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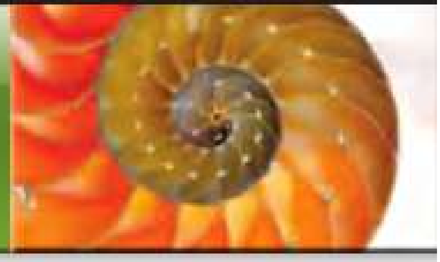

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Student Handbook

Symbols, Formulas, and Key Concepts

CHAPTER 1

Vectors

Then

In **previous grades,** you used trigonometry to solve triangles.

Now

In **this chapter,** you will:

- Represent and operate with vectors algebraically in the two- and three-dimensional coordinate systems.
- Find the projection of one vector onto another.
- Find cross products of vectors in space and find volumes of parallelepipeds.
- Find the dot products of and angles between vectors.

Why? ▲

ROWING Vectors are often used to model changes in direction due to water and air currents. For example, a vector can be used to determine the resultant speed and direction of a kayak that is traveling 12.9 kilometers per hour against a 4.8 mile-per-hour river current.

PREREAD Scan the lesson titles and Key Concept boxes in Chapter 1. Use this information to predict what you will learn in this chapter.

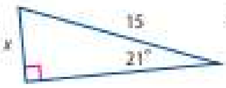
Get Ready for the Chapter

QuickCheck

Find the distance between each given pair of points and the midpoint of the segment connecting the given points. (Prerequisite Skill)

1. $(1, 4), (-2, 4)$
2. $(-5, 3), (-5, 8)$
3. $(2, -9), (-3, -7)$
4. $(-4, -1), (-6, -8)$

Find the value of x. Round to the nearest tenth if necessary.

5. 

6. 

7. 

8. 

9. **BALLOON** A hot air balloon is being held in place by two people holding ropes and standing 35 meters apart. The angle formed between the ground and the rope held by each person is 40°. Determine the length of each rope to the nearest tenth of a meter.

Find all solutions for the given triangle, if possible. If no solution exists, write *no solution*. Round side lengths to the nearest tenth and angle measures to the nearest degree.

10. $a = 10, b = 7, A = 128°$
11. $a = 15, b = 16, A = 127°$
12. $a = 15, b = 18, A = 52°$
13. $a = 30, b = 19, A = 91°$

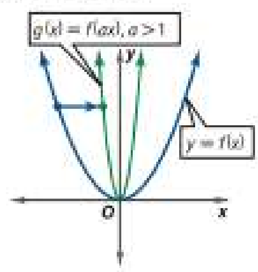
NewVocabulary

vector
initial point
terminal point
standard position
direction
magnitude
quadrant bearing
true bearing
parallel vectors
equivalent vectors
opposite vectors
resultant
zero vector
component form
unit vector
dot product
orthogonal
z-axis
octants
ordered triple
cross product
triple scalar product

ReviewVocabulary

scalar a quantity with magnitude only

dilation a transformation in which the graph of a function is compressed or expanded vertically or horizontally



Horizontal dilation

LESSON 1-1 Introduction to Vectors

Then

- You used trigonometry to solve triangles.

Now

1. Represent and operate with vectors geometrically.
2. Solve vector problems, and resolve vectors into their rectangular components.

Why?

- A successful goal attempt in football depends on several factors. While the speed of the ball after it is kicked is certainly important, the direction the ball goes is important as well. We can describe both of these factors using a single quantity called a *vector*.

NewVocabulary

vector
initial point
terminal point
standard position
direction
magnitude
quadrant bearing
true bearing
parallel vectors
equivalent vectors
opposite vectors
resultant
triangle method
parallelogram method
zero vector
components
rectangular components

1 Vectors Many physical quantities, such as speed, can be completely described by a single real number called a *scalar*. This number indicates the *magnitude* or *size* of the quantity. A **vector** is a quantity that has both magnitude and *direction*. The velocity of a ball is a vector that describes both the speed and direction of the ball .

Example 1 Identify Vector Quantities

State whether each quantity described is a *vector* quantity or a *scalar* quantity.

a. a boat traveling at 15 kilometers per hour

This quantity has a magnitude of 15 kilometers per hour, but no direction is given. Speed is a scalar quantity.

b. a hiker walking 25 paces due west

This quantity has a magnitude of 25 paces and a direction of due west. This directed distance is a vector quantity.

c. a person's weight on a bathroom scale

Weight is a vector quantity that is calculated using a person's mass and the downward pull due to gravity. (Acceleration due to gravity is also a vector.)

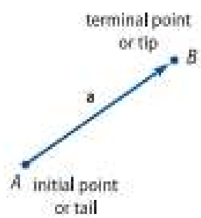
GuidedPractice

1A. a car traveling 60 kilometers per hour 15° east of south

1B. a parachutist falling straight down at 20.2 kilometers per hour

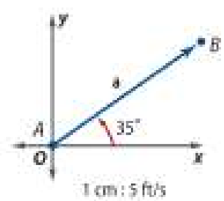
1C. a child pulling a sled with a force of 40 newtons

A vector can be represented geometrically by a directed line segment, or arrow diagram, that shows both magnitude and direction. Consider the directed line segment with an **initial point** A (also known as the *tail*) and **terminal point** B (also known as the *head* or *tip*) shown. This vector is denoted by $\overrightarrow{AB}$, $\vec{a}$, or $\mathbf{a}$.

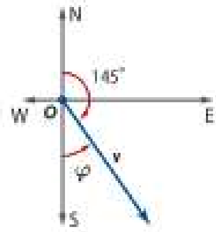


If a vector has its initial point at the origin, it is in **standard position**. The **direction** of a vector is the directed angle between the vector and the horizontal line that could be used to represent the positive x-axis. The direction of $\mathbf{a}$ is 35°.

The length of the line segment represents, and is proportional to, the **magnitude** of the vector. If the scale of the arrow diagram for $\mathbf{a}$ is 1 cm = 5 ft/s, then the magnitude of $\mathbf{a}$, denoted $|\mathbf{a}|$, is 2.6×5 or 13 feet per second.



The direction of a vector can also be given as a bearing. A **quadrant bearing** φ (the Greek letter *phi*) is a directional measurement between 0° and 90° east or west of the north-south line. The quadrant bearing of vector **v** shown is 35° east of south or southeast, written S35°E.



A **true bearing** is a directional measurement where the angle is measured clockwise from north. True bearings are always given using three digits. So, a direction that measures 25° clockwise from north would be written as a true bearing of 025°.

StudyTip

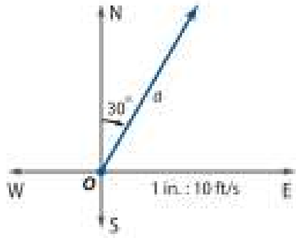
True Bearing When a degree measure is given without any additional directional components, it is assumed to be a true bearing. The true bearing of **v** is 145°.

Example 2 Represent a Vector Geometrically

Use a ruler and a protractor to draw an arrow diagram for each quantity described. Include a scale on each diagram.

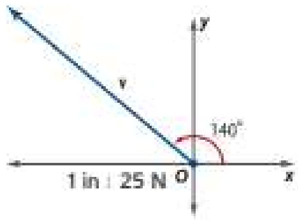
a. a = 20 feet per second at a bearing of 030°

Using a scale of 1 in: 10 ft/s, draw and label a 20 ÷ 10 or 2-inch arrow at an angle of 30° clockwise from the north.



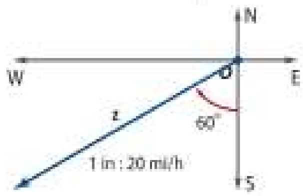
b. v = 75 newtons of force at 140° to the horizontal

Using a scale of 1 in : 25 N, draw and label a 75 ÷ 25 or 3-inch arrow in standard position at a 140° angle to the *x*-axis.



c. z = 30 miles per hour at a bearing of S60°W

Using a scale of 1 in : 20 mi/h, draw and label a 30 ÷ 20 or 1.5-inch arrow 60° west of south.



GuidedPractice

2A. **t** = 20 meters per second at a bearing of 065°

2B. **u** = 15 kilometers per hour at a bearing of S25°E

2C. **m** = 60 newtons of force at 80° to the horizontal

WatchOut!

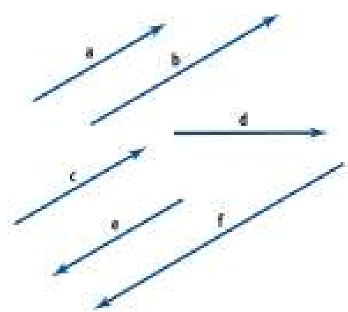
Magnitude The magnitude of a vector can represent distance, speed, or force. When a vector represents velocity, the length of the vector does not imply distance traveled.

In your operations with vectors, you will need to be familiar with the following vector types.

- **Parallel vectors** have the same or opposite direction but not necessarily the same magnitude. In the figure, **a** ∥ **b** ∥ **c** ∥ **e** ∥ **f**.

- **Equivalent vectors** have the same magnitude and direction. In the figure, **a** = **c** because they have the same magnitude and direction. Notice that **a** ≠ **b**, since |**a**| ≠ |**b**|, and **a** ≠ **d**, since **a** and **d** do not have the same direction.

- **Opposite vectors** have the same magnitude but opposite direction. The vector opposite **a** is written −**a**. In the figure, **e** = −**a**.



When two or more vectors are added, their sum is a single vector called the **resultant**. The resultant vector has the same effect as applying one vector after the other. Geometrically, the resultant can be found using either the **triangle method** or the **parallelogram method**.

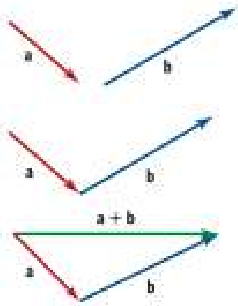
KeyConcept Finding Resultants

Triangle Method (Tip-to-Tail)

To find the resultant of **a** and **b**, follow these steps.

Step 1 Translate **b** so that the tail of **b** touches the tip of **a**.

Step 2 The resultant is the vector from the tail of **a** to the tip of **b**.



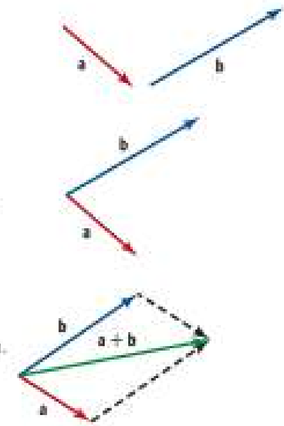
Parallelogram Method (Tail-to-Tail)

To find the resultant of **a** and **b**, follow these steps.

Step 1 Translate **b** so that the tail of **b** touches the tail of **a**.

Step 2 Complete the parallelogram that has **a** and **b** as two of its sides.

Step 3 The resultant is the vector that forms the indicated diagonal of the parallelogram.

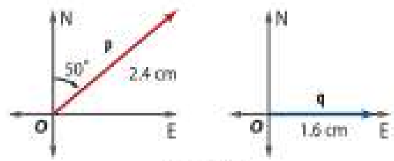


Real-World Example 3 Find the Resultant of Two Vectors

ORIENTEERING In an orienteering competition, Tia walks N50°E for 120 meters and then walks 80 meters due east. How far and at what quadrant bearing is Tia from her starting position?

Let **p** = walking 120 meters N50°E and **q** = walking 80 meters due east. Draw a diagram to represent **p** and **q** using a scale of 1 cm : 50 m.

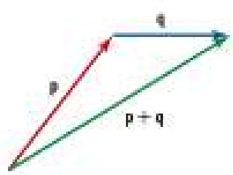
Use a ruler and a protractor to draw a 120 ÷ 50 or 2.4-centimeter arrow 50° east of north to represent **p** and an 80 ÷ 50 or 1.6-centimeter arrow due east to represent **q**.



1 cm : 50 m

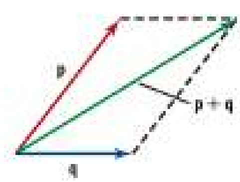
Method 1 Triangle Method

Translate **q** so that its tail touches the tip of **p**. Then draw the resultant vector **p** + **q** as shown.



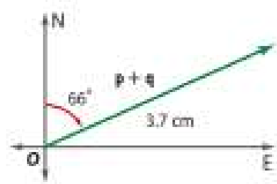
Method 2 Parallelogram Method

Translate **q** so that its tail touches the tail of **p**. Then complete the parallelogram and draw the diagonal, resultant **p** + **q**, as shown.



Both methods produce the same resultant vector **p** + **q**. Measure the length of **p** + **q** and then measure the angle this vector makes with the north-south line as shown.

The vector's length of approximately 3.7 centimeters represents 3.7 × 50 or 185 meters. Therefore, Tia is approximately 185 feet at a bearing of 66° east of north or N66°E from her starting position.



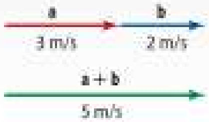
1 cm : 50 m

StudyTip

Resultants The parallelogram method must be repeated in order to find the resultant of more than two vectors. The triangle method, however, is easier to use when finding the resultant of three or more vectors. Continue to place the initial point of subsequent vectors at the terminal point of the previous vector.

StudyTip

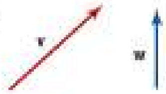
Parallel Vectors with Same Direction To add two or more parallel vectors with the same *direction*, add their magnitudes. The resultant has the same direction as the original vectors.



GuidedPractice

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant to the nearest centimeter and its direction relative to the horizontal.

3A.



3B.

3C. **PINBALL** A pinball is struck by flipper and is sent 310° at a velocity of 7 centimeters per second. The ball then bounces off of a bumper and heads 055° at a velocity of 4 centimeters per second. Find the resulting direction and velocity of the pinball.

When you add two opposite vectors, the resultant is the **zero vector** or *null* vector, denoted by $\vec{0}$ or $\mathbf{0}$, which has a magnitude of 0 and no specific direction. Subtracting vectors is similar to subtraction with integers. To find $\mathbf{p} - \mathbf{q}$, add the opposite of $\mathbf{q}$ to $\mathbf{p}$. That is, $\mathbf{p} - \mathbf{q} = \mathbf{p} + (-\mathbf{q})$.

$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

A vector can also be multiplied by a scalar.

KeyConcept Multiplying Vectors by a Scalar

If a vector $\mathbf{v}$ is multiplied by a real number scalar k, the scalar multiple $k\mathbf{v}$ has a magnitude of $|k|\,|\mathbf{v}|$. Its direction is determined by the sign of k.

- If $k > 0$, $k\mathbf{v}$ has the same direction as $\mathbf{v}$.
- If $k < 0$, $k\mathbf{v}$ has the opposite direction as $\mathbf{v}$.

Example 4 Operations with Vectors

Draw a vector diagram of $3\mathbf{x} - \frac{3}{4}\mathbf{y}$.

Rewrite the expression as the addition of two vectors: $3\mathbf{x} - \frac{3}{4}\mathbf{y} = 3\mathbf{x} + \left(-\frac{3}{4}\mathbf{y}\right)$. To represent $3\mathbf{x}$, draw a vector 3 times as long as $\mathbf{x}$ in the same direction as $\mathbf{x}$ (Figure 1.1.1). To represent $-\frac{3}{4}\mathbf{y}$, draw a vector $\frac{3}{4}$ the length of $\mathbf{y}$ in the opposite direction from $\mathbf{y}$ (Figure 1.1.2). Then use the triangle method to draw the resultant vector (Figure 1.1.3).

Figure 1.1.1

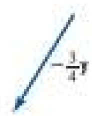


Figure 1.1.2

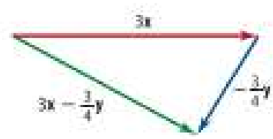
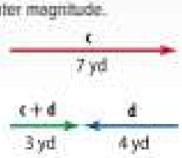


Figure 1.1.3

StudyTip

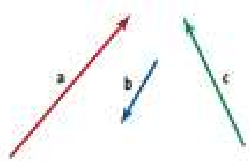
Parallel Vectors with Opposite Directions To add two parallel vectors with *opposite directions*, find the absolute value of the difference in their magnitudes. The resultant has the same direction as the vector with the greater magnitude.



GuidedPractice

Draw a vector diagram of each expression.

4A. $\mathbf{a} - \mathbf{c} + 2\mathbf{b}$



4B. $\mathbf{m} - \frac{1}{4}\mathbf{p}$

2 Vector Applications

Vector addition and trigonometry can be used to solve vector problems involving triangles which are often oblique.

In navigation, a *heading* is the direction in which a vessel, such as an airplane or boat, is steered to overcome other forces, such as wind or current. The *relative velocity* of the vessel is the resultant when the heading velocity and other forces are combined.

Real-World Example 5 Use Vectors to Solve Navigation Problems

AVIATION An airplane is flying with an airspeed of 310 knots on a heading of 050°. If a 78-knot wind is blowing from a true heading of 125°, determine the speed and direction of the plane relative to the ground.

Step 1 Draw a diagram to represent the heading and wind velocities (Figure 1.1.4). Translate the wind vector as shown in Figure 1.1.5, and use the triangle method to obtain the resultant vector representing the plane's ground velocity **g**. In the triangle formed by these vectors (Figure 1.1.6), $\gamma = 125° - 50°$ or 75°.

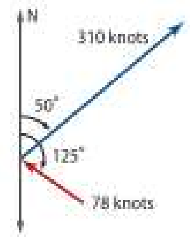


Figure 1.1.4

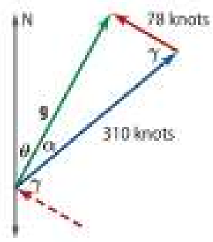


Figure 1.1.5

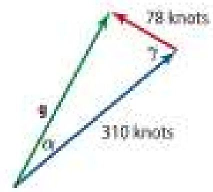


Figure 1.1.6

Step 2 Use the Law of Cosines to find $|\mathbf{g}|$, the plane's speed relative to the ground.

$c^2 = a^2 + b^2 - 2ab \cos \gamma$ — Law of Cosines

$|\mathbf{g}|^2 = 78^2 + 310^2 - 2(78)(310) \cos 75°$ — $c = |\mathbf{g}|$, $a = 78$, $b = 310$, and $\gamma = 75°$

$|\mathbf{g}| = \sqrt{78^2 + 310^2 - 2(78)(310) \cos 75°}$ — Take the positive square root of each side.

≈ 299.4 — Simplify.

The ground speed of the plane is about 299.4 knots.

Step 3 The heading of the resultant **g** is represented by angle θ, as shown in Figure 1.1.5. To find θ, first calculate α using the Law of Sines.

$\frac{\sin \alpha}{a} = \frac{\sin \gamma}{c}$ — Law of Sines

$\frac{\sin \alpha}{78} = \frac{\sin 75°}{299.4}$ — $c = |\mathbf{g}|$ or 299.4, $a = 78$, and $\gamma = 75°$

$\sin \alpha = \frac{78 \sin 75°}{299.4}$ — Solve for $\sin \alpha$.

$\alpha = \sin^{-1} \frac{78 \sin 75°}{299.4}$ — Apply the inverse sine function.

$\approx 14.6°$ — Simplify.

The measure of θ is $50° - \alpha$, which is $50° - 14.6°$ or 35.4°.

Therefore, the speed of the plane relative to the ground is about 299.4 knots at about 035°.

GuidedPractice

5. **SWIMMING** Ali rows due east at a speed of 3.5 feet per second across a river directly toward the opposite bank. At the same time, the current of the river is carrying him due south at a rate of 2 feet per second. Find Ali's speed and direction relative to the shore.

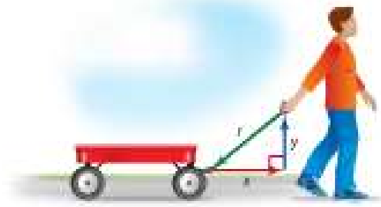
StudyTip

Alternate Interior Angles The translation of the tail of the wind vector to the tip of the vector representing the plane's heading produces two parallel vectors cut by a transversal. Since alternate interior angles of two parallel lines cut by a transversal are congruent, the angles made by these two vectors in both places in Figure 1.1.5 have the same measure.

WatchOut!

Wind Direction In Example 5, notice that the wind is blowing *from* a bearing of 125° and the vector is drawn so that the tip of the vector points *toward* the north-south line. Had the wind been blowing *at* a bearing of 125°, the vector would have pointed *away* from this line.

Two or more vectors with a sum that is a vector **r** are called **components** of **r**. While components can have any direction, it is often useful to express or *resolve* a vector into two perpendicular components. The **rectangular components** of a vector are horizontal and vertical.



In the diagram, the force **r** exerted to pull the wagon can be thought of as the sum of a horizontal component force **x** that moves the wagon forward and a vertical component force **y** that pulls the wagon upward.

Real-WorldLink

It takes a force of about 3 newtons to flip a light switch. The force due to gravity on a person is about 600 newtons. The force exerted by a weightlifter is about 2000 newtons.

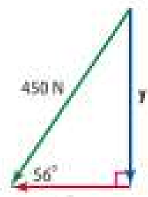
Source: *Contemporary College Physics*

Real-World Example 6 Resolve a Force into Rectangular Components

LAWN CARE **Hala is pushing the handle of a lawn mower with a force of 450 newtons at an angle of 56° with the ground.**

a. Draw a diagram that shows the resolution of the force that Hala exerts into its rectangular components.

Hala's push can be resolved into a horizontal push **x** forward and a vertical push **y** downward as shown.



b. Find the magnitudes of the horizontal and vertical components of the force.

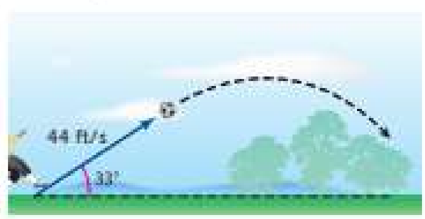
The horizontal and vertical components of the force form a right triangle. Use the sine or cosine ratios to find the magnitude of each force.

$\cos 56° = \frac{\|\mathbf{x}\|}{450}$	Right triangle definitions of cosine and sine	$\sin 56° = \frac{\|\mathbf{y}\|}{450}$
$\|\mathbf{x}\| = 450 \cos 56°$	Solve for *x* and *y*.	$\|\mathbf{y}\| = 450 \sin 56°$
$\|\mathbf{x}\| \approx 252$	Use a calculator.	$\|\mathbf{y}\| \approx 373$

The magnitude of the horizontal component is about 252 newtons, and the magnitude of the vertical component is about 373 newtons.

GuidedPractice

6. **FOOTBALL** A player kicks a football so that it leaves the ground with a velocity of 44 feet per second at an angle of 33° with the ground.



A. Draw a diagram that shows the resolution of this force into its rectangular components.

B. Find the magnitude of the horizontal and vertical components of the velocity.

Exercises

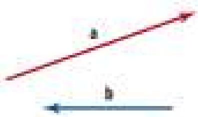
State whether each quantity described is a *vector* quantity or a *scalar* quantity. (Example 1)

1. a box being pushed with a force of 125 newtons
2. wind blowing at 20 knots
3. a deer running 15 meters per second due west
4. a baseball thrown with a speed of 85 miles per hour
5. a 3.75-kilogram stone hanging from a rope
6. a rock thrown straight up at a velocity of 50 feet per second

Use a ruler and a protractor to draw an arrow diagram for each quantity described. Include a scale on each diagram. (Example 2)

7. $\mathbf{h}$ = 13 centimeters per second at a bearing of 205°
8. $\mathbf{g}$ = 6 kilometers per hour at a bearing of N70°W
9. $\mathbf{j}$ = 5 meters per minute at 300° to the horizontal
10. $\mathbf{k}$ = 28 kilometers at 35° to the horizontal
11. $\mathbf{m}$ = 40 meters at a bearing of S55°E
12. $\mathbf{n}$ = 32 meter per second at a bearing of 030°

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant to the nearest tenth of a centimeter and its direction relative to the horizontal. (Example 3)

13.


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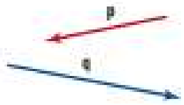

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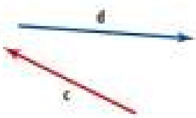

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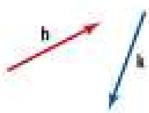

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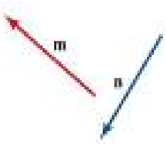
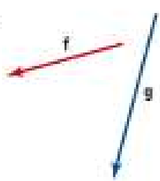


18.


19. **GOLF** While playing a golf video game, Omar hits a ball 35° above the horizontal at a speed of 64.4-kilometer per hour with a 8 kilometers per hour wind blowing, as shown. Find the resulting speed and direction of the ball. (Example 3)

20. **BOATING** A charter boat leaves port on a heading of N60°W for 12 nautical miles. The captain changes course to a bearing of N25°E for the next 15 nautical miles. Determine the ship's distance and direction from port to its current location. (Example 3)

21. **HIKING** Mazen and Ayoub hiked 3.75 kilometers to a lake 55° east of south from their campsite. Then they hiked 33° west of north to the nature center 5.6 kilometers from the lake. Where is the nature center in relation to their campsite? (Example 3)

Determine the magnitude and direction of the resultant of each vector sum. (Example 2)

22. 18 newtons directly forward and then 20 newtons directly backward
23. 100 meters due north and then 350 meters due south
24. 10 kilograms of force at a bearing of 025° and then 15 kilograms of force at a bearing of 045°
25. 17 kilometers east and then 16 kilometers south
26. 15 meters per second squared at a 60° angle to the horizontal and then 9.8 meters per second squared downward

Use the set of vectors to draw a vector diagram of each expression. (Example 4)

27. $\mathbf{m} - 2\mathbf{n}$
28. $\mathbf{n} - \frac{3}{4}\mathbf{m}$
29. $\frac{1}{2}\mathbf{p} + 3\mathbf{n}$
30. $4\mathbf{n} + \frac{4}{5}\mathbf{p}$
31. $\mathbf{p} + 2\mathbf{n} - \mathbf{m}$
32. $-\frac{1}{3}\mathbf{m} + \mathbf{p} - 2\mathbf{n}$
33. $3\mathbf{n} - \frac{1}{2}\mathbf{p} + \mathbf{m}$
34. $\mathbf{m} - 3\mathbf{n} + \frac{1}{4}\mathbf{p}$

35. **RUNNING** A runner's resultant velocity is 8 miles per hour due west running with a wind of 3 miles per hour N28°W. What is the runner's speed, to the nearest mile per hour, without the effect of the wind? (Example 5)

36. **GLIDING** A glider is traveling at an air speed of 15 kilometers per hour due west. If the wind is blowing at 5 kilometers per hour in the direction N60°E, what is the resulting ground speed of the glider? (Example 5)

37. **CURRENT** Sally is swimming due west at a rate of 1.5 meters per second. A strong current is flowing S20°E at a rate of 1 meter per second. Find Sally's resulting speed and direction. (Example 5)

Draw a diagram that shows the resolution of each vector into its rectangular components. Then find the magnitudes of the vector's horizontal and vertical components. (Example 6)

38. $2\frac{1}{8}$ centimeters at 310° to the horizontal

39. 1.5 centimeters at a bearing of N49°E

40. 3.2 centimeters per hour at a bearing of S78°W

41. $\frac{3}{4}$ centimeter per minute at a bearing of 255°

42. **AMERICAN FOOTBALL** For a field goal attempt, a ball is kicked with the velocity shown in the diagram below.

a. Draw a diagram that shows the resolution of this force into its rectangular components.

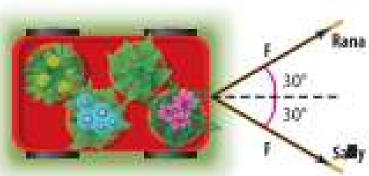
b. Find the magnitudes of the horizontal and vertical components. (Example 6)

43. **CLEANING** A push broom is pushed with a force of 190 newtons at an angle of 33° with the ground. (Example 6)

a. Draw a diagram that shows the resolution of this force into its rectangular components.

b. Find the magnitudes of the horizontal and vertical components.

44. **GARDENING** Rana and Sally are pulling a wagon full of plants. Each person pulls on the wagon with equal force at an angle of 30° with the axis of the wagon. The resultant force is 120 newtons.



a. How much force is each person exerting?

b. If each person exerts a force of 75 newtons, what is the resultant force?

c. How will the resultant force be affected if Rana and Sally move closer together?

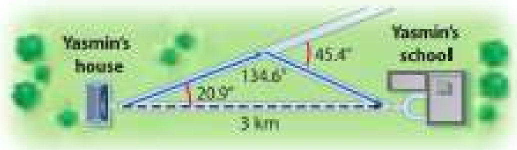
The magnitude and true bearings of three forces acting on an object are given. Find the magnitude and direction of the resultant of these forces.

45. 50 kg at 30°, 80 kg at 125°, and 100 kg at 220°

46. 8 newtons at 300°, 12 newtons at 45°, and 6 newtons at 120°

47. 18 kg at 190°, 3 kg at 20°, and 7 kg at 320°

48. **DRIVING** Yasmin's school is on a direct path three kilometers from her house. She drives on two different streets on her way to school. She travels at an angle of 20.9° with the path on the first street and then turns 45.4° onto the second street.



a. How far does Yasmin drive on the first street?

b. How far does she drive on the second street?

c. If it takes her 10 minutes to get to school and she averages 25 kilometers per hour on the first street, what speed does Yasmin average after she turns onto the second street?

49. **SLEDDING** Hamad is pulling his sister on a sled. The direction of his resultant force is 31°, and the horizontal component of the force is 86 newtons.

a. What is the vertical component of the force?

b. What is the magnitude of the resultant force?

50. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate multiplication of a vector by a scalar.

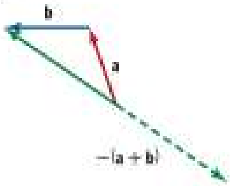
a. **GRAPHICAL** On a coordinate plane, draw a vector **a** so that the tail is located at the origin. Choose a value for a scalar k. Then draw the vector that results if you multiply the original vector by k on the same coordinate plane. Repeat the process for four additional vectors **b**, **c**, **d**, and **e**. Use the same value for k each time.

b. **TABULAR** Copy and complete the table below for each vector that you drew in part **a**.

Vector	Terminal Point of Vector	Terminal Point of Vector × k
a		
b		
c		
d		
e		

c. **ANALYTICAL** If the terminal point of a vector **a** is located at the point (a, b), what is the location of the terminal point of the vector k**a**?

An *equilibrant* vector is the opposite of a resultant vector. It balances a combination of vectors such that the sum of the vectors and the equilibrant is the zero vector. The equilibrant vector of a + b is −(a + b).



Find the magnitude and direction of the equilibrant vector for each set of vectors.

51. **a** = 15 kilometers per hour at a bearing of 125°
    b = 12 kilometers per hour at a bearing of 045°

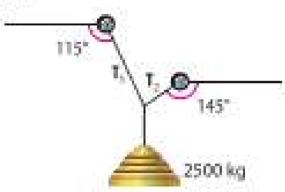
52. **a** = 4 meters at a bearing of N30W°
    b = 6 meters at a bearing of N20E°

53. **a** = 23 meters per second at a bearing of 205°
    b = 16 meters per second at a bearing of 345°

54. **MAGNITUDE** A round object is suspended from a ceiling by two wires of equal length as shown.

   a. Draw a vector diagram of the situation that indicates that two tension vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ with equal magnitude are keeping the object stationary or at equilibrium.
   b. Redraw the diagram using the triangle method to find $\mathbf{T}_1 + \mathbf{T}_2$.
   c. Use your diagram from part **b** and the fact that the equilibrant of the resultant $\mathbf{T}_1 + \mathbf{T}_2$ and the vector representing the weight of the object are equivalent vectors to calculate the magnitudes of $\mathbf{T}_1$ and $\mathbf{T}_2$.

55. **CABLE SUPPORT** Two cables with tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ are tied together to support a 2500-kilogram load at equilibrium.



   a. Write expressions to represent the horizontal and vertical components of $\mathbf{T}_1$ and $\mathbf{T}_2$.
   b. Given that the equilibrant of the resultant $\mathbf{T}_1 + \mathbf{T}_2$ and the vector representing the weight of the load are equivalent vectors, calculate the magnitudes of $\mathbf{T}_1$ and $\mathbf{T}_2$ to the nearest tenth of a kilogram.
   c. Use your answers from parts **a** and **b** to find the magnitudes of the horizontal and vertical components of $\mathbf{T}_1$ and $\mathbf{T}_2$ to the nearest tenth of a kilogram.

Find the magnitude and direction of each vector given its vertical and horizontal components and the range of values for the angle of direction θ to the horizontal.

56. horizontal: 0.32 cm, vertical: 2.28 cm, $90° < \theta < 180°$

57. horizontal: 3.1 m, vertical: 4.2 m, $0° < \theta < 90°$

58. horizontal: 2.6 cm, vertical: 9.7 cm, $270° < \theta < 360°$

59. horizontal: 2.9 m, vertical: 1.8 m, $180° < \theta < 270°$

Draw any three vectors a, b, and c. Show geometrically that each of the following vector properties holds using these vectors.

60. Commutative Property: **a** + **b** = **b** + **a**

61. Associative Property: (**a** + **b**) + **c** = **a** + (**b** + **c**)

62. Distributive Property: k(**a** + **b**) = k**a** + k**b**, for $k = 2, 0.5,$ and -2

H.O.T. Problems Use Higher-Order Thinking Skills

63. **OPEN ENDED** Consider a vector of 5 units directed along the positive x-axis. Resolve the vector into two perpendicular components in which no component is horizontal or vertical.

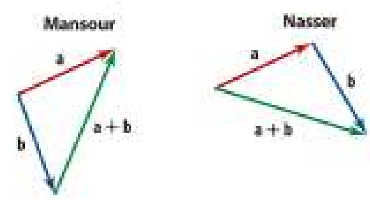
64. **REASONING** Is it *sometimes*, *always*, or *never* possible to find the sum of two parallel vectors using the parallelogram method? Explain your reasoning.

65. **REASONING** Why is it important to establish a common reference for measuring the direction of a vector, for example, from the positive x-axis?

66. **CHALLENGE** The resultant of **a** + **b** is equal to the resultant of **a** − **b**. If the magnitude of **a** is $4x$, what is the magnitude of **b**?

67. **REASONING** Consider the statement $|\mathbf{a}| + |\mathbf{b}| \geq |\mathbf{a} + \mathbf{b}|$.
   a. Express this statement using words.
   b. Is this statement true or false? Justify your answer.

68. **ERROR ANALYSIS** Mansour and Nasser are finding the resultant of vectors **a** and **b**. Is either of them correct? Explain your reasoning.



69. **REASONING** Is it possible for the sum of two vectors to equal one of the vectors? Explain.

70. **WRITING IN MATH** Compare and contrast the parallelogram and triangle methods of finding the resultant of two or more vectors.

Spiral Review

71. KICKBALL Suppose a kickball player kicks a ball at a 32° angle to the horizontal with an initial speed of 20 meters per second. How far away will the ball land?

72. Graph $(x')^2 + y' - 5 = 1$ if it has been rotated 45° from its position in the xy-plane.

Write an equation for a circle that satisfies each set of conditions. Then graph the circle.

73. center at (4, 5), radius 4

74. center at (1, −4), diameter 7

Determine the equation of and graph the parabola with the given focus F and vertex V.

75. $F(2, 4)$, $V(2, 3)$

76. $F(1, 5)$, $V(-7, 5)$

77. CRAFTS Majed is selling wood carvings. He sells large statues for $60, clocks for $40, dollhouse furniture for $25, and chess pieces for $5. He takes the following number of items to the fair: 12 large statues, 25 clocks, 45 pieces of dollhouse furniture, and 50 chess pieces.
   a. Write an inventory matrix for the number of each item and a cost matrix for the price of each item.
   b. Find Majed's total income if he sells all of the items.

Solve each equation for all values of x.

78. $4 \sin x \cos x - 2 \sin x = 0$

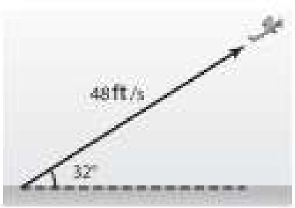
79. $\sin x - 2 \cos^2 x = -1$

Skills Review for Standardized Tests

80. SAT/ACT If town A is 12 kilometers from town B and town C is 18 kilometers from town A, then which of the following *cannot* be the distance from town B to town C?

A 5 km
B 7 km
C 10 km
D 12 km
E 18 km

81. A remote control airplane flew along an initial path of 32° to the horizontal at a velocity of 48 feet per second as shown. Which of the following represent the magnitudes of the horizontal and vertical components of the velocity?



F 25.4 ft/s, 40.7 ft/s
G 40.7 ft/s, 25.4 ft/s
H 56.6 ft/s, 90.6 ft/s
J 90.6 ft/s, 56.6 ft/s

82. REVIEW Triangle ABC has vertices $A(-4, 2)$, $B(-4, -3)$, and $C(3, -3)$. After a dilation, triangle $A'B'C'$ has vertices $A'(-12, 6)$, $B'(-12, -9)$, and $C'(9, -9)$. How many times as great is the area of $\triangle A'B'C'$ than the area of $\triangle ABC$?

A $\frac{1}{9}$
B $\frac{1}{3}$
C 3
D 9

83. REVIEW Halima is drawing a map of her neighborhood. Her house is represented by quadrilateral $ABCD$ with vertices $A(2, 2)$, $B(6, 2)$, $C(6, 6)$, and $D(2, 6)$. She wants to use the same coordinate system to make another map that is one half the size of the original map. What could be the new vertices of Halima's house?

F $A'(0, 0)$, $B'(2, 1)$, $C'(3, 3)$, $D'(0, 3)$
G $A'(0, 0)$, $B'(3, 1)$, $C'(2, 3)$, $D'(0, 2)$
H $A'(1, 1)$, $B'(3, 1)$, $C'(3, 3)$, $D'(1, 3)$
J $A'(1, 2)$, $B'(3, 0)$, $C'(2, 2)$, $D'(2, 3)$

LESSON 1-2 Vectors in the Coordinate Plane

Then	Now	Why?
You performed vector operations using scale drawings.	**1** Represent and operate with vectors in the coordinate plane. **2** Write a vector as a linear combination of unit vectors.	Wind can impact the ground speed and direction of an airplane. While pilots can use scale drawings to determine the heading a plane should take to correct for wind, these calculations are more commonly calculated using vectors in the coordinate plane.

New Vocabulary
- component form
- unit vector
- linear combination

1 Vectors in the Coordinate Plane

In Lesson 1-1, you found the magnitude and direction of the resultant of two or more forces geometrically by using a scale drawing. Since drawings can be inaccurate, an algebraic approach using a rectangular coordinate system is needed for situations where more accuracy is required or where the system of vectors is complex.

A vector $\overrightarrow{OP}$ in standard position on a rectangular coordinate system (as in Figure 1.2.1) can be uniquely described by the coordinates of its terminal point $P(x, y)$. We denote $\overrightarrow{OP}$ on the coordinate plane by $\langle x, y\rangle$. Notice that x and y are the rectangular components of $\overrightarrow{OP}$. For this reason, $\langle x, y\rangle$ is called the **component form** of a vector.

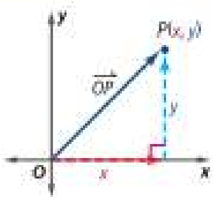


Figure 1.2.1

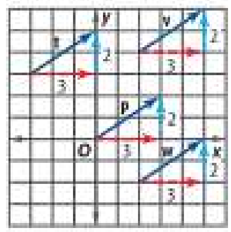


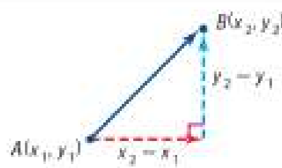
Figure 1.2.2

Since vectors with the same magnitude and direction are equivalent, many vectors can be represented by the same coordinates. For example, vectors **p**, **t**, **v**, and **w** in Figure 1.2.2 are *equivalent* because each can be denoted as $\langle 3, 2\rangle$. To find the component form of a vector that is not in standard position, you can use the coordinates of its initial and terminal points.

Key Concept Component Form of a Vector

The component form of a vector $\overline{AB}$ with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$ is given by

$$\langle x_2 - x_1, y_2 - y_1\rangle.$$



Example 1 Express a Vector in Component Form

Find the component form of $\overrightarrow{AB}$ with initial point $A(-4, 2)$ and terminal point $B(3, -5)$.

$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1\rangle$ — Component form

$= \langle 3 - (-4), -5 - 2\rangle$ — $(x_1, y_1) = (-4, 2)$ and $(x_2, y_2) = (3, -5)$

$= \langle 7, -7\rangle$ — Subtract.

Guided Practice

Find the component form of $\overrightarrow{AB}$ with the given initial and terminal points.

1A. $A(-2, -7)$, $B(6, 1)$

1B. $A(0, 8)$, $B(-9, -3)$

The magnitude of a vector in the coordinate plane is found by using the Distance Formula.

ReadingMath

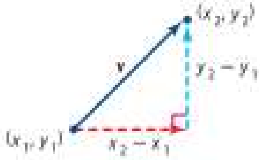
Norm The magnitude of a vector is sometimes called the *norm* of the vector.

KeyConcept Magnitude of a Vector in the Coordinate Plane

If **v** is a vector with initial point (x_1, y_1) and terminal point (x_2, y_2), then the magnitude of **v** is given by

$|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

If **v** has a component form of $\langle a, b\rangle$, then $|\mathbf{v}| = \sqrt{a^2 + b^2}$.



Example 2 Find the Magnitude of a Vector

Find the magnitude of $\overrightarrow{AB}$ with initial point $A(-4, 2)$ and terminal point $B(3, -5)$.

$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance Formula

$= \sqrt{[3 - (-4)]^2 + (-5 - 2)^2}$ $(x_1, y_1) = (-4, 2)$ and $(x_2, y_2) = (3, -5)$

$= \sqrt{98}$ or about 9.9 Simplify.

CHECK From Example 1, you know that $\overrightarrow{AB} = \langle 7, -7\rangle$. $|\overrightarrow{AB}| = \sqrt{7^2 + (-7)^2}$ or $\sqrt{98}$. ✓

GuidedPractice

Find the magnitude of $\overrightarrow{AB}$ with the given initial and terminal points.

2A. $A(-2, -7)$, $B(6, 1)$ **2B.** $A(0, 8)$, $B(-9, -3)$

Addition, subtraction, and scalar multiplication of vectors in the coordinate plane is similar to the same operations with matrices.

KeyConcept Vector Operations

If $\mathbf{a} = \langle a_1, a_2\rangle$ and $\mathbf{b} = \langle b_1, b_2\rangle$ are vectors and k is a scalar, then the following are true.

Vector Addition	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2\rangle$
Vector Subtraction	$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2\rangle$
Scalar Multiplication	$k\mathbf{a} = \langle ka_1, ka_2\rangle$

Example 3 Operations with Vectors

Find each of the following for $\mathbf{w} = \langle -4, 1\rangle$, $\mathbf{y} = \langle 2, 5\rangle$, and $\mathbf{z} = \langle -3, 0\rangle$.

a. $\mathbf{w} + \mathbf{y}$

$\mathbf{w} + \mathbf{y} = \langle -4, 1\rangle + \langle 2, 5\rangle$ Substitute.

$= \langle -4 + 2, 1 + 5\rangle$ or $\langle -2, 6\rangle$ Vector addition

b. $\mathbf{z} - 2\mathbf{y}$

$\mathbf{z} - 2\mathbf{y} = \mathbf{z} + (-2)\mathbf{y}$ Rewrite subtraction as addition.

$= \langle -3, 0\rangle + (-2)\langle 2, 5\rangle$ Substitute.

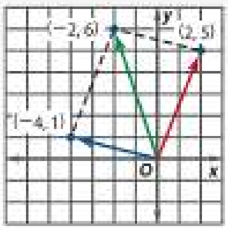
$= \langle -3, 0\rangle + \langle -4, -10\rangle$ or $\langle -7, -10\rangle$ Scalar multiplication and vector addition

GuidedPractice

3A. $4\mathbf{w} + \mathbf{z}$ **3B.** $-3\mathbf{w}$ **3C.** $2\mathbf{w} + 4\mathbf{y} - \mathbf{z}$

StudyTip

Check Graphically A graphical check of Example 3a using the parallelogram method is shown below.



Math HistoryLink

William Rowan Hamilton (1805–1865) An Irish mathematician, Hamilton developed the theory of quaternions and published *Lectures on Quaternions*. Many basic concepts of vector analysis have their basis in this theory.

2 Unit Vectors

A vector that has a magnitude of 1 unit is called a **unit vector**. It is sometimes useful to describe a nonzero vector **v** as a scalar multiple of a unit vector **u** with the same direction as **v**. To find **u**, divide **v** by its magnitude $|\mathbf{v}|$.

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \text{or} \quad \frac{1}{|\mathbf{v}|}\mathbf{v}$$

Example 4 Find a Unit Vector with the Same Direction as a Given Vector

Find a unit vector u with the same direction as $\mathbf{v} = \langle -2, 3\rangle$.

$\mathbf{u} = \frac{1}{\lvert\mathbf{v}\rvert}\mathbf{v}$	Unit vector with the same direction as v
$= \frac{1}{\lvert\langle -2, 3\rangle\rvert}\langle -2, 3\rangle$	Substitute.
$= \frac{1}{\sqrt{(-2)^2 + 3^2}}\langle -2, 3\rangle$	$\lvert\langle a, b\rangle\rvert = \sqrt{a^2 + b^2}$
$= \frac{1}{\sqrt{13}}\langle -2, 3\rangle$	Simplify.
$= \left\langle -\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$	Scalar multiplication
$= \left\langle -\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13} \right\rangle$	Rationalize denominators.

CHECK Since **u** is a scalar multiple of **v**, it has the same direction as **v**. Verify that the magnitude of **u** is 1.

$\lvert\mathbf{u}\rvert = \sqrt{\left(-\frac{2\sqrt{13}}{13}\right)^2 + \left(\frac{3\sqrt{13}}{13}\right)^2}$	Distance Formula
$= \sqrt{\frac{52}{169} + \frac{117}{169}}$	Simplify.
$= \sqrt{1}$ or 1 ✓	Simplify.

GuidedPractice

Find a unit vector with the same direction as the given vector.

4A. $\mathbf{w} = \langle 6, -2\rangle$

4B. $\mathbf{x} = \langle -4, -8\rangle$

WatchOut!

Unit Vector i Do not confuse the unit vector **i** with the imaginary number *i*. In this text, the unit vector is denoted by a bold, nonitalic letter **i**. The imaginary number is denoted by a bold italic letter ***i***.

The unit vectors in the direction of the positive *x*-axis and positive *y*-axis are denoted by $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$, respectively (Figure 1.2.3). Vectors **i** and **j** are called *standard unit vectors*.

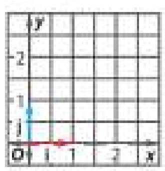


Figure 1.2.3

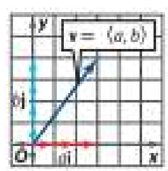


Figure 1.2.4

These vectors can be used to express any vector $\mathbf{v} = \langle a, b\rangle$ as $a\mathbf{i} + b\mathbf{j}$ as shown in Figure 1.2.4.

$\mathbf{v} = \langle a, b\rangle$	Component form of v
$= \langle a, 0\rangle + \langle 0, b\rangle$	Rewrite as the sum of two vectors.
$= a\langle 1, 0\rangle + b\langle 0, 1\rangle$	Scalar multiplication
$= a\mathbf{i} + b\mathbf{j}$	$\langle 1, 0\rangle = \mathbf{i}$ and $\langle 0, 1\rangle = \mathbf{j}$

The vector sum $a\mathbf{i} + b\mathbf{j}$ is called a **linear combination** of the vectors **i** and **j**.

Example 5 Write a Vector as a Linear Combination of Unit Vectors

Let $\overrightarrow{DE}$ be the vector with initial point $D(-2, 3)$ and terminal point $E(4, 5)$. Write $\overrightarrow{DE}$ as a linear combination of the vectors i and j.

First, find the component form of $\overrightarrow{DE}$.

$\overrightarrow{DE} = \langle x_2 - x_1, y_2 - y_1 \rangle$ — Component form

$= \langle 4 - (-2), 5 - 3 \rangle$ — $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (4, 5)$

$= \langle 6, 2 \rangle$ — Simplify.

Then rewrite the vector as a linear combination of the standard unit vectors.

$\overrightarrow{DE} = \langle 6, 2 \rangle$ — Component form

$= 6\mathbf{i} + 2\mathbf{j}$ — $\langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$

GuidedPractice

Let $\overrightarrow{DE}$ be the vector with the given initial and terminal points. Write $\overrightarrow{DE}$ as a linear combination of the vectors i and j.

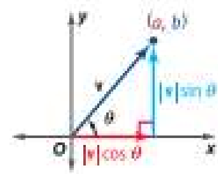
5A. $D(-6, 0), E(2, 5)$

5B. $D(-3, -8), E(-7, 1)$

StudyTip

Unit Vector From the statement that $\mathbf{v} = \langle |\mathbf{v}| \cos\theta, |\mathbf{v}| \sin\theta \rangle$, it follows that the unit vector in the direction of **v** has the form $\mathbf{v} = \langle 1 \cos\theta, 1 \sin\theta \rangle = \langle \cos\theta, \sin\theta \rangle$.

A way to specify the direction of a vector $\mathbf{v} = \langle a, b \rangle$ is to state the direction angle θ that **v** makes with the positive x-axis. From Figure 1.2.5, it follows that **v** can be written in component form or as a linear combination of **i** and **j** using the magnitude and direction angle of the vector.



$\mathbf{v} = \langle a, b \rangle$ — Component form

$= \langle |\mathbf{v}| \cos\theta, |\mathbf{v}| \sin\theta \rangle$ — Substitution

$= |\mathbf{v}| (\cos\theta)\mathbf{i} + |\mathbf{v}| (\sin\theta)\mathbf{j}$ — Linear combination of i and j

Example 6 Find Component Form

Find the component form of the vector v with magnitude 10 and direction angle 120°.

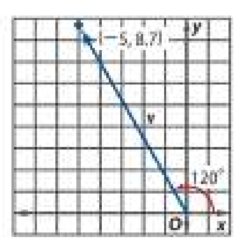
$\mathbf{v} = \langle |\mathbf{v}| \cos\theta, |\mathbf{v}| \sin\theta \rangle$ — Component form of v in terms of |v| and θ

$= \langle 10 \cos 120°, 10 \sin 120° \rangle$ — $|\mathbf{v}| = 10$ and $\theta = 120°$

$= \left\langle 10\left(-\frac{1}{2}\right), 10\left(\frac{\sqrt{3}}{2}\right) \right\rangle$ — $\cos 120° = -\frac{1}{2}$ and $\sin 120° = \frac{\sqrt{3}}{2}$

$= \langle -5, 5\sqrt{3} \rangle$ — Simplify.

CHECK Graph $\mathbf{v} = \langle -5, 5\sqrt{3} \rangle \approx \langle -5, 8.7 \rangle$. The measure of the angle **v** makes with the positive x-axis is about 120° as shown, and $|\mathbf{v}| = \sqrt{(-5)^2 + (5\sqrt{3})^2}$ or 10. ✔



GuidedPractice

Find the component form of v with the given magnitude and direction angle.

6A. $|\mathbf{v}| = 8, \theta = 45°$

6B. $|\mathbf{v}| = 24, \theta = 210°$

It also follows from Figure 1.2.5 on the previous page that the direction angle θ of vector $\mathbf{v} = \langle a, b \rangle$ can be found by solving the trigonometric equation $\tan \theta = \frac{|\mathbf{v}| \sin \theta}{|\mathbf{v}| \cos \theta}$ or $\tan \theta = \frac{b}{a}$.

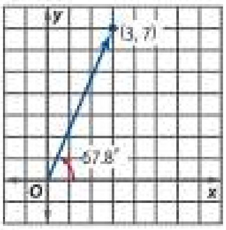


Figure 1.2.6

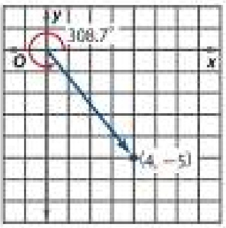


Figure 1.2.7

Example 7 Direction Angles of Vectors

Find the direction angle of each vector to the nearest tenth of a degree.

a. $\mathbf{p} = 3\mathbf{i} + 7\mathbf{j}$

$\tan \theta = \frac{b}{a}$ — Direction angle equation

$\tan \theta = \frac{7}{3}$ — $a = 3$ and $b = 7$

$\theta = \tan^{-1} \frac{7}{3}$ — Solve for θ.

$\theta \approx 66.8°$ — Use a calculator.

So, the direction angle of vector **p** is about 67.8° as shown in Figure 1.2.6.

b. $\mathbf{r} = \langle 4, -5 \rangle$

$\tan \theta = \frac{b}{a}$ — Direction angle equation

$\tan \theta = \frac{-5}{4}$ — $a = 4$ and $b = -5$

$\theta = \tan^{-1} \left(-\frac{5}{4}\right)$ — Solve for θ.

$\theta \approx -51.3°$ — Use a calculator.

Since **r** lies in Quadrant IV as shown in Figure 1.2.7, $\theta = 360 + (-51.3)$ or 308.7°.

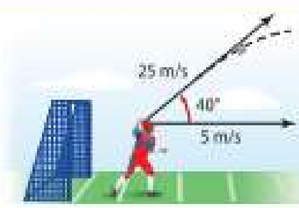
GuidedPractice

7A. $-6\mathbf{i} + 2\mathbf{j}$

7B. $\langle -3, -8 \rangle$

Real-World Example 8 Applied Vector Operations

FOOTBALL A goalkeeper running forward at 5 meters per second throws a ball with a velocity of 25 meters per second at an angle of 40° with the horizontal. What is the resultant speed and direction of the pass?



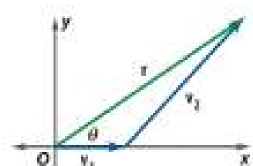
Since the goalkeeper moves straight forward, the component form of his velocity $\mathbf{v}_1$ is $\langle 5, 0 \rangle$. Use the magnitude and direction of the ball's velocity $\mathbf{v}_2$ to write this vector in component form.

$\mathbf{v}_2 = \langle |\mathbf{v}_2| \cos \theta, |\mathbf{v}_2| \sin \theta \rangle$ — Component form of $\mathbf{v}_2$

$= \langle 25 \cos 40°, 25 \sin 40° \rangle$ — $|\mathbf{v}_2| = 25$ and $\theta = 40°$

$\approx \langle 19.2, 16.1 \rangle$ — Simplify.

Add the algebraic vectors representing $\mathbf{v}_1$ and $\mathbf{v}_2$ to find the resultant velocity, vector **r**.



Note: Not drawn to scale.

$\mathbf{r} = \mathbf{v}_1 + \mathbf{v}_2$ — Resultant vector

$= \langle 5, 0 \rangle + \langle 19.2, 16.1 \rangle$ — Substitution

$= \langle 24.2, 16.1 \rangle$ — Vector Addition

The magnitude of this resultant is $|\mathbf{r}| = \sqrt{24.2^2 + 16.1^2}$ or about 29.1. Next find the resultant direction angle θ.

$\tan \theta = \frac{16.1}{24.2}$ — $\tan \theta = \frac{b}{a}$ where $\langle a, b \rangle = \langle 24.2, 16.1 \rangle$

$\theta = \tan^{-1} \frac{16.1}{24.2}$ or about 33.6° — Solve for θ.

Therefore, the resultant velocity of the pass is about 29.1 meters per second at an angle of about 33.6° with the horizontal.

GuidedPractice

8. FOOTBALL What would the resultant velocity of the ball be if the goalkeeper made the same pass running 5 meters per second backward?

Exercises

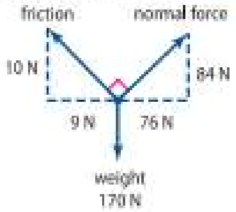
Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points. (Examples 1 and 2)

1. $A(-3, 1)$, $B(4, 5)$
2. $A(2, -7)$, $B(-6, 9)$
3. $A(10, -2)$, $B(3, -5)$
4. $A(-2, 7)$, $B(-9, -1)$
5. $A(-5, -4)$, $B(8, -2)$
6. $A(-2, 6)$, $B(1, 10)$
7. $A(2.5, -3)$, $B(-4, 1.5)$
8. $A(-4.3, 1.8)$, $B(9.4, -6.2)$
9. $A\left(\frac{1}{2}, -9\right)$, $B\left(6, \frac{5}{2}\right)$
10. $A\left(\frac{3}{5}, -\frac{2}{5}\right)$, $B(-1, 7)$

Find each of the following for $\mathbf{f} = \langle 8, 0 \rangle$, $\mathbf{g} = \langle -3, -5 \rangle$, and $\mathbf{h} = \langle -6, 2 \rangle$. (Example 3)

11. $4\mathbf{h} - \mathbf{g}$
12. $\mathbf{f} + 2\mathbf{h}$
13. $3\mathbf{g} - 5\mathbf{f} + \mathbf{h}$
14. $2\mathbf{f} + \mathbf{g} - 3\mathbf{h}$
15. $\mathbf{f} - 2\mathbf{g} - 2\mathbf{h}$
16. $\mathbf{h} - 4\mathbf{f} + 5\mathbf{g}$
17. $4\mathbf{g} - 3\mathbf{f} + \mathbf{h}$
18. $6\mathbf{h} + 5\mathbf{f} - 10\mathbf{g}$

19. **PHYSICS** In physics, force diagrams are used to show the effects of all the different forces acting upon an object. The following force diagram could represent the forces acting upon a child sliding down a slide. (Example 3)



a. Using the blue dot representing the child as the origin, express each force as a vector in component form.

b. Find the component form of the resultant vector representing the force that causes the child to move down the slide.

Find a unit vector u with the same direction as v. (Example 4)

20. $\mathbf{v} = \langle -2, 7 \rangle$
21. $\mathbf{v} = \langle 9, -3 \rangle$
22. $\mathbf{v} = \langle -8, -5 \rangle$
23. $\mathbf{v} = \langle 6, 3 \rangle$
24. $\mathbf{v} = \langle -2, 9 \rangle$
25. $\mathbf{v} = \langle -1, -5 \rangle$
26. $\mathbf{v} = \langle 1, 7 \rangle$
27. $\mathbf{v} = \langle 3, -4 \rangle$

Let $\overrightarrow{DE}$ be the vector with the given initial and terminal points. Write $\overrightarrow{DE}$ as a linear combination of the vectors i and j. (Example 5)

28. $D(4, -1)$, $E(5, -7)$
29. $D(9, -6)$, $E(-7, 2)$
30. $D(3, 11)$, $E(-2, -8)$
31. $D(9.5, 1)$, $E(0, -7.3)$
32. $D(-3, -5.7)$, $E(6, -8.1)$
33. $D(-4, -6)$, $E(9, 5)$
34. $D\left(\frac{1}{8}, 3\right)$, $E\left(-4, \frac{2}{7}\right)$
35. $D(-3, 1.5)$, $E(-3, 1.5)$

36. **COMMUTE** To commute to school, Lamis leaves her house and drives north on Al Nasr Street for 2.4 kilometers. She turns left on Freedom Street for 3.1 kilometers and then turns right on Hope Street for 5.8 kilometers. Express Lamis' commute as a linear combination of unit vectors **i** and **j**. (Example 5)

37. **ROWING** Najat is rowing across a river at a speed of 5 kilometers per hour perpendicular to the shore. The river has a current of 3 kilometers per hour heading downstream. (Example 5)

a. At what speed is she traveling?

b. At what angle is she traveling with respect to the shore?

Find the component form of v with the given magnitude and direction angle. (Example 6)

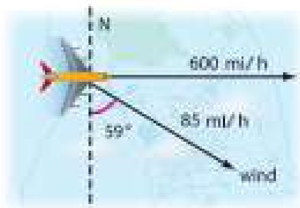
38. $|\mathbf{v}| = 12$, $\theta = 60°$
39. $|\mathbf{v}| = 4$, $\theta = 135°$
40. $|\mathbf{v}| = 6$, $\theta = 240°$
41. $|\mathbf{v}| = 16$, $\theta = 330°$
42. $|\mathbf{v}| = 28$, $\theta = 273°$
43. $|\mathbf{v}| = 15$, $\theta = 125°$

Find the direction angle of each vector to the nearest tenth of a degree. (Example 7)

44. $3\mathbf{i} + 6\mathbf{j}$
45. $-2\mathbf{i} + 5\mathbf{j}$
46. $8\mathbf{i} - 2\mathbf{j}$
47. $-4\mathbf{i} - 3\mathbf{j}$
48. $\langle -5, 9 \rangle$
49. $\langle 7, 7 \rangle$
50. $\langle -6, -4 \rangle$
51. $\langle 3, -8 \rangle$

52. **SLEDDING** Amna is pulling a sled with a force of 275 newtons by holding its rope at a 58° angle. Her brother is going to help by pushing the sled with a force of 320 newtons. Determine the magnitude and direction of the total resultant force on the sled. (Example 8)

53. **NAVIGATION** An airplane is traveling due east with a speed of 600 miles per hour. The wind blows at 85 miles per hour at an angle of S59°E. (Example 8)



a. Determine the speed of the airplane's flight.

b. Determine the angle of the airplane's flight.

54. **HEADING** A pilot needs to plot a course that will result in a velocity of 500 miles per hour in a direction of due west. If the wind is blowing 100 miles per hour from the directed angle of 192°, find the direction and the speed the pilot should set to achieve this resultant.

Determine whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ with the initial and terminal points given are equivalent. If so, prove that $\overrightarrow{AB} = \overrightarrow{CD}$. If not, explain why not.

55. $A(3, 5)$, $B(6, 9)$, $C(-4, -4)$, $D(-2, 0)$

56. $A(-4, -5)$, $B(-8, 1)$, $C(3, -3)$, $D(1, 0)$

57. $A(1, -3)$, $B(0, -10)$, $C(11, 8)$, $D(10, 1)$

58. **RAFTING** Hana's family is rafting across a river. Suppose that they are on a stretch of the river that is 150 meters wide, flowing south at a rate of 1.0 meter per second. In still water, their raft travels 5.0 meters per second.

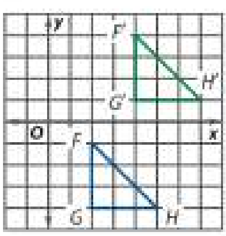
- **a.** What is the speed of the raft?
- **b.** How far downriver will the raft land?
- **c.** How long does it take them to travel from one bank to the other if they head directly across the river?

59. **NAVIGATION** A jet is flying with an air speed of 480 miles per hour at a bearing of N82°E. Because of the wind, the ground speed of the plane is 518 miles per hour at a bearing of N79°E.

- **a.** Draw a diagram to represent the situation.
- **b.** What are the speed and direction of the wind?
- **c.** If the pilot increased the air speed of the plane to 500 miles per hour, what would be the resulting ground speed and direction of the plane?

60. **TRANSLATIONS** You can translate a figure along a translation vector $\langle a, b\rangle$ by adding a to each x-coordinate and b to each y-coordinate. Consider the triangles shown below.

- **a.** Describe the translation from $\triangle FGH$ to $\triangle F'G'H'$ using a translation vector.
- **b.** Graph $\triangle F'G'H'$ and its translated image $\triangle F''G''H''$ along $\langle -3, -6\rangle$.
- **c.** Describe the translation from $\triangle FGH$ to $\triangle F''G''H''$ using a translation vector.

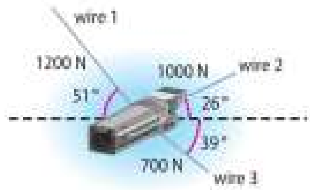


Given the initial point and magnitude of each vector, determine a possible terminal point of the vector.

61. $(-1, 4)$; $\sqrt{37}$

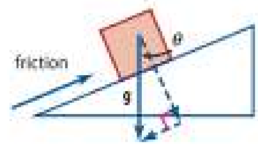
62. $(-3, -7)$; 10

63. **CAMERA** A video camera that follows the action at a sporting event is supported by three wires. The tension in each wire can be modeled by a vector.



- **a.** Find the component form of each vector.
- **b.** Find the component form of the resultant vector acting on the camera.
- **c.** Find the magnitude and direction of the resulting force.

64. **FORCE** A box is stationary on a ramp. Both gravity **g** and friction are exerted on the box. The components of gravity are shown in the diagram. What must be true of the force of friction for this scenario to be possible?



H.O.T. Problems Use Higher-Order Thinking Skills

65. **REASONING** If vectors **a** and **b** are parallel, write a vector equation relating **a** and **b**.

66. **CHALLENGE** To pull luggage, Ahmed exerts a force of 150 newtons at an angle of 58° with the horizontal. If the resultant force on the luggage is 72 newtons at an angle of 56.7° with the horizontal, what is the magnitude of the resultant of $\mathbf{F}_{friction}$ and $\mathbf{F}_{weight}$?

67. **REASONING** If given the initial point of a vector and its magnitude, describe the locus of points that represent possible locations for the terminal point.

68. **WRITING IN MATH** Explain how to find the direction angle of a vector in the fourth quadrant.

69. **CHALLENGE** The direction angle of $\langle x, y\rangle$ is $(4y)°$. Find x in terms of y.

PROOF **Prove each vector property. Let $\mathbf{a} = \langle x_1, y_1\rangle$, $\mathbf{b} = \langle x_2, y_2\rangle$, and $\mathbf{c} = \langle x_3, y_3\rangle$.**

70. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

71. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

72. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$, where k is a scalar

73. $|k\mathbf{a}| = |k|\,|\mathbf{a}|$, where k is a scalar

Spiral Review

74. TOYS Fahd is pulling a toy by exerting a force of 1.5 newtons on a string attached to the toy.

a. The string makes an angle of 52° with the floor. Find the horizontal and vertical components of the force.

b. If Fahd raises the string so that it makes a 78° angle with the floor, what are the magnitudes of the horizontal and vertical components of the force?

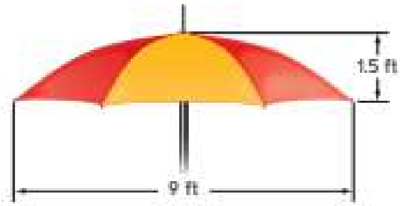
Write each pair of parametric equations in rectangular form.

75. $y = t^2 + 2, x = 3t - 6$

76. $y = t^2 - 5, x = 2t + 8$

77. $y = 7t, x = t^2 - 1$

78. UMBRELLAS A beach umbrella has an arch in the shape of a parabola. Write an equation to model the arch, assuming that the origin and the vertex are at the point where the pole and the canopy of the umbrella meet. Express y in terms of x.



Decompose each expression into partial fractions.

79. $\frac{5z - 11}{2z^2 + z - 6}$

80. $\frac{7x^2 + 18x - 1}{(x^2 - 1)(x + 2)}$

81. $\frac{9 - 9x}{x^2 - 9}$

Verify each identity.

82. $\sin(\theta + 180°) = -\sin\theta$

83. $\sin(60° + \theta) + \sin(60° - \theta) = \sqrt{3}\cos\theta$

Express each logarithm in terms of ln 3 and ln 7.

84. $\ln 189$

85. $\ln 5.\overline{4}$

86. $\ln 441$

87. $\ln \frac{9}{343}$

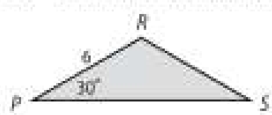
Find each $f(c)$ using synthetic substitution.

88. $f(x) = 6x^6 - 9x^4 + 12x^3 - 16x^2 + 8x + 24; c = 6$

89. $f(x) = 8x^5 - 12x^4 + 18x^3 - 24x^2 + 36x - 48, c = 4$

Skills Review for Standardized Tests

90. SAT/ACT If $PR = RS$, what is the area of triangle PRS?



A $9\sqrt{2}$

B $9\sqrt{3}$

C $18\sqrt{2}$

D $18\sqrt{3}$

E $36\sqrt{3}$

91. REVIEW Faleh has made a game for his younger sister's graduation celebration. The playing board is a circle divided evenly into 8 sectors. If the circle has a radius of 18 centimeters, what is the approximate area of one of the sectors? H

F 4 cm²

G 32 cm²

H 127 cm²

J 254 cm²

92. Paramedics Ibrahim and Ismail are moving a person on a stretcher. Ibrahim is pushing the stretcher from behind with a force of 135 newtons at 58° with the horizontal, while Ismail is pulling the stretcher from the front with a force of 214 newtons at 43° with the horizontal. What is the magnitude of the horizontal force exerted on the stretcher?

A 228 newtons

B 260 newtons

C 299 newtons

D 346 newtons

93. REVIEW Find the center and radius of the circle with equation $(x - 4)^2 + y^2 - 16 = 0$.

F $C(-4, 0); r = 4$ units

G $C(-4, 0); r = 16$ units

H $C(4, 0); r = 4$ units

J $C(4, 0); r = 16$ units

LESSON

1-3 Dot Products and Vector Projections

Then

- You found the magnitudes of and operated with algebraic vectors.

Now

1. Find the dot product of two vectors, and use the dot product to find the angle between them.
2. Find the projection of one vector onto another.

Why?

- The word *work* has different meanings in everyday life, but in physics, its definition is very specific. Work is the magnitude of a force applied to an object multiplied by the distance through which the object moves parallel to this applied force. Work, such as that done to push a car a specific distance, can also be calculated using a vector operation called a *dot product*.

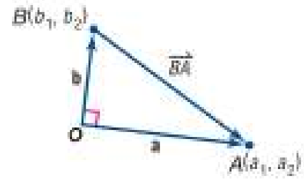
New Vocabulary

dot product
orthogonal
vector projection
work

1 Dot Product

In Lesson 1-2, you studied the vector operations of vector addition and scalar multiplication. In this lesson, you will use a third vector operation. Consider two perpendicular vectors in standard position **a** and **b**. Let $\overrightarrow{BA}$ be the vector between their terminal points as shown in the figure. By the Pythagorean Theorem, we know that

$$|\overrightarrow{BA}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2.$$



Using the definition of the magnitude of a vector, we can find $|\overrightarrow{BA}|^2$.

$|\overrightarrow{BA}| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$ — Definition of vector magnitude

$|\overrightarrow{BA}|^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2$ — Square each side.

$|\overrightarrow{BA}|^2 = a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2$ — Expand each binomial square.

$|\overrightarrow{BA}|^2 = (a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2(a_1b_1 + a_2b_2)$ — Group the squared terms.

$|\overrightarrow{BA}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2(a_1b_1 + a_2b_2)$ — $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$ so $|\mathbf{a}|^2 = a_1^2 + a_2^2$ and $|\mathbf{b}| = \sqrt{b_1^2 + b_2^2}$, so $|\mathbf{b}|^2 = b_1^2 + b_2^2$.

Notice that the expressions $|\mathbf{a}|^2 + |\mathbf{b}|^2$ and $|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2(a_1b_1 + a_2b_2)$ are equivalent if and only if $a_1b_1 + a_2b_2 = 0$. The expression $a_1b_1 + a_2b_2$ is called the **dot product** of **a** and **b**, denoted **a** · **b** and read as *a dot b*.

Key Concept Dot Product of Vectors in a Plane

The dot product of $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$.

Notice that unlike vector addition and scalar multiplication, the dot product of two vectors yields a scalar, not a vector. As demonstrated above, two nonzero vectors are perpendicular if and only if their dot product is 0. Two vectors with a dot product of 0 are said to be **orthogonal**.

Key Concept Orthogonal Vectors

The vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

The terms *perpendicular* and *orthogonal* have essentially the same meaning, except when **a** or **b** is the zero vector. The zero vector is orthogonal to any vector **a**, since $\langle 0, 0 \rangle \cdot \langle a_1, a_2 \rangle = 0a_1 + 0a_2$ or 0. However, since the zero vector has no magnitude or direction, it cannot be perpendicular to **a**.

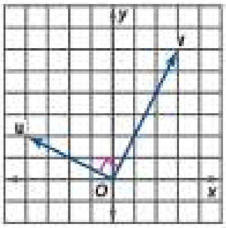


Figure 1.3.1

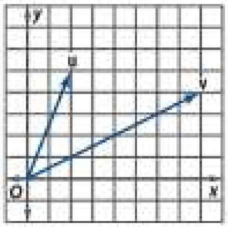


Figure 1.3.2

Example 1 Find the Dot Product to Determine Orthogonal Vectors

Find the dot product of u and v. Then determine if u and v are orthogonal.

a. $\mathbf{u} = \langle -4, 2 \rangle$, $\mathbf{v} = \langle 3, 6 \rangle$

$\mathbf{u} \cdot \mathbf{v} = -4(3) + 2(6)$
$= 0$

Since $\mathbf{u} \cdot \mathbf{v} = 0$, **u** and **v** are orthogonal, as illustrated in Figure 1.3.1.

b. $\mathbf{u} = \langle 2, 5 \rangle$, $\mathbf{v} = \langle 8, 4 \rangle$

$\mathbf{u} \cdot \mathbf{v} = 2(8) + 5(4)$
$= 36$

Since $\mathbf{u} \cdot \mathbf{v} \neq 0$, **u** and **v** are not orthogonal, as illustrated in Figure 1.3.2.

GuidedPractice

1A. $\mathbf{u} = \langle 3, -2 \rangle$, $\mathbf{v} = \langle -5, 1 \rangle$ **−17; not orthogonal**

1B. $\mathbf{u} = \langle -2, -3 \rangle$, $\mathbf{v} = \langle 9, -6 \rangle$ **0; orthogonal**

Dot products have the following properties.

KeyConcept Properties of the Dot Product

If **u**, **v**, and **w** are vectors and k is a scalar, then the following properties hold.

Commutative Property	$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
Distributive Property	$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
Scalar Multiplication Property	$k(\mathbf{u} \cdot \mathbf{v}) = k\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot k\mathbf{v}$
Zero Vector Dot Product Property	$\mathbf{0} \cdot \mathbf{u} = 0$
Dot Product and Vector Magnitude Relationship	$\mathbf{u} \cdot \mathbf{u} = \lvert\mathbf{u}\rvert^2$

Proof

Proof $\mathbf{u} \cdot \mathbf{u} = \lvert\mathbf{u}\rvert^2$

Let $\mathbf{u} = \langle u_1, u_2 \rangle$.

$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2$ — Dot product

$= \left(\sqrt{(u_1^2 + u_2^2)}\right)^2$ — Write as the square of the square root of $u_1^2 + u_2^2$.

$= \lvert\mathbf{u}\rvert^2$ — $\sqrt{u_1^2 + u_2^2} = \lvert\mathbf{u}\rvert$

ReadingMath

Inner and Scalar Products

The dot product is also called the *inner product* or the *scalar product*.

You will prove the first three properties in Exercises 70–72.

Example 2 Use the Dot Product to Find Magnitude

Use the dot product to find the magnitude of $\mathbf{a} = \langle -5, 12 \rangle$.

Since $\lvert\mathbf{a}\rvert^2 = \mathbf{a} \cdot \mathbf{a}$, then $\lvert\mathbf{a}\rvert = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

$\lvert\langle -5, 12 \rangle\rvert = \sqrt{\langle -5, 12 \rangle \cdot \langle -5, 12 \rangle}$ — $\mathbf{a} = \langle -5, 12 \rangle$

$= \sqrt{(-5)^2 + 12^2}$ or 13 — Simplify.

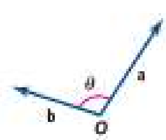
GuidedPractice

Use the dot product to find the magnitude of the given vector.

2A. $\mathbf{b} = \langle 12, 16 \rangle$ **20**

2B. $\mathbf{c} = \langle -1, -7 \rangle$ **$5\sqrt{2}$ or 7.07**

The angle θ between any two nonzero vectors **a** and **b** is the corresponding angle between these vectors when placed in standard position, as shown. This angle is always measured such that $0 \leq \theta \leq \pi$ or $0^\circ \leq \theta \leq 180^\circ$. The dot product can be used to find the angle between two nonzero vectors.



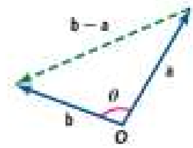
StudyTip

Parallel and Perpendicular Vectors Two vectors are perpendicular if the angle between them is 90°. Two vectors are parallel if the angle between them is 0° or 180°.

KeyConcept Angle Between Two Vectors

If θ is the angle between nonzero vectors **a** and **b**, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}.$$



Proof

Consider the triangle determined by **a**, **b**, and **b** − **a** in the figure above.

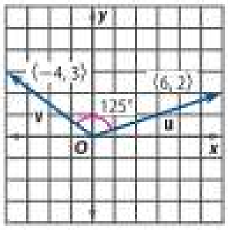
$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\| \cos \theta = \|\mathbf{b} - \mathbf{a}\|^2$	Law of Cosines
$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\| \cos \theta = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})$	$\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u}$
$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\| \cos \theta = \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a}$	Distributive Property for Dot Products
$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\| \cos \theta = \|\mathbf{b}\|^2 - 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{a}\|^2$	$\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$
$-2\|\mathbf{a}\|\,\|\mathbf{b}\| \cos \theta = -2\mathbf{a} \cdot \mathbf{b}$	Subtract $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2$ from each side.
$\cos \theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}$	Divide each side by $-2\|\mathbf{a}\|\,\|\mathbf{b}\|$.

Example 3 Find the Angle Between Two Vectors

Find the angle θ between vectors u and v to the nearest tenth of a degree.

a. $\mathbf{u} = \langle 6, 2 \rangle$ and $\mathbf{v} = \langle -4, 3 \rangle$

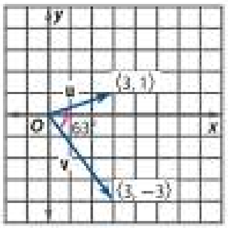
$\cos \theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$	Angle between two vectors
$\cos \theta = \dfrac{\langle 6, 2 \rangle \cdot \langle -4, 3 \rangle}{\|\langle 6, 2 \rangle\|\,\|\langle -4, 3 \rangle\|}$	$\mathbf{u} = \langle 6, 2 \rangle$ and $\mathbf{v} = \langle -4, 3 \rangle$
$\cos \theta = \dfrac{-24 + 6}{\sqrt{40}\,\sqrt{25}}$	Evaluate.
$\cos \theta = \dfrac{-9}{5\sqrt{10}}$	Simplify.
$\theta = \cos^{-1} \dfrac{-9}{5\sqrt{10}}$ or about 124.7°	Solve for θ.



The measure of the angle between **u** and **v** is about 124.7°.

b. $\mathbf{u} = \langle 3, 1 \rangle$ and $\mathbf{v} = \langle 3, -3 \rangle$

$\cos \theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$	Angle between two vectors
$\cos \theta = \dfrac{\langle 3, 1 \rangle \cdot \langle 3, -3 \rangle}{\|\langle 3, 1 \rangle\|\,\|\langle 3, -3 \rangle\|}$	$\mathbf{u} = \langle 3, 1 \rangle$ and $\mathbf{v} = \langle 3, -3 \rangle$
$\cos \theta = \dfrac{9 + (-3)}{\sqrt{10}\,\sqrt{18}}$	Evaluate.
$\cos \theta = \dfrac{1}{\sqrt{5}}$	Simplify.
$\theta = \cos^{-1} \dfrac{1}{\sqrt{5}}$ or about 63.4°	Solve for θ.



The measure of the angle between **u** and **v** is about 63.4°.

GuidedPractice

3A. $\mathbf{u} = \langle -5, -2 \rangle$ and $\mathbf{v} = \langle 4, 4 \rangle$ **156.8°**

3B. $\mathbf{u} = \langle 9, 5 \rangle$ and $\mathbf{v} = \langle -6, 7 \rangle$ **101.5°**

2 Vector Projection

In Lesson 1-1, you learned that a vector can be resolved or decomposed into two perpendicular components. While these components are often horizontal and vertical, it is sometimes useful instead for one component to be parallel to another vector.

