less than \$200,000) in order to be worth \$1 million at the end of 20 years. ■

To be fair, the numbers in example 2.7 (like most investment numbers) must be interpreted carefully. Of course, 20 years from now, \$1 million likely won't buy as much as \$1 million does today. For instance, the value of a million dollars adjusted for 8% annual inflation would be $\$1{,}000{,}000e^{-0.08(20)} \approx \$201{,}896$, which is not much larger than the \$181,943 initial investment required. However, if inflation is only 4%, then the value of a million dollars (in current dollars) is \$449,328. The lesson here is the obvious one: Be sure to invest money at an interest rate that exceeds the rate of inflation.

EXERCISES 4.2

WRITING EXERCISES

1. Discuss the differences between solving algebraic equations (e.g., $x^2 - 1 = 0$) and solving differential equations. Especially note what type of mathematical object you are solving for.

2. A differential equation is not separable if it can't be written in the form $g(y)y' = h(x)$. If you have an equation that you can't write in this form, how do you know whether it's really impossible or you just haven't figured it out yet? Discuss some general forms (e.g., $x + y$ and xy) that give you clues as to whether the equation is likely to separate or not.

3. The solution curves in Figures 4.6, 4.8a and 4.9 do not appear to cross. In fact, they never intersect. If solution curves crossed at the point (x_1, y_1), then there would be two solutions satisfying the initial condition $y(x_1) = y_1$. Explain why this does not happen. In terms of the logistic equation as a model of population growth, explain why it is important to know that this does not happen.

4. The logistic equation includes a term in the differential equation that slows population growth as the population increases. Discuss some of the reasons why this occurs in real populations (human, animal and plant).

In exercises 1–4, determine whether the differential equation is separable.

1. (a) $y' = (3x + 1)\cos y$ (b) $y' = (3x + y)\cos y$

2. (a) $y' = 2x(\cos y - 1)$ (b) $y' = 2x(y - x)$

3. (a) $y' = x^2 y + y \cos x$ (b) $y' = x^2 y - x \cos y$

4. (a) $y' = 2x \cos y - xy^3$ (b) $y' = x^3 - 2x + 1$

In exercises 5–16, the differential equation is separable. Find the general solution, in an explicit form if possible.

5. $y' = (x^2 + 1)y$ **6.** $y' = 2x(y - 1)$

7. $y' = 2x^2 y^2$ **8.** $y' = 2(y^2 + 1)$

9. $y' = \dfrac{6x^2}{y(1 + x^3)}$ **10.** $y' = \dfrac{3x}{y + 1}$

11. $y' = \dfrac{2x}{y} e^{y-x}$ **12.** $y' = \dfrac{\sqrt{1 - y^2}}{x \ln x}$

13. $y' = \dfrac{\cos x}{\sin y}$ **14.** $y' = x \cos^2 y$

15. $y' = \dfrac{xy}{1 + x^2}$ **16.** $y' = \dfrac{2}{xy + y}$

In exercises 17–20, find the general solution in an explicit form and sketch several members of the family of solutions.

17. $y' = -xy$ **18.** $y' = \dfrac{-x}{y}$

19. $y' = \dfrac{1}{y}$ **20.** $y' = 1 + y^2$

In exercises 21–28, solve the IVP, explicitly if possible.

21. $y' = 3(x + 1)^2 y,\ y(0) = 1$ **22.** $y' = \dfrac{x - 1}{y^2},\ y(0) = 2$

23. $y' = \dfrac{4x^2}{y},\ y(0) = 2$ **24.** $y' = \dfrac{x - 1}{y},\ y(0) = -2$

25. $y' = \dfrac{4y}{x + 3},\ y(-2) = 1$ **26.** $y' = \dfrac{3x}{4y + 1},\ y(1) = 4$

27. $y' = \dfrac{4x}{\cos y},\ y(0) = 0$ **28.** $y' = \dfrac{\tan y}{x},\ y(1) = \dfrac{\pi}{2}$

In exercises 29–34, use equation (2.6) to help solve the IVP.

29. $y' = 3y(2 - y),\ y(0) = 1$ **30.** $y' = y(3 - y),\ y(0) = 2$

31. $y' = 2y(5 - y),\ y(0) = 4$ **32.** $y' = y(2 - y),\ y(0) = 1$

33. $y' = y(1 - y), y(0) = \dfrac{3}{4}$ **34.** $y' = y(3 - y),\ y(0) = 0$

35. The logistic equation is sometimes written in the form $y'(t) = ry(t)(1 - y(t)/M)$. Show that this is equivalent to equation (2.4) with $r/M = k$. Biologists have measured the values of the carrying capacity M and growth rate r for a variety of fish. Just for the halibut, approximate values are $r = 0.71$ year^{-1} and $M = 8 \times 10^7$ kg. (a) If the initial biomass of halibut is $y(0) = 2 \times 10^7$ kg, find an equation for the biomass of halibut at any time. (b) Sketch a graph of the biomass as a function of time. (c) Estimate how long it will take for the biomass to get within 10% of the carrying capacity.

36. (a) Find the solution of equation (2.4) if $y(t) > M$. (b) If the halibut biomass (see exercise 35) explodes to 3×10^8 kg, how long will it take for the population to drop back to within 10% of the carrying capacity?

Exercises 37–40 relate to money investments.

37. (a) If continuous deposits are made into an account at the rate of \$2000 per year and interest is earned at 6% compounded continuously, find the size of the initial investment needed to reach \$1 million in 20 years. Comparing your answer to that of example 2.7, how much difference does interest rate make? (b) If \$10,000 is invested initially at 6% interest compounded continuously, find the (yearly) continuous deposit rate needed to reach \$1 million in 20 years. How much difference does an initial deposit make?

38. A house mortgage is a loan that is to be paid over a fixed period of time. Suppose \$150,000 is borrowed at 8% interest. If the monthly payment is \$$P$, then explain why the equation $A'(t) = 0.08A(t) - 12P$, $A(0) = 150{,}000$ is a model of the amount owed after t years. For a 30-year mortgage, the payment P is set so that $A(30) = 0$. (a) Find P. Then, compute the total amount paid and the amount of interest paid. (b) Rework part (a) with a 7.5% loan. Does the half-percent decrease in interest rate make a difference? (c) Rework part (a) with a 15-year mortgage. Compare the monthly payments and total amount paid. (d) Rework part (a) with a loan of \$125,000. How much difference does it make to add an additional \$25,000 down payment?

39. (a) A person contributes \$10,000 per year to a retirement fund continuously for 10 years until age 40 but makes no initial payment and no further payments. At 8% interest, what is the value of the fund at age 65? (b) A person contributes \$20,000 per year to a retirement fund from age 40 to age 65 but makes no initial payment. At 8% interest, what is the value of the fund at age 65? (c) Find the interest rate r at which the investors have equal retirement funds.

40. An endowment is seeded with \$1,000,000 invested with interest compounded continuously at 10%. Determine the amount that can be withdrawn (continuously) annually so that the endowment lasts 30 years.

41. (a) In example 2.3, find and graph the solution passing through (0, 0). (b) Notice that the initial value problem is $y' = \dfrac{x^2 + 7x + 3}{y^2}$ with $y(0) = 0$. If you substitute $y = 0$ into the differential equation, what is $y'(0)$? Describe what is happening graphically at $x = 0$. (c) Notice that $y'(x)$ does not exist at any x for which $y(x) = 0$. Given the solution of example 2.4, this occurs if $x^3 + \frac{21}{2}x^2 + 9x + 3c = 0$. Find the values c_1 and c_2 such that this equation has three real solutions if and only if $c_1 < c < c_2$.

42. (a) Graph the solution of $y' = \dfrac{x^2 + 7x + 3}{y^2}$ with $c = c_2$. (See exercise 41.) (b) For $c = c_2$ in exercise 41, argue that the solution to $y' = \dfrac{x^2 + 7x + 3}{y^2}$ with $y(0) = \sqrt[3]{3c_2}$ has two points with vertical tangent lines. (c) Estimate the locations of the three points with vertical tangent lines in exercise 41.

Exercises 43–46 relate to reversible bimolecular chemical reactions, where molecules A and B combine to form two other molecules C and D and vice versa. If $x(t)$ and $y(t)$ are the concentrations of C and D, respectively, and the initial concentrations of A, B, C and D are a, b, c and d, respectively, then the reaction is modeled by

$$x'(t) = k_1(a + c - x)(b + c - x) - k_{-1}x(d - c + x)$$

for rate constants k_1 and k_{-1}.

43. If $k_1 = 1$, $k_{-1} = 0.625$, $a + c = 0.4$, $b + c = 0.6$, $c = d$ and $x(0) = 0.2$, find the concentration $x(t)$. Graph $x(t)$ and find the eventual concentration level.

44. Repeat exercise 43 with (a) $x(0) = 0.3$ and (b) $x(0) = 0.6$. Briefly explain what is physically impossible about the initial condition in part (b).

45. For the bimolecular reaction with $k_1 = 0.6$, $k_{-1} = 0.4$, $a + c = 0.5$, $b + c = 0.6$ and $c = d$, write the differential equation for the concentration of C. For $x(0) = 0.2$, solve for the concentration at any time and graph the solution.

46. For the bimolecular reaction with $k_1 = 1.0$, $k_{-1} = 0.4$, $a + c = 0.6$, $b + c = 0.4$ and $d - c = 0.1$, write the differential equation for the concentration of C. For $x(0) = 0.2$, solve for the concentration at any time and graph the solution.

Exercises 47–50 relate to logistic growth with harvesting. Suppose that a population in isolation satisfies the logistic equation $y'(t) = ky(M - y)$. If the population is harvested (for example, by fishing) at the rate R, then the population model becomes $y'(t) = ky(M - y) - R$.

47. Suppose that a species of fish has population in hundreds of thousands that follows the logistic model with $k = 0.025$ and $M = 8$. (a) Determine the long-term effect on population if the initial population is 800,000 [$y(0) = 8$] and fishing removes fish at the rate of 20,000 per year. (b) Repeat if fish are removed at the rate of 60,000 per year.

48. For the fishing model $P'(t) = 0.025P(t)[8 - P(t)] - R$, the population is constant if $P'(t) = P^2 - 8P + 40R = 0$. The solutions are called equilibrium points. Compare the equilibrium points for parts (a) and (b) of exercise 47.

49. Solve the population model

$$\begin{aligned} P'(t) &= 0.05P(t)[8 - P(t)] - 0.6 \\ &= 0.4P(t)[1 - P(t)/8] - 0.6 \end{aligned}$$

with $P(0) > 2$ and determine the limiting amount $\lim_{t\to\infty} P(t)$. What happens if $P(0) < 2$?

50. The constant 0.4 in exercise 49 represents the natural growth rate of the species. Comparing answers to exercises 47 and 49, discuss how this constant affects the population size.

APPLICATIONS

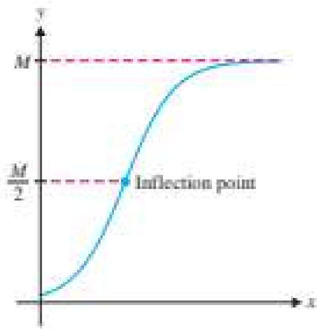
1. The resale value $r(t)$ of a machine decreases at a rate proportional to the difference between the current price and the scrap value S. Write a differential equation for r. If the machine sells new for $14,000, is worth $8000 in 4 years and has a scrap value of $1000, find an equation for the resale value at any time.