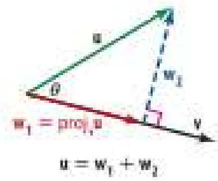
StudyTip

Perpendicular Component The vector $\mathbf{w}_2$ is called the *component of u perpendicular to v.*

KeyConcept Projection of u onto v

Let **u** and **v** be nonzero vectors, and let $\mathbf{w}_1$ and $\mathbf{w}_2$ be vector components of **u** such that $\mathbf{w}_1$ is parallel to **v** as shown. Then vector $\mathbf{w}_1$ is called the **vector projection** of **u** onto **v**, denoted $\text{proj}_\mathbf{v}\mathbf{u}$, and

$$\text{proj}_\mathbf{v}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}.$$



Proof

Since $\text{proj}_\mathbf{v}\mathbf{u}$ is parallel to **v**, it can be written as a scalar multiple of **v**. As a scalar multiple of a unit vector $\mathbf{v}_v$ with the same direction as **v**, $\text{proj}_\mathbf{v}\mathbf{u} = |\mathbf{w}_1|\, \mathbf{v}_v$. Use the right triangle formed by $\mathbf{w}_1$, $\mathbf{w}_2$, and **u** and the cosine ratio to find an expression for $|\mathbf{w}_1|$.

$\cos\theta = \frac{|\mathbf{w}_1|}{|\mathbf{u}|}$ — Cosine ratio

$|\mathbf{u}|\,|\mathbf{v}| \cos\theta = |\mathbf{u}|\,|\mathbf{v}|\,\frac{|\mathbf{w}_1|}{|\mathbf{u}|}$ — Multiply each side by the scalar quantity $|\mathbf{u}|\,|\mathbf{v}|$.

$\mathbf{u} \cdot \mathbf{v} = |\mathbf{v}|\,|\mathbf{w}_1|$ — $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$, so $|\mathbf{u}|\,|\mathbf{v}|\cos\theta = \mathbf{u} \cdot \mathbf{v}$.

$|\mathbf{w}_1| = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$ — Solve for $|\mathbf{w}_1|$.

Now use $\text{proj}_\mathbf{v}\mathbf{u} = |\mathbf{w}_1|\, \mathbf{v}_v$ to find $\text{proj}_\mathbf{v}\mathbf{u}$ as a scalar multiple of **v**.

$\text{proj}_\mathbf{v}\mathbf{u} = |\mathbf{w}_1|\, \mathbf{v}_v$

$= \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$ — $|\mathbf{w}_1| = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$ and $\mathbf{v}_v = \frac{\mathbf{v}}{|\mathbf{v}|}$

$= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$ — Multiply magnitudes.

Example 4 Find the Projection of u onto v

Find the projection of $\mathbf{u} = \langle 3, 2\rangle$ onto $\mathbf{v} = \langle 5, -5\rangle$. Then write u as the sum of two orthogonal vectors, one of which is the projection of u onto v.

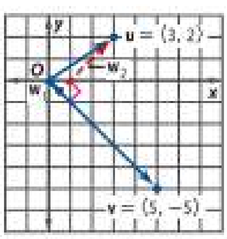


Figure 1.3.5

Step 1 Find the projection of **u** onto **v**.

$$\text{proj}_\mathbf{v}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

$$= \frac{\langle 3, 2\rangle \cdot \langle 5, -5\rangle}{|\langle 5, -5\rangle|^2}\langle 5, -5\rangle$$

$$= \frac{5}{50}\langle 5, -5\rangle$$

$$= \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle$$

Step 2 Find $\mathbf{w}_2$.

Since $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$$

$$= \mathbf{u} - \text{proj}_\mathbf{v}\mathbf{u}$$

$$= \langle 3, 2\rangle - \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle$$

$$= \left\langle \frac{5}{2}, \frac{5}{2}\right\rangle$$

Therefore, $\text{proj}_\mathbf{v}\mathbf{u}$ is $\mathbf{w}_1 = \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle$ as shown in Figure 1.3.5, and $\mathbf{u} = \left\langle \frac{1}{2}, -\frac{1}{2}\right\rangle + \left\langle \frac{5}{2}, \frac{5}{2}\right\rangle$.

GuidedPractice

4. Find the projection of $\mathbf{u} = \langle 1, 2\rangle$ onto $\mathbf{v} = \langle 8, 5\rangle$. Then write **u** as the sum of two orthogonal vectors, one of which is the projection of **u** onto **v**. $\text{proj}_\mathbf{v}\mathbf{u} = \left\langle \frac{144}{89}, \frac{90}{89}\right\rangle$; $\mathbf{u} = \left\langle \frac{144}{89}, \frac{90}{89}\right\rangle + \left\langle -\frac{55}{89}, \frac{88}{89}\right\rangle$

While the projection of **u** onto **v** is a vector parallel to **v**, this vector will not necessarily have the same direction as **v**, as illustrated in the next example.

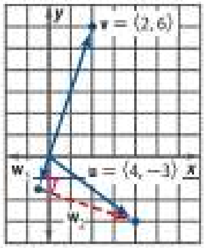


Figure 1.3.6

Example 5 Projection with Direction Opposite v

Find the projection of u = ⟨4, −3⟩ onto v = ⟨2, 6⟩. Then write u as the sum of two orthogonal vectors, one of which is the projection of u onto v.

Notice that the angle between **u** and **v** is obtuse, so the projection of **u** onto **v** lies on the vector opposite **v** or −**v**, as shown in Figure 1.3.6.

Step 1 Find the projection of **u** onto **v**.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

$$= \frac{\langle 4, -3\rangle \cdot \langle 2, 6\rangle}{|\langle 2, 6\rangle|^2}\langle 2, 6\rangle$$

$$= \frac{-10}{40}\langle 2, 6\rangle \text{ or } \left\langle -\frac{1}{2}, -\frac{3}{2}\right\rangle$$

Step 2 Find $\mathbf{w}_2$.

Since $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ or $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 4, -3\rangle - \left\langle -\frac{1}{2}, -\frac{3}{2}\right\rangle$$

$$= \left\langle \frac{9}{2}, -\frac{3}{2}\right\rangle$$

Therefore, $\text{proj}_{\mathbf{v}}\mathbf{u}$, $\mathbf{w}_1 = \left\langle -\frac{1}{2}, -\frac{3}{2}\right\rangle$ and $\mathbf{u} = \left\langle -\frac{1}{2}, -\frac{3}{2}\right\rangle + \left\langle \frac{9}{2}, -\frac{3}{2}\right\rangle$

GuidedPractice

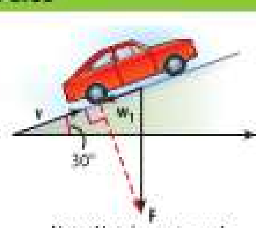
5. Find the projection of $\mathbf{u} = \langle -3, 4\rangle$ onto $\mathbf{v} = \langle 6, 1\rangle$. Then write **u** as the sum of two orthogonal vectors, one of which is the projection of **u** onto **v**.
$\text{proj}_{\mathbf{v}}\mathbf{u} = \left\langle -\frac{84}{37}, -\frac{14}{37}\right\rangle$; $\mathbf{u} = \left\langle -\frac{84}{37}, -\frac{14}{37}\right\rangle + \left\langle -\frac{27}{37}, \frac{162}{37}\right\rangle$

Figure 1.3.7

If the vector **u** represents a force, then $\text{proj}_{\mathbf{v}}\mathbf{u}$ represents the effect of that force acting in the direction of **v**. For example, if you push a box uphill (in the direction **v**) with a force **u** (Figure 1.3.7), the effective force is a component push in the direction of **v**, $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Real-World Example 6 Use a Vector Projection to Find a Force

CARS A 3,000-pound car sits on a hill inclined at 30° as shown. Ignoring the force of friction, what force is required to keep the car from rolling down the hill?



The weight of the car is the force exerted due to gravity, $\mathbf{F} = \langle 0, -3000\rangle$. To find the force $-\mathbf{w}_1$ required to keep the car from rolling down the hill, project **F** onto a unit vector **v** in the direction of the side of the hill.

Step 1 Find a unit vector **v** in the direction of the hill.

$\mathbf{v} = \langle |\mathbf{v}| \cos\theta, |\mathbf{v}| \sin\theta\rangle$ — Component form of v in terms of |v| and θ

$= \langle 1(\cos 30°), 1(\sin 30°)\rangle$ or $\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$ — $|\mathbf{v}| = 1$ and $\theta = 30°$

Step 2 Find $\mathbf{w}_1$, the projection of **F** onto unit vector **v**, $\text{proj}_{\mathbf{v}}\mathbf{F}$.

$\text{proj}_{\mathbf{v}}\mathbf{F} = \left(\frac{\mathbf{F} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$ — Projection of F onto v

$= (\mathbf{F} \cdot \mathbf{v})\mathbf{v}$ — Since v is a unit vector, $|\mathbf{v}| = 1$. Simplify.

$= \left(\langle 0, -3000\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle\right)\mathbf{v}$ — $\mathbf{F} = \langle 0, -3000\rangle$ and $\mathbf{v} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$

$= -1500\mathbf{v}$ — Find the dot product.

The force required is $-\mathbf{w}_1 = -(-1500\,\mathbf{v})$ or $1500\mathbf{v}$. Since **v** is a unit vector, this means that this force has a magnitude of 1500 pounds and is in the direction of the side of the hill.

GuidedPractice

6. **SLEDDING** Nisreen sits on a sled on the side of a hill inclined at 60°. What force is required to keep the sled from sliding down the hill if the weight of Nisreen and the sled is 125 kilograms?
about 108.3 kg

Another application of vector projection is the calculation of the work done by a force. Consider a constant force **F** acting on an object to move it from point A to point B as shown in Figure 1.3.8. If **F** is parallel to $\overrightarrow{AB}$, then the **work** W done by **F** is the magnitude of the force times the distance from A to B or $W = |\mathbf{F}||\overrightarrow{AB}|$.

Figure 1.3.8

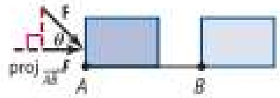


Figure 1.3.9

To calculate the work done by a constant force **F** in *any* direction to move an object from point A to B, as shown in Figure 1.3.9 you can use the vector projection of **F** onto $\overrightarrow{AB}$.

$W = |\text{proj}_{\overrightarrow{AB}}\mathbf{F}|\,|\overrightarrow{AB}|$ — Projection formula for work

$= |\mathbf{F}|\,(\cos\theta)\,|\overrightarrow{AB}|$ — $\cos\theta = \frac{|\text{proj}_{\overrightarrow{AB}}\mathbf{F}|}{|\mathbf{F}|}$, so $|\text{proj}_{\overrightarrow{AB}}\mathbf{F}| = |\mathbf{F}|\cos\theta$.

$= \mathbf{F} \cdot \overrightarrow{AB}$ — $\cos\theta = \frac{\mathbf{F}\cdot\overrightarrow{AB}}{|\mathbf{F}|\,|\overrightarrow{AB}|}$, so $|\mathbf{F}|\,|\overrightarrow{AB}|\cos\theta = \mathbf{F}\cdot\overrightarrow{AB}$.

Therefore, this work can be calculated by finding the dot product of the constant force **F** and the directed distance $\overrightarrow{AB}$.

StudyTip

Units for Work Work is measured in foot-pounds in the customary system of measurement and in newton-meters (N-m) or joules (J) in the metric system.

Real-World Example 7 Calculate Work

ROADSIDE ASSISTANCE A person pushes a car with a constant force of 120 newtons at a constant angle of 45° as shown. Find the work done in joules moving the car 10 meters.

Method 1 Use the projection formula for work.

The magnitude of the projection of **F** onto $\overrightarrow{AB}$ is $|\mathbf{F}|\cos\theta = 120\cos 45°$. The magnitude of the directed distance $\overrightarrow{AB}$ is 10.

$W = |\text{proj}_{\overrightarrow{AB}}\mathbf{F}||\overrightarrow{AB}|$ — Projection formula for work

$= (120\cos 45°)(10)$ or about 848.5 — Substitution

Method 2 Use the dot product formula for work.

The component form of the force vector **F** in terms of magnitude and direction angle given is $\langle 120\cos(-45°), 120\sin(-45°)\rangle$. The component form of the directed distance the car is moved is $\langle 10, 0\rangle$.

$W = \mathbf{F}\cdot\overrightarrow{AB}$ — Dot product formula for work

$= \langle 120\cos(-45°), 120\sin(-45°)\rangle \cdot \langle 10, 0\rangle$ — Substitution

$= [120\cos(-45°)](10)$ or about 848.5 — Dot product

Therefore, the person does about 848.5 joules of work pushing the car.

GuidedPractice

7. **CLEANING** Faris is pushing a vacuum cleaner with a force of 375 newtons. The handle of the vacuum cleaner makes a 60° angle with the floor. How much work in newton-meters does he do if he pushes the vacuum cleaner 2 meters? **375 N-m**

Exercises

Find the dot product of u and v. Then determine if u and v are orthogonal. (Example 1)

1. $\mathbf{u} = \langle 3, -5\rangle, \mathbf{v} = \langle 6, 2\rangle$ **8; not orthogonal**
2. $\mathbf{u} = \langle -10, -16\rangle, \mathbf{v} = \langle -8, 5\rangle$ **0; orthogonal**
3. $\mathbf{u} = \langle 9, -3\rangle, \mathbf{v} = \langle 1, 3\rangle$ **0; orthogonal**
4. $\mathbf{u} = \langle 4, -4\rangle, \mathbf{v} = \langle 7, 5\rangle$ **8; not orthogonal**
5. $\mathbf{u} = \langle 1, -4\rangle, \mathbf{v} = \langle 2, 8\rangle$ **−30; not orthogonal**
6. $\mathbf{u} = 11\mathbf{i} + 7\mathbf{j}; \mathbf{v} = -7\mathbf{i} + 11\mathbf{j}$ **0; orthogonal**
7. $\mathbf{u} = \langle -4, 6\rangle, \mathbf{v} = \langle -5, -2\rangle$ **8; not orthogonal**
8. $\mathbf{u} = 8\mathbf{i} + 6\mathbf{j}; \mathbf{v} = -\mathbf{i} + 2\mathbf{j}$ **4; not orthogonal**

9. **SPORTING GOODS** The vector $\mathbf{u} = \langle 406, 297\rangle$ gives the numbers of men's basketballs and women's basketballs, respectively, in stock at a sporting goods store. The vector $\mathbf{v} = \langle 27.5, 15\rangle$ gives the prices in dollars of the two types of basketballs, respectively. (Example 1)
   a. Find the dot product $\mathbf{u} \cdot \mathbf{v}$. **15,620**
   b. Interpret the result in the context of the problem. **The total revenue that can be made by selling all of the basketballs is $15,620.**

Use the dot product to find the magnitude of the given vector. (Example 2)

10. $\mathbf{m} = \langle -3, 11\rangle$ $\sqrt{130} \approx 1.4$
11. $\mathbf{r} = \langle -9, -4\rangle$ $\sqrt{97} \approx 9.8$
12. $\mathbf{n} = \langle 6, 12\rangle$ $6\sqrt{5} \approx 13.4$
13. $\mathbf{v} = \langle 1, -18\rangle$ $5\sqrt{13} \approx 18.0$
14. $\mathbf{p} = \langle -7, -2\rangle$ $\sqrt{53} \approx 7.3$
15. $\mathbf{t} = \langle 23, -16\rangle$ $\sqrt{785} \approx 28.0$

Find the angle θ between u and v to the nearest tenth of a degree. (Example 3)

16. $\mathbf{u} = \langle 0, -5\rangle, \mathbf{v} = \langle 1, -4\rangle$ **14.0°**
17. $\mathbf{u} = \langle 7, 10\rangle, \mathbf{v} = \langle 4, -4\rangle$ **100.0°**
18. $\mathbf{u} = \langle -2, 4\rangle, \mathbf{v} = \langle 2, -10\rangle$ **164.7°**
19. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}, \mathbf{v} = -4\mathbf{i} - 2\mathbf{j}$ **82.9°**
20. $\mathbf{u} = \langle -9, 0\rangle, \mathbf{v} = \langle -1, -1\rangle$ **45.0°**
21. $\mathbf{u} = -\mathbf{i} - 3\mathbf{j}, \mathbf{v} = -7\mathbf{i} - 3\mathbf{j}$ **48.4°**
22. $\mathbf{u} = \langle 6, 0\rangle, \mathbf{v} = \langle -10, 8\rangle$ **141.3°**
23. $\mathbf{u} = -10\mathbf{i} + \mathbf{j}, \mathbf{v} = 10\mathbf{i} - 5\mathbf{j}$ **159.1°**

24. **CAMPING** Omar and Ali set off from their campsite to search for firewood. The path that Omar takes can be represented by $\mathbf{u} = \langle 3, -5\rangle$. The path that Ali takes can be represented by $\mathbf{v} = \langle -7, 6\rangle$. Find the angle between the pair of vectors. (Example 3) **about 161.6°**

Find the projection of u onto v. Then write u as the sum of two orthogonal vectors, one of which is the projection of u onto v. (Examples 4 and 5) **25–32. See Answer Appendix.**

25. $\mathbf{u} = 3\mathbf{i} + 6\mathbf{j}, \mathbf{v} = -5\mathbf{i} + 2\mathbf{j}$
26. $\mathbf{u} = \langle 5, 7\rangle, \mathbf{v} = \langle -4, 4\rangle$
27. $\mathbf{u} = \langle 8, 2\rangle, \mathbf{v} = \langle -4, 1\rangle$
28. $\mathbf{u} = 6\mathbf{i} + \mathbf{j}, \mathbf{v} = -3\mathbf{i} + 9\mathbf{j}$
29. $\mathbf{u} = \langle 2, 4\rangle, \mathbf{v} = \langle -3, 8\rangle$
30. $\mathbf{u} = \langle -5, 9\rangle, \mathbf{v} = \langle 6, 4\rangle$
31. $\mathbf{u} = 5\mathbf{i} - 8\mathbf{j}, \mathbf{v} = 6\mathbf{i} - 4\mathbf{j}$
32. $\mathbf{u} = -2\mathbf{i} - 5\mathbf{j}, \mathbf{v} = 9\mathbf{i} + 7\mathbf{j}$

33. **WAGON** Eissa is pulling his sister in a wagon up a small slope at an incline of 15°. If the combined weight of Eissa's sister and the wagon is 344 newtons, what force is required to keep her from rolling down the slope? (Example 6) **about 89 N**

34. **SLIDE** Najla is going down a slide but stops herself when she notices that another student is lying hurt at the bottom of the slide. What force is required to keep her from sliding down the slide if the incline is 53° and she weighs 273 N? (Example 6) **about 218 N**

35. **PHYSICS** Ali is pushing a construction barrel up a ramp 1.5 meters long into the back of a truck. He is using a force of 534 newtons and the ramp is 25° from the horizontal. How much work in joules is Ali doing? (Example 7) **801 J**

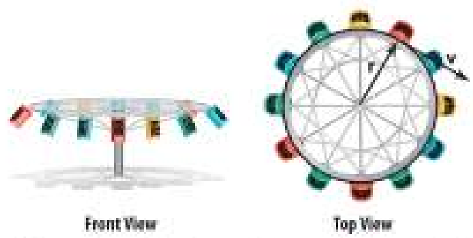
36. **SHOPPING** Suha is pushing a shopping cart with a force of 125 newtons at a downward angle, or angle of depression, of 52°. How much work in joules would Suha do if she pushed the shopping cart 200 meters? (Example 7) **about 15,391.5 J**

B

Find a vector orthogonal to each vector. **37–40. Sample answers given.**

37. $\langle -2, -8\rangle$ $\langle -12, 3\rangle$
38. $\langle 3, 5\rangle$ $\langle 10, -6\rangle$
39. $\langle 7, -4\rangle$ $\langle 8, 14\rangle$
40. $\langle -1, 6\rangle$ $\langle 6, 1\rangle$

41. **RIDES** For a circular amusement park ride, the position vector **r** is perpendicular to the tangent velocity vector **v** at any point on the circle, as shown below.



   a. If the radius of the ride is 20 feet and the speed of the ride is constant at 40 feet per second, write the component forms of the position vector **r** and the tangent velocity vector **v** when **r** is at a directed angle of 35°. $\langle 16.38, 11.47\rangle, \langle 22.94, -32.77\rangle$
   b. What method can be used to prove that the position vector and the velocity vector that you developed in part **a** are perpendicular? Show that the two vectors are perpendicular. **dot product;** $(20\cos 35°)(40\cos 55°) + (20\sin 35°)(-40\sin 55°) = 0$

Given v and u • v, find u. **42–45. Sample answers given.**

42. $\mathbf{v} = \langle 3, -6 \rangle$, $\mathbf{u} \cdot \mathbf{v} = 33$ **$\mathbf{u} = \langle 5, -3 \rangle$**

43. $\mathbf{v} = \langle 4, 6 \rangle$, $\mathbf{u} \cdot \mathbf{v} = 38$ **$\mathbf{u} = \langle -1, 7 \rangle$**

44. $\mathbf{v} = \langle -5, -1 \rangle$, $\mathbf{u} \cdot \mathbf{v} = -8$ **$\mathbf{u} = \langle 1, 3 \rangle$**

45. $\mathbf{v} = \langle -2, 7 \rangle$, $\mathbf{u} \cdot \mathbf{v} = -43$ **$\mathbf{u} = \langle 4, -5 \rangle$**

46. **SCHOOL** A student rolls her backpack from her Chemistry class to her Math class using a force of 175 newtons.

a. If she exerts 3060 joules to pull her backpack 31 meters, what is the angle of her force? **about 55.7°**

b. If she exerts 1315 joules at an angle of 60°, how far did she pull her backpack? **about 15 m**

Determine whether each pair of vectors are *parallel*, *perpendicular*, or *neither*. Explain your reasoning. **47–50. See margin for explanations.**

47. $\mathbf{u} = \left\langle -\frac{2}{3}, \frac{3}{4} \right\rangle$, $\mathbf{v} = \langle 9, 8 \rangle$ **perpendicular**

48. $\mathbf{u} = \langle -1, -4 \rangle$, $\mathbf{v} = \langle 3, 6 \rangle$ **neither**

49. $\mathbf{u} = \langle 5, 7 \rangle$, $\mathbf{v} = \langle -15, -21 \rangle$ **parallel**

50. $\mathbf{u} = \langle \sec\theta, \csc\theta \rangle$, $\mathbf{v} = \langle \csc\theta, -\sec\theta \rangle$ **perpendicular**

Find the angle between the two vectors in radians.

51. $\mathbf{u} = \cos\left(\frac{\pi}{3}\right)\mathbf{i} + \sin\left(\frac{\pi}{3}\right)\mathbf{j}$, $\mathbf{v} = \cos\left(\frac{3\pi}{4}\right)\mathbf{i} + \sin\left(\frac{3\pi}{4}\right)\mathbf{j}$ **$\frac{5\pi}{12}$**

52. $\mathbf{u} = \cos\left(\frac{7\pi}{6}\right)\mathbf{i} + \sin\left(\frac{7\pi}{6}\right)\mathbf{j}$, $\mathbf{v} = \cos\left(\frac{5\pi}{4}\right)\mathbf{i} + \sin\left(\frac{5\pi}{4}\right)\mathbf{j}$ **$\frac{\pi}{12}$**

53. $\mathbf{u} = \cos\left(\frac{\pi}{6}\right)\mathbf{i} + \sin\left(\frac{\pi}{6}\right)\mathbf{j}$, $\mathbf{v} = \cos\left(\frac{2\pi}{3}\right)\mathbf{i} + \sin\left(\frac{2\pi}{3}\right)\mathbf{j}$ **$\frac{\pi}{2}$**

54. $\mathbf{u} = \cos\left(\frac{\pi}{4}\right)\mathbf{i} + \sin\left(\frac{\pi}{4}\right)\mathbf{j}$, $\mathbf{v} = \cos\left(\frac{5\pi}{6}\right)\mathbf{i} + \sin\left(\frac{5\pi}{6}\right)\mathbf{j}$ **$\frac{7\pi}{12}$**

55. **WORK** Adnan lifts his nephew, who weighs 16 kilograms, a distance of 0.9 meter. The force of weight in newtons can be calculated using $F = mg$, where m is the mass in kilograms and g is 9.8 meters per second squared. How much work did Adnan do to lift his nephew? **about 141.1 J**

The vertices of a triangle on the coordinate plane are given. Find the measures of the angles of each triangle using vectors. Round to the nearest tenth of a degree.

56. (2, 3), (4, 7), (8, 1) **37.9°, 60.3°, 81.9°**

57. (−3, −2), (−3, −7), (3, −7) **39.8°, 50.2°, 90°**

58. (−4, −3), (−8, −2), (2, 1) **17.0°, 30.7°, 132.3°**

59. (1, 5), (4, −3), (−4, 0) **48.9°, 65.6°, 65.6°**

Given u, |v|, and θ, the angle between u and v, find possible values of v. Round to the nearest hundredth.

60. $\mathbf{u} = \langle 4, -2 \rangle$, $|\mathbf{v}| = 10$, 45° **⟨9.49, 3.17⟩ or ⟨3.16, −9.49⟩**

61. $\mathbf{u} = \langle 3, 4 \rangle$, $|\mathbf{v}| = \sqrt{29}$, 121° **⟨−5.36, 0.55⟩ or ⟨2.03, −4.99⟩**

62. $\mathbf{u} = \langle -1, -6 \rangle$, $|\mathbf{v}| = 7$, 96° **⟨−6.75, 1.87⟩ or ⟨6.99, −0.42⟩**

63. $\mathbf{u} = \langle -2, 5 \rangle$, $|\mathbf{v}| = 12$, 27° **⟨−9.03, 7.90⟩ or ⟨1.09, 11.95⟩**

64. **CARS** A car is stationary on a 9° incline. Assuming that the only forces acting on the car are gravity and the 275 newton force applied by the brakes, about how much does the car weigh? **about 1757.9 kg**

H.O.T. Problems Use Higher-Order Thinking Skills

65. **REASONING** Determine whether the statement below is *true* or *false*. Explain.

If |**d**|, |**e**|, and |**f**| form a Pythagorean triple, and the angles between **d** and **e** and between **e** and **f** are acute, then the angle between **d** and **f** must be a right angle. Explain your reasoning. **False; see margin for explanation.**

66. **ERROR ANALYSIS** Mahmoud and Mohammad are studying the properties of the dot product. Mahmoud concludes that the dot product is associative because it is commutative; that is, $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$. Mohammad disagrees. Is either of them correct? Explain your reasoning. **Mohammad; Sample answer: u • v is a scalar, so (u • v) • w is undefined.**

67. **REASONING** Determine whether the statement below is *true* or *false*.

If **a** and **b** are both orthogonal to **v** in the plane, then **a** and **b** are parallel. Explain your reasoning. **False; see margin for explanation.**

68. **CHALLENGE** If **u** and **v** are perpendicular, what is the projection of **u** onto **v**? **0**

69. **PROOF** Show that if the angle between vectors **u** and **v** is 90°, $\mathbf{u} \cdot \mathbf{v} = 0$ using the formula for the angle between two nonzero vectors. **See Answer Appendix.**

PROOF **Prove each dot product property. Let $\mathbf{u} = \langle u_1, u_2 \rangle$, $\mathbf{v} = \langle v_1, v_2 \rangle$, and $\mathbf{w} = \langle w_1, w_2 \rangle$.** **70–72. See Answer Appendix.**

70. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

71. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

72. $k(\mathbf{u} \cdot \mathbf{v}) = k\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot k\mathbf{v}$

73. **WRITING IN MATH** Explain how to find the dot product of two nonzero vectors. **See margin.**

Spiral Review

Find each of the following for $\mathbf{a} = \langle 10, 1 \rangle$, $\mathbf{b} = \langle -5, 2.8 \rangle$, and $\mathbf{c} = \left\langle \frac{3}{4}, -9 \right\rangle$.

74. $\mathbf{b} - \mathbf{a} + 4\mathbf{c}$ $\langle -12, -34.2 \rangle$

75. $\mathbf{c} - 3\mathbf{a} + \mathbf{b}$ $\left\langle -\frac{137}{4}, -9.2 \right\rangle$

76. $2\mathbf{a} - 4\mathbf{b} + \mathbf{c}$ $\left\langle \frac{163}{4}, -18.2 \right\rangle$

77. GOLF Yousif drives a golf ball with a velocity of 62.5 meters per second at an angle of 32° with the ground. On the same hole, Saeed drives a golf ball with a velocity of 57.9 meters per second at an angle of 41°. Find the magnitudes of the horizontal and vertical components for each force. **Yousif: about 52.9 m/s and about 33.1 m/s; Saeed: about 43.7 m/s and about 38 m/s**

Graph the hyperbola given by each equation. **78–80. See margin.**

78. $\frac{(x-5)^2}{48} - \frac{y^2}{5} = 1$

79. $\frac{x^2}{81} - \frac{y^2}{49} = 1$

80. $\frac{y^2}{36} - \frac{x^2}{4} = 1$

Find the exact value of each expression, if it exists.

81. $\arcsin\left(\sin \frac{\pi}{6}\right)$ $\frac{\pi}{6}$

82. $\arctan\left(\tan \frac{1}{2}\right)$ $\frac{1}{2}$

83. $\sin\left(\cos^{-1} \frac{3}{4}\right)$ $\frac{\sqrt{7}}{4}$

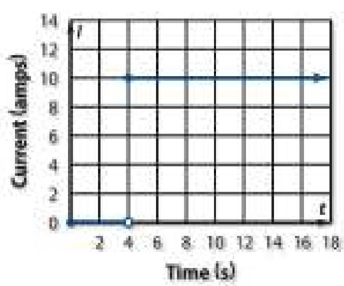
Solve each equation.

84. $\log_{12}(x^3 + 2) = \log_{12} 127$ **5**

85. $\log_2 x = \log_2 6 + \log_2 (x - 5)$ **6**

86. $e^{5x-4} = 70$ $\frac{\ln 70 + 4}{5} \approx 1.65$

87. ELECTRICITY A simple electric circuit contains only a power supply and a resistor. When the power supply is off, there is no current in the circuit. When the power supply is turned on, the current almost instantly becomes a constant value. This situation can be modeled by a graph like the one shown at the right. I represents current in amps, and t represents time in seconds.



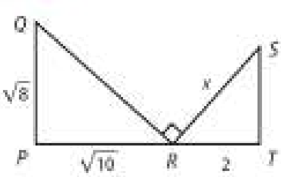
a. At what t-value is this function discontinuous? $t = 4$

b. When was the power supply turned on? **when $t = 4$**

c. If the person who turned on the power supply left and came back hours later, what would he or she measure the current in the circuit to be? **10 amps**

Skills Review for Standardized Tests

88. SAT/ACT In the figure below, $\triangle PQR \sim \triangle TRS$. What is the value of x? **C**



A $\sqrt{2}$ B $\sqrt{5}$ C 3   D $3\sqrt{2}$ E 6

89. REVIEW Consider $C(-9, 2)$ and $D(-4, -3)$. Which of the following is the component form and magnitude of $\overrightarrow{CD}$? **F**

F $\langle 5, -5 \rangle$, $5\sqrt{2}$

G $\langle 5, -5 \rangle$, $6\sqrt{2}$

H $\langle 6, -5 \rangle$, $5\sqrt{2}$

J $\langle 6, -6 \rangle$, $6\sqrt{2}$

90. A snow sled is pulled by exerting a force of 25 newtons on a rope that makes a 20° angle with the horizontal, as shown in the figure. What is the approximate work done in pulling the sled 50 meters? **C**

A 428 N·m

B 1093 N·m

C 1175 N·m

D 1250 N·m

91. REVIEW If $\mathbf{s} = \langle 4, -3 \rangle$ $\mathbf{t} = \langle -6, 2 \rangle$, which of the following represents $\mathbf{t} - 2\mathbf{s}$? **H**

F $\langle 14, 8 \rangle$

G $\langle 14, 6 \rangle$

H $\langle -14, 8 \rangle$

J $\langle -14, -8 \rangle$

CHAPTER 1

Mid-Chapter Quiz

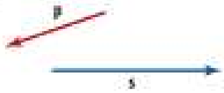
Lessons 1-1 through 1-3

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant in centimeters and its direction relative to the horizontal. (Lesson 1-1)

1.

2.

3.



4.

5. **SLEDDING** Ali pulls a sled through the snow with a force of 50 newtons at an angle of 35° with the horizontal. Find the magnitude of the horizontal and vertical components of the force. (Lesson 1-1)

6. Draw a vector diagram of $\frac{1}{2}\mathbf{c} - 3\mathbf{d}$. (Lesson 1-1)

Let $\overrightarrow{BC}$ be the vector with the given initial and terminal points. Write $\overrightarrow{BC}$ as a linear combination of the vectors i and j. (Lesson 1-2)

7. $B(3, -1)$, $C(4, -7)$

8. $B(10, -6)$, $C(-8, 2)$

9. $B(1, 12)$, $C(-2, -9)$

10. $B(4, -10)$, $C(4, -10)$

11. **MULTIPLE CHOICE** Which of the following is the component form of $\overrightarrow{AB}$ with initial point $A(-5, 3)$ and terminal point $B(2, -1)$? (Lesson 1-2)

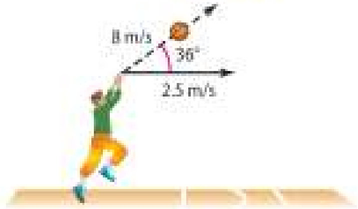
A $\langle 4, -1 \rangle$

B $\langle 7, -4 \rangle$

C $\langle 7, 4 \rangle$

D $\langle -6, 4 \rangle$

12. **BASKETBALL** With time running out in a game, Maysoun runs towards the basket at a speed of 2.5 meters per second and from half-court, launches a shot at a speed of 8 meters per second at an angle of 36° to the horizontal. (Lesson 1-2)



a. Write the component form of the vectors representing Maysoun's velocity and the path of the ball.

b. What is the resultant speed and direction of the shot?

Find the component form and magnitude of the vector with each initial and terminal point. (Lesson 1-2)

13. $A(-4, 2)$, $B(3, 6)$

14. $Q(1, -5)$, $R(-7, 8)$

15. $X(-3, -5)$, $Y(2, 5)$

16. $P(9, -2)$, $S(2, -5)$

Find the angle θ between u and v to the nearest tenth of a degree. (Lesson 1-3)

17. $\mathbf{u} = \langle 9, -4 \rangle$, $\mathbf{v} = \langle -1, -2 \rangle$

18. $\mathbf{u} = \langle 5, 2 \rangle$, $\mathbf{v} = \langle -4, 10 \rangle$

19. $\mathbf{u} = \langle 8, 4 \rangle$, $\mathbf{v} = \langle -2, 4 \rangle$

20. $\mathbf{u} = \langle 2, -2 \rangle$, $\mathbf{v} = \langle 3, 8 \rangle$

21. **MULTIPLE CHOICE** If $\mathbf{u} = \langle 2, 3 \rangle$, $\mathbf{v} = \langle -1, 4 \rangle$, and $\mathbf{w} = \langle 8, -5 \rangle$, find $(\mathbf{u} \cdot \mathbf{v}) + (\mathbf{w} \cdot \mathbf{v})$. (Lesson 1-3)

F −18

G −2

H 15

J 38

Find the dot product of u and v. Then determine if u and v are orthogonal. (Lesson 1-3)

22. $\langle 2, -5 \rangle \cdot \langle 4, 2 \rangle$

23. $\langle 4, -3 \rangle \cdot \langle 7, 4 \rangle$

24. $\langle 1, -6 \rangle \cdot \langle 5, 8 \rangle$

25. $\langle 3, -6 \rangle \cdot \langle 10, 5 \rangle$

26. **WAGON** Hamad uses a wagon to carry newspapers for his paper route. He is pulling the wagon with a force of 25 newtons at an angle of 30° with the horizontal. (Lesson 1-3)

a. How much work in joules is Hamad doing when he pulls the wagon 150 meters?

b. If the handle makes an angle of 40° with the ground and he pulls the wagon with the same distance and force, is Hamad doing more or less work? Explain your answer.

Find the projection of u onto v. Then write u as the sum of two orthogonal vectors, one of which is the projection of u onto v. (Lesson 1-3)

27. $\mathbf{u} = \langle 7, -3 \rangle$, $\mathbf{v} = \langle 2, 5 \rangle$

28. $\mathbf{u} = \langle 2, 4 \rangle$, $\mathbf{v} = \langle 1, 3 \rangle$

29. $\mathbf{u} = \langle 3, 8 \rangle$, $\mathbf{v} = \langle -9, 2 \rangle$

30. $\mathbf{u} = \langle -1, 4 \rangle$, $\mathbf{v} = \langle -6, 1 \rangle$

LESSON 1-4 Vectors in Three-Dimensional Space

Then

- You represented vectors both geometrically and algebraically in two-dimensions.

Now

1. Plot points and vectors in the three-dimensional coordinate system.
2. Express algebraically and operate with vectors in space.

Why?

- The direction of a rocket after takeoff is given in terms of both a two-dimensional bearing and an angle relative to the horizontal. Since directed distance, velocities, and forces are not restricted to the plane, the concept of vectors must be extended from two- to three-dimensional space.

NewVocabulary
three-dimensional coordinate system
z-axis
octant
ordered triple

1 Coordinates in Three Dimensions

The Cartesian plane is a two-dimensional coordinate system made up of the x- and y-axes that allows you to identify and locate points in a plane. We need a **three-dimensional coordinate system** to represent a point in space.

Start with the xy-plane and position it so that it gives the appearance of depth (Figure 1.4.1). Then add a third axis called the **z-axis** that passes through the origin and is perpendicular to both the x- and y-axes (Figure 1.4.2). The additional axis divides space into eight regions called **octants**. To help visualize the first octant, look at the corner of a room (Figure 1.4.3). The floor represents the xy-plane, and the walls represent the xz- and yz-planes.

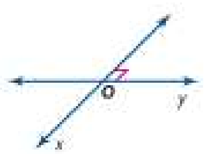


Figure 1.4.1

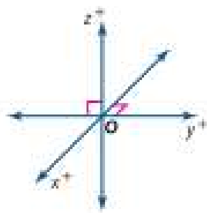


Figure 1.4.2

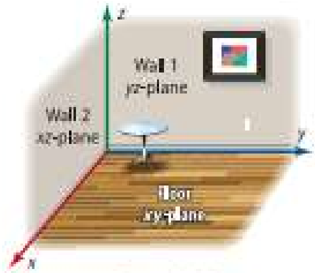


Figure 1.4.3

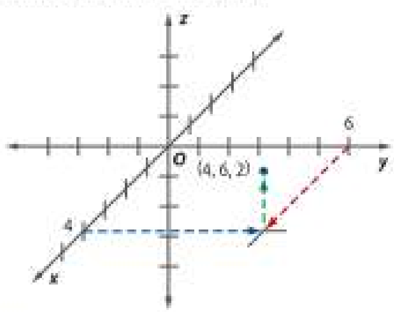
A point in space is represented by an **ordered triple** of real numbers (x, y, z). To plot such a point, first locate the point (x, y) in the xy-plane and move up or down parallel to the z-axis according to the directed distance given by z.

Example 1 Locate a Point in Space

Plot each point in a three-dimensional coordinate system.

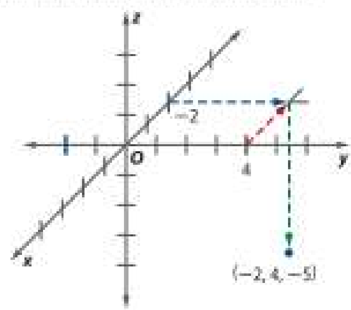
a. **(4, 6, 2)**

Locate (4, 6) in the xy-plane and mark it with a cross. Then plot a point **2** units up from this location parallel to the z-axis.



b. **(−2, 4, −5)**

Locate (−2, 4) in the xy-plane and mark it with a cross. Then plot a point **5** units down from this location parallel to the z-axis.



GuidedPractice

1A. (−3, −4, 2) **1B.** (3, 2, −3) **1C.** (5, −4, −1)

Finding the distance between points and the midpoint of a segment in space is similar to finding distance and a midpoint in the coordinate plane.

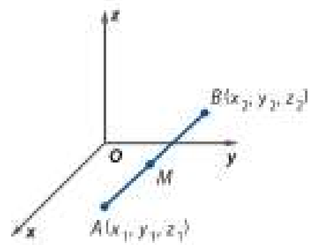
KeyConcept Distance and Midpoint Formulas in Space

The distance between points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The midpoint M of $\overline{AB}$ is given by

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$



You will prove these formulas in Exercise 66.

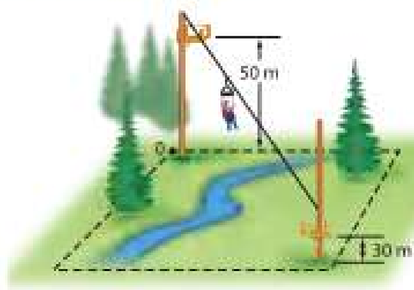
Real-WorldLink

A tour at Monteverde, Costa Rica, allows visitors to view nature from a system of trails, suspension bridges, and zip-lines. The zip-lines allow the guests to view the surroundings from as much as 456 feet above the ground.

Source: Monteverde Info

Real-World Example 2 Distance and Midpoint of Points in Space

ZIP-LINE A tour of the Sierra Madre Mountains in the Philippines lets guests experience nature by zip-lining from one platform to another over the scenic surroundings. Two platforms that are connected by a zip-line are represented by the coordinates (10, 12, 50) and (70, 92, 30), where the coordinates are given in meters.



a. Find the length of the zip-line needed to connect the two platforms.

Use the Distance Formula for points in space.

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ — Distance Formula

$= \sqrt{(70 - 10)^2 + (92 - 12)^2 + (30 - 50)^2}$ — $(x_1, y_1, z_1) = (10, 12, 50)$ and $(x_2, y_2, z_2) = (70, 92, 30)$

≈ 101.98 — Simplify.

The zip-line needs to be about 102 meters long to connect the two towers.

b. An additional platform is to be built halfway between the existing platforms. Find the coordinates of the new platform.

Use the Midpoint Formula for points in space.

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$ — Midpoint Formula

$= \left(\frac{10 + 70}{2}, \frac{12 + 92}{2}, \frac{50 + 30}{2}\right)$ or $(40, 52, 40)$ — $(x_1, y_1, z_1) = (10, 12, 50)$ and $(x_2, y_2, z_2) = (70, 92, 30)$

The coordinates of the new platform will be (40, 52, 40).

GuidedPractice

2. **AIRPLANES** Safety regulations require airplanes to be at least a half a mile apart when in the sky. Two planes are flying above Cleveland with the coordinates (300, 150, 30000) and (450, −250, 28000), where the coordinates are given in feet.
   - **A.** Are the two planes in violation of the safety regulations? Explain.
   - **B.** If a firework was launched and exploded directly in between the two planes, what are the coordinates of the firework explosion?

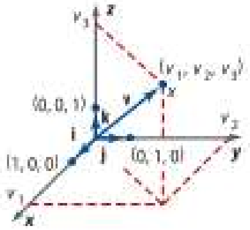


Figure 1.4.4

2 Vectors in Space

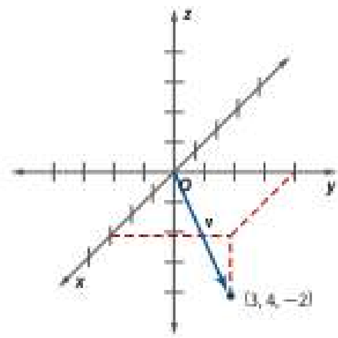
In space, a vector **v** in standard position with a terminal point located at (v_1, v_2, v_3) is denoted by $\langle v_1, v_2, v_3\rangle$. The zero vector is $\mathbf{0} = \langle 0, 0, 0\rangle$, and the standard unit vectors are $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$ as shown in Figure 1.4.4. The component form of **v** can be expressed as a linear combination of these unit vectors, $\langle v_1, v_2, v_3\rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

Example 3 Locate a Vector in Space

Locate and graph $\mathbf{v} = \langle 3, 4, -2\rangle$.

Plot the point $(3, 4, -2)$.

Draw **v** with terminal point at $(3, 4, -2)$.



GuidedPractice

Locate and graph each vector in space.

3A. $\mathbf{u} = \langle -4, 2, -3\rangle$

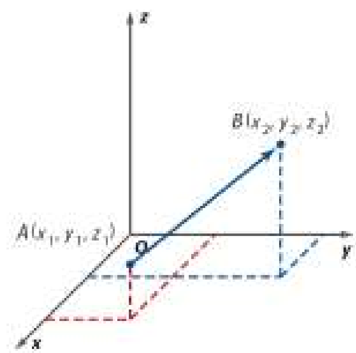
3B. $\mathbf{w} = -\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

As with two-dimensional vectors, to find the component form of the directed line segment from $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$, you subtract the coordinates of its initial point from its terminal point.

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$$

Then $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ or if $\overrightarrow{AB} = \langle a_1, a_2, a_3\rangle$, then $|\overrightarrow{AB}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

A unit vector **u** in the direction of $\overrightarrow{AB}$ is still $\mathbf{u} = \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$.



Example 4 Express Vectors in Space Algebraically

Find the component form and magnitude of $\overrightarrow{AB}$ with initial point $A(-4, -2, 1)$ and terminal point $B(3, 6, -6)$. Then find a unit vector in the direction of $\overrightarrow{AB}$.

$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$ — Component form of vector

$= \langle 3 - (-4), 6 - (-2), -6 - 1\rangle$ or $\langle 7, 8, -7\rangle$ — $(x_1, y_1, z_1) = (-4, -2, 1)$ and $(x_2, y_2, z_2) = (3, 6, -6)$

Using the component form, the magnitude of $\overrightarrow{AB}$ is

$|\overrightarrow{AB}| = \sqrt{7^2 + 8^2 + (-7)^2}$ or $9\sqrt{2}$ — $\overrightarrow{AB} = \langle 7, 8, -7\rangle$

Using this magnitude and component form, find a unit vector **u** in the direction of $\overrightarrow{AB}$.

$\mathbf{u} = \dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$ — Unit vector in the direction of $\overrightarrow{AB}$

$= \dfrac{\langle 7, 8, -7\rangle}{9\sqrt{2}}$ or $\left\langle \dfrac{7\sqrt{2}}{18}, \dfrac{4\sqrt{2}}{9}, -\dfrac{7\sqrt{2}}{18}\right\rangle$ — $\overrightarrow{AB} = \langle 7, 8, -7\rangle$ and $|\overrightarrow{AB}| = 9\sqrt{2}$

GuidedPractice

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points. Then find a unit vector in the direction of $\overrightarrow{AB}$.

4A. $A(-2, -5, -5)$, $B(-1, 4, -2)$

4B. $A(-1, 4, 6)$, $B(3, 3, 8)$

As with vectors in the plane, when vectors in space are in component form or expressed as a linear combination of unit vectors, they can be added, subtracted, or multiplied by a scalar.

KeyConcept Vector Operations in Space

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and any scalar k, then

Vector Addition	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$
Vector Subtraction	$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$
Scalar Multiplication	$k\mathbf{a} = \langle ka_1, ka_2, ka_3 \rangle$

StudyTip

Vector Operations The properties for vector operations in space are the same as those for operations in the plane.

Example 5 Operations with Vectors in Space

Find each of the following for $\mathbf{y} = \langle 3, -6, 2 \rangle$, $\mathbf{w} = \langle -1, 4, -4 \rangle$, and $\mathbf{z} = \langle -2, 0, 5 \rangle$.

a. $4\mathbf{y} + 2\mathbf{z}$

$4\mathbf{y} + 2\mathbf{z} = 4\langle 3, -6, 2 \rangle + 2\langle -2, 0, 5 \rangle$ — Substitute.

$= \langle 12, -24, 8 \rangle + \langle -4, 0, 10 \rangle$ or $\langle 8, -24, 18 \rangle$ — Scalar multiplication and vector addition

b. $2\mathbf{w} - \mathbf{z} + 3\mathbf{y}$

$2\mathbf{w} - \mathbf{z} + 3\mathbf{y} = 2\langle -1, 4, -4 \rangle - \langle -2, 0, 5 \rangle + 3\langle 3, -6, 2 \rangle$ — Substitute.

$= \langle -2, 8, -8 \rangle + \langle 2, 0, -5 \rangle + \langle 9, -18, 6 \rangle$ — Scalar multiplication

$= \langle 9, -10, -7 \rangle$ — Vector addition

GuidedPractice

5A. $4\mathbf{w} - 8\mathbf{z}$

5B. $3\mathbf{y} + 3\mathbf{z} - 6\mathbf{w}$

Real-World Example 6 Use Vectors in Space

ROCKETS After liftoff, a model rocket is headed due north and climbing at an angle of 75° relative to the horizontal at 200 kilometers per hour. If the wind blows from the northwest at 5 kilometers per hour, find a vector for the resultant velocity of the rocket relative to the point of liftoff.

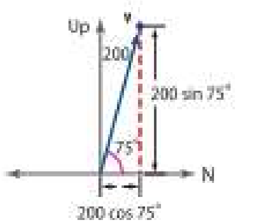


Figure 1.4.5

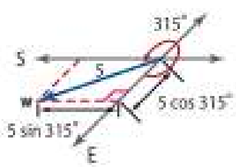
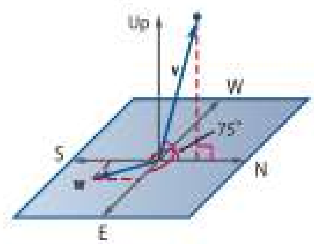


Figure 1.4.6

Let **i** point east, **j** point north, and **k** point up. Vector **v** representing the rocket's velocity and vector **w** representing the wind's velocity are shown. Notice that **w** points toward the southeast, since the wind is blowing *from* the northwest.



Since **v** has a magnitude of 200 and a direction angle of 75°, we can find the component form of **v**, as shown in Figure 1.4.5.

$\mathbf{v} = \langle 0, 200 \cos 75°, 200 \sin 75° \rangle$ or about $\langle 0, 51.8, 193.2 \rangle$

With east as the positive x-axis, **w** has direction angle of 315°. Since $|\mathbf{w}| = 5$, the component form of this vector is $\mathbf{w} = \langle 5 \cos 315°, 5 \sin 315°, 0 \rangle$ or about $\langle 3.5, -3.5, 0 \rangle$, as shown in Figure 1.4.6.

The resultant velocity of the rocket is $\mathbf{v} + \mathbf{w}$.

$\mathbf{v} + \mathbf{w} = \langle 0, 51.8, 193.2 \rangle + \langle 3.5, -3.5, 0 \rangle$

$= \langle 3.5, 48.3, 193.2 \rangle$ or $3.5\mathbf{i} + 48.3\mathbf{j} + 193.2\mathbf{k}$

GuidedPractice

6. AVIATION After takeoff, an airplane is headed east and is climbing at an angle of 18° relative to the horizontal. Its air speed is 250 kilometers per hour. If the wind blows from the northeast at 10 kilometers per hour, find a vector that represents the resultant velocity of the plane relative to the point of takeoff. Let **i** point east, **j** point north, and **k** point up.

Exercises

Plot each point in a three-dimensional coordinate system. (Example 1)

1. $(1, -2, -4)$
2. $(3, 2, 1)$
3. $(-5, -4, -2)$
4. $(-2, -5, 3)$
5. $(-5, 3, 1)$
6. $(2, -2, 3)$
7. $(4, -10, -2)$
8. $(-16, 12, -13)$

Find the length and midpoint of the segment with the given endpoints. (Example 2)

9. $(-4, 10, 4), (1, 0, 9)$
10. $(-6, 6, 3), (-9, -2, -2)$
11. $(6, 1, 10), (-9, -10, -4)$
12. $(8, 3, 4), (-4, -7, 5)$
13. $(-3, 2, 8), (9, 6, 0)$
14. $(-7, 2, -5), (-2, -5, -8)$

15. **VACATION** A family from Wichita, Kansas, is using a GPS device to plan a vacation to Castle Rock, Colorado. According to the device, the coordinates for the family's home are (37.7°, 97.2°, 433 m), and the coordinates to Castle Rock are (39.4°, 104.8°, 1981 m). Determine the longitude, latitude, and altitude of the halfway point between Wichita and Castle Rock. (Example 2)

16. **FIGHTER PILOTS** During a training session, the location of two F-18 fighter jets are represented by the coordinates (675, −121, 19,300) and (−289, 715, 16,100), where the coordinates are given in feet. (Example 2)
    a. Determine the distance between the two jets.
    b. To what location would one of the fighter pilots have to fly the F-18 in order to reduce the distance between the two jets by half?

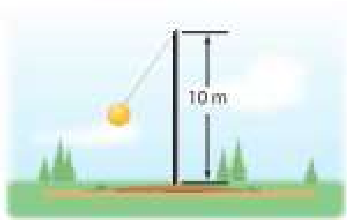
Locate and graph each vector in space. (Example 3)

17. $\mathbf{a} = \langle 0, -4, 4 \rangle$
18. $\mathbf{b} = \langle -3, -3, -2 \rangle$
19. $\mathbf{c} = \langle -1, 3, -4 \rangle$
20. $\mathbf{d} = \langle 4, -2, -3 \rangle$
21. $\mathbf{v} = 6\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}$
22. $\mathbf{w} = -10\mathbf{i} + 5\mathbf{k}$
23. $\mathbf{m} = 7\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}$
24. $\mathbf{n} = \mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points. Then find a unit vector in the direction of $\overrightarrow{AB}$. (Example 4)

25. $A(-5, -5, -9), B(11, -3, -1)$
26. $A(-4, 0, -3), B(-4, -8, 9)$
27. $A(3, 5, 1), B(0, 0, -9)$
28. $A(-3, -7, -12), B(-7, 1, 8)$
29. $A(2, -5, 4), B(1, 3, -6)$
30. $A(8, 12, 7), B(2, -3, 11)$
31. $A(3, 14, -5), B(7, -1, 0)$
32. $A(1, -18, -13), B(21, 14, 29)$
33. $A(-5, 12, 17), B(6, -11, 4)$
34. $A(9, 3, 7), B(-5, -7, 2)$

35. **TETHERBALL** In the game of tetherball, a ball is attached to a 3-meter pole by a length of rope. Two players hit the ball in opposing directions in an attempt to wind the entire length of rope around the pole. To serve, a certain player holds the ball so that its coordinates are (5, 3.6, 4.7) and the coordinates of the end of the rope connected to the pole are (0, 0, 9.8), where the coordinates are given in feet. Determine the magnitude of the vector representing the length of the rope. (Example 4)



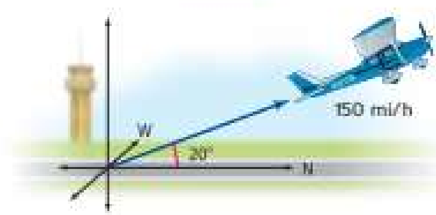
Find each of the following for $\mathbf{a} = \langle -5, -4, 3 \rangle$, $\mathbf{b} = \langle 6, -2, -7 \rangle$, and $\mathbf{c} = \langle -2, 2, 4 \rangle$. (Example 5)

36. $6\mathbf{a} - 7\mathbf{b} + 8\mathbf{c}$
37. $7\mathbf{a} - 5\mathbf{b}$
38. $2\mathbf{a} + 5\mathbf{b} - 9\mathbf{c}$
39. $6\mathbf{b} + 4\mathbf{c} - 4\mathbf{a}$
40. $8\mathbf{a} - 5\mathbf{b} - \mathbf{c}$
41. $-6\mathbf{a} + \mathbf{b} + 7\mathbf{c}$

Find each of the following for $\mathbf{x} = -9\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$, $\mathbf{y} = 6\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$, and $\mathbf{z} = -2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. (Example 5)

42. $7\mathbf{x} + 6\mathbf{y}$
43. $3\mathbf{x} - 5\mathbf{y} + 3\mathbf{z}$
44. $4\mathbf{x} + 3\mathbf{y} + 2\mathbf{z}$
45. $-8\mathbf{x} - 2\mathbf{y} + 5\mathbf{z}$
46. $-6\mathbf{y} - 9\mathbf{z}$
47. $-\mathbf{x} - 4\mathbf{y} - \mathbf{z}$

48. **AIRPLANES** An airplane is taking off headed due north with an air speed of 150 miles per hour at an angle of 20° relative to the horizontal. The wind is blowing with a velocity of 8 miles per hour from the southwest. Find a vector that represents the resultant velocity of the plane relative to the point of takeoff. Let **i** point east, **j** point north, and **k** point up. (Example 6)



49. **TRACK AND FIELD** Maysa throws a javelin due south at a speed of 18 miles per hour and at an angle of 48° relative to the horizontal. If the wind is blowing with a velocity of 12 miles per hour at an angle of S15°E, find a vector that represents the resultant velocity of the javelin. Let **i** point east, **j** point north, and **k** point up. (Example 6)

50. **SUBMARINE** A submarine heading due west dives at a speed of 25 knots and an angle of decline of 55°. The current is moving with a velocity of 4 knots at an angle of S20°W. Find a vector that represents the resultant velocity of the submarine relative to the initial point of the dive. Let **i** point east, **j** point north, and **k** point up. (Example 6)

B

If N is the midpoint of $\overline{MP}$, find P.

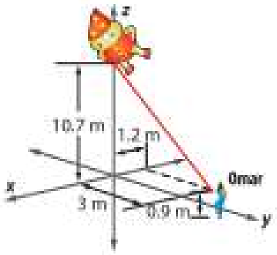
51. $M(3, 4, 5); N\left(\frac{7}{2}, 1, 2\right)$

52. $M(-1, -4, -9); N(-2, 1, -5)$

53. $M(7, 1, 5); N\left(5, -\frac{1}{2}, 6\right)$

54. $M\left(\frac{3}{2}, -5, 9\right); N\left(-2, -\frac{13}{2}, \frac{11}{2}\right)$

55. **VOLUNTEERING** Omar is volunteering to help guide a balloon in a parade. If the balloon is 10.7 meters high and he is holding the tether 0.9 meters above the ground as shown, how long is the tether to the nearest tenth of a meter?



Determine whether the triangle with the given vertices is *isosceles* or *scalene*.

56. $A(3, 1, 2), B(5, -1, 1), C(1, 3, 1)$

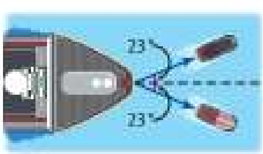
57. $A(4, 3, 4), B(4, 6, 4), C(4, 3, 6)$

58. $A(-1, 4, 3), B(2, 5, 1), C(0, -6, 6)$

59. $A(-2.2, 4.3, 5.6), B(0.7, 9.3, 15.6), C(3.6, 14.3, 5.6)$

60. **TUGBOATS** Two tugboats are pulling a disabled supertanker. One of the tow lines makes an angle 23° west of north and the other makes an angle 23° east of north. Each tug exerts a constant force of 2.5×10^6 newtons depressed 15° below the point where the lines attach to the supertanker. They pull the supertanker two miles due north.

C



a. Write a three-dimensional vector to describe the force from each tugboat.

b. Find the vector that describes the total force on the supertanker.

c. If each tow line is 300 feet long, about how far apart are the tugboats?

61. **SPHERES** Use the distance formula for two points in space to prove that the standard form of the equation of a sphere with center (h, k, ℓ) and radius r is $r^2 = (x - h)^2 + (y - k)^2 + (z - \ell)^2$.

Use the formula from Exercise 61 to write an equation for the sphere with the given center and radius.

62. center $= (-4, -2, 3)$; radius $= 4$

63. center $= (6, 0, -1)$; radius $= \frac{1}{2}$

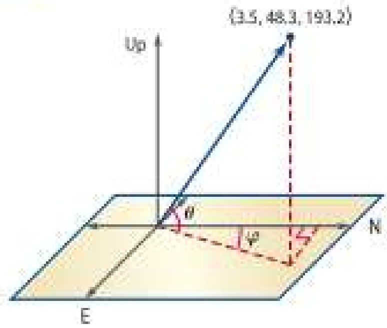
64. center $= (5, -3, 4)$; radius $= \sqrt{3}$

65. center $= (0, 7, -1)$; radius $= 12$

H.O.T. Problems Use Higher-Order Thinking Skills

66. **REASONING** Prove the Distance Formula in Space. (*Hint:* Use the Pythagorean Theorem twice.)

67. **CHALLENGE** Refer to Example 6.

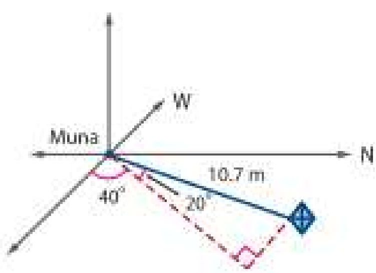


a. Calculate the resultant speed of the rocket.

b. Find the quadrant bearing φ of the rocket.

c. Calculate the resultant angle of incline θ of the rocket relative to the horizontal.

68. **CHALLENGE** Muna is standing in an open field facing N50°E. She is holding a kite on a 10.7-meter string that is flying at a 20° angle with the field. Find the components of the vector from Muna to the kite. (*Hint:* Use trigonometric ratios and two right triangles to find **x**, **y**, and **z**.)



69. **WRITING IN MATH** Describe a situation where it is more reasonable to use a two-dimensional coordinate system and one where it is more reasonable to use a three-dimensional coordinate system.

Spiral Review

Find the projection of u onto v. Then write u as the sum of two orthogonal vectors, one of which is the projection of u onto v.

70. $\mathbf{u} = \langle 6, 8 \rangle, \mathbf{v} = \langle 2, -1 \rangle$

71. $\mathbf{u} = \langle -1, 4 \rangle, \mathbf{v} = \langle 5, 1 \rangle$

72. $\mathbf{u} = \langle 5, 4 \rangle, \mathbf{v} = \langle 4, -2 \rangle$

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points.

73. $A(6, -4), B(-7, -7)$

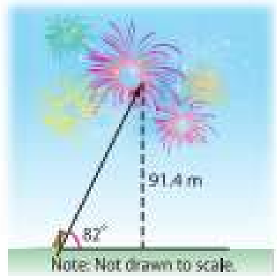
74. $A(-4, -8), B(1, 6)$

75. $A(-5, -12), B(1, 6)$

76. **ENTERTAINMENT** The UAE National Day fireworks at Burj Khalifa are fired at an angle of 82° with the horizontal. The technician firing the shells expects them to explode about 91.4 meters in the air 4.8 seconds after they are fired.

a. Find the initial speed of a shell fired from ground level.

b. Safety barriers will be placed around the launch area to protect the spectators. If the barriers are placed 91.4 meters from the point directly below the explosion of the shells, how far should the barriers be from the point where the fireworks are launched?



77. **CONSTRUCTION** A stone door that was designed as an arch in the shape of a semi-ellipse will have an opening with a height of 3 meters at the center and a width of 8 meters along the base. To sketch an outline of the door, a contractor uses a string tied to two thumbtacks.

a. At what locations should the thumbtacks be placed?

b. What length of string needs to be used? Explain your reasoning.

Solve each equation for all values of θ.

78. $\csc \theta + 2 \cot \theta = 0$

79. $\sec^2 \theta - 9 = 0$

80. $2 \csc \theta - 3 = 5 \sin \theta$

Sketch the graph of each function.

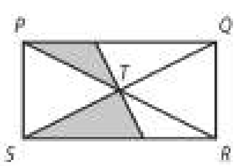
81. $y = \cos^{-1}(x - 2)$

82. $y = \arccos x + 3$

83. $y = \sin^{-1} 3x$

Skills Review for Standardized Tests

84. **SAT/ACT** What percent of the area of rectangle $PQRS$ is shaded?



A 22%
B 25%
C 30%
D $33\frac{1}{3}$%
E 35%

85. **REVIEW** A ship leaving port sails for 75 kilometers in a direction of 35° north of east. At that point, how far north of its starting point is the ship?

F 43 kilometers
G 55 kilometers
H 61 kilometers
J 72 kilometers

86. During a storm, the force of the wind blowing against a skyscraper can be expressed by the vector $\langle 132, 3454, -76 \rangle$, where the force of the wind is measured in newtons. What is the approximate magnitude of this force?

A 3457 N
B 3568 N
C 3692 N
D 3717 N

87. **REVIEW** An airplane is flying due west at a velocity of 100 meters per second. The wind is blowing from the south at 30 meters per second. What is the approximate magnitude of the airplane's resultant velocity?

F 4 m/s
G 95.4 m/s
H 100 m/s
J 104.4 m/s

EXTEND 1-4

Graphing Technology Lab

Vector Transformations with Matrices

Objective

- Use a graphing calculator to transform vectors using matrices.

In Lesson 1-4, you learned that a vector in space can be transformed when written in component form or when expressed as a linear combination. A vector in space can also be transformed when written as a 3 × 1 or 1 × 3 matrix.

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ or } [x \ y \ z]$$

Once in matrix form, a vector can be transformed using matrix-vector multiplication.

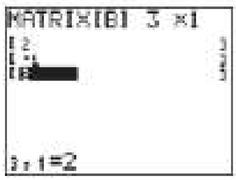
Activity Matrix-Vector Multiplication

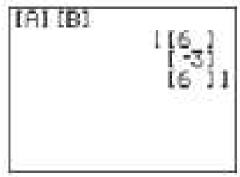
Multiply the vector $\mathbf{B} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ by the transformation matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, and graph both vectors.

Step 1 Write **B** as a matrix.

$$\mathbf{B} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Step 2 Enter **B** and A in a graphing calculator and find $A\mathbf{B}$. Convert to vector form.

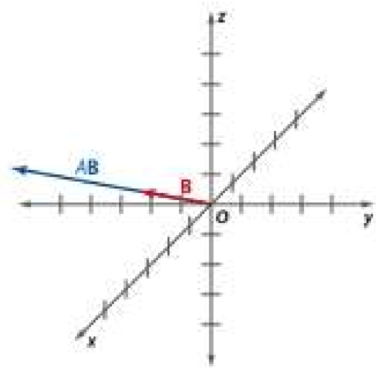




$A\mathbf{B} = 6\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$

Step 3 Graph **B** and $A\mathbf{B}$ on a coordinate plane.

$A\mathbf{B}$ is a dilation of **B** by a factor of 3.



Exercises

Multiply each vector by the transformation matrix. Graph both vectors.

1. $\mathbf{h} = 4\mathbf{i} + \mathbf{j} + 8\mathbf{k}$

   $B = \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$

2. $\mathbf{e} = 5\mathbf{i} + 3\mathbf{j} - 9\mathbf{k}$

   $V = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

3. $\mathbf{f} = \mathbf{i} + 7\mathbf{j} - 3\mathbf{k}$

   $W = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

4. **REASONING** Multiply $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ by the transformation matrix $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, and graph both vectors. Explain the type of transformation that was performed.

LESSON 1-5

Dot and Cross Products of Vectors in Space

Then	Now	Why?
• You found the dot product of two vectors in the plane.	• 1 Find dot products of and angles between vectors in space. 2 Find cross products of vectors in space, and use cross products to find area and volume.	• The tendency of a hinged door to rotate when pushed is affected by the distance between the location of the push and the hinge, the magnitude of the push, and the direction of the push. A quantity called *torque* measures how effectively a force applied to a lever causes rotation about an axis.

NewVocabulary
cross product
torque
parallelepiped
triple scalar product

1 Dot Products in Space

Calculating the dot product of two vectors in space is similar to calculating the dot product of two vectors in a plane. As with vectors in a plane, nonzero vectors in space are perpendicular if and only if their dot product equals zero.

KeyConcept Dot Product and Orthogonal Vectors in Space

The dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$. The vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Example 1 Find the Dot Product to Determine Orthogonal Vectors in Space

Find the dot product of u and v. Then determine if u and v are orthogonal.

a. $\mathbf{u} = \langle -7, 3, -3 \rangle$, $\mathbf{v} = \langle 5, 17, 5 \rangle$

$\mathbf{u} \cdot \mathbf{v} = -7(5) + 3(17) + (-3)(5)$
$= -35 + 51 + (-15)$ or 1

Since $\mathbf{u} \cdot \mathbf{v} \neq 0$, **u** and **v** are not orthogonal.

b. $\mathbf{u} = \langle 3, -3, 3 \rangle$, $\mathbf{v} = \langle 4, 7, 3 \rangle$

$\mathbf{u} \cdot \mathbf{v} = 3(4) + (-3)(7) + 3(3)$
$= 12 + (-21) + 9$ or 0

Since $\mathbf{u} \cdot \mathbf{v} = 0$, **u** and **v** are orthogonal.

GuidedPractice

1A. $\mathbf{u} = \langle 3, -5, 4 \rangle$, $\mathbf{v} = \langle 5, 7, 5 \rangle$

1B. $\mathbf{u} = \langle 4, -2, -3 \rangle$, $\mathbf{v} = \langle 1, 3, -2 \rangle$

As with vectors in a plane, if θ is the angle between nonzero vectors **a** and **b**, then $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$.

Example 2 Angle Between Two Vectors in Space

Find the angle θ between u and v to the nearest tenth of a degree if $\mathbf{u} = \langle 3, 2, -1 \rangle$ and $\mathbf{v} = \langle -4, 3, -2 \rangle$.

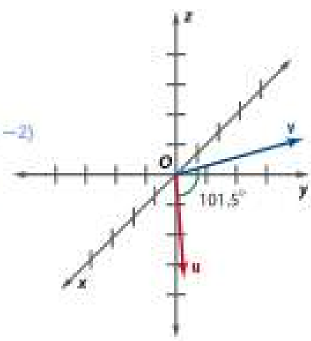
$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$ — Angle between two vectors

$\cos \theta = \frac{\langle 3, 2, -1 \rangle \cdot \langle -4, 3, -2 \rangle}{|\langle 3, 2, -1 \rangle|\,|\langle -4, 3, -2 \rangle|}$ — $\mathbf{u} = \langle 3, 2, -1 \rangle$ and $\mathbf{v} = \langle -4, 3, -2 \rangle$

$\cos \theta = \frac{-4}{\sqrt{14}\sqrt{29}}$ — Evaluate the dot product and magnitudes.

$\theta = \cos^{-1} \frac{-4}{\sqrt{406}}$ or about $101.5°$ — Simplify and solve for θ.

The measure of the angle between **u** and **v** is about 101.5°.

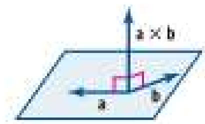


GuidedPractice

2. Find the angle between $\mathbf{u} = -4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 3\mathbf{k}$ to the nearest tenth of a degree.

2 Cross Products

Another important product involving vectors in space is the cross product. Unlike the dot product, the **cross product** of two vectors **a** and **b** in space, denoted **a** × **b** and read *a cross b*, is a vector, not a scalar. The vector **a** × **b** is perpendicular to the plane containing vectors **a** and **b**.



KeyConcept Cross Product of Vectors in Space

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, the cross product of **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

ReviewVocabulary

2 × 2 Determinant The determinant of the 2 × 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$.

If we apply the formula for calculating the determinant of a 3 × 3 matrix to the following *determinant form* involving **i**, **j**, **k**, and the components of **a** and **b**, we arrive at the same formula for **a** × **b**.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

← Put the unit vectors i, j, and k in Row 1.
← Put the components of a in Row 2.
← Put the components of b in Row 3.

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$ Apply the formula for a 3 × 3 determinant.

$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$ Compute each 2 × 2 determinant.

WatchOut!

Cross Product The cross product definition applies only to vectors in three-dimensional space. The cross product is not defined for vectors in the two-dimensional coordinate system.

Example 3 Find the Cross Product of Two Vectors

Find the cross product of u = ⟨3, −2, 1⟩ and v = ⟨−3, 3, 1⟩. Then show that u × v is orthogonal to both u and v.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ -3 & 3 & 1 \end{vmatrix}$$ $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

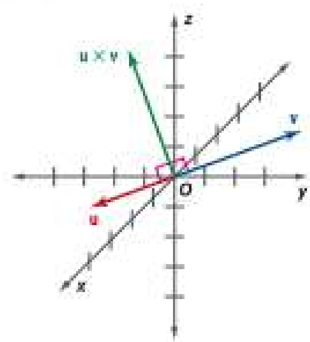
$$= \begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ -3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -2 \\ -3 & 3 \end{vmatrix}\mathbf{k}$$ Determinant of a 3 × 3 matrix

$= (-2 - 3)\mathbf{i} - [3 - (-3)]\mathbf{j} + (9 - 6)\mathbf{k}$ Determinants of 2 × 2 matrices

$= -5\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$ Simplify.

$= \langle -5, -6, 3 \rangle$ Component form

In the graph of **u**, **v**, and **u** × **v**, **u** × **v** is orthogonal to **u** and **u** × **v** is orthogonal to **v**.



To show that **u** × **v** is orthogonal to both **u** and **v**, find the dot product of **u** × **v** with **u** and **u** × **v** with **v**.

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$
$= \langle -5, -6, 3 \rangle \cdot \langle 3, -2, 1 \rangle$
$= -5(3) + (-6)(-2) + 3(1)$
$= -15 + 12 + 3$ or 0 ✔

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}$
$= \langle -5, -6, 3 \rangle \cdot \langle -3, 3, 1 \rangle$
$= -5(-3) + (-6)(3) + 3(1)$
$= 15 + (-18) + 3$ or 0 ✔

Because both dot products are zero, the vectors are orthogonal.

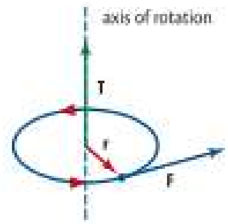
GuidedPractice

Find the cross product of u and v. Then show that u × v is orthogonal to both u and v.

3A. $\mathbf{u} = \langle 4, 2, -1 \rangle$, $\mathbf{v} = \langle 5, 1, 4 \rangle$

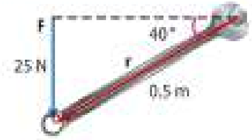
3B. $\mathbf{u} = \langle -2, -1, -3 \rangle$, $\mathbf{v} = \langle 5, 1, 4 \rangle$

You can use the cross product to find a vector quantity called **torque**. Torque measures how effectively a force applied to a lever causes rotation along the axis of rotation. The torque vector $\mathbf{T}$ is perpendicular to the plane containing the directed distance $\mathbf{r}$ from the axis of rotation to the point of the applied force and the applied force $\mathbf{F}$ as shown. Therefore, the torque vector is $\mathbf{T} = \mathbf{r} \times \mathbf{F}$ and is measured in newton-meters (N • m).



Real-World Example 4 Torque Using Cross Product

AUTO REPAIR Abdulkarim uses a lug wrench to tighten a lug nut. The wrench he uses is 50 centimeters or 0.5 meter long. Find the magnitude and direction of the torque about the lug nut if he applies a force of 25 newtons straight down to the end of the handle when it is 40° below the positive x-axis as shown.



Step 1 Graph each vector in standard position (Figure 1.5.1).

Step 2 Determine the component form of each vector.

The component form of the vector representing the directed distance from the axis of rotation to the end of the handle can be found using the triangle in Figure 1.5.2 and trigonometry. Vector $\mathbf{r}$ is therefore $\langle 0.5 \cos 40°, 0, -0.5 \sin 40° \rangle$ or about $\langle 0.38, 0, -0.32 \rangle$. The vector representing the force applied to the end of the handle is 25 newtons straight down, so $\mathbf{F} = \langle 0, 0, -25 \rangle$.

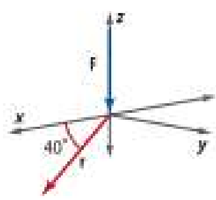


Figure 1.5.1

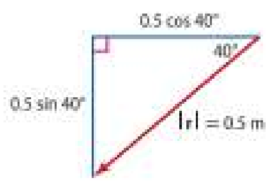


Figure 1.5.2

Step 3 Use the cross product of these vectors to find the vector representing the torque about the lug nut.

$\mathbf{T} = \mathbf{r} \times \mathbf{F}$ — Torque Cross Product Formula

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.38 & 0 & -0.32 \\ 0 & 0 & -25 \end{vmatrix}$ — Cross product of r and F

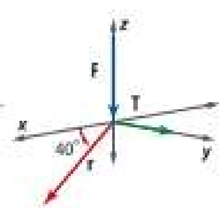
$= \begin{vmatrix} 0 & -0.32 \\ 0 & -25 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0.38 & -0.32 \\ 0 & -25 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0.38 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{k}$ — Determinant of a 3 × 3 matrix

$= 0\mathbf{i} - (-9.5)\mathbf{j} + 0\mathbf{k}$ — Determinants of 2 × 2 matrices

$= \langle 0, 9.5, 0 \rangle$ — Component form

Step 4 Find the magnitude and direction of the torque vector.

The component form of the torque vector $\langle 0, 9.5, 0 \rangle$ tells us that the magnitude of the vector is about 9.5 newton-meters parallel to the positive y-axis as shown.



GuidedPractice

4. **AUTO REPAIR** Find the magnitude of the torque if Abdulkarim applied the same amount of force to the end of the handle straight down when the handle makes a 40° angle above the positive x-axis as shown in Figure 1.5.3.

Real-WorldCareer

Automotive Mechanic
Automotive mechanics perform repairs ranging from simple mechanical problems to high-level, technology-related repairs. They should have good problem-solving skills, mechanical aptitude, and knowledge of electronics and mathematics. Most mechanics complete a vocational training program in automotive service technology.

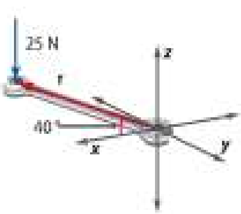


Figure 1.5.3

The cross product of two vectors has several geometric applications. One is that the magnitude of $\mathbf{u} \times \mathbf{v}$ represents the area of the parallelogram that has $\mathbf{u}$ and $\mathbf{v}$ as its adjacent sides (Figure 1.5.4).

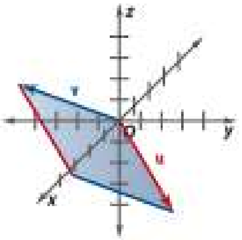


Figure 1.5.4

Example 5 Area of a Parallelogram in Space

Find the area of the parallelogram with adjacent sides $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.

Step 1 Find $\mathbf{u} \times \mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -3 \\ 1 & -5 & 3 \end{vmatrix} \qquad \mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} \text{ and } \mathbf{v} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$$

$$= \begin{vmatrix} 4 & -3 \\ -5 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -3 \\ 1 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 4 \\ 1 & -5 \end{vmatrix}\mathbf{k} \qquad \text{Determinant of a } 3 \times 3 \text{ matrix}$$

$$= -3\mathbf{i} - 9\mathbf{j} - 14\mathbf{k} \qquad \text{Determinants of } 2 \times 2 \text{ matrices}$$

Step 2 Find the magnitude of $\mathbf{u} \times \mathbf{v}$.

$$|\mathbf{u} \times \mathbf{v}| = \sqrt{(-3)^2 + (-9)^2 + (-14)^2} \qquad \text{Magnitude of a vector in space}$$

$$= \sqrt{286} \text{ or about } 16.9 \qquad \text{Simplify.}$$

The area of the parallelogram shown in Figure 1.5.4 is about 16.9 square units.

GuidedPractice

5. Find the area of the parallelogram with adjacent sides $\mathbf{u} = -6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Three vectors that lie in different planes but share the same initial point determine the adjacent edges of a **parallelepiped**, a polyhedron with faces that are all parallelograms (Figure 1.5.5). The absolute value of the **triple scalar product** of these vectors represents the volume of the parallelepiped.

StudyTip

Triple Scalar Product Notice that to find the triple scalar product of $\mathbf{t}$, $\mathbf{u}$, and $\mathbf{v}$, you write the determinant representing $\mathbf{u} \times \mathbf{v}$ and replace the top row with the values for the vector $\mathbf{t}$.

KeyConcept Triple Scalar Product

If $\mathbf{t} = t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k}$, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, the triple scalar product is given by $\mathbf{t} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} t_1 & t_2 & t_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$.

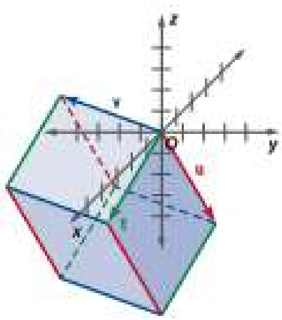


Figure 1.5.5

Example 6 Volume of a Parallelepiped

Find the volume of the parallelepiped with adjacent edges $\mathbf{t} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$, and $\mathbf{v} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.

$$\mathbf{t} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} 4 & -2 & -2 \\ 2 & 4 & -3 \\ 1 & -5 & 3 \end{vmatrix} \qquad \mathbf{t} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k},\ \mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} \text{ and } \mathbf{v} = \mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$$

$$= \begin{vmatrix} 4 & -3 \\ -5 & 3 \end{vmatrix}(4) - \begin{vmatrix} 2 & -3 \\ 1 & 3 \end{vmatrix}(-2) + \begin{vmatrix} 2 & 4 \\ 1 & -5 \end{vmatrix}(-2) \qquad \text{Determinant of a } 3 \times 3 \text{ matrix}$$

$$= -12 + 18 + 28 \text{ or } 34 \qquad \text{Simplify.}$$

The volume of the parallelepiped shown in Figure 1.5.5 is $|\mathbf{t} \cdot (\mathbf{u} \times \mathbf{v})|$ or 34 cubic units.

GuidedPractice

6. Find the volume of the parallelepiped with adjacent edges $\mathbf{t} = 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{u} = -6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Exercises

Find the dot product of u and v. Then determine if u and v are orthogonal. (Example 1)

1. $\mathbf{u} = \langle 3, -9, 6\rangle$, $\mathbf{v} = \langle -8, 2, 7\rangle$
2. $\mathbf{u} = \langle 5, 0, -4\rangle$, $\mathbf{v} = \langle 6, -1, 4\rangle$
3. $\mathbf{u} = \langle 2, -8, -7\rangle$, $\mathbf{v} = \langle 5, 9, -7\rangle$
4. $\mathbf{u} = \langle -7, -3, 1\rangle$, $\mathbf{v} = \langle -4, 5, -13\rangle$
5. $\mathbf{u} = \langle 11, 4, -2\rangle$, $\mathbf{v} = \langle -1, 3, 8\rangle$
6. $\mathbf{u} = 6\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$
7. $\mathbf{u} = 3\mathbf{i} - 10\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 7\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
8. $\mathbf{u} = 9\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$, $\mathbf{v} = 6\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$

9. **CHEMISTRY** A water molecule, in which the oxygen atom is centered at the origin, has one hydrogen atom centered at $(55.5, 55.5, -55.5)$ and the second hydrogen atom centered at $(-55.5, -55.5, -55.5)$. Determine the bond angle between the vectors formed by the oxygen-hydrogen bonds. (Example 2)

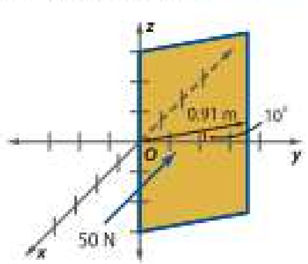
Find the angle θ between vectors u and v to the nearest tenth of a degree. (Example 2)

10. $\mathbf{u} = \langle 3, -2, 2\rangle$, $\mathbf{v} = \langle 1, 4, -7\rangle$
11. $\mathbf{u} = \langle 6, -5, 1\rangle$, $\mathbf{v} = \langle -8, -9, 5\rangle$
12. $\mathbf{u} = \langle -8, 1, 12\rangle$, $\mathbf{v} = \langle -6, 4, 2\rangle$
13. $\mathbf{u} = \langle 10, 0, -8\rangle$, $\mathbf{v} = \langle 3, -1, -12\rangle$
14. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} + 9\mathbf{k}$, $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j} - 10\mathbf{k}$
15. $\mathbf{u} = -6\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{v} = -4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

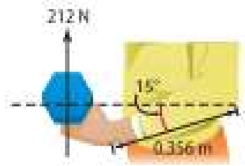
Find the cross product of u and v. Then show that u × v is orthogonal to both u and v. (Example 3)

16. $\mathbf{u} = \langle -1, 3, 5\rangle$, $\mathbf{v} = \langle 2, -6, -3\rangle$
17. $\mathbf{u} = \langle 4, 7, -2\rangle$, $\mathbf{v} = \langle -5, 9, 1\rangle$
18. $\mathbf{u} = \langle 3, -6, 2\rangle$, $\mathbf{v} = \langle 1, 5, -8\rangle$
19. $\mathbf{u} = \langle 5, -8, 0\rangle$, $\mathbf{v} = \langle -4, -2, 7\rangle$
20. $\mathbf{u} = -2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{v} = 7\mathbf{i} + \mathbf{j} - 6\mathbf{k}$
21. $\mathbf{u} = -4\mathbf{i} + \mathbf{j} + 8\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$

22. **RESTAURANTS** A restaurant server applies 50 newtons of force to open a door. Find the magnitude and direction of the torque about the door's hinge. (Example 4)



23. **WEIGHTLIFTING** A weightlifter doing bicep curls applies 212 newtons of force to lift the dumbbell. The weightlifter's forearm is 0.356 meters long and she begins the bicep curl with her elbow bent at a 15° angle below the horizontal in the direction of the positive x-axis. (Example 4)



a. Find the vector representing the torque about the weightlifter's elbow in component form.

b. Find the magnitude and direction of the torque.

Find the area of the parallelogram with adjacent sides u and v. (Example 5)

24. $\mathbf{u} = \langle 2, -5, 3\rangle$, $\mathbf{v} = \langle 4, 6, -1\rangle$
25. $\mathbf{u} = \langle -9, 1, 2\rangle$, $\mathbf{v} = \langle 6, -5, 3\rangle$
26. $\mathbf{u} = \langle 4, 3, -1\rangle$, $\mathbf{v} = \langle 7, 2, -2\rangle$
27. $\mathbf{u} = 6\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{v} = 5\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$
28. $\mathbf{u} = \mathbf{i} + 4\mathbf{j} - 8\mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$
29. $\mathbf{u} = -3\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = 4\mathbf{i} - \mathbf{j} + 6\mathbf{k}$

Find the volume of the parallelepiped having t, u, and v as adjacent edges. (Example 6)

30. $\mathbf{t} = \langle -1, -9, 2\rangle$, $\mathbf{u} = \langle 4, -7, -5\rangle$, $\mathbf{v} = \langle 3, -2, 6\rangle$
31. $\mathbf{t} = \langle -6, 4, -8\rangle$, $\mathbf{u} = \langle -3, -1, 6\rangle$, $\mathbf{v} = \langle 2, 5, -7\rangle$
32. $\mathbf{t} = \langle 2, -3, -1\rangle$, $\mathbf{u} = \langle 4, -6, 3\rangle$, $\mathbf{v} = \langle -9, 5, -4\rangle$
33. $\mathbf{t} = -4\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{u} = 5\mathbf{i} + 7\mathbf{j} - 6\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$
34. $\mathbf{t} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - 6\mathbf{j} + 8\mathbf{k}$
35. $\mathbf{t} = 5\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$, $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$, $\mathbf{v} = 8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

B

Find a vector that is orthogonal to each vector.

36. $\langle 3, -8, 4\rangle$
37. $\langle -1, -2, 5\rangle$
38. $\left\langle 6, -\frac{1}{3}, -3\right\rangle$
39. $\langle 7, 0, 8\rangle$

Given v and u · v, find u.

40. $\mathbf{v} = \langle 2, -4, -6\rangle$, $\mathbf{u} \cdot \mathbf{v} = -22$
41. $\mathbf{v} = \left\langle \frac{1}{2}, 0, 4\right\rangle$, $\mathbf{u} \cdot \mathbf{v} = \frac{31}{2}$
42. $\mathbf{v} = \langle -2, -6, -5\rangle$, $\mathbf{u} \cdot \mathbf{v} = 35$

Determine whether the points are collinear.

43. $(-1, 7, 7)$, $(-3, 9, 11)$, $(-5, 11, 13)$
44. $(11, 8, -1)$, $(17, 5, -7)$, $(8, 11, 5)$

Determine whether each pair of vectors are parallel.

45. $\mathbf{m} = \langle 2, -10, 6 \rangle$, $\mathbf{n} = \langle 3, -15, 9 \rangle$

46. $\mathbf{a} = \langle 6, 3, -7 \rangle$, $\mathbf{b} = \langle -4, -2, 3 \rangle$

47. $\mathbf{w} = \left\langle -\frac{3}{2}, \frac{3}{4}, -\frac{9}{8} \right\rangle$, $\mathbf{z} = \langle -4, 2, -3 \rangle$

Write the component form of each vector.

48. $\mathbf{u}$ lies in the yz-plane, has a magnitude of 8, and makes a 60° angle above the positive y-axis.

49. $\mathbf{v}$ lies in the xy-plane, has a magnitude of 11, and makes a 30° angle to the left of the negative x-axis.

Given the four vertices, determine whether quadrilateral *ABCD* is a parallelogram. If it is, find its area, and determine whether it is a rectangle.

50. $A(3, 0, -2)$, $B(0, 4, -1)$, $C(0, 2, 5)$, $D(3, 2, 4)$

51. $A(7, 5, 5)$, $B(4, 4, 4)$, $C(4, 6, 2)$, $D(7, 7, 3)$

52. **AIR SHOWS** In an air show, two airplanes take off at the same time. The first plane starts at the position $(0, -2, 0)$ and is at the position $(6, -10, 15)$ after three seconds. The second plane starts at the position $(0, 2, 0)$ and is at the position $(6, 10, 15)$ after three seconds. Are the paths of the two planes parallel? Explain.

For $\mathbf{u} = \langle 3, 2, -2 \rangle$ and $\mathbf{v} = \langle -4, 4, 5 \rangle$, find each of the following, if possible.

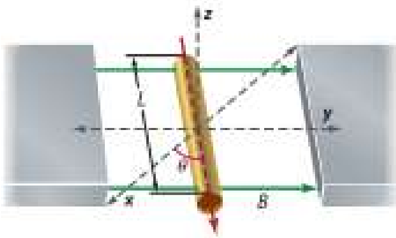
53. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$

54. $\mathbf{v} \times (\mathbf{u} \cdot \mathbf{v})$

55. $\mathbf{u} \times \mathbf{u} \times \mathbf{v}$

56. $\mathbf{v} \cdot \mathbf{v} \cdot \mathbf{u}$

57. **ELECTRICITY** When a wire carrying an electric current is placed in a magnetic field, the force on the wire in newtons is given by $\vec{F} = I\,\vec{L} \times \vec{B}$, where I represents the current flowing through the wire in amps, $\vec{L}$ represents the vector length of the wire pointing in the direction of the current in meters, and $\vec{B}$ is the force of the magnetic field in teslas. In the figure below, the wire is rotated through an angle θ in the xy-plane.

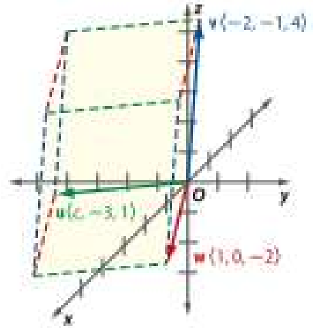


a. If the force of a magnetic field is 1.1 teslas, find the magnitude of the force on a wire in the xy-plane that is 0.15 meter in length carrying a current of 25 amps at an angle of 60°.

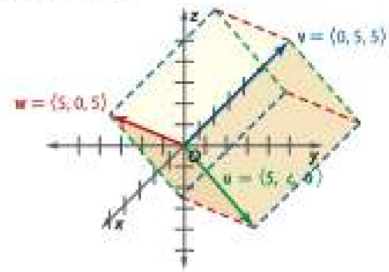
b. If the force on the wire is $\vec{F} = \langle 0, 0, -0.63 \rangle$, what is the angle of the wire?

Given v, w, and the volume of the parallelepiped having adjacent edges u, v, and w, find *c*.

58. $\mathbf{v} = \langle -2, -1, 4 \rangle$, $\mathbf{w} = \langle 1, 0, -2 \rangle$, $\mathbf{u} = \langle c, -3, 1 \rangle$, and $V = 7$ cubic units

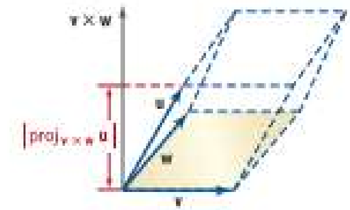


59. $\mathbf{v} = \langle 0, 5, 5 \rangle$, $\mathbf{w} = \langle 5, 0, 5 \rangle$, $\mathbf{u} = \langle 5, c, 0 \rangle$, and $V = 250$ cubic units



H.O.T. Problems Use Higher-Order Thinking Skills

60. **PROOF** Verify the formula for the volume of a parallelepiped. (*Hint*: Use the projection of $\mathbf{u}$ onto $\mathbf{v} \times \mathbf{w}$.)



61. **REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain.

For any two nonzero, nonparallel vectors in space, there is a vector that is perpendicular to both.

62. **REASONING** If $\mathbf{u}$ and $\mathbf{v}$ are parallel in space, then how many vectors are perpendicular to both? Explain.

63. **CHALLENGE** Given $\mathbf{u} = \langle 4, 6, c \rangle$ and $\mathbf{v} = \langle -3, -2, 5 \rangle$, find the value of c for which $\mathbf{u} \times \mathbf{v} = 34\mathbf{i} - 26\mathbf{j} + 10\mathbf{k}$.

64. **REASONING** Explain why the cross product is not defined for vectors in the two-dimensional coordinate system.

65. **WRITING IN MATH** Compare and contrast the methods for determining whether vectors in space are parallel or perpendicular.

Spiral Review

Find the length and midpoint of the segment with the given endpoints.

66. $(1, 10, 13), (-2, 22, -6)$

67. $(12, -1, -14), (21, 19, -23)$

68. $(-22, 24, -9), (10, 10, 2)$

Find the dot product of u and v. Then determine if u and v are orthogonal.

69. $\langle -8, -7 \rangle \cdot \langle 1, 2 \rangle$

70. $\langle -4, -6 \rangle \cdot \langle 7, 5 \rangle$

71. $\langle 6, -3 \rangle \cdot \langle -3, 5 \rangle$

72. **BAKERY** Abdulaziz's bakery has racks that can hold up to 900 bagels and muffins. Due to costs, the number of bagels produced must be less than or equal to 300 plus twice the number of muffins produced. The demand for bagels is at least three times that of muffins. Abdulaziz makes a profit of \$3 per muffin sold and \$1.25 per bagel sold. How many of each item should he make to maximize profit?

73. Decompose $\frac{2m + 16}{m^2 - 16}$ into partial fractions.

Verify each identity.

74. $\tan^2 \theta + \cos^2 \theta + \sin^2 \theta = \sec^2 \theta$

75. $\sec^2 \theta \cot^2 \theta - \cot^2 \theta = 1$

76. $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$

Solve each triangle. Round side lengths to the nearest tenth and angle measures to the nearest degree.

77. $a = 20, c = 24, B = 47°$

78. $A = 25°, B = 78°, a = 13.7$

79. $a = 21.5, b = 16.7, c = 10.3$

Write each decimal degree measure in DMS form and each DMS measure in decimal degree form to the nearest thousandth.

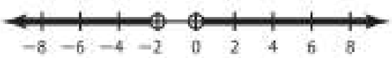
80. $-72.775°$

81. $29° 6' 6''$

82. $132° 18' 31''$

Skills Review for Standardized Tests

83. **SAT/ACT** The graph represents the set of all possible solutions to which of the following statements?



A $|x - 1| > 1$

B $|x - 1| < 1$

C $|x + 1| < 1$

D $|x + 1| > 1$

84. What is the cross product of $\mathbf{u} = \langle 3, 8, 0 \rangle$ and $\mathbf{v} = \langle -4, 2, 6 \rangle$? F

F $48\mathbf{i} - 18\mathbf{j} + 38\mathbf{k}$

G $48\mathbf{i} - 22\mathbf{j} + 38\mathbf{k}$

H $46\mathbf{i} - 22\mathbf{j} + 38\mathbf{k}$

J $46\mathbf{i} - 18\mathbf{j} + 38\mathbf{k}$

85. **FREE RESPONSE** A batter hits a ball at a 30° angle with the ground at an initial speed of 90 feet per second.

a. Find the magnitude of the horizontal and vertical components of the velocity.

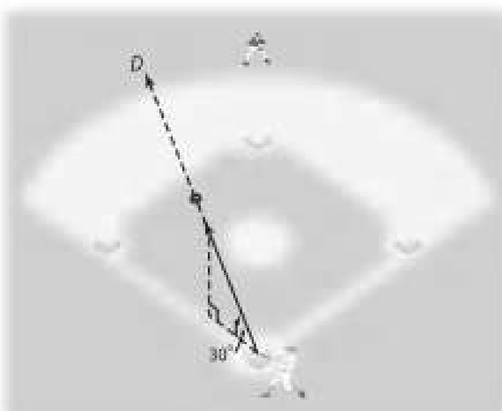
b. Are the values in part **a** vectors or scalars?

c. Assume that the ball is not caught and the player hit it one yard off the ground. How far will it travel in the air?

d. Assume that home plate is at the origin and second base lies due north. If the ball is hit at a bearing of N20°W and lands at point D, find the component form of $\overrightarrow{CD}$.

e. Determine the unit vector of $\overrightarrow{CD}$.

f. The fielder is standing at (0, 150) when the ball is hit. At what quadrant bearing should the fielder run in order to meet the ball where it will hit the ground?



Study Guide and Review

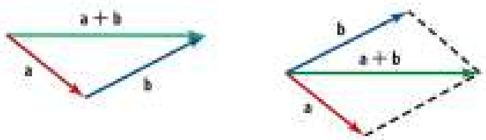
Chapter Summary

KeyConcepts

Introduction to Vectors (Lesson 1-1)

- The direction of a vector is the directed angle between the vector and a horizontal line. The magnitude of a vector is its length.
- When two or more vectors are combined, their sum is a single vector called the resultant, which can be found using the triangle or parallelogram method.

Triangle Method **Parallelogram Method**



Vectors in the Coordinate Plane (Lesson 1-2)

- The component form of a vector with rectangular components x and y is $\langle x, y \rangle$.
- The component form of a vector that is not in standard position, with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$, is given by $\langle x_2 - x_1, y_2 - y_1 \rangle$.
- The magnitude of a vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is given by $|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ are vectors and k is a scalar, then $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, $\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$, and $k\mathbf{a} = \langle ka_1, ka_2 \rangle$.
- The standard unit vectors $\mathbf{i}$ and $\mathbf{j}$ can be used to express any vector $\mathbf{v} = \langle a, b \rangle$ as $a\mathbf{i} + b\mathbf{j}$.

Dot Products (Lesson 1-3)

- The dot product of $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$.
- If θ is the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then $\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$.

Vectors in Three-Dimensional Space (Lesson 1-4)

- The distance between $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- The midpoint of $\overline{AB}$ is given by $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

Dot and Cross Products of Vectors in Space (Lesson 1-5)

- The dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.
- If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{a} \times \mathbf{b} =$ $(a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$.

KeyVocabulary

component form
components
cross product
direction
dot product
equivalent vectors
initial point
linear combination
magnitude
octants
opposite vectors
ordered triple
orthogonal
parallelepiped
parallelogram method
parallel vectors
quadrant bearing
rectangular components
resultant
standard position
terminal point
three-dimensional coordinate system
torque
triangle method
triple scalar product
true bearing
unit vector
vector
vector projection
work
z-axis
zero vector

VocabularyCheck

Determine whether each statement is *true* or *false*. If false, replace the underlined term or expression to make the statement true.

1. The terminal point of a vector is where the vector <u>begins</u>.
2. If $\mathbf{a} = \langle -4, 1 \rangle$ and $\mathbf{b} = \langle 3, 2 \rangle$, the dot product is calculated by <u>$-4(1) + 3(2)$</u>.
3. The midpoint of $\overline{AB}$ with $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by <u>$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$</u>.
4. The <u>magnitude</u> of $\mathbf{r}$ if the initial point is $A(-1, 2)$ and the terminal point is $B(2, -4)$ is $\langle 3, -6 \rangle$.
5. Two vectors are <u>equal</u> only if they have the same direction and magnitude.
6. When two nonzero vectors are orthogonal, the angle between them is <u>180°</u>.
7. The <u>component</u> of $\mathbf{u}$ onto $\mathbf{v}$ is the vector with direction that is parallel to $\mathbf{v}$ *and* with length that is the component of $\mathbf{u}$ along $\mathbf{v}$.
8. To find at least one vector orthogonal to any two vectors in space, calculate the <u>cross product</u> of the two original vectors.
9. When a vector is subtracted, it is equivalent to adding the opposite vector.
10. If $\mathbf{v}$ is a unit vector in the same direction as $\mathbf{u}$, then <u>$\mathbf{v} = \frac{|\mathbf{u}|}{\mathbf{u}}$</u>.

Lesson-by-Lesson Review

1-1 Introduction to Vectors

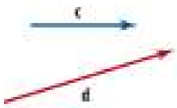
State whether each quantity described is a *vector* quantity or a *scalar* quantity.

11. a car driving 50 kilometers an hour due east

12. a gust of wind blowing 5 meters per second

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant to the nearest tenth of a centimeter and its direction relative to the horizontal.

13.



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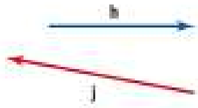

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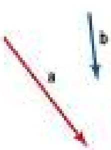

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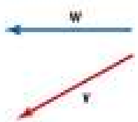


Determine the magnitude and direction of the resultant of each vector sum.

17. 70 meters due west and then 150 meters due east

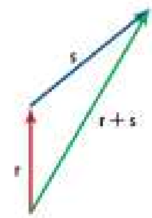
18. 8 newtons directly backward and then 12 newtons directly backward

Example 1

Find the resultant of r and s using either the triangle or parallelogram method. State the magnitude of the resultant in centimeters and its direction relative to the horizontal.

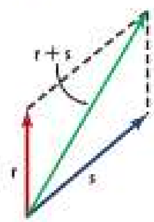
Triangle Method

Translate **r** so that the tip of **r** touches the tail of **s**. The resultant is the vector from the tail of **r** to the tip of **s**.



Parallelogram Method

Translate **s** so that the tail of **s** touches the tail of **r**. Complete the parallelogram that has **r** and **s** as two of its sides. The resultant is the vector that forms the indicated diagonal of the parallelogram.



The magnitude of the resultant is 3.4 cm and the direction is 59°.

1-2 Vectors in the Coordinate Plane

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points.

19. $A(-1, 3)$, $B(5, 4)$

20. $A(7, -2)$, $B(-9, 6)$

21. $A(-8, -4)$, $B(6, 1)$

22. $A(2, -10)$, $B(3, -5)$

Find each of the following for $\mathbf{p} = \langle 4, 0\rangle$, $\mathbf{q} = \langle -2, -3\rangle$, and $\mathbf{t} = \langle -4, 2\rangle$.

23. $2\mathbf{q} - \mathbf{p}$

24. $\mathbf{p} + 2\mathbf{t}$

25. $\mathbf{t} - 3\mathbf{p} + \mathbf{q}$

26. $2\mathbf{p} + \mathbf{t} - 3\mathbf{q}$

Find a unit vector u with the same direction as v.

27. $\mathbf{v} = \langle -7, 2\rangle$

28. $\mathbf{v} = \langle 3, -3\rangle$

29. $\mathbf{v} = \langle -5, -8\rangle$

30. $\mathbf{v} = \langle 9, 3\rangle$

Example 2

Find the component form and magnitude of $\overrightarrow{AB}$ with initial point $A(3, -2)$ and terminal point $B(4, -1)$.

$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1\rangle$ — Component form

$= \langle 4 - 3, -1 - (-2)\rangle$ — Substitute.

$= \langle 1, 1\rangle$ — Subtract.

Find the magnitude using the Distance Formula.

$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ — Distance Formula

$= \sqrt{(4 - 3)^2 + [-1 - (-2)]^2}$ — Substitute.

$= \sqrt{2}$ or about 1.4 — Simplify.

1-3 Dot Products and Vector Projections

Find the dot product of **u** and **v**. Then determine if **u** and **v** are orthogonal.

31. $\mathbf{u} = \langle -3, 5 \rangle, \mathbf{v} = \langle 2, 1 \rangle$

32. $\mathbf{u} = \langle 4, 4 \rangle, \mathbf{v} = \langle 5, 7 \rangle$

33. $\mathbf{u} = \langle -1, 4 \rangle, \mathbf{v} = \langle 8, 2 \rangle$

34. $\mathbf{u} = \langle -2, 3 \rangle, \mathbf{v} = \langle 1, 3 \rangle$

Find the angle θ between **u** and **v** to the nearest tenth of a degree.

35. $\mathbf{u} = \langle 5, -1 \rangle, \mathbf{v} = \langle -2, 3 \rangle$

36. $\mathbf{u} = \langle -1, 8 \rangle, \mathbf{v} = \langle 4, 2 \rangle$

Example 3

Find the dot product of $x = \langle 2, -5 \rangle$ and $\mathbf{y} = \langle -4, 7 \rangle$. Then determine if x and y are orthogonal.

$x \cdot \mathbf{y} = x_1 y_1 + x_2 y_2$ — Dot product

$= 2(-4) + -5(7)$ — Substitute.

$= -8 + (-35)$ or -43 — Simplify.

Since $\mathbf{x} \cdot \mathbf{y} \neq 0$, **x** and **y** are not orthogonal.

1-4 Vectors in Three-Dimensional Space

Plot each point in a three-dimensional coordinate system.

37. $(1, 2, -4)$

38. $(3, 5, 3)$

39. $(5, -3, -2)$

40. $(-2, -3, -2)$

Find the length and midpoint of the segment with the given endpoints.

41. $(-4, 10, 4), (2, 0, 8)$

42. $(-5, 6, 4), (-9, -2, -2)$

43. $(3, 2, 0), (-9, -10, 4)$

44. $(8, 3, 2), (-4, -6, 6)$

Locate and graph each vector in space.

45. $\mathbf{a} = \langle 0, -3, 4 \rangle$

46. $\mathbf{b} = -3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

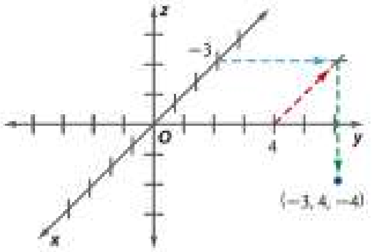
47. $\mathbf{c} = -2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

48. $\mathbf{d} = \langle -4, -5, -3 \rangle$

Example 4

Plot $(-3, 4, -4)$ in a three-dimensional coordinate system.

Locate the point $(-3, 4)$ in the xy-plane and mark it with a cross. Then plot a point 4 units down from this location parallel to the z-axis.



1-5 Dot and Cross Products of Vectors in Space

Find the dot product of **u** and **v**. Then determine if **u** and **v** are orthogonal.

49. $\mathbf{u} = \langle 2, 5, 2 \rangle, \mathbf{v} = \langle 8, 2, -13 \rangle$

50. $\mathbf{u} = \langle 5, 0, -6 \rangle, \mathbf{v} = \langle -6, 1, 3 \rangle$

Find the cross product of **u** and **v**. Then show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v**.

51. $\mathbf{u} = \langle 1, -3, -2 \rangle, \mathbf{v} = \langle 2, 4, -3 \rangle$

52. $\mathbf{u} = \langle 4, 1, -2 \rangle, \mathbf{v} = \langle 5, -4, -1 \rangle$

Example 5

Find the cross product of $\mathbf{u} = \langle -4, 2, -3 \rangle$ and $\mathbf{v} = \langle 7, 11, 2 \rangle$. Then show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both u and v.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} 2 & -3 \\ 11 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & -3 \\ 7 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & 2 \\ 7 & 11 \end{vmatrix}\mathbf{k}$$

$$= \langle 37, -13, -58 \rangle$$

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \langle 37, -13, -58 \rangle \cdot \langle -4, 2, -3 \rangle$

$= -148 - 26 + 174$ or 0 ✓

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \langle 37, -13, -58 \rangle \cdot \langle 7, 11, 2 \rangle$

$= 259 - 143 - 116$ or 0 ✓

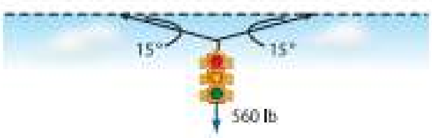
Study Guide and Review *Continued*

Applications and Problem Solving

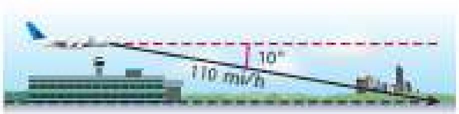
53. BASEBALL A player throws a baseball with an initial velocity of 16.7 meters per second at an angle of 25° above the horizontal, as shown below. Find the magnitude of the horizontal and vertical components. (Lesson 1-1)

54. STROLLER Laila is pushing a stroller with a force of 200 newtons at an angle of 20° below the horizontal. Find the magnitude of the horizontal and vertical components of the force. (Lesson 1-1)

55. LIGHTS A traffic light at an intersection is hanging from two wires of equal length at 15° below the horizontal as shown. If the traffic light weighs 560 pounds, what is the tension in each wire keeping the light at equilibrium? (Lesson 1-1)



56. AIRPLANE An airplane is descending at a speed of 110 miles per hour at an angle of 10° below the horizontal. Find the component form of the vector that represents the velocity of the airplane. (Lesson 1-2)



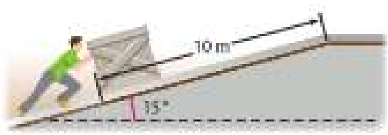
57. LIFEGUARD A lifeguard at a wave pool swims at a speed of 4 kilometers per hour at a 60° angle to the side of the pool as shown. (Lesson 1-2)

a. At what speed is the lifeguard traveling if the current in the pool is 2 kilometers per hour parallel to the side of the pool as shown?

b. At what angle is the lifeguard traveling with respect to the starting side of the pool?

58. TRAFFIC A 680.4-kilogram car is stopped in traffic on a hill that is at an incline of 10°. Determine the force that is required to keep the car from rolling down the hill. (Lesson 1-3)

59. WORK At a warehouse, Jassim pushes a box on sliders with a constant force of 80 newtons up a ramp that has an incline of 15° with the horizontal. Determine the amount of work in joules that Jassim does if he pushes the dolly 10 meters. (Lesson 1-3)



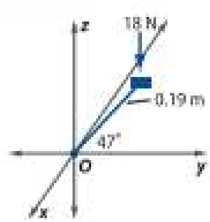
60. SATELLITES The positions of two satellites that are in orbit can be represented by the coordinates (28,625, 32,461, −38,426) and (−31,613, −29,218, 43,015), where (0, 0, 0) represents the center of Earth and the coordinates are given in miles. The radius of Earth is about 3963 miles. (Lesson 1-4)

a. Determine the distance between the two satellites.

b. If a third satellite were to be placed directly between the two satellites, what would the coordinates be?

c. Can a third satellite be placed at the coordinates found in part **b**? Explain your reasoning.

61. BICYCLES A bicyclist applies 18 newtons of force down on the pedal to put the bicycle in motion. The pedal has an initial position of 47° above the *y*-axis, and a length of 0.19 meters to the pedal's axle, as shown. (Lesson 1-5)



a. Find the vector representing the torque about the axle of the bicycle pedal in component form.

b. Find the magnitude and direction of the torque.

CHAPTER 1 Practice Test

Find the resultant of each pair of vectors using either the triangle or parallelogram method. State the magnitude of the resultant to the nearest tenth of a centimeter and its direction relative to the horizontal.

1.

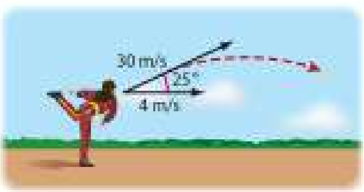
2.

Find the component form and magnitude of $\overrightarrow{AB}$ with the given initial and terminal points.

3. $A(1, -3), B(-5, 1)$

4. $A\left(\frac{1}{2}, \frac{3}{2}\right), B(-1, 7)$

5. SOFTBALL A batter on the opposing softball team hits a ground ball that rolls out to Lamya in left field. She runs toward the ball at a velocity of 4 meters per second, scoops it, and proceeds to throw it to the catcher at a speed of 30 meters per second and at an angle of 25° with the horizontal in an attempt to throw out a runner. What is the resultant speed and direction of the throw?



Find a unit vector in the same direction as u.

6. $\mathbf{u} = \langle -1, 4 \rangle$

7. $\mathbf{u} = \langle 6, -3 \rangle$

Find the dot product of u and v. Then determine if u and v are orthogonal.

8. $\mathbf{u} = \langle 2, -5 \rangle, \mathbf{v} = \langle -3, 2 \rangle$

9. $\mathbf{u} = \langle 4, -3 \rangle, \mathbf{v} = \langle 6, 8 \rangle$

10. $\mathbf{u} = 10\mathbf{i} - 3\mathbf{j}, \mathbf{v} = \mathbf{i} + 8\mathbf{j}$

11. MULTIPLE CHOICE Write **u** as the sum of two orthogonal vectors, one of which being the projection of **u** onto **v** if $\mathbf{u} = \langle 1, 3 \rangle$ and $\mathbf{v} = \langle -4, 2 \rangle$.

A $\mathbf{u} = \left\langle \frac{2}{5}, -\frac{3}{5} \right\rangle + \left\langle \frac{3}{5}, \frac{18}{5} \right\rangle$

B $\mathbf{u} = \left\langle \frac{2}{5}, \frac{3}{5} \right\rangle + \left\langle \frac{3}{5}, \frac{12}{5} \right\rangle$

C $\mathbf{u} = \left\langle -\frac{4}{5}, \frac{2}{5} \right\rangle + \left\langle \frac{9}{5}, \frac{13}{5} \right\rangle$

D $\mathbf{u} = \left\langle -\frac{2}{5}, \frac{1}{5} \right\rangle + \left\langle \frac{7}{5}, \frac{14}{5} \right\rangle$

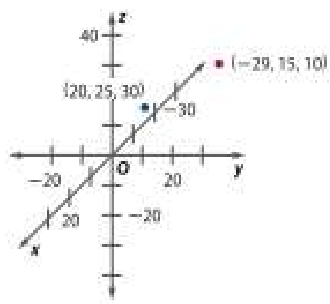
12. MOVING Lamis is pushing a box along a level floor with a force of 54.4 newtons at an angle of depression of 20°. Determine how much work is done if the box is moved 25 meters.

Find each of the following for $\mathbf{a} = \langle 2, 4, -3 \rangle$, $\mathbf{b} = \langle -5, -7, 1 \rangle$, and $\mathbf{c} = \langle 8, 5, -9 \rangle$.

13. $2\mathbf{a} + 5\mathbf{b} - 3\mathbf{c}$

14. $\mathbf{b} - 6\mathbf{a} + 2\mathbf{c}$

15. HOT AIR BALLOONS During a festival, twelve hot air balloons take off. A few minutes later, the coordinates of the first two balloons are (20, 25, 30) and (−29, 15, 10) as shown, where the coordinates are given in feet.



a. Determine the distance between the first two balloons that took off.

b. A third balloon is halfway between the first two balloons. Determine the coordinates of the third balloon.

c. Find a unit vector in the direction of the first balloon if it took off at (0, 0, 0).

Find the angle θ between vectors u and v to the nearest tenth of a degree.

16. $\mathbf{u} = \langle -2, 4, 6 \rangle, \mathbf{v} = \langle 3, 7, 12 \rangle$

17. $\mathbf{u} = -9\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}, \mathbf{v} = -5\mathbf{i} - 7\mathbf{j} - 6\mathbf{k}$

Find the cross product of u and v. Then show that u × v is orthogonal to both u and v.

18. $\mathbf{u} = \langle 1, 7, 3 \rangle, \mathbf{v} = \langle 9, 4, 11 \rangle$

19. $\mathbf{u} = -6\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{v} = 5\mathbf{i} - 3\mathbf{j} - 2k$

20. BOATING The tiller is a lever that controls the position of the rudder on a boat. When force is applied to the tiller, the boat will turn. Suppose the tiller on a certain boat is 0.75 meter in length and is currently resting in the *xy*-plane at a 15° angle from the positive *x*-axis. Find the magnitude of the torque that is developed about the axle of the tiller if 50 newtons of force is applied in a direction parallel to the positive *y*-axis.

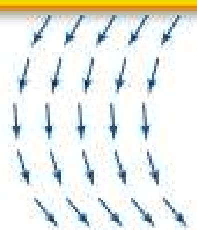
CHAPTER 1

Connect to AP Calculus
Vector Fields

Objective

- Graph vectors in and identify graphs of vector fields.

In this chapter, you examined the effects that wind and water currents have on a moving object. The force produced by the wind and current was represented by a single vector. However, we know that the current in a body of water or the force produced by wind is not necessarily constant; instead it differs from one region to the next. If we want to represent the entire current or air flow in an area, we would need to assign a vector to each point in space, thus creating a *vector field*.



Vector fields are commonly used in engineering and physics to model air resistance, magnetic and gravitational forces, and the motion of liquids. While these applications of vector fields require multiple dimensions, we will analyze vector fields in only two dimensions.

A vector field $\mathbf{F}(x, y)$ is a function that converts two-dimensional coordinates into sets of two-dimensional vectors.

$$\mathbf{F}(x, y) = \langle f_1(x, y), f_2(x, y) \rangle, \text{ where } f_1(x, y) \text{ and } f_2(x, y) \text{ are scalar functions.}$$

To graph a vector field, evaluate $\mathbf{F}(x, y)$ at (x, y) and graph the vector using (x, y) as the initial point. This is done for several points.

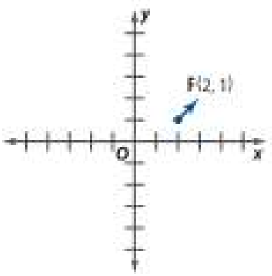
Activity 1 Vector Fields

Evaluate $\mathbf{F}(2, 1)$, $\mathbf{F}(-1, -1)$, $\mathbf{F}(1.5, -2)$, and $\mathbf{F}(-3, 2)$ for the vector field $\mathbf{F}(x, y) = \langle y^2, x - 1 \rangle$. Graph each vector using (x, y) as the initial point.

Step 1 To evaluate $\mathbf{F}(2, 1)$, let $x = 2$ and $y = 1$.

$$\begin{aligned} \langle y^2, x - 1 \rangle &= \langle 1^2, 2 - 1 \rangle \\ &= \langle 1, 1 \rangle \end{aligned}$$

Step 2 To graph, let $(2, 1)$ represent the initial point of the vector.
This makes $(2 + 1, 1 + 1)$ or $(3, 2)$ the terminal point.

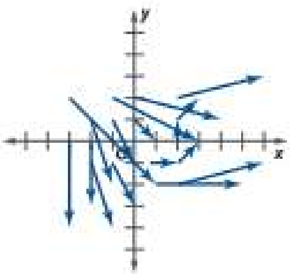


Step 3 Repeat Steps 1–2 for $\mathbf{F}(-1, -1)$, $\mathbf{F}(1.5, -2)$ and $\mathbf{F}(-3, 2)$.

Analyze the Results

1. Are the magnitudes and directions of the vectors the *same* or *different*?
2. Make a conjecture as to why a vector field can be defined as a function.
3. Is it possible to graph every vector in a vector field? Explain your reasoning.

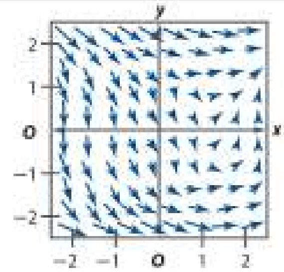
A graph of a vector field $\mathbf{F}(x, y)$ should include a variety of vectors all with initial points at (x, y). Graphing devices are typically used to graph vector fields because sketching vector fields by hand is often too difficult.



StudyTip

Graphs of Vector Fields Every point in a plane has a corresponding vector. The graphs of vector fields only show a selection of points.

To keep vectors from overlapping each other and to prevent the graph from looking too jumbled, the graphing devices proportionally reduce the lengths of the vectors and only construct vectors at certain intervals. For example, if we continue to graph more vectors for the vector field from Activity 1, the result would be the graph on the right.



Analyze the component functions of a vector field to identify the type of graph it will produce.

Activity 2 Vector Fields

Match each vector field to its graph.

$\mathbf{F}(x, y) = \langle 2, 1 + 2xy \rangle$ $\mathbf{G}(x, y) = \langle e^y, x \rangle$ $\mathbf{H}(x, y) = \langle e^y, y \rangle$

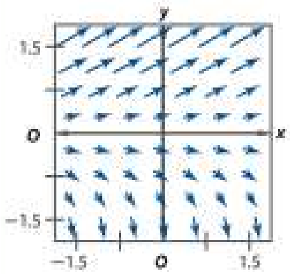


Figure 1

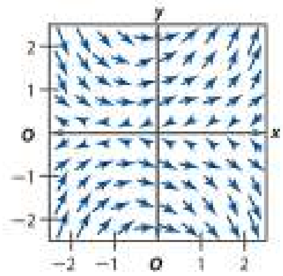


Figure 2

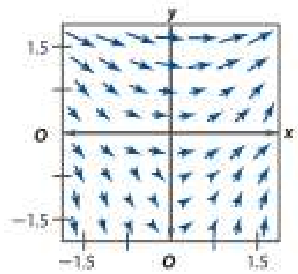


Figure 3

Step 1 Start by analyzing the components that make up $\mathbf{F}(x, y)$. The second component $(1 + 2xy)$ will produce a positive outcome when x and y have the same sign. The vertical component of the vectors in Quadrants I and III is positive, which makes the vectors in these quadrants point upward.

Step 2 The graph that has vectors pointing upward in Quadrants I and III is Figure 2.

Step 3 Repeat Steps 1–2 for the remaining vector fields.

Analyze the Results

4. Suppose the vectors in a vector field represent a force. What is the relationship between the force, the magnitude, and the length of a vector?

5. Representing wind by a single vector assumed that the force created remained constant for an entire area. If the force created by wind is represented by multiple vectors in a vector field, what assumption would have to be made about the third dimension?

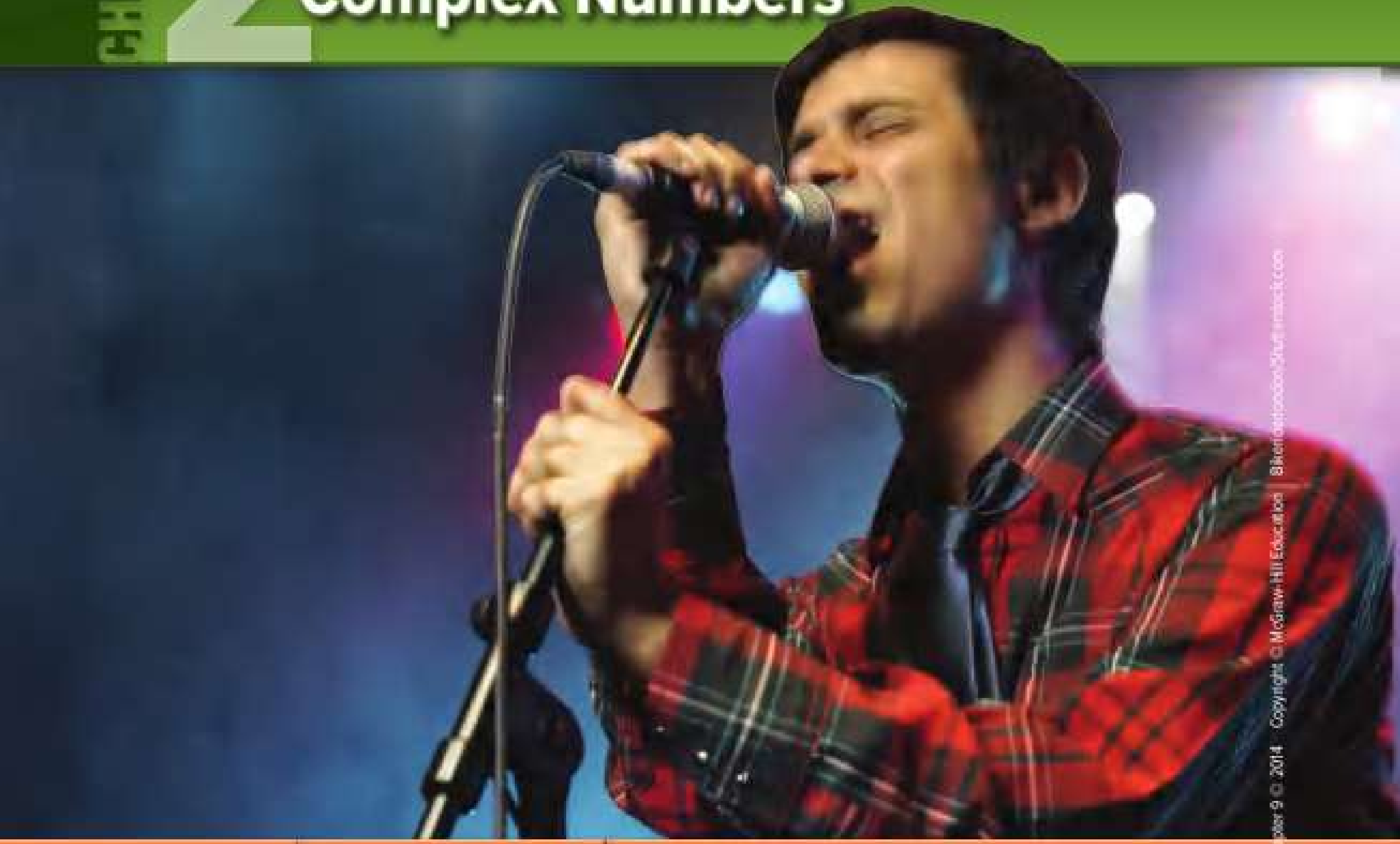
Model and Apply

6. Complete the table for the vector field $\mathbf{F}(x, y) = \langle -y, x \rangle$. Then graph each vector.

(x, y)	$\langle -y, x \rangle$	(x, y)	$\langle -y, x \rangle$
(2, 0)		(−2, 1)	
(1, 2)		(−2, 0)	
(2, 1)		(−1, −2)	
(0, 2)		(0, −2)	
(−1, 2)		(1, −2)	
(−2, −1)		(2, −1)	

CHAPTER 2

Polar Coordinates and Complex Numbers



Then

In **Grade 10**, you identified and graphed rectangular equations of conic sections.

Now

In **Chapter 2**, you will:

- Graph polar coordinates and equations.
- Convert between polar and rectangular coordinates and equations.
- Identify polar equations of conic sections.
- Convert complex numbers between polar and rectangular form.

Why? ▲

CONCERTS Polar equations can be used to model sound patterns to help determine stage arrangement, speaker and microphone placement, and volume and recording levels. Polar equations can also be used with lighting and camera angles when concerts are filmed.

PREREAD Use the Lesson Openers in Chapter 2 to make two or three predictions about what you will learn in this chapter.

Get Ready for the Chapter

QuickCheck

Graph each function using a graphing calculator. Analyze the graph to determine whether each function is *even*, *odd*, or *neither*. Confirm your answer algebraically. If odd or even, describe the symmetry of the graph of the function.

1. $f(x) = x^2 + 10$

2. $f(x) = -2x^3 + 5x$

3. $g(x) = \sqrt{x + 9}$

4. $h(x) = \sqrt{x^2 - 3}$

5. $g(x) = 3x^5 - 7x$

6. $h(x) = \sqrt{x^2} - 5$

7. **BALLOON** The distance in meters between a balloon and a person can be represented by $d(t) = \sqrt{t^2 + 3000}$, where t represents time in seconds. Analyze the graph to determine whether the function is *even*, *odd*, or *neither*.

Approximate to the nearest hundredth the relative or absolute extrema of each function. State the *x*-values where they occur.

8. $f(x) = 4x^2 - 20x + 24$

9. $g(x) = -2x^2 + 9x - 1$

10. $f(x) = -x^3 + 3x - 2$

11. $f(x) = x^3 + x^2 - 5x$

12. **ROCKET** A rocket is fired into the air. The function $h(t) = -16t^2 + 35t + 15$ represents the height h of the rocket in feet after t seconds. Find the extrema of this function.

Identify all angles that are coterminal with the given angle. Then find and draw one positive and one negative angle coterminal with the given angle.

13. 165°

14. 205°

15. −10°

16. $\frac{\pi}{6}$

17. $\frac{4\pi}{3}$

18. $-\frac{\pi}{4}$

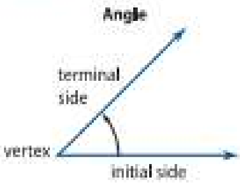
NewVocabulary

polar coordinate system
pole
polar axis
polar coordinates
polar equation
polar graph
limaçon
cardioid
rose
lemniscate
spiral of Archimedes
complex plane
real axis
imaginary axis
Argand plane
absolute value of a complex number
polar form
trigonometric form
modulus
argument

ReviewVocabulary

initial side of an angle the starting position of the ray

terminal side of an angle the ray's position after rotation



measure of an angle the amount and direction of rotation necessary to move from the initial side to the terminal side of the angle

LESSON 2-1

Polar Coordinates

::Then

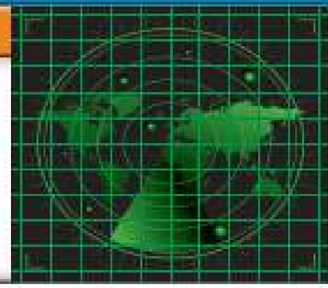
- You drew positive and negative angles given in degrees and radians in standard position.

::Now

1. Graph points with polar coordinates.
2. Graph simple polar equations.

::Why?

- To provide safe routes and travel, air traffic controllers use advanced radar systems to direct the flow of airplane traffic. This ensures that airplanes keep a sufficient distance from other aircraft and landmarks. The radar uses angle measure and directional distance to plot the positions of aircraft. Controllers can then relay this information to the pilots.



NewVocabulary

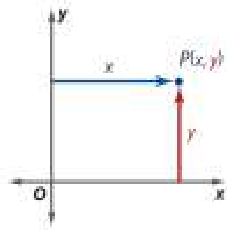
- polar coordinate system
- pole
- polar axis
- polar coordinates
- polar equation
- polar graph

1 Graph Polar Coordinates

To this point, you have graphed equations in a rectangular coordinate system. When air traffic controllers record the locations of airplanes using distances and angles, they are using a **polar coordinate system** or polar plane.

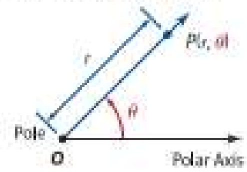
In a rectangular coordinate system, the x- and y-axes are horizontal and vertical lines, respectively, and their point of intersection O is called the origin. The location of a point P is identified by rectangular coordinates of the form (x, y), where x and y are the horizontal and vertical *directed distances*, respectively, to the point. For example, the point $(3, -4)$ is 3 units to the right of the y-axis and 4 units below the x-axis.

Rectangular Coordinate System



In a polar coordinate system, the origin is a fixed point O called the **pole**. The **polar axis** is an initial ray from the pole, usually horizontal and directed toward the right. The location of a point P in the polar coordinate system can be identified by **polar coordinates** of the form (r, θ), where r is the directed distance from the pole to the point and θ is the *directed angle* from the polar axis to $\overrightarrow{OP}$.

Polar Coordinate System



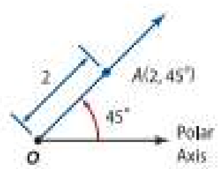
To graph a point given in polar coordinates, remember that a positive value of θ indicates a counterclockwise rotation from the polar axis, while a negative value indicates a clockwise rotation. If r is positive, then P lies on the terminal side of θ. If r is negative, P lies on the ray opposite the terminal side of θ.

Example 1 Graph Polar Coordinates

Graph each point.

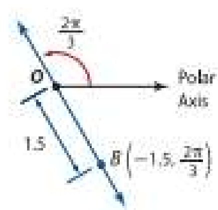
a. $A(2, 45°)$

Because $\theta = 45°$, sketch the terminal side of a 45° angle with the polar axis as its initial side. Because $r = 2$, plot a point 2 units from the pole along the terminal side of this angle.



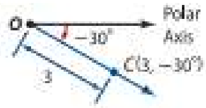
b. $B\left(-1.5, \frac{2\pi}{3}\right)$

Because $\theta = \frac{2\pi}{3}$, sketch the terminal side of a $\frac{2\pi}{3}$ angle with the polar axis as its initial side. Because r is negative, extend the terminal side of the angle in the *opposite* direction and plot a point 1.5 units from the pole along this extended ray.



c. $C(3, -30°)$

Because $\theta = -30°$, sketch the terminal side of a $-30°$ angle with the polar axis as its initial side. Because $r = 3$, plot a point 3 units from the pole along the terminal side of this angle.



GuidedPractice

Graph each point.

1A. $D\left(-1, \frac{\pi}{2}\right)$ **1B.** $E(2.5, 240°)$ **1C.** $F\left(4, -\frac{5\pi}{6}\right)$

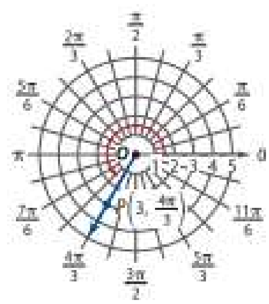
Just as rectangular coordinates are graphed on a rectangular grid, polar coordinates are graphed on a circular or *polar* grid representing the polar plane.

Example 2 Graph Points on a Polar Grid

Graph each point on a polar grid.

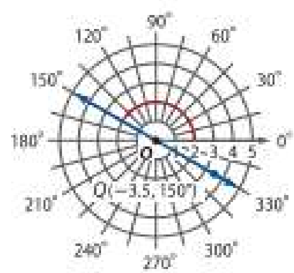
a. $P\left(3, \frac{4\pi}{3}\right)$

Because $\theta = \frac{4\pi}{3}$, sketch the terminal side of a $\frac{4\pi}{3}$ angle with the polar axis as its initial side. Because $r = 3$, plot a point 3 units from the pole along the terminal side of the angle.



b. $Q(-3.5, 150°)$

Because $\theta = 150°$, sketch the terminal side of a $150°$ angle with the polar axis as its initial side. Because r is negative, extend the terminal side of the angle in the *opposite* direction and plot a point 3.5 units from the pole along this extended ray.



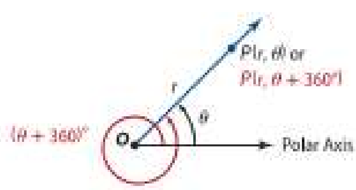
GuidedPractice

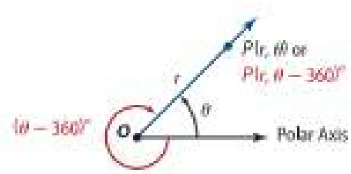
2A. $R\left(1.5, -\frac{7\pi}{6}\right)$ **2B.** $S(-2, -135°)$

StudyTip

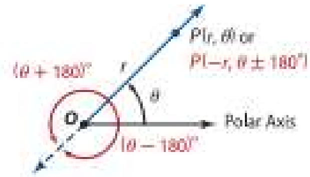
Pole The pole can be represented by $(0, \theta)$, where θ is any angle.

In a rectangular coordinate system, each point has a unique set of coordinates. This is *not* true in a polar coordinate system.A, you learned that a given angle has infinitely many coterminal angles. As a result, if a point has polar coordinates (r, θ), then it also has polar coordinates $(r, \theta \pm 360°)$ or $(r, \theta \pm 2\pi)$ as shown.





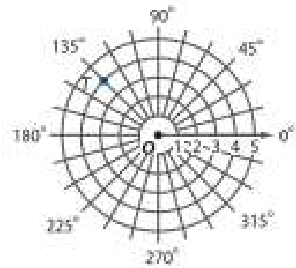
Additionally, because r is a directed distance, (r, θ) and $(-r, \theta \pm 180°)$ or $(-r, \theta \pm \pi)$ represent the same point as shown.



In general, if n is any integer, the point with polar coordinates (r, θ) can also be represented by polar coordinates of the form $(r, \theta + 360n°)$ or $(-r, \theta + (2n + 1)180°)$. Likewise, if θ is given in radians and n is any integer, the other representations of (r, θ) are of the form $(r, \theta + 2n\pi)$ or $(-r, \theta + (2n + 1)\pi)$.

Example 3 Multiple Representations of Polar Coordinates

Find four different pairs of polar coordinates that name point T if $-360° \leq \theta \leq 360°$.



One pair of polar coordinates that name point T is $(4, 135°)$. The other three representations are as follows.

$(4, 135°) = (4, 135° - 360°)$ Subtract 360° from θ.
$= (4, -225°)$

$(4, 135°) = (-4, 135° + 180°)$ Replace r with $-r$ and add 180° to θ.
$= (-4, 315°)$

$(4, 135°) = (-4, 135° - 180°)$ Replace r with $-r$ and subtract 180° from θ.
$= (-4, -45°)$

GuidedPractice

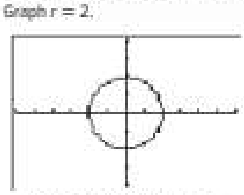
Find three additional pairs of polar coordinates that name the given point if $-360° \leq \theta \leq 360°$ or $-2\pi \leq \theta \leq \pi$.

3A. $(5, 240°)$ **3B.** $\left(2, \frac{\pi}{6}\right)$

2 Graphs of Polar Equations

An equation expressed in terms of polar coordinates is called a **polar equation**. For example, $r = 2 \sin \theta$ is a polar equation. A **polar graph** is the set of all points with coordinates (r, θ) that satisfy a given polar equation.

You already know how to graph equations in the Cartesian, or *rectangular*, coordinate system. Graphs of equations involving constants like $x = 2$ and $y = -3$ are considered basic in the Cartesian coordinate system. Similarly, the graphs of the polar equations $r = k$ and $\theta = k$, where k is a constant, are considered basic in the polar coordinate system.

> **Technology Tip**
>
> **Graphing Polar Equations**
> To graph the polar equation $r = 2$ on a graphing calculator, first press MODE and change the graphing setting from FUNC to POL. When you press Y=, notice that the dependent variable has changed from Y to r and the independent variable from x to θ. Graph $r = 2$.
>
> 
>
> $[0, 2\pi]$ scl: $\frac{\pi}{16}$ by $[-6, 6]$ scl: 1 by $[-4, 4]$ scl: 1

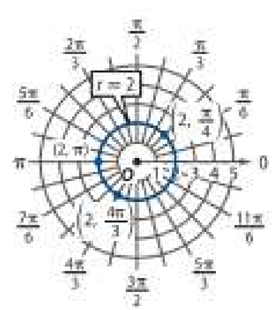
Example 4 Graph Polar Equations

Graph each polar equation.

a. $r = 2$

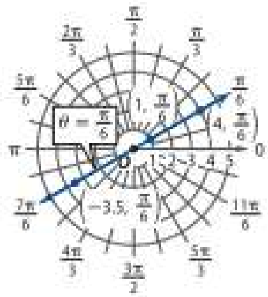
The solutions of $r = 2$ are ordered pairs of the form $(2, \theta)$, where θ is any real number.

The graph consists of all points that are 2 units from the pole, so the graph is a circle centered at the origin with radius 2.



b. $\theta = \frac{\pi}{6}$

The solutions of $\theta = \frac{\pi}{6}$ are ordered pairs of the form $\left(r, \frac{\pi}{6}\right)$, where r is any real number. The graph consists of all points on the line that makes an angle of $\frac{\pi}{6}$ with the positive polar axis.



GuidedPractice

Graph each polar equation.

4A. $r = 3$

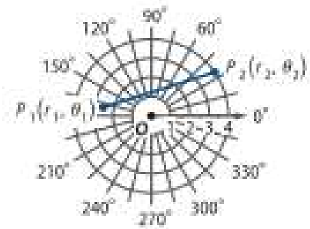
4B. $\theta = \frac{2\pi}{3}$

The distance between two points in the polar plane can be found using the following formula.

KeyConcept Polar Distance Formula

If $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ are two points in the polar plane, then the distance P_1P_2 is given by

$$\sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}.$$



You will prove this formula in Exercise 63.

WatchOut!

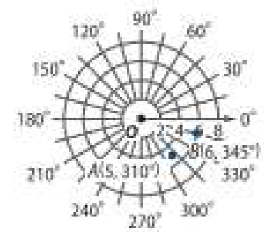
Mode When using the Polar Distance Formula, if θ is given in degrees, make sure your graphing calculator is set in degree mode.

Real-World Example 5 Find the Distance Between Polar Coordinates

AIR TRAFFIC An air traffic controller is tracking two airplanes that are flying at the same altitude. The coordinates of the planes are $A(5, 310°)$ and $B(6, 345°)$, where the directed distance is measured in kilometers.

a. Sketch a graph of this situation.

Airplane A is located 5 kilometers from the pole on the terminal side of the angle 310°, and airplane B is located 6 kilometers from the pole on the terminal side of the angle 345°, as shown.



b. If regulations prohibit airplanes from passing within three kilometers of each other, are these airplanes in violation? Explain.

Use the Polar Distance Formula.

$AB = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}$ Polar Distance Formula

$= \sqrt{5^2 + 6^2 - 2(5)(6)\cos(345° - 310°)}$ or about 3.44 $(r_2, \theta_2) = (6, 345°)$ and $(r_1, \theta_1) = (5, 310°)$

The planes are about 3.44 kilometers apart, so they are not in violation of this regulation.

GuidedPractice

5. BOATS A naval radar is tracking two aircraft carriers. The coordinates of the two carriers are $(8, 150°)$ and $(3, 65°)$, with r measured in kilometers.

A. Sketch a graph of this situation.

B. What is the distance between the two aircraft carriers?

Real-WorldLink

Germany developed a radar device in 1936 that could detect planes in a 128-kilometer radius. The following year, Germany was credited with supplying a battleship, the *Graf Spee*, with the first radar system.

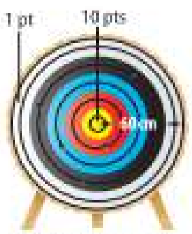
Source: A History of the World Semiconductor Industry

Exercises

Graph each point on a polar grid. (Examples 1 and 2)

1. $R(1, 120°)$
2. $T(-2.5, 330°)$
3. $F\left(-2, \frac{2\pi}{3}\right)$
4. $A\left(3, \frac{\pi}{6}\right)$
5. $Q\left(4, -\frac{5\pi}{6}\right)$
6. $B(5, -60°)$
7. $D\left(-1, -\frac{5\pi}{3}\right)$
8. $G\left(3.5, -\frac{11\pi}{6}\right)$
9. $C(-4, \pi)$
10. $M(0.5, 270°)$
11. $P(4.5, -210°)$
12. $W(-1.5, 150°)$

13. **ARCHERY** The target in competitive target archery consists of 10 evenly spaced concentric circles with score values from 1 to 10 points from the outer circle to the center. Suppose an archer using a target with a 60-centimeter radius shoots arrows at (57, 45°), (41, 315°), and (15, 240°). (Examples 1 and 2)

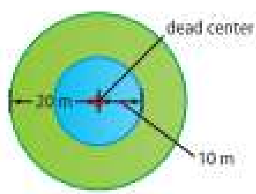


   a. Plot the points where the archer's arrows hit the target on a polar grid.
   b. How many points did the archer earn?

Find three different pairs of polar coordinates that name the given point if $-360° \le \theta \le 360°$ or $-2\pi \le \theta \le 2\pi$. (Example 3)

14. $(1, 150°)$
15. $(-2, 300°)$
16. $\left(4, -\frac{7\pi}{6}\right)$
17. $\left(-3, \frac{2\pi}{3}\right)$
18. $\left(5, \frac{11\pi}{6}\right)$
19. $\left(-5, -\frac{4\pi}{3}\right)$
20. $(2, -30°)$
21. $(-1, -240°)$

22. **SKYDIVING** In competitive accuracy landing, skydivers attempt to land as near as possible to "dead center," the center of a target marked by a disk 2 meters in diameter. (Example 4)



   a. Write polar equations representing the three target boundaries.
   b. Graph the equations on a polar grid.

Graph each polar equation. (Example 4)

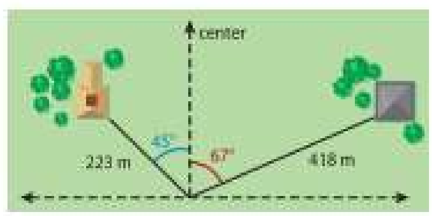
23. $r = 4$
24. $\theta = 225°$
25. $r = 1.5$
26. $\theta = -\frac{7\pi}{6}$
27. $\theta = -15°$
28. $r = -3.5$

29. **DARTBOARD** A certain dartboard has a radius of 225 millimeters. The bull's-eye has two sections. The 50-point section has a radius of 6.3 millimeters. The 25-point section surrounds the 50-point section for an additional 9.7 millimeters. (Example 4)
   a. Write and graph polar equations representing the boundaries of the dartboard and these sections.
   b. What percentage of the dartboard's area does the bull's-eye comprise?

Find the distance between each pair of points. (Example 5)

30. $(2, 30°), (5, 120°)$
31. $\left(3, \frac{\pi}{2}\right), \left(8, \frac{4\pi}{3}\right)$
32. $(6, 45°), (-3, 300°)$
33. $\left(7, -\frac{\pi}{3}\right), \left(1, \frac{2\pi}{3}\right)$
34. $\left(-5, \frac{7\pi}{6}\right), \left(4, \frac{\pi}{6}\right)$ 1
35. $(4, -315°), (1, 60°)$
36. $(-2, -30°), (8, 210°)$
37. $\left(-3, \frac{11\pi}{6}\right), \left(-2, \frac{5\pi}{6}\right)$
38. $\left(1, -\frac{\pi}{4}\right), \left(-5, \frac{7\pi}{6}\right)$
39. $(7, -90°), (-4, -330°)$
40. $\left(8, -\frac{2\pi}{3}\right), \left(4, -\frac{3\pi}{4}\right)$
41. $(-5, 135°), (-1, 240°)$

42. **SURVEYING** A surveyor mapping out the land where a new housing development will be built identifies a landmark 223 meters away and 45° left of center. A second landmark is 418 meters away and 67° right of center. Determine the distance between the two landmarks. (Example 5)



43. B **SURVEILLANCE** A mounted surveillance camera oscillates and views part of a circular region determined by $-60° \le \theta \le 150°$ and $0 \le r \le 40$, where r is in meters.
   a. Sketch a graph of the region that the security camera can view on a polar grid.
   b. Find the area of the region.

Find a different pair of polar coordinates for each point such that $0 \le \theta \le 180°$ or $0 \le \theta \le \pi$.

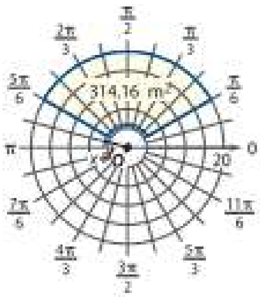
44. $(5, 960°)$
45. $\left(-2.5, \frac{5\pi}{2}\right)$
46. $\left(4, \frac{11\pi}{4}\right)$
47. $(1.25, -920°)$
48. $\left(-1, -\frac{21\pi}{8}\right)$
49. $(-6, -1460°)$

50. **AMPHITHEATER** Suppose a singer is performing at an amphitheater. We can model this situation with polar coordinates by assuming that the singer is standing at the pole and is facing the direction of the polar axis. The seats can then be described as occupying the area defined by $-45° \le \theta \le 45°$ and $30 \le r \le 240$, where r is measured in meters.

a. Sketch a graph of this region on a polar grid.

b. If each person needs 5 square meters of space, how many seats can fit in the amphitheater?

51. **SECURITY** A security light that is mounted above a house illuminates part of a circular region defined by $\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$ and $x \le r \le 20$, where r is measured in meters. If the total area of the region is approximately 314.16 square meters, find x.



C

Find a value for the missing coordinate that satisfies the following condition.

52. $P_1 = (3, 35°)$; $P_2 = (r, 75°)$; $P_1P_2 = 4.174$

53. $P_1 = (5, 125°)$; $P_2 = (2, \theta)$; $P_1P_2 = 4$; $0 \le \theta \le 180°$

54. $P_1 = (3, \theta)$; $P_2 = \left(4, \frac{7\pi}{9}\right)$; $P_1P_2 = 5$; $0 \le \theta \le \pi$

55. $P_1 = (r, 120°)$; $P_2 = (4, 160°)$; $P_1P_2 = 3.297$

56. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the relationship between polar coordinates and rectangular coordinates.

a. **GRAPHICAL** Plot points $A\left(2, \frac{\pi}{3}\right)$ and $B\left(4, \frac{5\pi}{6}\right)$ on a polar grid. Sketch a rectangular coordinate system on top of the polar grid so that the origins coincide and the x-axis aligns with the polar axis. The y-axis should align with the line $\theta = \frac{\pi}{2}$. Form one right triangle by connecting point A to the origin and perpendicular to the x-axis. Form another right triangle by connecting point B to the origin and perpendicular to the x-axis.

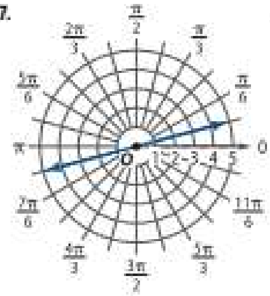
b. **NUMERICAL** Calculate the lengths of the legs of each triangle.

c. **ANALYTICAL** How do the lengths of the legs relate to rectangular coordinates for each point?

d. **ANALYTICAL** Explain the relationship between the polar coordinates (r, θ) and the rectangular coordinates (x, y).

Write an equation for each polar graph.

57.



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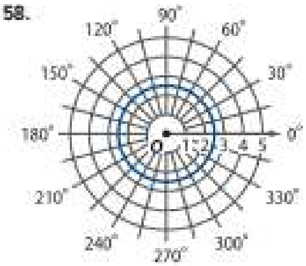

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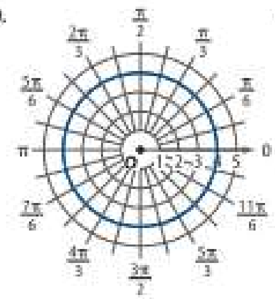
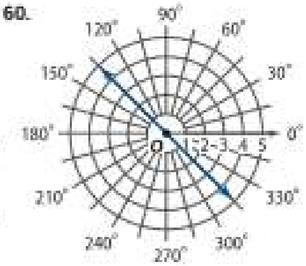


60.



H.O.T. Problems Use Higher-Order Thinking Skills

61. **REASONING** Explain why the order of the points used in the Polar Distance Formula is not important. That is, why can you choose either point to be P_1 and the other to be P_2?

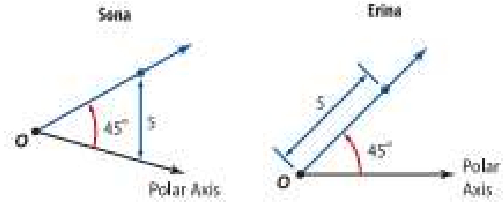
62. **CHALLENGE** Find an ordered pair of polar coordinates to represent the point with rectangular coordinates $(-3, -4)$. Round the angle measure to the nearest degree.

63. **PROOF** Prove that the distance between two points with polar coordinates $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ is

$$P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}.$$

64. **REASONING** Describe what happens to the Polar Distance Formula when $\theta_2 - \theta_1 = \frac{\pi}{2}$. Explain this change.

65. **ERROR ANALYSIS** Sona and Suhaila both graphed the polar coordinates (5, 45°). Is either of them correct? Explain your reasoning.



66. **WRITING IN MATH** Make a conjecture as to why having the polar coordinates for an aircraft is not enough to determine its exact position.

Spiral Review

Find the dot product of u and v. Then determine if u and v are orthogonal.

67. $\mathbf{u} = \langle 4, 10, 1 \rangle, \mathbf{v} = \langle -5, 1, 7 \rangle$

68. $\mathbf{u} = \langle -5, 4, 2 \rangle, \mathbf{v} = \langle -4, -9, 8 \rangle$

69. $\mathbf{u} = \langle -8, -3, 12 \rangle, \mathbf{v} = \langle 4, -6, 0 \rangle$

Find each of the following for $\mathbf{a} = \langle -4, 3, -2 \rangle$, $\mathbf{b} = \langle 2, 5, 1 \rangle$, and $\mathbf{c} = \langle 3, -6, 5 \rangle$.

70. $3\mathbf{a} + 2\mathbf{b} + 8\mathbf{c}$

71. $-2\mathbf{a} + 4\mathbf{b} - 5\mathbf{c}$

72. $5\mathbf{a} - 9\mathbf{b} + \mathbf{c}$

For each equation, identify the vertex, focus, axis of symmetry, and directrix. Then graph the parabola.

73. $-14(x - 2) = (y - 7)^2$

74. $(x - 7)^2 = -32(y - 12)$

75. $y = \frac{1}{2}x^2 - 3x + \frac{19}{2}$

76. **STATE FAIR** If Hareb and Zayed each purchased the number of game and ride tickets shown below, what was the price for each type of ticket?

Person	Ticket Type	Tickets	Total ($)
Hareb	game ride	6 15	93
Zayed	game ride	7 12	81

Write the augmented matrix for the system of linear equations.

77. $12w + 14x - 10y = 23$
$4w - 5y + 6z = 33$
$11w - 13x + 2z = -19$
$19x - 6y + 7z = -25$

78. $-6x + 2y + 5z = 18$
$5x - 7y + 3z = -8$
$y - 12z = -22$
$8x - 3y + 2z = 9$

79. $x + 8y - 3z = 25$
$2x - 5y + 11z = 13$
$-5x + 8z = 26$
$y - 4z = 17$

Solve each equation for all values of x.

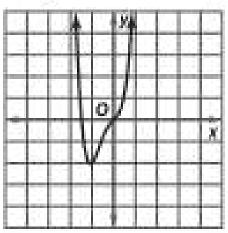
80. $2\cos^2 x + 5\sin x - 5 = 0$

81. $\tan^2 x + 2\tan x + 1 = 0$

82. $\cos^2 x + 3\cos x = -2$

Skills Review for Standardized Tests

83. **SAT/ACT** If the figure shows part of the graph of $f(x)$, which of the following could be the range of $f(x)$?



A $\{y \mid y \geq -2\}$

B $\{y \mid y \leq -2\}$

C $\{y \mid -2 < y < 1\}$

D $\{y \mid -2 \leq y \leq 1\}$

E $\{y \mid y > -2\}$

84. **REVIEW** Which of the following is the component form of $\overline{RS}$ with initial point $R(-5, 3)$ and terminal point $S(2, -7)$?

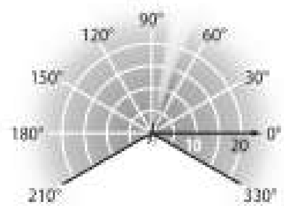
F $\langle 7, -10 \rangle$

G $\langle -3, 10 \rangle$

H $\langle -7, 10 \rangle$

J $\langle -3, -10 \rangle$

85. The lawn sprinkler shown can cover the part of the circular region determined by the polar inequalities $-30° \leq \theta \leq 210°$ and $0 \leq r \leq 20$, where r is measured in meters. What is the approximate area of this region?



A 821 m^2

B 838 m^2

C 852 m^2

D 866 m^2

86. **REVIEW** What type of conic is represented by $25y^2 = 400 + 16x^2$?

F circle

G ellipse

H hyperbola

J parabola

EXPLORE 2-2

Graphing Technology Lab
Investigate Graphs of Polar Equations

Objective

- Use a graphing calculator to explore the shape and symmetry of graphs of polar equations.

StudyTip

Square the Window To view the graphs in this activity without any distortion, square the window by selecting ZSquare under the ZOOM menu.

In Lesson 2-1, you graphed polar coordinates and simple polar equations on the polar coordinate system. Now you will explore the shape and symmetry of more complex graphs of polar equations by using a graphing calculator.

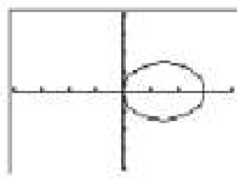
Activity Graph Polar Equations

Graph each equation. Then describe the shape and symmetry of the graph.

a. $r = 3 \cos \theta$

First, change the graph mode to polar and the angle mode to radians. Next, enter $r = 3 \cos \theta$ for r_1 in the Y= list. Use the viewing window shown.

The graph of $r = 3 \cos \theta$ is a circle with a center at (1.5, 0) and radius 1.5 units. The graph is symmetric with respect to the polar axis.

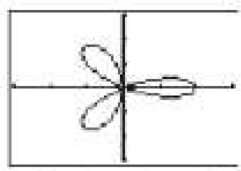


$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-4, 4]$ scl: 1 by $[-4, 4]$ scl: 1

b. $r = 2 \cos 3\theta$

Clear the equation from part **a** in the Y= list and insert $r = 2 \cos 3\theta$. Use the window shown.

The graph of $r = 2 \cos 3\theta$ is a classic polar curve called a rose, which will be covered in Lesson 2-2. The graph has 3 petals and is symmetric with respect to the polar axis.

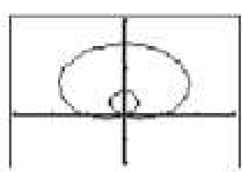


$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-3, 3]$ scl: 1 by $[-3, 3]$ scl: 1

c. $r = 1 + 2 \sin \theta$

Clear the equation from part **b** in the Y= list, and enter $r = 1 + 2 \sin \theta$. Adjust the window to display the entire graph.

The graph of $r = 1 + 2 \sin \theta$ is a classic polar curve called a *limaçon,* which will be covered in Lesson 2-2. The graph has a curve with an inner loop and is symmetric with respect to the line $\theta = \frac{\pi}{2}$.



$[0, 2\pi]$ scl: $\frac{\pi}{24}$ by $[-3, 3]$ scl: 1 by $[-2, 4]$ scl: 1

Exercises

Graph each equation. Then describe the shape and symmetry of the graph.

1. $r = -3 \cos \theta$
2. $r = 3 \sin \theta$
3. $r = -3 \sin \theta$
4. $r = 2 \cos 4\theta$
5. $r = 2 \cos 5\theta$
6. $r = 2 \cos 6\theta$
7. $r = 2 + 4 \sin \theta$
8. $r = 1 - 3 \sin \theta$
9. $r = 1 + 2 \sin (-\theta)$

Analyze the Results

10. **ANALYTICAL** Explain how each value affects the graph of the given equation.
    a. the value of n in $r = a \cos n\theta$
    b. the value of $|a|$ in $r = b \pm a \sin m\theta$

11. **MAKE A CONJECTURE** Without graphing $r = 10 \cos 24\theta$, describe the shape and symmetry of the graph. Explain your reasoning.

LESSON 2-2 Graphs of Polar Equations

::Then	::Now	::Why?
• You graphed functions in the rectangular coordinate system.	• **1** Graph polar equations. **2** Identify and graph classical curves.	• To reduce background noise, networks that broadcast sporting events use directional microphones to capture the sounds of the game. Directional microphones have the ability to pick up sound primarily from one direction or region. The sounds that these microphones can detect can be expressed as polar functions.

NewVocabulary
limaçon
cardioid
rose
lemniscate
spiral of Archimedes

1 Graphs of Polar Equations

When you graphed equations on a rectangular coordinate system, you began by using an equation to obtain a set of ordered pairs. You then plotted these coordinates as points and connected them with a smooth curve. In this lesson, you will approach the graphing of polar equations in a similar manner.

Example 1 Graph Polar Equations by Plotting Points

Graph each equation.

a. $r = \cos\theta$

Make a table of values to find the r-values corresponding to various values of θ on the interval $[0, 2\pi]$. Round each r-value to the nearest tenth.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$r = \cos\theta$	1	0.9	0.5	0	−0.5	−0.9	−1	−0.9	−0.5	0	0.5	0.9	1

Graph the ordered pairs (r, θ) and connect them with a smooth curve. It appears that the graph shown in Figure 2.2.1 is a circle with center at (0.5, 0) and radius 0.5 unit.

b. $r = \sin\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$r = \sin\theta$	0	0.5	0.9	1	0.9	0.5	0	−0.5	−0.9	−1	−0.9	−0.5	0

Graph the ordered pairs and connect them with a smooth curve. It appears that the graph shown in Figure 2.2.2 is a circle with center at $\left(0.5, \frac{\pi}{2}\right)$ and radius 0.5 unit.

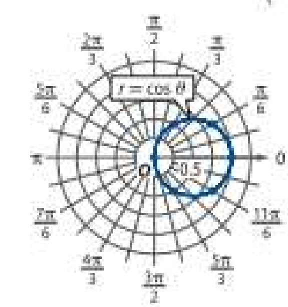


Figure 2.2.1

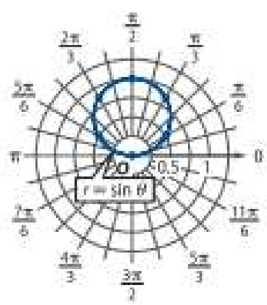


Figure 2.2.2

GuidedPractice

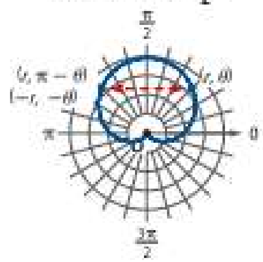
1A. $r = -\sin\theta$ **1B.** $r = 2\cos\theta$ **1C.** $r = \sec\theta$

Notice that as θ increases on $[0, 2\pi]$, each graph above is traced twice. This is because the polar coordinates obtained on $[0, \pi]$ represent the same points as those obtained on $[\pi, 2\pi]$.