2. A granary is filled with 6000 kg of grain. The grain is shipped out at a constant rate of 1000 kg per month. Storage costs equal 2 cents per kg per month. Let $S(t)$ be the total storage charge for t months. Write a differential equation for S with $0 \leq t \leq 6$. Solve the initial value problem for $S(t)$. What is the total storage bill for 6 months?

3. For the logistic equation $y'(t) = ky(M - y)$, show that a graph of $\frac{1}{y}y'$ as a function of y produces a linear graph. Given the slope m and intercept b of this line, explain how to compute the model parameters k and M. Use the following data to estimate k and M for a fish population. Predict the eventual population of the fish.

t	2	3	4	5
y	1197	1291	1380	1462

4. It is an interesting fact that the inflection point in the solution of a logistic equation (see figure) occurs at $y = \frac{1}{2}M$. To verify this, you do not want to compute two derivatives of equation (2.6) and solve $y'' = 0$. This would be quite ugly and would give you the solution in terms of t, instead of y. Instead, a more abstract approach works well. Start with the differential equation $y' = ky(M - y)$ and take derivatives of both sides.



(Hint: Use the product and chain rules on the right-hand side.) You should find that $y'' = ky'(M - 2y)$. Then, $y'' = 0$ if and only if $y' = 0$ or $y = \frac{1}{2}M$. Rule out $y' = 0$ by describing how the solution behaves at the equilibrium values.

5. The downward velocity of a falling object is modeled by the differential equation $\frac{dv}{dt} = 9.8 - 0.002v^2$. If $v(0) = 0$ m/s, the velocity will increase to a **terminal velocity**. The terminal velocity is an equilibrium solution where the upward air drag exactly cancels the downward gravitational force. Find the terminal velocity.

6. Suppose that f is a function such that $f(x) \geq 0$ and $f'(x) < 0$ for $x > 0$. Show that the area of the triangle with sides $x = 0$, $y = 0$ and the tangent line to $y = f(x)$ at $x = a > 0$ is $A(a) = -\frac{1}{2}\left\{a^2 f'(a) - 2af(a) + [f(a)]^2/f'(a)\right\}$. To find a curve such that this area is the same for any choice of $a > 0$, solve the equation $\frac{dA}{da} = 0$.

EXPLORATORY EXERCISES

1. An object traveling through the air is acted on by gravity (acting vertically), air resistance (acting in the direction opposite velocity) and other forces (such as a motor). An equation for the horizontal motion of a jet plane is $v' = c - f(v)/m$, where c is the thrust of the motor and $f(v)$ is the air resistance force. For some ranges of velocity, the air resistance actually *drops* substantially for higher velocities as the air around the object becomes turbulent. For example, suppose that $v' = 32{,}000 - f(v)$, where $f(v) = \begin{cases} 0.8v^2 & \text{if } 0 \le v \le 100 \\ 0.2v^2 & \text{if } 100 < v \end{cases}$. To solve the initial value problem $v' = 32{,}000 - f(v)$, $v(0) = 0$, start with the initial value problem $v' = 32{,}000 - 0.8v^2$, $v(0) = 0$. Solve this IVP and determine the time t such that $v(t) = 100$. From this time forward, the equation becomes $v' = 32{,}000 - 0.2v^2$. Solve the IVP $v' = 32{,}000 - 0.2v^2$, $v(0) = 100$. Put this solution together with the previous solution to piece together a solution valid for all time.

2. Solve the initial value problems $\frac{dy}{dt} = 2(1-y)(2-y)(3-y)$ with (a) $y(0) = 0$, (b) $y(0) = 1.5$, (c) $y(0) = 2.5$ and (d) $y(0) = 4$. State as completely as possible how the limit $\lim_{t\to\infty} y(t)$ depends on $y(0)$.

4.3 FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

A first-order linear differential equation can be expressed in general form as

$$y' + P(x)y = Q(x) \tag{3.1}$$

where $P(x)$ and $Q(x)$ are continuous functions of x. Equation (3.1) is *linear with respect to y* (not x) because it does not involve any non-linear terms such as $y \cdot y'$, y'', $\sin y$, *or* $\ln y'$. Thus, y and its derivative y′ are to the first power; no multiplication, division and higher powers will be allowed in a linear DE.

EXAMPLE 3.1 A Linear DE of the First Order

Indicate whether $x^2 y' - xy = 2 \ln x$ is linear or not. If it is, put it in standard form and identify $P(x)$ and $Q(x)$.

Solution Clearly, the above equation is linear in terms of y as it does not contain any of the elements mentioned before. Dividing by x^2 we get,

$$y' - \frac{1}{x}y = \frac{2}{x^2}\ln x$$

Then, $P(x) = -\frac{1}{x}$ *and* $Q(x) = \frac{2}{x^2}\ln x$ ■

EXAMPLE 3.2 A Non-Linear DE

Indicate whether $y' + e^x \sin y = 3y^2$ is linear or not.

Solution The above differential equation is not linear since the terms "sin y" and "y^2" are not allowed in a linear differential equation ■

General Solution of Linear Differential Equations of the First Order

$$y' + P(x)y = Q(x)$$

The solution of the linear differential equation could be obtained easily if we can express the left-hand side of the equation as a derivative of a single term. For that, we will multiply both sides of the equation by a positive function $\eta(x)$. Equation (3.1) will be

$$\eta(x)y' + \eta(x)P(x)y = \eta(x)Q(x) \tag{3.2}$$

We will show you how we get $v(x)$ in a while. For now, we will focus on how to solve the equation. We would like you to observe what happens to the left-hand side of the equation keeping in mind the derivative of a product of two functions $(u{\cdot}v = u{\cdot}v' + v{\cdot}u')$

$$\eta(x)y' + \eta(x)P(x)y = \frac{d}{dx}(\eta(x) \cdot y) \tag{3.3}$$

Substituting in equation (3.2) we get,

$$\frac{d}{dx}(\eta(x) \cdot y) = \eta(x)Q(x)$$

When integrating with respect to x, the equation will be

$$\eta(x) \cdot y = \int \eta(x)Q(x)dx$$

then,

$$y = \frac{1}{\eta(x)}\int \eta(x)Q(x)dx \tag{3.4}$$

Equation (3.4) is the solution of the linear differential equation that we've started with. The presence of $\eta(x)$ was crucial for in order to integrate easily and obtain the solution. For this reason we call $\eta(x)$ the integrating factor. Still, one part of the puzzle remains a mystery. Observe that in equation (3.3) the hidden assumption was that:

$$\eta'(x) = P(x)\eta(x)$$

$$\frac{d\eta}{dx} = P \cdot \eta$$

a simple separation of variables yields to

$$\frac{d\eta}{\eta} = P\,dx$$

$$\ln \eta = \int P\,dx$$

then,

$$\eta = e^{\int P\,dx} \tag{3.5}$$

In sum, the general solution of a linear differential equation of the first order is given by

$$y = \frac{1}{\eta(x)}\int \eta(x)Q(x)dx$$

where

$\eta(x) = e^{\int P(x)dx}$ is called the integrating factor.

EXAMPLE 3.3 Solving a Linear Equation

Solve the differential equation

$$y' + \frac{1}{x}y = 4x^2 \qquad x > 0$$

Solution The first thing to do is to make sure that our DE is in standard form. The next step is to identify $P(x)$ and $Q(x)$.

$$P(x) = \frac{1}{x}$$

$$Q(x) = 4x^2$$

The integrating factor $\eta(x) = e^{\int P(x)\,dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$ (remember the properties of logarithms).

Equation (3.4) gives the solution

$$y = \frac{1}{x}\int x \cdot 4x^2\,dx = \frac{1}{x}\int 4x^3\,dx = \frac{1}{x}\,(x^4 + c)$$

$$y = x^3 + \frac{c}{x} \quad \blacksquare$$

EXAMPLE 3.4 Solution That Involves Trigonometric Forms

Solve the particular solution of the differential equation $\sin xy' - 1 = -\cos xy$

Solution The DE is not in standard form. We divide by $\sin x$ and re-order to identify $P(x)$ and $Q(x)$.

$$y' + \frac{\cos x}{\sin x}y = \frac{1}{\sin x}$$

then,

$$P(x) = \cot x \text{ and } Q(x) = \frac{1}{\sin x}$$

$$\eta(x) = e^{\int P(x)\,dx} = e^{\int \cot x\,dx} = e^{\ln(\sin x)} = \sin x$$

$$y = \frac{1}{\sin x} \cdot \int \sin x \cdot \frac{1}{\sin x}dx = \frac{1}{\sin x}[x + C]$$

$$y = \frac{x}{\sin x} + \frac{C}{\sin x} \quad \blacksquare$$

EXAMPLE 3.5 A Particular Solution That Requires Integration By Parts

Solve the particular solution of the differential equation

$$xy' + 2y = \cos 2x \qquad y\left(\frac{\pi}{2}\right) = 0$$

Solution The DE is not in standard form. We divide by x in order to identify $P(x)$ and $Q(x)$.

$$y' + \frac{2}{x}y = \frac{\cos 2x}{x}$$

Therefore,

$$P(x) = \frac{2}{x}$$

$$Q(x) = \frac{\cos 2x}{x}$$

The integrating factor

$$\eta(x) = e^{\int P(x)\,dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

Equation (3.4) gives the solution

$$y = \frac{1}{x^2}\int x^2 \cdot \frac{\cos 2x}{x}dx = \frac{1}{x^2}\int x \cdot \cos 2x\,dx$$

Use integration by parts (tabular method) for the right-hand side

$$y = \frac{1}{x^2}\left(\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c\right)$$

$$y = \frac{1}{2x}\sin 2x + \frac{1}{4x^2}\cos 2x + \frac{c}{x^2}$$

When $x = \frac{\pi}{2}$ and $y = 0$, the above equation will be

$$0 = \frac{1}{\pi}\sin\pi + \frac{1}{\pi^2}\cos\pi + \frac{4C}{\pi^2}$$

$$c = \frac{1}{4}$$

The particular solution is $y = \frac{1}{2x}\sin 2x + \frac{1}{4x^2}\cos 2x + \frac{1}{4x^2}$ ■

EXERCISES 4.3

WRITING EXERCISES

1. A differential equation of the first order is not linear if it can't be written in the form $y' + P(x)y = Q(x)$. Discuss some of the useful strategies that you've picked up from this lesson.

2. Discuss the importance of dividing the constant of integration in your general solution by the integrating factor. Is this something that we can do without?

In exercises 1–10, solve the given linear differential equations.

1. $xy' + 4y = 5x$
2. $y' - \frac{2}{x}y = 4x^4$
3. $y' - e^{-x} = -y$
4. $y' = 3y + e^{3x}$
5. $\frac{dy}{dx} = \frac{xy - 2}{x^2}$
6. $\frac{dy}{dx} = \frac{x^2\ln x - xy}{x^2}$

Use integration by parts to evaluate the integral

$$y = x\left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C\right] = \frac{1}{2}x^3\ln x - \frac{1}{4}x^3 + Cx$$

7. $y' - 2y = 3x^2$
8. $x^2y' = -3xy + \cos 2x$
9. $\sin x\frac{dy}{dx} + y\cos x = \tan x$
10. $x\frac{dy}{dx} + (3 + x)y = \frac{4}{x}$

In exercises 11–16, solve the initial value problems.

11. $xy' + 3y = 4x$ $\quad y(1) = 3$
12. $y' - \frac{2}{x}y = 3x^4$ $\quad y(2) = 64$
13. $\frac{dy}{dx} = e^{-x} - y$ $\quad y(0) = e$
14. $\sin xy' + \cos xy = 2$ $\quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$
15. $\frac{dr}{d\theta} = 2\cos\theta(1 - y\csc\theta)$ $\quad r\left(\frac{\pi}{2}\right) = 1$
16. $\theta\frac{dr}{d\theta} - 3r = \theta^4\sec\theta\tan\theta$ $\quad r\left(\frac{\pi}{2}\right) = 1$

4.4 DIRECTION FIELDS AND EULER'S METHOD

> **HISTORICAL NOTES**
>
> **Leonhard Euler (1707–1783)** A Swiss mathematician regarded as the most prolific mathematician of all time. Euler's complete works fill over 100 large volumes, with much of his work being completed in the last 17 years of his life after losing his eyesight. Euler made important and lasting contributions in numerous research fields, including calculus, number theory, calculus of variations, complex variables, graph theory and differential geometry.

In sections 4.2 and 4.3, we saw how to solve some simple first-order differential equations, namely, those that are separable and those that are linear. While there are numerous other special cases of differential equations whose solutions are known (you will encounter many of these in any beginners' course in differential equations), the vast majority cannot be solved exactly. For instance, the equation

$$y' = x^2 + y^2 + 1$$

is not separable nor is it linear, and cannot be solved using our current techniques. Nevertheless, *some* information about the solution(s) can be determined. In particular, since $y' = x^2 + y^2 + 1 > 0$, we can conclude that every solution is an increasing function. This type of information is called *qualitative*, since it tells us about some quality of the solution without providing any specific quantitative information.

In this section, we examine first-order differential equations in a more general setting. We consider any first-order equation of the form

$$y' = f(x, y). \tag{4.1}$$

While we cannot solve all such equations, it turns out that there are many numerical methods available for approximating the solution of such problems. In this section, we will study one such method, called Euler's method.

Mathematics author George Simmons calls Euler, "the Shakespeare of mathematics —universal, richly detailed and inexhaustible."[1] Several excellent books about Euler were published around 2007, celebrating the 300th birthday of this giant of mathematics.

[1] Simmons, G.F. (2016). *Differential Equations with Applications and Historical Notes* (London: CRC Press).

We begin by observing that any solution of equation (4.1) is a function $y = y(x)$ whose slope at any particular point (x, y) is given by $f(x, y)$. To get an idea of what a solution curve looks like, we draw a short line segment through each of a sequence of points (x, y), with slope $f(x, y)$, respectively. This collection of line segments is called the **direction field** or **slope field** of the differential equation. Notice that if a particular solution curve passes through a given point (x, y), then its slope at that point is $f(x, y)$. Thus, the direction field gives an indication of the behavior of the family of solutions of a differential equation.

EXAMPLE 4.1 Constructing a Direction Field

Construct the direction field for

$$y' = \frac{1}{2}y. \tag{4.2}$$

Solution All that needs to be done is to plot a number of points and then through each point (x, y), draw a short line segment with slope $f(x, y)$. For example, at the point $(0, 1)$, draw a short line segment with slope

$$y'(0) = f(0, 1) = \frac{1}{2}(1) = \frac{1}{2}.$$

Draw corresponding segments at 25 to 30 points. While this is a bit tedious to do by hand, a good graphing utility can do this for you with minimal effort. See Figure 4.11a for the direction field for equation (4.2). Notice that equation (4.2) is separable. We leave it as an exercise to produce the general solution

$$y = A\,e^{\frac{1}{2}x}.$$

We plot a number of the curves in this family of solutions in Figure 4.11b using the same graphing window we used for Figure 4.11a. Notice that if you connected some of the line segments in Figure 4.11a, you would obtain a close approximation to the exponential curves depicted in Figure 4.11b. It is significant to note that the direction field was constructed using only elementary algebra, *without* ever solving the differential equation. That is, by constructing the direction field, we obtain a reasonably good picture of how the solution curves behave. Such qualitative information about the solution gives us a graphical idea of how solutions behave, but no details, such as the value of a solution at a specific point. We'll see later in this section that we can obtain approximate values of the solution of an IVP numerically.

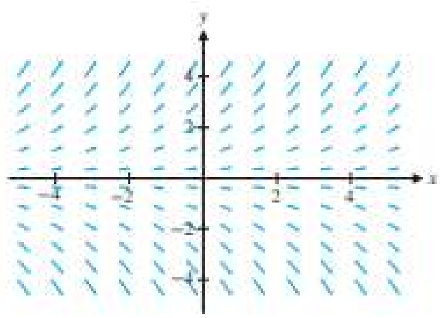


FIGURE 4.11a
Direction field for $y' = \frac{1}{2}y$

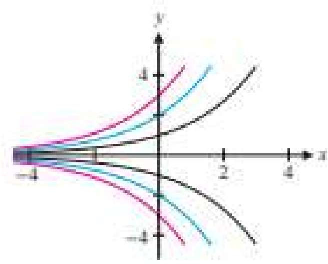


FIGURE 4.11b
Several solutions of $y' = \frac{1}{2}y$

As we have already seen, differential equations are used to model a wide variety of phenomena in science and engineering. For instance, differential equations are used to find flow lines or equipotential lines for electromagnetic fields. In such cases, it is very helpful to visualize solutions graphically, so as to gain an intuitive understanding of the behavior of such solutions and the physical phenomena they are describing.

EXAMPLE 4.2 Using a Direction Field to Visualize the Behavior of Solutions

Construct the direction field for

$$y' = x + e^{-y}.$$

Solution There's really no trick to this; just draw a number of line segments with the correct slope. Again, we let our CAS do this for us and obtained the direction field in Figure 4.12a. Unlike example 4.1, you do not know how to solve this differential equation exactly. Even so, you should be able to clearly see from the direction field how solutions behave. For example, solutions that start out in the second quadrant initially decrease very rapidly, may dip into the third quadrant and then get pulled into the first quadrant and increase quite rapidly toward infinity. This is quite a bit of information to have determined using little more than elementary algebra. In Figure 4.12b, we have plotted the solution of the differential equation that also satisfies the initial condition $y(-4) = 2$. We'll see how to generate such an approximate solution later in this section. Note how well this corresponds with what you get by connecting a number of the line segments in Figure 4.12a.

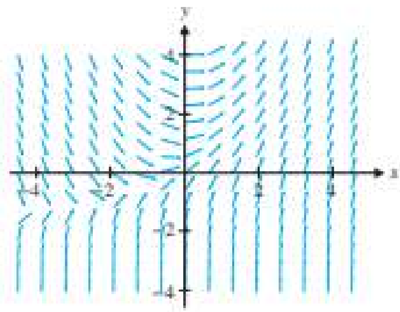


FIGURE 4.12a
Direction field for $y' = x + e^{-y}$

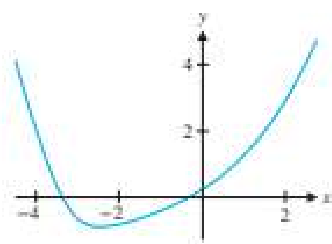


FIGURE 4.12b
Solution of $y' = x + e^{-y}$ passing through $(-4, 2)$

■

You have already seen (in sections 4.1 and 4.2) how differential equation models can provide important information about how populations change over time. A model of population growth that includes a **critical threshold** is

$$P'(t) = -2[1 - P(t)][2 - P(t)]P(t),$$

where $P(t)$ represents the size of a population at time t.

A simple context in which you might encounter a critical threshold is with the case of a sudden infestation of pests. For instance, suppose that you have some method for removing ants from your home. As long as the reproductive rate of the ants is lower than your removal rate, the ant population will stay under control. However, as soon as your removal rate becomes less than the ant reproductive rate (i.e., the removal rate crosses a critical threshold), you won't be able to keep up with the extra ants and you will suddenly be faced with a big ant problem. We see this type of behavior in example 4.3.

EXAMPLE 4.3 Population Growth with a Critical Threshold

Draw the direction field for

$$P'(t) = -2[1 - P(t)][2 - P(t)]P(t)$$

and discuss the eventual size of the population.

Solution The direction field is particularly easy to sketch here since the right-hand side depends on P, but not on t. If $P(t) = 0$, then $P'(t) = 0$, also, so that the direction field is horizontal. The same is true for $P(t) = 1$ and $P(t) = 2$. If $0 < P(t) < 1$, then $P'(t) < 0$ and the solution decreases. If $1 < P(t) < 2$, then $P'(t) > 0$ and the solution increases. Finally, if $P(t) > 2$, then $P'(t) < 0$ and the solution decreases. Putting all of these pieces together, we get the direction field seen in Figure 4.13. The constant solutions $P(t) = 0$, $P(t) = 1$ and $P(t) = 2$ are called **equilibrium solutions**. The solution $P(t) = 1$ is called an **unstable equilibrium**, since populations that start near 1 don't remain close to 1. Similarly, the solutions $P(t) = 0$ and $P(t) = 2$ are called **stable equilibria**, since populations either rise to 2 or drop to 0 (extinction), depending on which side of the critical threshold $P(t) = 1$ they are on. (Look again at Figure 4.13.)

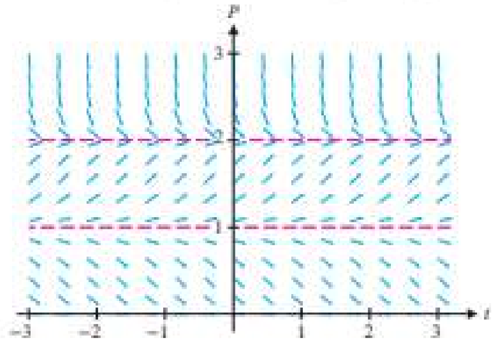


FIGURE 4.13
Direction field for $P'(t) = -2[1 - P(t)][2 - P(t)]P(t)$

In cases where you are interested in finding a specific solution, the numerous arrows of a direction field can be distracting. Euler's method, developed below, enables you to approximate a single solution curve. The method is quite simple, based essentially on the idea of a direction field. Although Euler's method does not provide particularly accurate approximations, related and more accurate methods will be explored in the exercises.