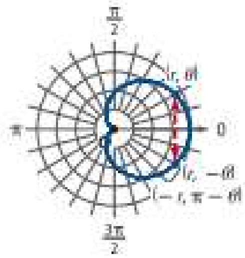
Like knowing whether a graph in the rectangular coordinate system has symmetry with respect to the x-axis, y-axis, or origin, knowing whether the graph of a polar equation is symmetric can help reduce the number of points needed to sketch its graph. Graphs of polar equations can be symmetric with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole, as shown below.

KeyConcept Symmetry of Polar Graphs

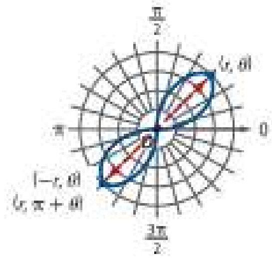
Symmetry with Respect to the Line $\theta = \frac{\pi}{2}$



Symmetry with Respect to Polar Axis



Symmetry with Respect to the Pole



The graphical definitions above provide a way of testing a polar equation for symmetry. For example, if replacing (r, θ) in a polar equation with $(r, -\theta)$ or $(-r, \pi - \theta)$ produces an equivalent equation, then its graph is symmetric with respect to the polar axis. If an equation passes one of the symmetry tests, this is sufficient to guarantee that the equation has that type of symmetry. The converse, however, is *not* true. If a polar equation fails all of these tests, the graph may still have symmetry.

Example 2 Polar Axis Symmetry

Use symmetry to graph $r = 1 - 2\cos\theta$.

Replacing (r, θ) with $(r, -\theta)$ yields $r = 1 - 2\cos(-\theta)$. Because cosine is an even function, $\cos(-\theta) = \cos\theta$, so this equation simplifies to $r = 1 - 2\cos\theta$. Because the replacement produced an equation equivalent to the original equation, the graph of this equation is symmetric with respect to the polar axis.

Because of this symmetry, you need only make a table of values to find the r-values corresponding to θ on the interval $[0, \pi]$.

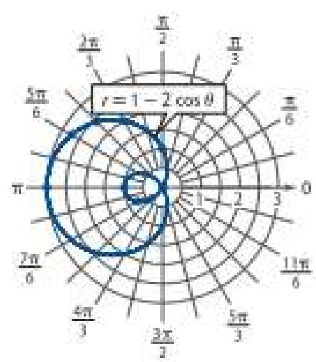
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$r = 1 - 2\cos\theta$	−1	−0.7	−0.4	0	1	2	2.4	2.7	3

StudyTip

Graphing Polar Equations It is customary to graph polar functions in radians, rather than in degrees.

Plotting these points and using polar axis symmetry, you obtain the graph shown.

The type of curve is called a **limaçon**. Some limaçons have inner loops like this one. Other limaçons come to a point, have a dimple, or just curve outward.



GuidedPractice

Use symmetry to graph each equation.

2A. $r = 1 - \cos\theta$

2B. $r = 2 + \cos\theta$

In Examples 1 and 2, notice that the graphs of $r = \cos\theta$ and $r = 1 - 2\cos\theta$ are symmetric with respect to the polar axis, while the graph of $r = \sin\theta$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$. These observations can be generalized as follows.

KeyConcept Quick Tests for Symmetry in Polar Graphs

Words The graph of a polar equation is symmetric with respect to

- the polar axis if it is a function of $\cos\theta$, and
- the line $\theta = \frac{\pi}{2}$ if it is a function of $\sin\theta$.

Example The graph of $r = 3 + \sin\theta$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

You will justify these tests for specific cases in Exercises 65–66.

Symmetry can be used to graph polar functions that model real-world situations.

Real-WorldLink

Live Aid was a 1985 rock concert held in an effort to raise $1 million for Ethiopian aid. Concerts in London, Philadelphia, and other cities were televised and viewed by about 1.9 billion people in 150 countries. The event raised a total of $190 million.

Source: CNN

Real-World Example 3 Symmetry with Respect to the Line $\theta = \frac{\pi}{2}$

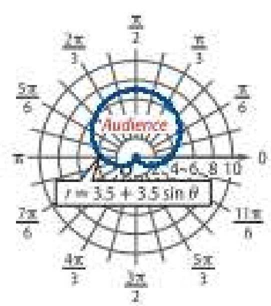
AUDIO TECHNOLOGY During a concert, a directional microphone was placed facing the audience from the center of stage to capture the crowd noise for a live recording. The area of sound the microphone captures can be represented by $r = 3.5 + 3.5\sin\theta$. Suppose the front of the stage faces due north.

a. Graph the polar pattern of the microphone.

Because this polar equation is a function of the sine function, it is symmetric with respect to the line $\theta = \frac{\pi}{2}$. Therefore, make a table and calculate the values of r on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$r = 3.5 + 3.5\sin\theta$	0	0.5	1.0	1.8	3.5	5.25	6.0	6.5	7

Plotting these points and using symmetry with respect to the line $\theta = \frac{\pi}{2}$, you obtain the graph shown. This type of curve is called a **cardioid** (CAR-dee-oid). A cardioid is a special limaçon that has a heart shape.



b. Describe what the polar pattern tells you about the microphone.

The polar pattern indicates that the microphone will pick up sounds up to 7 units away directly in front of the microphone and up to 3.5 units away directly to the left or right of the microphone.

WatchOut!

Graphing over the Period

Usually the period of the trigonometric function used in a polar equation is sufficient to trace the entire graph, but sometimes it is not. The best way to know if you have graphed enough to discern a pattern is to plot more points.

GuidedPractice

3. **VIDEOTAPING** A high school teacher is videotaping presentations performed by her students using a stationary video camera positioned in the back of the room. The area of sound captured by the camera's microphone can be represented by $r = 5 + 2\sin\theta$. Suppose the front of the classroom is due north of the camera.

   A. Graph the polar pattern of the microphone.

   B. Describe what the polar pattern tells you about the microphone.

Previously, you used maximum and minimum points along with zeros to aid in graphing trigonometric functions. On the graph of a polar function, r is at its maximum for a value of θ when the distance between that point (r, θ) and the pole is maximized. To find the maximum point(s) on the graph of a polar equation, find the θ-values for which $|r|$ is maximized. Additionally, if $r = 0$ for some value of θ, you know that the graph intersects the pole.

Example 4 Symmetry, Zeros, and Maximum r-Values

Use symmetry, zeros, and maximum r-values to graph $r = 2 \cos 3\theta$.

Determine the symmetry of the graph.

This function is symmetric with respect to the polar axis, so you can find points on the interval $[0, \pi]$ and then use polar axis symmetry to complete the graph.

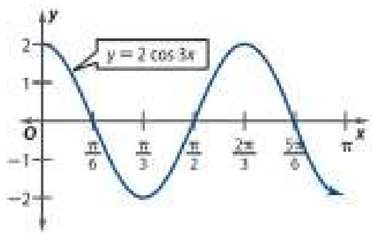
StudyTip

Alternative Method Solving the rectangular function $y = 2 \cos 3x$, we find that the function has extrema when $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, or π. Similarly, the function has zeros when $x = \frac{\pi}{6}, \frac{\pi}{2}$, or $\frac{5\pi}{6}$.

Find the zeros and the maximum r-value.

Sketch the graph of the rectangular function $y = 2 \cos 3x$ on the interval $[0, \pi]$.

From the graph, you can see that $|y| = 2$ when $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, and π and $y = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}$, and $\frac{5\pi}{6}$.



Interpreting these results in terms of the polar equation $r = 2 \cos 3\theta$, we can say that $|r|$ has a maximum value of 2 when $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, or π and $r = 0$ when $\theta = \frac{\pi}{6}, \frac{\pi}{2}$, or $\frac{5\pi}{6}$.

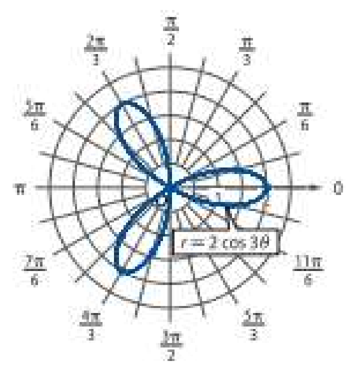
Graph the function.

Use these and a few additional points to sketch the graph of the function.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
$r = 2 \cos 3\theta$	2	1.4	0	−1	−2	−1.4	0	1.4	2	1.4	0	−1.4	−2

Notice that polar axis symmetry can be used to complete the graph after plotting points on $\left[0, \frac{\pi}{2}\right]$.

This type of curve is called a **rose**. Roses can have three or more equal loops.



GuidedPractice

Use symmetry, zeros, and maximum r-values to graph each function.

4A. $r = 3 \sin 2\theta$

4B. $r = \cos 5\theta$

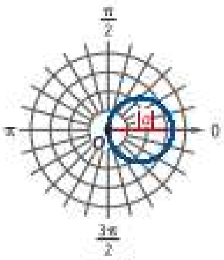
2 Classic Polar Curves

Circles, limaçons, cardioids, and roses are examples of classic curves. The forms and model graphs of these and other classic curves are summarized below.

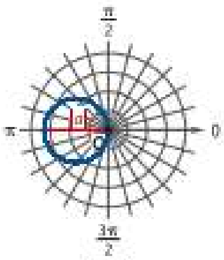
ConceptSummary Special Types of Polar Graphs

Circles

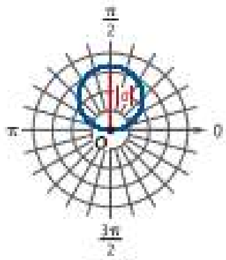
$r = a\cos\theta$ or $r = a\sin\theta$



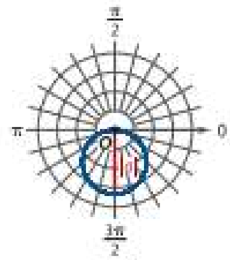
$a > 0$
$r = a\cos\theta$



$a < 0$
$r = a\cos\theta$



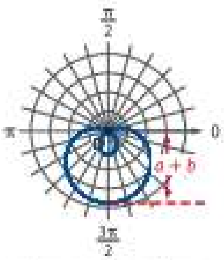
$a > 0$
$r = a\sin\theta$



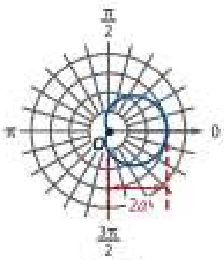
$a < 0$
$r = a\sin\theta$

Limaçons

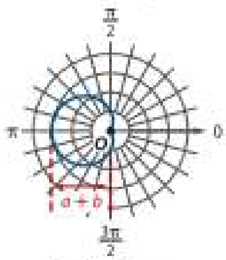
$r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$, where a and b are both positive



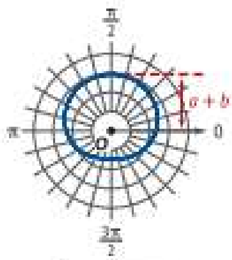
Limaçon with inner loop
$a < b$
$r = a - b\sin\theta$



Cardioid
$a = b$
$r = a + b\cos\theta$



Dimpled limaçon
$b < a < 2b$
$r = a - b\cos\theta$

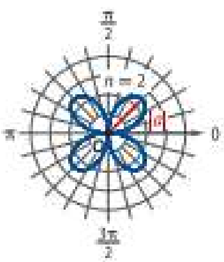


Convex limaçon
$a \geq 2b$
$r = a + b\sin\theta$

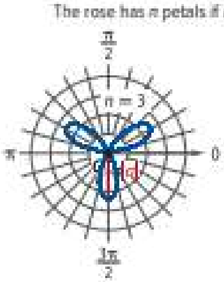
Roses

$r = a\cos n\theta$ or $r = a\sin n\theta$, where $n \geq 2$ is an integer

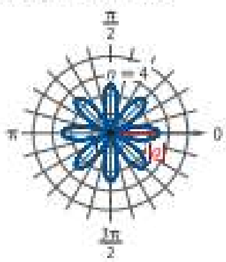
The rose has n petals if n is odd and $2n$ petals if n is even.



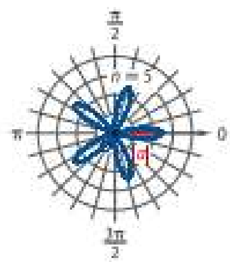
$r = a\sin n\theta$



$r = a\sin n\theta$



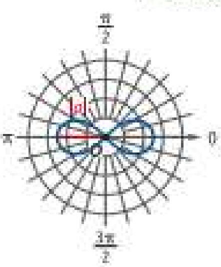
$r = a\cos n\theta$



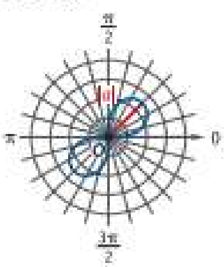
$r = a\cos n\theta$

Lemniscates (LEM-nis-keyts)

$r^2 = a^2\cos 2\theta$ or $r^2 = a^2\sin 2\theta$



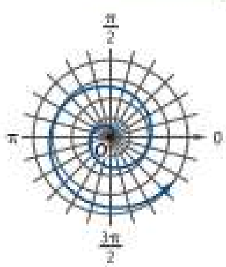
$r^2 = a^2\cos 2\theta$



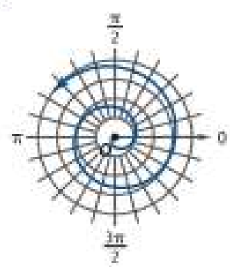
$r^2 = a^2\sin 2\theta$

Spirals of Archimedes (ahr-kuh-MEE-deez)

$r = a\theta + b$



$\theta > 0, r = a\theta + b$



$\theta < 0, r = a\theta + b$

Example 5 Identify and Graph Classic Curves

Identify the type of curve given by each equation. Then use symmetry, zeros, and maximum r-values to graph the function.

a. $r^2 = 16 \sin 2\theta$

Type of Curve and Symmetry

The equation is of the form $r^2 = a^2 \sin 2\theta$, so its graph is a lemniscate. Replacing (r, θ) with $(-r, \theta)$ yields $(-r)^2 = 16 \sin 2\theta$ or $r^2 = 16 \sin 2\theta$. Therefore, the function has symmetry with respect to the pole.

Maximum r-Value and Zeros

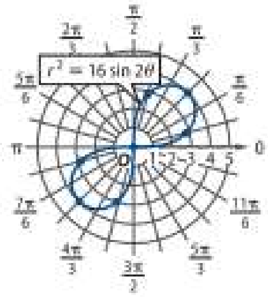
The equation $r^2 = 16 \sin 2\theta$ is equivalent to $r = \pm 4\sqrt{\sin 2\theta}$, which is undefined when $\sin 2\theta < 0$. Therefore, the domain of the function is restricted to the intervals $\left[0, \frac{\pi}{2}\right]$ or $\left[\pi, \frac{3\pi}{2}\right]$.

Because you can use pole symmetry, you need only graph points in the interval $\left[0, \frac{\pi}{2}\right]$. The function attains a maximum r-value of $|a|$ or 4 when $\theta = \frac{\pi}{4}$ and zero r-value when $x = 0$ and $\frac{\pi}{2}$.

Graph

Use these points and the indicated symmetry to sketch the graph of the function.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	0	±2.8	±3.7	±4	±3.7	±2.8	0



b. $r = 3\theta$

Type of Curve and Symmetry

The equation is of the form $r = a\theta + b$, so its graph is a spiral of Archimedes. Replacing (r, θ) with $(-r, -\theta)$ yields $(-r) = 3(-\theta)$ or $r = 3\theta$. Therefore, the function has symmetry with respect to the line $\theta = \frac{\pi}{2}$.

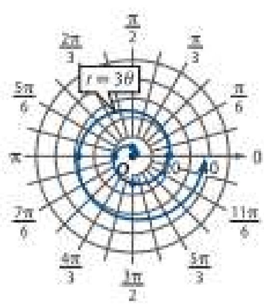
Maximum r-Value and Zeros

Spirals are unbounded. Therefore, the function has no maximum r-values and only one zero when $\theta = 0$.

Graph

Use points on the interval $[0, 4\pi]$ to sketch the graph of the function. To show symmetry, points on the interval $[-4\pi, 0]$ should also be graphed.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	3π	4π
r	0	2.4	4.7	9.4	14.1	18.8	28.3	37.7



Technology Tip

Window Settings θmin and θmax determine the values of θ that will be graphed. Normal settings for these are θmin=0 and θmax=2π, although it may be necessary to change these values to obtain a complete graph. θstep determines the interval for plotting points. The smaller this value is, the smoother the look of the graph.

Guided Practice

5A. $r^2 = 9 \cos 2\theta$

5B. $r = 3 \sin 5\theta$

Exercises

Graph each equation by plotting points. (Example 1)

1. $r = -\cos\theta$
2. $r = \csc\theta$
3. $r = \frac{1}{2}\cos\theta$
4. $r = 3\sin\theta$
5. $r = -\sec\theta$
6. $r = \frac{1}{3}\sin\theta$
7. $r = -4\cos\theta$
8. $r = -\csc\theta$

Use symmetry to graph each equation. (Examples 2 and 3)

9. $r = 3 + 3\cos\theta$
10. $r = 1 + 2\sin\theta$
11. $r = 4 - 3\cos\theta$
12. $r = 2 + 4\cos\theta$
13. $r = 2 - 2\sin\theta$
14. $r = 3 - 5\cos\theta$
15. $r = 5 + 4\sin\theta$
16. $r = 6 - 2\sin\theta$

Use symmetry, zeros, and maximum r-values to graph each function. (Example 4)

17. $r = \sin 4\theta$
18. $r = 2\cos 2\theta$
19. $r = 5\cos 3\theta$
20. $r = 3\sin 2\theta$
21. $r = \frac{1}{2}\sin 3\theta$
22. $r = 4\cos 5\theta$
23. $r = 2\sin 5\theta$
24. $r = 3\cos 4\theta$

25. **MARINE BIOLOGY** Rose curves can be observed in marine wildlife. Determine the symmetry, zeros, and maximum r-values of each function modeling a marine species for $0 \le \theta \le \pi$. Then use the information to graph the function. (Example 4)

a. The pores forming the petal pattern of a sand dollar (Figure 2.2.3) can be modeled by $r = 3\cos 5\theta$.

b. The outline of the body of a crown-of-thorns sea star (Figure 2.2.4) can be modeled by $r = 20\cos 8\theta$.

Figure 2.2.3

Figure 2.2.4

Identify the type of curve given by each equation. Then use symmetry, zeros, and maximum r-values to graph the function. (Example 5)

26. $r = \frac{1}{3}\cos\theta$
27. $r = 4\theta + 1;\ \theta > 0$
28. $r = 2\sin 4\theta$
29. $r = 6 + 6\cos\theta$
30. $r^2 = 4\cos 2\theta$
31. $r = 5\theta + 2;\ \theta > 0$
32. $r = 3 - 2\sin\theta$
33. $r^2 = 9\sin 2\theta$

34. **FIGURE SKATING** The original focus of figure skating was to carve figures, known as *compulsory figures*, into the ice. The shape of one of these figures can be modeled by $r^2 = 25\cos 2\theta$. (Example 5)

a. Which classic curve does the figure model?

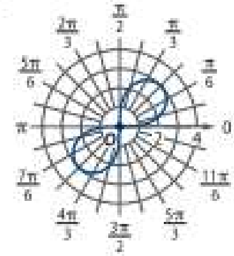
b. Graph the model.

B

Write an equation for each graph.

35.

36.



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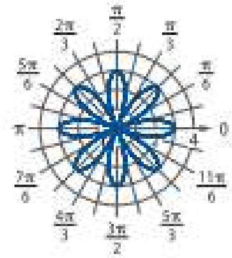

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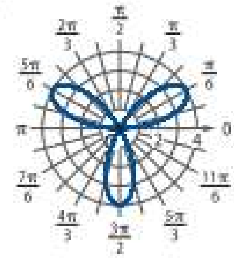

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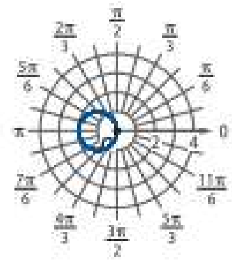

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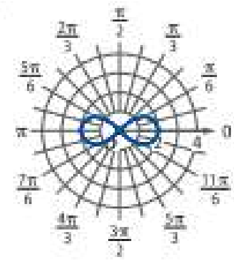


41. **FAN** A ceiling fan has a central motor with five blades that each extend 4 units from the center. The shape of the fan can be represented by a rose curve.

a. Write two polar equations that can be used to represent the fan.

b. Sketch two graphs of the fan using the equations that you wrote.

Use one of the three tests to prove the specified symmetry.

42. $r = 3 + \sin\theta$, symmetric about the line $\theta = \frac{\pi}{2}$
43. $r^2 = 4\sin 2\theta$, symmetric about the pole
44. $r = 3\sin 2\theta$, symmetric about the polar axis
45. $r = 5\cos 8\theta$, symmetric about the line $\theta = \frac{\pi}{2}$
46. $r = 2\sin 4\theta$, symmetric about the pole

47. **FOUR-LEAF CLOVER** The shape of a certain type of clover can be represented using a rose curve. Write a polar equation for the clover if it has:

a. 5 petals with a length of 2 units each.

b. 4 petals with a length of 7 units each.

c. 8 petals with a length of 6 units each.

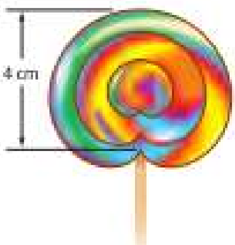
48. **CONCERT** For a concert, a circular stage is constructed and placed in the center so fans can completely surround the musicians. To record the sound of the crowd, two directional microphones are placed next to each other on the stage, one facing due east and the other facing due west. The patterns of the microphones can be represented by the polar equations $r = 2.5 + 2.5 \cos \theta$ and $r = -2.5 - 2.5 \cos \theta$.

a. Identify the type of curve given by each polar equation.

b. Sketch the graph of each microphone pattern on the same polar grid.

c. Describe what the graph tells you about the area covered by the microphones.

49. **CANDY** Write an equation that can model this lollipop in the shape of a limaçon if it is symmetric with respect to the line $\theta = \frac{\pi}{2}$ and measures 4 centimeters from the top of the lollipop to where the candy meets the stick.



Match each equation with its graph.

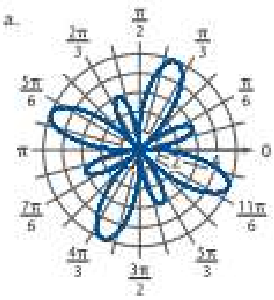
50. $r = 1 + 4 \cos 3\theta$

51. $r = 1 - 4 \sin 4\theta$

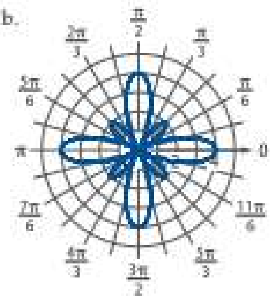
52. $r = 1 - 3 \sin 3\theta$

53. $r = 1 + 3 \cos 4\theta$

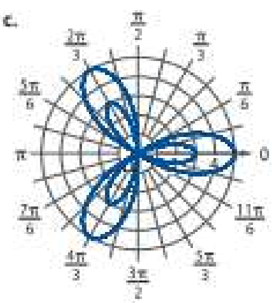
a.



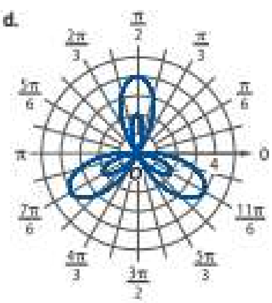
b.



c.



d.



Find x for the interval $0 \leq \theta \leq x$ so that x is a minimum and the graph is complete.

54. $r = 3 + 2 \cos \theta$

55 $r = 2 - \sin 2\theta$

56. $r = 1 + \cos \frac{\theta}{3}$

Match each equation with an equation that produces an equivalent graph.

57. $r = 5 + 4 \cos \theta$ | **a.** $r = 5 + 4 \sin \theta$

58. $r = -5 + 4 \sin \theta$ | **b.** $r = -5 + 4 \cos \theta$

59. $r = 5 - 4 \sin \theta$ | **c.** $r = 5 - 4 \cos \theta$

60. $r = -5 - 4 \cos \theta$ | **d.** $r = -5 - 4 \sin \theta$

61. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate a spiral of Archimedes.

a. GRAPHICAL Sketch separate graphs of $r = \theta$ for the intervals $0 \leq \theta \leq 3\pi$, $-3\pi \leq \theta \leq 0$, and $-3\pi \leq \theta \leq 3\pi$.

b. VERBAL Make a conjecture as to the symmetry of $r = \theta$. Explain your reasoning.

c. ANALYTICAL Prove your conjecture from part **b** by using one of the symmetry tests discussed in this lesson.

d. VERBAL How does changing the interval for θ affect the other classic curves? How does this differ from how the interval affects a spiral of Archimedes? Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

62. **ERROR ANALYSIS** Halima and Eiman are graphing polar equations. Eiman says that $r = 7 \sin 2\theta$ is not a function because it does not pass the vertical line test. Halima says the vertical line test does not apply in a polar grid. Is either of them correct? Explain your reasoning.

63. **REASONING** Sketch the graphs of $r_1 = \cos \theta$, $r_2 = \cos\left(\theta - \frac{\pi}{2}\right)$, and $r_3 = \cos(\theta - \pi)$ on the same polar grid. Describe the relationship between the three graphs. Make a conjecture as to the change in a graph when a value d is subtracted from θ.

64. **CHALLENGE** Solve the following system of polar equations algebraically on $[0, 2\pi]$. Graph the system and compare the points of intersection with the solutions that you found. Explain any discrepancies.

$r = 1 + 2 \sin \theta$

$r = 4 \sin \theta$

65. **PROOF** Prove that the graph of $r = a + b \cos 2\theta$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

66. **PROOF** Prove that the graph of $r = a \sin 2\theta$ is symmetric with respect to the polar axis.

67. **WRITING IN MATH** Describe the effect of a in the graph of $r = a \cos \theta$.

68. **OPEN ENDED** Sketch the graph of a rose with 8 petals. Then write the equation for your graph.

Spiral Review

Graph each polar equation. (Lesson 2-1)

69. $r = 3.5$

70. $\theta = -\frac{\pi}{3}$

71. $\theta = 225°$

Find the angle θ between vectors u and v to the nearest tenth of a degree.

72. $\mathbf{u} = \langle 4, -3, 5\rangle, \mathbf{v} = \langle 2, 6, -8\rangle$

73. $\mathbf{u} = 2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}, \mathbf{v} = 5\mathbf{i} + 6\mathbf{j} - 11\mathbf{k}$

74. $\mathbf{u} = \langle -1, 1, 5\rangle, \mathbf{v} = \langle 7, -6, 9\rangle$

Let $\overrightarrow{DE}$ be the vector with the given initial and terminal points. Write $\overrightarrow{DE}$ as a linear combinatioin of the vectors i and j.

75. $D\left(-5, \frac{2}{3}\right), E\left(-\frac{4}{5}, 0\right)$

76. $D\left(-\frac{1}{2}, \frac{4}{7}\right), E\left(-\frac{3}{4}, \frac{5}{7}\right)$

77. $D(9.7, -2.4), E(-6.1, -8.5)$

78. **YARDWORK** Ahmed is pushing a wheelbarrow full of leaves with a force of 525 newtons at a 48° angle with the ground.

a. Draw a diagram that shows the resolution of the force that Kyle is exerting into its rectangular components.

b. Find the magnitudes of the horizontal and vertical components of the force.

Graph the hyperbola given by each equation.

79. $\frac{x^2}{9} - \frac{y^2}{25} = 1$

80. $\frac{(y-4)^2}{16} - \frac{(x+2)^2}{9} = 1$

81. $\frac{(x+1)^2}{4} - \frac{(y+3)^2}{9} = 1$

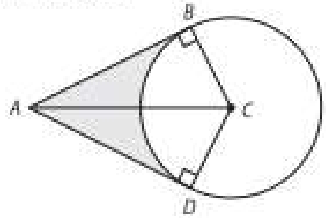
Write an equation for and graph each parabola with focus F and the given characteristics.

82. $F(-5, 8)$; opens right; contains $(-5, 12)$

83. $F(-1, -5)$; opens left; contains $(-1, 5)$

Skills Review for Standardized Tests

84. **SAT/ACT** In the figure, C is the center of the circle, $AC = 12$, and $m\angle BAD = 60°$. What is the perimeter of the shaded region?



A $12 + 3\pi$

B $6\sqrt{3} + 4\pi$

C $6\sqrt{3} + 3\pi$

D $12\sqrt{3} + 3\pi$

E $12\sqrt{3} + 4\pi$

85. **REVIEW** While mapping a level site, a surveyor identifies a landmark 450 meters away and 30° left of center and another landmark 600 meters away and 50° right of center. What is the approximate distance between the two landmarks?

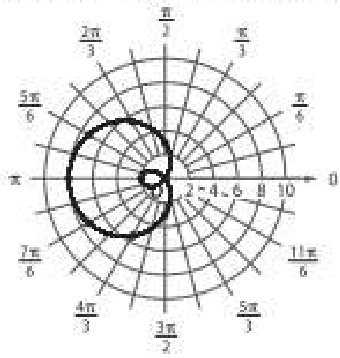
F 672 m

G 685 m

H 691 m

J 703 m

86. Which type of curve does the figure represent?



A lemniscate

B limaçon

C rose

D cardioid

87. **REVIEW** An air traffic controller is tracking two jets at the same altitude. The coordinates of the jets are (5, 310°) and (6, 345°), with r measured in kilometers. What is the approximate distance between the jets?

F 2.97 km

G 3.25 km

H 3.44 km

J 3.71 km

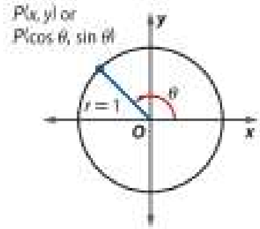
LESSON 2-3 Polar and Rectangular Forms of Equations

::Then	::Now	::Why?
• You used a polar coordinate system to graph points and equations. (Lessons 2-1 and 2-2)	• **1** Convert between polar and rectangular coordinates. **2** Convert between polar and rectangular equations.	• An ultrasonic sensor attached to a robot emits an outward beam that rotates through a full circle. The sensor receives a return signal when the beam intercepts an object, and it calculates the position of the object in terms of its distance r and the angle measure θ relative to the front of the robot. The sensor relays these polar coordinates to the robot, which converts them to rectangular coordinates so it can plot the object on an internal map.

1 Polar and Rectangular Coordinates

The coordinates of a point $P(x, y)$ corresponding to an angle θ on a unit circle with radius 1 can be written in terms of θ as $P(\cos\theta, \sin\theta)$ because

$$\cos\theta = \frac{x}{r} = \frac{x}{1} \text{ or } x \qquad \text{and} \qquad \sin\theta = \frac{y}{r} = \frac{y}{1} \text{ or } y.$$



If we let r take on any real value, we can write a point $P(x, y)$ in terms of both r and θ.

$$\cos\theta = \frac{x}{r} \qquad \text{and} \qquad \sin\theta = \frac{y}{r}$$

$$r\cos\theta = x \qquad\qquad r\sin\theta = y \qquad \text{Multiply each side by } r.$$

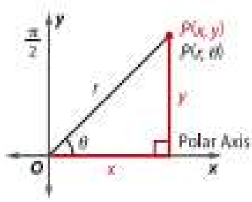
If we let the polar axis and pole in the polar coordinate system coincide with the positive x-axis and origin in the rectangular coordinate system, respectively, we now have a means of converting polar coordinates to rectangular coordinates.

KeyConcept Convert Polar to Rectangular Coordinates

If a point P has polar coordinates (r, θ), then the rectangular coordinates (x, y) of P are given by

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

That is, $(x, y) = (r\cos\theta, r\sin\theta)$.



Example 1 Polar Coordinates to Rectangular Coordinates

Find the rectangular coordinates for each point with the given polar coordinates.

a. $P\left(4, \frac{\pi}{6}\right)$

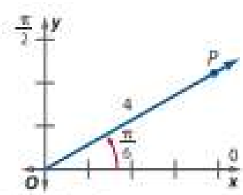
For $P\left(4, \frac{\pi}{6}\right)$, $r = 4$ and $\theta = \frac{\pi}{6}$.

$$x = r\cos\theta \qquad \text{Conversion formula} \qquad y = r\sin\theta$$

$$= 4\cos\frac{\pi}{6} \qquad r = 4 \text{ and } \theta = \frac{\pi}{6} \qquad = 4\sin\frac{\pi}{6}$$

$$= 4\left(\frac{\sqrt{3}}{2}\right) \qquad \text{Simplify.} \qquad = 4\left(\frac{1}{2}\right)$$

$$= 2\sqrt{3} \qquad\qquad = 2$$

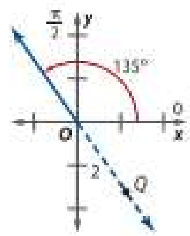


The rectangular coordinates of P are $(2\sqrt{3}, 2)$ or approximately (3.46, 2) as shown.

b. $Q(-2, 135°)$

For $Q(-2, 135°)$, $r = -2$ and $\theta = 135°$.

$x = r \cos \theta$	Conversion formula	$y = r \sin \theta$
$= -2 \cos 135°$	$r = -2$ and $\theta = 135°$	$= -2 \sin 135°$
$= -2\left(-\frac{\sqrt{2}}{2}\right)$	Simplify.	$= -2\left(\frac{\sqrt{2}}{2}\right)$
$= \sqrt{2}$		$= -\sqrt{2}$

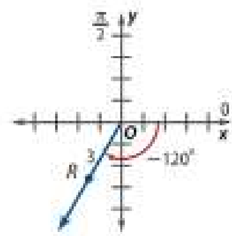


The rectangular coordinates of Q are $(\sqrt{2}, -\sqrt{2})$ or approximately $(1.41, -1.41)$ as shown.

c. $V(3, -120°)$

For $V(3, -120°)$, $r = 3$ and $\theta = -120°$.

$x = r \cos \theta$	Conversion formula	$y = r \sin \theta$
$= 3 \cos -120°$	$r = 3$ and $\theta = -120°$	$= 3 \sin -120°$
$= 3\left(-\frac{1}{2}\right)$	Simplify.	$= 3\left(-\frac{\sqrt{3}}{2}\right)$
$= -\frac{3}{2}$		$= -\frac{3\sqrt{3}}{2}$



The rectangular coordinates of V are $\left(-\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$ or approximately $(-1.5, -2.6)$ as shown.

GuidedPractice

1A. $R(-6, -120°)$ **1B.** $S\left(5, \frac{\pi}{3}\right)$ **1C.** $T(-3, 45°)$

StudyTip

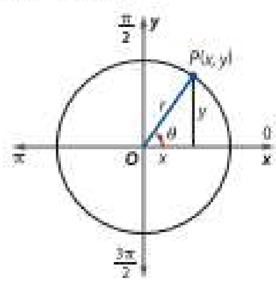
Coordinate Conversions The process for converting rectangular coordinates to polar coordinates is the same as the process used to determine the magnitude and direction of vectors.

To write a pair of rectangular coordinates in polar form, you need to find the distance r a point (x, y) is from the origin or pole and the angle measure θ that point is from the x- or polar axis.

To find the distance r from the point (x, y) to the origin, use the Pythagorean Theorem.

$r^2 = x^2 + y^2$ Pythagorean Theorem

$r = \sqrt{x^2 + y^2}$ Take the positive square root of each side.

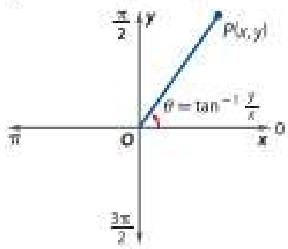


The angle θ is related to x and y by the tangent function.

$\tan \theta = \frac{y}{x}$ Tangent Ratio

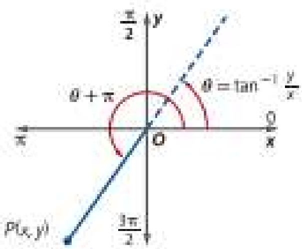
$\theta = \tan^{-1} \frac{y}{x}$ Definition of inverse tangent function

Recall that the inverse tangent function is only defined on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $[-90°, 90°]$. In the rectangular coordinate system, this refers to θ-values in Quadrants I and IV or when $x > 0$, as shown in Figure 2.3.1. If a point is located in Quadrant II or III, which is when $x < 0$, you must add π or 180° to the angle measure given by the inverse tangent function, as shown in Figure 2.3.2.



When $x > 0$, $\theta = \tan^{-1} \frac{y}{x}$.

Figure 2.3.1



When $x < 0$, $\theta = \tan^{-1} \frac{y}{x} + \pi$ or $\theta = \tan^{-1} \frac{y}{x} + 180°$.

Figure 2.3.2

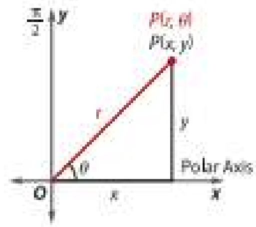
KeyConcept Convert Rectangular to Polar Coordinates

If a point P has rectangular coordinates (x, y) then the polar coordinates (r, θ) of P are given by

$r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$, when $x > 0$

$\theta = \tan^{-1}\frac{y}{x} + \pi$ or

$\theta = \tan^{-1}\frac{y}{x} + 180°$, when $x < 0$.



Recall that polar coordinates are not unique. The conversion from rectangular coordinates to polar coordinates results in just *one* representation of the polar coordinates. There are, however, infinitely many polar representations for a point given in rectangular form.

> **TechnologyTip**
>
> **Coordinate Conversions**
> To convert rectangular coordinates to polar coordinates using a calculator, press [2nd] [APPS] to view the ANGLE menu. Select R►Pr(and enter the coordinates. This will calculate the value of r.
> To calculate θ, repeat this process but select R►Pθ(.

Example 2 Rectangular Coordinates to Polar Coordinates

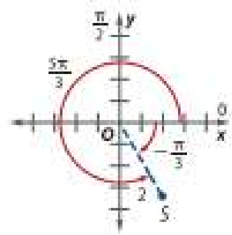
Find two pairs of polar coordinates for each point with the given rectangular coordinates.

a. $S(1, -\sqrt{3})$

For $S(x, y) = (1, -\sqrt{3})$, $x = 1$ and $y = -\sqrt{3}$. Because $x > 0$, use $\tan^{-1}\frac{y}{x}$ to find θ.

$r = \sqrt{x^2 + y^2}$	Conversion formula	$\theta = \tan^{-1}\frac{y}{x}$
$= \sqrt{1^2 + (-\sqrt{3})^2}$	$x = 1$ and $y = -\sqrt{3}$	$= \tan^{-1}\frac{-\sqrt{3}}{1}$
$= \sqrt{4}$ or 2	Simplify.	$= -\frac{\pi}{3}, \frac{5\pi}{3}$

One set of polar coordinate for S is $\left(2, -\frac{\pi}{3}\right)$. Another representation that uses a positive θ-value is $\left(2, -\frac{\pi}{3} + 2\pi\right)$ or $\left(2, \frac{5\pi}{3}\right)$, as shown.



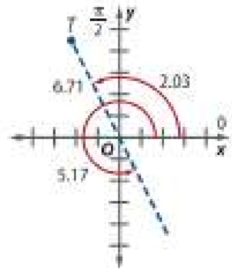
b. $T(-3, 6)$

For $T(x, y) = (-3, 6)$, $x = -3$ and $y = 6$.

Because $x < 0$, use $\tan^{-1}\frac{y}{x} + \pi$ to find θ.

$r = \sqrt{x^2 + y^2}$	Conversion formula	$\theta = \tan^{-1}\frac{y}{x} + \pi$
$= \sqrt{(-3)^2 + 6^2}$	$x = -3$ and $y = 6$	$= \tan^{-1}\left(-\frac{6}{3}\right) + \pi$
$= \sqrt{45}$ or about 6.71	Simplify.	$= \tan^{-1}(-2) + \pi$ or about 2.03

One set of polar coordinates for T is approximately (6.71, 2.03). Another representation that uses a negative r-value is $(-6.71, 2.03 + \pi)$ or $(-6.71, 5.17)$, as shown.



GuidedPractice

Find two pairs of polar coordinates for each point with the given rectangular coordinates. Round to the nearest hundredth, if necessary.

2A. $V(8, 10)$

2B. $W(-9, -4)$

For some real-world phenomena, it is useful to be able to convert between polar coordinates and rectangular coordinates.

Real-World Example 3 Conversion of Coordinates

ROBOTICS Refer to the beginning of the lesson. Suppose the robot is facing due east and its sensor detects an object at (5, 295°).

a. What are the rectangular coordinates that the robot will need to calculate?

$x = r\cos\theta$	Conversion formula	$y = r\sin\theta$
$= 5\cos 295°$	$r = 5$ and $\theta = 295°$	$= 5\sin 295°$
≈ 2.11	Simplify.	≈ -4.53

The object is located at the rectangular coordinates (2.11, −4.53).

b. If a previously detected object has rectangular coordinates of (3, 7), what are the distance and angle measure of the object relative to the front of the robot?

$r = \sqrt{x^2 + y^2}$	Conversion formula	$\theta = \tan^{-1}\frac{y}{x}$
$= \sqrt{3^2 + 7^2}$	$x = 3$ and $y = 7$	$= \tan^{-1}\frac{7}{3}$
≈ 7.62	Simplify.	$\approx 66.8°$

The object is located at the polar coordinates (7.62, 66.8°).

Real-WorldLink

NASA's Special Purpose Dexterous Manipulator, or Dextre, is a 1,542-kilogram robot that stands 3.7 meters tall with an arm span of 3.4 meters. Dextre is responsible for performing jobs in space that previously required astronauts.

Source: *The New York Times*

GuidedPractice

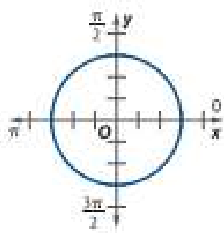
3. **FISHING** A fish finder is a type of radar that is used to locate fish under water. Suppose a boat is facing due east, and a fish finder gives the polar coordinates of a school of fish as (6, 125°).
   - **A.** What are the rectangular coordinates for the school of fish?
   - **B.** If a previously detected school of fish had rectangular coordinates of (−2, 6), what are the distance and angle measure of the school relative to the front of the boat?

2 Polar and Rectangular Equations

In calculus, you will sometimes need to convert from the rectangular form of an equation to its polar form and vice versa to facilitate some calculations. Some complicated rectangular equations have much simpler polar equations. Consider the rectangular and polar equations of the circle graphed below.

Rectangular Equation

$x^2 + y^2 = 9$



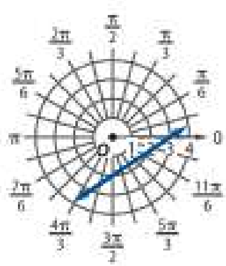
Polar Equation

$r = 3$

Likewise, some polar equations have much simpler rectangular equations, such as the line graphed below.

Polar Equation

$r = \frac{6}{2\cos\theta - 3\sin\theta}$



Rectangular Equation

$2x - 3y = 6$

The conversion of a rectangular equation to a polar equation is fairly straightforward. Replace x with $r\cos\theta$ and y with $r\sin\theta$, and then simplify the resulting equation using algebraic manipulations and trigonometric identities.

Example 4 Rectangular Equations to Polar Equations

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation.

a. $(x-4)^2+y^2=16$

The graph of $(x-4)^2+y^2=16$ is a circle with radius 4 centered at (4, 0). To find the polar form of this equation, replace x with $r\cos\theta$ and y with $r\sin\theta$. Then simplify.

$(x-4)^2+y^2=16$	Original equation
$(r\cos\theta-4)^2+(r\sin\theta)^2=16$	$x=r\cos\theta$ and $y=r\sin\theta$
$r^2\cos^2\theta-8r\cos\theta+16+r^2\sin^2\theta=16$	Multiply.
$r^2\cos^2\theta-8r\cos\theta+r^2\sin^2\theta=0$	Subtract 16 from each side.
$r^2\cos^2\theta+r^2\sin^2\theta=8r\cos\theta$	Isolate the squared terms.
$r^2(\cos^2\theta+\sin^2\theta)=8r\cos\theta$	Factor.
$r^2(1)=8r\cos\theta$	Pythagorean Identity
$r=8\cos\theta$	Divide each side by r.

The graph of this polar equation (Figure 2.3.3) is a circle with radius 4 centered at (4, 0).

StudyTip

Trigonometric Identities You will find it helpful to review trigonometric identities you to help you simplify the polar forms of rectangular equations. A summary of these identities is found inside the back cover of this text.

b. $y=x^2$

The graph of $y=x^2$ is a parabola with vertex at the origin that opens up.

$y=x^2$	Original equation
$r\sin\theta=(r\cos\theta)^2$	$x=r\cos\theta$ and $y=r\sin\theta$
$r\sin\theta=r^2\cos^2\theta$	Multiply.
$\frac{\sin\theta}{\cos^2\theta}=r$	Divide each side by $r\cos^2\theta$.
$\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}=r$	Rewrite.
$\tan\theta\sec\theta=r$	Quotient and Reciprocal Identities

The graph of the polar equation $r=\tan\theta\sec\theta$ (Figure 2.3.4) is a parabola with vertex at the pole that opens up.

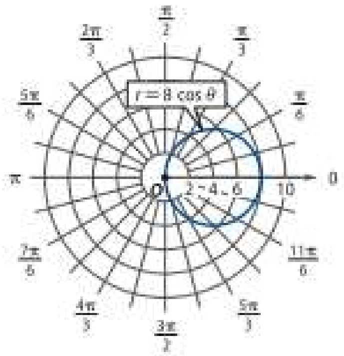


Figure 2.3.3

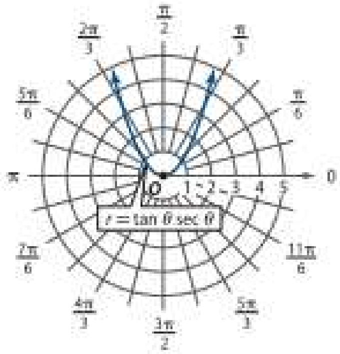


Figure 2.3.4

GuidedPractice

4A. $x^2+(y-3)^2=9$

4B. $x^2-y^2=1$

To write a polar equation in rectangular form, you also make use of the relationships $r^2 = x^2 + y^2$, $x = r\cos\theta$, and $y = r\sin\theta$, as well as the relationship $\tan\theta = \frac{y}{x}$. The process, however, is not as straightforward as converting from rectangular to polar form.

Study Tip

Alternative Method Two points on the line $\theta = \frac{\pi}{6}$ are $\left(2, \frac{\pi}{6}\right)$ and $\left(4, \frac{\pi}{6}\right)$. In rectangular form, these points are $(\sqrt{3}, 1)$ and $(2\sqrt{3}, 2)$. The equation of the line through these points is $y = \frac{\sqrt{3}}{3}x$.

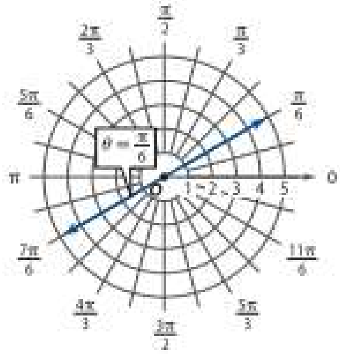
Example 5 Polar Equations to Rectangular Equations

Write each equation in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation.

a. $\theta = \frac{\pi}{6}$

$\theta = \frac{\pi}{6}$	Original equation
$\tan\theta = \frac{\sqrt{3}}{3}$	Find the tangent of each side.
$\frac{y}{x} = \frac{\sqrt{3}}{3}$	$\tan\theta = \frac{y}{x}$
$y = \frac{\sqrt{3}}{3}x$	Multiply each side by x.

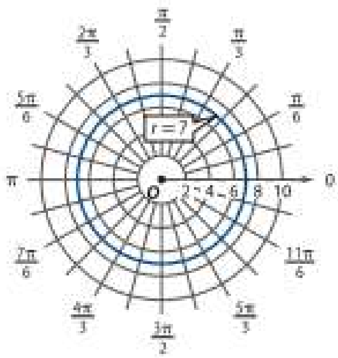
The graph of this equation is a line through the origin with slope $\frac{\sqrt{3}}{3}$ or about $\frac{2}{3}$, as supported by the graph of $\theta = \frac{\pi}{6}$ shown.



b. $r = 7$

$r = 7$	Original equation
$r^2 = 49$	Square each side.
$x^2 + y^2 = 49$	$r^2 = x^2 + y^2$

The graph of this rectangular equation is a circle with center at the origin and radius 7, supported by the graph of $r = 7$ shown.



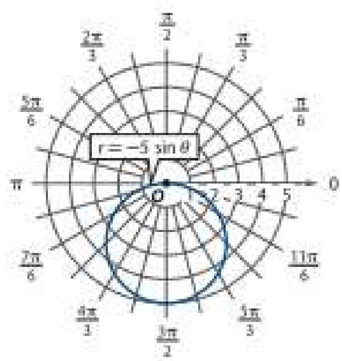
Study Tip

Converting to Rectangular Form Other useful substitutions are variations of the equations $x = r\cos\theta$ and $y = r\sin\theta$, such as $r = \frac{x}{\cos\theta}$ and $r = \frac{y}{\sin\theta}$.

c. $r = -5\sin\theta$

$r = -5\sin\theta$	Original equation
$r^2 = -5r\sin\theta$	Multiply each side by r.
$x^2 + y^2 = -5y$	$r^2 = x^2 + y^2$ and $y = r\sin\theta$
$x^2 + y^2 + 5y = 0$	Add $5y$ to each side.

Because in standard form, $x^2 + (y + 2.5)^2 = 6.25$, you can identify the graph of this equation as a circle centered at $(0, -2.5)$ with radius 2.5, as supported by the graph of $r = -5\sin\theta$.



Guided Practice

5A. $r = -3$ **5B.** $\theta = \frac{\pi}{3}$ **5C.** $r = 3\cos\theta$

Exercises

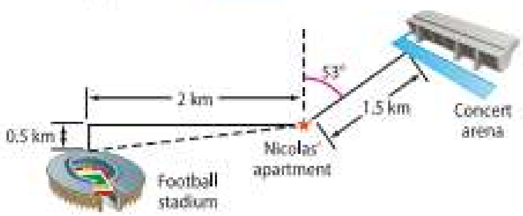
Find the rectangular coordinates for each point with the given polar coordinates. Round to the nearest hundredth, if necessary. (Example 1)

1. $\left(2, \frac{\pi}{4}\right)$
2. $\left(\frac{1}{4}, \frac{\pi}{2}\right)$
3. $(5, 240°)$
4. $(2.5, 250°)$
5. $\left(-2, \frac{4\pi}{3}\right)$
6. $(-13, -70°)$
7. $\left(3, \frac{\pi}{2}\right)$
8. $\left(\frac{1}{2}, \frac{3\pi}{4}\right)$
9. $(-2, 270°)$
10. $(4, 210°)$
11. $\left(-1, -\frac{\pi}{6}\right)$
12. $\left(5, \frac{\pi}{3}\right)$

Find two pairs of polar coordinates for each point with the given rectangular coordinates if $0 \leq \theta \leq 2\pi$. Round to the nearest hundredth, if necessary. (Example 2)

13. $(7, 10)$
14. $(-13, 4)$
15. $(-6, -12)$
16. $(4, -12)$
17. $(2, -3)$
18. $(0, -173)$
19. $(a, 3a)$, $a > 0$
20. $(-14, 14)$
21. $(52, -31)$
22. $(3b, -4b)$, $b > 0$
23. $(1, -1)$
24. $(2, \sqrt{2})$

25. **DISTANCE** Standing on top of his apartment building, Nicolas determines that a concert arena is 53° east of north. Suppose the arena is exactly 1.5 kilometers from Nicolas' apartment. (Example 3)



a. How many kilometers north and east will Nicolas have to travel to reach the arena?

b. If a football stadium is 2 kilometers west and 0.5 kilometer south of Nicolas' apartment, what are the polar coordinates of the stadium if Nicolas' apartment is at the pole?

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation. (Example 4)

26. $x = -2$
27. $(x + 5)^2 + y^2 = 25$
28. $y = -3$
29. $x = y^2$
30. $(x - 2)^2 + y^2 = 4$
31. $(x - 1)^2 - y^2 = 1$
32. $x^2 + (y + 3)^2 = 9$
33. $y = \sqrt{3}x$
34. $x^2 + (y + 1)^2 = 1$
35. $x^2 + (y - 8)^2 = 64$

Write each equation in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation. (Example 5)

36. $r = 3 \sin \theta$
37. $\theta = -\frac{\pi}{3}$
38. $r = 10$
39. $r = 4 \cos \theta$
40. $\tan \theta = 4$
41. $r = 8 \csc \theta$
42. $r = -4$
43. $\cot \theta = -7$
44. $\theta = \frac{3\pi}{4}$
45. $r = \sec \theta$

46. **EARTHQUAKE** An equation to model the seismic waves of an earthquake is $r = 12.6 \sin \theta$, where r is measured in kilometers. (Example 5)

a. Graph the polar pattern of the earthquake.

b. Write an equation in rectangular form to model the seismic waves.

c. Find the rectangular coordinates of the epicenter of the earthquake, and describe the area that is affected by the earthquake.

47. **MICROPHONE** The polar pattern for a directional microphone at a football game is given by $r = 2 + 2 \cos \theta$. (Example 5)

a. Graph the polar pattern.

b. Will the microphone detect a sound that originates from the point with rectangular coordinates $(-2, 0)$? Explain.

B

Write each equation in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation.

48. $r = \frac{1}{\cos \theta + \sin \theta}$
49. $r = 10 \csc \left(\theta + \frac{7\pi}{4}\right)$
50. $r = 3 \csc \left(\theta - \frac{\pi}{2}\right)$
51. $r = -2 \sec \left(\theta - \frac{11\pi}{6}\right)$
52. $r = 4 \sec \left(\theta - \frac{4\pi}{3}\right)$
53. $r = \frac{5 \cos \theta + 5 \sin \theta}{\cos^2 \theta - \sin^2 \theta}$
54. $r = 2 \sin \left(\theta + \frac{\pi}{3}\right)$
55. $r = 4 \cos \left(\theta + \frac{\pi}{2}\right)$

56. **ASTRONOMY** Polar equations are used to model the paths of satellites or other orbiting bodies in space. Suppose the path of a satellite is modeled by $r = \frac{4}{4 + 3 \sin \theta}$, where r is measured in tens of thousands of kilometers, with Earth at the pole.

a. Sketch a graph of the path of the satellite.

b. Determine the minimum and maximum distances the satellite is from Earth at any time.

c. Suppose a second satellite passes through a point with rectangular coordinates $(1.5, -3)$. Are the two satellites at risk of ever colliding at this point? Explain.

C

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation.

57. $6x - 3y = 4$

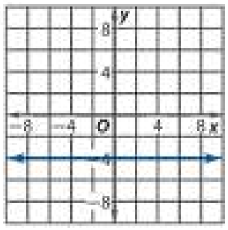
58. $2x + 5y = 12$

59. $(x - 6)^2 + (y - 8)^2 = 100$

60. $(x + 3)^2 + (y - 2)^2 = 13$

Write rectangular and polar equations for each graph.

61.



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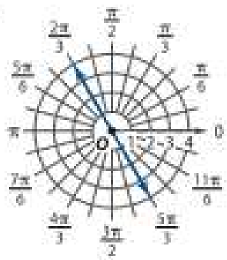

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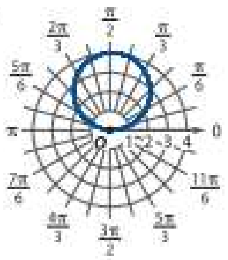

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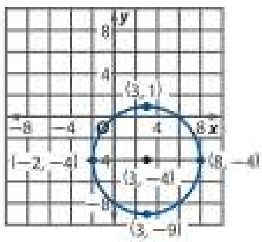
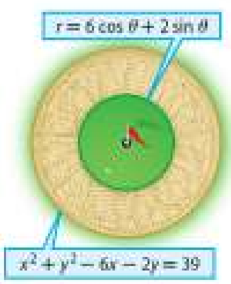
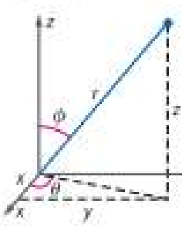


65. **GOLF** On the 18th hole at Hilly Pines Golf Course, the circular green is surrounded by a ring of sand as shown in the figure. Find the area of the region covered by sand assuming the hole acts as the pole for both equations and units are given in meters.



66. **CONSTRUCTION** Boom cranes operate on three-dimensional counterparts of polar coordinates called *spherical coordinates*. A point in space has spherical coordinates (r, θ, ϕ), where r represents the distance from the pole, θ represents the angle of rotation about the vertical axis, and ϕ represents the polar angle from the positive vertical axis. Given a point in spherical coordinates (r, θ, ϕ) find the rectangular coordinates (x, y, z) in terms of r, θ, and ϕ.



67. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the relationship between complex numbers and polar coordinates.

a. **GRAPHICAL** The complex number $a + bi$ can be plotted on a complex plane using the ordered pair (a, b), where the x-axis is the real axis R and the y-axis is the imaginary axis i. Graph the complex number $6 + 8i$.

b. **NUMERICAL** Find polar coordinates for the complex number using the rectangular coordinates plotted in part **a** if $0 < \theta < 360°$. Graph the coordinates on a polar grid.

c. **GRAPHICAL** Graph the complex number $-3 + 3i$ on a rectangular coordinate system.

d. **GRAPHICAL** Find polar coordinates for the complex number using the rectangular coordinates plotted in part **c** if $0 < \theta < 360°$. Graph the coordinates on a polar grid.

e. **ANALYTICAL** For a complex number $a + bi$, find an expression for converting to polar coordinates.

H.O.T. Problems Use Higher-Order Thinking Skills

68. **ERROR ANALYSIS** Usama and Saleh are writing the polar equation $r = \sin\theta$ in rectangular form. Saleh believes that the answer is $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$. Usama believes that the answer is simply $y = \sin x$. Is either of them correct? Explain your reasoning.

69. **CHALLENGE** The equation for a circle is $r = 2a\cos\theta$. Write this equation in rectangular form. Find the center and radius of the circle.

70. **REASONING** Given a set of rectangular coordinates (x, y) and a value for r, write expressions for finding θ in terms of sine and in terms of cosine. (*Hint:* You may have to write multiple expressions for each function, similar to the expressions given in this lesson using tangent.)

71. **WRITING IN MATH** Make a conjecture about when graphing an equation is made easier by representing the equation in polar form rather than rectangular form and vice versa.

72. **PROOF** Use $x = r\cos\theta$ and $y = r\sin\theta$ to prove that $r = x\sec\theta$ and $r = y\csc\theta$.

73. **CHALLENGE** Write $r^2(4\cos^2\theta + 3\sin^2\theta) + r(-8a\cos\theta + 6b\sin\theta) = 12 - 4a^2 - 3b^2$ in rectangular form. (*Hint:* Distribute before substituting values for r^2 and r. The rectangular equation should be a conic.)

74. **WRITING IN MATH** Use the definition of a polar axis given in Lesson 2-1 to explain why it was necessary to state that the robot in Example 3 was facing due east. How can the use of quadrant bearings help to eliminate this?

Spiral Review

Use symmetry to graph each equation. (Lesson 2-2)

75. $r = 1 - 2\sin\theta$

76. $r = -2 - 2\sin\theta$

77. $r = 2\sin 3\theta$

Find three different pairs of polar coordinates that name the given point if $-360° < \theta \le 360°$ or $-2\pi < \theta \le 2\pi$. (Lesson 2-1)

78. $T(1.5, 180°)$

79. $U\left(-1, \frac{\pi}{3}\right)$

80. $V(4, 315°)$

Find the angle θ between u and v to the nearest tenth of a degree.

81. $\mathbf{u} = \langle 6, -4\rangle$, $\mathbf{v} = \langle -5, -7\rangle$

82. $\mathbf{u} = \langle 2, 3\rangle$, $\mathbf{v} = \langle -9, 6\rangle$

83. $\mathbf{u} = \langle 1, 10\rangle$, $\mathbf{v} = \langle 8, -2\rangle$

Write each pair of parametric equations in rectangular form. Then graph and state any restrictions on the domain.

84. $y = t + 6$ and $x = \sqrt{t}$

85. $y = \frac{t}{2} + 1$ and $x = \frac{t^2}{4}$

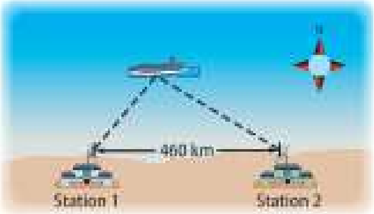
86. $y = -3\sin t$ and $x = 3\cos t$

87. **NAVIGATION** Two LORAN broadcasting stations are located 460 kilometers apart. A ship receives signals from both stations and determines that it is 108 kilometers farther from Station 2 than Station 1.

a. Determine the equation of the hyperbola centered at the origin on which the ship is located.

b. Graph the equation, indicating on which branch of the hyperbola the ship is located.

c. Find the coordinates of the location of the ship on the coordinate grid if it is 110 kilometers from the x-axis.



88. **BICYCLES** Woodland Bicycles makes two models of off-road bicycles: the Adventure, which sells for $250, and the Grande Venture, which sells for $350. Both models use the same frame. The painting and assembly time required for the Adventure is 2 hours, while the time is 3 hours for the Grande Venture. If there are 175 frames and 450 hours of labor available for production, how many of each model should be produced to maximize revenue? What is the maximum revenue?

Solve each system of equations using Gauss-Jordan elimination.

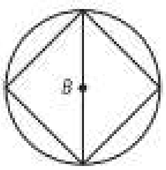
89. $3x + 9y + 6z = 21$
$4x - 10y + 3z = 15$
$-5x + 12y - 2z = -6$

90. $x + 5y - 3z = -14$
$2x - 4y + 5z = 18$
$-7x - 6y - 2z = 1$

91. $2x - 4y + z = 20$
$5x + 2y - 2z = -4$
$6x + 3y + 5z = 23$

Skills Review for Standardized Tests

92. **SAT/ACT** A square is inscribed in circle B. If the circumference of the circle is 50π, what is the length of the diagonal of the square?



A $10\sqrt{2}$
B 25
C $25\sqrt{2}$
D 50
E $50\sqrt{2}$

93. **REVIEW** Which of the following could be an equation for a rose with three petals?

F $r = 3\sin\theta$
G $r = \sin 3\theta$
H $r = 6\sin\theta$
J $r = \sin 6\theta$

94. What is the polar form of $x^2 + (y - 2)^2 = 4$?

A $r = \sin\theta$
B $r = 2\sin\theta$
C $r = 4\sin\theta$
D $r = 8\sin\theta$

95. **REVIEW** Which of the following could be an equation for a spiral of Archimedes that passes through the point $A\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$?

F $r = \frac{\sqrt{2}\pi}{2}\cos\theta$
G $r = \theta$
H $r = \frac{3}{4}$
J $r = \frac{\theta}{2}$

Mid-Chapter Quiz

Lessons 2-1 through 2-3

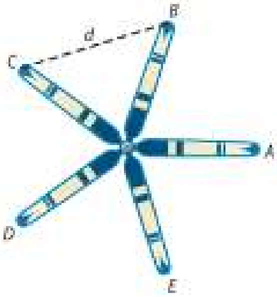
Graph each point on a polar grid. (Lesson 2-1)

1. $A(-2, 45°)$
2. $D(1, 315°)$
3. $C\left(-1.5, -\frac{4\pi}{3}\right)$
4. $B\left(3, -\frac{5\pi}{6}\right)$

Graph each polar equation. (Lesson 2-1)

5. $r = 3$
6. $\theta = -\frac{3\pi}{4}$
7. $\theta = 60°$
8. $r = -1.5$

9. **HELICOPTERS** A toy helicopter rotor consists of five equally spaced blades. Each blade is 11.5 centimeters long. (Lesson 2-1)



   a. If the angle blade A makes with the polar axis is 3°, write an ordered pair to represent the tip of each blade on a polar grid. Assume that the rotor is centered at the pole.

   b. What is the distance d between the tips of the helicopter blades to the nearest centimeter?

Graph each equation. (Lesson 2-2)

10. $r = \frac{1}{4} \sec \theta$
11. $r = \frac{1}{3} \cos \theta$
12. $r = 3 \csc \theta$
13. $r = 4 \sin \theta$

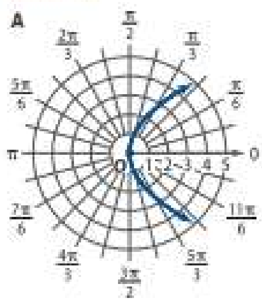
14. **STAINED GLASS** A rose window is a circular window seen in gothic architecture. The pattern of the window radiates from the center. The window shown can be approximated by the equation $r = 3 \sin 6\theta$. Use symmetry, zeros, and maximum r-values of the function to graph the function. (Lesson 2-2)

Identify and graph each classic curve. (Lesson 2-2)

15. $r = \frac{1}{2} \sin \theta$
16. $r = \frac{1}{3}\theta + 3, \theta \geq 0$
17. $r = 1 + 2 \cos \theta$
18. $r = 5 \sin 3\theta$

19. **MULTIPLE CHOICE** Identify the polar graph of $y^2 = \frac{1}{2}x$. (Lesson 2-3)

A



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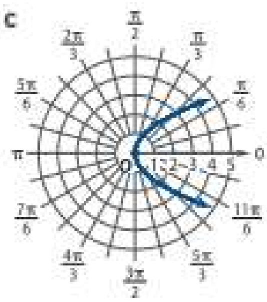

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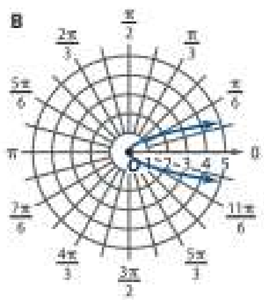

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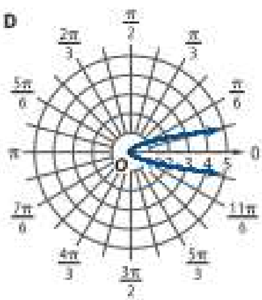


Find the rectangular coordinates for each point with the given polar coordinates. (Lesson 2-3)

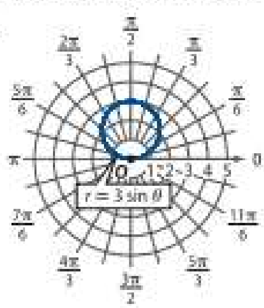
20. $\left(4, \frac{2\pi}{3}\right)$
21. $\left(-2, -\frac{\pi}{4}\right)$
22. $(-1, 210°)$
23. $(3, 30°)$

Find two pairs of polar coordinates for each point with the given rectangular coordinates if $0 \leq \theta \leq 2\pi$. Round to the nearest hundredth. (Lesson 2-3)

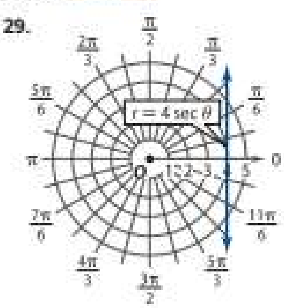
24. $(-3, 5)$
25. $(8, 1)$
26. $(7, -6)$
27. $(-4, -10)$

Write a rectangular equation for each graph. (Lesson 2-3)

28.



29.



LESSON 2-4

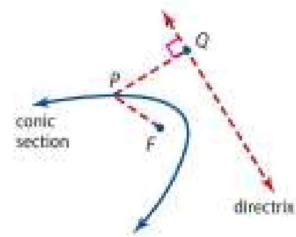
Polar Forms of Conic Sections

::Then	::Now	::Why?
• You defined conic sections.	• **1** Identify polar equations of conics. **2** Write and graph the polar equation of a conic given its eccentricity and the equation of its directrix.	• Polar equations of conic sections can be used to model orbital motion, such as the orbit of a planet around the Sun or the orbit of a satellite around a planet.

1 Use Polar Equations of Conics

Previously, you defined conic sections in terms of the distance between a focus and directrix (parabola) or between two foci (ellipse and hyperbola). Alternatively, we can define all of these curves using the focus-directrix definition of a parabola.

In general, a conic section can be defined as the locus of points such that the distance from a point P to the focus and the distance from the point to a fixed line not containing P (the directrix) is a constant ratio. This constant ratio $\frac{PF}{PQ}$ represents the eccentricity of a conic and is denoted e.



e as Constant Ratio | **e as Constant Multiplier**

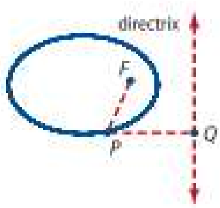
$$e = \frac{PF}{PQ} \quad \text{or} \quad PF = e \cdot PQ$$

Recall that for a parabola, $PF = PQ$. Therefore, a parabola has eccentricity $\frac{PQ}{PQ}$ or 1. Other values of e give us other conics. These eccentricities are summarized below.

ConceptSummary Eccentricities of Conics

Ellipse

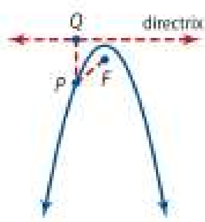
$0 < e < 1$



$0 < \frac{PF}{PQ} < 1$

Parabola

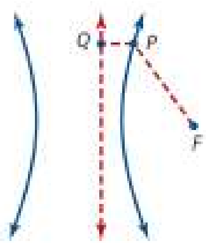
$e = 1$



$\frac{PF}{PQ} = 1$

Hyperbola

$e > 1$



$\frac{PF}{PQ} > 1$

Recall too that when the center of a conic section lies at the origin, the rectangular equations of conics take on a simpler form.

Ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ or } \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

Parabolas

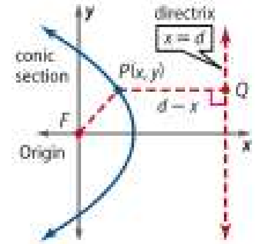
$$x^2 = 4py \text{ or } y^2 = 4px$$

Hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ or } \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Using the focus-directrix definition, the equation of a conic in polar form is simplified if a *focus* of the conic lies at the origin.

Consider a conic with its focus located at the origin and its directrix to the right at $x = d$. For any point $P(x, y)$ on the curve, the distance PF is given by $\sqrt{x^2 + y^2}$, and the distance PQ is given by $d - x$. We can substitute these expressions in the definition of a conic section.



$PF = e \cdot PQ$ — Definition of a conic section

$\sqrt{x^2 + y^2} = e(d - x)$ — $PF = \sqrt{x^2 + y^2}$ and $PQ = d - x$

StudyTip

Other Conics When defining conics in terms of their eccentricity, *e* is a strictly positive constant. There are no circles, lines, or other degenerate conics.

The expression $\sqrt{x^2 + y^2}$ should make you think of polar coordinates. In fact, the equation above has a simpler form in the polar coordinate system.

$\sqrt{x^2 + y^2} = e(d - x)$	Rectangular form of conic defined in terms of its eccentricity e
$r = e(d - r\cos\theta)$	$r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta$
$r = ed - er\cos\theta$	Distributive Property
$r + er\cos\theta = ed$	Isolate r-terms.
$r(1 + e\cos\theta) = ed$	Factor.
$r = \dfrac{ed}{1 + e\cos\theta}$	Solve for r.

This last equation is the polar form of an equation for the conic sections with focus at the pole and vertical directrix and center or vertex to the right of the pole. Different orientations of the focus and directrix can produce different forms of this polar equation as summarized below.

ReadingMath

Eccentricity In each of these polar equations, the letter *e* is a variable that represents the eccentricity of the conic. It should *not* be confused with the transcendental number *e*, which is a constant.

KeyConcept Polar Equations of Conics

The conic section with eccentricity $e > 0$, $d > 0$, and focus at the pole has the polar equation:

- $r = \dfrac{ed}{1 + e\cos\theta}$ if the directrix is the vertical line $x = d$ (Figure 2.4.1),
- $r = \dfrac{ed}{1 - e\cos\theta}$ if the directrix is the vertical line $x = -d$ (Figure 2.4.2),
- $r = \dfrac{ed}{1 + e\sin\theta}$ if the directrix is the horizontal line $y = d$ (Figure 2.4.3), and
- $r = \dfrac{ed}{1 - e\sin\theta}$ if the directrix is the horizontal line $y = -d$ (Figure 2.4.4).

In each of the examples below, $e = 1$, so the conic takes the form of a parabola.

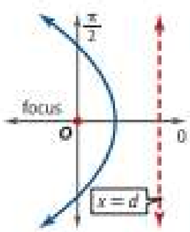


Figure 2.4.1

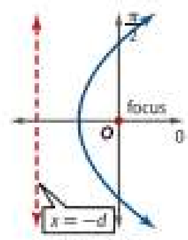


Figure 2.4.2

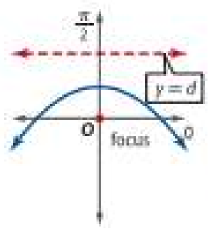


Figure 2.4.3

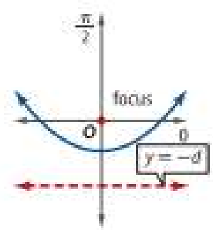


Figure 2.4.4

You will derive the last three of these equations in Exercises 50–52.

Notice that for $r = \frac{ed}{1 - e\cos\theta}$, the directrix of the conic is to the left of the pole. For $r = \frac{ed}{1 + e\sin\theta}$, the directrix is above the pole. For $r = \frac{ed}{1 - e\sin\theta}$, the directrix is below the pole.

To analyze the polar equation of a conic, begin by writing the equation in standard form, $r = \frac{ed}{1 \pm e \cos \theta}$ or $r = \frac{ed}{1 \pm e \sin \theta}$. In this form, determine the eccentricity and use this value to identify the type of conic the equation represents. Then determine the equation of the directrix, and use it to describe the orientation of the conic.

Example 1 Identify Conics from Polar Equations

Determine the eccentricity, type of conic, and equation of the directrix for each polar equation.

a. $r = \frac{9}{3 + 2.25 \cos \theta}$

Write the equation in standard form, $r = \frac{ed}{1 + e \cos \theta}$.

$r = \frac{9}{3 + 2.25 \cos \theta}$ Original equation

$r = \frac{3(3)}{3(1 + 0.75 \cos \theta)}$ Factor the numerator and denominator.

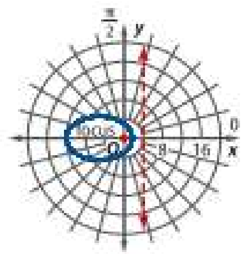
$r = \frac{3}{1 + 0.75 \cos \theta}$ Divide the numerator and denominator by 3.

In this form, you can see from the denominator that $e = 0.75$. Therefore, the conic is an ellipse. For polar equations of this form, the equation of the directrix is $x = d$. From the numerator, we know that $ed = 3$, so $d = 3 \div 0.75$ or 4. Therefore, the equation of the directrix is $x = 4$.

StudyTip

Focus-Directrix Pairs While a parabola has one focus and one directrix, ellipses and hyperbolas have two foci-directrix pairs. Either focus-directrix pair can be used to generate the conic.

CHECK Sketch the graph of $r = \frac{9}{3 + 2.25 \cos \theta}$ and its directrix $x = 4$ using either the techniques shown in Lesson 2-2 or a graphing calculator. The graph is an ellipse with its directrix to the right of the pole. ✔



b. $r = \frac{-16}{4 \sin \theta - 2}$

Write the equation in standard form.

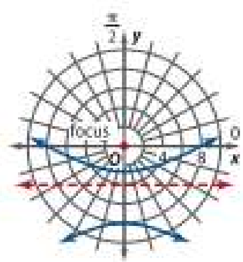
$r = \frac{-16}{4 \sin \theta - 2}$ Original equation

$r = \frac{-2(8)}{-2(1 - 2 \sin \theta)}$ Factor the numerator and denominator.

$r = \frac{8}{1 - 2 \sin \theta}$ Divide the numerator and denominator by −2.

The equation is of the form $r = \frac{ed}{1 - e \sin \theta}$, so $e = 2$. Therefore, the conic is a hyperbola. For polar equations of this form, the equation of the directrix is $y = -d$. Because $ed = 8$, $d = 8 \div 2$ or 4. Therefore, the equation of the directrix is $y = -4$.

CHECK Sketch the graph of $r = \frac{-16}{4 \sin \theta - 2}$ and its directrix $y = -4$. The graph is a hyperbola with one focus at the origin, above the directrix. ✔



GuidedPractice

1A. $r = \frac{-6}{3 \cos \theta - 1}$

1B. $r = \frac{9}{3 + 3 \sin \theta}$

1C. $r = \frac{1}{6 + 1.2 \cos \theta}$

2 Write Polar Equations of Conics

You can write the polar equation of a conic given its eccentricity and the equation of the directrix or its eccentricity and some other characteristics.

Example 2 Write Polar Equations of Conics

Write and graph a polar equation and directrix for the conic with the given characteristics.

a. $e = 2$; **directrix:** $y = 4$

Because $e = 2$, the conic is a hyperbola. The directrix $y = 4$ is above the pole, so the equation is of the form $r = \frac{ed}{1 + e\sin\theta}$. Use the values for e and d to write the equation.

$r = \frac{ed}{1 + e\sin\theta}$ — Polar form of conic with directrix $y = d$

$r = \frac{2(4)}{1 + 2\sin\theta}$ or $\frac{8}{1 + 2\sin\theta}$ — $e = 2$ and $d = 4$

Sketch the graph of this polar equation and its directrix. The graph is a hyperbola with its directrix above the pole.

StudyTip

Effects of Various Eccentricities You will investigate the effects of various eccentricities for a fixed directrix and various directrixes for a fixed eccentricity in Exercise 49.

b. $e = 0.5$; **vertices at** $(-4, 0)$ **and** $(12, 0)$

Because $e = 0.5$, the conic is an ellipse. The center of the ellipse is at $(4, 0)$, the midpoint of the segment between the given vertices. This point is to the right of the pole. Therefore, the directrix will be to the left of the pole at $x = -d$. The polar equation of a conic with this directrix is $r = \frac{ed}{1 - e\cos\theta}$.

Use the value of e and the polar form of a point on the conic to find the value of d. The vertex point $(12, 0)$ has polar coordinates $(r, \theta) = \left(\sqrt{12^2 + 0^2}, \tan^{-1}\frac{0}{12}\right)$ or $(12, 0)$.

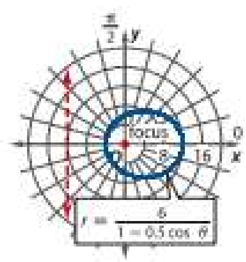
$r = \frac{ed}{1 - e\cos\theta}$ — Polar form of conic with directrix $x = -d$

$12 = \frac{0.5d}{1 - 0.5\cos 0}$ — $e = 0.5$, $r = 12$, and $\theta = 0$

$12 = \frac{0.5d}{0.5}$ — $\cos 0 = 1$

$12 = d$ — Simplify.

Therefore, the equation for the ellipse is $r = \frac{0.5 \cdot 12}{1 - 0.5\cos\theta}$ or $r = \frac{6}{1 - 0.5\cos\theta}$. Because $d = 12$, the equation of the directrix is $x = -12$. The graph is an ellipse with vertices at $(-4, 0)$ and $(12, 0)$.



GuidedPractice

2A. $e = 1$; directrix: $x = 2$

2B. $e = 2.5$; vertices at $(0, -3)$ and $(0, -7)$

Previously, you analyzed the rectangular equations of conics in standard form to describe the geometric properties of parabolas, ellipses, and hyperbolas. You can use the geometric analysis of the graph of a conic given in polar form to write the equation in rectangular form.

Example 3 Write the Polar Form of Conics in Rectangular Form

Write each polar equation in rectangular form.

a. $r = \dfrac{4}{1 - \sin\theta}$

Step 1 Analyze the polar equation.

For this equation, $e = 1$ and $d = 4$. The eccentricity and form of the equation determine that this is a parabola that opens vertically with focus at the pole and a directrix $y = -4$. The general equation of such a parabola in rectangular form is $(x - h)^2 = 4p(y - k)$.