Consider the IVP

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

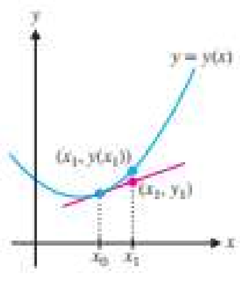


FIGURE 4.14
Tangent line approximation

Once again, we must emphasize that, assuming there is a solution $y = y(x)$, the differential equation tells us that the slope of the tangent line to the solution curve at any point (x, y) is given by $f(x, y)$. Remember that the tangent line to a curve stays close to that curve near the point of tangency. Notice that we already know one point on the graph of $y = y(x)$, namely, the initial point (x_0, y_0). Referring to Figure 4.14, if we would like to approximate the value of the solution at $x = x_1$ [i.e., $y(x_1)$] and if x_1 is not too far from x_0, then we could follow the tangent line at (x_0, y_0) to the point corresponding to $x = x_1$ and use the y-value at that point (call it y_1) as an approximation to $y(x_1)$. This is virtually the same thinking we employed when we devised Newton's method and differential (tangent line) approximations in Chapter 3. The equation of the tangent line at $x = x_0$ is

$$y = y_0 + y'(x_0)(x - x_0).$$

Thus, an approximation to the value of the solution at $x = x_1$ is the y-coordinate of the point on the tangent line corresponding to $x = x_1$, that is,

$$y(x_1) \approx y_1 = y_0 + y'(x_0)(x_1 - x_0). \tag{4.3}$$

You have only to glance at Figure 4.14 to realize that this approximation is valid only when x_1 is close to x_0. In solving an IVP, though, we are usually interested in finding the value of the solution on an interval $[a, b]$ of the x-axis. With Euler's method, we settle for finding an approximate solution at a sequence of points in the interval $[a, b]$. First, we partition the interval $[a, b]$ into n equal-sized pieces (a *regular partition;* where did you see this notion before?):

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

where

$$x_{i+1} - x_i = h,$$

for all $i = 0, 1, \ldots, n - 1$. We call h the **step size**. From the tangent line approximation (4.3), we already have

$$\begin{aligned} y(x_1) \approx y_1 &= y_0 + y'(x_0)(x_1 - x_0) \\ &= y_0 + hf(x_0, y_0), \end{aligned}$$

where we have replaced $(x_1 - x_0)$ by the step size, h, and used the differential equation to write $y'(x_0) = f(x_0, y_0)$. To approximate the value of $y(x_2)$, we could use the tangent line at the point $(x_1, y(x_1))$ to produce a tangent line approximation.

$$\begin{aligned} y(x_2) &\approx y(x_1) + y'(x_1)(x_2 - x_1) \\ &= y(x_1) + hf(x_1, y(x_1)), \end{aligned}$$

where we have used the differential equation to replace $y'(x_1)$ by $f(x_1, y(x_1))$ and used the fact that $x_2 - x_1 = h$. This is not usable, though, since we do not know the value of $y(x_1)$. However, we can approximate $y(x_1)$ by the approximation obtained in the previous step, y_1, to obtain

$$\begin{aligned} y(x_2) &\approx y(x_1) + hf(x_1, y(x_1)) \\ &\approx y_1 + hf(x_1, y_1) = y_2. \end{aligned}$$

Continuing in this way, we obtain the sequence of approximate values:

EULER'S METHOD

$$y(x_{i+1}) \approx y_{i+1} = y_i + hf(x_i, y_i), \quad \text{for } i = 0, 1, 2 \ldots. \tag{4.4}$$

This tangent line method of approximation is called **Euler's method**.

EXAMPLE 4.4 Using Euler's Method

Use Euler's method to approximate the solution of the IVP

$$y' = y, \quad y(0) = 1.$$

Solution You can probably solve this equation by inspection, but if not, notice that it's separable and that the solution of the IVP is $y = y(x) = e^x$. We will use this exact solution to evaluate the performance of Euler's method. From (4.4) with $f(x, y) = y$ and taking $h = 1$, we have

$$\begin{aligned} y(x_1) &\approx y_1 = y_0 + hf(x_0, y_0) \\ &= y_0 + h\,y_0 = 1 + 1(1) = 2. \end{aligned}$$

Likewise, for further approximations, we have

$$\begin{aligned} y(x_2) &\approx y_2 = y_1 + hf(x_1, y_1) \\ &= y_1 + h\,y_1 = 2 + 1(2) = 4, \\ y(x_3) &\approx y_3 = y_2 + hf(x_2, y_2) \\ &= y_2 + h\,y_2 \\ &= 4 + 1(4) = 8 \end{aligned}$$

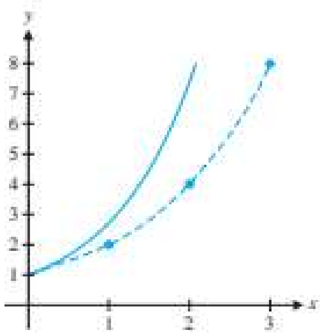


FIGURE 4.15
Exact solution versus the approximate solution (dashed line)

and so on. In this way, we construct a sequence of approximate values of the solution function. In Figure 4.15, we have plotted the exact solution (solid curve) against the approximate solution obtained from Euler's method (dashed curve). Notice how the error grows as x gets farther and farther from the initial point. This is characteristic of Euler's method (and other similar methods). This growth in error becomes even more apparent if we look at a table of values of the approximate and exact solutions together. We display these in the table that follows, where we have used $h = 0.1$ (values are displayed to seven digits).

h	*Euler*	*Exact*	*Error = Exact − Euler*
0.1	1.1	1.1051709	0.0051709
0.2	1.21	1.2214028	0.0114028
0.3	1.331	1.3498588	0.0188588
0.4	1.4641	1.4918247	0.0277247
0.5	1.61051	1.6487213	0.0382113
0.6	1.771561	1.8221188	0.0505578
0.7	1.9487171	2.0137527	0.0650356
0.8	2.1435888	2.2255409	0.0819521
0.9	2.3579477	2.4596031	0.1016554
1.0	2.5937425	2.7182818	0.1245393

As you might expect from our development of Euler's method, the smaller we make h, the more accurate the approximation at a given point tends to be. As well, the smaller the value of h, the more steps it takes to reach a given value of x. In the following table, we display the Euler's method approximation, the error and the number of steps needed to reach $x = 1.0$. Here, the exact value of the solution is $y = e^1 \approx 2.718281828459$.

h	*Euler*	*Error = Exact − Euler*	*Number of Steps*
1.0	2	0.7182818	1
0.5	2.25	0.4682818	2
0.25	2.4414063	0.2768756	4
0.125	2.5657845	0.1524973	8
0.0625	2.6379285	0.0803533	16
0.03125	2.6769901	0.0412917	32
0.015625	2.697345	0.0209369	64
0.0078125	2.707739	0.0105428	128
0.00390625	2.7129916	0.0052902	256

From the table, observe that each time the step size h is cut in half, the error is also cut roughly in half. This increased accuracy, though, comes at the cost of the additional steps of Euler's methods required to reach a given point (doubled each time h is halved). ■

TODAY IN MATHEMATICS

Kay McNulty (1921–2006) An Irish mathematician who became one of the first computer software designers. In World War II, before computers, she approximated solutions of projectile differential equations. McNulty said, "We did have desk calculators at that time, mechanical and driven with electric motors, that could do simple arithmetic. You'd do a multiplication and when the answer appeared, you had to write it down to re-enter it into the machine to do the next calculation. We were preparing a firing table for each gun, with maybe 1800 simple trajectories. To hand-compute just one of these trajectories took 30 or 40 hours of sitting at a desk with paper and a calculator. . . . Actually, my title working for the ballistics project was 'computer'. . . . ENIAC made me, one of the first 'computers,' obsolete."[2]

[2] Petzinger, T. (1996). History of Software Begins with the Work of Some Brainy Women. *Wall Street Journal*, November 15.

The point of having a numerical method, of course, is to find meaningful approximations to the solution of problems that we do not know how to solve exactly. Example 4.5 is of this type.

EXAMPLE 4.5 Finding an Approximate Solution

Find an approximate solution of the IVP

$$y' = x^2 + y^2, \; y(-1) = -\frac{1}{2}.$$

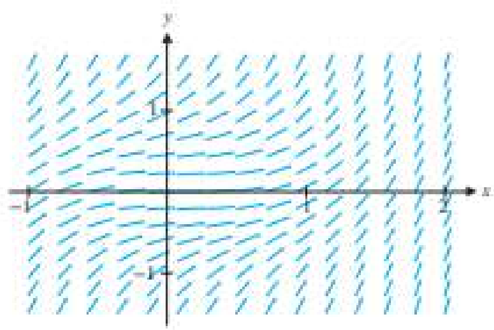


FIGURE 4.16
Direction field for $y' = x^2 + y^2$

Solution First, let's take a look at the direction field, so that we can see how solutions to this differential equation should behave. (See Figure 4.16.) Using Euler's method with $h = 0.1$, we get

$$\begin{aligned} y(x_1) \approx y_1 &= y_0 + hf(x_0, y_0) \\ &= y_0 + h(x_0^2 + y_0^2) \\ &= -\frac{1}{2} + 0.1\left[(-1)^2 + \left(-\frac{1}{2}\right)^2\right] = -0.375 \end{aligned}$$

and
$$\begin{aligned} y(x_2) \approx y_2 &= y_1 + hf(x_1, y_1) \\ &= y_1 + h(x_1^2 + y_1^2) \\ &= -0.375 + 0.1[(-0.9)^2 + (-0.375)^2] = -0.2799375 \end{aligned}$$

and so on. Continuing in this way, we generate the table of values that follows.

x	*Euler*	*x*	*Euler*	*x*	*Euler*
−0.9	−0.375	0.1	−0.0575822	1.1	0.3369751
−0.8	−0.2799375	0.2	−0.0562506	1.2	0.4693303
−0.7	−0.208101	0.3	−0.0519342	1.3	0.6353574
−0.6	−0.1547704	0.4	−0.0426645	1.4	0.8447253
−0.5	−0.116375	0.5	−0.0264825	1.5	1.1120813
−0.4	−0.0900207	0.6	−0.0014123	1.6	1.4607538
−0.3	−0.0732103	0.7	0.0345879	1.7	1.9301340
−0.2	−0.0636743	0.8	0.0837075	1.8	2.5916757
−0.1	−0.0592689	0.9	0.1484082	1.9	3.587354
0.0	−0.0579176	1.0	0.2316107	2.0	5.235265

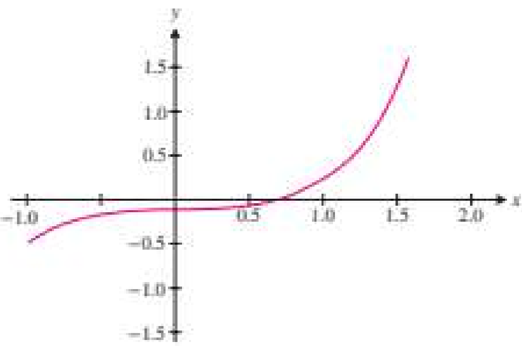


FIGURE 4.17a
Approximate solution of $y' = x^2 + y^2$, passing through $(-1, -\frac{1}{2})$

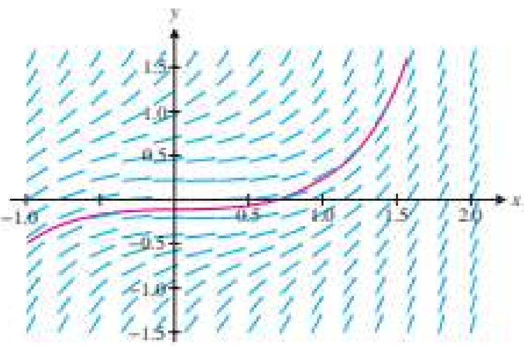


FIGURE 4.17b
Approximate solution superimposed on the direction field

In Figure 4.17a, we display a smooth curve connecting the data points in the preceding table. Take particular note of how well this corresponds with the direction field in Figure 4.16. To make this correspondence more apparent, we show a graph of the approximate solution superimposed on the direction field in Figure 4.17b. Since this corresponds so well with the behavior you expect from the direction field, you should expect that there are no gross errors in this approximate solution. (Certainly, there is always some level of round-off and other numerical errors.) ■

We can expand on the concept of equilibrium solution, which we introduced briefly in example 4.3. More generally, we say that the constant function $y = c$ is an **equilibrium solution** of the differential equation $y' = f(t, y)$ if $f(t, c) = 0$ for all t. In simple terms, this says that $y = c$ is an equilibrium solution of the differential equation $y' = f(t, y)$ if the substitution $y = c$ reduces the equation to simply $y' = 0$. Observe that this, in turn, says that $y(t) = c$ is a (constant) solution of the differential equation. In example 4.6, notice that finding equilibrium solutions requires only basic algebra.