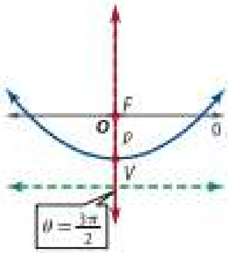


Figure 2.4.5

Step 2 Determine values for h, k, and p.

The vertex lies between the focus F and directrix of the parabola, occurring when $\theta = \frac{3\pi}{2}$, as shown in Figure 2.4.5. Evaluating the function at this value, we find that the vertex lies at polar coordinates $\left(2, \frac{3\pi}{2}\right)$, which correspond to rectangular coordinates $(0, -2)$. So, $(h, k) = (0, -2)$. The distance p from the vertex at $(0, -2)$ to the focus at $(0, 0)$ is 2.

Step 3 Substitute the values for h, k, and p into the standard form of an equation for a parabola.

$(x - h)^2 = 4p(y - k)$ — Standard form of a parabola

$(x - 0)^2 = 4(2)[y - (-2)]$ — $h = 0$, $k = -2$, and $p = 2$

$x^2 = 8y + 16$ — Simplify.

b. $r = \dfrac{3.2}{1 - 0.6\cos\theta}$

Step 1 Analyze the polar equation.

For this equation, $e = 0.6$ and $d \approx 5.3$. The eccentricity and form of the equation determine that this is an ellipse with directrix $x = -5.3$. Therefore, the major axis of the ellipse lies along the polar or x-axis. The general equation of such an ellipse in rectangular form is $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$.

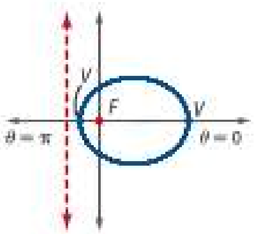


Figure 2.4.6

Step 2 Determine values for h, k, a, and b.

The vertices are the endpoints of the major axis and occur when $\theta = 0$ and π as shown in Figure 2.4.6. Evaluating the function at these values, we find that the vertices have polar coordinates $(8, 0)$ and $(2, \pi)$, which correspond to rectangular coordinates $(8, 0)$ and $(-2, 0)$. The ellipse's center is the midpoint of the segment between the vertices, so $(h, k) = (3, 0)$.

The distance a between the center and each vertex is 5. The distance c from the center to the focus at $(0, 0)$ is 3. By the Pythagorean relation $b = \sqrt{a^2 - c^2}$, $b = \sqrt{5^2 - 3^2}$ or 4.

Step 3 Substitute the values for h, k, a, and b into the standard form of an equation for an ellipse.

$\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$ — Standard form of an ellipse

$\dfrac{(x - 3)^2}{5^2} + \dfrac{(y - 0)^2}{4^2} = 1$ — $h = 3$, $k = 0$, $a = 5$, and $b = 4$

$\dfrac{(x - 3)^2}{25} + \dfrac{y^2}{16} = 1$ — Simplify.

GuidedPractice

3A. $r = \dfrac{2.5}{1 - 1.5\cos\theta}$

3B. $r = \dfrac{5}{1 + \sin\theta}$

Exercises

Determine the eccentricity, type of conic, and equation of the directrix for each polar equation. (Example 1)

1. $r = \frac{20}{4 + 4\sin\theta}$

2. $r = \frac{18}{2 - 6\cos\theta}$

3. $r = \frac{21}{3\cos\theta + 1}$

4. $r = \frac{24}{4\sin\theta + 8}$

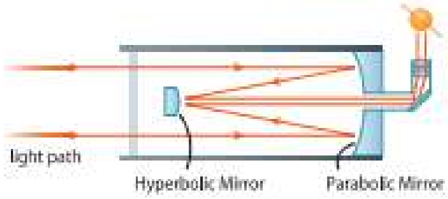
5. $r = \frac{-12}{6\cos\theta - 6}$

6. $r = \frac{9}{4 - 3\sin\theta}$

7. $r = \frac{-8}{\sin\theta - 0.25}$

8. $r = \frac{10}{2.5 + 2.5\cos\theta}$

9. **TELESCOPES** The Cassegrain Telescope, invented in 1692, produces an image by reflecting light off of parabolic and hyperbolic mirrors. Determine the eccentricity, type of conic, and the equation of the directrix for each equation modeling a mirror in the telescope. (Example 1)



a. $r = \frac{7}{2\sin\theta + 2}$

b. $r = \frac{28}{12.5\cos\theta + 5}$

Write and graph a polar equation and directrix for the conic with the given characteristics. (Example 2)

10. $e = 1$; directrix: $y = 6$

11. $e = 0.75$; directrix: $x = -8$

12. $e = 5$; directrix: $x = 2$

13. $e = 0.1$; directrix: $y = 8$

14. $e = 6$; directrix: $y = -7$

15. $e = 1$; directrix: $x = -1.5$

16. $e = 0.8$; vertices at $(-36, 0)$ and $(4, 0)$

17. $e = 1.5$; vertices at $(-3, 0)$ and $(-15, 0)$

Write each polar equation in rectangular form. (Example 3)

18. $r = \frac{4.8}{1 + \sin\theta}$

19. $r = \frac{30}{4 + \cos\theta}$

20. $r = \frac{5}{1 - 1.5\cos\theta}$

21. $r = \frac{5.1}{1 + 0.7\sin\theta}$

22. $r = \frac{12}{1 - \cos\theta}$

23. $r = \frac{6}{0.25 - 0.75\sin\theta}$

24. $r = \frac{4.5}{1 + 1.25\sin\theta}$

25. $r = \frac{8.4}{1 - 0.4\cos\theta}$

B

GRAPHING CALCULATOR **Determine the type of conic for each equation, then graph.**

26. $r = \frac{2}{2 + \sin\left(\theta + \frac{\pi}{3}\right)}$

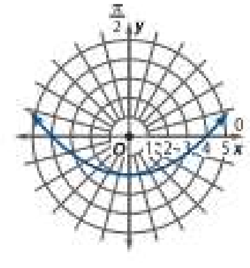
27. $r = \frac{3}{1 + \cos\left(\theta - \frac{\pi}{4}\right)}$

28. $r = \frac{2}{1 - \cos\left(\theta + \frac{\pi}{6}\right)}$

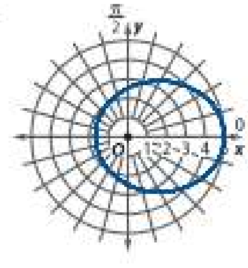
29. $r = \frac{4}{1 + 2\sin\left(\theta + \frac{3\pi}{4}\right)}$

Match each polar equation with its graph.

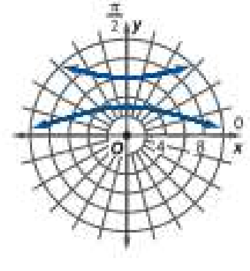
a.



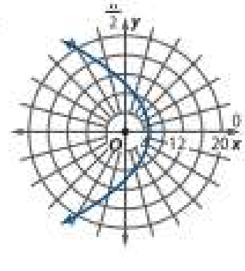
b.



c.



d.



30. $r = \frac{10}{1 + \cos\theta}$

31. $r = \frac{4}{1 - \sin\theta}$

32. $r = \frac{5}{2 - \cos\theta}$

33. $r = \frac{12}{1 + 3\sin\theta}$

Determine the eccentricity, type of conic, and equation of the directrix for each polar equation. Then sketch the graph of the equation, and label the directrix.

34. $r = \frac{12}{2 - 0.75\cos\theta}$

35. $r = \frac{1}{0.2 - 0.2\sin\theta}$

36. $r = \frac{6}{1.2\sin\theta + 0.3}$

37. $r = \frac{8}{\cos\theta + 5}$

38. **ASTRONOMY** The comet Borrelly travels in an elliptical orbit around the Sun with eccentricity $e = 0.624$. The point in a comet's orbit nearest to the Sun is defined as the *perihelion*, while the farthest point from the Sun is defined as the *aphelion*. The aphelion occurs at a distance of 5.83 AU (astronomical units, based on the distance between Earth and the Sun) from the Sun and the perihelion occurs at a distance of 1.35 AU. The diameter of the Sun is about 0.0093 AU.

a. Write a polar equation for the elliptical orbit of the comet Borrelly, and graph the equation.

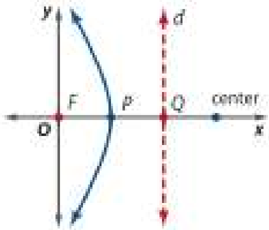
b. Determine the distance in kilometers between the comet Borrelly and the Sun at the aphelion and perihelion if 1 AU ≈ 149.7 million kilometers.

PROOF **Prove each of the following.**

39. $b = a\sqrt{1 - e^2}$ for an ellipse

40. $b = a\sqrt{e^2 - 1}$ for a hyperbola

41. **PROOF** Use the definition for the eccentricity of a conic, $PF = ePQ$, and the drawing of the hyperbola shown below, to verify that $d = \frac{a(e^2 - 1)}{e}$ for any hyperbola.



Write each rectangular equation in polar form.

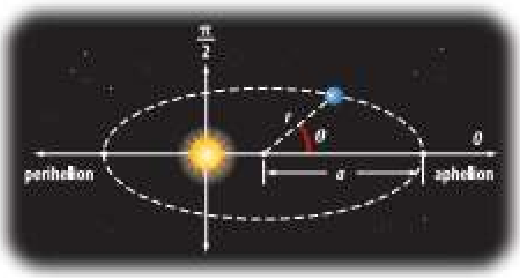
42. $x^2 = 4y + 4$

43. $-10y + 25 = x^2$

44. $\frac{(x-2)^2}{16} + \frac{y^2}{12} = 1$

45. $\frac{(x+4)^2}{64} + \frac{y^2}{48} = 1$

46. **ASTRONOMY** The planets travel around the Sun in approximately elliptical orbits with the Sun at one focus, as shown below.



a. Show that the polar equation of the planets' orbit can be written as $r = \frac{a(1 - e^2)}{(1 - e\cos\theta)}$.

b. Prove that the perihelion distance of any planet is $a(1 - e)$, and the aphelion distance is $a(1 + e)$.

c. Use the formulas from part **a** to find the perihelion and aphelion distances for each of the planets.

Planet	a	e	Planet	a	e
Earth	1.000	0.017	Neptune	30.06	0.009
Jupiter	5.203	0.048	Saturn	9.539	0.056
Mars	1.524	0.093	Uranus	19.18	0.047
Mercury	0.387	0.206	Venus	0.723	0.007

d. For which planet is the distance between the perihelion and aphelion the smallest? the greatest?

Write each equation in polar form. (*Hint*: **Translate each conic so that a focus lies on the pole.)**

47. $\frac{(x-2)^2}{64} - \frac{y^2}{36} = 1$

48. $3(x + 5)^2 + 4y^2 = 192$

49. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the effects of varying the eccentricity and the directrix on graphs of conic sections.

a. **NUMERICAL** Write an equation for a conic section with focus (0, 0) and directrix $x = 3$ for $e = 0.4, 0.6, 1, 1.6$, and 2. Then identify the type of conic that each equation represents.

b. **GRAPHICAL** Graph and label the eccentricity for each of the equations that you found in part **a** on the same coordinate plane.

c. **VERBAL** Describe the changes in the graphs from part **b** as e approaches 2.

d. **NUMERICAL** Write an equation for a conic section with focus (0, 0) and eccentricity $e = 0.5$ for $d = 0.25, 1$, and 4.

e. **GRAPHICAL** Graph each of the equations on the same coordinate plane.

f. **VERBAL** Describe the relationship between the value of d and the distances between the vertices and the foci of the graphs from part **e**.

Derive each of the following polar equations of conics for the equation $r = \frac{ed}{1 + e\cos\theta}$. Include a diagram with each derivation.

50. $r = \frac{ed}{1 - e\cos\theta}$

51. $r = \frac{ed}{1 + e\sin\theta}$

52. $r = \frac{ed}{1 - e\sin\theta}$

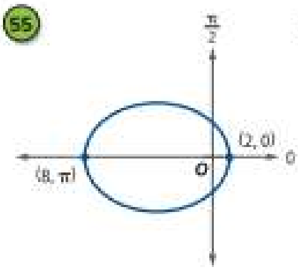
H.O.T. Problems Use Higher-Order Thinking Skills

53. **WRITING IN MATH** Describe two definitions that can be used to define a conic section.

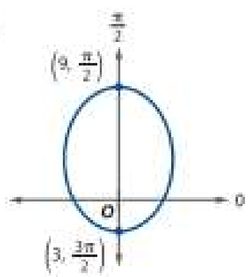
54. **REASONING** Explain why $r = \frac{ed}{1 + e\sin\theta}$ does not produce a true circle for any value of e.

CHALLENGE **Determine a polar equation for the ellipse with the given vertices if one focus is at the pole.**

55.



56.



57. **WRITING IN MATH** Explain how you can find the distance from the focus at (0, 0) to any point on the conic when the rectangular coordinates, polar coordinates, or θ is provided.

Spiral Review

Find two pairs of polar coordinates for each point with the given rectangular coordinates if $0 \leq \theta \leq 2\pi$. If necessary, round to the nearest hundredth. (Lesson 2-3)

58. $(-\sqrt{2}, \sqrt{2})$

59. $(-2, -5)$

60. $(8, -12)$

Identify and graph each classic curve. (Lesson 2-2)

61. $r = 3 + 3 \cos \theta$

62. $r = -2 \sin 3\theta$

63. $r = \frac{5}{2}\theta, \theta \geq 0$

Determine an equation of an ellipse with each set of characteristics.

64. co-vertices (5, 8), (5, 0); foci (8, 4), (2, 4)

65. major axis (−2, 4) to (8, 4); minor axis (3, 1) to (3, 7)

66. foci (1, −1), (9, −1); length of minor axis equals 6

67. **OLYMPICS** In the Olympic Games, team standings are determined according to each team's total points. Each type of Olympic medal earns a team a given number of points. Use the information to determine the Olympics in which the United States earned the most points.

Olympics	Gold	Silver	Bronze
1996	44	32	25
2000	37	24	31
2004	35	39	29
2008	36	38	36

Medal	Points
gold	3
silver	2
bronze	1

Find the values of sin 2θ, cos 2θ, and tan 2θ for the given value and interval.

68. $\sin \theta = \frac{2}{3}, (0°, 90°)$

69. $\tan \theta = -\frac{24}{7}, \left(\frac{\pi}{2}, \pi\right)$

70. $\sin \theta = -\frac{4}{5}, \left(\pi, \frac{3\pi}{2}\right)$

Locate the vertical asymptotes, and sketch the graph of each function.

71. $y = \sec\left(x + \frac{\pi}{3}\right)$

72. $y = 4 \cot \frac{x}{2}$

73. $y = 2 \cot\left[\frac{2}{3}\left(x - \frac{\pi}{2}\right)\right] + 0.75$

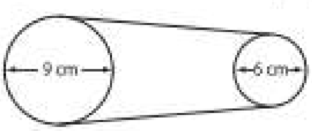
Find the exact values of the five remaining trigonometric functions of θ.

74. $\sec \theta = 2$, where $\sin \theta > 0$ and $\cos \theta > 0$

75. $\csc \theta = \sqrt{5}$, where $\sin \theta > 0$ and $\cos \theta > 0$

Skills Review for Standardized Tests

76. **SAT/ACT** A pulley with a 9-centimeter diameter is belted to a pulley with a 6-centimeter diameter, as shown in the figure. If the larger pulley runs at 120 rpm (revolutions per minute), how fast does the smaller pulley run?



A 80 rpm
B 120 rpm
C 160 rpm
D 180 rpm
E 200 rpm

77. What type of conic is given by $r = \frac{3}{2 - 0.5 \cos \theta}$?

F circle
G ellipse
H parabola
J hyperbola

78. **REVIEW** Which of the following includes the component form and magnitude of $\overrightarrow{AB}$ with initial point $A(3, 4, -2)$ and terminal point $B(-5, 2, 1)$?

A $\langle -8, -2, 3 \rangle, \sqrt{77}$
B $\langle 8, -2, 3 \rangle, \sqrt{77}$
C $\langle -8, -2, 3 \rangle, \sqrt{109}$
D $\langle 8, -2, 3 \rangle, \sqrt{109}$

79. **REVIEW** What is the eccentricity of the ellipse described by $\frac{y^2}{47} + \frac{(x - 12)^2}{34} = 1$?

F 0.38
G 0.41
H 0.53
J 0.62

LESSON 2-5 Complex Numbers and De Moivre's Theorem

::Then

- You performed operations with complex numbers written in rectangular form.

::Now

1. Convert complex numbers from rectangular to polar form and vice versa.
2. Find products, quotients, powers, and roots of complex numbers in polar form.

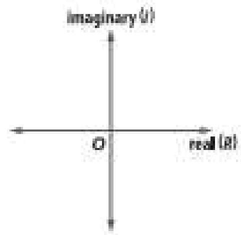
::Why?

- Electrical engineers use complex numbers to describe certain relationships of electricity. Voltage *E*, impedance *Z*, and current *I* are the three quantities related by the equation $E = I \cdot Z$ used to describe alternating current. Each variable can be written as a complex number in the form $a + bj$, where *j* is an imaginary number (engineers use *j* to not be confused with current *I*). For impedance, the real part *a* represents the opposition to current flow due to resistors, and the imaginary part *b* is related to the opposition due to inductors and capacitors.

NewVocabulary
- complex plane
- real axis
- imaginary axis
- Argand plane
- absolute value of a complex number
- polar form
- trigonometric form
- modulus
- argument
- *p*th roots of unity

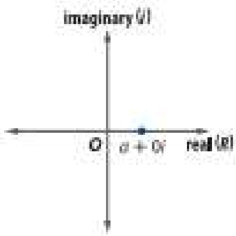
1 Polar Forms of Complex Numbers

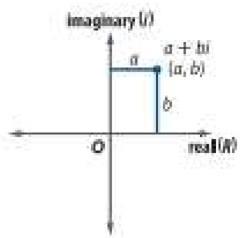
A complex number given in rectangular form, $a + bi$, has a real component *a* and an imaginary component *bi*. You can graph a complex number on the **complex plane** by representing it with the point (a, b). Similar to a coordinate plane, we need two axes to graph a complex number. The real component is plotted on the horizontal axis called the **real axis**, and the imaginary component is plotted on the vertical axis called the **imaginary axis**. The complex plane may also be referred to as the **Argand Plane** (ar GON).



Complex Plane

Consider a complex number where $b = 0$, $a + 0i$. The result is a real number *a* that can be graphed using just a real number line or the real axis. When $b \neq 0$, the imaginary axis is needed to represent the imaginary component.



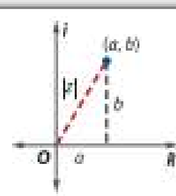


Recall that the absolute value of a real number is its distance from zero on the number line. Similarly, the **absolute value of a complex number** is its distance from zero in the complex plane. When $a + bi$ is graphed in the complex plane, the distance from zero can be calculated using the Pythagorean Theorem.

KeyConcept Absolute Value of a Complex Number

The absolute value of the complex number $z = a + bi$ is

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$

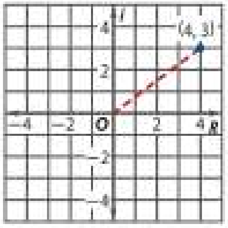


Example 1 Graphs and Absolute Values of Complex Numbers

Graph each number in the complex plane, and find its absolute value.

a. $z = 4 + 3i$

$(a, b) = (4, 3)$



$|z| = \sqrt{a^2 + b^2}$ Absolute value formula

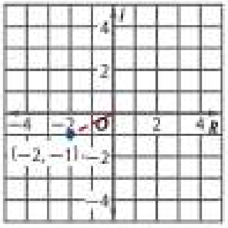
$= \sqrt{4^2 + 3^2}$ $a = 4$ and $b = 3$

$= \sqrt{25}$ or 5 Simplify.

The absolute value of $4 + 3i = 5$.

b. $z = -2 - i$

$(a, b) = (-2, -1)$



$|z| = \sqrt{a^2 + b^2}$ Absolute value formula

$= \sqrt{(-2)^2 + (-1)^2}$ $a = -2$ and $b = -1$

$= \sqrt{5}$ or 2.24 Simplify.

The absolute value of $-2 - i \approx 2.24$.

GuidedPractice

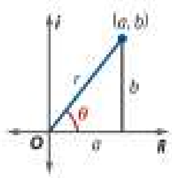
1A. $5 + 2i$

1B. $-3 + 4i$

WatchOut!

Polar Form *The polar form of a complex number* should not be confused with *polar coordinates of a complex number*. The polar form of a complex number is another way to represent a complex number. Polar coordinates of a complex number will be discussed later in this lesson.

Just as rectangular coordinates (x, y) can be written in polar form, so can the coordinates that represent the graph of a complex number in the complex plane. The same trigonometric ratios that were used to find values of x and y can be applied to represent values for a and b.



$\cos\theta = \frac{a}{r}$ and $\sin\theta = \frac{b}{r}$

$r\cos\theta = a$ $\quad r\sin\theta = b$ Multiply each side by r.

Substituting the polar representations for a and b, we can calculate the **polar form** or **trigonometric form** of a complex number.

$z = a + bi$ Original complex number

$= r\cos\theta + (r\sin\theta)i$ $a = r\cos\theta$ and $b = r\sin\theta$

$= r(\cos\theta + i\sin\theta)$ Factor.

In the case of a complex number, r represents the absolute value, or **modulus**, of the complex number and can be found using the same process you used when finding the absolute value, $r = |z| = \sqrt{a^2 + b^2}$. The angle θ is called the **argument** of the complex number. Similar to finding θ with rectangular coordinates (x, y), when using a complex number, $\theta = \tan^{-1}\frac{b}{a}$ or $\theta = \tan^{-1}\frac{b}{a} + \pi$ if $a < 0$.

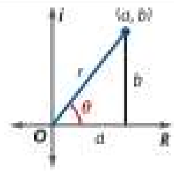
StudyTip

Argument The argument of a complex number is also called the *amplitude*. Just as in polar coordinates, θ is not unique, although it is normally given in the interval $-2\pi < \theta < 2\pi$.

KeyConcept Polar Form of a Complex Number

The polar or trigonometric form of the complex number $z = a + bi$ is
$z = r(\cos\theta + i\sin\theta)$, where

$r = |z| = \sqrt{a^2 + b^2}$, $a = r\cos\theta$, $b = r\sin\theta$, and $\theta = \tan^{-1}\frac{b}{a}$ for $a > 0$ or $\theta = \tan^{-1}\frac{b}{a} + \pi$ for $a < 0$.



Example 2 Complex Numbers in Polar Form

Express each complex number in polar form.

a. $-6 + 8i$

Find the modulus r and argument θ.

$r = \sqrt{a^2 + b^2}$ Conversion formula $\theta = \tan^{-1}\frac{b}{a} + \pi$

$= \sqrt{(-6)^2 + 8^2}$ or 10 $a = -6$ and $b = 8$ $= \tan^{-1}-\frac{8}{6} + \pi$ or about 2.21

The polar form of $-6 + 8i$ is about $10(\cos 2.21 + i \sin 2.21)$.

b. $4 + \sqrt{3}i$

Find the modulus r and argument θ.

$r = \sqrt{a^2 + b^2}$ Conversion formula $\theta = \tan^{-1}\frac{b}{a}$

$= \sqrt{4^2 + (\sqrt{3})^2}$ $a = 4$ and $b = \sqrt{3}$ $= \tan^{-1}\frac{\sqrt{3}}{4}$

$= \sqrt{19}$ or about 4.36   Simplify. ≈ 0.41

The polar form of $4 + \sqrt{3}i$ is about $4.36(\cos 0.41 + i \sin 0.41)$.

GuidedPractice

2A. $9 + 7i$ **2B.** $-2 - 2i$

ReadingMath

Polar Form $r(\cos\theta + i\sin\theta)$ is often abbreviated as $r \text{ cis } \theta$. In Example 2a, $-6 + 8i$ can also be expressed as 10 cis 2.21, where $10 = \sqrt{(-6)^2 + 8^2}$ and $2.21 = \tan^{-1} -\frac{8}{6}$.

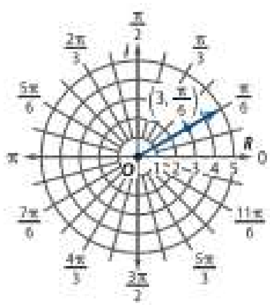
You can use the polar form of a complex number to graph the number on a polar grid by using the r and θ values as your polar coordinates (r, θ). You can also take a complex number written in polar form and convert it to rectangular form by evaluating.

Example 3 Graph and Convert the Polar Form of a Complex Number

Graph $z = 3\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$ on a polar grid. Then express it in rectangular form.

The value of r is 3, and the value of θ is $\frac{\pi}{6}$.

Plot the polar coordinates $\left(3, \frac{\pi}{6}\right)$.



To express the number in rectangular form, evaluate the trigonometric values and simplify.

$3\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$ Polar form

$= 3\left[\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right]$ Evaluate for cosine and sine.

$= \frac{3\sqrt{3}}{2} + \frac{3}{2}i$ Distributive Property

The rectangular form of $z = 3\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$ is $z = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$.

GuidedPractice

Graph each complex number on a polar grid. Then express it in rectangular form.

3A. $5\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)$ **3B.** $4\left(\cos\frac{5\pi}{3} + i \sin\frac{5\pi}{3}\right)$

TechnologyTip

Complex Number Conversions You can convert a complex number in polar form to rectangular form by entering the expression in polar form, then selecting ENTER. To be in polar mode, select MODE then a + bi.

```
3(cos(π/6)+isin(
π/6))
2.598076211+1.5i
```

2 Products, Quotients, Powers, and Roots of Complex Numbers

The polar form of complex numbers, along with the sum and difference formulas for cosine and sine, greatly aid in the multiplication and division of complex numbers. The formula for the product of two complex numbers in polar form can be derived by performing the multiplication.

$z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)$ — Original equation

$= r_1r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)$ — FOIL

$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2)]$ — $i^2 = -1$ and group imaginary terms.

$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$ — Factor out i.

$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$ — Sum identities for cosine and sine

KeyConcept Product and Quotient of Complex Numbers in Polar Form

Given the complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$:

Product Formula $z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

Quotient Formula $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$, where z_2 and $r_2 \neq 0$

You will prove the Quotient Formula in Exercise 77.

ReadingMath

Plural Forms *Moduli* is the plural of *modulus*.

Notice that when multiplying complex numbers, you multiply the moduli and add the arguments. When dividing, you divide the moduli and subtract the arguments.

Example 4 Product of Complex Numbers in Polar Form

Find $2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)\cdot 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ in polar form. Then express the product in rectangular form.

$2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)\cdot 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ — Original expression

$= 2(4)\left[\cos\left(\frac{5\pi}{3} + \frac{\pi}{6}\right) + i\sin\left(\frac{5\pi}{3} + \frac{\pi}{6}\right)\right]$ — Product Formula

$= 8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$ — Simplify.

Now find the rectangular form of the product.

$8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$ — Polar form

$= 8\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$ — Evaluate.

$= 4\sqrt{3} - 4i$ — Distributive Property

The polar form of the product is $8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$. The rectangular form of the product is $4\sqrt{3} - 4i$.

GuidedPractice

Find each product in polar form. Then express the product in rectangular form.

4A. $3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\cdot 5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

4B. $-6\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\cdot 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

As mentioned at the beginning of this lesson, quotients of complex numbers can be used to show relationships in electricity.

Real-World Example 5 Quotient of Complex Numbers in Polar Form

ELECTRICITY If a circuit has a voltage E of 150 volts and an impedance Z of $6 - 3j$ ohms, find the current I amps in the circuit in rectangular form. Use $E = I \cdot Z$.

Express each number in polar form.

$150 = 150(\cos 0 + j \sin 0)$ — $r = \sqrt{150^2 + 0^2}$ or 150; $\theta = \tan^{-1} \frac{0}{150}$ or 0

$6 - 3j = 3\sqrt{5}[\cos(-0.46) + j \sin(-0.46)]$ — $r = \sqrt{6^2 + (-3)^2}$ or $3\sqrt{5}$; $\theta = \tan^{-1} -\frac{3}{6}$ or about -0.46

Solve for the current I in $I \cdot Z = E$.

$I \cdot Z = E$	Original equation
$I = \frac{E}{Z}$	Divide each side by Z.
$I = \frac{150(\cos 0 + j \sin 0)}{3\sqrt{5}[\cos(-0.46) + j \sin(-0.46)]}$	$E = 150(\cos 0 + j \sin 0)$ and $Z = 3\sqrt{5}[\cos(-0.46) + j \sin(-0.46)]$
$I = \frac{150}{3\sqrt{5}}\{\cos [0 - (-0.46)] + j \sin [0 - (-0.46)]\}$	Quotient Formula
$I = 10\sqrt{5}(\cos 0.46 + j \sin 0.46)$	Simplify.

Now convert the current to rectangular form.

$I = 10\sqrt{5}(\cos 0.46 + j \sin 0.46)$	Original equation
$= 10\sqrt{5}(0.90 + 0.44j)$	Evaluate.
$= 20.12 + 9.84j$	Distributive Property

The current is about $20.12 + 9.84j$ amps.

Real-WorldCareer

Electrical Engineers Electrical engineers design and create new technology used to manufacture global positioning systems, giant generators that power entire cities, turbine engines used in jet aircrafts, and radar and navigation systems. They also work on improving various products such as cell phones, cars, and robots.

GuidedPractice

5. **ELECTRICITY** If a circuit has a voltage of 120 volts and a current of $8 + 6j$ amps, find the impedance of the circuit in rectangular form.

Before calculating the powers and roots of complex numbers, it may be helpful to express the complex numbers in polar form. Abraham De Moivre is credited with discovering a useful pattern for evaluating powers of complex numbers.

We can use the formula for the product of complex numbers to help visualize the pattern that De Moivre discovered.

First, find z^2 by taking the product of $z \cdot z$.

$z \cdot z = r(\cos \theta + i \sin \theta) \cdot r(\cos \theta + i \sin \theta)$	Multiply.
$z^2 = r^2[\cos (\theta + \theta) + i \sin (\theta + \theta)]$	Product Formula
$z^2 = r^2(\cos 2\theta + i \sin 2\theta)$	Simplify.

Now find z^3 by calculating $z^2 \cdot z$.

$z^2 \cdot z = r^2(\cos 2\theta + i \sin 2\theta) \cdot r(\cos \theta + i \sin \theta)$	Multiply.
$z^3 = r^3[\cos (2\theta + \theta) + i \sin (2\theta + \theta)]$	Product Formula
$z^3 = r^3(\cos 3\theta + i \sin 3\theta)$	Simplify.

Notice that when calculating these powers of a complex number, you take the nth power of the modulus and multiply the argument by n.

This pattern is summarized below.

Math HistoryLink

Abraham De Moivre (1667–1754) A French mathematician, De Moivre is known for the theorem named for him and his book on probability theory, *The Doctrine of Chances*. He is credited with being one of the pioneers of analytic geometry and probability.

KeyConcept De Moivre's Theorem

If the polar form of a complex number is $z = r(\cos\theta + i\sin\theta)$, then for positive integers n

$$z^n = [r(\cos\theta + i\sin\theta)]^n \text{ or } r^n(\cos n\theta + i\sin n\theta).$$

You will prove De Moivre's Theorem in Lesson 3-4.

Example 6 De Moivre's Theorem

Find $(4 + 4\sqrt{3}i)^6$, and express it in rectangular form.

First, write $4 + 4\sqrt{3}i$ in polar form.

$r = \sqrt{a^2 + b^2}$ — Conversion formula — $\theta = \tan^{-1}\frac{b}{a}$

$= \sqrt{4^2 + (4\sqrt{3})^2}$ — $a = 4$ and $b = 4\sqrt{3}$ — $= \tan^{-1}\frac{4\sqrt{3}}{4}$

$= \sqrt{16 + 48}$ — Simplify. — $= \tan^{-1}\sqrt{3}$

$= 8$ — Simplify. — $= \frac{\pi}{3}$

The polar form of $4 + 4\sqrt{3}i$ is $8\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

Now use De Moivre's Theorem to find the sixth power.

$(4 + 4\sqrt{3}i)^6 = \left[8\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^6$ — Original equation

$= 8^6\left[\cos 6\left(\frac{\pi}{3}\right) + i\sin 6\left(\frac{\pi}{3}\right)\right]$ — De Moivre's Theorem

$= 262{,}144(\cos 2\pi + i\sin 2\pi)$ — Simplify.

$= 262{,}144(1 + 0i)$ — Evaluate.

$= 262{,}144$ — Simplify.

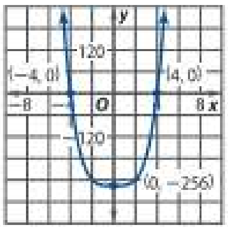
Therefore, $(4 + 4\sqrt{3}i)^6 = 262{,}144$.

GuidedPractice

Find each power, and express it in rectangular form.

6A. $(1 + \sqrt{3}i)^4$ **6B.** $(2\sqrt{3} - 2i)^8$

In the real number system, $x^4 = 256$ has two solutions, 4 and -4. The graph of $y = x^4 - 256$ shows that there are two real zeros at $x = 4$ and $x = -4$. In the complex number system, however, there are two real solutions and two complex solutions.



Through the Fundamental Theorem of Algebra polynomials of degree n have exactly n zeros in the complex number system. Therefore, the equation $x^4 = 256$, rewritten as $x^4 - 256 = 0$, has exactly four solutions, or roots: 4, -4, $4i$, and $-4i$. In general, all nonzero complex numbers have p distinct pth roots. That is, they each have two square roots, three cube roots, four fourth roots, and so on.

ReviewVocabulary

Fundamental Theorem of Algebra
A polynomial function of degree n, where $n > 0$, has at least one zero (real or imaginary) in the complex number system.

To find all of the roots of a polynomial, we can use De Moivre's Theorem to arrive at the following expression.

KeyConcept Distinct Roots

For a positive integer p, the complex number $r(\cos\theta + i\sin\theta)$ has p distinct pth roots. They are found by

$$r^{\frac{1}{p}}\left(\cos\frac{\theta + 2n\pi}{p} + i\sin\frac{\theta + 2n\pi}{p}\right),$$

where $n = 0, 1, 2, \ldots, p - 1$.

We can use this formula for the different values of n, but we can stop when $n = p - 1$. When n equals or exceeds p, the roots repeat as the following shows.

$$\frac{\theta + 2\pi p}{p} = \frac{\theta}{p} + 2\pi$$ Coterminal with $\frac{\theta}{p}$, when $n = 0$

Example 7 *p*th Roots of a Complex Number

Find the fourth roots of $-4 - 4i$.

First, write $-4 - 4i$ in polar form.

$-4 - 4i = 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$ — $r = \sqrt{(-4)^2 + (-4)^2}$ or $4\sqrt{2}$; $\theta = \tan^{-1}\frac{-4}{-4} + \pi$ or $\frac{5\pi}{4}$

Now write an expression for the fourth roots.

$(4\sqrt{2})^{\frac{1}{4}}\left(\cos\frac{\frac{5\pi}{4} + 2n\pi}{4} + i\sin\frac{\frac{5\pi}{4} + 2n\pi}{4}\right)$ — $\theta = \frac{5\pi}{4}$, $p = 4$, and $r^{\frac{1}{p}} = (4\sqrt{2})^{\frac{1}{4}}$

$= \sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{n\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{n\pi}{2}\right)\right]$ — Simplify.

Let $n = 0, 1, 2,$ and 3 successively to find the fourth roots.

Let $n = 0$. $\sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{(0)\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{(0)\pi}{2}\right)\right]$ — Distinct Roots

$= \sqrt[8]{32}\left(\cos\frac{5\pi}{16} + i\sin\frac{5\pi}{16}\right)$ or $0.86 + 1.28i$ — First fourth root

Let $n = 1$. $\sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{(1)\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{(1)\pi}{2}\right)\right]$

$= \sqrt[8]{32}\left(\cos\frac{13\pi}{16} + i\sin\frac{13\pi}{16}\right)$ or $-1.28 + 0.86i$ — Second fourth root

Let $n = 2$. $\sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{(2)\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{(2)\pi}{2}\right)\right]$

$= \sqrt[8]{32}\left(\cos\frac{21\pi}{16} + i\sin\frac{21\pi}{16}\right)$ or $-0.86 - 1.28i$ — Third fourth root

Let $n = 3$. $\sqrt[8]{32}\left[\cos\left(\frac{5\pi}{16} + \frac{(3)\pi}{2}\right) + i\sin\left(\frac{5\pi}{16} + \frac{(3)\pi}{2}\right)\right]$

$= \sqrt[8]{32}\left(\cos\frac{29\pi}{16} + i\sin\frac{29\pi}{16}\right)$ or $1.28 - 0.86i$ — Fourth fourth root

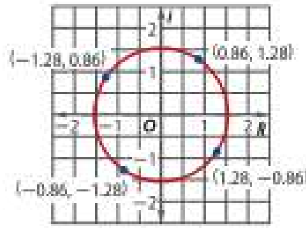
The fourth roots of $-4 - 4i$ are approximately $0.86 + 1.28i$, $-1.28 + 0.86i$, $-0.86 - 1.28i$, and $1.28 - 0.86i$.

GuidedPractice

7A. Find the cube roots of $2 + 2i$.

7B. Find the fifth roots of $4\sqrt{3} - 4i$.

We can make observations about the distinct roots of a number by graphing the roots on a coordinate plane. As shown at the right, the four fourth roots found in Example 7 lie on a circle. If we look at the polar form of each complex number, each has the same modulus of $\sqrt[4]{32}$, which acts as the radius of the circle. The roots are also equally spaced around the circle as a result of the arguments differing by $\frac{\pi}{2}$.



A special case of finding roots occurs when finding the pth roots of 1. When 1 is written in polar form, $r = 1$. As mentioned in the previous paragraph, the modulus of our roots is the radius of the circle that is formed from plotting the roots on a coordinate plane. Thus, the pth roots of 1 lie on the unit circle. Finding the pth roots of 1 is referred to as finding the **pth roots of unity**.

StudyTip

The pth Roots of a Complex Number Each root will have the same modulus of $r^{\frac{1}{p}}$. The argument of the first root is $\frac{\theta}{p}$, and each successive root is found by repeatedly adding $\frac{2\pi}{p}$ to the argument.

Example 8 The pth Roots of Unity

Find the eighth roots of unity.

First, write 1 in polar form.

$1 = 1 \cdot (\cos 0 + i \sin 0)$ — $r = \sqrt{1^2 + 0^2}$ or 1 and $\theta = \tan^{-1}\frac{0}{1}$ or 0

Now write an expression for the eighth roots.

$1\left(\cos \frac{0 + 2n\pi}{8} + i \sin \frac{0 + 2n\pi}{8}\right)$ — $\theta = 0$, $p = 8$, and $r^{\frac{1}{p}} = 1^{\frac{1}{8}}$ or 1

$= \cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4}$ — Simplify.

Let $n = 0$ to find the first root of 1.

$n = 0 \qquad \cos \frac{(0)\pi}{4} + i \sin \frac{(0)\pi}{4}$ — Distinct Roots

$= \cos 0 + i \sin 0$ or 1 — First root

Notice that the modulus of each complex number is 1. The arguments are found by $\frac{n\pi}{4}$, resulting in θ increasing by $\frac{\pi}{4}$ for each successive root. Therefore, we can calculate the remaining roots by adding $\frac{\pi}{4}$ to each previous θ.

$\cos 0 + i \sin 0$ or 1 — 1st root

$\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$ or $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ — 2nd root

$\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$ or i — 3rd root

$\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}$ or $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ — 4th root

$\cos \pi + i \sin \pi$ or -1 — 5th root

$\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}$ or $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ — 6th root

$\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$ or $-i$ — 7th root

$\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}$ or $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ — 8th root

The eighth roots of 1 are 1, $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, i, $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, -1, $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, $-i$, and $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ as shown in Figure 2.5.1.

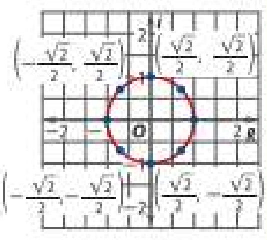


Figure 2.5.1

GuidedPractice

8A. Find the cube roots of unity.

8B. Find the seventh roots of unity.

Exercises

Graph each number in the complex plane, and find its absolute value. (Example 1)

1. $z = 4 + 4i$
2. $z = -3 + i$
3. $z = -4 - 6i$
4. $z = 2 - 5i$
5. $z = 3 + 4i$
6. $z = -7 + 5i$
7. $z = -3 - 7i$
8. $z = 8 - 2i$

9. **VECTORS** The force on an object is given by $z = 10 + 15i$, where the components are measured in newtons (N). (Example 1)
   a. Represent z as a vector in the complex plane.
   b. Find the magnitude and direction angle of the vector.

Express each complex number in polar form. (Example 2)

10. $4 + 4i$
11. $-2 + i$
12. $4 - \sqrt{2}i$
13. $2 - 2i$
14. $4 + 5i$
15. $-2 + 4i$
16. $-1 - \sqrt{3}i$
17. $3 + 3i$

Graph each complex number on a polar grid. Then express it in rectangular form. (Example 3)

18. $10(\cos 6 + i \sin 6)$
19. $2(\cos 3 + i \sin 3)$
20. $4\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$
21. $3\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$
22. $\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$
23. $2\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$
24. $-3(\cos 180° + i \sin 180°)$
25. $\frac{3}{2}(\cos 360° + i \sin 360°)$

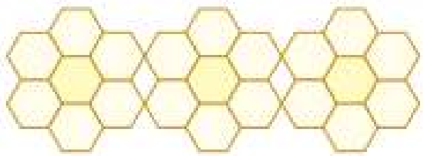
Find each product or quotient, and express it in rectangular form. (Examples 4 and 5)

26. $6\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) \cdot 4\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$
27. $5(\cos 135° + i \sin 135°) \cdot 2(\cos 45° + i \sin 45°)$
28. $3\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right) \div \frac{1}{2}(\cos \pi + i \sin \pi)$
29. $2(\cos 90° + i \sin 90°) \cdot 2(\cos 270° + i \sin 270°)$
30. $3\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right) \div 4\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$
31. $4\left(\cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4}\right) \div 2\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$
32. $\frac{1}{2}(\cos 60° + i \sin 60°) \cdot 6(\cos 150° + i \sin 150°)$
33. $6\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right) \div 2\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$
34. $5(\cos 180° + i \sin 180°) \cdot 2(\cos 135° + i \sin 135°)$
35. $\frac{1}{2}\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \div 3\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$

Find each power, and express it in rectangular form. (Example 6)

36. $(2 + 2\sqrt{3}i)^6$
37. $(12i - 5)^3$
38. $\left[4\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)\right]^4$
39. $(\sqrt{3} - i)^3$
40. $(3 - 5i)^4$
41. $(2 + 4i)^4$
42. $(3 - 6i)^4$
43. $(2 + 3i)^2$
44. $\left[3\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)\right]^3$
45. $\left[2\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)\right]^4$

46. **DESIGN** Suha works for an advertising agency. She wants to incorporate a design comprised of regular hexagons as the artwork for one of her proposals. Suha can locate the vertices of one of the central regular hexagons by graphing the solutions to $x^6 - 1 = 0$ in the complex plane. Find the vertices of this hexagon. (Example 7)



Find all of the distinct *p*th roots of the complex number. (Examples 7 and 8)

47. sixth roots of i
48. fifth roots of $-i$
49. fourth roots of $4\sqrt{3} - 4i$
50. cube roots of $-117 + 44i$
51. fifth roots of $-1 + 11\sqrt{2}i$
52. square root of $-3 - 4i$
53. find the square roots of unity
54. find the fourth roots of unity

55. **ELECTRICITY** The impedance in one part of a series circuit is $5(\cos 0.9 + j \sin 0.9)$ ohms. In the second part of the circuit, it is $8(\cos 0.4 + j \sin 0.4)$ ohms.
   B
   a. Convert each expression to rectangular form.
   b. Add your answers from part **a** to find the total impedance in the circuit.
   c. Convert the total impedance back to polar form.

Find each product. Then repeat the process by multiplying the polar forms of each pair of complex numbers using the Product Formula.

56. $(1 - i)(4 + 4i)$
57. $(3 + i)(3 - i)$
58. $(4 + i)(3 - i)$
59. $(-6 + 5i)(2 - 3i)$
60. $(\sqrt{2} + 2i)(1 + i)$
61. $(3 - 2i)(1 + \sqrt{3}i)$

62. **FRACTALS** A *fractal* is a geometric figure that is made up of a pattern that is repeated indefinitely on successively smaller scales, as shown below.

In this problem, you will generate a fractal through iterations of $f(z) = z^2$. Consider $z_0 = 0.8 + 0.5i$.

a. Calculate z_1, z_2, z_3, z_4, z_5, z_6, and z_7 where $z_1 = f(z_0)$, $z_2 = f(z_1)$, and so on.

b. Graph each of the numbers on the complex plane.

c. Predict the location of z_{100}. Explain.

63. **TRANSFORMATIONS** There are certain operations with complex numbers that correspond to geometric transformations in the complex plane. Describe the transformation applied to point z to obtain point w in the complex plane for each of the following operations.

a. $w = z + (3 - 4i)$

b. w is the complex conjugate of z.

c. $w = i \cdot z$

d. $w = 0.25z$

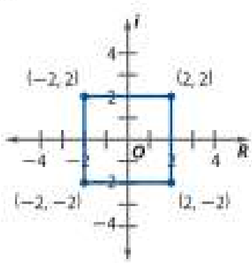
C

Find z and the pth roots of z given each of the following.

64. $p = 3$, one cube root is $\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

65. $p = 4$, one fourth root is $-1 - i$

66. **GRAPHICS** By representing each vertex by a complex number in polar form, a programmer dilates and then rotates the square below 45° counterclockwise so that the new vertices lie at the midpoints of the sides of the original square.



a. By what complex number should the programmer multiply each number to produce this transformation?

b. What happens if the numbers representing the original vertices are multiplied by the square of your answer to part **a**?

Use the Distinct Roots Formula to find all of the solutions of each equation. Express the solutions in rectangular form.

67. $x^3 = i$

68. $x^3 + 3 = 128$

69. $x^4 = 81i$

70. $x^5 - 1 = 1023$

71. $x^3 + 1 = i$

72. $x^4 - 2 + i = -1$

H.O.T. Problems Use Higher-Order Thinking Skills

73. **ERROR ANALYSIS** Alma and Bilal are evaluating $\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^5$. Alma uses De Moivre's Theorem and gets an answer of $\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}$. Bilal tells her that she has only completed part of the problem. Is either of them correct? Explain your reasoning.

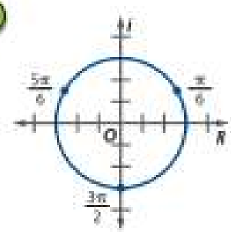
74. **REASONING** Suppose $z = a + bi$ is one of the 29th roots of 1.

a. What is the maximum value of a?

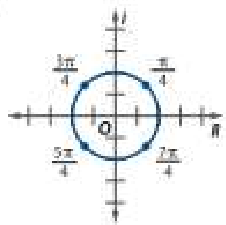
b. What is the maximum value of b?

CHALLENGE **Find the roots shown on each graph and write them in polar form. Then identify the complex number with the given roots.**

75.



76.



77. **PROOF** Given $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$, where $r_2 \neq 0$, prove that $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)]$.

REASONING **Determine whether each statement is *sometimes*, *always*, or *never* true. Explain your reasoning.**

78. The pth roots of a complex number z are equally spaced around the circle centered at the origin with radius $r^{\frac{1}{p}}$.

79. The complex conjugate of $z = a + bi$ is $\bar{z} = a - bi$. For any z, $z + \bar{z}$ and $z \cdot \bar{z}$ are real numbers.

80. **OPEN ENDED** Find two complex numbers $a + bi$ in which $a \neq 0$ and $b \neq 0$ with an absolute value of $\sqrt{17}$.

81. **WRITING IN MATH** Explain why the sum of the imaginary parts of the p distinct pth roots of anypositive real number must be zero. (*Hint*: Theroots are the vertices of a regular polygon.)

Spiral Review

Write each polar equation in rectangular form. (Lesson 2-4)

82. $r = \frac{15}{1 + 4\cos\theta}$

83. $r = \frac{14}{2\cos\theta + 2}$

84. $r = \frac{-6}{\sin\theta - 2}$

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation. (Lesson 2-3)

85. $(x - 3)^2 + y^2 = 9$

86. $x^2 - y^2 = 1$

87. $x^2 + y^2 = 2y$

Graph the conic given by each equation.

88. $y = x^2 + 3x + 1$

89. $y^2 - 2x^2 - 16 = 0$

90. $x^2 + 4y^2 + 2x - 24y + 33 = 0$

Find the center, foci, and vertices of each ellipse.

91. $\frac{(x + 8)^2}{9} + \frac{(y - 7)^2}{81} = 1$

92. $25x^2 + 4y^2 + 150x + 24y = -161$

93. $4x^2 + 9y^2 - 56x + 108y = -484$

Solve each system of equations using Gauss-Jordan elimination.

94. $x + y + z = 12$
$6x - 2y - z = 16$
$3x + 4y + 2z = 28$

95. $9g + 7h = -30$
$8h + 5j = 11$
$-3g + 10j = 73$

96. $2k - n = 2$
$3p = 21$
$4k + p = 19$

97. POPULATION In the beginning of 2008, the world's population was about 6.7 billion. If the world's population grows continuously at a rate of 2%, the future population P, in billions, can be predicted by $P = 6.5e^{0.02t}$, where t is the time in years since 2008.

a. According to this model, what was the world's population in 2018?

b. Some experts have estimated that the world's food supply can support a population of at most 18 billion people. According to this model, for how many more years will the food supply be able to support the trend in world population growth?

Skills Review for Standardized Tests

98. SAT/ACT The graph on the xy-plane of the quadratic function g is a parabola with vertex at $(3, -2)$. If $g(0) = 0$, then which of the following must also equal 0?

A $g(2)$

B $g(3)$

C $g(4)$

D $g(6)$

E $g(7)$

99. Which of the following expresses the complex number $20 - 21i$ in polar form?

F $29(\cos 5.47 + i\sin 5.47)$

G $29(\cos 5.52 + i\sin 5.52)$

H $32(\cos 5.47 + i\sin 5.47)$

J $32(\cos 5.52 + i\sin 5.52)$

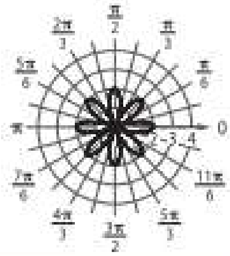
100. FREE RESPONSE Consider the graph at the right.

a. Give a possible equation for the function.

b. Describe the symmetries of the graph.

c. Give the zeroes of the function over the domain $0 \le \theta \le 2\pi$.

d. What is the minimum value of r over the domain $0 \le \theta \le 2\pi$?



CHAPTER 2

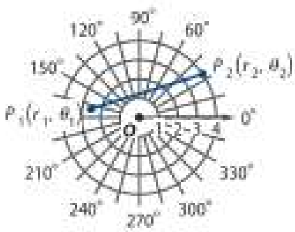
Study Guide and Review

Chapter Summary

KeyConcepts

Polar Coordinates (Lesson 2-1)

- In the polar coordinate system, a point (r, θ) is located using its directed distance r and directed angle θ.
- The distance between $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ in the polar plane is $P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}$.



Graphs of Polar Equations (Lesson 2-2)

- Circle: $r = a \cos \theta$ or $r = a \sin \theta$
- Limaçon: $r = a \pm b \cos \theta$ or $r = a \pm b \sin \theta$; $a > 0, b > 0$
- Rose: $r = a \cos n\theta$ or $r = a \sin n\theta$, $n \geq 2$, $n \in \mathbb{Z}$
- Lemniscate: $r^2 = a^2 \cos 2\theta$ or $r^2 = a^2 \sin 2\theta$
- Spirals of Archimedes: $r = a\theta + b$, $\theta \geq 0$

Polar and Rectangular Forms of Equations (Lesson 2-3)

- A point $P(r, \theta)$ has rectangular coordinates $(r \cos \theta, r \sin \theta)$.
- To convert a point $P(x, y)$ from rectangular to polar coordinates, use the equations $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \frac{y}{x}$, when $x > 0$ or $\theta = \tan^{-1} \frac{y}{x} + \pi$, when $x < 0$.

Polar Forms of Conic Sections (Lesson 2-4)

- The polar equation of a conic section is of the form $r = \frac{ed}{1 \pm e \cos \theta}$ or $r = \frac{ed}{1 \pm e \sin \theta}$, depending on the location and orientation of the directrix.

Complex Numbers and De Moivre's Theorem (Lesson 2-5)

- The polar or trigonometric form of the complex number $a + bi$ is $r(\cos \theta + i \sin \theta)$.
- The product formula for two complex numbers z_1 and z_2 is $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$.
- The quotient formula for two complex numbers z_1 and z_2 is $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$, where z_2 and $r_2 \neq 0$.
- De Moivre's Theorem states that if the polar form of a complex number is $z = r(\cos \theta + i \sin \theta)$, then $z^n = r^n(\cos n\theta + i \sin n\theta)$ for positive integers n.

KeyVocabulary

absolute value of a complex number
Argand plane
argument
cardioid
complex plane
imaginary axis
lemniscate
limaçon
modulus
polar axis
polar coordinate system
polar coordinates
polar equation
polar form
polar graph
pole
*p*th roots of unity
real axis
rose
spiral of Archimedes
trigonometric form

VocabularyCheck

Choose the correct term from the list above to complete each sentence.

1. A(n) ________ is the set of all points with coordinates (r, θ) that satisfy a given polar equation.
2. A plane that has an axis for the real component and an axis for the imaginary component is a(n) ________.
3. The location of a point in the ________ is identified using the directed distance from a fixed point and the angle from a fixed axis.
4. A special type of limaçon with equation of the form $r = a + b \cos \theta$ where $a = b$ is called a(n) ________.
5. The ________ is the angle θ of a complex number written in the form $r(\cos \theta + i \sin \theta)$.
6. The origin of a polar coordinate system is called the ________.
7. The absolute value of a complex number is also called the ________.
8. The ________ is another name for the complex plane.
9. The graph of a polar equation of the form $r^2 = a^2 \cos 2\theta$ or $r^2 = a^2 \sin 2\theta$ is called a(n) ________.
10. The ________ is an initial ray from the pole, usually horizontal and directed toward the right.

Lesson-by-Lesson Review

2-1 Polar Coordinates

Graph each point on a polar grid.

11. $W(-0.5, 210°)$

12. $X\left(1.5, \frac{7\pi}{4}\right)$

13. $Y(4, -120°)$

14. $Z\left(-3, \frac{5\pi}{6}\right)$

Graph each polar equation.

15. $\theta = -60°$

16. $r = \frac{9}{2}$

17. $r = 7$

18. $\theta = \frac{11\pi}{6}$

Find the distance between each pair of points.

19. $\left(5, \frac{\pi}{2}\right), \left(2, -\frac{7\pi}{6}\right)$

20. $(-3, 60°), (4, 240°)$

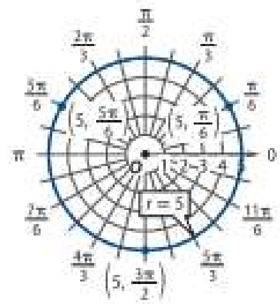
21. $(-1, -45°), (6, 270°)$

22. $\left(7, \frac{5\pi}{6}\right), \left(2, \frac{4\pi}{3}\right)$

Example 1

Graph $r = 5$.

The solutions of $r = 5$ are ordered pairs of the form $(5, \theta)$ where θ is any real number. The graph consists of all points that are 5 units from the pole, so the graph is a circle centered at the pole with radius 5.



2-2 Graphs of Polar Equations

Use symmetry, zeros, and maximum r-values to graph each function.

23. $r = \sin 3\theta$

24. $r = 2\cos\theta$

25. $r = 5\cos 2\theta$

26. $r = 4\sin 4\theta$

27. $r = 2 + 2\cos\theta$

28. $r = 1.5\theta, \theta \geq 0$

Use symmetry to graph each equation.

29. $r = 2 - \sin\theta$

30. $r = 1 + 5\cos\theta$

31. $r = 3 - 2\cos\theta$

32. $r = 4 + 4\sin\theta$

33. $r = -3\sin\theta$

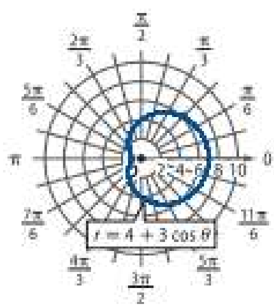
34. $r = -5 + 3\cos\theta$

Example 2

Use symmetry to graph $r = 4 + 3\cos\theta$.

Replacing (r, θ) with $(r, -\theta)$ yields $r = 4 + 3\cos(-\theta)$, which simplifies to $r = 4 + 3\cos\theta$ because cosine is even. The equations are equivalent, so the graph of this equation is symmetric with respect to the polar axis. Therefore, you can make a table of values to find the r-values corresponding to θ on the interval $[0, \pi]$.

θ	r
0	7
$\frac{\pi}{4}$	$\frac{8+3\sqrt{2}}{2}$
$\frac{\pi}{2}$	4
$\frac{3\pi}{4}$	$\frac{8-3\sqrt{2}}{2}$
π	1



By plotting these points and using polar axis symmetry, you obtain the graph shown.

2-3 Polar and Rectangular Forms

Find two pairs of polar coordinates for each point with the given rectangular coordinates if $0 \leq \theta \leq 2\pi$. Round to the nearest hundredth.

35. $(-1, 5)$

36. $(3, 7)$

37. $(2a, 0)$, $a > 0$

38. $(4b, -6b)$, $b > 0$

Write each equation in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation.

39. $r = 5$

40. $r = -4 \sin \theta$

41. $r = 6 \sec \theta$

42. $r = \frac{1}{3} \csc \theta$

Example 3

Write $r = 2 \cos \theta$ in rectangular form, and then identify its graph. Support your answer by graphing the polar form of the equation.

$r = 2 \cos \theta$	Original equation
$r^2 = 2r \cos \theta$	Multiply each side by r.
$x^2 + y^2 = 2x$	$r^2 = x^2 + y^2$ and $x = r \cos \theta$
$x^2 + y^2 - 2x = 0$	Subtract $2x$ from each side.

In standard form, $(x - 1)^2 + y^2 = 1$, you can identify the graph of this equation as a circle centered at (1, 0) with radius 1, as supported by the graph of $r = 2 \cos \theta$.

2-4 Polar Forms of Conic Sections

Determine the eccentricity, type of conic, and equation of the directrix for each polar equation.

43. $r = \frac{3.5}{1 + \sin \theta}$

44. $r = \frac{1.2}{1 + 0.3 \cos \theta}$

45. $r = \frac{14}{1 - 2 \sin \theta}$

46. $r = \frac{6}{1 - \cos \theta}$

Write and graph a polar equation and directrix for the conic with the given characteristics.

47. $e = 0.5$; vertices at $(0, -2)$ and $(0, 6)$

48. $e = 1.5$; directrix $x = 5$

Write each polar equation in rectangular form.

49. $r = \frac{1.6}{1 - 0.2 \sin \theta}$

50. $r = \frac{5}{1 + \cos \theta}$

Example 4

Determine the eccentricity, type of conic, and equation of the directrix for $r = \frac{7}{3.5 - 3.5 \cos \theta}$.

Write the equation in standard form, $r = \frac{ed}{1 + e \cos \theta}$.

$r = \frac{7}{3.5 - 3.5 \cos \theta}$	Original equation
$r = \frac{3.5(2)}{3.5(1 - \cos \theta)}$	Factor the numerator and denominator.
$r = \frac{2}{1 - \cos \theta}$	Divide the numerator and denominator by 3.5.

In this form, you can see from the denominator that $e = 1$; therefore, the conic is a parabola. For polar equations of this form, the equation of the directrix is $x = -d$. From the numerator, we know that $ed = 2$, so $d = 2 \div 1$ or 2. Therefore, the equation of the directrix is $x = -2$.

2-5 Complex Numbers and De Moivre's Theorem

Graph each number in the complex plane, and find its absolute value.

51. $z = 3 - i$

52. $z = 4i$

53. $z = -4 + 2i$

54. $z = 6 - 3i$

Express each complex number in polar form.

55. $3 + \sqrt{2}i$

56. $-5 + 8i$

57. $-4 - \sqrt{3}i$

58. $\sqrt{2} + \sqrt{2}i$

Graph each complex number on a polar grid. Then express it in rectangular form.

59. $z = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

60. $z = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

61. $z = -2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

62. $z = 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

Find each product or quotient, and express it in rectangular form.

63. $-2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \cdot -4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

64. $8(\cos 225° + i\sin 225°) \cdot \frac{1}{2}(\cos 120° + i\sin 120°)$

65. $5\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \div \frac{1}{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

66. $6(\cos 210° + i\sin 210°) \div 3(\cos 150° + i\sin 150°)$

Find each power, and express it in rectangular form.

67. $(4 - i)^5$

68. $(\sqrt{2} + 3i)^4$

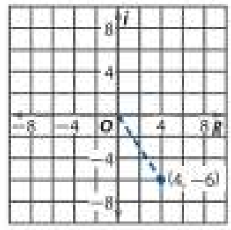
Find all of the distinct *p*th roots of the complex number.

69. cube roots of $6 - 4i$

70. fourth roots of $1 + i$

Example 5

Graph $4 - 6i$ in the complex plane and express in polar form.



Find the modulus.

$r = \sqrt{a^2 + b^2}$ — Conversion formula

$= \sqrt{4^2 + (-6)^2}$ or $2\sqrt{13}$ — $a = 4$ and $b = -6$

Find the argument.

$\theta = \tan^{-1}\frac{b}{a}$ — Conversion formula

$= \tan^{-1}\left(-\frac{6}{4}\right)$ — $a = 4$ and $b = -6$

$= -0.98$ — Simplify.

The polar form of $4 - 6i$ is approximately $2\sqrt{13}\,[\cos(-0.98) + i\sin(-0.98)]$.

Example 6

Find $-3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \cdot 5\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$ in polar form. Then express the product in rectangular form.

$-3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \cdot 5\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$ — Original expression

$= (-3 \cdot 5)\left[\cos\left(\frac{\pi}{4} + \frac{7\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{7\pi}{6}\right)\right]$ — Product Formula

$= -15\left[\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right]$ — Simplify.

Now find the rectangular form of the product.

$-15\left[\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right]$ — Polar form

$= -15[-0.26 + i(-0.97)]$ — Evaluate.

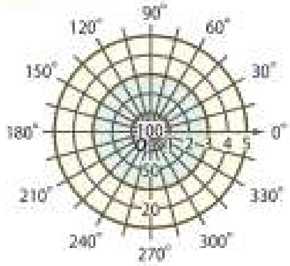
$= 3.9 + 14.5i$ — Distributive Property

The polar form of the product is $-15\left[\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right]$

The rectangular form of the product is $3.9 + 14.5i$.

Applications and Problem Solving

71. GAMES An arcade game consists of rolling a ball up an incline at a target. The region in which the ball lands determines the number of points earned. The model shows the point value for each region. (Lesson 2-1)



a. If, on a turn, a player rolls the ball to the point (3.5, 165°), how many points does he get?

b. Give two possible locations that a player will receive 50 points.

72. LANDSCAPING A landscaping company uses an adjustable lawn sprinkler that can rotate 360° and can cover a circular region with a radius of 20 meters. (Lesson 2-1)

a. Graph the dimensions of the region that the sprinkler can cover on a polar grid if it is set to rotate 360°.

b. Find the area of the region that the sprinkler covers if the rotation is adjusted to $-30° \le \theta \le 210°$.

73. BIOLOGY The pattern on the shell of a snail can be modeled using $r = \frac{1}{3}\theta + \frac{1}{2}$, $\theta \ge 0$. Identify and graph the classic curve that models this pattern. (Lesson 2-2)

74. RIDES The path of a Ferris wheel can be modeled by $r = 50 \sin \theta$, where r is given in meters. (Lesson 2-3)

a. What are the polar coordinates of a rider located at $\theta = \frac{\pi}{12}$? Round to the nearest tenth, if necessary.

b. What are the rectangular coordinates of the rider's location? Round to the nearest tenth, if necessary.

c. What is the rider's distance above the ground if the polar axis represents the ground?

75. ORIENTEERING Orienteering requires participants to make their way through an area using a topographic map. One orienteer starts at Checkpoint A and walks 5000 meters at an angle of 35° measured clockwise from due east. A second orienteer starts at Checkpoint A and walks 3,000 meters due west and then 2,000 meters due north. How far, to the nearest meter, are the two orienteers from each other? (Lesson 2-3)

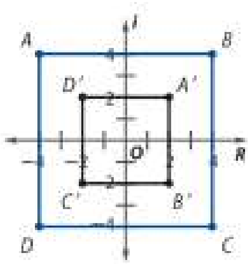
76. SATELLITE The orbit of a satellite around Earth has eccentricity of 0.05, and the distance from a vertex of the path to the center of Earth is 32,082 kilometers. Write a polar equation that can be used to model the path of the satellite if Earth is located at the focus closest to the given vertex. (Lesson 2-4)

77. ELECTRICITY Most circuits in Europe are designed to accommodate 220 volts. For parts **a** and **b**, use $E = I \cdot Z$, where voltage E is measured in volts, impedance Z is measured in ohms, and current I is measured in amps. Round to the nearest tenth. (Lesson 2-5)

a. If the circuit has a current of $2 + 5j$ amps, what is the impedance?

b. If a circuit has an impedance of $1 - 3j$ ohms, what is the current?

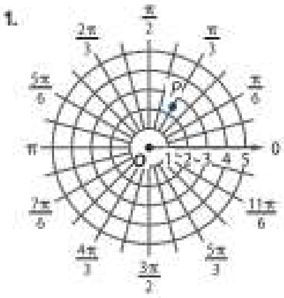
78. COMPUTER GRAPHICS Geometric transformation of figures can be performed using complex numbers. If a programmer starts with square $ABCD$, as shown below, each of the vertices can be represented by a complex number in polar form. Multiplication can then be used to rotate and dilate the square, producing the square $A'B'C'D'$. By what complex number should the programmer multiply each number to produce this transformation? (Lesson 2-5)



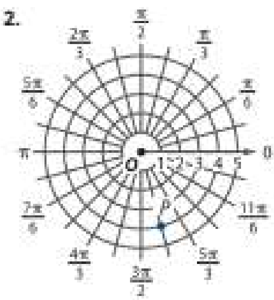
CHAPTER 2 Practice Test

Find four different pairs of polar coordinates that name point P if $-2\pi \leq \theta \leq 2\pi$.

1.



2.



Graph each polar equation.

3. $\theta = 30°$

4. $r = 1$

5. $r = 2.5$

6. $\theta = \frac{5\pi}{3}$

7. $r = \frac{2}{3}\sin\theta$

8. $r = -\frac{1}{2}\sec\theta$

9. $r = -4\csc\theta$

10. $r = 2\cos\theta$

Identify and graph each classic curve.

11. $r = 1.5 + 1.5\cos\theta$

12. $r^2 = 6.25\sin 2\theta$

13. RADAR An air traffic controller is tracking an airplane with a current location of (66, 115°). The value of r is given in kilometers.

a. What are the rectangular coordinates of the airplane? Round to the nearest tenth of a kilometer.

b. If a second plane is located at the point (50, −75), what are the polar coordinates of the plane if $r > 0$ and $0 \leq \theta \leq 360°$? Round to the nearest kilometer and the nearest tenth of a degree, if necessary.

c. What is the distance between the two planes? Round to the nearest kilometer.

Identify the graph of each rectangular equation. Then write the equation in polar form. Support your answer by graphing the polar form of the equation.

14. $(x - 7)^2 + y^2 = 49$

15. $y = 3x^2$

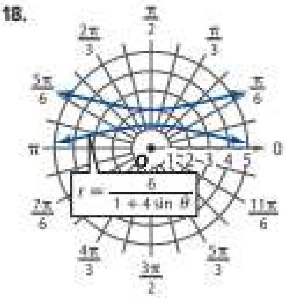
Determine the eccentricity, type of conic, and equation of the directrix for each polar equation.

16. $r = \frac{2}{1 - 0.4\sin\theta}$

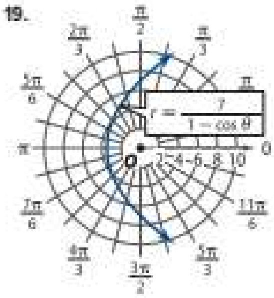
17. $r = \frac{6}{2\cos\theta + 1}$

Write the equation for each polar graph in rectangular form.

18.



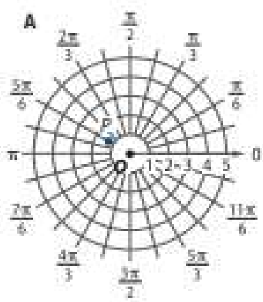
19.



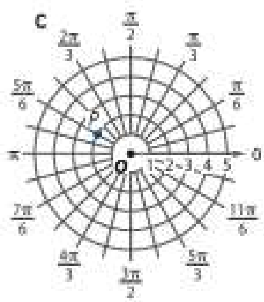
20. ELECTRICITY If a circuit has a voltage E of 135 volts and a current I of $3 - 4j$ amps, find the impedance Z of the circuit in ohms in rectangular form. Use the equation $E = I \cdot Z$.

21. MULTIPLE CHOICE Identify the graph of point P with complex coordinates $(-\sqrt{3}, -1)$ on the polar coordinate plane.

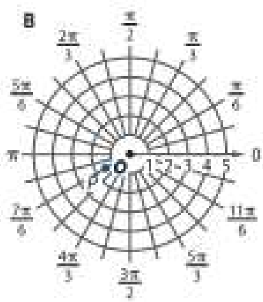
A



C



B



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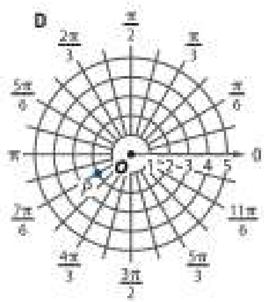


Find each power, and express it in rectangular form. Round to the nearest tenth.

22. $(-1 + 4i)^3$

23. $(-7 - 3i)^5$

24. $(6 + i)^4$

25. $(2 - 5i)^6$

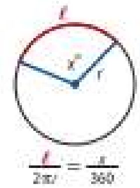
CHAPTER 2

Connect to AP Calculus
Arc Length

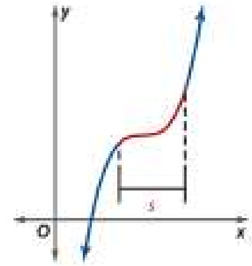
Objective

- Approximate the arc length of a curve.

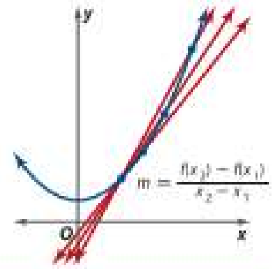
You can find the length of a line segment by using the Distance Formula. You can find the length of an arc by using proportions. In calculus, you will need to calculate many lengths that are not represented by line segments or sections of a circle.



Integral calculus focuses on areas, volumes, and lengths. It can be used to find the length of a curve for which we do not have a standard equation, such as a curve defined by a quadratic, cubic, or polar function. *Riemann sums* and *definite integrals*, two concepts that you will learn more about in the following chapters, are needed to calculate the exact length of a curve, or *arc length*, denoted *s*.



In this lesson, we will approximate the arc length of a curve using a process similar to a method used to approximate the rate of change at a point. You can calculate the slopes of secant lines to approximate the rates of change for graphs at specific points. Decreasing the distance between the two points on the secant lines increases the accuracy of the approximations as shown in the graph at the right.



Activity 1 Approximate Arc Length

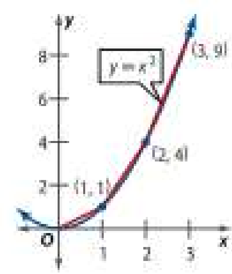
Approximate the arc length of the graph of $y = x^2$ for $0 \leq x \leq 3$.

Step 1 Graph $y = x^2$ for $0 \leq x \leq 3$ as shown.

Step 2 Graph points on the curve at $x = 1$, 2, and 3. Connect the points using line segments as shown.

Step 3 Use the Distance Formula to find the length of each line segment.

Step 4 Approximate the length of the arc by finding the sum of the lengths of the line segments.

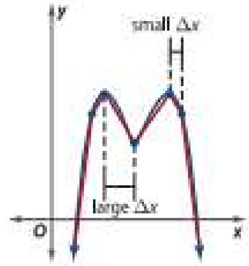


Analyze the Results

1. Is your approximation *greater* or *less* than the actual length? Explain your reasoning.
2. Approximate the arc length a second time using 6 line segments that are formed by the points $x = 0$, 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0. Include a sketch of the graph with your approximation.
3. Describe what happens to the approximation for the arc length as shorter line segments are used.
4. For the two approximations, the endpoints of the line segments were equally spaced along the x-axis. Do you think this will always produce the most accurate approximation? Explain your reasoning.

Notice that for the first activity, the endpoints of the line segments were equally spaced 0.5 units apart along the x-axis. When using advanced methods of calculus to find *exact* arc length, a constant difference between a pair of endpoints along the x-axis is essential. This difference is denoted Δx.

Accurately approximating arc length by using a constant Δx to create the line segments may not always be the most efficient method. The shape of the arc will dictate the spacing of the endpoints, thus creating different values for Δx. For example, if a graph shows an increase or decrease over a large interval for x, a large line segment may be used for the approximation. If a graph includes a turning point, it is better to use small line segments to account for the curve in the graph.



Previously, you learned how to calculate the distance between polar coordinates. This formula can be used to approximate the arc length of a curve represented by a polar equation.

Activity 2 Approximate Arc Length

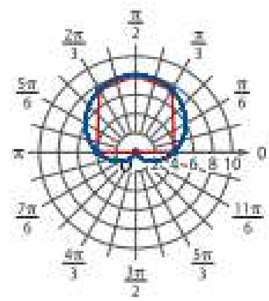
Approximate the arc length of the graph of $r = 4 + 4 \sin \theta$ for $0 \leq x \leq 2\pi$.

Step 1 Graph $r = 4 + 4 \sin \theta$ for $0 \leq x \leq 2\pi$ as shown.

Step 2 Draw 6 points on the curve at $\theta = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$, and $\frac{3\pi}{2}$. Connect the points using line segments as shown.

Step 3 Use the Polar Distance Formula to find the length of each line segment.

Step 4 Approximate the length of the arc by finding the sum of the lengths of the line segments.



Study Tip

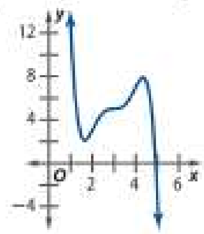
Polar Graphs Create a table of values for r and θ when calculating the arc length for a polar graph. This will help to reduce errors created by functions that produce negative values for r.

Analyze the Results

5. Explain how symmetry can be used to reduce the number of calculations in Step 3.
6. Approximate the arc length using at least 10 segments. Include a sketch of the graph.
7. Let n be the number of line segments used in an approximation and $\Delta\theta$ be a constant difference in θ between the endpoints of a line segment. Make a conjecture regarding the relationship between n, θ, and the approximation for an arc length.

Model and Apply

Approximate the arc length for each graph. Include a sketch of your graph.

8. 

$y = -(x-3)^5 + 3(x-3)^3 + 5$ for $1 \leq x \leq 5$

9. 

$r = \theta$ for $0 \leq \theta \leq 2\pi$

CHAPTER 3

Sequences and Series

Then

In **previous chapters**, you modeled data using various types of functions.

Now

In **Chapter 3**, you will:

- Relate sequences and functions.
- Represent and calculate sums of series with sigma notation.
- Use arithmetic and geometric sequences and series.
- Prove statements by using mathematical induction.
- Expand powers by using the Binomial Theorem.

Why? ▲

MARCHING BAND Sequences and series can be used to predict patterns. For example, arithmetic sequences can be used to determine the number of band members in a specified row of a pyramid formation.

PREREAD Use the text on this page to predict the organization of Chapter 3.

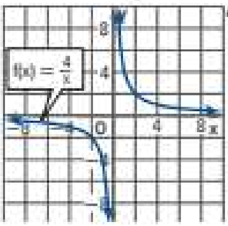
Get Ready for the Chapter

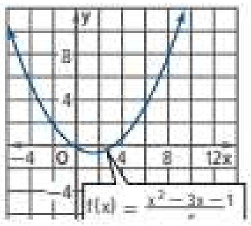
QuickCheck

Expand each binomial.

1. $(x + 3)^3$
2. $(2x - 1)^4$

Use the graph of each function to describe its end behavior. Support the conjecture numerically.

3. 

4. 

Sketch and analyze the graph of each function. Describe its domain, range, intercepts, asymptotes, end behavior, and where the function is increasing or decreasing.

5. $f(x) = 3^{-x}$
6. $r(x) = 5^{-x}$
7. $h(x) = 0.1^{x+2}$
8. $k(x) = -2^x$

Evaluate each expression.

9. $\log_2 16$
10. $\log_{10} 10$
11. $\log_6 \frac{1}{216}$

12. **MUSIC** The table shows the type and number of CDs that Adnan and Lindsay bought. Write and solve a system of equations to determine the price of each type of CD.

Buyer	New CD	Used CD	Price ($)
Adnan	2	5	49
Lindsay	3	4	56

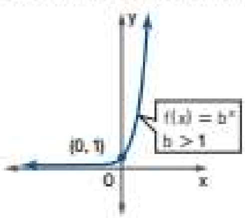
NewVocabulary

sequence
term
finite sequence
infinite sequence
recursive sequence
explicit sequence
Fibonacci sequence
converge
diverge
series
finite series
nth partial sum
infinite series
sigma notation
arithmetic sequence
common difference
arithmetic series
common ratio
geometric means
geometric series
binomial coefficients
power series

ReviewVocabulary

exponential function a function in which the base is a constant and the exponent is a variable

Exponential Growth Function



LESSON 3-1 Sequences, Series, and Sigma Notation

Then

- You used functions to generate ordered pairs and used graphs to analyze end behavior.

Now

1. Investigate several different types of sequences.
2. Use sigma notation to represent and calculate sums of series.

Why?

- Wafa developed a website where students at her high school can post their own social networking web pages. A student at the high school is given a free page if he or she refers the website to five friends. The site starts with one page created by Wafa, who in turn, refers five friends that each create a page. Those five friends refer five more people each, all of whom develop pages, and so on.

NewVocabulary

sequence
term
finite sequence
infinite sequence
recursive sequence
explicit sequence
Fibonacci sequence
converge
diverge
series
finite series
sigma notation
infinite series
*n*th partial sum

1 Sequences

In mathematics, a **sequence** is an ordered list of numbers. Each number in the sequence is known as a **term**. A **finite sequence**, such as 1, 3, 5, 7, 9, 11, contains a finite number of terms. An **infinite sequence**, such as 1, 3, 5, 7, ..., contains an infinite number of terms.

Each term of a sequence is a function of its position. Therefore, an infinite sequence is a function whose domain is the set of natural numbers and can be written as $f(1) = a_1$, $f(2) = a_2$, $f(3) = a_3$, ..., $f(n) = a_n$, ..., where a_n denotes the *n*th term. If the domain of the function is only the first *n* natural numbers, the sequence is finite.

Infinitely many sequences exist with the same first few terms. To sufficiently define a *unique* sequence, a formula for the *n*th term or other information *must* be given. When defined *explicitly*, an **explicit formula** gives the *n*th term a_n as a function of *n*.

Example 1 Find Terms of Sequences

a. Find the next four terms of the sequence 2, 7, 12, 17, … .

The *n*th term of this sequence is not given. One possible pattern is that each term is 5 greater than the previous term. Therefore, a sample answer for the next four terms is 22, 27, 32, and 37.

b. Find the next four terms of the sequence 2, 5, 10, 17, … .

The *n*th term of this sequence is not given. If we subtract each term from the term that follows, we start to see a possible pattern.

$a_2 - a_1 = 5 - 2$ or 3 $\qquad a_3 - a_2 = 10 - 5$ or 5 $\qquad a_4 - a_3 = 17 - 10$ or 7

It appears that each term is generated by adding the next successive odd number. However, looking at the pattern, it may also be determined that each term is 1 more than each perfect square, or $a_n = n^2 + 1$. Using either pattern, a sample answer for the next four terms is 26, 37, 50, and 65.

c. Find the first four terms of the sequence given by $a_n = 2n(-1)^n$.

Use the explicit formula given to find a_n for $n = 1, 2, 3$, and 4.

$a_1 = 2 \cdot 1 \cdot (-1)^1$ or -2 $\quad n = 1$ $\qquad a_2 = 2 \cdot 2 \cdot (-1)^2$ or 4 $\quad n = 2$

$a_3 = 2 \cdot 3 \cdot (-1)^3$ or -6 $\quad n = 3$ $\qquad a_4 = 2 \cdot 4 \cdot (-1)^4$ or 8 $\quad n = 4$

The first four terms in the sequence are −2, 4, −6, and 8.

GuidedPractice

Find the next four terms of each sequence. 1A–1B. Sample answers are given.

1A. 32, 16, 8, 4, … $2, 1, \frac{1}{2}, \frac{1}{4}$

1B. 1, 2, 4, 7, 11, 16, 22, … 29, 37, 46, 56

1C. Find the first four terms of the sequence given by $a_n = n^3 - 10$. −9, −2, 17, 54

Sequences can also be defined *recursively*. Recursively defined sequences give one or more of the first few terms and then define the terms that follow using those previous terms. The formula defining the nth term of the sequence is called a **recursive formula** or a *recurrence relation*.

StudyTip

Notation The term denoted a_n represents the nth term of a sequence. The term denoted a_{n-1} represents the term immediately before a_n. The term a_{n-2} represents the term two terms before a_n.

Example 2 Recursively Defined Sequences

Find the fifth term of the recursively defined sequence $a_1 = 1$, $a_n = a_{n-1} + 2n - 1$, where $n \geq 2$.

Since the sequence is defined recursively, all the terms before the fifth term must be found first. Use the given first term, $a_1 = 2$, and the recursive formula for a_n.

$a_2 = a_{2-1} + 2(2) - 1$ $\quad n = 2$

$= a_1 + 3$ $\quad$ Simplify.

$= 2 + 3$ or 5 $\quad a_1 = 2$

$a_3 = a_{3-1} + 2(3) - 1$ $\quad n = 3$

$= a_2 + 5$ or 10 $\quad a_2 = 5$

$a_4 = a_{4-1} + 2(4) - 1$ $\quad n = 4$

$= a_3 + 7$ or 17 $\quad a_3 = 10$

$a_5 = a_{5-1} + 2(5) - 1$ $\quad n = 5$

$= a_4 + 9$ or 26 $\quad a_4 = 17$

GuidedPractice

Find the sixth term of each sequence.

2A. $a_1 = 3$, $a_n = (-2)a_{n-1}$, $n \geq 2$ **−96**

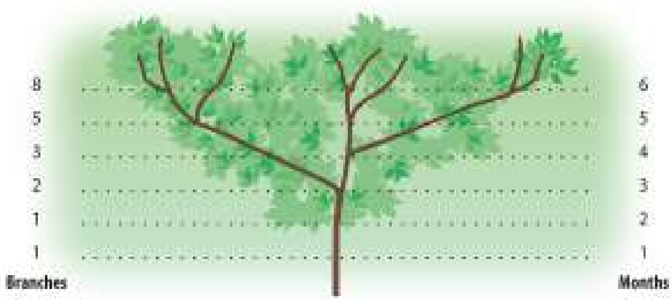
2B. $a_1 = 8$, $a_n = 2a_{n-1} - 7$, $n \geq 2$ **39**

The **Fibonacci sequence** describes many patterns found in nature. This sequence is often defined recursively.

Real-World Example 3 Fibonacci Sequence

NATURE Suppose that when a plant first starts to grow, the stem has to grow for two months before it is strong enough to support branches. At the end of the second month, it sprouts a new branch and will continue to sprout one new branch each month. The new branches also each grow for two months and then start to sprout one new branch each month. If this pattern continues, how many branches will the plant have after 10 months?

During the first two months, there will only be one branch, the stem. At the end of the second month, the stem will produce a new branch, making the total for the third month two branches. The new branch will grow and develop two months before producing a new branch of its own, but the original branch will now produce a new branch each month.



Real-WorldLink

Along with being found in flower petals, sea shells, and the bones in a human hand, Fibonacci sequences can also be found in pieces of art, music, poetry, and architecture.

Source: *Universal Principles of Design*

WatchOut!

Notation The first term of a sequence is occasionally denoted as a_0. When this occurs, the domain of the function describing the sequence is the set of whole numbers.

The following table shows the pattern.

Month	1	2	3	4	5	6	7	8	9	10
Branches	1	1	2	3	5	8	13	21	34	55

Each term is the sum of the previous two terms. This pattern can be written as the recursive formula $a_0 = 1$, $a_1 = 1$, $a_n = a_{n-2} + a_{n-1}$, where $n \geq 2$.

GuidedPractice

3. **NATURE** How many branches will a plant like the one described in Example 3 have after 15 months if no branches are removed? **610**

Previously, you examined the end behavior of the graphs of functions. You learned that as the domains of some functions approach ∞, the ranges approach a unique number called a limit. As a function, an infinite sequence may also have a limit. If a sequence has a limit such that the terms approach a unique number, then it is said to **converge**. If not, the sequence is said to **diverge**.

TechnologyTip

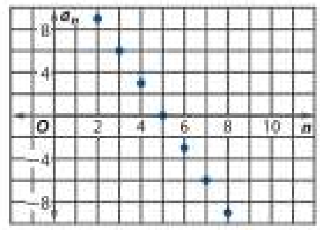
Convergent or Divergent Sequences If an explicit formula for a sequence is known, you can enter the formula in the Y= menu of a graphing calculator and graph the related function. Analyzing the end behavior of the graph can help you to determine whether the sequence is convergent or divergent.

Example 4 Convergent and Divergent Sequences

Determine whether each sequence is *convergent* or *divergent*.

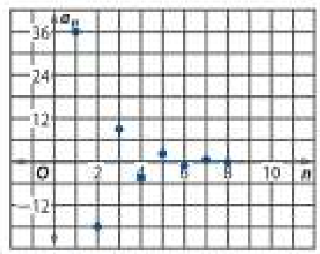
a. $a_n = -3n + 12$

The first eight terms of this sequence are 12, 9, 6, 3, 0, −3, −6, and −9. From the graph at the right, you can see that a_n does not approach a finite number. Therefore, this sequence is divergent.



b. $a_1 = 36$, $a_n = -\frac{1}{2}a_{n-1}$, $n \geq 2$

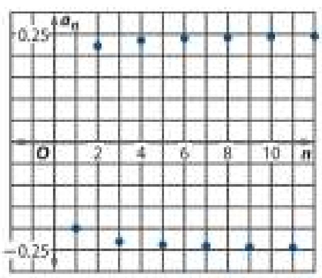
The first eight terms of this sequence are 36, −18, 9, −4.5, 2.25, −1.125, 0.5625, −0.28125, and 0.140625. From the graph at the right, you can see that a_n approaches 0 as n increases. This sequence has a limit and is therefore convergent.



c. $a_n = \dfrac{(-1)^n \cdot n}{4n + 1}$

The first twelve terms of this sequence are given or approximated below.

$a_1 = -0.2$	$a_2 \approx 0.222$
$a_3 \approx -0.231$	$a_4 \approx 0.235$
$a_5 \approx -0.238$	$a_6 = 0.24$
$a_7 \approx -0.241$	$a_8 \approx 0.242$
$a_9 \approx -0.243$	$a_{10} \approx 0.244$
$a_{11} \approx -0.244$	$a_{12} \approx 0.245$



It appears that when n is odd, a_n approaches $-\frac{1}{4}$, and when n is even, a_n approaches $\frac{1}{4}$. Since a_n does not approach one particular value, the sequence has no limit. Therefore, the sequence is divergent.

GuidedPractice

4A. $a_n = \frac{64}{2n}$ **convergent**

4B. $a_1 = 9$, $a_n = a_{n-1} + 4$ **divergent**

4C. $a_n = 3(-1)^n$ **divergent**

2 Series

A **series** is the indicated sum of all of the terms of a sequence. Like sequences, series can be finite or infinite. A **finite series** is the indicated sum of all the terms of a finite sequence, and an **infinite series** is the indicated sum of all the terms of an infinite sequence.

	Sequence	Series
Finite	1, 3, 5, 7, 9	1 + 3 + 5 + 7 + 9
Infinite	1, 3, 5, 7, 9, ...	1 + 3 + 5 + 7 + 9 + ...

The sum of the first n terms of a series is called the **nth partial sum** and is denoted S_n. The nth partial sum of any series can be found by calculating each term up to the nth term and then finding the sum of those terms.

Example 5 The nth Partial Sum

a. Find the fourth partial sum of $a_n = (-2)^n + 3$.

Find the first four terms.

$a_1 = (-2)^1 + 3$ or 1 $\quad n = 1$

$a_2 = (-2)^2 + 3$ or 7 $\quad n = 2$

$a_3 = (-2)^3 + 3$ or -5 $\quad n = 3$

$a_4 = (-2)^4 + 3$ or 19 $\quad n = 4$

The fourth partial sum is $S_4 = 1 + 7 + (-5) + 19$ or 22.

b. Find S_3 of $a_n = \frac{4}{10^n}$.

Find the first three terms.

$a_1 = \frac{4}{10^1}$ or 0.4 $\quad n = 1$

$a_2 = \frac{4}{10^2}$ or 0.04 $\quad n = 2$

$a_3 = \frac{4}{10^3}$ or 0.004 $\quad n = 3$

The third partial sum is $S_3 = 0.4 + 0.04 + 0.004$ or 0.444.

GuidedPractice

5A. Find the sixth partial sum of $a_1 = 8$, $a_n = 0.5(a_{n-1})$, $n \geq 2$. **15.75**

5B. Find the seventh partial sum of $a_n = 3\left(\frac{1}{10}\right)^n$. **0.3333333**

> **StudyTip**
> **Converging Infinite Sequences** While it is necessary for an infinite sequence to converge to 0 in order for the corresponding infinite series to have a sum, it is not sufficient. Some infinite sequences converge to 0 and the corresponding infinite series still do not have sums.

Since an infinite series does not have a finite number of terms, you might assume that an infinite series has no sum S. This is true for the series below.

Infinite Sequence	Infinite Series	Sequence of First Four Partial Sums
1, 4, 7, 10, ...	1 + 4 + 7 + 10 + ...	1, 5, 12, 22, ...

However, some infinite series do have sums. For an infinite series to have a fixed sum S, the infinite sequence associated with this series must converge to 0. Notice the sequence of partial sums in the infinite series below appears to approach a sum of $0.\overline{1}$ or $\frac{1}{9}$.

Infinite Sequence	Infinite Series	Sequence of First Three Partial Sums
0.1, 0.01, 0.001, ...	0.1 + 0.01 + 0.001 + ...	0.1, 0.11, 0.111, ...

We will take a closer look at sums of infinite sequences in Lesson 3-2.

Series are often more conveniently notated using the uppercase Greek letter sigma Σ. A series written using this letter is said to be expressed using *summation notation* or **sigma notation**.

ReadingMath

Sigma Notation $\sum_{n=1}^{k} a_n$ is read *the summation from $n = 1$ to k of a sub n.*

KeyConcept Sigma Notation

For any sequence $a_1, a_2, a_3, a_4, \ldots$, the sum of the first k terms is denoted

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \ldots + a_k,$$

where n is the index of summation, k is the upper bound of summation, and 1 is the lower bound of summation.

In this notation, the lower bound indicates where to begin summing the terms of the sequence and the upper bound indicates where to end the sum. If the upper bound is given as ∞, the sigma notation represents an infinite series.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$$

WatchOut!

Variations in Sigma Notation The index of summation does not have to be the letter n. It can be represented by any variable. For example, the summation in Example 6a could also be written as $\sum_{i=1}^{5} (4i - 3)$.

Example 6 Sums in Sigma Notation

Find each sum.

a. $\sum_{n=1}^{5} (4n - 3)$

$$\sum_{n=1}^{5} (4n - 3) = [4(1) - 3] + [4(2) - 3] + [4(3) - 3] + [4(4) - 3] + [4(5) - 3]$$
$$= 1 + 5 + 9 + 13 + 17 \text{ or } 45$$

b. $\sum_{n=3}^{7} \frac{6n - 3}{2}$

$$\sum_{n=3}^{7} \frac{6n - 3}{2} = \frac{6(3) - 3}{2} + \frac{6(4) - 3}{2} + \frac{6(5) - 3}{2} + \frac{6(6) - 3}{2} + \frac{6(7) - 3}{2}$$
$$= 7.5 + 10.5 + 13.5 + 16.5 + 19.5 \text{ or } 67.5$$

c. $\sum_{n=1}^{\infty} \frac{7}{10^n}$

$$\sum_{n=1}^{\infty} \frac{7}{10^n} = \frac{7}{10^1} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \frac{7}{10^5} + \ldots$$
$$= 0.7 + 0.07 + 0.007 + 0.0007 + 0.00007 + \ldots$$
$$= 0.77777\ldots \text{ or } \frac{7}{9}$$

GuidedPractice

6A. $\sum_{n=1}^{5} \frac{n^2 - 1}{2}$ **25**

6B. $\sum_{n=7}^{13} (n^3 - n^2)$ **7112**

6C. $\sum_{n=1}^{\infty} \frac{6}{10^n}$ $\frac{2}{3}$

Note that while the lower bound of a summation is often 1, a sum can start with any term p in a sequence as long as $p < k$. In Example 6b, the summation started with the 3rd term of the sequence and ended with the 7th term.

Exercises

1–6. Sample answers given.

Find the next four terms of each sequence. (Example 1)

1. 1, 8, 15, 22, … 29, 36, 43, 50
2. 3, −6, 12, −24, … 48, −96, 192, −384
3. 81, 27, 9, 3, … $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$
4. 1, 3, 7, 13, … 21, 31, 43, 57
5. −2, −15, −28, −41… −54, −67, −80, −93
6. 1, 4, 10, 19, … 31, 46, 64, 85

Find the first four terms of each sequence. (Example 1)

7. $a_n = n^2 - 1$ 0, 3, 8, 15
8. $a_n = -2^n + 7$ 5, 3, −1, −9
9. $a_n = \frac{n + 7}{9 - n}$ $1, \frac{9}{7}, \frac{5}{3}, \frac{11}{5}$
10. $a_n = (-1)^{n+1} + n$ 2, 1, 4, 3

11. **AUTOMOBILE LEASES** Lease agreements often contain clauses that limit the number of kilometers driven per year by charging a per-kilometer fee over that limit. For the car shown below, the lease requires that the number of kilometers driven each year must be no more than 15,000. (Example 2)

a. Write the sequence describing the maximum number of allowed kilometers on the car at the end of every 12 months of the lease if the car has 1,350 kilometers at the beginning of the lease. 16,350; 31,350; 46,350

b. Write the first 4 terms of the sequence that gives the cumulative cost of the lease for a given month. 2098, 2497, 2896, 3295

c. Write an explicit formula to represent the sequence in part b. $a_n = 1699 + 399n$

d. Determine the total amount of money paid by the end of the lease. $16,063

Find the specified term of each sequence. (Example 2)

12. 4th term, $a_1 = 5$, $a_n = -3a_{n-1} + 10$, $n \geq 2$ −65
13. 7th term, $a_1 = 14$, $a_n = 0.5a_{n-1} + 3$, $n \geq 2$ 6.125
14. 4th term, $a_1 = 0$, $a_n = 3^{a_{n-1}}$, $n \geq 2$ 27
15. 3rd term, $a_1 = 3$, $a_n = (a_{n-1})^2 - 5a_{n-1} + 4$, $n \geq 2$ 18

16. **WEB SITE** Wafa, the student from the beginning of the lesson, had great success expanding her website. Each student who received a referral developed a web page and referred five more students to Wafa's site. (Example 3)

a. List the first five terms of a sequence modeling the number of new web pages created through Wafa's site. 1, 5, 25, 125, 625

b. Suppose the school has 1576 students. After how many rounds of referrals did the entire student body have a web page? 5

17. **BEES** Female honeybees come from fertilized eggs (male and female parent), while male honeybees come from unfertilized eggs (one female parent). (Example 3)

a. Draw a family tree showing the 3 previous generations of a male honeybee (parents only). See margin.

b. Determine the number of parent bees in the 11th previous generation of a male honeybee. 144

Determine whether each sequence is *convergent* or *divergent*. (Example 4) 18–27. See margin.

18. $a_1 = 4$, $1.5a_{n-1}$, $n \geq 2$
19. $a_n = \frac{5}{10^n}$
20. $a_n = -n^2 - 8n + 106$
21. $a_1 = -64$, $\frac{3}{4}a_{n-1}$, $n \geq 2$
22. $a_1 = 1$, $a_n = 4 - a_{n-1}$, $n \geq 2$
23. $a_n = n^2 - 3n + 1$
24. $a_n = \frac{n^2 + 4}{3 + n}$
25. $a_1 = 9$, $a_n = \frac{a_{n-1} + 3}{2}$, $n \geq 2$
26. $a_n = \frac{5n + 6}{n}$
27. $a_n = \frac{5n}{5^n} + 1$

Find the indicated sum for each sequence. (Example 5)

28. 5th partial sum of $a_n = n(n - 4)(n - 3)$ 20
29. 6th partial sum of $a_n = \frac{-5n + 3}{n}$ −22.65
30. S_8 of $a_1 = 1$, $a_n = a_{n-1} + (18 - n)$, $n \geq 2$ 428
31. S_4 of $a_1 = 64$, $a_n = -\frac{3}{4}a_{n-1}$, $n \geq 2$ 25
32. 11th partial sum of $a_1 = 4$, $a_n = (-1)^{n-1}(|a_{n-1}| + 3)$, $n \geq 2$ 19
33. S_9 of $a_1 = -35$, $a_n = a_{n-1} + 8$, $n \geq 2$ −27
34. 4th partial sum of $a_1 = 3$, $a_n = (a_{n-1} - 2)^3$, $n \geq 2$ −24
35. S_4 of $a_n = \frac{(-3)^n}{10}$ 6

Find each sum. (Example 6)

36. $\sum_{n=1}^{8} (6n - 11)$ 128
37. $\sum_{n=4}^{11} (30 - 4n)$ 0
38. $\sum_{n=1}^{7} [n^2(n - 5)]$ 84
39. $\sum_{n=2}^{7} (n^2 - 6n + 1)$ −17
40. $\sum_{n=8}^{15} \left(\frac{n}{4} - 7\right)$ −33
41. $\sum_{n=1}^{10} [(n - 4)^2(n - 5)]$ 300
42. $\sum_{n=0}^{6} [(-2)^n - 9]$ −20
43. $\sum_{n=1}^{3} 7\left(\frac{1}{10}\right)^{2n}$ $\frac{70,707}{1,000,000}$
44. $\sum_{n=1}^{\infty} 5\left(\frac{1}{10^n}\right)$ $\frac{5}{9}$
45. $\sum_{n=1}^{\infty} \frac{8}{10^n}$ $\frac{8}{9}$

46. **FINANCIAL LITERACY** Mazen's bank account had an initial deposit of $380, earning 3.5% interest per year compounded annually.

B

a. Find the balance each year for the first five years. $393.30, $407.07, $421.31, $436.06, $451.32

b. Write a recursive and an explicit formula defining his account balance. b–c. See margin.

c. For very large values of n, which formula gives a more accurate balance? Explain.

47. INVESTING Hiyam invests \$200 every 3 months. The investment pays an annual percentage rate of 8%, and the interest is compounded quarterly. If Hiyam makes each payment at the beginning of the quarter and the interest is posted at the end of the quarter, what will the total value of the investment be after 2 years? **\$1750.93**

48. RIDES The table shows the number of riders of the Mean Streak each year from 1998 to 2007. This ridership data can be approximated by $a_n = -\frac{1}{20}n + 1.3$, where $n = 1$ represents 1998, $n = 2$ represents 1999, and so on.

Mean Streak Roller Coaster

Year	Number of Riders (millions)	Year	Number of Riders (millions)
1998	1.31	2003	0.99
1999	1.15	2004	0.95
2000	1.14	2005	0.89
2001	1.09	2006	0.81
2002	1.05	2007	0.82

Source: *Cedar Fair Entertainment Company*

a–b. See margin.

a. Sketch a graph of the number of riders from 1998 to 2007. Then determine whether the sequence appears to be *convergent* or *divergent*. Does this make sense in the context of the situation? Explain your reasoning.

b. Use the table to find the total number of riders from 1998 to 2005. Then use the explicit sequence to find the 8th partial sum of a_n. Compare the results.

c. If the sequence continues, find a_{14}. What does this number represent? **0.6 million; the number of riders in 2011**

Copy and complete the table. **49–56. See margin.**

	Recursive Formula	Sequence	Explicit Formula
49.		6, 8, 10, 12, ...	
50.	$a_1 = 15, a_n = a_{n-1} - 1, n \geq 2$		
51.		7, 21, 63, 189, ...	
52.			$a_n = 10(-2)^n$
53.			$a_n = 8n - 3$
54.	$a_1 = 2, a_n = 4a_{n-1}, n \geq 2$		
55.	$a_1 = 3, a_n = a_{n-1} + 2n - 1, n \geq 2$		
56.			$a_n = n^2 + 1$

Write each series in sigma notation. The lower bound is given.

57. $-2 - 1 + 0 + 1 + 2 + 3 + 4 + 5$; $n = 1$ **$\sum_{n=1}^{8}(n-3)$**

58. $\frac{1}{20} + \frac{1}{25} + \frac{1}{30} + \frac{1}{35} + \frac{1}{40} + \frac{1}{45}$; $n = 4$ **$\sum_{n=4}^{9}\frac{1}{5n}$**

59. $8 + 27 + 64 + \cdots + 1000$; $n = 2$ **$\sum_{n=2}^{10}n^3$**

60. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{128}$; $n = 1$ **$\sum_{n=1}^{7}\frac{1}{2^n}$**

61. $-8 + 16 - 32 + 64 - 128 + 256 - 512$; $n = 3$ **$\sum_{n=3}^{9}(-2)^n$**

62. $8\left(-\frac{1}{3}\right) + 8\left(\frac{1}{9}\right) + 8\left(-\frac{1}{27}\right) + \cdots + 8\left(-\frac{1}{243}\right)$; $n = 1$ **$\sum_{n=1}^{5}8\left(-\frac{1}{3}\right)^n$**

Determine whether each sequence is *convergent* or *divergent*. Then find the fifth partial sum of the sequence.

63. $a_n = \sin\frac{n\pi}{2}$ **divergent; 1**

64. $a_n = n\cos\pi$ **divergent; −15**

65. $a_n = e^{-\frac{n}{2}}\cos\pi n$ **convergent; ≈−0.4**

66. WATER PRESSURE The pressure exerted on the human body at sea level is 101,353 newton per square meter (N/m²). For each additional meter below sea level, the pressure is about 9,205 N/m² greater, as shown.

a–b. See margin.

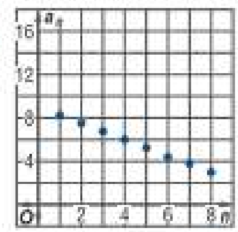
a. Write a recursive formula to represent a_n, the pressure at n meters below sea level. (*Hint*: Let $a_0 = 14.7$.)

b. Write the first three terms of the sequence and describe what they represent.

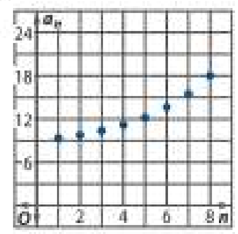
c. Scuba divers cannot safely dive deeper than 100 meters. Write an explicit formula to represent a_n. Then use the formula to find the water pressure at 100 meters below sea level. **$a_n = 9{,}205n + 101{,}353$; 408,170 N/m²**

Match each sequence with its graph.

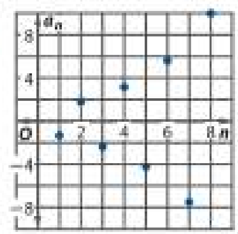
67. **b**



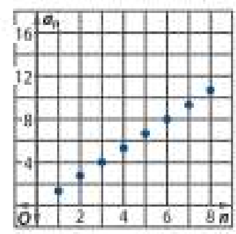
68. **f**



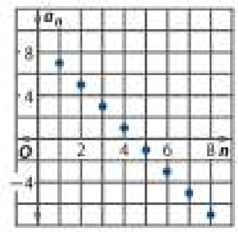
69. **c**



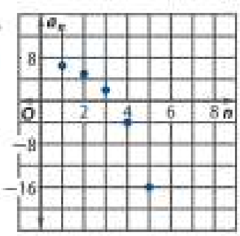
70. **a**



71. **e**



72. **d**



a. $a_n = \frac{4}{3}n$

b. $a_n = -\frac{3}{4}n + 9$

c. $a_n = \left(-\frac{4}{3}\right)^n$

d. $a_n = 8 - \frac{3}{4}(2^n)$

e. $a_n = 9 - 2n$

f. $a_n = \left(\frac{4}{3}\right)^n + 8$

73. **GOLDEN RATIO** Consider the Fibonacci sequence 1, 1, 2, 3, …, $a_{n-2} + a_{n-1}$. **a–d. See margin.**

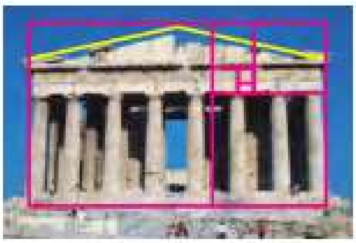
a. Find $\frac{a_n}{a_{n-1}}$ for the second through eleventh terms of the Fibonacci sequence.

b. Sketch a graph of the terms found in part **a.** Let $n - 1$ be the x-coordinate and $\frac{a_n}{a_{n-1}}$ be the y-coordinate.

c. Based on the graph found in part **b,** does this sequence appear to be convergent? If so, describe the limit to three decimal places. If not, explain why not.

d. In a *golden rectangle*, the ratio of the length to the width is about 1.61803399. This is called the *golden ratio*. How does the limit of the sequence $\frac{a_n}{a_{n-1}}$ compare to the golden ratio?

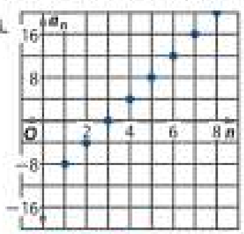
e. Golden rectangles are common in art and architecture. The Parthenon, in Greece, is an example of how golden rectangles are used in architecture.



Research golden rectangles and find two more examples of golden rectangles in art or architecture. **Sample answer: the Mona Lisa and the Taj Mahal**

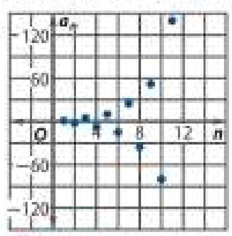
Determine whether each sequence is *convergent* or *divergent*.

74.



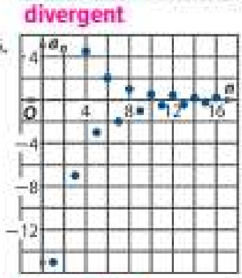
divergent

75.



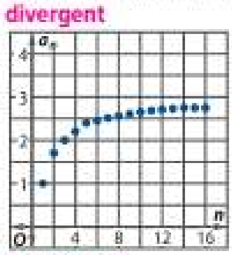
divergent

76.



convergent

77.



convergent

Write an explicit formula for each recursively defined sequence.

78. $a_1 = 10;\ a_n = a_{n-1} + 5$ $\quad a_n = 5n + 5$

79. $a_1 = 1.25;\ a_n = a_{n-1} - 0.5$ $\quad a_n = -0.5n + 1.75$

80. $a_1 = 128;\ a_n = 0.5a_{n-1}$ $\quad a_n = 128(0.5)^{n-1}$

81. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate sums of infinite series.

a. **NUMERICAL** Calculate the first five terms of the infinite sequence $a_n = \frac{4}{10^n}$. **0.4, 0.04, 0.004, 0.0004, 0.00004**

b. **GRAPHICAL** Use a graphing calculator to sketch $y = \frac{4}{10^x}$. **See Answer Appendix.**

c. **VERBAL** Describe what is happening to the terms of the sequence as $n \to \infty$. **As $n \to \infty$, the terms $\to 0$.**

d. **NUMERICAL** Find the sum of the first 5, 7, and 9 terms of the series. **0.44444; 0.4444444; 0.444444444**

e. **VERBAL** Describe what is happening to the partial sums S_n as n increases. **e–f. See Answer Appendix.**

f. **VERBAL** Predict the sum of the first n terms of the series. Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

82. **CHALLENGE** Consider the recursive sequence below.

$a_n = a_{n - a_{n-1}} + a_{n - a_{n-2}}$ for $a_1 = 1,\ a_2 = 1,\ n \geq 3$

a. Find the first eight terms of the sequence. **a–b. See margin.**

b. Describe the similarities and differences between this sequence and the other recursive sequences in this lesson.

83. **OPEN ENDED** Write a sequence either recursively or explicitly that has the following characteristics. **See Answer Appendix.**

a. converges to 0

b. converges to 3

c. diverges

84. **WRITING IN MATH** Describe why an infinite sequence must not only converge, but converge to 0, in order for there to be a sum. **See margin.**

REASONING Determine whether each statement is *true* or *false*. Explain your reasoning. 85–86. See Answer Appendix.

85. $\sum_{n=1}^{5}(n^2 + 3n) = \sum_{n=1}^{5} n^2 + 3\sum_{n=1}^{5} n$

86. $\sum_{n=1}^{5} 3^n = \sum_{n=3}^{7} 3^{n-2}$

87. **CHALLENGE** Find the sum of the first 60 terms of the sequence below. Explain how you determined your answer. **See margin.**

15, 17, 2, −15, −17, …,
where $a_n = a_{n-1} - a_{n-2}$ for $n \geq 3$

88. **WRITING IN MATH** Make an outline that could be used to describe the steps involved in finding the 300th partial sum of the infinite sequence $a_n = 2n - 3$. Then explain how to express the sum using sigma notation. **See students' work.**

Spiral Review

Graph each complex number on a polar grid. Then express it in rectangular form. 89–91. See margin.

89. $2\left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right)$ **90.** $2.5(\cos 1 + i \sin 1)$ **91.** $5(\cos 0 + i \sin 0)$

Determine the eccentricity, type of conic, and equation of the directrix given by each polar equation.

92. $r = \frac{3}{2 - 0.5 \cos \theta}$ $e = 0.25$; ellipse; $x = -6$

93. $r = \frac{6}{1.2 \sin \theta + 0.3}$ $e = 4$; hyperbola; $y = 5$

94. $r = \frac{1}{0.2 - 0.2 \sin \theta}$ $e = 1$; parabola; $y = -5$

Determine whether the points are collinear. Write *yes* **or** *no*.

95. $(-3, -1, 4), (3, 8, 1), (5, 12, 0)$ no

96. $(4, 8, 6), (0, 6, 12), (8, 10, 0)$ yes

97. $(0, -4, 3), (8, -10, 5), (12, -13, 2)$ no

98. $(-7, 2, -1), (-9, 3, -4), (-5, 1, 2)$ yes

Find the length and the midpoint of the segment with the given endpoints.

99. $(2, -15, 12), (1, -11, 15)$ $\sqrt{26} \approx 5.10$; $(1.5, -13, 13.5)$

100. $(-4, 2, 8), (9, 6, 0)$ $\sqrt{249} \approx 15.78$; $(2.5, 4, 4)$

101. $(7, 1, 5), (-2, -5, -11)$ $\sqrt{373} \approx 19.31$; $(2.5, -2, -3)$

102. TIMING The path traced by the tip of the hour-hand of a clock can be modeled by a circle with parametric equations $x = 6 \sin t$ and $y = 6 \cos t$.

a. Find an interval for t in radians that can be used to describe the motion of the tip as it moves from 12 o'clock noon to 12 o'clock noon the next day. $0 \leq t \leq 4\pi$

b. Simulate the motion described in part **a** by graphing the equation in parametric mode on a graphing calculator. See margin.

c. Write an equation in rectangular form that models the motion of the hour-hand. Find the radius of the circle traced out by the hour-hand if x and y are given in inches. $x^2 + y^2 = 36$; 6 in.

Find the exact value of each expression.

103. $\tan \frac{\pi}{12}$ $2 - \sqrt{3}$ **104.** $\sin 75°$ $\frac{\sqrt{6} + \sqrt{2}}{4}$ **105.** $\cos 165°$ $-\frac{\sqrt{2} + \sqrt{6}}{4}$

Find the partial fraction decomposition of each rational expression.

106. $\frac{10x^2 - 11x + 4}{2x^2 - 3x + 1}$ $5 - \frac{2}{2x - 1} + \frac{3}{x - 1}$

107. $\frac{1}{2x^2 + x}$ $\frac{1}{x} - \frac{2}{2x + 1}$

108. $\frac{x + 1}{x^3 + x}$ $\frac{1}{x} - \frac{x - 1}{x^2 + 1}$

Skills Review for Standardized Tests

109. SAT/ACT The first term in a sequence is -5, and each subsequent term is 6 more than the term that immediately precedes it. What is the value of the 104th term? B

A 607
B 613
C 618
D 619
E 615

110. REVIEW Find the exact value of $\cos 2\theta$ if $\sin \theta = -\frac{\sqrt{5}}{3}$ and $180° < \theta < 270°$. J

F $-\frac{\sqrt{6}}{6}$ H $-\frac{\sqrt{30}}{6}$

G $-\frac{4\sqrt{5}}{9}$ J $-\frac{1}{9}$

111. The first four terms of a sequence are 144, 72, 36, and 18. What is the tenth term in the sequence? C

A 0 C $\frac{9}{32}$

B $\frac{9}{64}$ D $\frac{9}{16}$

112. REVIEW How many 5-centimeter cubes can be stacked inside a box that is 10 centimeters long, 15 centimeters wide, and 5 centimeters tall? G

F 5
G 6
H 15
J 20

LESSON 3-2

Arithmetic Sequences and Series

::Then	::Now	::Why?
You found terms of sequences and sums of series.	1 Find nth terms and arithmetic means of arithmetic sequences. 2 Find sums of n terms of arithmetic series.	With cross country season approaching, Meg decides to train every day until the first day of practice. She plans to run 1 mile the first day, 1.25 miles the second day, 1.5 miles the third day, and so on. Her goal is to run a total of 100 miles before the first day of practice.

NewVocabulary
arithmetic sequence
common difference
arithmetic means
first difference
second difference
arithmetic series

1 Arithmetic Sequences

A sequence in which the difference between successive terms is a constant is called an **arithmetic sequence**. The constant is referred to as the **common difference**, denoted d. To find the common difference of an arithmetic sequence, subtract any term from its succeeding term. To find the next term in the sequence, add the common difference to the given term.

Example 1 Arithmetic Sequences

Determine the common difference, and find the next four terms of the arithmetic sequence 17, 12, 7, … .

First, find the common difference.

$a_2 - a_1 = 12 - 17$ or -5

$a_3 - a_2 = 7 - 12$ or -5

Find the difference between two pairs of consecutive terms to verify the common difference.

The common difference is -5. Add -5 to the third term to find the fourth term, and so on.

$a_4 = 7 + (-5)$ or 2 $\quad a_5 = 2 + (-5)$ or -3 $\quad a_6 = -3 + (-5)$ or -8 $\quad a_7 = -8 + (-5)$ or -13

The next four terms are 2, -3, -8, and -13.

GuidedPractice

Determine the common difference, and find the next four terms of each arithmetic sequence.

1A. $-129, -98, -67, \ldots$ **1B.** $244, 187, 130, \ldots$

Each term in an arithmetic sequence is found by adding the common difference to the preceding term. Therefore, $a_n = a_{n-1} + d$. You can use this recursive formula to develop an explicit formula for generating an arithmetic sequence. Consider the arithmetic sequence in which $a_1 = 6$ and $d = 3$.

first term	a_1	a_1	6
second term	a_2	$a_1 + d$	$6 + 1(3) = 9$
third term	a_3	$a_1 + 2d$	$6 + 2(3) = 12$
fourth term	a_4	$a_1 + 3d$	$6 + 3(3) = 15$
fifth term	a_5	$a_1 + 4d$	$6 + 4(3) = 18$
nth term	a_n	$a_1 + (n - 1)d$	$6 + (n - 1)3$

The pattern formed leads to the following formula for finding the nth term of an arithmetic sequence.

KeyConcept The nth Term of an Arithmetic Sequence

Words	The nth term of an arithmetic sequence with first term a_1 and common difference d is given by $a_n = a_1 + (n - 1)d$.
Example	The 16th term of 2, 5, 8, … is $a_{16} = 2 + (16 - 1) \cdot 3$ or 47.

StudyTip

Explicit Formulas If a term other than a_1 is given, the explicit formula for finding the nth term of a sequence needs to be adjusted. This can be done by subtracting the number of the term given from n. For example, if a_5 is given, the equation would become $a_n = a_5 + (n-5)d$, or if a_0 is given, then $a_n = a_0 + nd$.

Example 2 Explicit and Recursive Formulas

Find both an explicit formula and a recursive formula for the nth term of the arithmetic sequence 12, 21, 30, … .

First, find the common difference.

$a_2 - a_1 = 21 - 12$ or 9 — Find the difference between two pairs of consecutive terms to verify the common difference.

$a_3 - a_2 = 30 - 21$ or 9

For an explicit formula, substitute $a_1 = 12$ and $d = 9$ in the formula for the nth term of an arithmetic sequence.

$a_n = a_1 + (n-1)d$ — nth term of an arithmetic sequence

$= 12 + (n-1)9$ — $a_1 = 12$ and $d = 9$

$= 12 + 9(n-1)$ or $9n + 3$ — Simplify.

For a recursive formula, state the first term a_1 and then indicate that the next term is the sum of the previous term a_{n-1} and d.

$a_1 = 12,\ a_n = a_{n-1} + 9$

GuidedPractice

2. Find both an explicit formula and a recursive formula for the nth term of the arithmetic sequence 35, 23, 11, … .

The formula for the nth term of an arithmetic sequence can be used to find a specific term in a sequence.

StudyTip

Rate of Change Arithmetic sequences have a constant rate of change which is equivalent to the common difference d.

Example 3 nth Terms

a. Find the 68th term of the arithmetic sequence 25, 17, 9, … .

First, find the common difference.

$a_2 - a_1 = 17 - 25$ or -8 — Find the difference between two pairs of consecutive terms to verify the common difference.

$a_3 - a_2 = 9 - 17$ or -8

Use the formula for the nth term of an arithmetic sequence to find a_{68}.

$a_n = a_1 + (n-1)d$ — nth term of an arithmetic sequence

$a_{68} = 25 + (68-1)(-8)$ — $n = 68$, $a_1 = 25$, and $d = -8$

$a_{68} = -511$ — Simplify.

b. Find the first term of the arithmetic sequence for which $a_{25} = 139$ and $d = \frac{3}{4}$.

Substitute $a_{25} = 139$, $n = 25$, and $d = \frac{3}{4}$ in the formula for the nth term of an arithmetic sequence to find a_1.

$a_n = a_1 + (n-1)d$ — nth term of an arithmetic sequence

$139 = a_1 + (25-1)\frac{3}{4}$ — $n = 25$, $a_n = 139$, and $d = \frac{3}{4}$

$a_1 = 121$ — Simplify.

GuidedPractice

3A. Find the 38th term of the arithmetic sequence $-29, -2, 25, \ldots$.

3B. Find d of the arithmetic sequence for which $a_1 = 75$ and $a_{38} = 56.5$.

If two nonconsecutive terms of an arithmetic sequence are known, the terms between them can be calculated. These terms are called **arithmetic means**. In the sequence below, 17, 27, and 37 are the arithmetic means between 7 and 47.

$$-3, 7, 17, 27, 37, 47, 57$$

StudyTip

Alternative Method An alternative method to find d would be to subtract the first term from the last term and divide by the total number of terms minus 1.

Example 4 Arithmetic Means

Write an arithmetic sequence that has four arithmetic means between 4.3 and 12.8.

The sequence will resemble 4.3, __?__, __?__, __?__, __?__, 12.8. Note that 12.8 is the sixth term of the sequence or a_6.

First, find the common difference using $a_6 = 12.8$, $a_1 = 4.3$, and $n = 6$.

$a_n = a_1 + (n - 1)d$	nth term of an arithmetic sequence
$12.8 = 4.3 + (6 - 1)d$	$a_n = 12.8$, $a_1 = 4.3$, and $n = 6$
$12.8 = 4.3 + 5d$	Simplify.
$d = 1.7$	Solve for d.

Then determine the arithmetic means by using $d = 1.7$.

$a_2 = 4.3 + 1.7$ or 6

$a_3 = 6 + 1.7$ or 7.7

$a_4 = 7.7 + 1.7$ or 9.4

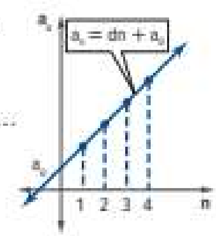
$a_5 = 9.4 + 1.7$ or 11.1

The sequence is 4.3, 6, 7.7, 9.4, 11.1, 12.8.

GuidedPractice

4. Write a sequence that has six arithmetic means between 12.4 and −24.7.

The **first differences** of a sequence are found by subtracting each term from its successive term.



Sequence a_1, a_2, a_3, a_4, a_5, …

1st differences d d d d

When the first differences are all the same, the sequence is arithmetic and the nth term can be modeled by a linear function of the form $a_n = dn + a_0$, as shown.

If the first differences are not the same, the sequence is not arithmetic. However, the differences may still help to identify the type of function that can be used to model the sequence. Consecutive first differences may be subtracted from one another, thus producing **second differences**.

Sequence 12, 20, 30, 42, 56, …

1st differences 8 10 12 14

2nd differences 2 2 2

If the second differences are constant, then the nth term of the sequence can be modeled by a quadratic function. This function can be found by solving a system of equations, as demonstrated in Example 5.

Example 5 Second Differences

Find a quadratic model for the sequence 12, 20, 30, 42, 56, ...

The nth term can be represented by a quadratic equation of the form $a_n = an^2 + bn + c$. Substitute values for a_n and n into the equation.

$12 = a(1)^2 + b(1) + c$ — $a_n = 12$ and $n = 1$

$20 = a(2)^2 + b(2) + c$ — $a_n = 20$ and $n = 2$

$30 = a(3)^2 + b(3) + c$ — $a_n = 30$ and $n = 3$

This yields a system of linear equations in three variables.

$12 = a + b + c$ — Simplified first equation

$20 = 4a + 2b + c$ — Simplified second equation

$30 = 9a + 3b + c$ — Simplified third equation

Solving for a, b, and c gives $a = 1$, $b = 5$, and $c = 6$. Substituting these values in the equation for a_n, the model for the sequence is $a_n = n^2 + 5n + 6$, as shown in Figure 10.2.1.

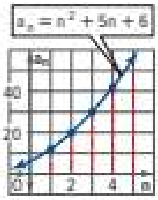


Figure 3.2.1

GuidedPractice

5. Find a quadratic model for the sequence −14, −8, 0, 10, 22, 36, ...

StudyTip

Higher Differences The number of variables in the standard form of the equation dictates the number of equations needed in the system formed.

If calculating second differences does not result in a constant difference, higher differences may be found. This process is similar to the process needed for finding a quadratic equation. The function that will model a sequence is dependent upon how many computed differences are necessary before finding a constant difference.

Differences	Model
first	linear
second	quadratic
third	cubic
fourth	quartic
fifth	quintic

Higher differences may never result in constant differences. In this case, there may not be a polynomial model that can be used to describe the sequence.

2 Arithmetic Series

An **arithmetic series** is the indicated sum of the terms of an arithmetic sequence.

Arithmetic Sequence	Arithmetic Series
−6, −3, 0, 3, 6	−6 + (−3) + 0 + 3 + 6
4.25, 4, 3.75, 3.5, 3.25	4.25 + 4 + 3.75 + 3.5 + 3.25
$a_1, a_2, a_3, a_4, \cdots, a_n$	$a_1 + a_2 + a_3 + a_4 + \cdots + a_n$

To develop a formula for finding the sum of a finite arithmetic series, start by looking at the series S_n that has terms created by adding multiples of d to a_1. If we combine this with the same series written in reverse order, we can find a formula for calculating the sum of a finite arithmetic series.

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - 2d) + (a_n - d) + a_n$$
$$(+)\ S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1$$
$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n)$$
$$2S_n = n(a_1 + a_n)$$

There are n terms in the series, all of which are $(a_1 + a_n)$.

Therefore, $S_n = \frac{n}{2}(a_1 + a_n)$. When the value of the last term is unknown, you can still determine the nth partial sum of the series by combining the nth term of an arithmetic sequence formula and the sum of a finite arithmetic series formula.

$S_n = \frac{n}{2}(a_1 + a_n)$ — Sum of a finite arithmetic series formula

$S_n = \frac{n}{2}[a_1 + [a_1 + (n - 1)d]]$ — $a_n = a_1 + (n - 1)d$, nth term of an arithmetic sequence formula

$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ — Simplify.

KeyConcept Sum of a Finite Arithmetic Series

The sum of a finite arithmetic series with n terms or the nth partial sum of an arithmetic series can be found using one of two related formulas.

Formula 1 $S_n = \frac{n}{2}(a_1 + a_n)$

Formula 2 $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

StudyTip

Arithmetic Series All infinite arithmetic sequences diverge except for those in which $d = 0$. As a result, only a finite arithmetic series or the nth partial sum of an infinite arithmetic series can be calculated.

Example 6 Sum of Arithmetic Series

Find the indicated sum of each arithmetic series.

a. $-5 + 2 + 9 + \cdots + 317$

In this sequence, $a_1 = -5$, $a_n = 317$, and $d = 2 - (-5)$ or 7. Use the nth term formula to find the number of terms in the sequence n.

$a_n = a_1 + (n-1)d$ — nth term of an arithmetic sequence

$317 = -5 + (n-1)7$ — $a_n = 317$, $a_1 = -5$, and $d = 7$

$47 = n$ — Simplify.

Now use Formula 1 to find the sum of the series.

$S_n = \frac{n}{2}(a_1 + a_n)$ — Formula 1

$S_{47} = \frac{47}{2}(-5 + 317)$ — $n = 47$, $a_1 = -5$, and $a_n = 317$

$= 7332$ — Simplify.

b. the 28th partial sum of $27 + 14 + 1 + \cdots$

In this sequence, $a_1 = 27$ and $d = 14 - 27$ or -13. Use Formula 2 to find the 28th partial sum.

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$ — Formula 2

$S_{28} = \frac{28}{2}[2(27) + (28-1)(-13)]$ — $n = 28$, $a_1 = 27$, and $d = -13$

$= -4158$ — Simplify.

c. $\sum_{n=6}^{28} (5n - 17)$

$$\sum_{n=6}^{28} (5n - 17) = [5(6) - 17] + [5(7) - 17] + \cdots + [5(28) - 17]$$

$$= 13 + 18 + \cdots + 123$$

The first term of this series is 13 and the last term is 123. The number of terms is equal to the upper bound minus the lower bound plus one, which is $28 - 6 + 1$ or 23. Therefore $a_1 = 13$, $a_n = 123$, and $n = 23$. Use Formula 1 to find the sum of the series.

$S_n = \frac{n}{2}(a_1 + a_n)$ — Formula 1

$S_{23} = \frac{23}{2}(13 + 123)$ — $n = 23$, $a_1 = 13$, and $a_n = 123$

$= 1564$ — Simplify.

GuidedPractice

6A. $211 + 193 + 175 + \cdots + (-455)$

6B. the 19th partial sum of $-19 + 23 + 65 + \cdots$

6C. $\sum_{n=23}^{37} (2n + 3)$

6D. $\sum_{n=12}^{18} (-2n + 57)$

Real-WorldCareer

Software Engineer Most video game programmers are software engineers who plan and write game software. Most programmers have a bachelor's degree in computer science, information systems, or mathematics. Some also obtain technical or professional certification.

Arithmetic series have many useful real-life applications.

Real-World Example 7 Sum of an Arithmetic Series

VIDEO GAMES A video game tournament, in which gamers compete in multiple games and accumulate an overall score, pays the top 20 finishers. First place receives $5000, second place receives $4800, third place receives $4600, and so on. How much total prize money is awarded?

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$ — Formula 2

$S_{20} = \frac{20}{2}[2(5000) + (20-1)(-200)]$ — $n = 20$, $a_1 = 5000$, and $d = -200$

$= 62{,}000$ — Simplify.

The total prize money awarded is $62,000.

GuidedPractice

7. **VIDEO GAMES** Selma is playing a video game. She scores 50 points if she clears the first level. Each following level is worth 50 more points than the previous level. Thus, she scores 100 points for clearing the second level, 150 for the third, and so on. What is the total amount of points Selma will score after she clears the ninth level?

The formula for the sum of a finite arithmetic series can also be used to solve for values of n.

Real-World Example 8 Sum of an Arithmetic Series

BASEBALL Carter has been collecting baseball cards since his father gave him a 20-card collection. During each month, Carter's father gives him 5 more cards than the previous month. In how many months will Carter reach 1000 cards?

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$ — Formula 2

$1000 = \frac{n}{2}[2(20) + (n-1)5]$ — $S_n = 1000$, $a_1 = 20$, and $d = 5$

$2000 = n(5n + 35)$ — Multiply each side by 2 and simplify.

$0 = 5n^2 + 35n - 2000$ — Distribute and subtract 2000 from each side.

$0 = n^2 + 7n - 400$ — Divide each side by 5.

$n = \frac{-7 \pm \sqrt{7^2 - 4(1)(-400)}}{2(1)}$ — Use the Quadratic Formula.

$n \approx 16.8$ and -23.8 — Simplify.

Because time cannot be negative, Carter will reach 1000 cards in 17 months.

CHECK $20 + 25 + \cdots + 100 = \frac{17}{2}(20 + 100)$

$= 1020$ ✔

In seventeen months, Carter will have 1020 baseball cards, which is more than 1000.

GuidedPractice

8. **LAWN SERVICE** Kevin runs a lawn mowing service. He currently has 14 clients. He has gained 2 new clients at the beginning of each of the past three years. Each year, he mows each client's lawn an average of 15 times. Starting now, if Kevin continues to gain 2 clients each year and if he charges $30 per lawn, after how many years will he earn a total of $51,300?

Exercises

Determine the common difference, and find the next four terms of each arithmetic sequence.

1. 20, 17, 14, …

2. 3, 16, 29, …

3. 117, 108, 99, …

4. −83, −61, −39, …

5. −3, 1, 5, …

6. 4, 21, 38, …

7. −4.5, −9.5, −14.5, …

8. −97, −29, 39, …

9 MARCHING BAND A marching band begins its performance in a pyramid formation. The first row has 1 band member, the second row has 3 band members, the third row has 5 band members, and so on.

a. Find the number of band members in the 8th row.

b. Write an explicit formula and a recursive formula for finding the number of band members in the nth row.

Find both an explicit formula and a recursive formula for the nth term of each arithmetic sequence.

10. 2, 5, 8, …

11. −6, 5, 16, …

12. −9, −16, −23, …

13. 4, 19, 34, …

14. 25, 11, −3, …

15. 7, −3.5, −14, …

16. −18, 4, 26, …

17. 1, 37, 73, …

Find the specified value for the arithmetic sequence with the given characteristics.

18. If $a_{14} = 85$ and $d = 9$, find a_1.

19. Find d for −24, −31, −38, ….

20. If $a_n = 14$, $a_1 = -36$, and $d = 5$, find n.

21. If $a_1 = 47$ and $d = -5$, find a_{12}.

22. If $a_{22} = 95$ and $a_1 = 11$, find d.

23. Find a_6 for 84, 5, −74, ….

24. If $a_n = -20$, $a_1 = 46$, and $d = -11$, find n.

25. If $a_{35} = -63$ and $a_1 = 39$, find d.

26. CONSTRUCTION Each 8-foot section of a wooden fence contains 14 pickets. Let a_n represent the number of pickets in n sections.

a. Find the first 5 terms of the sequence.

b. Write a recursive formula for the sequence in part **a**.

c. If 448 pickets were used to fence in the customer's backyard, how many feet of fencing was used?

Find the indicated arithmetic means for each set of nonconsecutive terms.

27. 3 means; 19 and −5

28. 5 means; −62 and −8

29. 4 means; 3 and 88

30. 8 means; −5.5 and 23.75

31. 7 means; −4.5 and 7.5

32. 10 means; 6 and 259

Find a quadratic model for each sequence.

33. 12, 19, 28, 39, 52, 67, …

34. −11, −9, −5, 1, 9, 19, …

35. 8, 3, −6, −19, −36, −57, …

36. −7, −2, 9, 26, 49, 78, …

37. 6, −2, −12, −24, −38, −54, …

38. −3, 1, 13, 33, 61, 97, …

Find the indicated sum of each arithmetic series.

39. 26th partial sum of $3 + 15 + 27 + \cdots + 303$

40. $-28 + (-19) + (-10) + \cdots + 242$

41. 42nd partial sum of $120 + 114 + 108 + \cdots$

42. 54th partial sum of $213 + 205 + 197 + \cdots$

43. $-17 + 1 + 19 + \cdots + 649$

44. $89 + 58 + 27 + \cdots + (-562)$

45. RUNNING Refer to the beginning of the lesson.

a. Determine the number of miles Meg will run on her 12th day of training.

b. During which day of training will Meg reach her goal of 100 total miles?

Find the indicated sum of each arithmetic series.

46. $\sum_{n=1}^{20} (3 + 2n)$

47. $\sum_{n=1}^{28} (100 - 4n)$

48. $\sum_{n=7}^{18} (-9n - 26)$

49. $\sum_{n=6}^{52} (7n + 1)$

50. $\sum_{n=7}^{42} (84 - 3n)$

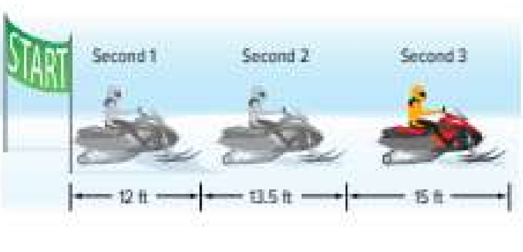
51. $\sum_{n=1}^{13} [32 + 4(n - 1)]$

52. $\sum_{n=20}^{24} \left(\frac{n}{2} - 9\right)$

53. $\sum_{n=2}^{9} (-15n - 12)$

54. CONSTRUCTION A crew is tiling a hotel lobby with a trapezoidal mosaic pattern. The shorter base of the trapezoid begins with a row of 8 tiles. Each row has two additional tiles until the 20th row. Determine the number of tiles needed to create the mosaic design.

55. SNOWMOBILING A snowmobiling competitor travels 12 feet in the first second of a race. If the competitor travels 1.5 additional feet each subsequent second, how many feet did the competitor travel in 64 seconds?



56. **FUNDRAISING** Lalana organized a charity walk. In the first year, the walk generated \$3000. She hopes to increase this amount by \$900 each year for the next several years. If her goal is met, in how many years will the walk have generated a total of at least \$65,000?

57. Find a_n if $S_n = 490$, $a_1 = -5$, and $n = 100$.

58. If $S_n = 51.7$, $n = 22$, $a_n = -11.3$, find a_1.

59. Find n for $-7 + (-5.5) + (-4) + \cdots$ if $S_n = -14$ and $a_n = 3.5$.

60. Find a_1 if $S_n = 1287$, $n = 22$, and $d = 5$.

61. If $S_{26} = 1456$, and $a_1 = -19$, find d.

62. If $S_{12} = 174$, $a_{12} = 39$, find d.

Write each arithmetic series in sigma notation. The lower bound is given.

63. $6 + 12 + 18 + \cdots + 66$; $n = 1$

64. $-1 + 0 + 1 + \cdots + 7$; $n = 1$

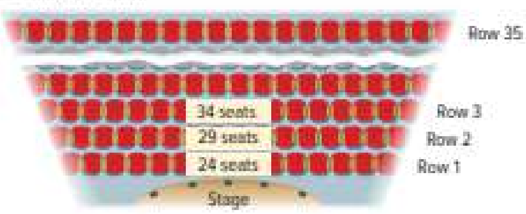
65. $17 + 21 + 25 + \cdots + 61$; $n = 4$

66. $1 + 0 + (-1) + (-2) + \cdots + (-13)$; $n = 6$

67. $-\frac{13}{5} + \left(-\frac{12}{5}\right) + \left(-\frac{11}{5}\right) + \cdots + \left(-\frac{3}{5}\right)$; $n = 2$

68. $9.25 + 8.5 + 7.75 + \cdots - 2$; $n = 1$

69. **CONCERTS** The seating in a concert auditorium is arranged as shown below.



a. Write a series in sigma notation to represent the number of seats in the auditorium, if the seating pattern shown in the first 3 rows continues for each successive row.

b. Find the total number of seats in the auditorium.

c. Another auditorium has 32 rows with 18 seats in the first row and 4 more seats in each of the successive rows. How many seats are there in this auditorium?

Write a function that can be used to model the *n*th term of each sequence.

70. 2, 5, 8, 11, 14, 17, …

71. 8, 13, 20, 29, 40, 53, …

72. 2, 2, 4, 8, 14, 22, …

73. 5, 31, 97, 221, 421, 715, …

74. −6, −8, −6, 6, 34, 84, …

75. 0, 23, 134, 447, 1124, 2375, …

Find each common difference.

76. $\sum_{n=1}^{100} (6n + 2)$

77. $\sum_{n=21}^{65} \left(8 - \frac{2n}{3}\right)$

78. $a_{12} = 63$, $a_{19} = 7$

79. $a_8 = -4$, $a_{27} = \frac{7}{3}$

80. **CALCULUS** The area between the graph of a continuous function and the x-axis can be approximated using sequences. Consider $f(x) = x^2$ on the interval [1, 3].

a. Write the sequence x_n formed when there are 5 arithmetic means between 1 and 3.

b. Write the sequence y_n formed when $y_n = f(x_n)$.

c. Write the sequence p_n defined by $d \cdot y_n$.

d. The *left-hand* approximation of the area is given by $L_n = \sum_{k=1}^{n} p_k$. Find L_6.

e. The *right-hand* approximation of the area is given by $R_n = \sum_{k=2}^{n+1} p_k$. Find R_6.

H.O.T. Problems Use Higher-Order Thinking Skills

81. **ERROR ANALYSIS** Peter and Candace are given the arithmetic sequence 2, 9, 16, … . Peter wrote the explicit formula $a_n = 2 + 7(n - 1)$ for the sequence. Candace's formula is $a_n = 7n - 5$. Is either of them correct? Explain.

82. **OPEN ENDED** You have learned that the nth term of an arithmetic sequence can be modeled by a linear function. Can the sequence of partial sums of an arithmetic series also be modeled by a linear function? If yes, provide an example. If no, how can the sequence be modeled? Explain.

83. **CHALLENGE** Prove that for an arithmetic sequence, $a_n = a_k + (n - k)d$ for integers k in the domain of the sequence.

REASONING Determine whether each statement is *true* or *false* for finite arithmetic series. Explain.

84. If you know the sum and d, you can solve for a_1.

85. If you only know the first and last terms, then you can find the sum.

86. If the first three terms of a sequence are positive, then all of the terms of the sequence are positive or the sum of the series is positive.

87. **CHALLENGE** Consider the arithmetic sequence of odd natural numbers.

a. Find S_7 and S_9.

b. Make a conjecture about the pattern that results from the sums of the corresponding arithmetic series.

c. Write an algebraic proof verifying the conjecture that you made in part **b**.

88. **WRITING IN MATH** Explain why the arithmetic series $25 + 20 + 15 + \ldots$ does not have a sum.

Spiral Review

Find the next four terms of each sequence.

89. 12, 16, 20, …

90. 3, 1, −1, …

91. 31, 24, 17, …

Find each product or quotient and express it in rectangular form.

92. $6\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] \cdot 3\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$

93. $3\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right) \div \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

Find the dot product of u and v. Then determine if u and v are orthogonal.

94. $\mathbf{u} = \langle 4, -1\rangle, \mathbf{v} = \langle 1, 5\rangle$

95. $\mathbf{u} = \langle 8, -3\rangle, \mathbf{v} = \langle 4, 2\rangle$

96. $\mathbf{u} = \langle 4, 6\rangle, \mathbf{v} = \langle 9, -5\rangle$

Find the direction angle of each vector to the nearest tenth of a degree.

97. $-\mathbf{i} - 3\mathbf{j}$

98. $\langle -9, 5\rangle$

99. $\langle -7, 7\rangle$

100. MANUFACTURING A cam in a punch press is shaped like an ellipse with the equation $\frac{x^2}{81} + \frac{y^2}{36} = 1$. The camshaft goes through the focus on the positive axis.

a. Graph a model of the cam.

b. Find an equation that translates the model so that the camshaft is at the origin.

c. Find the equation of the model in part **b** when the cam is rotated to an upright position.

101. Use the graph of $f(x) = \ln x$ to describe the transformation that results in the graph of $g(x) = 3\ln(x - 1)$. Then sketch the graphs of f and g.

Skills Review for Standardized Tests

102. SAT/ACT What is the units digit of 3^{36}?

A 0

B 1

C 3

D 7

E 9

103. Using the table, which formula can be used to determine the nth term of the sequence?

n	a_n
1	6
2	10
3	14
4	18

F $a_n = 6n$

G $a_n = n + 5$

H $a_n = 2n + 1$

J $a_n = 4n + 2$

104. REVIEW If $a_1 = 3$, $a_2 = 5$, and $a_n = a_{n-2} + 3n$, find a_{10}.

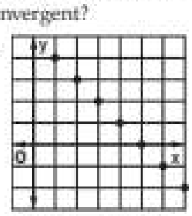
A 59

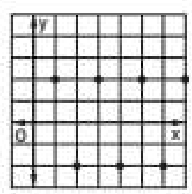
B 75

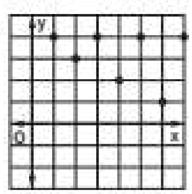
C 89

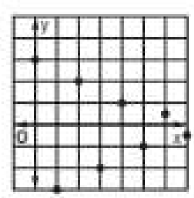
D 125

105. REVIEW Which of the sequences shown below is convergent?

F


H


G


J


LESSON 3-3 Geometric Sequences and Series

::Then	::Now	::Why?
• You found terms and means of arithmetic sequences and sums of arithmetic series.	• **1** Find *n*th terms and geometric means of geometric sequences. **2** Find sums of *n* terms of geometric series and the sums of infinite geometric series.	• The first summer X Games took place in Rhode Island in 1995 and included 27 events. Due to their growing popularity, the winter X Games were introduced at Big Bear Lake, California, in 1997. With an immense fan base, the annual X Games now receive live 24-hour network coverage. Since its inaugural year, the event has seen an average growth of 13% in revenue each year.

NewVocabulary
geometric sequence
common ratio
geometric means
geometric series

1 Geometric Sequences

A sequence in which the ratio between successive terms is a constant is called a **geometric sequence**. The constant is referred to as the **common ratio**, denoted r. To find the common ratio of a geometric sequence, divide any term following the first term by the preceding term. Given a term of the sequence, to find the next term of the sequence, multiply the given term by the common ratio. While the rate of change of an arithmetic sequence is constant, the rate of change of a geometric sequence can either increase or decrease.

Example 1 Geometric Sequences

Determine the common ratio, and find the next three terms of each geometric sequence.

a. $8, -2, \frac{1}{2}, \ldots$

First, find the common ratio.

$a_2 \div a_1 = -2 \div 8$ or $-\frac{1}{4}$ — Find the ratio between two pairs of consecutive terms to verify the common ratio.

$a_3 \div a_2 = \frac{1}{2} \div -2$ or $-\frac{1}{4}$

The common ratio is $-\frac{1}{4}$. Multiply the third term by $-\frac{1}{4}$ to find the fourth term, and so on.

$a_4 = \frac{1}{2}\left(-\frac{1}{4}\right)$ or $-\frac{1}{8}$ $\qquad$ $a_5 = -\frac{1}{8}\left(-\frac{1}{4}\right)$ or $\frac{1}{32}$ $\qquad$ $a_6 = \frac{1}{32}\left(-\frac{1}{4}\right)$ or $-\frac{1}{128}$

The next three terms are $-\frac{1}{8}, \frac{1}{32}$, and $-\frac{1}{128}$.

b. $w + 3, 2w + 6, 4w + 12, \ldots$

First, find the common ratio.

$a_2 \div a_1 = \frac{2w + 6}{w + 3}$ — $a_2 = 2w + 6$ and $a_1 = w + 3$

$= \frac{2(w + 3)}{w + 3}$

$= 2$

$a_3 \div a_2 = \frac{4w + 12}{2w + 6}$ — $a_3 = 4w + 12$ and $a_2 = 2w + 6$

$= \frac{4(w + 3)}{2(w + 3)}$ — Factor.

$= 2$ — Simplify.

The common ratio is 2. Multiply the third term by 2 to find the fourth term, and so on.

$a_4 = 2(4w + 12)$ or $8w + 24$

$a_5 = 2(8w + 24)$ or $16w + 48$

$a_6 = 2(16w + 48)$ or $32w + 96$

The next three terms are $8w + 24$, $16w + 48$, and $32w + 96$.

GuidedPractice

1A. $4, 11, 30.25, \ldots$ $\qquad$ **1B.** $64r - 128, -16r + 32, 4r - 8, \ldots$

Brian Bahr/Staff/Getty Images Sport/Getty Images

WatchOut!

Type of Sequence Remember that if a sequence is not arithmetic, it does not necessarily mean that the sequence is geometric. Test several terms for a common ratio before determining that the sequence is indeed geometric.

In Lesson 3-2, you learned that arithmetic sequences can be defined both recursively and explicitly. This also applies to geometric sequences. A geometric sequence can be expressed recursively, where a term a_n is found by taking the product of the previous term a_{n-1} and r, or $a_n = a_{n-1}r$, as illustrated by the previous example. To develop an explicit formula for a geometric sequence, consider the pattern created by the geometric sequence for which $a_1 = 3$ and $r = 4$.

	Term	Expanded Form	Exponential Form	Example
first term	a_1	a_1	a_1	3
second term	a_2	$a_1 \cdot r$	a_1r^1	$3 \cdot 4 = 12$
third term	a_3	$a_1 \cdot r \cdot r$	a_1r^2	$3 \cdot 4^2 = 48$
fourth term	a_4	$a_1 \cdot r \cdot r \cdot r$	a_1r^3	$3 \cdot 4^3 = 192$
fifth term	a_5	$a_1 \cdot r \cdot r \cdot r \cdot r$	a_1r^4	$3 \cdot 4^4 = 768$
nth term	a_n	$\underbrace{a_1 \cdot r \cdot r \cdot r \cdot \ldots \cdot r}_{n-1 \text{ factors}}$	a_1r^{n-1}	$3 \cdot 4^{n-1}$

KeyConcept The nth Term of a Geometric Sequence

Words The nth term of a geometric sequence with first term a_1 and common ratio r is given by $a_n = a_1r^{n-1}$.

Example The 9th term of 2, 10, 50, … is $a_9 = 2 \cdot 5^{9-1}$ or 781,250.

Example 2 Explicit and Recursive Formulas

Write an explicit formula and a recursive formula for finding the nth term of the geometric sequence given in Example 1a.

For an explicit formula, substitute $a_1 = 8$ and $r = -0.25$ in the nth term formula.

$a_n = a_1r^{n-1}$ — nth term of a geometric sequence

$= 8(-0.25)^{n-1}$ — $a_1 = 8$ and $r = -0.25$

For a recursive formula, state the first term a_1. Then indicate that the next term is the product of the previous term a_{n-1} and r.

$a_1 = 8$, $a_n = (-0.25)a_{n-1}$

GuidedPractice

2. Write an explicit formula and a recursive formula for finding the nth term in the sequence 2, 25, 312.5, … .

Finding the nth term of a geometric sequence is simplified by explicit formulas.

Example 3 nth Terms

Find the 27th term of the geometric sequence 189, 151.2, 120.96, … .

First, find the common ratio.

$a_2 \div a_1 = 151.2 \div 189$ or 0.8 — Find the ratio between two pairs of consecutive terms to verify the common ratio.

$a_3 \div a_2 = 120.96 \div 151.2$ or 0.8

Use the formula for the nth term of a geometric sequence.

$a_n = a_1r^{n-1}$ — nth term of a geometric sequence

$a_{27} = 189(0.8)^{27-1}$ — $n = 27$, $a_1 = 189$, and $r = 0.8$

$a_{27} \approx 0.57$ — Simplify.

GuidedPractice

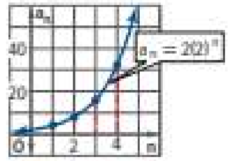
Find the specified term of each geometric sequence or sequence with the given characteristics.

3A. a_9 for 4, 14, 49, ...

3B. a_{12} if $a_3 = 32$ and $r = -4$

Just as arithmetic sequences are linear functions with restricted domains, geometric sequences are also functions. Consider the exponential function $f(x) = 2(2)^x$ and the explicit formula for the geometric sequence $a_n = 2(2)^n$.

Notice that the graphs of the terms of the geometric sequence lie on a curve, as shown. A geometric sequence can be modeled by an exponential function in which the domain is restricted to the natural numbers.



Real-WorldLink

The value of a newly purchased vehicle can depreciate by as much as 30–35% in its first year. Each year after, the value continues to depreciate by 7–12%, depending on the make and model. After a five-year period, on average, cars are worth 35% of the original sticker price, making car buying costly.

Source: *Kelly Blue Book*

Real-World Example 4 nth Term of a Geometric Sequence

AUTOMOBILE Damian purchased a late-model car for $15,000. At the end of each year, the value of the car depreciates 11%.

a. Write an explicit formula for the value of Damian's car after n years.

If the car's value depreciates at a rate of 11% per year, it retains 100% − 11% or 89% of its original value. Note that the original value given represents the a_0 and not the a_1 term, so we need to use an adjusted formula for the nth term of this geometric sequence.

first term	$a_1 = a_0r$
second term	$a_2 = a_0r^2$
⋮	⋮
nth term	$a_n = a_0r^n$

Use this adjusted formula to find an explicit formula for the value of the car after n years.

$a_n = a_0r^n$ — Adjusted nth term of a geometric sequence

$a_n = 15{,}000(0.89)^n$ — $a_0 = 15{,}000$, $r = 0.89$

b. What is the value of Damian's car at the end of the seventh year?

Evaluate the formula found in part **a** for $n = 7$.

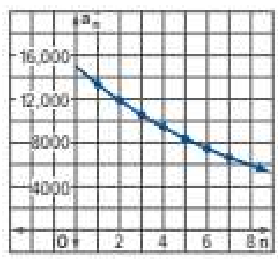
$a_n = 15{,}000(0.89)^n$ — Original equation

$= 15{,}000(0.89)^7$ — $n = 7$

≈ 6634.70 — Simplify.

The value of the car at the beginning of the seventh year is about $6634.70.

The value of the car at each year is shown as a point on the graph. The function connecting the points represents exponential decay.



GuidedPractice

4. WATERCRAFT Rohan purchased a personal watercraft for $9000. Assume that by the end of each year, the value of the watercraft depreciates 30%.

A. Write an explicit formula for finding the value of Rohan's watercraft after n years.

B. What is the value of Rohan's watercraft after 5 years?

Similar to arithmetic sequences, if two nonconsecutive terms of a geometric sequence are known, the terms between them can be calculated. These terms are called **geometric means**.

Example 5 Geometric Means

Write a sequence that has two geometric means between 480 and −7.5.

The sequence will resemble 480, __?__, __?__, −7.5.

Note that $a_1 = 480$, $n = 4$, and $a_4 = -7.5$. Find the common ratio using the nth term for a geometric sequence formula.

$$a_4 = a_1 r^{n-1}$$ *nth term of a geometric sequence*

$$-7.5 = 480r^{4-1}$$ $a_4 = -7.5$, $a_1 = 480$, and $n = 4$.

$$-\frac{1}{64} = r^3$$ Simplify and divide each side by 480.

$$-\frac{1}{4} = r$$ Take the cube root of each side.

The common ratio is $-\frac{1}{4}$. Use r to find the geometric means.

$a_2 = 480(-0.25)$ or -120 $\qquad$ $a_3 = -120(-0.25)$ or 30

Therefore, a sequence with two geometric means between 480 and −7.5, is 480, −120, 30, −7.5.

StudyTip

Geometric Means Sometimes, more than one set of geometric means are possible. For example, the three geometric means between 3 and 48 can be 6, 12, and 24 or −6, 12, and −24.

GuidedPractice

Find the indicated geometric means for each pair of nonconsecutive terms.

5A. −4 and 13.5; 2 means

5B. 10 and 0.016; 3 means

2 Geometric Series

A **geometric series** is the sum of the terms of a geometric sequence.

Geometric Sequence	Geometric Series
2, 4, 8, 16, 32	2 + 4 + 8 + 16 + 32
$27, 9, 3, 1, \frac{1}{3}$	$27 + 9 + 3 + 1 + \frac{1}{3}$
$a_1, a_2, a_3, a_4, \cdots, a_n$	$a_1 + a_2 + a_3 + a_4 + \cdots + a_n$

A formula for the sum S_n of the first n terms of a finite geometric series can be developed by looking at the series S_n and rS_n. To create the terms for rS_n, each term in S_n is multiplied by r.
These series are then aligned so that similar terms are grouped together and then rS_n is subtracted from S_n.

$$S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1}$$

$$(-)\ rS_n = a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1} + a_1r^n$$

$$S_n - rS_n = a_1 - a_1r^n$$ Subtract.

$$S_n(1 - r) = a_1 - a_1r^n$$ Factor.

$$S_n = \frac{a_1 - a_1r^n}{1 - r}$$ Divide each side by $1 - r$.

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$ Factor.

Therefore, $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$.

If the value of n is not provided, the sum of a finite geometric series can still be found. If we take a look at the next-to-last step of the proof, we can substitute for a_1r^n.

$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$ — Sum of a finite geometric series formula

$S_n = \frac{a_1 - a_1r^n}{1-r}$ — Multiply.

$= \frac{a_1 - a_1r^{n-1} \cdot r}{1-r}$ — Factor one r from a_1r^n.

$= \frac{a_1 - a_nr}{1-r}$ — $a_1r^{n-1} = a_n$, nth term of a geometric sequence formula

KeyConcept Sum of a Finite Geometric Series

The sum of a finite geometric series with n terms or the nth partial sum of a geometric series can be found using one of two related formulas.

Formula 1 $S_n = a_1\left(\frac{1-r^n}{1-r}\right)$

Formula 2 $S_n = \frac{a_1 - a_nr}{1-r}$

StudyTip

Infinite vs. Finite Geometric Series Notice that the series given in Example 6a is an infinite geometric series. Because you are asked to find the sum of the *first six terms* of the series, you are actually finding the sum of a finite series.

Example 6 Sums of Geometric Series

a. Find the sum of the first six terms of the geometric series 8 + 14 + 24.5 + … .

First, find the common ratio.

$a_2 \div a_1 = 14 \div 8$ or 1.75

$a_3 \div a_2 = 24.5 \div 14$ or 1.75

Find the ratio between two pairs of consecutive terms to verify the common ratio.

The common ratio is 1.75. Use Formula 1 to find the sum of the series.

$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$ — Formula 1

$S_6 = 8\left(\frac{1-1.75^6}{1-1.75}\right)$ — $n = 6$, $a_1 = 8$, and $r = 1.75$

$S_6 \approx 295.71$ — Simplify.

The sum of the first six terms of the geometric series is about 295.71.

CHECK The next three terms of the related sequence are 42.875, 75.03125, and 131.3046875.

$8 + 14 + 24.5 + 42.875 + 75.03125 + 131.3046875 \approx 295.71$ ✓

b. Find the sum of the first n terms of a geometric series with $a_1 = 3$, $a_n = 768$, and $r = -2$.

Use Formula 2 for the sum of a finite geometric series.

$S_n = \frac{a_1 - a_nr}{1-r}$ — Formula 2

$= \frac{3 - 768(-2)}{1-(-2)}$ — $a_1 = 3$, $a_n = 768$, and $r = -2$

$= 513$ — Simplify.

The sum of the first n terms of the geometric series is 513.

GuidedPractice

6A. Find the sum of the first 11 terms of the geometric series $7 + (-24.5) + 85.75 + \dots$.

6B. Find the sum of the first n terms of a geometric series with $a_1 = -8$, $a_n = 131{,}072$, and $r = -4$.

Geometric series may also be represented in sigma notation.

Example 7 Geometric Sum in Sigma Notation

Find $\sum_{n=2}^{7} 3(5)^{n-1}$.

Find n, a_1, and r.

$n = 7 - 2 + 1$ or 6 — Upper bound minus lower bound plus 1

$a_1 = 3(5)^{2-1}$ or 15 — $n = 2$

$r = 5$ — r is the base of the exponential function.

Method 1 Substitute $n = 6$, $a_1 = 15$, and $r = 5$ into Formula 1.

$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$ — Formula 1

$S_6 = 15\left(\frac{1 - 5^6}{1 - 5}\right)$ — $a_1 = 15$, $r = 5$, and $n = 6$

$S_6 = 58,590$ — Simplify.

Method 2 Find a_n.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$= 15(5)^{6-1}$ — $a_1 = 15$, $r = 5$, and $n = 6$

$= 46,875$ — Simplify.

Substitute $a_1 = 15$, $a_n = 46,875$, and $r = 5$ into Formula 2.

$S_n = \frac{a_1 - a_n r}{1 - r}$ — Formula 2

$S_6 = \frac{15 - (46,875)(5)}{1 - 5}$ — $a_1 = 15$, $a_n = 46,875$, and $r = 5$

$S_6 = 58,590$ — Simplify.

Therefore, $\sum_{n=2}^{7} 3(5)^{n-1} = 58,590$.

Guided Practice

7A. $\sum_{n=16}^{31} 0.5(2)^{n-1}$

7B. $\sum_{n=4}^{11} 120(0.5)^{n-1}$

Study Tip

Infinite Series If the sequence of partial sums S_n has a limit, then the corresponding infinite series has a sum and the nth term a_n of the series approaches 0 as $n \to \infty$. However, if the nth term of the series approaches 0, the infinite series does not necessarily have a sum. For example, the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ does not have a sum.

In Lesson 3-1, you learned that calculating the sums of infinite series may be possible if the sequence of terms converges to 0. For this reason, the sums of infinite arithmetic series cannot be found.

The formula for the sum of a finite geometric series can be used to develop a formula for the sum of an infinite geometric series. If $|r| > 1$, then $|r^n|$ increases without limit as $n \to \infty$. However, when $|r| < 1$, r^n approaches 0 as $n \to \infty$. Thus,

$S = a_1\left(\frac{1 - r^n}{1 - r}\right)$ — Sum of a finite geometric series formula

$= a_1\left(\frac{1 - 0}{1 - r}\right)$ — r^n approaches 0 as $n \to \infty$.

$= \frac{a_1}{1 - r}$ — Simplify and multiply.

Key Concept The Sum of an Infinite Geometric Series

The sum S of an infinite geometric series for which $|r| < 1$ is given by

$$S = \frac{a_1}{1 - r}.$$

The formula for the sum of an infinite geometric series involves three variables: S, a_1, and r. If any two of the three variables are known, you can solve for the third.

Example 8 Sums of Infinite Geometric Series

If possible, find the sum of each infinite geometric series.

a. $9 + 3 + 1 + \ldots$

First, find the common ratio.

$a_2 \div a_1 = 3 \div 9$ or $\frac{1}{3}$ — Find the ratio between two pairs of consecutive terms to verify the common ratio.

$a_3 \div a_2 = 1 \div 3$ or $\frac{1}{3}$

StudyTip

Common Ratio Recall that $|r| < 1$ is equivalent to $-1 < r < 1$.

The common ratio r is $\frac{1}{3}$, and $\left|\frac{1}{3}\right| < 1$. This infinite geometric series has a sum. Use the formula for the sum of an infinite geometric series.

$S = \frac{a_1}{1 - r}$ — Sum of an infinite geometric series formula

$= \frac{9}{1 - \frac{1}{3}}$ — $a_1 = 9$ and $r = \frac{1}{3}$

$= 13.5$ — Simplify.

The sum of the infinite series is 13.5.

b. $0.25 + (-1.25) + 6.25 + \ldots$

First, find the common ratio.