EXAMPLE 4.6 Finding Equilibrium Solutions

Find all equilibrium solutions of (a) $y'(t) = k[y(t) - 70]$ and (b) $y'(t) = 2y(t)[4 - y(t)]$.

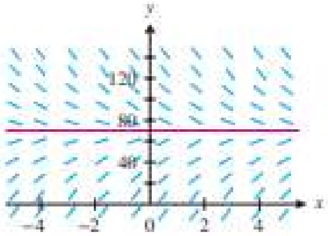


FIGURE 4.18a
Direction field

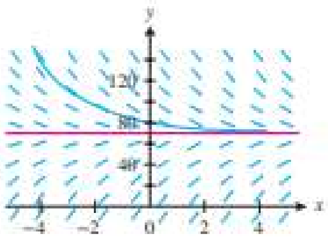


FIGURE 4.18b
Solution curve starting above $y = 70$

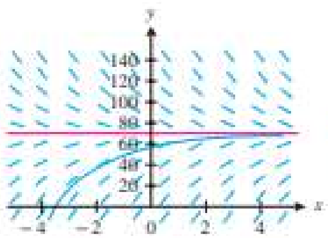


FIGURE 4.18c
Solution curve starting below $y = 70$

Solution An equilibrium solution is a constant solution that reduces the equation to $y'(t) = 0$. For part (a), this gives us

$$0 = y'(t) = k[y(t) - 70] \quad \text{or} \quad 0 = y(t) - 70.$$

The only equilibrium solution is then $y = 70$. For part (b), we want

$$0 = 2y(t)[4 - y(t)] \quad \text{or} \quad 0 = y(t)[4 - y(t)].$$

So, in this case, there are two equilibrium solutions: $y = 0$ and $y = 4$. ■

There is some special significance to an equilibrium solution, which we describe from a sketch of the direction field. Start with the differential equation $y'(t) = k[y(t) - 70]$ for some negative constant k. Notice that if $y(t) > 70$, then $y'(t) = k[y(t) - 70] < 0$ (since k is negative). Of course, $y'(t) < 0$ means that the solution is decreasing. Similarly, when $y(t) < 70$, we have that $y'(t) = k[y(t) - 70] > 0$, so that the solution is increasing. Observe that the direction field sketched in Figure 4.18a suggests that $y(t) \to 70$ as $t \to \infty$, since all arrows point toward the line $y = 70$. More precisely, if a solution curve lies slightly above the line $y = 70$, notice that the solution decreases, toward $y = 70$, as indicated in Figure 4.18b. Similarly, if the solution curve lies slightly below $y = 70$, then the solution increases toward $y = 70$, as shown in Figure 4.18c. You should observe that we obtained this information without solving the differential equation.

We say that an equilibrium solution is **stable** if solutions close to the equilibrium solution tend to approach that solution as $t \to \infty$. Observe that this is the behavior indicated in Figures 4.18a to 4.18c, so that the solution $y = 70$ is a stable equilibrium. Alternatively, an equilibrium solution is **unstable** if solutions close to the equilibrium solution tend to get further away from that solution as $t \to \infty$.

In example 4.6, part (b), we found that $y'(t) = 2y(t)[4 - y(t)]$ has the two equilibrium solutions $y = 0$ and $y = 4$. We now use a direction field to determine whether these solutions are stable or unstable.

EXAMPLE 4.7 Determining the Stability of Equilibrium Solutions

Draw a direction field for $y'(t) = 2y(t)[4 - y(t)]$ and determine the stability of all equilibrium solutions.

Solution We determined in example 4.6 that the equilibrium solutions are $y = 0$ and $y = 4$. We superimpose the horizontal lines $y = 0$ and $y = 4$ on the direction field in Figure 4.19.

Observe that the behavior is distinctly different in each of three regions in this diagram: $y > 4$, $0 < y < 4$ and $y < 0$. We analyze each separately. First, observe that if $y(t) > 4$, then $y'(t) = 2y(t)[4 - y(t)] < 0$ (since $2y$ is positive, but $4 - y$ is negative). Next, if $0 < y(t) < 4$, then $y'(t) = 2y(t)[4 - y(t)] > 0$ (since $2y$ and $4 - y$ are both positive in this case). Finally, if $y(t) < 0$, then $y'(t) = 2y(t)[4 - y(t)] < 0$. In Figure 4.19, the arrows on either side of the line $y = 4$ all point toward $y = 4$. This indicates that $y = 4$ is stable. By contrast, the arrows on either side of $y = 0$ point away from $y = 0$, indicating that $y = 0$ is an unstable equilibrium. ■

Notice that the direction field in example 4.7 gives strong evidence that if $y(0) > 0$, then the limiting value is $\lim_{t \to \infty} y(t) = 4$. [Think about why the condition $y(0) > 0$ is needed here.]

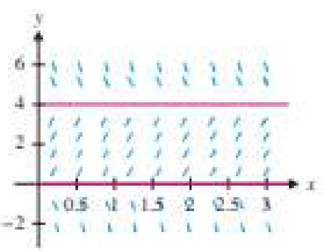


FIGURE 4.19
Direction field for $y' = 2y(4 - y)$

BEYOND FORMULAS

Numerical approximations of solutions of differential equations are basic tools of the trade for modern engineers and scientists. Euler's method, presented in this section, is one of the least accurate methods in use today, but its simplicity makes it useful in a variety of applications. Since most differential equations cannot be solved exactly, we need reliable numerical methods to obtain approximate values of the solution. What other types of calculations have you seen that typically must be approximated?

EXERCISES 4.4

WRITING EXERCISES

1. For Euler's method, explain why using a smaller step size should produce a better approximation.
2. Look back at the direction field in Figure 4.16 and the Euler's method solution in Figure 4.17a. Describe how the direction field gives you a more accurate sense of the exact solution. Given this, explain why Euler's method is important. (Hint: How would you get a table of approximate values of the solution from a direction field?)
3. In the situation of example 4.3, if you only needed to know the stability of an equilibrium solution, explain why a qualitative method is preferred over trying to solve the differential equation. Discuss the extra information provided by a solution.
4. Imagine superimposing solution curves over Figure 4.12a. Explain why the Euler's method approximation takes you from one solution curve to a nearby one. Use one of the examples in this section to describe how such a small error could lead to very large errors in approximations for large values of x.

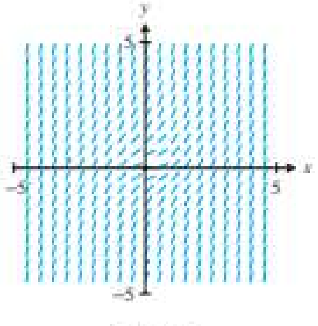
In exercises 1–6, construct four of the direction field arrows by hand and use your CAS or calculator to do the rest. Describe the general pattern of solutions.

1. $y' = x + 4y$
2. $y' = \sqrt{x^2 + y^2}$
3. $y' = 2y - y^2$
4. $y' = y^3 - 1$
5. $y' = 2xy - y^2$
6. $y' = y^3 - x$

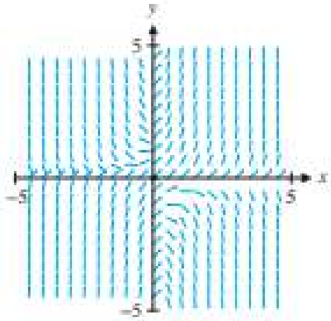
In exercises 7–12, match each differential equation to the correct direction field.

7. $y' = 2 - xy$ 8. $y' = 1 + 2xy$ 9. $y' = x\cos 3y$

10. $y' = y\cos 3x$ 11. $y' = \sqrt{x^2 + y^2}$ 12. $y' = \ln(x^2 + y^2)$



FIELD A



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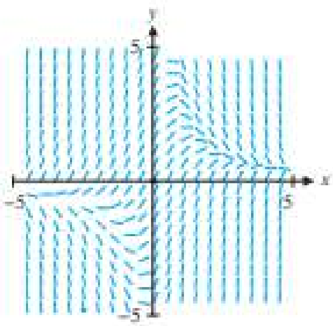

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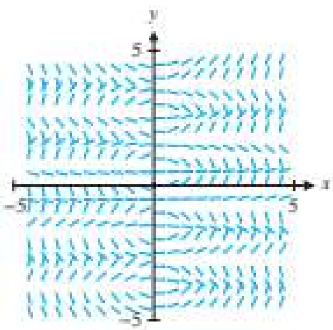
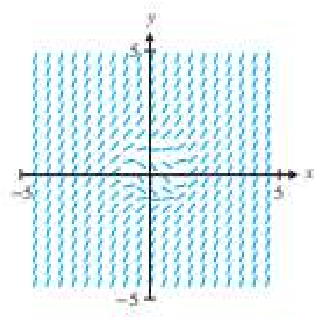
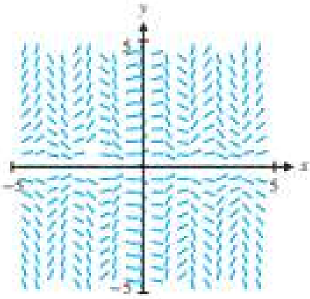


FIELD D



FIELD E



FIELD F

In exercises 13–20, use Euler's method with $h = 0.1$ and $h = 0.05$ to approximate $y(1)$ and $y(2)$. Show the first two steps by hand.

13. $y' = 2xy, y(0) = 1$
14. $y' = x/y, y(0) = 2$
15. $y' = 4y - y^2, y(0) = 1$
16. $y' = x/y2, y(0) = 2$
17. $y' = 1 - y + e^{-x}, y(0) = 3$
18. $y' = \sin y - x^2, y(0) = 1$
19. $y' = \sqrt{x + y}, y(0) = 1$
20. $y' = \sqrt{x^2 + y^2}, y(0) = 4$

21. Find the exact solutions in exercises 13 and 14, and compare $y(1)$ and $y(2)$ to the Euler's method approximations.

22. Find the exact solutions in exercises 15 and 16, and compare $y(1)$ and $y(2)$ to the Euler's method approximations.

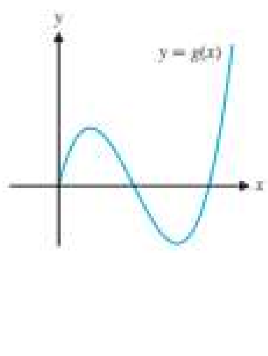
23. Sketch the direction fields for exercises 17 and 18, highlight the curve corresponding to the given initial condition and compare the Euler's method approximations to the location of the curve at $x = 1$ and $x = 2$.

24. Sketch the direction fields for exercises 19 and 20, highlight the curve corresponding to the given initial condition and compare the Euler's method approximations to the location of the curve at $x = 1$ and $x = 2$.