$a_2 \div a_1 = -1.25 \div 0.25$ or -5 — Find the ratio between two pairs of consecutive terms to verify the common ratio.

$a_3 \div a_2 = 6.25 \div (-1.25)$ or -5

The common ratio r is -5, and $|-5| > 1$. Therefore, this infinite geometric series has no sum.

c. $\sum_{n=4}^{\infty} 4(0.2)^{n-1}$

The common ratio r is 0.2, and $|0.2| < 1$. Therefore, this infinite geometric series has a sum. Find a_1.

$a_1 = 4(0.2)^{4-1}$ — Lower bound = 4

$= 0.032$ — Simplify.

Use the formula for the sum of an infinite geometric series to find the sum.

$S = \frac{a_1}{1 - r}$ — Sum of an infinite geometric series formula

$= \frac{0.032}{1 - 0.2}$ — $a_1 = 0.032$ and $r = 0.2$

$= 0.04$ — Simplify.

The sum of the infinite series is 0.04.

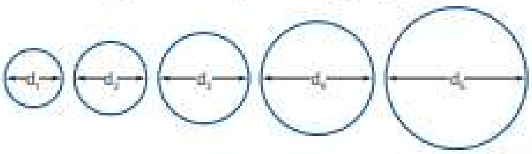
GuidedPractice

8A. $10 + (-5) + 2.5 + \ldots$ **8B.** $20 + 15 + 10 + \ldots$ **8C.** $\sum_{n=1}^{\infty} 120(0.8)^{n-1}$

Exercises

Determine the common ratio, and find the next three terms of each geometric sequence.

1. $-\frac{1}{4}, \frac{1}{2}, -1, \ldots$
2. $\frac{1}{2}, -\frac{3}{8}, \frac{9}{32}, \ldots$
3. $0.5, 0.75, 1.125, \ldots$
4. $8, 20, 50, \ldots$
5. $2x, 10x, 50x, \ldots$
6. $64x, 16x, 4x, \ldots$
7. $x + 5, 3x + 15, 9x + 45, \ldots$
8. $-9 - y, 27 + 3y, -81 - 9y, \ldots$
9. **GEOMETRY** Consider a sequence of circles with diameters that form a geometric sequence: d_1, d_2, d_3, d_4, d_5.

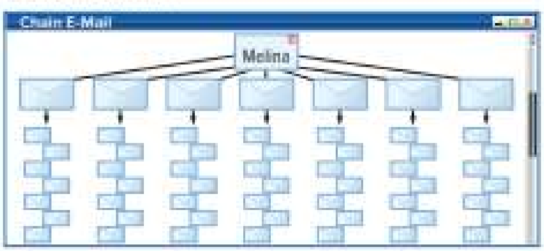


   a. Show that the sequence of circumferences of the circles is also geometric. Identify r.

   b. Show that the sequence of areas of the circles is also geometric. Identify the common ratio.

Write an explicit formula and a recursive formula for finding the nth term of each geometric sequence.

10. $36, 12, 4, \ldots$
11. $64, 16, 4, \ldots$
12. $-2, 10, -50, \ldots$
13. $4, -12, 36, \ldots$
14. $4, 8, 16, \ldots$
15. $20, 30, 45, \ldots$
16. $15, 5, \frac{5}{3}, \ldots$
17. $\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \ldots$
18. **CHAIN E-MAIL** Melina receives a chain e-mail that she forwards to 7 of her friends. Each of her friends forwards it to 7 of their friends.



   a. Write an explicit formula for the pattern.

   b. How many will receive the e-mail after 6 forwards?

19. **BIOLOGY** A certain bacteria divides every 15 minutes to produce two complete bacteria.

   a. If an initial colony contains a population of b_0 bacteria, write an equation that will determine the number of bacteria b_t present after t hours.

   b. Suppose a Petri dish contains 12 bacteria. Use the equation found in part **a** to determine the number of bacteria present 4 hours later.

Find the specified term for each geometric sequence or sequence with the given characteristics.

20. a_9 for $60, 30, 15, \ldots$
21. a_4 for $7, 14, 28, \ldots$
22. a_5 for $3, 1, \frac{1}{3}, \ldots$
23. a_6 for $540, 90, 15, \ldots$
24. a_7 if $a_3 = 24$ and $r = 0.5$
25. a_6 if $a_3 = 32$ and $r = -0.5$
26. a_6 if $a_1 = 16{,}807$ and $r = \frac{3}{7}$
27. a_8 if $a_1 = 4096$ and $r = \frac{1}{4}$
28. **ACCOUNTING** Julian Rockman is an accountant for a small company. On January 1, 2014, the company purchased \$50,000 worth of computers, printers, scanners, and hardware. Because this equipment is a company asset, Mr. Rockman needs to determine how much the computer equipment is presently worth. He estimates that the computer equipment depreciates at a rate of 45% per year. What value should Mr. Rockman assign the equipment in his 2019 year-end accounting report?
29. Find the sixth term of a geometric sequence with a first term of 9 and a common ratio of 2.
30. If $r = 4$ and $a_8 = 100$, what is the first term of the geometric sequence?
31. **X GAMES** Refer to the beginning of the lesson. The X Games netted approximately \$40 million in revenue in 2002. If the X Games continue to generate 13% more revenue each year, how much revenue will the X Games generate in 2020?

Find the indicated geometric means for each pair of nonconsecutive terms.

32. 4 and 256; 2 means
33. 256 and 81; 3 means
34. $\frac{4}{7}$ and 7; 1 mean
35. -2 and 54; 2 means
36. 1 and 27; 2 means
37. 48 and -750; 2 means
38. i and -1; 4 means
39. t^8 and t^{-7}; 4 means

Find the sum of each geometric series described.

40. first six terms of $3 + 9 + 27 + \cdots$
41. first nine terms of $0.5 + (-1) + 2 + \cdots$
42. first eight terms of $2 + 2\sqrt{3} + 6 + \cdots$
43. first n terms of $a_1 = 4$, $a_n = 2000$, $r = -3$
44. first n terms of $a_1 = 5$, $a_n = 1{,}310{,}720$, $r = 4$
45. first n terms of $a_1 = 3$, $a_n = 46{,}875$, $r = -5$
46. first n terms of $a_1 = -8$, $a_n = -256$, $r = 2$
47. first n terms of $a_1 = -36$, $a_n = 972$, $r = 7$

Find each sum.

48. $\sum_{n=1}^{6} 5(2)^{n-1}$

49. $\sum_{n=1}^{7} -4(3)^{n-1}$

50. $\sum_{n=1}^{5} (-3)^{n-1}$

51. $\sum_{n=1}^{6} 2(1.4)^{n-1}$

52. $\sum_{n=1}^{6} 100\left(\frac{1}{2}\right)^{n-1}$

53. $\sum_{n=1}^{9} \frac{1}{27}(-3)^{n-1}$

54. $\sum_{n=1}^{7} 144\left(-\frac{1}{2}\right)^{n-1}$

55. $\sum_{n=1}^{20} 3(2)^{n-1}$

If possible, find the sum of each infinite geometric series.

56. $\frac{1}{20} + \frac{1}{40} + \frac{1}{80} + \cdots$

57. $\frac{2}{7} + \frac{4}{7} + \frac{8}{7} + \cdots$

58. $18 + (-27) + 40.5 + \cdots$

59. $12 + (-7.2) + 4.32 + \cdots$

60. $\sum_{n=1}^{\infty} 6(-0.4)^{n-1}$

61. $\sum_{n=1}^{\infty} 40\left(\frac{3}{5}\right)^{n-1}$

62. $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{8}\right)^{n-1}$

63. $\sum_{n=1}^{\infty} 35\left(-\frac{3}{4}\right)^{n-1}$

64. **BUNGEE JUMPING** A bungee jumper falls 35 meters before his cord causes him to spring back up. He rebounds $\frac{2}{5}$ of the distance after each fall.

 a. Find the first five terms of the infinite sequence representing the vertical distance traveled by the bungee jumper. Include each drop and rebound distance as separate terms.

 b. What is the total vertical distance the jumper travels before coming to rest? (*Hint*: Rewrite the infinite sequence suggested by part **a** as two infinite geometric sequences.)

Find the missing quantity for the geometric sequence with the given characteristics.

65. Find a_1 if $S_{12} = 1365$ and $r = 2$.

66. If $S_6 = 196.875$, $a_1 = 100$, $r = 0.5$, find a_6.

67. Find r if $a_1 = 0.12$, $S_n = 590.52$, and $a_n = 787.32$.

68. Find n for $4.1 + 8.2 + 16.4 + \cdots$ if $S_n = 61.5$.

69. If $15 - 18 + 21.6 - \cdots$, $S_n = 23.784$, find a_n.

70. If $r = -0.4$, $S_5 = 144.32$, and $a_1 = 200$, find a_5.

71. Find a_1 if $S_n = 468$, $a_n = 375$, and $r = 5$.

72. If $S_n = \frac{61}{40}$, $\frac{5}{8} + \frac{1}{2} + \frac{2}{5} + \cdots$, find n.

73. **LOANS** Marc is making monthly payments on a loan. Suppose instead of the same monthly payment, the bank requires a low initial payment that grows at an exponential rate each month. The total cost of the loan is represented by $\sum_{n=1}^{k} 5(1.1)^{n-1}$.

 a. What is Marc's initial payment and at what rate is this payment increasing?

 b. If the sum of Marc's payments at the end of the loan is \$7052, how many payments did Marc make?

Find the common ratio for the geometric sequence with the given terms.

74. $a_3 = 12$, $a_6 = 187.5$

75. $a_2 = -6$, $a_7 = -192$

76. $a_4 = -28$, $a_6 = -1372$

77. $a_5 = 6$, $a_8 = -0.048$

78. **ADVERTISING** Word-of-mouth advertising can be an effective form of marketing, or it can be very harmful. Consider a new restaurant that serves 27 customers on its opening night.

 a. Of the 27 customers, 25 found the experience enjoyable and each told 3 friends over the next month. This group each told 3 friends over the next month, and so on, for 6 months. Assuming that no one heard twice, how many people have had a positive experience or heard positive reviews of the restaurant?

 b. Suppose the 2 unhappy customers each told 6 friends over the next month about the experience. This group then each told 6 friends, and so on, for 6 months. Assuming that no one heard a review twice, how many people have had a negative experience or heard a negative review?

Write the first 3 terms of the infinite geometric series with the given characteristics.

79. $S = 12$, $r = \frac{1}{2}$

80. $S = -25$, $r = 0.2$

81. $S = 44.8$, $a_1 = 56$

82. $S = \frac{2}{3}$, $a_1 = \frac{8}{9}$

83. $S = -60$, $r = 0.4$

84. $S = -126.25$, $a_1 = -50.5$

85. $S = -115$, $a_1 = -138$

86. $S = \frac{891}{20}$, $r = -\frac{1}{9}$

87. $\sum_{n=1}^{\infty} 12\left(-\frac{1}{4}\right)^{n-1}$

88. $\sum_{n=1}^{\infty} \frac{1}{6}\left(\frac{2}{3}\right)^{n-1}$

Determine whether each sequence is *arithmetic*, *geometric*, or *neither*. Then find the next three terms of the sequence.

89. $\frac{1}{4}, \frac{2}{6}, \frac{3}{8}, \frac{4}{10}, \ldots$

90. $\frac{9}{2}, \frac{17}{4}, 4, \frac{15}{4}, \ldots$

91. $12, 24, 36, 48, \ldots$

92. $128, 96, 72, 54, \ldots$

93. $36k, 49k, 64k, 81k, \ldots$

94. $7.2y, 9.1y, 11y, 12.9y, \ldots$

95. $3\sqrt{5}, 15, 15\sqrt{5}, 75, \ldots$

96. $2\sqrt{3}, 2\sqrt{6}, 2\sqrt{9}, 2\sqrt{12}, \ldots$

Write each geometric series in sigma notation.

97. $3 + 12 + 48 + \cdots + 3072$

98. $9 + 18 + 36 + \cdots + 1152$

99. $50 + 85 + 144.5 + \cdots + 417.605$

100. $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - \cdots + 8$

101. $0.2 - 1 + 5 - \cdots - 625$

102. **HORSES** For each of the first few months after a horse is born, the amount of weight that it gains is about 120% of the previous month's weight gain. In the first month, a horse has gained 30 pounds.

a. Write a geometric series in sigma notation that can be used to model the horse's weight gain for the first five months.

b. About how much weight did the horse gain in the fourth month?

c. If the horse weighed 150 pounds at birth, about how much did it weigh after 5 months?

d. Will the horse continue to grow at this rate indefinitely? Explain.

103. **MEDICINE** A newly developed and researched medicine has a half-life of about 1.5 hours after it is administered. The medicine is given to patients in doses of d milligrams every 6 hours.

a. What fraction of the first dose will be left in the patient's system when the second dose is taken?

b. Find the first four terms of the sequence that represents the amount of medicine in the patient's system after the first 4 doses.

c. Write a recursive formula that can be used to determine the amount of medicine in the patient's system after the nth dose.

104. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the limits of $\frac{1-r^n}{1-r}$.

a. **GRAPHICAL** Graph $S_n = \frac{1-r^n}{1-r}$ for $r = 0.2$, 0.5, and 0.9 on the same graph.

b. **TABULAR** Copy and complete the table shown below.

n	$S_n, r = 0.2$	$S_n, r = 0.5$	$S_n, r = 0.9$
0			
4			
8			
12			
16			
20			
24			

c. **ANALYTICAL** For each graph in part **a**, describe the values of S_n as $n \to \infty$.

d. **GRAPHICAL** Graph $S_n = \frac{1-r^n}{1-r}$ for $r = 1.2$, 2.5, and 4 on the same graph.

e. **ANALYTICAL** For each graph in part **d**, describe the values of S_n as $n \to \infty$.

f. **ANALYTICAL** Make a conjecture about what happens to S_n as $n \to \infty$ for $S_n = \frac{1-8.6^n}{1-8.6}$.

H.O.T. Problems Use Higher-Order Thinking Skills

105. **ERROR ANALYSIS** Emilio believes that the sum of the infinite geometric series 16 + 4 + 1 + 0.25 + ... can be calculated. Annie disagrees. Is either of them correct? Explain your reasoning.

106. **CHALLENGE** A ball is dropped from a height of 5 meters. On each bounce, the ball rises to 65% of the height it reached on the previous bounce.

a. Approximate the total vertical distance the ball travels, until it stops bouncing.

b. The ball makes its first complete bounce in 2 seconds, that is, from the moment it first touches the ground until it next touches the ground. Each complete bounce that follows takes 0.8 times as long as the preceding bounce. Estimate the total amount of time that the ball bounces.

107. **WRITING IN MATH** Explain why an infinite geometric series will not have a sum if $|r| > 1$.

REASONING **Determine whether each statement is *true* or *false*. Explain your reasoning.**

108. If the first two terms of a geometric sequence are positive, then the third term is positive.

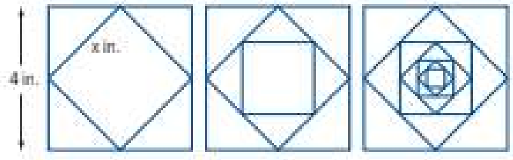
109. If you know r and the sum of a finite geometric series, you can find the last term.

110. If r is negative, then the geometric sequence converges.

111. **REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

If all of the terms of an infinite geometric series are negative, then the series has a sum that is a negative number.

112. **CHALLENGE** The midpoints of the sides of a square are connected so that a new square is formed. Suppose this process is repeated indefinitely.



a. What is the perimeter of the square with side lengths of x inches?

b. What is the sum of the perimeters of all the squares?

c. What is the sum of the areas of all the squares?

Spiral Review

Find each sum.

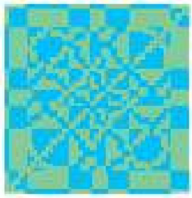
113. $\sum_{n=1}^{7}(2n+1)$

114. $\sum_{n=3}^{7}(3n+4)$

115. $\sum_{n=1}^{150}(11+2n)$

116. **TOURIST ATTRACTIONS** To prove that objects of different weights fall at the same rate, Marlene dropped two objects with different weights from the Leaning Tower of Pisa in Italy. The objects hit the ground at the same time. When an object is dropped from a tall building, it falls about 16 feet in the first second, 48 feet in the second second, and 80 feet in the third second, regardless of its weight. If this pattern continues, how many feet would an object fall in the sixth second?

117. **TEXTILES** Patterns in fabric can often be created by modifying a mathematical graph. The pattern at the right can be modeled by a lemniscate.



a. Suppose the designer wanted to begin with a lemniscate that was 6 units from end to end. What polar equation could have been used?

b. What polar equation could have been used to generate a lemniscate that was 8 units from end to end?

Graph each polar equation on a polar grid.

118. $\theta = -\frac{\pi}{4}$

119. $r = 1.5$

120. $\theta = -150°$

Find the cross product of u and v. Then show that u × v is orthogonal to both u and v.

121. $\mathbf{u} = \langle 1, 9, -1\rangle, \mathbf{v} = \langle -2, 6, -4\rangle$

122. $\mathbf{u} = \langle -3, 8, 2\rangle, \mathbf{v} = \langle 1, -5, -7\rangle$

123. $\mathbf{u} = \langle 9, 0, -4\rangle, \mathbf{v} = \langle -6, 2, 5\rangle$

Find the component form and magnitude of $\overline{AB}$ with the given initial and terminal points. Then find a unit vector in the direction of $\overline{AB}$.

124. $A(6, 7, 9), B(18, 21, 18)$

125. $A(24, -6, 16), B(8, 12, -4)$

126. $A(3, -5, 9), B(-1, 15, -7)$

Skills Review for Standardized Tests

127. **SAT/ACT** In the geometric sequence $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \ldots$, each term after the first is equal to the previous term times a constant. What is the value of the 13th term?

A 2^7

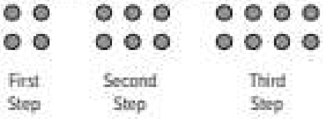
B 2^8

C 2^9

D 2^{10}

E 2^{11}

128. **REVIEW** The pattern of dots shown below continues infinitely, with more dots being added at each step.



Which expression can be used to determine the number of dots at the nth step?

F $2n$

G $n(n+2)$

H $n(n+1)$

J $2(n+1)$

129. The first term of a geometric series is -1, and the common ratio is -3. How many terms are in the series if its sum is 182?

A 6

B 7

C 8

D 9

130. **REVIEW** Cora begins a phone tree to notify her friends about a party. In step 1, she calls 3 friends. In step 2, each of those friends calls 3 new friends. In step 3, each of those friends calls 3 more new friends. After step 3, how many people know about the party, including Cora?

F 12

G 13

H 39

J 40

EXTEND 3-3

Graphing Technology Lab
Continued Fractions

Objective

- Use a graphing calculator to represent continued fractions.

An expression of the following form is called a *continued fraction.* Continued fractions can be used to write sequences that approach limits.

$$a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \ldots}}}$$

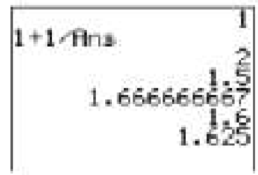
Activity 1 $a = b = 1$

Evaluate $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}$.

Step 1 On the home screen, press 1 and [ENTER].

Step 2 Then type in 1 [+] 1 [÷] [2nd] [ANS] [ENTER].

Step 3 Press [ENTER] four more times.



StudyTip
Memory You may need to clear the calculator's memory to eliminate any previously stored values.

Analyze the Results

1A. Write the expression that was evaluated to generate each of the values obtained in Steps 2–3.

1B. Press [ENTER] 20 more times and record the result to approximate the limit of this sequence. This number is an approximation for the *golden ratio*.

1C. Solve $x = 1 + \frac{1}{x}$. How do you think this relates to the golden ratio?

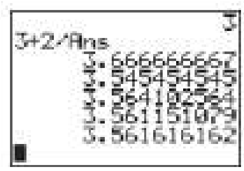
Activity 2 a = 3 and b = 2

Evaluate $3 + \cfrac{2}{3 + \cfrac{2}{3 + \cfrac{2}{3 + \ldots}}}$.

Step 1 On the home screen, press 3 and [ENTER].

Step 2 Then type in 3 [+] 2 [÷] [2nd] [ANS] [ENTER].

Step 3 Press [ENTER] four more times.



Analyze the Results

2A. Write the expression that was evaluated to generate each of the values obtained in Steps 2–3.

2B. Press [ENTER] 20 more times. What is the approximation for the limit of this sequence?

2C. **MAKE A CONJECTURE** Use $a = 3$ and the approximation found in Exercise 2B to develop an expression for both a and b. (*Hint:* Solve for the radicand, and determine how b can be used to make it equal.) This is the general expression for the limit of the continued fraction sequence for any values of a and b.

StudyTip
Finding Expressions When developing the expression in 2C, think of different ways in which b can make the expression equivalent to the limit of the sequence. Use different values of a and b to confirm your answer.

Exercises

Approximate the value of the continued fraction with the given values for a and b.

1. $a = 4$ and $b = 3$
2. $a = 5$ and $b = 2$
3. $a = 3$ and $b = 1$

CHAPTER 3 Mid-Chapter Quiz

Lessons 3-1 through 3-3

Find the next four terms of each sequence.

1. 109, 97, 85, 73, ...
2. 2, 6, 14, 30, ...
3. 0, 1, 5, 14, ...
4. −2187, 729, −243, 81, ...
5. **NATURE** A petting zoo starts a population of rabbits with one newborn male and one newborn female. Assuming that each adult pair will produce one male and one female offspring per month starting at two months, how many rabbits will there be after 6 months?

Determine whether each sequence is convergent or divergent.

6. 3, 5, 8, 12, ...
7. $a_1 = 15$, $a_n = \frac{a_{n-1} - 1}{3}$
8. 48, 24, 12, 6, ...
9. $a_n = n^2 + 5n$

Find each sum.

10. $\sum_{n=0}^{5} \frac{n^2}{4}$
11. $\sum_{n=-5}^{0} (n^3 + 7)$
12. $\sum_{n=1}^{6} (2^n - 4)$
13. $\sum_{n=8}^{13} (4n - 10)$
14. **GOLF** In a charity golf tournament, each of the top ten finishers wins a donation to the charity of his or her choice. The amount of the donation follows the arithmetic sequence shown below. What is the total amount of money donated to charity as a result of the tournament?

Write an explicit formula and a recursive formula for finding the *n*th term of each arithmetic sequence.

15. −11, −15, −19, −23, ...
16. −96, −84, −72, −60, ...
17. 7, 10, 13, 16, ...
18. 32, 30, 28, 26, ...
19. **JEWELRY** Mary Anne is hosting a jewelry party. For each guest who buys an item of jewelry, she gets a hostess bonus in the amount shown. She receives a larger amount for each guest making a purchase.

Guests Purchasing Jewelry	Amount Mary Anne Receives ($)
first	10
second	15
third	20

   a. How much will Mary Anne receive for the 12th guest who makes a purchase?

   b. If she wants a total hostess bonus of $100, how many guests need to make a purchase?

Write an explicit formula and a recursive formula for finding the *n*th term of each geometric sequence.

20. $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}, \ldots$
21. $9, -3, 1, -\frac{1}{3}, \ldots$
22. 3, 18, 108, 648, ...
23. −4, 20, −100, 500, ...
24. **POPULATION** The population of Sandy Shores is currently 55,000 and is decreasing at a rate of 3% annually.

   a. Write an explicit formula for finding the population of Sandy Shores during the *n*th year.

   b. What do you predict will be the population of Sandy Shores after 10 years?

   c. After how many years do you predict the population of Sandy shores will reach 37,000?

25. **MULTIPLE CHOICE** If possible, find the sum of the geometric series $12 + 3 + \frac{3}{4} + \frac{3}{16} + \ldots$.

   A 13.5

   B 16

   C 18

   D not possible

LESSON 3-4 Mathematical Induction

Then

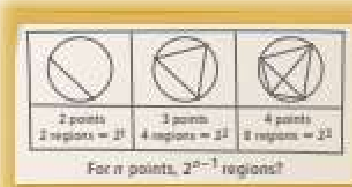
- You found the next term in a sequence or series.

Now

- **1** Use mathematical induction to prove summation formulas and properties of divisibility involving a positive integer n.
- **2** Use extended mathematical induction.

Why?

- Raini draws points on a circle and connects every pair of points with a chord, dividing the circle into regions. After drawing circles with 2, 3, and 4 points, Raini conjectures that if there are n points on a circle, then connecting each pair of points will divide the circle into 2^{n-1} regions. While his conjecture holds for $n = 2$, 3, and 4, are these three examples sufficient to prove that his conjecture is true?



NewVocabulary
principle of mathematical induction
anchor step
inductive hypothesis
inductive step
extended principle of mathematical induction

1 Mathematical Induction

When looking for patterns and making conjectures, it is often tempting to assume that if a conjecture holds for several cases, then it is true in all cases. In the situation above, Raini may be convinced that his conjecture is true once he shows that it holds for $n = 5$ because connecting 5 points does form 16 or 2^4 regions. "Proof by example," however, is *not* a logically valid method of proof because it does not show that a conjecture is true for *all* cases. In fact, you can show that Raini's conjecture fails when $n = 6$.

While all that is required to prove mathematical conjectures false is a counterexample, proving one true requires a more formal method. One such method uses the **principle of mathematical induction.** The essential idea behind the principle of mathematical induction is that a conjecture can be proven true if you can:

1. show that something works for the first case (base or **anchor step**),
2. assume that it works for any particular case (**inductive hypothesis**), and then
3. show that it works for the next case (**inductive step**).

This principle, described more formally below, is a powerful tool for proving many conjectures about positive integers.

KeyConcept The Principle of Mathematical Induction

Let P_n be a statement about a positive integer n. Then P_n is true for all positive integers n if and only if

- P_1 is true, and
- for every positive integer k, if P_k is true, then P_{k+1} is true.

To understand why the principle of mathematical induction works, imagine a ladder with an infinite number of rungs (Figure 3.4.1). If you can get on the ladder (anchor step) and then move from one rung to the next (inductive hypothesis and step), you can climb the whole ladder. Similarly, imagine an unending line of dominos (Figure 3.4.2) arranged so that if any kth domino falls, the $(k + 1)$th domino will also fall (inductive hypothesis and step). By pushing over the first domino (anchor step), you start a chain reaction that knocks down the whole line.

Figure 3.4.1

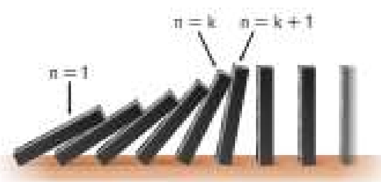


Figure 3.4.2

To apply the principle of mathematical induction, follow these steps.

Step 1 Verify that a conjecture P_n is true for $n = 1$. (Anchor Step)

Step 2 Assume that P_n is true for $n = k$. (Inductive Hypothesis)

Step 3 Use this assumption to prove that P_n is also true for $n = k + 1$. (Inductive Step)

Example 1 Prove a Summation Formula

Use mathematical induction to prove that the sum of the first n even positive integers is $n^2 + n$. That is, prove that $2 + 4 + 6 + \cdots + 2n = n^2 + n$ is true for all positive integers n.

Conjecture Let P_n be the statement that $2 + 4 + 6 + \cdots + 2n = n^2 + n$.

Anchor Step Verify that P_n is true for $n = 1$.

P_n: $2 + 4 + 6 + \cdots + 2n = n^2 + n$ — Original statement P_n

P_1: $2 = 1^2 + 1$ — P_n for $n = 1$, the first partial sum

Because $2 = 1^2 + 1$ is a true statement, P_n is true for $n = 1$.

Inductive Hypothesis Assume that P_n is true for $n = k$.

To write the inductive hypothesis, replace n with k in P_n. That is, assume that P_k: $2 + 4 + 6 + \cdots + 2k = k^2 + k$ is true.

Study Tip

Representing the Next Term To determine the $(k + 1)$th term, substitute the quantity $k + 1$ for k in the expression for the general form of the next term in the series. In Example 1, because $2k$ represents the kth term, $2(k + 1)$ represents the $(k + 1)$th term.

Inductive Step Use the inductive hypothesis to prove that P_n is true for $n = k + 1$.

To prove that P_n is true for $n = k + 1$, we need to show that P_{k+1} must be true. Start with your inductive hypothesis and then add the next term, the $(k + 1)$th term, to each side.

$2 + 4 + 6 + \cdots + 2k = k^2 + k$ — Inductive hypothesis

$2 + 4 + 6 + \cdots + 2k + 2(k + 1) = k^2 + k + 2(k + 1)$ — Add the $(k + 1)$th term to each side.

$2 + 4 + 6 + \cdots + 2k + 2(k + 1) = k^2 + k + 2k + 2$ — Simplify the right-hand side.

$2 + 4 + 6 + \cdots + 2k + 2(k + 1) = (k^2 + 2k + 1) + (k + 1)$ — Rewrite 2 as 1 + 1 and regroup.

$2 + 4 + 6 + \cdots + 2k + 2(k + 1) = (k + 1)^2 + (k + 1)$ — Factor $k^2 + 2k + 1$.

This final statement is exactly the statement for P_{k+1}, so P_{k+1} is true. It follows that if P_n is true for $n = k$, then P_n is also true for $n = k + 1$.

Watch Out!

Use the Inductive Hypothesis To show that P_n is true for $n = k + 1$, you do not substitute $k + 1$ for n on each side of the equation for P_n. Doing so would give you nothing to prove. To complete the inductive step, you *must* use the inductive hypothesis.

Conclusion Because P_n is true for $n = 1$ and P_k implies P_{k+1}, P_n is true for $n = 2$, $n = 3$, and so on. That is, by the principle of mathematical induction, P_n: $2 + 4 + 6 + \cdots + 2n = n^2 + n$ is true for all positive integers n.

Guided Practice

1. Use mathematical induction to prove that the sum of the first n even positive integers is n^2. That is, prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ is true for all positive integers n.

In Example 1, notice that you are *not* trying to prove that P_n is true for $n = k$. Instead, you *assume* that P_n is true for $n = k$ and use that assumption to show that P_n is true for the next number, $n = k + 1$. If while trying to complete either the anchor step or the inductive step you arrive at a contradiction, then the assumption you made in your inductive hypothesis *may be* false. For example, if there is a contradiction in the anchor step, then you know it does not work for *that value only*. The inductive hypothesis may still be true.

Mathematical induction can be used to prove divisibility. Recall that an integer p is *divisible* by an integer q if $p = qr$ for some integer r.

Example 2 Prove Divisibility

Prove that $3^n - 1$ is divisible by 2 for all positive integers n.

Conjecture and Anchor Step Let P_n be the statement that $3^n - 1$ is divisible by 2. P_1 is the statement that $3^1 - 1$ is divisible by 2. P_1 is true because $3^1 - 1$ is 2, which is divisible by 2.

Tip
Inductive Step There is no "fixed" way of completing the inductive step for a proof by mathematical induction. Each problem has its own special characteristics that require a different technique to complete the proof.

Inductive Hypothesis and Step Assume that $3^k - 1$ is divisible by 2. That is, assume that $3^k - 1 = 2r$ for some integer r. Use this hypothesis to show that $3^{k+1} - 1$ is divisible by 2.

$3^k - 1 = 2r$ — Inductive hypothesis

$3^k = 2r + 1$ — Add 1 to each side.

$3 \cdot 3^k = 3(2r + 1)$ — Multiply each side by 3.

$3^{k+1} = 6r + 3$ — Simplify.

$3^{k+1} - 1 = 6r + 2$ — Subtract 1 from each side.

$3^{k+1} - 1 = 2(3r + 1)$ — Factor.

Because r is an integer, $3r + 1$ is an integer and $2(3r + 1)$ is divisible by 2. Therefore, $3^{k+1} - 1$ is divisible by 2.

Conclusion Because P_n is true for $n = 1$ and P_k implies P_{k+1}, P_n is true for $n = 2$, $n = 3$, and so on. By the principle of mathematical induction, $3^n - 1$ is divisible by 2 for all positive integers n.

Guided Practice

2. Prove that $4^n - 1$ is divisible by 3 for all positive integers n.

You can also prove statements of inequality using mathematical induction.

Example 3 Prove Statements of Inequality

Study Tip
Proofs of Inequalities The approach of showing that the difference between some quantity and $k + 1$ is greater than or less than zero, along with the Transitive Property of Inequality, is used in many inequality proofs by mathematical induction.

Prove that $n < 2^n$ for all positive integers n.

Conjecture and Anchor Step Let P_n be the statement $n < 2^n$. P_1 and P_2 are true, since $1 < 2^1$ and $2 < 2^2$ are true inequalities. Showing P_2 to be true provides the anchor for the second part of our inductive hypothesis below.

Inductive Hypothesis and Step Assume that $k < 2^k$ is true for a positive $k > 1$. Use both parts of this inductive hypothesis to show that $k + 1 < 2^{k+1}$ is true.

$k < 2^k$ — Inductive hypothesis — $k > 1$

$2 \cdot k < 2 \cdot 2^k$ — $k - 1 > 0$

$2k < 2^{k+1}$ — $2k - k - 1 > 0$

$2k - (k + 1) > 0$

$2k > k + 1$

$k + 1 < 2k$

By the Transitive Property of Inequality, if $k + 1 < 2k$ and $2k < 2^{k+1}$, then $k + 1 < 2^{k+1}$.

Conclusion Because P_n is true for $n = 1$ and 2 and P_k implies P_{k+1} for $k \geq 2$, P_n is true for $n = 3$, $n = 4$, and so on. By the principle of mathematical induction, $n < 2^n$ is true for all positive integers n.

Guided Practice

3. Prove that $2n < 3^n$ for all positive integers n.

2 Extended Mathematical Induction

Sometimes you will be asked to prove a statement that is true for an arbitrary value greater than 1. In situations like this, you can use a variation on the principle of mathematical induction called the **extended principle of mathematical induction.** Instead of verifying that P_n is true for $n = 1$, you can instead verify that P_n is true for the first possible case.

Example 4 Use Extended Mathematical Induction

Prove that $n! > 2^n$ for integer values of $n \geq 4$.

Conjecture and Anchor Step Let P_n be the statement $n! > 2^n$. P_4 is true since $4! > 2^4$ or $24 > 16$ is a true statement.

Inductive Hypothesis and Step Assume that $k! > 2^k$ is true for a positive integer $k \geq 4$. Show that $(k+1)! > 2^{k+1}$ is true. Use this inductive hypothesis and its restriction that $k \geq 4$.

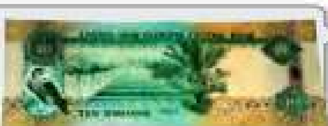
$k! > 2^k$	Inductive hypothesis
$(k+1) \cdot k! > (k+1) \cdot 2^k$	Multiply each side by $k + 1$.
$(k+1)! > (k+1) \cdot 2^k$	$(k+1) \cdot k! = (k+1)!$
$(k+1)! > (k+1) \cdot 2^k > 2 \cdot 2^k$	$k + 1 > 2$ is true for $k \geq 4$; therefore by the Multiplication Property of Inequality $(k+1) \cdot 2^k > 2 \cdot 2^k$.
$(k+1)! > 2 \cdot 2^k$	Transitive Property of Inequality
$(k+1)! > 2^{k+1}$	Simplify.

Therefore, $(k+1)! > 2^{k+1}$ is true.

Conclusion Because P_n is true for $n = 4$ and P_k implies P_{k+1} for $k \geq 4$, P_n is true for $n = 5$, $n = 6$, and so on. That is, by the extended principle of mathematical induction, $n! > 2^n$ is true for integer values of $n \geq 4$.

GuidedPractice

4. Prove that $n! > 3^n$ for integer values of $n \geq 7$.



Real-WorldLink

Most automatic teller machines dispense withdrawals in $10 or $20 increments, but a few can go as low as $5.

Source: *The Economist*

Real-World Example 5 Apply Extended Mathematical Induction

MONEY **Prove that all multiples of $10 greater than $40 can be formed using only $20 and $50 bills.**

Conjecture and Anchor Step P_n: There exists a set of $20 and $50 bills that adds to $10n for $n > 4$. For $n = 5$, the first possible case, the conjecture is true because $10(5) = $20(0) + $50(1).

Inductive Hypothesis and Step Assume that there exists a set of $20 and/or $50 bills that adds to $10k. Show that this implies the existence of a set of $20 and/or $50 bills that adds to $10(k + 1).

Case 1 The set contains at least one $50 bill. Replace one $50 bill in the set with three $20 bills and the value of the set is increased by $10 to $10k + 10 or $10(k + 1), which is exactly P_{k+1}.

Case 2 The set contains no $50 bills. The set must contain at least three $20 bills because the value of the set must be greater than $40. Replace two of the $20 bills with a $50 bill and the value of the set is increased by $10, to $10k + 10 or $10(k + 1), which is exactly P_{k+1}.

Conclusion In both cases, P_n is true for $n = k + 1$. Because P_n is true for $n = 5$ and P_k implies P_{k+1} for $k \geq 5$, P_n is true for $n = 6$, $n = 7$, and so on. That is, by the extended principle of mathematical induction, all multiples of $10 greater than $40 can be formed using just $20 and $50 bills.

GuidedPractice

5. AMUSEMENT Prove that all rides at the fair requiring more than 7 tickets can be paid for using only 3-ticket and 5-ticket vouchers offered by the school for donations of canned goods.

Exercises

Use mathematical induction to prove that each conjecture is true for all positive integers *n*.

1. $3 + 5 + 7 + \cdots + (2n + 1) = n(n + 2)$

2. $1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$

3. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$

4. $3 + 7 + 11 + \cdots + (4n - 1) = 2n^2 + n$

5. $1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n - 1)}{2}$

6. $1 + 8 + 27 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4}$

7. $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$

8. $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}$

9. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

10. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}$

11. **TRIANGULAR NUMBERS** Triangular numbers are numbers that can be represented by a triangular array of dots, with *n* dots on each side. The first three triangular numbers are 1, 3, and 6.

   a. Find the next five triangular numbers.

   b. Write a general formula for the *n*th term of this sequence.

   c. Prove that the sum of first *n* triangular numbers can be found using $S_n = \frac{n(n + 1)(n + 2)}{6}$.

12. **ICEBREAKER** At freshman orientation, students are separated into groups to play an icebreaker game. The game requires each student in a group to have one individual interaction with every other student in the group.

   a. Develop a formula to calculate the total number of interactions taking place during the icebreaker for a group of *n* students.

   b. Prove that the formula is true for all positive integer values of *n*.

   c. Determine the length of the icebreaker game in minutes for a group of 12 students if each interaction is allotted 30 seconds.

Prove that each conjecture is true for all positive integers *n*.

13. $9^n - 1$ is divisible by 8.

14. $6^n + 4$ is divisible by 5.

15. $2^{3n} - 1$ is divisible by 7.

16. $5^n - 2^n$ is divisible by 3.

Prove that each inequality is true for the indicated values of *n*.

17. $3^n \geq 3n, n \geq 1$

18. $n! > 4^n, n \geq 9$

19. $2^n > 2n, n \geq 3$

20. $9n < 3^n, n \geq 4$

21. $3n < 4^n, n \geq 1$

22. $2^n > 10n + 7, n \geq 10$

23. $2n < 1.5^n, n \geq 7$

24. $1.5^n > 10 + 0.5n, n \geq 7$

25. **POSTAGE** Prove that all postage greater than 20¢ can be formed using just 5¢ and 6¢ stamps.

26. **ENTERTAINMENT** All of the activities at the Family Fun Entertainment Center, such as video games, paintball, and go-kart racing, require tokens worth 25¢, 50¢, 75¢, and so on. Prove that all of the activities costing more than $1.50 can be paid for using just 50¢ and 75¢ tokens.

27. **OBLONG NUMBERS** Oblong numbers are numbers that can be represented by a rectangular array having one more column than rows.

   Prove that the sum of the first *n* oblong numbers is given by $S_n = \frac{n^3 + 3n^2 + 2n}{3}$.

Use mathematical induction to prove that each conjecture is true for all positive integers *n*.

28. $\sum_{a=1}^{n} (4a - 3) = n(2n - 1)$

29. $\sum_{a=1}^{n} (3a - 2) = \frac{n}{2}(3n - 1)$

30. $\sum_{a=1}^{n} (a^2 + a) = \frac{n(n + 1)(n + 2)}{3}$

31. $\sum_{a=1}^{n} \frac{1}{4a^2 - 1} = \frac{n}{2n + 1}$

32. $\sum_{a=1}^{n} \frac{1}{2a(a + 1)} = \frac{n}{2(n + 1)}$

33. $\sum_{a=1}^{n} \frac{1}{(a + 1)(a + 2)} = \frac{n}{2(n + 2)}$

Prove that each statement is true for all positive integers n or find a counterexample.

34. $1 + 6 + 11 + \cdots + (5n - 4) = n(2n - 1)$

35. $\frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \frac{1}{6 \cdot 8} + \cdots + \frac{1}{2n(2n + 2)} = \frac{n}{2(2n + 2)}$

36. $n^2 - n + 5$ is prime.

37. $3^n + 4n + 1$ is divisible by 4.

38. $4^n + 6n - 1$ is divisible by 9.

39. $2^{2n+1} + 3^{2n+1}$ is divisible by 5.

Prove the inequality for the indicated integer values of n and indicated values of a, b, and x.

40. $\left(\frac{a}{b}\right)^n > \left(\frac{a}{b}\right)^{n+1}$, $n \geq 1$ and $0 < a < b$

41. $(x + 1)^n \geq nx$, $n \geq 1$ and $x \geq 1$

42. $(1 + a)^n > 1 + na$, $n \geq 2$ and $a > 0$

Find a formula for the sum S_n of the first n terms of each sequence. Then prove that your formula is true using mathematical induction.

43. $2, 6, 10, 14, \ldots, (4n - 2)$

44. $2, 7, 12, 17, \ldots, (5n - 3)$

45. $-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{16}, \ldots, -\frac{1}{2^n}$

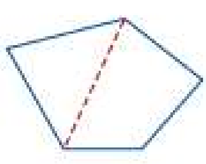
46. $\frac{1}{6}, \frac{1}{18}, \frac{1}{36}, \frac{1}{60}, \ldots, \frac{1}{3n(n + 1)}$

47. **FIBONACCI NUMBERS** In the Fibonacci sequence, 1, 1, 2, 3, 5, 8, ..., each element after the first two is found by adding the previous two terms. Numbers in the Fibonacci sequence occur throughout nature. For example, the number of scales in the clockwise and counterclockwise spirals on a pinecone are Fibonacci numbers. If f_n represents the nth Fibonacci number, prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.

48. **COMPLEX NUMBERS** Prove that DeMoivre's Theorem for finding the power of a complex number written in polar form is true for any positive integer n.

$$[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)$$

49. **GEOMETRY** According to the Interior Angle Sum Formula, the sum of the measures of the interior angles of a convex polygon with n sides is $180(n - 2)$ degrees. Use extended mathematical induction to prove this formula for $n \geq 3$.



Use mathematical induction to prove that each conjecture is true for all positive integers n.

50. $(xy)^n = x^n y^n$

51. $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

52. $x^{-n} = \frac{1}{x^n}$

53. $\cos n\pi = (-1)^n$

Use mathematical induction to prove each formula for the sum of the first n terms in a series.

54. $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$

55. $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$

H.O.T. Problems — Use Higher-Order Thinking Skills

56. **CHALLENGE** Prove that $n! < n^n$ when $n > 1$.

57. **OPEN ENDED** Consider the following statement.

$a^n + b$ is divisible by c.

a. Make a divisibility conjecture by replacing a, b, and c with positive integers.

b. Use mathematical induction to prove that the conjecture you found in part **a** is true. If it is not true, find a counterexample.

58. **REASONING** Describe the error in the proof by mathematical induction shown below.

Conjecture and Anchor Step
Let P_n be the statement that in a room with n students, all of the students have the same birthday. When $n = 1$, P_1 is true since one student has only one birthday.

Inductive Hypothesis and Step
Assume that in a room with k students, all of the students have the same birthday. Suppose $k + 1$ students are in a room. If one student leaves, then the remaining k students must have the same birthday, according to the inductive hypothesis. Then if the first student returns and another student leaves, then that group (one) of k students must have the same birthday. So, P_n is true for $n = k + 1$. Therefore, P_n is true for all positive integers n. That is, in a room with n students, all of the students have the same birthday.

59. **CHALLENGE** If a_n is represented by the sequence

$$\sqrt{6}, \sqrt{6 + \sqrt{6}}, \sqrt{6 + \sqrt{6 + \sqrt{6}}}, \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}, \ldots,$$

prove that the nth term in this sequence is always less than 3.

60. **WRITING IN MATH** In the inductive hypothesis step of mathematical induction, you assume that P_n is true for $n = k$. Explain why you cannot simply assume that P_n is true for n.

Spiral Review

Find the specified nth term of each geometric sequence.

61. a_9 for $a_1 = \frac{1}{5}, 1, 5, \ldots$

62. $a_4 = 16, r = 0.5, n = 8$

63. $a_6 = 3, r = 2, n = 12$

64. **GAMES** In a game, the first person in a group states his or her name and an interesting fact about himself or herself. The next person must repeat the first person's name and fact and then say his or her own information. Each person must repeat the information for all those who preceded him or her. If there are 20 people in a group, what is the total number of times the names and facts will be stated?

Graph each number in the complex plane, and find its absolute value.

65. $z = 5 - 3i$

66. $z = -9 - 8i$

67. $z = 2 + 6i$

Find rectangular coordinates for each point with the given polar coordinates.

68. $\left(3, -\frac{5\pi}{4}\right)$

69. $\left(2, \frac{7\pi}{6}\right)$

70. $(-4, 1.4)$

Write the component form of each vector.

71. **w** lies in the xz plane, has a magnitude of 2, and makes a 45° angle to the left of the positive z-axis.

72. **u** lies in the yz plane, has a magnitude of 9, and makes a 30° angle to the right of the negative z-axis.

For each equation, identify the vertex, focus, axis of symmetry, and directrix. Then graph the parabola.

73. $(x - 4)^2 = -(y + 7)$

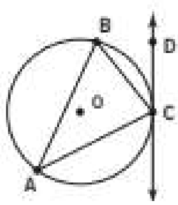
74. $6(x + 6) = (y - 4)^2$

75. $(y - 6)^2 = 4x$

76. **BUSINESS** A factory is making skirts and dresses from the same fabric. Each skirt requires 1 hour of cutting and 1 hour of sewing. Each dress requires 2 hours of cutting and 3 hours of sewing. The cutting and sewing departments can work up to 120 and 150 hours each week, respectively. If profits are \$12 for each skirt and \$18 for each dress, how many of each item should the factory make for maximum profit?

Skills Review for Standardized Tests

77. **SAT/ACT** Triangle ABC is inscribed in circle O. $\overleftrightarrow{CD}$ is tangent to circle O at point C. If $m\angle BCD = 40°$, find $m\angle A$.



A 60°

B 50°

C 40°

D 30°

E 25°

78. Which of the following is divisible by 2 for all positive integers n?

F $1^n - 1$

G $2^n - 1$

H $3^n - 1$

J $4^n - 1$

79. **REVIEW** What is the first term in the arithmetic sequence?

$\_\_\_, 8\frac{1}{3}, 7, 5\frac{2}{3}, 4\frac{1}{3}, \ldots$

A 3

B $9\frac{2}{3}$

C $10\frac{1}{3}$

D 11

80. **REVIEW** What is the tenth term in the arithmetic sequence that begins 10, 5.6, 1.2, −3.2, ...?

F −39.6

G −29.6

H 29.6

J 39.6

LESSON 3-5 The Binomial Theorem

Then

- You represented infinite series using sigma notation.

Now

- **1** Use Pascal's triangle to write binomial expansions.
- **2** Use the Binomial Theorem to write and find the coefficients of specified terms in binomial expansions.

Why?

- Suppose a biologist studying an endangered species of gibbon has found that on average 80% of gibbon offspring are female and 20% are male. Zoo workers anticipate that their captive gibbons will produce n offspring and want to know the probability that none of these offspring will be male. The biologist can use a term from the binomial expansion of $(0.8 + 0.2)^n$ to solve this problem.

NewVocabulary
binomial coefficients
Pascal's triangle
Binomial Theorem

1 Pascal's Triangle

Recall that a *binomial* is an algebraic expression involving the sum of two unlike terms. An important series is generated by the expansion of a binomial raised to an integer power. Examine this series generated by the expansion of $(a + b)^n$ for several nonnegative integer values of n.

$$(a+b)^0 = 1a^0b^0$$
$$(a+b)^1 = 1a^1b^0 + 1a^0b^1$$
$$(a+b)^2 = 1a^2b^0 + 2a^1b^1 + 1a^0b^2$$
$$(a+b)^3 = 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$$
$$(a+b)^4 = 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$$
$$(a+b)^5 = 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5$$

Observe the following patterns in the expansions of $(a + b)^n$ above.

- Each expansion has $n + 1$ terms.
- The first term is a^n, and the last term is b^n.
- In successive terms, the powers of a decrease by 1 and the powers of b increase by 1.
- The sum of the exponents in each term is n.
- The coefficients, in red above, increase and then decrease in a symmetric pattern.

If just the coefficients of these expansions, called the **binomial coefficients**, are extracted and arranged in a triangular array, they form a pattern called **Pascal's triangle**, named after the French mathematician Blaise Pascal. The top row in this triangle is called the *zeroth row* because it corresponds to the binomial expansion of $(a + b)^0$.

1	0th row
1 1	1st row
1 2 1	2nd row
1 3 3 1	3rd row
1 4 6 4 1	
1 5 10 10 5 1	
1 6 15 20 15 6 1	

Notice that the first and last numbers in each row are 1 and every other number is formed by adding the two numbers immediately above that number in the previous row. Pascal's triangle can be extended indefinitely using the recursive relationship that the coefficients in the $(n - 1)$th row can be used to determine the coefficients in the nth row.

By extending Pascal's triangle and using the patterns observed in the first 5 expansions of $(a + b)^n$, you can expand a binomial raised to any whole number power.

Example 1 Power of a Binomial Sum

Use Pascal's triangle to expand each binomial.

a. $(a + b)^7$

Step 1 Write the series for $(a + b)^7$, omitting the coefficients. Because the power is 7, this series should have 7 + 1 or 8 terms. Use the pattern of increasing and decreasing exponents to complete the series.

$$a^7b^0 + a^6b^1 + a^5b^2 + a^4b^3 + a^3b^4 + a^2b^5 + a^1b^6 + a^0b^7$$

Exponents of a decrease from 7 to 0.
Exponents of b increase from 0 to 7.

Step 2 Use the numbers in the seventh row of Pascal's triangle as the coefficients of the terms. To find these numbers, extend Pascal's triangle to the 7th row.

	1		6		15		20		15		6		1		6th row
		+		+		+		+		+		+			
1		7		21		35		35		21		7		1	7th row

$$(a + b)^7 = 1a^7b^0 + 7a^6b^1 + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7a^1b^6 + 1a^0b^7$$

$$= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7 \quad \text{Simplify.}$$

b. $(3x + 2)^4$

Step 1 Write the series for $(a + b)^4$, omitting the coefficients and replacing a with $3x$ and b with 2. The series has 4 + 1 or 5 terms.

$$(3x)^4(2)^0 + (3x)^3(2)^1 + (3x)^2(2)^2 + (3x)^1(2)^3 + (3x)^0(2)^4$$

Exponents of $3x$ decrease from 4 to 0.
Exponents of 2 increase from 0 to 4.

Step 2 The numbers in the 4th row of Pascal's triangle are 1, 4, 6, 4, and 1. Use these numbers as the coefficients of the terms in the series. Then simplify.

$$(3x + 2)^4 = 1(3x)^4(2)^0 + 4(3x)^3(2)^1 + 6(3x)^2(2)^2 + 4(3x)^1(2)^3 + 1(3x)^0(2)^4$$

$$= 81x^4 + 216x^3 + 216x^2 + 96x + 16$$

Guided Practice

1A. $(a + b)^8$

1B. $(2x + 3y)^5$

> **Study Tip**
> **Finding the Correct Row** The second number in any row of Pascal's triangle is always the same as the power to which the binomial is raised. For example, the second number in the 7th row of Pascal's triangle is 7.

To expand a binomial difference, first rewrite the expression as a binomial sum.

Example 2 Power of a Binomial Difference

Use Pascal's triangle to expand $(x - 4y)^5$.

Because $(x - 4y)^5 = [x + (-4y)]^5$, write the series for $(a + b)^5$, replacing a with x and b with $-4y$. Use the numbers in the 5th row of Pascal's triangle, 1, 5, 10, 10, 5, and 1, as the binomial coefficients. Then simplify.

$$(x - 4y)^5 = 1x^5(-4y)^0 + 5x^4(-4y)^1 + 10x^3(-4y)^2 + 10x^2(-4y)^3 + 5x^1(-4y)^4 + 1x^0(-4y)^5$$

$$= x^5 - 20x^4y + 160x^3y^2 - 640x^2y^3 + 1280xy^4 - 1024y^5$$

Guided Practice

Use Pascal's triangle to expand each binomial.

2A. $(2x - 7)^3$

2B. $(2x - 3y)^4$

> **Study Tip**
> **Alternating Signs** Notice that when expanding a power of a binomial difference, the signs of the terms in the series alternate.

2 The Binomial Theorem

While it is possible to expand any binomial using Pascal's triangle, the recursive method of computing the binomial coefficients makes expansions of $(a+b)^n$ for large values of n time consuming. An explicit formula for computing each binomial coefficient is developed by considering $(a+b)^n$ as the product of n factors in which each factor contributes an a or a b to each product in the expansion.

$$(a+b)^n = \underbrace{(a+b)(a+b)(a+b)(a+b)}_{n \text{ factors}}\cdots(a+b) = \ldots + \underbrace{a^{n-r}b^r}_{} + \ldots$$

If there are r letter b's, then there are $(n-r)$ letter a's.

Consider $(a+b)^3$. There are three ways to choose 1 a and 2 b's from each of its three factors to form the product ab^2, and 3 is the binomial coefficient of the ab^2 term in the expansion.

$(a+b)(a+b)(a+b) = \ldots + ab^2 + \ldots$

$(a+b)(a+b)(a+b) = \ldots + ab^2 + \ldots$

$(a+b)(a+b)(a+b) = \ldots + ab^2 + \ldots$

3 ways → $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

ReadingMath

Combinations The notations ${}_nC_r$ and $\binom{n}{r}$ are both read as *the combination of n things taken r at a time.*

Because those factors that do not contribute a b will by default contribute an a, the number of ways to form the product $a^{n-r}b^r$ can be more simply thought of as the number of ways to choose r factors to contribute a b to the product from the n factors available. This is the combination given by ${}_nC_r = \frac{n!}{(n-r)!\,r!}$, also written $\binom{n}{r}$.

KeyConcept Formula for the Binomial Coefficients of $(a+b)^n$

Words The binomial coefficient of the $a^{n-r}b^r$ term in the expansion of $(a+b)^n$ is given by ${}_nC_r = \frac{n!}{(n-r)!\,r!}$.

Example

$$(a+b)^3 = {}_3C_0a^3b^0 + {}_3C_1a^2b^1 + {}_3C_2a^1b^2 + {}_3C_3a^0b^3$$

$$= \frac{3!}{(3-0)!\,0!}a^3 + \frac{3!}{(3-1)!\,1!}a^2b + \frac{3!}{(3-2)!\,2!}ab^2 + \frac{3!}{(3-3)!\,3!}b^3$$

$$= 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

In the example above, notice that for the first term $r = 0$, for the second term $r = 1$, for the third term $r = 2$, and so on. In general, to find the binomial coefficient of the kth term in an expansion of the form $(a+b)^n$, use the formula ${}_nC_r$ and let $r = k - 1$.

Example 3 Find Binomial Coefficients

Find the coefficient of the 5th term in the expansion of $(a+b)^7$.

To find the coefficient of the 5th term, evaluate ${}_nC_r$ for $n = 7$ and $r = 5 - 1$ or 4.

${}_7C_4 = \frac{7!}{(7-4)!\,4!}$ ${}_nC_r = \frac{n!}{(n-r)!\,r!}$

$= \frac{7!}{3!\,4!}$ Subtract.

$= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{3!\,4!}$ Rewrite 7! as $7 \cdot 6 \cdot 5 \cdot 4!$ and divide out common factorials.

$= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1}$ or 35   Simplify.

The coefficient of the 5th term in the expansion of $(a+b)^7$ is 35.

CHECK From Example 1a, you know the 5th term in the expansion of $(a+b)^7$ is $35a^3b^4$. ✓

GuidedPractice

Find the coefficient of the indicated term in each binomial expansion.

3A. $(x+y)^9$, 6th term

3B. $(a-b)^{13}$, 3rd term

Example 4 Binomials with Coefficients Other than 1

Find the coefficient of the x^7y^2 term in the expansion of $(4x - 3y)^9$.

For $(4x - 3y)^9$ to have the form $(a + b)^n$, let $a = 4x$ and $b = -3y$. The coefficient of the term containing $a^{n-r}b^r$ in the expansion of $(a + b)^n$ is given by ${}_nC_r$. So, to find the binomial coefficient of the term containing a^7b^2 in the expansion of $(a + b)^9$, evaluate ${}_nC_r$ for $n = 9$ and $r = 2$.

$${}_9C_2 = \frac{9!}{(9-2)!\,2!} \quad {}_nC_r = \frac{n!}{(n-r)!\,r!}$$

$$= \frac{9!}{7!\,2!} \quad \text{Subtract.}$$

$$= \frac{9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}\,2!} \quad \text{Rewrite } 9! \text{ as } 9 \cdot 8 \cdot 7! \text{ and divide out common factorials.}$$

$$= \frac{9 \cdot 8}{2 \cdot 1} \text{ or } 36 \quad \text{Simplify.}$$

Thus, the binomial coefficient of the a^7b^2 term in $(a + b)^9$ is 36. Substitute $4x$ for a and $-3y$ for b to find the coefficient of the x^7y^2 term in the original binomial expansion.

$$36a^7b^2 = 36(4x)^7(-3y)^2 \quad a = 4x \text{ and } b = -3y$$

$$= 5{,}308{,}416x^7y^2 \quad \text{Simplify.}$$

Therefore, the coefficient of the x^7y^2 term in the expansion of $(4x - 3y)^9$ is 5,308,416.

GuidedPractice

Find the coefficient of the indicated term in each binomial expansion.

4A. $(2x - 3y)^8$, x^3y^5 term

4B. $(2p + 1)^{15}$, 11th term

You can use the coefficients of binomial expansions to solve real-world problems in which there are only two outcomes for an event. Problems that can be solved using a binomial expansion are called *binomial experiments*. Such experiments occur if and only if: (1) the experiment consists of n identical trials, (2) each trial results in *one* of two outcomes, and (3) the trials are independent.

For n independent trials of an experiment, if the probability of a success is p and the probability of a failure is $q = 1 - p$, then the term ${}_nC_x\, p^xq^{n-x}$ in the expansion of $(p + q)^n$ gives the probability of x successes for those n trials.

Real-World Example 5 Use Binomial Coefficients

BASEBALL **The probability that Andres gets a hit when at bat is $\frac{1}{5}$. What is the probability that Andres gets exactly 4 hits during his next 10 at bats?**

A success in this situation is Andres getting a hit, so $p = \frac{1}{5}$ and $q = 1 - \frac{1}{5}$ or $\frac{4}{5}$. Each at bat represents a trial, so $n = 10$. You want to find the probability that Andres succeeds 4 times out of those 10 trials, so let $x = 4$. To find this probability, find the value of the term ${}_nC_x\, p^xq^{n-x}$ in the expansion of $(p + q)^n$.

$${}_nC_x\, p^xq^{n-x} = {}_{10}C_4\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^{10-4} \quad p = \frac{1}{5},\ q = \frac{4}{5},\ n = 10, \text{ and } x = 4$$

$$= \frac{10!}{(10-4)!\,4!}\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^6 \quad {}_nC_r = \frac{n!}{(n-r)!\,r!}$$

$$\approx 0.088 \quad \text{Use a calculator.}$$

So, the probability that Andres gets 4 hits during his next 10 at bats is about 0.088 or 8.8%.

GuidedPractice

5. **COIN TOSS** A fair coin is flipped 8 times. Find the probability of each outcome.

A. exactly 3 heads

B. exactly 6 tails

Real-WorldLink

An at bat in baseball is any time that a batter faces a pitcher except when the player "(i) hits a sacrifice bunt or sacrifice fly; (ii) is awarded first base on four called balls; (iii) is hit by a pitched ball; or (iv) is awarded first base because of interference or obstruction."

Source: Major League Baseball

The formula for finding the coefficients of a binomial expansion leads us to a theorem about expanding powers of binomials called the **Binomial Theorem**.

KeyConcept Binomial Theorem

For any positive integer n, the expansion of $(a + b)^n$ is given by

$$(a+b)^n = {}_nC_0 a^n b^0 + {}_nC_1 a^{n-1} b^1 + {}_nC_2 a^{n-2} b^2 + \cdots + {}_nC_r a^{n-r} b^r + \cdots + {}_nC_n a^0 b^n,$$

where $r = 0, 1, 2, \ldots, n$.

You will prove the Binomial Theorem in Exercise 75.

TechnologyTip

Combinations To evaluate ${}_{10}C_4$ using a calculator, enter 10, select nCr from the MATH ► PRB menu, and then enter 4.

Example 6 Expand a Binomial Using the Binomial Theorem

Use the Binomial Theorem to expand each binomial.

a. $(3x - y)^4$

Apply the Binomial Theorem to expand $(a + b)^4$, where $a = 3x$ and $b = -y$.

$$\begin{aligned}(3x-y)^4 &= {}_4C_0(3x)^4(-y)^0 + {}_4C_1(3x)^3(-y)^1 + {}_4C_2(3x)^2(-y)^2 + \\ &\quad {}_4C_3(3x)^1(-y)^3 + {}_4C_4(3x)^0(-y)^4 \\ &= 1(81x^4)(1) + 4(27x^3)(-y) + 6(9x^2)(y^2) + 4(3x)(-y^3) + 1(1)(y^4) \\ &= 81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4\end{aligned}$$

b. $(2p + q^2)^5$

Apply the Binomial Theorem to expand $(a + b)^5$, where $a = 2p$ and $b = q^2$.

$$\begin{aligned}(2p+q^2)^5 &= {}_5C_0(2p)^5(q^2)^0 + {}_5C_1(2p)^4(q^2)^1 + {}_5C_2(2p)^3(q^2)^2 + {}_5C_3(2p)^2(q^2)^3 + {}_5C_4(2p)^1(q^2)^4 + \\ &\quad {}_5C_5(2p)^0(q^2)^5 \\ &= 1(32p^5)(1) + 5(16p^4)(q^2) + 10(8p^3)(q^4) + 10(4p^2)(q^6) + 5(2p)(q^8) + 1(1)(q^{10}) \\ &= 32p^5 + 80p^4q^2 + 80p^3q^4 + 40p^2q^6 + 10pq^8 + q^{10}\end{aligned}$$

GuidedPractice

6A. $(5m + 4)^3$

6B. $(8x^2 - 2y)^4$

Because a binomial expansion is a sum, the Binomial Theorem is often written using sigma notation. In addition, the notation ${}_nC_r$ is usually replaced by $\binom{n}{r}$.

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

Example 7 Write a Binomial Expansion Using Sigma Notation

Represent the expansion of $(5x - 7y)^{20}$ using sigma notation.

Apply the Binomial Theorem to represent the expansion of $(a + b)^{20}$ using sigma notation, where $a = 5x$ and $b = -7y$.

$$(5x-7y)^{20} = \sum_{r=0}^{20} \binom{20}{r} (5x)^{20-r}(-7y)^r$$

GuidedPractice

7. Represent the expansion of $(3a + 12b)^{30}$ using sigma notation.

Exercises

Use Pascal's triangle to expand each binomial.

1. $(2 + x)^4$
2. $(n + m)^5$
3. $(4a - b)^3$
4. $(x + y)^6$
5. $(3x + 2y)^7$
6. $(n - 4)^6$
7. $(3c - d)^4$
8. $(m - a)^5$
9. $(a - b)^3$
10. $(3p - 2q)^4$

Find the coefficient of the indicated term in each expansion.

11. $(x - 2)^{10}$, 5th term
12. $(4m + 1)^8$, 3rd term
13. $(x + 3y)^{10}$, 8th term
14. $(2c - d)^{12}$, 6th term
15. $(a + b)^8$, 4th term
16. $(2a + 3b)^{10}$, 5th term
17. $(x - y)^9$, 6th term
18. $(x + y)^{12}$, 7th term
19. $(x + 2)^7$, 4th term
20. $(a - 3)^8$, 5th term
21. $(2a + 3b)^{10}$, a^6b^4 term
22. $(2x + 3y)^9$, x^6y^3 term
23. $\left(x + \frac{1}{3}\right)^7$, 4th term
24. $\left(x - \frac{1}{2}\right)^{10}$, 6th term
25. $(x + 4y)^7$, x^2y^5 term
26. $(3x + 5y)^{10}$, x^6y^4 term

27. **TESTING** Alfonso is taking a test that contains a section of 16 true-false questions.
    a. How many of the possible sets of answers to these questions have exactly 12 correct answers of false?
    b. How many of the possible sets of answers to these questions have exactly 8 correct answers of true?

28. **BUSINESS** The probability of a certain sales representative successfully making a sale is $\frac{1}{5}$. The sales representative has 12 appointments this week.
    a. Find the probability that the sales representative makes no sales this week.
    b. What is the probability that the sales representative makes exactly 3 sales this week?
    c. Find the probability that the sales representative will make 10 sales this week.

29. **BIOLOGY** Refer to the beginning of the lesson. Assume that the zoo workers expect 30 gibbon offspring this year.
    a. What is the probability that there will be no male gibbon offspring this year?
    b. Find the probability that there will be exactly 2 male gibbon offspring this year.
    c. What is the probability that there will be 23 female gibbon offspring this year?

30. **BOWLING** Carol averages 2 strikes every 10 frames. What is the probability that Carol will get exactly 4 strikes in the next 10 frames?

Bowling Castle Scorecard

	1	2	3	4	5	6	7	8	9	10	Total
Carol	7 /	X	5 4	9 /	8 /	X	X	7 0	X	8 1	165
	20	39	48	66	86	113	130	137	156	165	
	1	2	3	4	5	6	7	8	9	10	Total

Use the Binomial Theorem to expand each binomial.

31. $(4t + 3)^5$
32. $(8 - 5y)^3$
33. $(2m - n)^6$
34. $(9h + 2j)^4$
35. $(3p + q)^7$
36. $(a^2 - 2b)^8$
37. $(7c^2 + 3d)^5$
38. $(2w - 4x^3)^7$

Represent the expansion of each expression using sigma notation.

39. $(2q + 3)^{15}$
40. $(m - 8n)^{25}$
41. $(11x + y)^{31}$
42. $(4a + 7b)^{19}$
43. $\left(3f - \frac{3}{4}g\right)^{22}$
44. $\left(\frac{1}{2}s - 5t\right)^{36}$

45. **COMPUTER GAMES** In a computer game, when a treasure chest is opened, it contains either gold coins or rocks. The probability that it contains gold coins is $\frac{5}{6}$.
    a. The treasure chest is opened 15 times per game. In one game, how many different ways is it possible to open the chest and find gold coins exactly 9 times?
    b. What is the probability that a person playing the game will find gold in the chest more than 12 times?

46. **COMMUNITY OUTREACH** At a food bank, canned goods are received and distributed to people in the community who are in need. Volunteers check the quality of the food before distribution. The probability that a canned good received at a food bank is distributed to the needy is $\frac{4}{5}$.
    a. A volunteer checks 30 canned goods per hour. In one hour, how many different ways is it possible to check a canned good and donate it exactly 23 times?
    b. What is the probability that a volunteer checking canned goods will find themselves throwing out items less than 4 times?

Use the Binomial Theorem to expand and simplify each expression.

47. $(2d + \sqrt{5})^4$
48. $(\sqrt{a} - \sqrt{b})^5$
49. $\left(4s + \frac{1}{2}t\right)^5$
50. $\left(\frac{1}{y} - 3z\right)^6$

Find the coefficient of the indicated term in each expansion.

51. $(k - \sqrt{5})^9$, 5th term

52. $(\sqrt{2} + 2c)^{10}$, middle term

53. $\left(\frac{1}{4}p + q\right)^{11}$, 7th term

54. $\left(\sqrt{h} - 3\sqrt{j}\right)^{11}$, 6th term

Use the Binomial Theorem to expand and simplify each power of a complex number.

55. $(i + 2)^4$

56. $(i - 3)^3$

57. $(1 - 4i)^5$

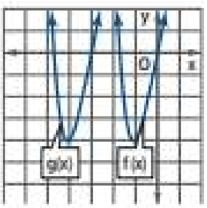
58. $(2 + \sqrt{7}i)^4$

59. $\left(\frac{\sqrt{2}}{2}i - \frac{1}{2}\right)^3$

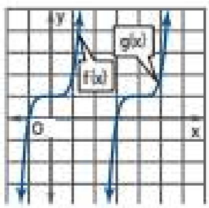
60. $\left(\sqrt{-16}i + 3\right)^5$

The graph of $g(x)$ is a translation of the graph of $f(x)$. Use the Binomial Theorem to find the polynomial function for $g(x)$ in standard form.

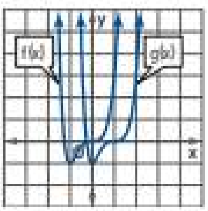
61. $f(x) = x^4 + 5x$
$g(x) = f(x + 3)$



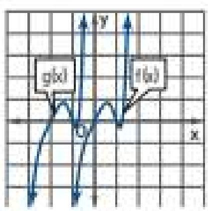
62. $f(x) = x^5 + 1$
$g(x) = f(x - 4)$



63. $f(x) = x^6 + 2x^3$
$g(x) = f(x - 1)$



64. $f(x) = x^7 - 3x^4 + 2x$
$g(x) = f(x + 2)$



65. **MULTIPLE REPRESENTATIONS** In this problem, you will use the Binomial Theorem to investigate the difference quotient $\frac{f(x + h) - f(x)}{h}$ for power functions.

a. **ANALYTICAL** Use the Binomial Theorem to expand and simplify the difference quotient for $f(x) = x^3$, $f(x) = x^4$, $f(x) = x^5$, $f(x) = x^6$, and $f(x) = x^7$. Use the pattern to simplify the difference quotient for $f(x) = x^n$.

b. **TABULAR** Evaluate each expression in part **a** for $h = 0.1$, 0.01, 0.001, and 0.0001 and record your results in a table. What do you observe?

c. **GRAPHICAL** Graph the set of resulting functions from part **b** for $f(x) = x^3$ on the same coordinate plane. What do you observe?

d. **ANALYTICAL** As h approaches 0, write an expression for the difference quotient when $f(x) = x^n$, where n is a positive integer.

66. In the expansion of $(ax + b)^5$ the numerical coefficient of the second term is 400 and the numerical coefficient of the third term is 2000. Find the values of a and b.

67. **RESEARCH** Although Pascal's triangle is named for Blaise Pascal, other mathematicians applied their knowledge of the triangle hundreds of years before Pascal. Use the internet or another source to research at least one other person who used the properties of the triangle before Pascal was born. Then describe other patterns found in Pascal's triangle that are not described in this lesson.

H.O.T. Problems Use Higher-Order Thinking Skills

68. **ERROR ANALYSIS** Jena and Gil are finding the 6th term of the expansion of $(x + y)^{14}$. Jena says that the coefficient of the term is 3003. Gil thinks that it is 2002. Is either of them correct? Explain your reasoning.

69. **CHALLENGE** Describe a strategy that uses the Binomial Theorem to expand $(x + y + z)^n$. Then write and simplify an expansion for the expression.

70. **PROOF** The sums of the coefficients in the first five rows of Pascal's triangle are shown below.

row 0	1	$= 2^0$
row 1	$1 + 1$	$= 2^1$
row 2	$1 + 2 + 1$	$= 2^2$
row 3	$1 + 3 + 3 + 1$	$= 2^3$
row 4	$1 + 4 + 6 + 4 + 1$	$= 2^4$
row 5	$1 + 5 + 10 + 10 + 5 + 1$	$= 2^5$

Prove that the sum of the coefficients in the nth row of Pascal's triangle is 2^n. (*Hint:* Write 2^n as $(1 + 1)^n$. Then use the Binomial Theorem to expand.)

71. **WRITING IN MATH** Describe how to find the numbers in each row of Pascal's triangle. Then write a few sentences to describe how the expansions of $(a + b)^{n-1}$ and $(a - b)^n$ are different from the expansion of $(a + b)^n$.

72. **REASONING** Determine whether the following statement is *sometimes*, *always*, or *never* true. Justify your reasoning.

If a binomial is raised to the power 5, the two middle terms of the expansion have the same coefficients.

73. **CHALLENGE** Explain how you could find a term in the expansion of $\left(\frac{1}{2v} + 6v^7\right)^8$ that does not contain the variable v. Then find the term.

74. **PROOF** Use the principle of mathematical induction to prove the Binomial Theorem.

Spiral Review

Prove that each statement is true for all positive integers n or find a counterexample.

75. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(3n-1)}{2}$

76. $10^{2n-1} + 1$ is divisible by 11.

77. **GENEALOGY** In the book *Roots*, author Alex Haley traced his family history back many generations. If you could trace your family back for 15 generations, starting with your parents, how many ancestors would there be?

78. Let $\overrightarrow{DE}$ be the vector with initial point $D(5, -12)$ and terminal point $E(8, -17)$. Write $\overrightarrow{DE}$ as a linear combination of the vectors **i** and **j**.

Write each equation in standard form. Identify the related conic.

79. $x^2 + y^2 - 16x + 10y + 64 = 0$

80. $y^2 + 16x - 10y + 57 = 0$

81. $x^2 + y^2 + 2x + 24y + 141 = 0$

Find the partial fraction decomposition of each rational expression.

82. $\dfrac{5x^2 - 14}{(x^2 - 2)^2}$

83. $\dfrac{x^3 - 8x^2 + 21x - 22}{x^2 - 8x + 15}$

84. $\dfrac{3x}{(x-3)^2}$

Determine whether each matrix is in row-echelon form.

85. $\left[\begin{array}{ccc|c} 1 & 10 & -5 & 3 \\ 0 & 1 & 14 & -2 \\ 0 & 1 & 9 & 6 \end{array}\right]$

86. $\left[\begin{array}{ccc|c} 1 & 3 & -7 & 11 \\ 0 & 1 & -13 & 18 \\ 0 & 0 & 0 & 1 \end{array}\right]$

87. $\left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{array}\right]$

Find the exact value of each trigonometric expression.

88. $\tan 195°$

89. $\csc \dfrac{5\pi}{12}$

90. $\sin \dfrac{\pi}{12}$

91. **SAVINGS** Janet's father deposited \$30 into a bank account for her. They forgot about the money and made no further deposits or withdrawals. The table shows the account balance for several years.

a. Make a scatter plot of the data.

b. Find an exponential function to model the data.

c. Use the function to predict the balance of the account in 41 years.

Elapsed Time (yr)	Balance (\$)
0	30.00
5	41.10
10	56.31
15	77.16
20	105.71
25	144.83
30	198.43

Skills Review for Standardized Tests

92. **SAT/ACT** In the figure below, $ABCD$ is a parallelogram. What are the coordinates of point C?

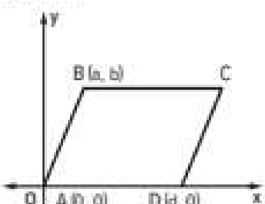
A $(d + a, y)$

B $(d - a, b)$

C $(d + x, b)$

D $(d + a, b)$

E $(d + b, a)$



93. **REVIEW** What is the value of $\dfrac{12!}{8!\,4!}$?

F 495

G 500

H 660

J 710

94. Mrs. Thomas is giving a four-question multiple-choice quiz. Each question can be answered A, B, C, or D. How many ways could a student answer the questions using each answer A, B, C, or D once?

A 20

B 22

C 24

D 26

95. **REVIEW** Which expression is equivalent to $(2x - 2)^4$?

F $16x^4 + 64x^3 - 96x^2 - 64x + 16$

G $16x^4 - 32x^3 - 192x^2 - 64x + 16$

H $16x^4 - 64x^3 + 96x^2 - 64x + 16$

J $16x^4 + 32x^3 - 192x^2 - 64x + 16$

LESSON

3-6 Functions as Infinite Series

::Then	::Now	::Why?
You found the *n*th term of an infinite series expressed using sigma notation.	**1** Use a power series to represent a rational function. **2** Use power series representations to approximate values of transcendental functions.	The music to which you listen on a digital audio player is first performed by an artist. The waveform of each sound in that performance is then broken down into its component parts and stored digitally. These parts are then retrieved and combined to reproduce each original sound of the performance. The analysis of a special series is an essential ingredient in this process.

NewVocabulary
power series
exponential series
Euler's Formula

1 Power Series

Earlier in this chapter, you saw how some series of numbers can be expressed as functions. In this lesson, you will see that some functions can be broken down into infinite series of component functions.

In Lesson 3-3, you learned that the sum of an infinite geometric series,

$$1 + r + r^2 + \cdots + r^n + \cdots, a_1 = 1$$

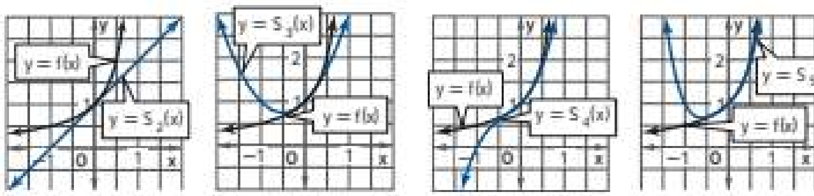
with common ratio r, converges to a sum of $\frac{a_1}{1-r}$ if $|r| < 1$. Replacing r with x,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}, \text{ for } |x| < 1.$$

It follows that $f(x) = \frac{1}{1-x}$ can be expressed as an infinite series. That is,

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ or } 1 + x + x^2 + \cdots + x^n + \cdots \text{ for } |x| < 1.$$

The figures below show the graph of $f(x) = \frac{1}{1-x}$ and the second through fifth partial sums $S_n(x)$ of the series: $S_2(x) = 1 + x$, $S_3(x) = 1 + x + x^2$, $S_4(x) = 1 + x + x^2 + x^3$, and $S_5(x) = 1 + x + x^2 + x^3 + x^4$.



Notice that as n increases, the graph of $S_n(x)$ appears to come closer and closer to the graph of $f(x)$ on the interval $(-1, 1)$ or $|x| < 1$. Notice too that each of the partial sums of the series is a polynomial function, so the series can be thought of as an "infinite" polynomial. An infinite series of this type is called a **power series**.

KeyConcept Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

where x and a_n can take on any values for $n = 0, 1, 2, \ldots$, is called a power series in x.

If you know the power series representation of one function, you can use it to find the power series representations of other related functions.

Example 1 Power Series Representation of a Rational Function

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x) = \frac{1}{3-x}$. Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the sixth partial sum of its power series.

To find the transformation that relates $f(x)$ to $g(x)$, use u-substitution. Substitute u for x in f, equate the two functions, and solve for u as shown.

$$g(x) = f(u)$$
$$\frac{1}{3-x} = \frac{1}{1-u}$$
$$1 - u = 3 - x$$
$$-u = 2 - x$$
$$u = x - 2$$

Therefore, $g(x) = f(x-2)$. Replacing x with $x - 2$ in $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$ yields

$$f(x-2) = \sum_{n=0}^{\infty} (x-2)^n \text{ for } |x-2| < 1.$$

Therefore, $g(x) = \frac{1}{3-x}$ can be represented by the power series $\sum_{n=0}^{\infty} (x-2)^n$.

This series converges for $|x-2| < 1$, which is equivalent to $-1 < x - 2 < 1$ or $1 < x < 3$.

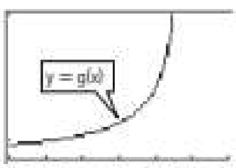
WatchOut!

When finding the kth partial sum of a series where the lower bound starts at 0 use the series $\sum_{n=0}^{k-1}$. For instance in Example 1, the sixth partial sum is called for, but since the lower bound is 0, the upper bound is 6 − 1 or 5, not 6.