In exercises 25–30, find the equilibrium solutions and identify each as stable, unstable or neither.

25. $y' = 2y - y^2$
26. $y' = y^3 - 1$
27. $y' = y^2 - y^4$
28. $y' = e^{-y} - 1$
29. $y' = (1 - y)\sqrt{1 + y^2}$
30. $y' = \sqrt{1 - y^2}$

31. Given the graph of g, sketch a direction field for $y' = g$.



32. Given the graph of g, sketch a direction field for $y' = g'$.

33. Apply Euler's method with $h = 0.1$ to the initial value problem $y' = y^2 - 1, y(0) = 3$ and estimate $y(0.5)$. Repeat with $h = 0.05$ and $h = 0.01$. In general, Euler's method is more accurate with smaller h-values. Conjecture how the exact solution behaves in this example. (This is explored further in exercises 34–36.)

34. Show that $f(x) = \dfrac{2 + e^{2x}}{2 - e^{2x}}$ is a solution of the initial value problem in exercise 33. Compute $f(0.1)$, $f(0.2)$, $f(0.3)$, $f(0.4)$ and $f(0.5)$, and compare to the approximations in exercise 33.

35. Graph the solution of $y' = y^2 - 1, y(0) = 3$, given in exercise 34. Find an equation of the vertical asymptote. Explain why Euler's method would be "unaware" of the existence of this asymptote and would therefore provide very unreliable approximations.

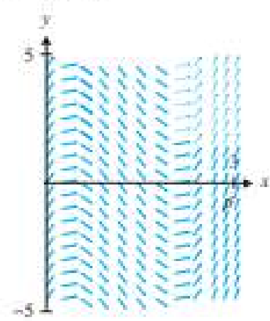
36. In exercises 33–35, suppose that x represents time (in hours) and y represents the force (in newtons) exerted on an arm of a robot. Explain what happens to the arm. Given this, explain why the negative function values in exercise 34 are irrelevant and, in some sense, the Euler's method approximations in exercise 33 give useful information.

37. Apply Euler's method with (a) $h = 1$ and (b) $h = 0.1$ to $y' = \frac{1}{3}y(8 - y), y(0) = 1$. Discuss the behavior of the approximations and the actual solution as t increases.

38. This exercise develops a continuous Newton's method. (See John Neuberger's article in the May 2007 issue of *American Mathematical Monthly.*[3]) Instead of following the tangent line to the x-axis, the tangent line is followed an infinitesimally small distance, producing a curve $(x(t), y(t))$ that satisfies $x'(t) = -f(x(t))/f'(x(t))$. (a) For $f(x) = x^2 - 2, x(0) = 1$, show that $x(t) = \sqrt{2 - e^{-2t}}$ and $\lim_{t \to \infty} x(t) = \sqrt{2}$. (b) Show that Euler's method with $h = 1$ produces Newton's method.

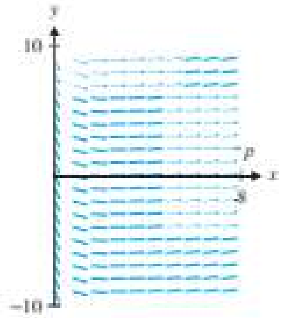
In exercises 39–42, use the direction field to sketch solution curves and estimate the initial value $y(0)$ such that the solution curve would pass through the given point P. In exercises 39 and 40, solve the equation and determine how accurate your estimate is. In exercises 41 and 42, use a CAS if available to determine how accurate your estimate is.

39. $y' = x^2 - 4x + 2, P(3, 0)$

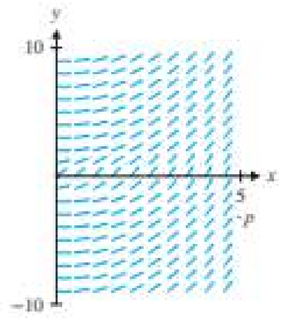


[3] Neuberger, J. (2007). Continuous Newton's Method, Inverse Function and Nash-Moser. *American Mathematical Monthly*, May.

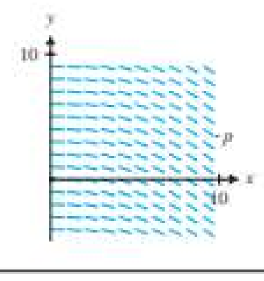
40. $y' = \dfrac{x-3}{4x+1}$, $P(8, 1)$



41. $y' = 0.2x + e^{-y^2}$, $P(5, -3)$

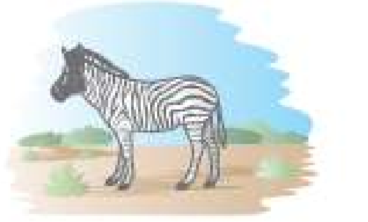


42. $y' = -0.1x - 0.1e^{-y^2/20}$, $P(10, 4)$



APPLICATIONS

1. Zebra stripes and patterns on butterfly wings are thought to be the result of gene-activated chemical processes. Suppose $g(t)$ is the amount of a gene that is activated at time t. The differential equation $g' = -g + \dfrac{3g^2}{1+g^2}$ has been used to model the process. Show that there are three equilibrium solutions: 0 and two positive solutions a and b, with $a < b$. Show that $g' > 0$ if $a < g < b$ and $g' < 0$ if $0 < g < a$ or $g > b$. Explain why $\lim_{t\to\infty} g(t)$ could be 0 or b, depending on the initial amount of activated gene. Suppose that a patch of zebra skin extends from $x = 0$ to $x = 4\pi$ with an initial activated-gene distribution $g(0) = \frac{3}{2} + \frac{3}{2}\sin x$ at location x. If black corresponds to an eventual activated-gene level of 0 and white corresponds to an eventual activated-gene level of b, show what the zebra stripes will look like.



2. Many species of trees are plagued by sudden infestations of worms. Let $x(t)$ be the population of a species of worm on a particular tree. For some species, a model for population change is $x' = 0.1x(1 - x/k) - x^2/(1 + x^2)$ for some positive constant k. If $k = 10$, show that there is only one positive equilibrium solution. If $k = 50$, show that there are three positive equilibrium solutions. Sketch the direction field for $k = 50$. Explain why the middle equilibrium value is called a *threshold*. An outbreak of worms corresponds to crossing the threshold for a large value of k (k is determined by the resources available to the worms).

EXPLORATORY EXERCISES

1. In this exercise, we use an alternative form of Euler's method to derive a method known as the **Improved Euler's method.** Start with the differential equation $y'(x) = f(x, y(x))$ and integrate both sides from $x = x_n$ to $x = x_{n+1}$. Show that $y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x))\,dx$. Given $y(x_n)$, then, you can estimate $y(x_{n+1})$ by estimating the integral $\int_{x_n}^{x_{n+1}} f(x, y(x))\,dx$. One such estimate is a Riemann sum using left-endpoint evaluation, given by $f(x_n, y(x_n))\,\Delta x$. Show that with this estimate you get Euler's method. There are numerous ways of getting better estimates of the integral. One is to use the Trapezoidal Rule, $\int_{x_n}^{x_{n+1}} f(x, y(x))\,dx \approx \dfrac{f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))}{2}\Delta x$. The drawback with this estimate is that you know $y(x_n)$ but you do *not* know $y(x_{n+1})$. The way out is to use Euler's method; you can approximate $y(x_{n+1})$ by $y(x_{n+1}) \approx y(x_n) + hf(x_n, y(x_n))$. Put all of this together to get the Improved Euler's method:

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))].$$

Use the Improved Euler's method for the IVP $y' = y$, $y(0) = 1$ with $h = 0.1$ to compute y_1, y_2 and y_3. Compare to the exact values and the Euler's method approximations given in example 4.4.

2. As in exercise 1, derive a numerical approximation method based on (a) the Midpoint Rule and (b) Simpson's Rule. Compare your results to those obtained in example 4.4 and exercise 1.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Differential equation	Doubling time	Half-life
Newton's Law of Cooling	Equilibrium solution	Stable
Separable equation	Logistic growth	Euler's method
Direction field		System of equations
		Phase portrait
		Predator-prey systems

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. For exponential growth and decay, the rate of change is constant.
2. For logistic growth, the rate of change is proportional to the amount present.
3. Any separable equation can be solved for y as a function of x.
4. The direction field of a differential equation is tangent to the solution.
5. The smaller h is, the more accurate Euler's method is.
6. An equilibrium point of a system of two equations and unknown functions x and y is any value of x such that $x' = 0$ or $y' = 0$.
7. A phase portrait shows several solutions on the same graph.

In exercises 1–6, solve the IVP.

1. $y' = 2y$, $y(0) = 3$
2. $y' = -3y$, $y(0) = 2$
3. $y' = \frac{2x}{y}$, $y(0) = 2$
4. $y' = -3xy^2$, $y(0) = 4$
5. $y' = \sqrt{xy}$, $y(1) = 4$
6. $y' = x + y^2x$, $y(0) = 1$

7. A bacterial culture has an initial population 10^4 and doubles every 2 hours. Find an equation for the population at any time t and determine when the population reaches 10^6.
8. An organism has population 100 at time $t = 0$ and population 140 at time $t = 2$. Find an equation for the population at any time and determine the population at time $t = 6$.
9. The half-life of nicotine in the human bloodstream is 2 hours. If there is initially 2 mg of nicotine present, find an equation for the amount at any time t and determine when the nicotine level reaches 0.1 mg.
10. If the half-life of a radioactive material is 3 hours, what percentage of the material will be left after 9 hours? 11 hours?
11. If you invest \$2000 at 8% compounded continuously, how long will it take the investment to double?
12. If you invest \$4000 at 6% compounded continuously, how much will the investment be worth in 10 years?
13. A cup of French press coffee is prepared at 93°C in a room with temperature 20°C. After 1 minute, the temperature has dropped to 80°C. Find an equation for the temperature at any time and determine when the temperature will reach 50°C.
14. A glass of iced tea is served at 4°C in a room with temperature 30°C. After 4 minutes, the temperature has increased to 9°C. Find an equation for the temperature at any time and determine when the temperature will reach 15°C.

In exercises 15–18, solve each separable equation, explicitly if possible.

15. $y' = 2x^3 y$
16. $y' = \frac{y}{\sqrt{1 - x^2}}$
17. $y' = \frac{4}{(y^2 + y)(1 + x^2)}$
18. $y' = e^{x+y}$

In exercises 19–22, find all equilibrium solutions and determine which are stable and which are unstable.

19. $y' = 3y(2 - y)$
20. $y' = y(1 - y^2)$
21. $y' = -y\sqrt{1 + y^2}$
22. $y' = y + \frac{2y}{1 - y}$

In exercises 23–26, sketch the direction field.

23. $y' = -x(4 - y)$
24. $y' = 4x - y^2$
25. $y' = 2xy - y^2$
26. $y' = 4x - y$

27. Suppose that the concentration x of a chemical in a bimolecular reaction satisfies the differential equation $x'(t) = (0.3 - x)(0.4 - x) - 0.25x^2$. For (a) $x(0) = 0.1$ and (b) $x(0) = 0.4$, find the concentration at any time. Graph the solutions. Explain what is physically impossible about problem (b).
28. For exercise 27, find equilibrium solutions and use a slope diagram to determine the stability of each equilibrium.
29. In the second-order chemical reaction $x' = r(a - x)(b - x)$, suppose that A and B are the same (thus, $a = b$). Identify the values of x that are possible. Draw the direction field and determine the limiting amount $\lim_{t \to \infty} x(t)$. Verify your answer by solving for x. Interpret the physical significance of a in this case.

Review Exercises

30. In an **autocatalytic reaction,** a substance reacts with itself. Explain why the concentration would satisfy the differential equation $x' = rx(1 - x)$. Identify the values of x that are possible. Draw the direction field and determine the limiting amount $\lim_{t\to\infty} x(t)$. Verify your answer by solving for x.

31. Suppose that \$100,000 is invested initially and continuous deposits are made at the rate of \$20,000 per year. Interest is compounded continuously at 10%. How much time will it take for the account to reach \$1 million?

32. Rework exercise 31 with the \$20,000 payment made at the end of each year instead of continuously.

In exercises 33–36, identify the system of equations as a predator-prey model or a competing species model. Find and interpret all equilibrium points.

33. $\begin{cases} x' = 0.1x - 0.1x^2 - 0.2xy \\ y' = -0.1y + 0.1xy \end{cases}$

34. $\begin{cases} x' = 0.2x - 0.1x^2 - 0.2xy \\ y' = 0.1y - 0.1y^2 - 0.1xy \end{cases}$

35. $\begin{cases} x' = 0.5x - 0.1x^2 - 0.2xy \\ y' = 0.4y - 0.1y^2 - 0.2xy \end{cases}$

36. $\begin{cases} x' = 0.4x - 0.1x^2 - 0.2xy \\ y' = -0.2y + 0.1xy \end{cases}$

37. Use direction fields to determine the stability of each equilibrium point in exercise 33.

38. Use direction fields to determine the stability of each equilibrium point in exercise 35.

39. Write the second-order equation $y'' - 4x^2y' + 2y = 4xy - 1$ as a system of first-order equations.

40. If you have a CAS that can solve systems of equations, sketch solutions of the system of exercise 33 with the initial conditions (a) $x = 0.4$, $y = 0.1$; (b) $x = 0.1$, $y = 0.4$.

EXPLORATORY EXERCISE

1. In this exercise, we compare two models of the vertical velocity of a falling object. Forces acting on the object are gravity and air drag. From experience, you know that the faster an object moves, the more air drag there is. But, is the drag force proportional to velocity v or the square of velocity v^2? It turns out that the answer depends on the shape and speed of the object. The goal of this exercise is to explore how much difference it makes which model is used. Define the following models for a falling object with $v \le 0$ (units of meters and seconds):

$$\text{Model 1: } \frac{dv}{dt} = -9.8 + 0.7v$$

$$\text{Model 2: } \frac{dv}{dt} = -9.8 + 0.05v^2.$$

Solve each equation with the initial condition $v(0) = 0$. Graph the two solutions on the same axes and discuss similarities and differences. Show that in both cases the limiting velocity is $\lim_{t\to\infty} v(t) = 14$ m/s. In each case, determine the time required to reach 4 m/s and the time required to reach 13 m/s. Summarizing, discuss how much difference it makes which model you use.

Student Handbook

Symbols, Formulas, and Key Concepts

Symbols EM-1

Measures EM-2

Arithmetic Operations and Relations EM-3

Algebraic Formulas and Key Concepts EM-3

Geometric Formulas and Key Concepts EM-5

Trigonometric Functions and Identities EM-6

Parent Functions and Function Operations EM-7

Calculus EM-7

Statistics Formulas and Key Concepts EM-8

Symbols

Algebra

$\neq$	is not equal to
$\approx$	is approximately equal to
$\sim$	is similar to
$>, \geq$	is greater than, is greater than or equal to
$<, \leq$	is less than, is less than or equal to
$-a$	opposite or additive inverse of *a*
$\lvert a \rvert$	absolute value of *a*
$\sqrt{a}$	principal square root of *a*
$a : b$	ratio of *a* to *b*
(x, y)	ordered pair
(x, y, z)	ordered triple
i	the imaginary unit
$b^{\frac{1}{n}} = \sqrt[n]{b}$	nth root of b
$\mathbb{Q}$	rational numbers
$\mathbb{I}$	irrational numbers
$\mathbb{Z}$	integers
$\mathbb{W}$	whole numbers
$\mathbb{N}$	natural numbers
∞	infinity
$-\infty$	negative infinity
[ ]	endpoint included
( )	endpoints not included
$\log_b x$	logarithm base b of x
$\log x$	common logarithm of x
$\ln x$	natural logarithm of x
ω	omega, angular speed
α	alpha, angle measure
β	beta, angle measure
γ	gamma, angle measure
θ	theta, angle measure
λ	lambda, wavelength
ϕ	phi, angle measure
$\mathbf{a}$	vector **a**
$\lvert \mathbf{a} \rvert$	magnitude of vector **a**

Sets and Logic

$\in$	is an element of
$\subset$	is a subset of
$\cap$	intersection
$\cup$	union
$\varnothing$	empty set
$\sim p$	negation of *p*, not *p*
$p \wedge q$	conjunction of *p* and *q*
$p \vee q$	disjunction of *p* and *q*
$p \longrightarrow q$	conditional statement, if *p* then *q*
$p \longleftrightarrow q$	biconditional statement, *p* if and only if *q*

Geometry

$\angle$	angle
$\triangle$	triangle
$^\circ$	degree
π	pi
$\measuredangle$	angles
$m\angle A$	degree measure of $\angle A$
$\overleftrightarrow{AB}$	line containing points *A* and *B*
$\overline{AB}$	segment with endpoints *A* and *B*
$\overrightarrow{AB}$	ray with endpoint *A* containing *B*
AB	measure of $\overline{AB}$, distance between points *A* and *B*
$\parallel$	is parallel to
$\nparallel$	is not parallel to
$\perp$	is perpendicular to
$\triangle$	triangle
$\square$	parallelogram
n-gon	polygon with *n* sides
$\vec{a}$	vector *a*
$\overrightarrow{AB}$	vector from *A* to *B*
$\lvert \overrightarrow{AB} \rvert$	magnitude of the vector from *A* to *B*
A'	the image of preimage *A*
$\rightarrow$	is mapped onto
$\odot A$	circle with center *A*
$\widehat{AB}$	minor arc with endpoints *A* and *B*
$\widehat{ABC}$	major arc with endpoints *A* and *C*
$m\widehat{AB}$	degree measure of arc *AB*

Trigonometry

$\sin x$	sine of *x*
$\cos x$	cosine of *x*
$\tan x$	tangent of *x*
$\sin^{-1} x$	Arcsin *x*
$\cos^{-1} x$	Arccos *x*
$\tan^{-1} x$	Arctan *x*

Symbols

Functions

$f(x)$	f of x, the value of f at x
$f(x) = \{$	piecewise-defined function
$f(x) = \lvert x \rvert$	absolute value function
$f(x) = [\![x]\!]$	function of greatest integer not greater than x
$f(x, y)$	f of x and y, a function with two variables, x and y
$[f \circ g](x)$	f of g of x, the composition of functions f and g
$f^{-1}(x)$	inverse of $f(x)$

Calculus

$\lim_{x \to c}$	limit as x approaches c
m_{sec}	slope of a secant line
$f'(x)$	derivative of $f(x)$
Δ	delta, change
$\int$	indefinite integral
$\int_a^b$	definite integral
$F(x)$	antiderivative of $f(x)$

Probability and Statistics

$P(a)$	probability of a
$P(n, r)$ or ${}_nP_r$	permutation of n objects taken r at a time
$C(n, r)$ or ${}_nC_r$	combination of n objects taken r at a time
$P(A)$	probability of A
$P(A\|B)$	the probability of A given that B has already occurred
$n!$	factorial of n (n being a natural number)
Σ	sigma (uppercase), summation
μ	mu, population mean
σ	sigma (lowercase), population standard deviation
σ^2	population variance
s	sample standard deviation
s^2	sample variance
$\sum_{n=1}^{k}$	summation from $n = 1$ to k
$\bar{x}$	x-bar, sample mean
H_0	null hypothesis
H_a	alternative hypothesis

Measures

Metric	Customary
Length	
1 kilometer (km) = 1000 meters (m) 1 meter = 100 centimeters (cm) 1 centimeter = 10 millimeters (mm)	1 mile (mi) = 1760 yards (yd) 1 mile = 5280 feet (ft) 1 yard = 3 feet 1 foot = 12 inches (in) 1 yard = 36 inches
Volume and Capacity	
1 liter (L) = 1000 milliliters (mL) 1 kiloliter (kL) = 1000 liters	1 gallon (gal) = 4 quarts (qt) 1 gallon = 128 fluid ounces (fl oz) 1 quart = 2 pints (pt) 1 pint = 2 cups (c) 1 cup = 8 fluid ounces
Weight and Mass	
1 kilogram (kg) = 1000 grams (g) 1 gram = 1000 milligrams (mg) 1 metric ton (t) = 1000 kilograms	1 ton (T) = 2000 pounds (lb) 1 pound = 16 ounces (oz)

Arithmetic Operations and Relations

Identity	For any number a, $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
Substitution (=)	If $a = b$, then a may be replaced by b.
Reflexive (=)	$a = a$
Symmetric (=)	If $a = b$, then $b = a$.
Transitive (=)	If $a = b$ and $b = c$, then $a = c$.
Commutative	For any numbers a and b, $a + b = b + a$ and $a \cdot b = b \cdot a$.
Associative	For any numbers a, b, and c, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
Distributive	For any numbers a, b, and c, $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$.
Additive Inverse	For any number a, there is exactly one number $-a$ such that $a + (-a) = 0$.
Multiplicative Inverse	For any number $\frac{a}{b}$, where $a, b \neq 0$, there is exactly one number $\frac{b}{a}$ such that $\frac{a}{b} \cdot \frac{b}{a} = 1$.
Multiplicative (0)	For any number a, $a \cdot 0 = 0 \cdot a = 0$.
Addition (=)	For any numbers a, b, and c, if $a = b$, then $a + c = b + c$.
Subtraction (=)	For any numbers a, b, and c, if $a = b$, then $a - c = b - c$.
Multiplication and Division (=)	For any numbers a, b, and c, with $c \neq 0$, if $a = b$, then $ac = bc$ and $\frac{a}{c} = \frac{b}{c}$.
Addition (>)*	For any numbers a, b, and c, if $a > b$, then $a + c > b + c$.
Subtraction (>)*	For any numbers a, b, and c, if $a > b$, then $a - c > b - c$.
Multiplication and Division (>)*	For any numbers a, b, and c, 1. if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$. 2. if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
Zero Product	For any real numbers a and b, if $ab = 0$, then $a = 0$, $b = 0$, or both a and b equal 0.

** These properties are also true for $<$, $\geq$, and $\leq$.*

Algebraic Formulas and Key Concepts

Matrices			
Adding	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$	**Multiplying by a Scalar**	$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$
Subtracting	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$	**Multiplying**	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$
Polynomials			
Quadratic Formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$	**Square of a Difference**	$(a - b)^2 = (a - b)(a - b)$ $= a^2 - 2ab + b^2$
Square of a Sum	$(a + b)^2 = (a + b)(a + b)$ $= a^2 + 2ab + b^2$	**Product of Sum and Difference**	$(a + b)(a - b) = (a - b)(a + b)$ $= a^2 - b^2$
Logarithms			
Product Property	$\log_x ab = \log_x a + \log_x b$	**Power Property**	$\log_b m^p = p \log_b m$
Quotient Property	$\log_x \frac{a}{b} = \log_x a - \log_x b, b \neq 0$	**Change of Base**	$\log_a n = \frac{\log_b n}{\log_b a}$

Algebraic Formulas and Key Concepts

Exponential and Logarithmic Functions			
Compound Interest	$A = P\left(1 + \frac{r}{n}\right)^{nt}$	**Exponential Growth or Decay**	$N = N_0(1 + r)^t$
Continuous Compound Interest	$A = Pe^{rt}$	**Continuous Exponential Growth or Decay**	$N = N_0 e^{kt}$
Product Property	$\log_b xy = \log_b x + \log_b y$	**Power Property**	$\log_b x^p = p \log_b x$
Quotient Property	$\log_b \frac{x}{y} = \log_b x - \log_b y$	**Change of Base**	$\log_b x = \frac{\log_a x}{\log_a b}$
Logistic Growth	$f(t) = \frac{c}{1 + ae^{-bt}}$		

Sequences and Series			
***n*th term, Arithmetic**	$a_n = a_1 + (n - 1)d$	***n*th term, Geometric**	$a_n = a_1 r^{n-1}$
Sum of Arithmetic Series	$S_n = n\left(\frac{a_1 + a_n}{2}\right)$ or $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$	**Sum of Geometric Series**	$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ or $S_n = \frac{a_1 - a_n r}{1 - r}, r \neq 1$
Sum of Infinite Geometric Series	$S = \frac{a_1}{1 - r}, \lvert r \rvert < 1$	**Euler's Formula**	$e^{i\theta} = \cos \theta + i \sin \theta$
Power Series	$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$	**Exponential Series**	$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
Binomial Theorem	$(a + b)^n = {}_nC_0\, a^n b^0 + {}_nC_1\, a^{n-1} b^1 + {}_nC_2\, a^{n-2} b^2 + \cdots + {}_nC_r\, a^{n-r} b^r + \cdots + {}_nC_n\, a^0 b^n$		
Cosine and Sine Power Series	$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$		$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Vectors			
Addition in Plane	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$	**Addition in Space**	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
Subtraction in Plane	$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$	**Subtraction in Space**	$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$
Scalar Multiplication in Plane	$k\mathbf{a} = \langle ka_1, ka_2 \rangle$	**Scalar Multiplication in Space**	$k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$
Dot Product in Plane	$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$	**Dot Product in Space**	$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
Angle Between Two Vectors	$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$	**Projection of u onto v**	$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\lvert\mathbf{v}\rvert^2}\right)\mathbf{v}$
Magnitude of a Vector	$\lvert\mathbf{v}\rvert = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	**Triple Scalar Product**	$\mathbf{t} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} t_1 & t_2 & t_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

Equations of a Line on a Coordinate Plane	
Slope-intercept form of a line	$y = mx + b$
Point-slope form of a line	$y - y_1 = m(x - x_1)$

Algebraic Formulas and Key Concepts

Conic Sections			
Parabola	$(x-h)^2 = 4p(y-k)$ or $(y-k)^2 = 4p(x-h)$	**Circle**	$x^2 + y^2 = r^2$ or $(x-h)^2 + (y-k)^2 = r^2$
Ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$	**Hyperbola**	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Rotation of Conics	$x' = x \cos \theta + y \sin \theta$ and $y' = y \cos \theta - x \sin \theta$		

Parametric Equations			
Vertical Position	$y = tv_0 \sin \theta - \frac{1}{2}gt^2 + h_0$	**Horizontal Distance**	$x = tv_0 \cos \theta$

Complex Numbers			
Product Formula	$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + \boldsymbol{i} \sin(\theta_1 + \theta_2)]$	**Quotient Formula**	$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + \boldsymbol{i} \sin(\theta_1 - \theta_2)]$
Distinct Roots Formula	$r^{\frac{1}{p}}\left(\cos \frac{\theta + 2n\pi}{p} + \boldsymbol{i} \sin \frac{\theta + 2n\pi}{p}\right)$	**De Moivre's Theorem**	$z^n = [r(\cos \theta + \boldsymbol{i} \sin \theta)]^n$ or $r^n(\cos n\theta + \boldsymbol{i} \sin n\theta)$

Geometric Formulas and Key Concepts

Coordinate Geometry			
Slope	$m = \frac{y_2 - y_1}{x_2 - x_1}, x_2 \neq x_1$	**Distance on a number line**	$d = \|a - b\|$
Distance on a coordinate plane	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	**Arc length**	$\ell = \frac{x}{360} \cdot 2\pi r$
Midpoint on a coordinate plane	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$	**Midpoint on a number line**	$M = \frac{a + b}{2}$
Midpoint in space	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$	**Pythagorean Theorem**	$a^2 + b^2 = c^2$
Distance in space	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$		

Perimeter and Circumference					
Square	$P = 4s$	**Rectangle**	$P = 2\ell + 2w$	**Circle**	$C = 2\pi r$ or $C = \pi d$

Lateral Surface Area			
Prism	$L = Ph$	**Pyramid**	$L = \frac{1}{2}P\ell$
Cylinder	$L = 2\pi rh$	**Cone**	$L = \pi r\ell$

Total Surface Area					
Prism	$S = Ph + 2B$	**Cone**	$S = \pi r\ell + \pi r^2$	**Cylinder**	$S = 2\pi rh + 2\pi r^2$
Pyramid	$S = \frac{1}{2}P\ell + B$	**Sphere**	$S = 4\pi r^2$	**Cube**	$S = 6s^2$

Volume					
Prism	$V = Bh$	**Cone**	$V = \frac{1}{3}\pi r^2 h$	**Cylinder**	$V = \pi r^2 h$
Pyramid	$V = \frac{1}{3}Bh$	**Sphere**	$V = \frac{4}{3}\pi r^3$	**Cube**	$V = s^3$
Rectangular prism	$V = \ell wh$				

Trigonometric Functions and Identities

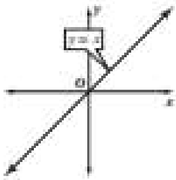
Trigonometric Functions			
Trigonometric Functions	$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin\theta}$	$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\cos\theta}$	$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\cos\theta}{\sin\theta}$
Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos A$	$b^2 = a^2 + c^2 - 2ac\cos B$	$c^2 = a^2 + b^2 - 2ab\cos C$
Law of Sines	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	**Heron's Formula**	$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$
Linear Speed	$v = \frac{s}{t}$	**Angular Speed**	$\omega = \frac{\theta}{t}$

Trigonometric Identities			
Reciprocal	$\sin\theta = \frac{1}{\csc\theta}$ $\csc\theta = \frac{1}{\sin\theta}$	$\cos\theta = \frac{1}{\sec\theta}$ $\sec\theta = \frac{1}{\cos\theta}$	$\tan\theta = \frac{1}{\cot\theta}$ $\cot\theta = \frac{1}{\tan\theta}$
Pythagorean	$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$\cot^2\theta + 1 = \csc^2\theta$
Cofunction	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$	$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$	$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$
Odd-Even	$\sin(-\theta) = -\sin\theta$ $\csc(-\theta) = -\csc\theta$	$\cos(-\theta) = \cos\theta$ $\sec(-\theta) = \sec\theta$	$\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$
Sum & Difference	$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$	$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$	
Double-Angle	$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = 2\cos^2\theta - 1$ $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$	$\cos 2\theta = 1 - 2\sin^2\theta$
Power-Reducing	$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$	$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$	$\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$
Half-Angle	$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}$ $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}$	$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$ $\tan\frac{\theta}{2} = \frac{1 - \cos\theta}{\sin\theta}$	$\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta}$
Product-to-Sum	$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$ $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$	$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$ $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$	
Sum-to-Product	$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$ $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$	$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$ $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$	

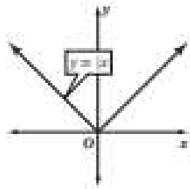
Parent Functions and Function Operations

Parent Functions

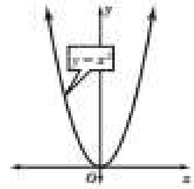
Linear Functions



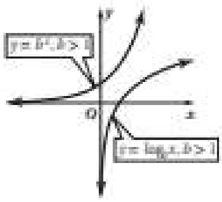
Absolute Value Functions



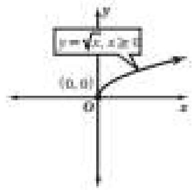
Quadratic Functions



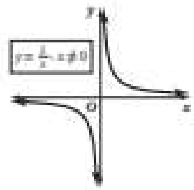
Exponential and Logarithmic Functions



Square Root Functions



Reciprocal and Rational Functions



Function Operations

Addition	$(f+g)(x) = f(x) + g(x)$	**Multiplication**	$(f \cdot g)(x) = f(x) \cdot g(x)$
Subtraction	$(f-g)(x) = f(x) - g(x)$	**Division**	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$

Calculus

Limits

Sum	$\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$	**Difference**	$\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
Scalar Multiple	$\lim_{x \to c} [k\, f(x)] = k \lim_{x \to c} f(x)$	**Product**	$\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
Quotient	$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, if $\lim_{x \to c} g(x) \neq 0$	**Power**	$\lim_{x \to c} [f(x)^n] = \left[\lim_{x \to c} f(x)\right]^n$
***n*th Root**	$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$, if $\lim_{x \to c} f(x) > 0$ when n is even	**Velocity**	Average: $v_{avg} = \frac{f(b) - f(a)}{b - a}$ Instantaneous: $v(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$

Derivatives

Power Rule	If $f(x) = x^n$, $f'(x) = nx^{n-1}$	**Sum or Difference**	If $f(x) = g(x) \pm h(x)$, then $f'(x) = g'(x) \pm h'(x)$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$	**Quotient Rule**	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Integrals

Indefinite Integral	$\int f(x)\, dx = F(x) + C$	**Fundamental Theorem of Calculus**	$\int_a^b f(x)\, dx = F(b) - F(a)$

Statistics Formulas and Key Concepts

z-Values	$z = \frac{X - \mu}{\sigma}$	**z-Value of a Sample Mean**	$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$
Binomial Probability	$P(X) = {}_nC_x\, p^x q^{n-x} = \frac{n!}{(n-x)!\,x!} p^x q^{n-x}$	**Maximum Error of Estimate**	$E = z \cdot \sigma_{\bar{x}}$ or $z \cdot \frac{\sigma}{\sqrt{n}}$
Confidence Interval, Normal Distribution	$CI = \bar{x} \pm E$ or $\bar{x} \pm z \cdot \frac{\sigma}{\sqrt{n}}$	**Confidence Interval, t-Distribution**	$CI = \bar{x} \pm t \cdot \frac{s}{\sqrt{n}}$
Correlation Coefficient	$r = \frac{1}{n-1}\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$	**t-Test for the Correlation Coefficient**	$t = r\sqrt{\frac{n-2}{1-r^2}}$, degrees of freedom: $n - 2$

Cover label guide

Cycle 03 Color

Icon indicates the book type (**Example, Course book**)

Number 3 has been shaded to represent the **volume of Course book**

Number 9 Shows grade level

Shows stream type

- Course book
- Activity book (workbook)
- Reading book (Arabic, social)
- Interactive Student Guide (Math)
- Activity Lab Manual (Science)
- Teacher Guide

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