The sixth partial sum $S_6(x)$ of this series is

$$\sum_{n=0}^{5} (x-2)^n \text{ or } 1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4 + (x-2)^5.$$

The graphs of $g(x) = \frac{1}{3-x}$ and $S_6(x) = 1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4 + (x-2)^5$ are shown. Notice that on the interval (1, 3), the graph of $S_6(x)$ comes close to the graph of $g(x)$.



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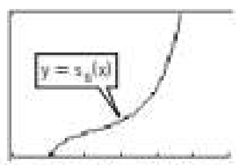

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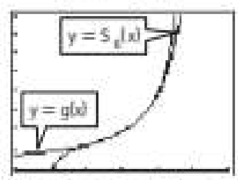

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StudyTip

Graphs of Series Notice that the graphs of $f(x)$ and $S_n(x)$ only converge on an interval. The graphs may differ greatly outside of that interval.

GuidedPractice

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x)$. Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the sixth partial sum of its power series.

1A. $g(x) = \frac{1}{1-2x}$

1B. $g(x) = \frac{2}{1-x}$

In calculus, power series representations are often easier to use in calculations than other representations of functions when determining functions called *derivatives* and *integrals*. A more immediate application can be seen by looking at the power series representations of transcendental functions such as $f(x) = e^x$, $f(x) = \sin x$, and $f(x) = \cos x$.

2 Transcendental Functions as Power Series

In a previous lesson, you learned that the transcendental number e is given by $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$. Thus, $e^x = \lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^{nx}$. We can use this definition along with the Binomial Theorem to derive a power series representation for $f(x) = e^x$.

ReadingMath

Euler Number The Swiss mathematician Leonhard Euler (pronounced OY ler), published a work in which he developed this irrational number, called e, the Euler number.

If we let $u = \frac{1}{n}$ and $k = nx$, then $\left(1 + \frac{1}{n}\right)^{nx}$ becomes $(1 + u)^k$. Applying the Binomial Theorem,

$$
\begin{aligned}
(1+u)^k &= {}_kC_0\,(1)^k u^0 + {}_kC_1\,(1)^{k-1}u + {}_kC_2\,(1)^{k-2}u^2 + {}_kC_3\,(1)^{k-3}u^3 + \cdots \\
&= \frac{k!}{(k-0)!\,0!}(1) + \frac{k!}{(k-1)!\,1!}(1)u + \frac{k!}{(k-2)!\,2!}(1)u^2 + \frac{k!}{(k-3)!\,3!}(1)u^3 + \cdots \\
&= 1 + \frac{k(k-1)!}{(k-1)!}u + \frac{k(k-1)(k-2)!}{(k-2)!\,2!}u^2 + \frac{k(k-1)(k-2)(k-3)!}{(k-3)!\,3!}u^3 + \cdots \\
&= 1 + ku + \frac{k(k-1)}{2!}u^2 + \frac{k(k-1)(k-2)}{3!}u^3 + \cdots
\end{aligned}
$$

Now replace u with $\frac{1}{n}$ and k with nx and find the limit as n approaches infinity. Use the fact that as n approaches infinity, the fraction $\frac{1}{n}$ gets increasingly smaller, so $\lim_{n \to \infty} \frac{1}{n} = 0$.

$$
\begin{aligned}
\lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^{nx} &= 1 + (nx)\frac{1}{n} + \frac{nx(nx-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{nx(nx-1)(nx-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots \\
&= 1 + x + \frac{x\left(x - \frac{1}{n}\right)}{2!} + \frac{x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right)}{3!} + \cdots \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
\end{aligned}
$$

This series is often called the **exponential series**.

StudyTip

Defining e The exponential series provides yet another way to define e. When $x = 1$, $e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!}$ or $\sum_{n=0}^{\infty} \frac{1}{n!}$.

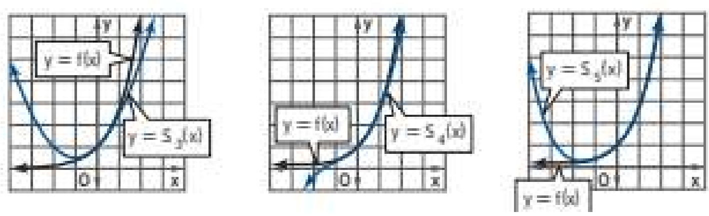
KeyConcept Exponential Series

The power series representing e^x is called the exponential series and is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

which is convergent for all x.

The graph of $f(x) = e^x$ and the partial sums $S_3(x)$, $S_4(x)$, and $S_5(x)$ of the exponential series are shown below.



You can see from the graphs that the partial sums of the exponential series approximate the graph of $f(x) = e^x$ on increasingly wider intervals of the domain for increasingly greater values of n.

Notice that the calculations involved in the exponential series are relatively simple: multiplications (for powers and factorials), divisions, and additions. Because of this, calculators and computer programs use partial sums of the exponential series to evaluate e^x to desired degrees of accuracy.

WatchOut!

Evaluating e^x The fifth partial sum of the exponential series only gives reasonably good approximations of e^x for x on $[-1.5, 2.5]$. Subsequent partial sums, such as the sixth and seventh partial sums, are more accurate for wider intervals of x-values.

Example 2 Exponential Series

Use the fifth partial sum of the exponential series to approximate the value of $e^{1.5}$. Round to three decimal places.

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \qquad e^x = \sum_{n=0}^{4} \frac{x^n}{n!}$$

$$e^{1.5} \approx 1 + 1.5 + \frac{1.5^2}{2!} + \frac{1.5^3}{3!} + \frac{1.5^4}{4!} \qquad x = 1.5$$

$$\approx 4.398 \qquad \text{Simplify.}$$

CHECK A calculator, using a partial sum of the exponential series with many more terms, returns an approximation of 4.48 for $e^{1.5}$. Therefore, an approximation of 4.398 is reasonable. ✔

GuidedPractice

Use the fifth partial sum of the exponential series to approximate each value. Round to three decimal places.

2A. $e^{-0.75}$ **2B.** $e^{0.25}$

Math HistoryLink

Mādhava of Sangamagramma (1340–1425)
An Indian mathematician born near Cochin, Mādhava discovered the series equivalent to the expansions of sin x, cos x, and arctan x around 1400, two hundred years before their discovery in Europe.

Other transcendental functions have power series representations as well. Calculators and computers use **power series** to approximate the values of cosine and sine functions.

KeyConcept Power Series for Cosine and Sine

The power series representations for cos x and sin x are given by

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots, \text{ and}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots,$$

which are convergent for all x.

By replacing x with any angle measure expressed in radians and carrying out the computations, approximate values of the cosine and sine functions can be found to any desired degree of accuracy.

Example 3 Trigonometric Series

a. Use the fifth partial sum of the power series for cosine to approximate the value of $\cos \frac{\pi}{7}$. Round to three decimal places.

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \qquad \cos x = \sum_{n=0}^{4} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos \frac{\pi}{7} \approx 1 - \frac{(0.449)^2}{2!} + \frac{(0.449)^4}{4!} - \frac{(0.449)^6}{6!} + \frac{(0.449)^8}{8!} \qquad x = \frac{\pi}{7} \text{ or about } 0.449$$

$$\approx 0.901 \qquad \text{Simplify.}$$

CHECK A calculator, using a partial sum of the power series for cosine with many more terms, returns an approximation of 0.901, to three decimal places, for $\cos \frac{\pi}{7}$. Therefore, an approximation of 0.901 is reasonable. ✔

Study Tip

Fifth Partial Sum While additional partial sums provide a better approximation, the fifth partial sum typically is accurate to three decimal places.

b. **Use the fifth partial sum of the power series for sine to approximate the value of $\sin \frac{\pi}{5}$. Round to three decimal places.**

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \qquad \sin x = \sum_{n=0}^{5} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin \frac{\pi}{5} \approx 0.628 - \frac{(0.628)^3}{3!} + \frac{(0.628)^5}{5!} - \frac{(0.628)^7}{7!} + \frac{(0.628)^9}{9!} \qquad x = \frac{\pi}{5} \text{ or about } 0.628$$

$$\approx 0.588 \qquad \text{Simplify.}$$

CHECK Using a calculator, $\sin \frac{\pi}{5} \approx 0.588$. Therefore, an approximation of 0.588 is reasonable. ✓

Guided Practice

Use the fifth partial sum of the power series for cosine or sine to approximate each value. Round to three decimal places.

3A. $\sin \frac{\pi}{11}$

3B. $\cos \frac{2\pi}{17}$

You may have noticed similarities in the power series representations of $f(x) = e^x$ and the power series representations of $f(x) = \sin x$ and $f(x) = \cos x$. A relationship is derived by replacing x by $i\theta$ in the exponential series, where i is the imaginary unit and θ is the measure of an angle in radians.

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots \qquad e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots \qquad i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, i^6 = -1, i^7 = -i$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - i\frac{\theta^7}{7!} + \cdots\right) \qquad \text{Group real and imaginary terms.}$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \qquad \text{Distributive Property}$$

$$= \cos \theta + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \qquad \cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$= \cos \theta + i \sin \theta \qquad \sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \cdots$$

This relationship is called **Euler's Formula**.

Key Concept Euler's Formula

For any real number θ, $e^{i\theta} = \cos \theta + i \sin \theta$.

From your work in Lesson 2-5, you should recognize the right-hand side of this equation as being part of the polar form of a complex number. Applying Euler's Formula to the polar form of a complex number yields the following result.

$$a + bi = r(\cos \theta + i \sin \theta) \qquad \text{Polar form of a complex number}$$

$$= re^{i\theta} \qquad \text{Euler's Formula}$$

Therefore, Euler's Formula gives us a way of expressing a complex number in exponential form.

Key Concept Exponential Form of a Complex Number

The exponential form of a complex number $a + bi$ is given by

$$a + bi = re^{i\theta},$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$ for $a > 0$ and $\theta = \tan^{-1}\frac{b}{a} + \pi$ for $a < 0$.

Example 4 Write a Complex Number in Exponential Form

Write $-\sqrt{3} + i$ in exponential form.

Write the polar form of $-\sqrt{3} + i$. In this expression, $a = -\sqrt{3}$, $b = 1$, and $a < 0$. Find r.

$r = \sqrt{(-\sqrt{3})^2 + 1^2}$ — $r = \sqrt{a^2 + b^2}$

$= \sqrt{4}$ or 2 — Simplify.

Now find θ.

$\theta = \tan^{-1} \frac{1}{-\sqrt{3}} + \pi$ — $\theta = \tan^{-1} \frac{b}{a} + \pi$ for $a < 0$

$= -\frac{\pi}{6} + \pi$ or $\frac{5\pi}{6}$ — Simplify.

Therefore, because $a + bi = re^{i\theta}$, the exponential form of $-\sqrt{3} + i$ is $2e^{i\frac{5\pi}{6}}$.

GuidedPractice

Write each complex number in exponential form.

4A. $1 + \sqrt{3}i$

4B. $\sqrt{2} + \sqrt{2}i$

From your study of logarithms in Grade 10, you know that no *real* number can be the logarithm of a negative number. We can use Euler's Formula to show that the natural logarithm of a negative number does exist in the *complex* number system.

$e^{i\theta} = \cos\theta + i \sin\theta$ — Euler's Formula

$e^{i\pi} = \cos\pi + i\sin\pi$ — Let $\theta = \pi$.

$e^{i\pi} = -1 + i(0)$ — $\cos\pi = -1$ and $\sin\pi = 0$

$e^{i\pi} = -1$ — Simplify.

$\ln e^{i\pi} = \ln(-1)$ — Take the natural logarithm of each side.

$i\pi = \ln(-1)$ — Power Property of Logarithms

This result indicates that the natural logarithm of -1 exists and is the complex number $i\pi$. You can use this result to find the natural logarithm of any negative number $-k$, for $k > 0$.

$\ln(-k) = \ln[(-1)k]$ — $-k = (-1)k$

$= \ln(-1) + \ln k$ — Product Property of Logarithms

$= i\pi + \ln k$ — $\ln(-1) = i\pi$

$= \ln k + i\pi$ — Write in the form $a + bi$.

> **TechnologyTip**
>
> Complex Numbers You can use your calculator to evaluate the natural logarithm of a negative number by changing from REAL to $a + bi$ under MODE.

Example 5 Natural Logarithm of a Negative Number

Find the value of $\ln(-5)$ in the complex number system.

$\ln(-5) = \ln 5 + i\pi$ — $\ln(-k) = \ln k + i\pi$

$\approx 1.609 + i\pi$ — Use a calculator to compute ln 5.

GuidedPractice

Find the value of each natural logarithm in the complex number system.

5A. $\ln(-8)$

5B. $\ln(-6.24)$

Exercises

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x)$.

Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the sixth partial sum of its power series.

1. $g(x) = \frac{4}{1-x}$

2. $g(x) = \frac{3}{1-2x}$

3. $g(x) = \frac{2}{1-x^2}$

4. $g(x) = \frac{3}{2-x}$

5. $g(x) = \frac{2}{5-3x}$

6. $g(x) = \frac{4}{3-2x^2}$

Use the fifth partial sum of the exponential series to approximate each value. Round to three decimal places.

7. $e^{0.5}$

8. $e^{-0.25}$

9. $e^{-2.5}$

10. $e^{0.8}$

11. $e^{-0.3}$

12. $e^{3.5}$

13. **ECOLOGY** The population density P per square meter of zebra mussels in the Upper Mississippi River can be modeled by $P = 3.5e^{0.08t}$, where t is measured in weeks. Use the fifth partial sum of the exponential series to estimate the zebra mussel population density after 4 weeks, 12 weeks, and 1 year.

Use the fifth partial sum of the power series for cosine or sine to approximate each value. Round to three decimal places.

14. $\sin \frac{\pi}{9}$

15. $\cos \frac{2\pi}{13}$

16. $\sin \frac{5\pi}{13}$

17. $\cos \frac{3\pi}{10}$

18. $\cos \frac{2\pi}{9}$

19. $\sin \frac{3\pi}{19}$

20. **AMUSEMENT PARK** A ride at an amusement park is in the shape of a giant pendulum that swings riders back and forth in a 240° arc to a maximum height of 137 feet. The pendulum is supported by a tower that is 85 feet tall and dips below ground-level into a pit when swinging below the tower. Use the fifth partial sum of the power series for cosine or sine to approximate the length of the pendulum.

Write each complex number in exponential form.

21. $\sqrt{3} + i$

22. $\sqrt{3} - i$

23. $\sqrt{2} - \sqrt{2}i$

24. $-\sqrt{3} - i$

25. $1 - \sqrt{3}i$

26. $-1 + \sqrt{3}i$

27. $-\sqrt{2} + \sqrt{2}i$

28. $-1 - \sqrt{3}i$

Find the value of each natural logarithm in the complex number system.

29. $\ln(-6)$

30. $\ln(-3.5)$

31. $\ln(-2.45)$

32. $\ln(-7)$

33. $\ln(-4.36)$

34. $\ln(-9.12)$

35. **POWER SERIES** Use the power series representations of $\sin x$ and $\cos x$ to answer each of the following questions.

- **a.** Graph $f(x) = \sin x$ and the third partial sum of the power series representing $\sin x$. Repeat for the fourth and fifth partial sums. Describe the interval of convergence for each.
- **b.** Repeat part **a** for $f(x) = \cos x$ and the third, fourth, and fifth partial sums of the power series representing $\cos x$. Describe the interval of convergence for each.
- **c.** Describe how the interval of convergence changes as n increases. Then make a conjecture as to the relationship between each trigonometric function and its related power series as $n \to \infty$.

Solve for z over the complex numbers. Round to three decimal places.

36. $2e^z + 5 = 0$

37. $e^{2z} + 12 = 0$

38. $4e^{2z} + 7 = 6$

39. $3(e^z - 1) + 5 = -2$

40. $e^{2z} - e^z = 2$

41. $10e^{2z} + 17e^z = -3$

42. **ECONOMICS** The total value of an investment of P dollars compounded continuously at an annual interest rate of r over t years is Pe^{rt}. Use the first five terms of the exponential series to approximate the value of an investment of $10,000 compounded continuously at 5.25% for 5 years.

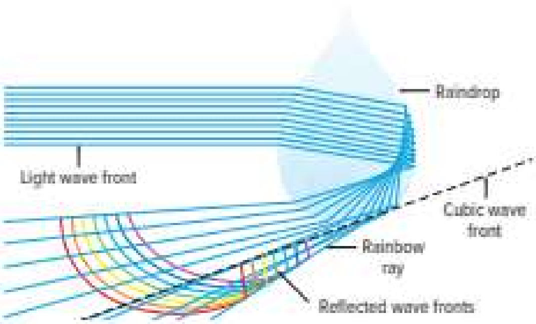
43. **RELATIVE ERROR** *Relative error* is the absolute error in estimating a quantity divided by its true value. The relative error of an approximation a of a quantity b is given by $\frac{|b-a|}{b}$. Find the relative error in approximating $e^{2.1}$ using two, three, and six terms of the exponential series.

Approximate the value of each expression using the first four terms of the power series for sine and cosine. Then find the expected value of each.

44. $\sin^2 \frac{1}{2} + \cos^2 \frac{1}{2}$

45. $\sec^2 1 - \tan^2 1$

46. **RAINBOWS** *Airy's equation*, which is used in physics to model the diffraction of light, can also be used to explain how a light wave front is converted into a curved wave front in forming rainbows.

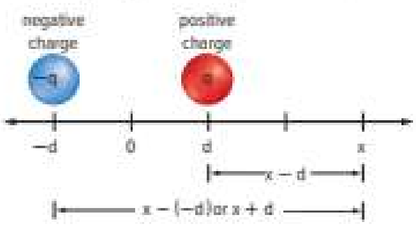


This equation can be represented by the power series shown below.

$$f(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(2 \cdot 3)(5 \cdot 6) \cdots [(3k-1) \cdot (3k)]}$$

Use the fifth partial sum of the series to find $f(3)$. Round to the nearest hundredth.

47. **ELECTRICITY** When an electric charge is accompanied by an equal and opposite charge nearby, such an object is called an *electric dipole*. It consists of charge q at the point $x = d$ and charge $-q$ at $x = -d$, as shown below.



Along the x-axis, the electric field strength at x is the sum of the electric fields from each of the two charges. This is given by $E(x) = \frac{kq}{(x-d)^2} - \frac{kq}{(x+d)^2}$. Find a power series representing $E(x)$ if k is a constant and $d = 1$.

48. **SOUND** The *Fourier Series* represents a periodic function of time $f(t)$ as a summation of sine waves and cosine waves with frequencies that start at 0 and increase by integer multiples. The series below represents a sound wave from the digital data fed from a CD into a CD player.

$$f(t) = 0.7 + \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n} \cos 270.6n\pi t + \frac{1}{2n-1} \sin 270.6n\pi t \right)$$

Graph the series for $n = 4$. Then analyze the graph.

IDENTITIES **Use power series representations from this lesson to verify each trigonometric identity.**

49. $\sin(-x) = -\sin x$

50. $\cos(-x) = \cos x$

51. **APPROXIMATIONS** The infinite series for the inverse tangent function $f(x) = \tan^{-1} x$, is given by $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$. However, this series is only valid for values of x on the interval $(-1, 1)$.

a. Write the first five terms of the infinite series representation for $f(x) = \tan^{-1} x$.

b. Use the first five terms of the series to approximate $\tan^{-1} 0.1$.

c. On the same coordinate plane, graph $f(x) = \tan^{-1} x$ and the third partial sum of the power series representing $f(x) = \tan^{-1} x$. On another coordinate plane, graph $f(x)$ and the fourth partial sum. Then graph $f(x)$ and the fifth partial sum.

d. Describe what happens on the interval $(-1, 1)$ and in the regions $x \geq 1$ or $x \leq -1$.

H.O.T. Problems Use Higher-Order Thinking Skills

52. **WRITING IN MATH** Describe how using additional terms in the approximating series for e^x affects the outcome.

53. **REASONING** Use the power series for sine to explain why, for x-values on the interval $[-0.1, 0.1]$, a close approximation of $\sin x$ is x.

54. **CHALLENGE** Prove that $2 \sin \theta \cos \theta = \frac{e^{2i\theta} - e^{-2i\theta}}{2i}$.

55. **REASONING** For what values of α and β does $e^{i\alpha} = e^{i\beta}$? Explain.

PROOF **Show that for all real numbers x, the following are true.**

56. $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

57. $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

58. **CHALLENGE** The hyperbolic sine and hyperbolic cosine functions are analogs of the trigonometric functions that you studied in Grade 10. Just as the points $(\cos x, \sin x)$ form a unit circle, the points $(\cosh t, \sinh t)$ form the right half of an equilateral hyperbola. An equilateral hyperbola has perpendicular asymptotes. The hyperbolic sine (sinh) and hyperbolic cosine (cosh) functions are defined below. Find the power series representations for these functions.

$\sinh x = \frac{e^x - e^{-x}}{2}$ $\quad\quad$ $\cosh x = \frac{e^x + e^{-x}}{2}$

Spiral Review

Use Pascal's triangle to expand each binomial.

59. $(3m + \sqrt{2})^4$

60. $\left(\frac{1}{2}n + 2\right)^5$

61. $(p^2 + q)^8$

62. Prove that $4 + 7 + 10 + \cdots + (3n + 1) = \frac{n(3n + 5)}{2}$ for all positive integers n.

Find each power, and express it in rectangular form.

63. $(-2 + 2i)^3$

64. $(1 + \sqrt{3}i)^4$

65. $(\sqrt{2} + \sqrt{2}i)^{-2}$

66. Given $\mathbf{t} = \langle -9, -3, c \rangle$, $\mathbf{u} = \langle 8, -4, 3 \rangle$, $\mathbf{v} = \langle 2, 5, -6 \rangle$, and that the volume of the parallelepiped having adjacent edges $\mathbf{t}$, $\mathbf{u}$, and $\mathbf{v}$ is 93 cubic units, find c.

Use an inverse matrix to solve each system of equations, if possible.

67. $x - 8y = -7$
$2x + 5y = 28$

68. $4x + 7y = 22$
$-9x + 11y = 4$

69. $w + 2x + 3y = 18$
$4w - 8x + 7y = 41$
$-w + 9x - 2y = -4$

Determine whether A and B are inverse matrices.

70. $A = \begin{bmatrix} 1 & -2 \\ 7 & -6 \end{bmatrix}, B = \begin{bmatrix} -6 & 2 \\ -7 & 1 \end{bmatrix}$

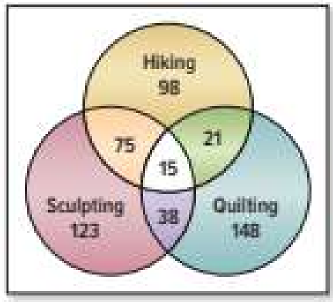
71. $A = \begin{bmatrix} -11 & -5 \\ 9 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 \\ -9 & -11 \end{bmatrix}$

72. $A = \begin{bmatrix} 6 & 2 \\ -2 & 8 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 \\ -3 & -5 \end{bmatrix}$

73. **CONFERENCE** A university sponsored a conference for 680 women. The Venn diagram shows the number of participants in three of the activities offered. Suppose women who attended the conference were randomly selected for a survey.

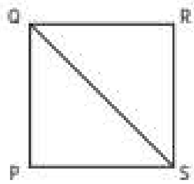
a. What is the probability that a woman selected participated in hiking or sculpting?

b. Describe a set of women such that the probability of being selected is about 0.39.



Skills Review for Standardized Tests

74. **SAT/ACT** $PQRS$ is a square. What is the ratio of the length of diagonal $\overline{QS}$ to the length of side $\overline{RS}$?



A 2

B $\sqrt{2}$

C 1

D $\frac{\sqrt{2}}{2}$

E $\frac{\sqrt{3}}{2}$

75. **REVIEW** What is the sum of the infinite geometric series $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \cdots$?

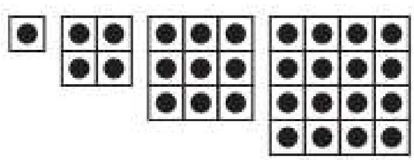
F $\frac{2}{3}$

G 1

H $1\frac{1}{3}$

J $1\frac{2}{3}$

76. **FREE RESPONSE** Consider the pattern of dots shown.



a. Draw the next figure in this sequence.

b. Write the sequence, starting with 1, that represents the number of dots that must be added to each figure in the sequence to get the number of dots in the next figure.

c. Find the expression for the nth term of the sequence found in part **b**.

d. Find the expression for the number of dots in the nth figure in the original sequence.

e. Prove, through mathematical induction, that the sum of the sequence found in part **b** is equal to the expression found in part **d**.

EXTEND

3-6 Spreadsheet Lab: Detecting Patterns in Data

Objective

- Organize and display data using spreadsheets to detect patterns and departures from patterns.

In Chapter 3, you learned how to detect patterns in a sequence and describe them by using functions.

Pattern in Data Sequence	Pattern in Graph of Data Sequence	Type of Sequence	Function Describing Sequence
common 1st differences	data in a linear pattern	arithmetic	linear
common ratio	data in an exponential pattern	geometric	exponential

In this lab, you will use a spreadsheet to organize and display paired data in order to look for such patterns.

Activity 1 Detect Patterns

DOGS **A certain golden retriever had a mass of 1.45 kilograms at birth. The table shows the puppy's mass in the first 70 days of its growth. Use a spreadsheet to find a pattern in the data.**

Days after Birth	10	20	30	40	50	60	70
Mass (kg)	1.91	2.46	3.17	4.10	5.22	6.81	8.63

Step 1 Enter the data into the spreadsheet.

Step 2 To determine if the sequence of masses is arithmetic, enter a formula in the next column to find the average daily rate of change in the puppy's mass.

Step 3 To determine if the sequence is geometric, enter the formula shown in the next column to find the average ratio of change in the puppy's mass each day.

	A	B	C	D
1	*Days after Birth*	*Mass (kg)*	*Average Rate of Change*	*Average Ratio of Change*
2	0	1.45		
3	10	1.91	=(B3-B2)/(A3-A2)	=(B3/B2)^(1/(A3-A2))-1
4	20	2.46		
5	30	3.17		
6	40	4.1		
7	50	5.22		
8	60	6.81		
9	70	8.63		

Sheet 1 | Sheet 2 | Sheet 3

Analyze the Results

1. Explain the formulas used in the spreadsheet.
2. Describe any pattern you see in the data. What type of sequence approximates the data? Explain.
3. Use the chart tool to create a scatter plot of the data. Does this graph support your answer to Exercise 2? Explain.
4. Write an equation approximating the dog's mass y after x days.
5. Use your equation to predict the dog's mass 25 days after birth and 365 days after birth. Are these predictions reasonable? Explain.

You can also use a spreadsheet to detect and analyze departures from patterns.

Activity 2 Detect Departures from Patterns

HOMEWORK Adrian recorded the number of precalculus problems and how long he worked on them for eight nights. Look for a pattern in the data and any departures from that pattern.

Number of Problems	0	3	5	6	8	9	10	15
Time (min)	0	27	44	70	72	82	95	140

Step 1 Enter the data into the spreadsheet.

Step 2 Enter formulas in the adjacent columns to detect whether the sequence of is arithmetic or geometric. Then copy these formulas into the cells below.

Step 3 Look for patterns. Notice that all but two of the rates of change cluster around 9.

	A	B	C
1	Number of Problems	Time (min)	Average Rate of Change
2	0	0	
3	3	27	9.00
4	5	44	8.50
5	6	70	26.00
6	8	72	1.00
7	9	81	9.00
8	10	89	8.00
9	15	136	9.40

Sheet 1 Sheet 2 Sheet 3

Study Tip

Series in Data To investigate series in data, you can use the Auto Sum tool. For Activity 2, enter =B2 in cell D2 and =SUM(B2,B3) in cell D3. Copy this second formula into the remaining cells in the column to create a sequence of partial sums.

Analyze the Results

6. Where does the departure in the pattern occur?

7. Write a spreadsheet formula that could model the data if this data value were removed.

8. Use the Chart Wizard to create a scatter plot that shows the actual data and the model of the data. Does this graph support your answer to Exercise 7? Explain.

9. Use your formula from Exercise 7 to predict how long it would take Adrian to complete 12 problems and 20 problems. Are these predictions reasonable? Explain.

Exercises

Use a spreadsheet to organize and identify a pattern or departure from a pattern in each set of data. Then use a calculator to write an equation to model the data.

10. **INTERNET** The table shows the number of times the main page of a popular blog is read (hits) each month.

Month	2	4	6	8	10	12	15	20
Hits	83	171	266	368	479	732	1405	4017

11. **COLLEGE** The table shows the composite ACT scores and grade-point averages (GPA) of 20 students after their first semester in college. (*Hint:* First use the Sort Ascending tool to organize the data.)

ACT Score	27	16	15	22	20	21	25
College GPA	3.9	2.9	2.7	3.6	3.2	3.4	3.1
ACT Score	26	18	23	19	29	28	17
College GPA	4.0	3.1	3.6	2.6	4.0	3.9	3.0

Study Guide and Review

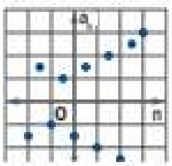
Chapter Summary

KeyConcepts

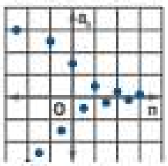
Sequences, Series, and Sigma Notation

- A *finite sequence* is a sequence with a set number of terms. An *infinite sequence* has infinitely many terms.
- A sequence with a limit is said to *converge*. A sequence with no limit is said to *diverge*.

Divergent Sequence



Convergent Series



- A *series* is the sum of all of the terms of a sequence.

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_k$$

Arithmetic Sequences and Series

- The nth term of an arithmetic sequence with first term a_1 and common difference d is given by $a_n = a_1 + (n - 1)d$.
- The sum of a finite arithmetic series is given by $S_n = \frac{n}{2}(a_1 + a_n)$ or $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$.

Geometric Sequences and Series

- The nth term of a geometric sequence with first term a_1 and common ratio r is given by $a_n = a_1r^{n-1}$.
- The sum of the first n terms of a geometric series is given by $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$ or $S_n = \frac{a_1 - a_nr}{1 - r}$.
- The sum of an infinite geometric series is given by $S = \frac{a_1}{1 - r}$, for $|r| < 1$.

Mathematical Induction

- If P_n is a statement about a positive integer n, then P_n is true for all positive integers n if and only if P_1 is true, and for every positive integer k, if P_k is true, then P_{k+1} is true.

The Binomial Theorem

- The expression $(a + b)^n$ can be expanded using the nth row of Pascal's triangle to determine the coefficients of each term.
- The binomial coefficient of the $a^{n-r}b^r$ term in the expansion of $(a + b)^n$ is given by ${}_nC_r = \frac{n!}{(n - r)!\,r!}$.

Functions as Infinite Series

- A power series in x is an infinite series of the form $\sum_{n=0}^{\infty} a_nx^n$.
- Euler's Formula states that for any real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$.

KeyVocabulary

anchor step	geometric means
arithmetic means	geometric sequence
arithmetic sequence	geometric series
arithmetic series	inductive hypothesis
binomial coefficients	inductive step
Binomial Theorem	infinite sequence
common difference	infinite series
common ratio	nth partial sum
converge	power series
diverge	recursive formula
Euler's Formula	second difference
exponential series	sequence
Fibonacci sequence	series
finite sequence	sigma notation
finite series	term
first difference	

VocabularyCheck

State whether each sentence is true or false. If false, replace the underlined term to make a true sentence.

1. In mathematical induction, the assumption that a conjecture works for any particular case is called the inductive hypothesis.
2. A sequence that has a set number of terms is called an infinite sequence.
3. A sequence a_n defined as a function of n is defined recursively.
4. The step in which you show that something works for the first case is called the inductive step in mathematical induction.
5. Explicitly defined sequences give one or more of the first few terms and then define the terms that follow using those previous terms.
6. The sum of the first n terms of a finite or infinite sequence is called a finite series.
7. The difference between the terms of an arithmetic sequence is called the common ratio.
8. A geometric series is the sum of the terms of a geometric sequence.
9. If a sequence does not have a limit, it is said to converge.
10. The Binomial Theorem is a recursive sequence that describes many patterns found in nature.

Lesson-by-Lesson Review

3-1 Sequences, Series, and Sigma Notation

Find the next four terms of each sequence.

11. 1, 9, 17, 25, ...

12. −1, 2, 7, 14, ...

Find the indicated sum for each sequence.

13. fourth partial sum of $a_n = 2n - 10$

14. S_7 of $a_n = -n^3$

Find each sum.

15. $\sum_{n=2}^{9} \frac{4n-6}{3}$

16. $\sum_{n=1}^{6} 7n - 4$

Example 1

Find the sum of $\sum_{n=1}^{4} n^2 - 5$.

Find the values for a_n for $n = 1, 2, 3$, and 4.

$a_1 = (1)^2 - 5$ or -4 $n = 1$

$a_2 = (2)^2 - 5$ or -1 $n = 2$

$a_3 = (3)^2 - 5$ or 4 $n = 3$

$a_4 = (4)^2 - 5$ or 11 $n = 4$

Therefore, $\sum_{n=1}^{4} n^2 - 5 = -4 + (-1) + 4 + 11$ or 10.

3-2 Arithmetic Sequences and Series

Write an explicit formula and a recursive formula for finding the *n*th term of each arithmetic sequence.

17. −6, −1, 4, ...

18. 23, 15, 7, ...

Find each sum.

19. 50th partial sum of 55 + 62 + 69 + ... + 398

20. 37th partial sum of 9 + 3 + (−3) + ...

Find each sum.

21. $\sum_{n=24}^{35} -3n - 11$

22. $\sum_{n=8}^{27} 4n + 14$

Example 2

Find the 43rd partial sum of the arithmetic series 104 + 100 + 96 + ...

Use the second sum of a finite arithmetic series formula.

$S_n = \frac{n}{2}[2a_1 + (n-1)d]$ Sum of a finite arithmetic series formula

$S_{43} = \frac{43}{2}[2(104) + (43 - 1)(-4)]$ $n = 43$, $a_1 = 104$, and $d = -4$

$= 860$ Simplify.

The 43rd partial sum is $S_{43} = 860$.

3-3 Geometric Sequences and Series

Determine the common ratio, and find the next three terms of each geometric sequence.

23. 5, 7.5, 11.25, ...

24. $3 + a$, $-12 - 4a$, $48 + 16a$, ...

Write an explicit formula and a recursive formula for finding the *n*th term of each geometric sequence.

25. 10, −20, 40, ...

26. 162, 54, 18, ...

Find each sum.

27. first seven terms of $80 + 32 + \frac{64}{5} + ...$

28. $a_1 = 11$, $a_n = 360{,}448$, $r = 8$

Example 3

Determine the common ratio, and find the next three terms of 27, −9, 3, ...

First, find the common ratio.

$a_2 \div a_1 = -9 \div 27$ or $-\frac{1}{3}$

$a_3 \div a_2 = 3 \div -9$ or $-\frac{1}{3}$

Find the ratio between two pairs of consecutive terms to verify the common ratio.

The common ratio is $-\frac{1}{3}$. Multiply the third term by $-\frac{1}{3}$ to find the fourth term, and so on.

$a_4 = 3 \cdot -\frac{1}{3}$ or -1 $a_5 = -1 \cdot -\frac{1}{3}$ or $\frac{1}{3}$

$a_6 = \frac{1}{3} \cdot -\frac{1}{3}$ or $-\frac{1}{9}$

The next three terms are $-1, \frac{1}{3}$, and $-\frac{1}{9}$.

3-4 Mathematical Induction

Use mathematical induction to prove that each conjecture is true for all positive integers n.

29. $6^n - 9$ is divisible by 3.

30. $7^n - 5$ is divisible by 2.

31. $5^n + 3$ is divisible by 4.

32. $2 \cdot 3 + 4 \cdot 5 + 6 \cdot 7 + \cdots + 2n(2n+1) = \frac{n(n+1)(4n+5)}{3}$

33. $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

Prove each inequality for the indicated values of n.

34. $4^n \geq 4n$, for all positive integers n

35. $5n < 6^n$, for all positive integers n

Example 4

Prove that $5^n - 1$ is divisible by 4 for all positive integers n.

Conjecture and Anchor Step Let P_n be the statement that $5^n - 1$ is divisible by 4. When $n = 1$, $5^n - 1 = 5^1 - 1$ or 4. Since 4 is divisible by 4, the statement P_1 is true.

Inductive Hypothesis and Step Assume that $5^k - 1$ is divisible by 4. That is, assume that $5^k - 1 = 4r$ for some integer r. Use this inductive hypothesis to show that $5^{k+1} - 1$ is divisible by 4.

$5^k - 1 = 4r$	Inductive hypothesis
$5^k = 4r + 1$	Add 1 to each side.
$5 \cdot 5^k = 5(4r + 1)$	Multiply each side by 5.
$5^{k+1} = 20r + 5$	Simplify.
$5^{k+1} - 1 = 20r + 4$	Subtract 1 from each side.
$5^{k+1} - 1 = 4(5r + 1)$	Factor.

Since r is an integer, $5r + 1$ is an integer and $4(5r + 1)$ is divisible by 4. Therefore, $5^{k+1} - 1$ is divisible by 4.

Conclusion Since P_n is true for $n = 1$ and P_k implies P_{k+1}, P_n is true for $n = 2$, $n = 3$, and so on. By the principle of mathematical induction, $5^n - 1$ is divisible by 4 for all positive integers n.

3-5 The Binomial Theorem

Use Pascal's triangle to expand each binomial.

36. $(4x + 6)^5$

37. $(m - 5n)^6$

Find the coefficient of the indicated term in each expansion.

38. $(6x - 3y)^{10}$, x^4y^6 term

39. $(2y + 3)^{13}$, 8^{th} term

Use the Binomial Theorem to expand each binomial.

40. $(2p^2 - 7)^4$

41. $(4m + 3n)^7$

Example 5

Use the Binomial Theorem to expand $(3x + 10)^5$.

Apply the Binomial Theorem to expand $(a + b)^5$, where $a = 3x$ and $b = 10$.

$$(3x + 10)^5 = {}_5C_0(3x)^5(10)^0 + {}_5C_1(3x)^4(10)^1 + {}_5C_2(3x)^3(10)^2 + {}_5C_3(3x)^2(10)^3 + {}_5C_4(3x)^1(10)^4 + {}_5C_5(3x)^0(10)^5$$

$$= 1(243x^5)(1) + 5(81x^4)(10) + 10(27x^3)(100) + 10(9x^2)(1000) + 5(3x)(10{,}000) + 1(1)(100{,}000)$$

$$= 243x^5 + 4050x^4 + 27{,}000x^3 + 90{,}000x^2 + 150{,}000x + 100{,}000$$

3-6 Functions as Infinite Series

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of g (x).

Indicate the interval on which the series converges. Use a graphing calculator to graph g (x) and the 6th partial sum of its power series.

42. $g(x) = \frac{1}{1-5x}$

43. $g(x) = \frac{3}{1-2x}$

Use the fifth partial sum of the exponential series to approximate each value. Round to three decimal places.

44. $e^{\frac{1}{4}}$

45. $e^{-1.5}$

Find the value of each natural logarithm in the complex number system.

46. ln (−4)

47. ln (−7.15)

Example 6

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x) = \frac{4}{1-x}$. Indicate the interval on which the series converges.

A geometric series converges to $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. Replace x with $\frac{x+3}{4}$ since $g(x)$ is a transformation of $f(x)$ and: $g(x) = f\left(\frac{x+3}{4}\right)$. The result is

$$f\left(\frac{x+3}{4}\right) = \sum_{n=0}^{\infty}\left(\frac{x+3}{4}\right)^n \text{ for } \left|\frac{x+3}{4}\right| < 1.$$

Therefore, $g(x) = \frac{4}{1-x}$ can be represented by $\sum_{n=0}^{\infty}\left(\frac{x+3}{4}\right)^n$. This series converges for $\left|\frac{x+3}{4}\right| < 1$,

which is equivalent to $-1 < \frac{x+3}{4} < 1$ or $-7 < x < 1$.

Applications and Problem Solving

48. RETAIL A chain of retail coffee stores has a business plan that includes opening 6 new stores nationwide annually. If there were 480 stores on January 1, 2012, how many stores will there be at the end of the year in 2018?

49. DANCE Mara's dance team is performing at a recital. In one formation, there are three dancers in the front row and two additional dancers in each row behind the first row.

a. How many dancers are in the fourth row?

b. If there are eight rows, how many members does the dance team have?

50. AGRICULTURE The population of a herd of cows increases as shown over the course of four years.

Year	1	2	3	4
Population	47	51	56	61

a. Write an explicit formula for finding the population of the herd after n years. Assume that the sequence shown above is geometric.

b. What will the population of the herd be after 7 years?

c. About how many years will it take the population of the herd to exceed 85?

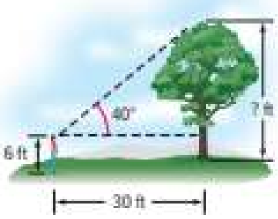
51. NUMBER THEORY Consider the statement $0.\overline{9} = 1$.

a. Prove that $0.9 + 0.09 + 0.009 + \cdots + \frac{9}{10^n} = \frac{10^n - 1}{10^n}$ for any positive integer n.

b. Use your understanding of limits and the statement you proved in part a to explain why $0.\overline{9} = 1$ is true.

52. BASKETBALL Julie usually makes 4 out of every 6 free throws that she attempts. What is the probability that Julie will make 5 out of 6 of the next free throws that she attempts?

53. HEIGHT Lina is estimating the height of a tree. She stands 30 feet from the base and estimates that her angle of sight to the top of the tree is 40°. If she uses the fifth partial sum of the trigonometric series for cosine and sine approximated to three decimal places to calculate the height of the tree, what is Lina's estimate?



CHAPTER 3 Practice Test

Find the specified term of each sequence.

1. ninth term, $a_n = \frac{n^3}{n+3}$
2. sixth term, $a_1 = 156$, $a_n = \frac{a_{n-1} - 4}{2}$

Find the indicated sum for each series.

3. fifth partial sum of $a_n = 3^n + 4$
4. S_8 of $-2, 3, 8, 13, \ldots$

Find the indicated arithmetic means for each set of nonconsecutive terms.

5. 3 means; -10 and -2
6. 4 means; -4 and 56

7. **EXERCISE** While training for a 5-mile fun run, Kirsten ran 1 mile each workout during the first week of training and added 0.5 mile per week to each run so that she ran 5 miles each workout the week before the run. If Kirsten had training three times per week, how many miles did she run during her training?

8. **FIREWORKS** If the year to year increase in attendance at the fireworks show is constant, what was the attendance each year from 2003 to 2007?


Show a success

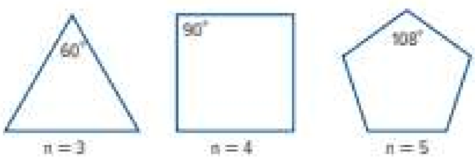
Attendance at annual fireworks show increases from 12,000 in 2002 to 13,122 in 2008.

If possible, find the sum of each infinite geometric series.

9. $\frac{4}{10}, \frac{4}{5}, \frac{8}{5}, \ldots$

10. $\sum_{n=3}^{\infty} -2(0.6)^{n-1}$

11. **GEOMETRY** The measure of each interior angle a of a regular polygon with n sides is $a_n = \frac{180(n-2)}{n}$, where $n \geq 3$.



a. Find the measure of an exterior angle of a regular triangle, square, pentagon, and hexagon.

b. Write a general formula for the measure of an exterior angle of a regular polygon with n sides.

c. Determine whether the sequence is convergent or divergent. Does this make sense in the context of the situation? Explain your reasoning.

Prove each inequality for the indicated values of n.

12. $n! > 5^n$, $n \geq 12$

13. $4n < 7^n$, $n \geq 1$

14. $3^n > n + 8$, $n \geq 3$

15. $3n < \left(\frac{5}{2}\right)^n$, $n \geq 2$

Use Pascal's triangle to expand each binomial.

16. $(2x + 3y)^4$

17. $(x - 6)^7$

18. **MULTIPLE CHOICE** Ellis' basketball scoring statistics are shown below. Based on his statistics, what is the probability that he will score a 2- or 3-point field goal on 3 of his next 7 shots?

Shots on Goal	2-point Shots Scored	3-point Shots Scored
20	11	3

A 0.10 %

B 0.24 %

C 9.72 %

D 23.88 %

Use the Binomial Theorem to expand and simplify each expression.

19. $(x - 4y)^4$

20. $(3a + b^3)^5$

Use $\sum_{n=0}^{\infty} x^n$ to find a power series representation of $g(x)$.

Indicate the interval on which the series converges. Use a graphing calculator to graph $g(x)$ and the 6th partial sum of its power series.

21. $g(x) = \frac{3}{1-x}$

22. $g(x) = \frac{2}{1-4x}$

23. **SKATEBOARDING** A skateboarding ramp has an incline of 24°. Use the 5th partial sum of the trigonometric series for cosine or sine approximated to three decimal places to find the length of the ramp.

Write each complex number in exponential form.

24. $-\sqrt{3} - i$

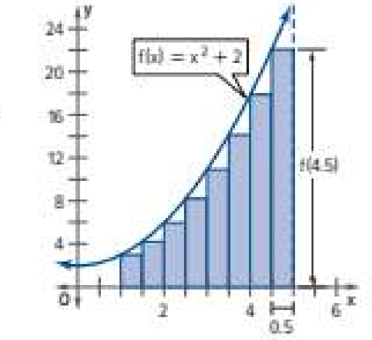
25. $1 - \sqrt{3}i$

CHAPTER 3 Connect to AP Calculus
Riemann Sum

Objective

- Develop notation for approximating the area of a region bound by a curve and the x-axis.

You can approximate the area between a curve and the *x*-axis. You can divide the area into rectangles, find the area of each individual rectangle, and then calculate the sum of the areas. In calculus, this process is assigned special notation and is studied further in an effort to calculate exact areas. We will analyze the components of this process to better understand the notation.



The area *A* of the region shown above can be approximated as follows.

$A = 0.5 \cdot f(1) + 0.5 \cdot f(1.5) + 0.5 \cdot f(2.0) + 0.5 \cdot f(2.5) + 0.5 \cdot f(3.0) + 0.5 \cdot f(3.5) + 0.5 \cdot f(4.0) + 0.5 \cdot f(4.5)$

Notice that we can factor out the width.

$A = \underbrace{0.5}_{\text{width}} \cdot \underbrace{[f(1) + f(1.5) + f(2.0) + f(2.5) + f(3.0) + f(3.5) + f(4.0) + f(4.5)]}_{\text{sum of the heights}}$

The approximation is equal to the product of the width of the rectangles and the sum of their heights. We will examine both of these components separately.

The first component used to approximate the area of a region is the width of the rectangles. The width of the rectangles, denoted Δx, is the difference between the left endpoint and the right endpoint of a rectangle, such as $2.5 - 2$ or 0.5. Generally, we are not given any of the *x*-coordinates of our rectangles. Instead, we get the *lower bound a* and the *upper bound b* of the interval $[a, b]$ and the number of rectangles *n*.

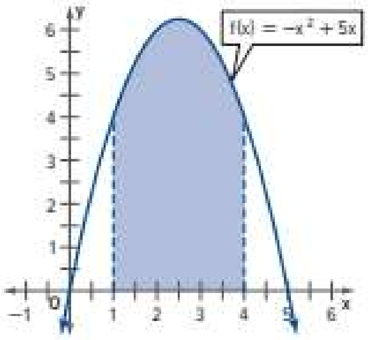
Activity 1 Find Δx

Find Δx if we want to approximate the area between the graph of $f(x) = -x^2 + 5x$ and the x-axis on the interval [1, 4] using 6, 12, and 24 rectangles.

Step 1 Find the total length of the interval by calculating $b - a$.

Step 2 Divide the answer from Step 1 by 6.

Step 3 Repeat Step 2 for 12 and 24 rectangles.



Analyze the Results

1. As the number of rectangles increases, what is happening to Δx? How would this affect your approximation for the area?
2. Find Δx if we want to approximate the area between a curve and the x-axis on the interval $[a, b]$ using n rectangles.
3. As n approaches ∞, what is happening to Δx?

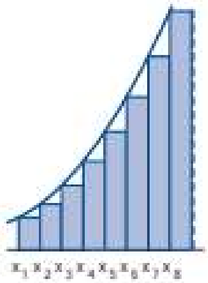


Figure 3.CAP.1

The second component needed to approximate the area of a region is the sum of the heights of the rectangles. The sum of the heights resembles the sum of a series. For the example presented in Activity 1, this sum is

$$f(1) + f(1.5) + f(2.0) + f(2.5) + f(3.0) + f(3.5) + f(4.0) + f(4.5).$$

Since there were 8 rectangles, $f(x)$ is evaluated for 8 values of x. We can write this series as

$$f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7) + f(x_8),$$

where $x_1, x_2, \ldots, x_8$ are the x-coordinates used to find the heights of the rectangles, as shown in the figure. We can represent this series using sigma notation as $\sum_{i=1}^{8} f(x_i)$. For example, the sum of the heights for $f(x)$ can be written in *expanded form* as

$$\sum_{i=1}^{8} f(x_i) = \overset{x_1}{f(1)} + \overset{x_2}{f(1.5)} + \overset{x_3}{f(2.0)} + \overset{x_4}{f(2.5)} + \overset{x_5}{f(3.0)} + \overset{x_6}{f(3.5)} + \overset{x_7}{f(4.0)} + \overset{x_8}{f(4.5)}.$$

In general, the sum of the heights for n rectangles can be described as $\sum_{i=1}^{n} f(x_i)$.

You now have the two components for approximating the area of a region using n rectangles.

We can multiply our width by our expression for the sum of the heights to develop the notation $\sum_{i=1}^{n} f(x_i) \cdot \Delta x$. This expression is called a *Riemann sum*.

Activity 2 Approximate Area Under a Curve

Approximate the area between the graph of $f(x) = -x^2 + 5x$ and the x-axis on the interval [1, 4] using 6 rectangles. Let the left endpoint of each rectangle represent the height.

Step 1 Let $a = 1$, $b = 4$, and $n = 6$. Calculate Δx.

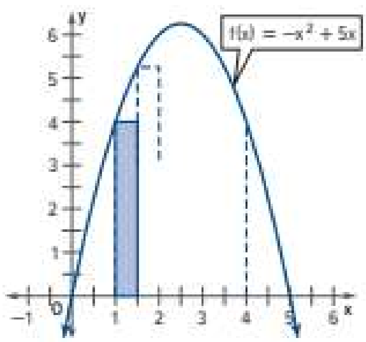
Step 2 Write the approximation in sigma notation. Substitute the value found in Step 1 for Δx and let $n = 6$.

$$\sum_{i=1}^{6} f(x_i) \cdot 0.5$$

Step 3 Write the expression in expanded form.
$0.5\,[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)]$

Step 4 Find each value for x. x_1 will start at 1. Each successive value for x can be found by adding Δx to each previous value. For example, $x_2 = 1 + 0.5$, $x_3 = x_2 + 0.5$, and so on.

Step 5 Calculate the area of the rectangles.



Study Tip

Notation When having to evaluate the sum, it may be easier to represent the approximation for the area of a region as $\Delta x \sum_{i=1}^{n} f(x_i)$.

Analyze the Results

4. If n increased to 12, how would it change the expression in expanded form? What would happen to the x-values found in Step 4?

5. As n approaches ∞, what happens to the calculation?

Model and Apply

Given n and an interval $[a, b]$, find Δx. Then write the approximation for finding the area between the graph of $f(x) = -x^2 + 10x$ and the x-axis in sigma notation. Calculate the area. Let the left endpoint of each rectangle represent the height.

6. $n = 4$; [1, 2]

7. $n = 10$; [6, 10]

8. $n = 24$; [3, 9]

CHAPTER 4

Inferential Statistics

Then

- You found measures of center and spread and organized statistical data.

Now

- In Chapter 4, you will:
  - Use the shape of a distribution to select appropriate descriptive statistics.
  - Construct and use probability distributions.
  - Use the Central Limit Theorem.
  - Find and use confidence intervals, and perform hypothesis testing.
  - Analyze and predict using bivariate data.

Why? ▲

- **ENVIRONMENTAL ENGINEERING** Statistics are extremely important in engineering. In environmental engineering, hypothesis testing can be used to determine if a change in an emission level for a chemical has a significant impact on overall pollution. Also, confidence intervals can be used to help suggest restrictions on by-product wastes in ground water.

PREREAD Scan the study guide and review and use it to make two or three predictions about what you will learn in Chapter 4.

Get Ready for the Chapter

QuickCheck

Find each value.

1. ${}_5P_2$ **2.** ${}_9P_4$ **3.** ${}_8C_3$

4. **INTERNET** The table shows the survey results of 18 high school students who were asked how many hours they spent on the Internet the previous week.

Hours Spent on the Internet					
2	3.5	1	8	2.5	7.5
10	4	5.5	3.5	7.5	1.5
4.5	11	3.5	5	8	6.5

a. Make a histogram of the data.

b. Were there more students on the Internet for fewer than 3 hours or more than 6 hours?

For Exercises 5 and 6, complete each step.

a. Linearize the data according to the given model.

b. Graph the linearized data, and find the linear regression equation.

c. Use the linear model to find a model for the original data.

5. exponential

x	y
0	11.1
1	40.7
2	149.5
3	548.4
4	2012.1
5	7383.1

6. quadratic

x	y
0	2.0
1	0.9
2	6.0
3	17.3
4	34.8
5	58.5

NewVocabulary

percentiles
random variable
probability distribution
binomial distribution
normal distribution
z-value
standard error of the mean
inferential statistics
confidence level
critical values
confidence interval
t-distribution
hypothesis test
level of significance
p-value
correlation coefficient
regression line
residual

ReviewVocabulary

statistics the science of collecting, analyzing, interpreting, and presenting data

histogram numerical data organized into equal intervals and displayed using bars

LESSON 4-1

Descriptive Statistics

Then	Now	Why?
● You found measures of central tendency and standard deviations.	● 1 Identify the shapes of distributions in order to select more appropriate statistics. 2 Use measures of position to compare two sets of data.	● A high school newspaper reports that according to a random survey of students, the mean and median number of unexcused tardies received by students last year were 7 and 5, respectively. While both of these values can be used to describe the center of the survey data, only a graph of the data can reveal which measure best represents the typical number of student tardies.

NewVocabulary

- univariate
- negatively skewed distribution
- symmetrical distribution
- positively skewed distribution
- resistant statistic
- cluster
- bimodal distribution
- percentiles
- percentile graph

1 Describing Distributions

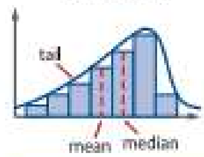
You described distributions of **univariate** or one-variable data numerically. You did this by calculating and reporting a distribution's

- center using either the mean or median and
- spread or variability using either the standard deviation or five-number summary (quartiles).

To determine which summary statistics you should choose to best describe the center and spread of a data set, you must identify the shape of the distribution. Three common shapes are given below.

KeyConcept Symmetric and Skewed Distributions

Negative or Left-Skewed Distribution

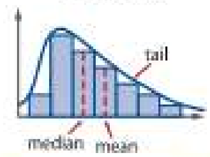


In a **negatively skewed distribution**, the mean is less than the median, the majority of the data is on the right, and the tail extends to the left.

Symmetrical Distribution

In a **symmetrical distribution**, the data are evenly distributed on both sides of the mean. The mean and median are approximately equal.

Positive or Right-Skewed Distribution



In a **positively skewed distribution**, the mean is greater than the median, the majority of the data is on the left, and the tail extends to the right.

When a distribution is reasonably symmetrical, the mean and median are close together. In skewed distributions, however, the mean is located closer to the tail than the median. Outliers, which are extremely high or low values in a data set, will cause the mean to drift even farther toward the tail. The median is less affected by the presence of outliers. For these reasons, the median is called a **resistant statistic** and the mean a *nonresistant statistic.*

Since standard deviation measures the spread of a distribution by how far data values are from the mean, this statistic is also nonresistant to the effects of outliers. This leads to the following guidelines about choosing summary statistics to describe a distribution.

KeyConcept Choosing Summary Statistics

When choosing measures of center and spread to describe a distribution, first examine the shape of the distribution.

- If the distribution is reasonably symmetrical and free of outliers, use the mean and standard deviation.
- If the distribution is skewed or has strong outliers, the five-number summary (minimum, quartile 1, median, quartile 3, maximum) usually provides a better summary of the overall pattern in the data.

When identifying the shape of a distribution, focus on major peaks in the graph instead of minor ups and downs.

Example 1 Skewed Distribution

REAL ESTATE The table shows the selling prices for a sample of new homes in a community.

New Home Selling Prices (thousands of dollars)				
248	219	234	250	225
299	205	212	215	245
257	228	221	233	212
220	213	231	212	266
238	249	292	223	235
218	227	209	242	217

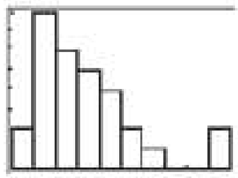
Technology Tip

Bin Width On a graphing calculator, each bar is called a *bin*. The bin widths are chosen by the calculator when you use the ZoomStat feature. The bin width can be adjusted by changing the Xscl parameter under WINDOW. A bin width that is too narrow or too wide will affect the apparent shape of a distribution.

a. Construct a histogram and use it to describe the shape of the distribution.

On a graphing calculator, press STAT Edit and input the data into L1. Then turn on Plot1 under the STAT PLOT menu and choose the histogram icon. Graph the histogram by pressing ZoomStat or by pressing GRAPH and adjusting the window manually.

The graph shown has a single peak. Using the TRACE feature, you can determine that this peak represents selling prices from $210 to $220 thousand.



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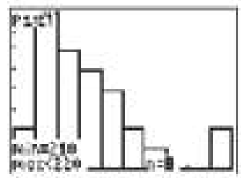


[200, 300] scl: 10 by [0, 8] scl: 1

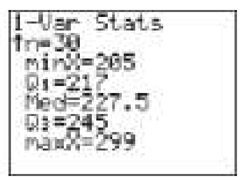
The graph is positively skewed. Most of the selling prices appear to fall between $210 and $250 thousand, but a few were much higher, so the tail of the distribution trails off to the right.

Watch Out!

Skew Direction The tail of the distribution indicates in which direction a distribution is skewed, not the peak.

b. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

Since the distribution is skewed, use the five-number summary instead of the mean and standard deviation to summarize the center and spread of the data. To display this summary, press STAT, select 1-Var Stats under the CALC submenu, and scroll down.



The five-number summary (minX, Q1, Med, Q3, and maxX) indicates that while the prices range from $205 to $299 thousand, the median selling price was $227.5 thousand and half of the prices were between $217 and $245 thousand.

Guided Practice

1. **LAB GRADES** The laboratory grades of all of the students in a biology class are shown.

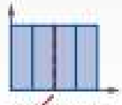
   A. Construct a histogram, and use it to describe the shape of the distribution.

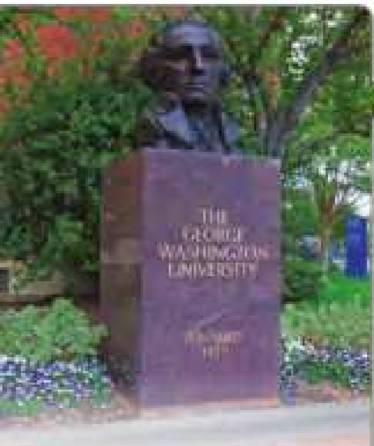
   B. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

Lab Grades (percent)					
72	84	67	80	75	87
86	76	89	91	96	74
68	83	80	76	63	98
92	73	80	88	94	78

StudyTip

Uniform Distribution In another type of distribution, known as a *uniform distribution*, each value has the same relative frequency, as shown below.





Real-WorldLink

In 2008, George Washington University had the highest tuition costs in the U.S. at $37,826 per year. This was roughly 82% of the median annual family income of $46,326.

Source: Forbes Magazine

Distributions of data are not always symmetrical or skewed. Sometimes data will fall into subgroups or **clusters**. If a distribution has a gap in the middle, two separate clusters of data may result. A distribution of data that has two modes, and therefore two peaks, is known as a **bimodal distribution**.

In data that represent a reported preference about a topic, a bimodal distribution can indicate a polarization of opinions. Often, however, a bimodal distribution indicates that the sample data comes from two or more overlapping distributions.

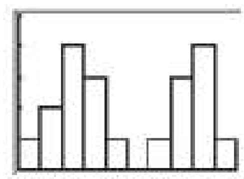
Real-World Example 2 Bimodal Distribution

TUITION **The annual cost of tuition for a sample of 20 colleges at a college fair are shown.**

College Tuition Costs ($)				
32,000	10,100	31,000	11,000	31,500
5,500	35,000	10,800	3600	11,500
7,400	15,100	18,200	25,600	33,100
36,200	32,000	30,400	14,300	12,400

a. Construct a histogram, and use it to describe the shape of the distribution.

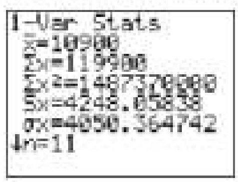
The histogram of the data has not one but two major peaks. Therefore, the distribution is neither symmetrical nor skewed but bimodal. The two separate clusters suggest that two types of colleges are mixed in the data set. It is likely that the 11 colleges with less expensive tuitions are public colleges, and the 9 colleges with more expensive tuitions are private colleges.

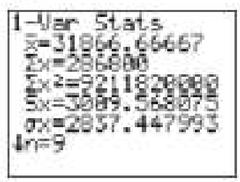


[0, 40,000] scl: 4000 by [0, 6] scl: 1

b. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

Since the distribution is bimodal, an overall summary of center and spread would give an inaccurate depiction of the data. Instead, summarize the center and spread of each cluster. Since each cluster appears fairly symmetrical, enter each cluster separately and summarize the data using the mean and standard deviation of each cluster.





The mean cost of Cluster 1 is $10,900 with a standard deviation of about $4050, while the mean cost of the Cluster 2 is $31,866 with a standard deviation of about $2837.

GuidedPractice

2. **TRACK** The numbers of minutes that 30 members of a high school cross-country team ran during a practice session are shown.

Practice Session Times (min)									
26	36	31	58	51	29	56	23	61	46
30	50	45	22	64	49	34	42	53	55
41	37	28	54	32	50	59	48	62	39

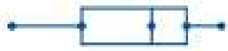
A. Construct a histogram, and use it to describe the shape of the distribution.

B. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

You can also examine a box-and-whisker plot or *box plot* of a set of data to identify the shape of a distribution. To determine symmetry or skewness from a box plot, you must consider both the position of the line representing the median and the length of each "whisker."

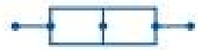
KeyConcept Symmetric and Skewed Box Plots

Negatively Skewed



The left whisker is longer than right whisker, and the line representing the median is closer to Q_3 than to Q_1.

Symmetrical



The whiskers are the same length, and the line representing the median is exactly between Q_1 and Q_3.

Positively Skewed

The right whisker is longer than left whisker, and the line representing the median is closer to Q_1 than to Q_3.

Example 3 Describe a Distribution Using a Box Plot

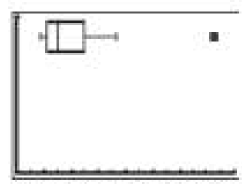
POPULATION The table shows the populations, in thousands of people, during a recent year for fifteen cities in Florida.

Population (Thousands)				
151	95	303	89	186
362	137	109	152	118
102	226	139	736	248

a. Construct a box plot, and use it to describe the shape of the distribution.

Input the data into L1 on a graphing calculator. Then turn on Plot1 under the STAT PLOT menu and choose the box plot icon. Graph the box plot by pressing ZoomStat or by pressing WINDOW and adjusting the window manually.

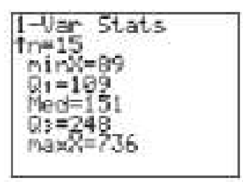
Since the right whisker is longer than the left whisker and the line representing the median is closer to Q_1 than to Q_3, the distribution is positively skewed. Notice that the distribution has an outlier at 736.



[0, 800] scl: 50 by [0, 1] scl: 0.5

b. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

Since the distribution is skewed, use the five-number summary. This summary indicates that while the populations ranged from 89,000 to 736,000, the median population was 151,000. Populations in the middle half of the data varied by 248,000 − 109,000 or 139,000 people, which is the interquartile range.



ReviewVocabulary

interquartile range the difference between the upper quartile and lower quartile of a data set

GuidedPractice

3. TRUCKS The costs on a used car website for twelve trucks that are the same make, model, and year are shown.

A. Construct a box plot, and use it to describe the shape of the distribution.

B. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

Used Truck Costs ($)		
9,000	8,200	9,200
7,800	8,900	8,500
6,500	7,500	7,800
8,000	6,400	5,500

2 Measures of Position

The quartiles given by the five-number summary specify the positions of data values within a distribution. For this reason, box plots are most useful for side-by-side comparisons of two or more distributions.

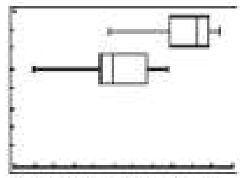
Example 4 Compare Position Using Box Plots

BASKETBALL The number of games won by the Boston Celtics during two different 15-year periods are shown. Construct side-by-side box plots of the data sets. Then use this display to compare the distributions.

1st 15-Year Period				
49	52	59	57	60
60	54	62	59	58
54	48	34	44	56

2nd 15-Year Period				
48	32	35	33	15
36	19	36	35	49
44	36	45	33	24

Input the data into L1 and L2. Then turn on Plot1 and Plot2 under the STAT PLOT menu, choose , and graph the box plots by pressing ZoomStat or by pressing GRAPH and adjusting the window manually.



[10, 65] scl: 5 by [0, 1] scl: 0.5

Compare Measures of Position

The median number of games won each season in the first 15-year period is greater than those won every season in the second 15-year period. The first quartile for the first 15-year period is approximately equal to the maximum value for the second 15-year period. This means that 75% of the data values for the first 15-year period are greater than any of the values in the second 15-year period. Therefore, we can conclude that the Celtics had significantly more successful seasons during the first 15-year period than during the second 15-year period.

Compare Spreads

The spread of the middle half of the data, represented by the box, is roughly the same in each distribution. Therefore, the variability in the number of games won each season in those 15-year periods was about the same.

GuidedPractice

4. BASEBALL The number of home runs hit in Major League Baseball in 1927 and 2007 by the top 20 home run hitters is shown. Construct side-by-side box plots of the data sets. Then use this display to compare the distributions.

1927				
19	10	13	30	16
14	18	14	12	60
30	14	47	14	15
12	20	18	26	17

2007				
40	32	47	31	34
34	33	35	30	46
32	54	31	33	32
36	31	34	50	35

StudyTip

Fractiles Quartiles and percentiles are two types of *fractiles*—numbers that divide an ordered set of data into equal groups. *Deciles* divide a data set into ten equal groups.

In addition to quartiles, you can also use *percentiles* to indicate the relative position of an individual value within a data set. **Percentiles** divide a distribution into 100 equal groups and are symbolized by $P_1, P_2, P_3, \ldots, P_{99}$. The *n*th percentile or P_n, is the value such that n% of the data are less than P_n and $(100 - n)$% of the data are equal to or greater than P_n. The highest percentile that a data value can be is the 99^{th} percentile.

A **percentile graph** uses the same values as a cumulative relative frequency graph, except that the proportions are instead expressed as percents.

You can use a percentile graph to approximate the percentile rank of a given value for a variable.

Example 5 Construct and Use a Percentile Graph

GPA The table gives the frequency distribution of the GPAs of the 200 students at Ashlyn's high school.

Class Boundaries	f	Class Boundaries	f
2.00–2.25	10	3.00–3.25	36
2.25–2.50	28	3.25–3.50	32
2.50–2.75	30	3.50–3.75	26
2.75–3.00	32	3.75–4.00	6

WatchOut!

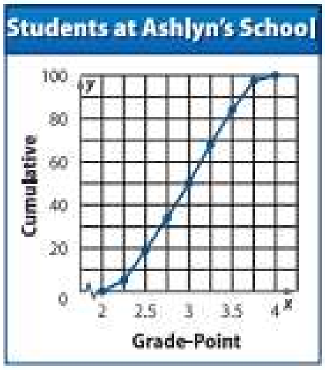
Percentage Versus Percentile

Percentages are not the same as *percentiles*. If a student gets 85 problems correct out of a possible 100, he obtains a percentage score of 85. This does not indicate whether the grade was high or low compared to the rest of the class.

a. Construct a percentile graph of the data.

First, find the cumulative frequencies. Then find the cumulative percentages by expressing the cumulative frequencies as percents. The calculations for the first two classes are shown.

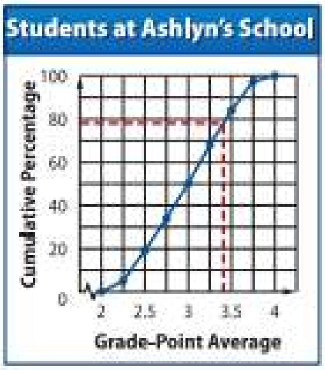
Class Boundaries	f	Cumulative Frequency	Cumulative Percentages
2.00–2.25	10	10	$\frac{10}{200}$ or 5%
2.25–2.50	28	10 + 28 or 38	$\frac{38}{200}$ or 19%
2.50–2.75	30	68	34%
2.75–3.00	32	100	50%
3.00–3.25	36	136	68%
3.25–3.50	32	168	84%
3.50–3.75	26	194	97%
3.75–4.00	6	200	100%



Finally, graph the data with the class boundaries along the x-axis and the cumulative percentages along the y-axis, as shown.

b. Estimate the percentile rank a GPA of 3.4 would have in this distribution, and interpret its meaning.

Find 3.4 on the x-axis and draw a vertical line to the graph. This point on the graph corresponds to approximately the 78th percentile. Therefore, a student with a GPA of 3.4 has a better grade-point average than about 78% of the students at Ashlyn's school.



GuidedPractice

5. HEIGHT The table gives the frequency distribution of the heights of girls in Mrs. Khawla Math classes.

   A. Construct a percentile graph of the data.

   B. Estimate the percentile rank a girl with a height of 169 centimeters would have in this distribution, and interpret its meaning.

Class Boundaries	Frequency (f)
146.5–154	11
154–161.5	15
161.5–169	15
169–176.5	12
176.5–184	7

WatchOut!

Understanding Percentiles

Saying that a girl's height is at the 75th percentile does *not* mean that her height is 75% of some ideal height. Instead, her height is greater than 75% of all girls in the precalculus class.

Exercises

For Exercises 1–4, complete each step.

a. Construct a histogram, and use it to describe the shape of the distribution.

b. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice. (Examples 1 and 2)

1. **AVIATION** The landing speeds in kilometers per hour of 20 commercial airplane flights at a certain airport are shown.

Landing Speeds (kmph)			
150	157	153	145
155	158	158	162
149	142	138	154
156	161	146	148
158	144	151	152

2. **COMPUTERS** The retail prices of laptop and desktop computers at a certain electronics store are shown.

Computer Prices (AED)		
950	1,000	975
1,150	450	1,075
675	1,250	540
1,025	1,180	925
580	950	890

3. **BOWLING** Bowling scores range from 0 to 300. The scores for randomly selected players at a certain bowling alley are shown.

Bowling Scores				
116	81	234	173	75
61	205	92	219	156
134	259	273	53	241
105	190	94	127	235
228	248	271	46	112
99	223	142	217	68

4. **SALARIES** The starting salary for an employee at a certain new company ranges from $20,000 to $90,000. Starting salary depends in part on the employee's years of previous experience and the level of the position for which they were hired. The starting salaries for all the company's new hires last year are shown.

Salaries (thousands of dollars)			
24	40	34	59
48	52	65	54
68	26	85	32
36	42	33	45
38	89		

For Exercises 5–6, complete each step.

a. Construct a box plot, and use it to describe the shape of the distribution.

b. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice. (Example 3)

5. **VIDEO GAMES** The amount of time that a sample of students at East High School spends playing video games each week is shown.

Time Spent Playing Video Games (hours)					
1.5	2.5	0	4.5	12.5	1
2.5	4	2	8.5	1.5	9
1	0	2	1.5	5.5	2

6. **ACT** The students in Mrs. Eiman's homeroom class recently took the ACT. The score for each student is shown.

ACT Scores							
32	21	24	35	28	29	28	30
28	25	29	19	24	23	25	22
23	29	27	24	27	29	21	18

For Exercises 7–8, complete each step.

a. Construct side-by-side box plots of the data sets.

b. Use this display to compare the two distributions. (Example 4)

7. **HYBRID CARS** The fuel efficiency in miles per gallon for 18 hybrid cars manufactured during two recent years are shown.

Year 1								
23	48	31	27	28	35	27	28	24
15	16	28	33	22	16	28	40	24
Year 2								
29	34	25	33	26	35	27	40	27
22	48	29	34	21	24	29	21	34

8. **EARTHQUAKES** The Richter scale magnitudes of 18 earthquakes that occurred in recent years in Alaska and California are shown.

Alaska								
6.6	6.6	6.4	7.2	6.5	6.7	4.8	6.8	6.8
7.8	6.9	7.1	6.6	7.9	6.7	5.3	7.9	7.7
California								
5.4	5.4	5.6	4.4	4.2	4.3	5.2	4.5	4.7
6.6	4.9	7.2	5.2	4.1	6.0	3.0	6.6	3.5

9. **MARINE BIOLOGY** The table gives the frequency distribution of the weights, in kilograms, of 40 adult female sea otters in Washington. (Example 5)

a. Construct a percentile graph of the data.

b. Estimate the percentile rank a weight of 25 kilograms would have in this distribution, and interpret its meaning.

Class Boundaries	f
18–20.5	4
20.5–23	5
23–25.5	7
25.5–28	12
28–30.5	9
30.5–33	3

10. **RAINFALL** The table gives the frequency distribution of the average annual rainfall for all 50 U.S. states. (Example 5)

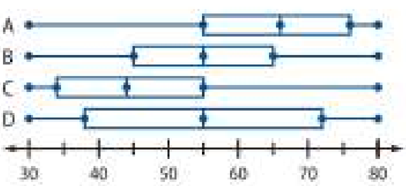
Class Boundaries	f
0–20	3
20–40	8
40–60	4
60–80	14
80–100	16
100–120	5

a. Construct a percentile graph of the data.

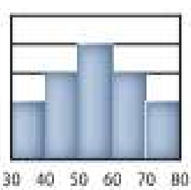
b. Estimate the percentile rank an average rainfall of 100 centimeters would have in this distribution, and interpret its meaning.

B

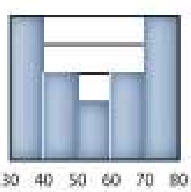
Write the letter of the box plot that corresponds to each of the following histograms.



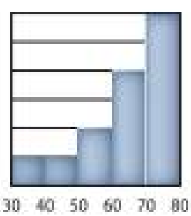
11.



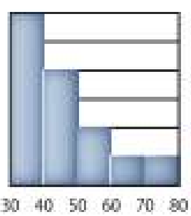
12.



13.



14.



15. **ATTENDANCE** The average number of New York Yankee home game attendees in thousands of people, per season, from 1979 to 2008 is shown.

Home Game Attendance					
31.7	32.4	30.2	25.2	27.9	22.5
27.5	28.0	30.0	32.7	27.0	24.8
23.0	21.6	29.8	29.7	23.5	27.8
31.9	36.5	40.7	38.0	40.8	42.7
42.8	47.8	50.5	51.9	52.7	53.1

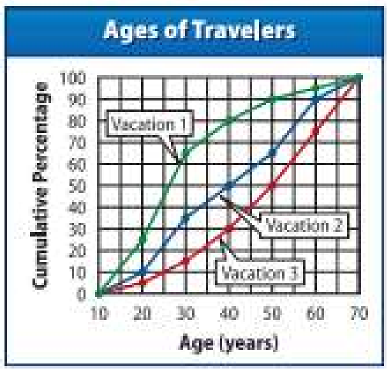
a. Construct a histogram and box plot, and use the graphs to describe the shape of the distribution.

b. Find the average number of people who attended home games during the past 30 years.

c. Which of the graphs would be best to use when estimating the average? Explain your reasoning.

d. Can either of the graphs from part **a** be used to describe any trends in home game attendance during that period? Explain your reasoning.

16. **VACATION** The percentile graph represents the ages of people who went on three different two-week vacations.



a. Describe the shape of each of the distributions.

b. Which vacation had younger travelers? older travelers? Explain your reasoning.

17. **MANUFACTURING** The lifetimes, measured in number of charging cycles, for two brands of rechargeable batteries are shown.

Brand A				
998	950	1,020	1,003	990
942	1,115	973	1018	981
1,047	1,002	997	1,110	1,003
Brand B				
892	1,044	1,001	999	903
950	998	993	1,002	995
990	1,000	1,005	997	1,004

a. Construct a histogram of each data set.

b. Which of the brands has a greater variation in lifetime?

18. **BASKETBALL** The heights in meters for the players on the U.S. men's and women's national basketball teams during the 2008 Olympics are shown.

Men's Heights					
2.06	1.93	2.03	1.91	1.98	1.93
1.98	2.11	2.08	1.83	2.06	2.03
Women's Heights					
1.75	1.85	1.75	1.73	1.85	1.96
1.83	1.88	1.83	1.98	1.80	1.93

a. Construct a percentile graph of the data.

b. Estimate the percentile ranks that a male and a female player with a height of 1.91 meters would have in each distribution. Interpret their meaning.

c. Suppose the 1.98-meter-tall women's player is replaced with a 1.88-meter-tall player. What percentile rank would the new player have in the corresponding distribution?

Another measure of center known as the *midquartile* is given by $\frac{Q_1 + Q_3}{2}$. Find Q_1, Q_2, Q_3, and the midquartile for each set of data.

19.

0.12	0.25	0.19	0.38	0.28	0.16
0.41	0.29	0.32	0.11	0.04	0.25
0.29	0.07	0.26	0.09	0.31	0.23

20.

112	101	138	200	176	199
105	127	146	128	116	154
167	202	191	143	205	130

21. **ENERGY** Petroleum consumption from 1988 to 2007 for the United States and for North America is shown.

United States (thousands of barrels/day)				
16,700	17,300	17,300	17,000	16,700
17,000	17,200	17,700	17,700	18,300
18,600	18,900	19,500	19,700	19,600
19,800	20,000	20,700	20,800	20,700
North America (thousands of barrels/day)				
19,900	20,600	20,800	20,000	20,200
20,600	20,800	21,400	21,300	22,000
22,400	22,800	23,500	23,800	23,700
23,800	24,200	25,000	25,200	25,000

a. Construct side-by-side box plots and histograms.

b. Compare the average petroleum consumption for the U.S. and North America.

c. Which of the graphs is easier to use when comparing measures of center and spread?

d. On average, what percent of petroleum consumption in North America can be attributed to the U.S.? Round to the nearest percent.

22. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate how a linear transformation affects the shape, center, and spread of a distribution of data. Consider the table shown.

52	37	59	31	45
23	48	42	65	39
40	53	14	49	56
68	32	77	44	28

a. **GRAPHICAL** Construct a histogram and use it to describe the shape of the distribution.

b. **NUMERICAL** Find the mean and standard deviation of the data set.

c. **TABULAR** Perform each of the following linear transformations of the form $X' = a + bX$, where X is the initial data value and X' is the transformed data value. Record each set of transformed data values (**i–iii**) in a separate table.

i. $a = 3, b = 5$ **ii.** $a = 10, b = 1$ **iii.** $a = 0, b = 5$

d. **GRAPHICAL** Repeat parts **a** and **b** for each set of transformed data values that you found in part **c**. Adjust the bin width for each appropriately.

e. **VERBAL** Describe how a linear transformation affects the shape, center, and spread of a distribution of data.

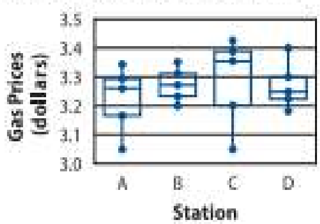
f. **ANALYTICAL** If every value in a data set is multiplied by a constant c, what will happen to the mean and standard deviation of the distribution?

H.O.T. Problems Use Higher-Order Thinking Skills

23. **WRITING IN MATH** Explain why using the range can be an ineffective method for measuring the spread of a distribution of data.

24. **CHALLENGE** Suppose 20% of a data set lies between 35 and 55. If 10 is added to each value in the set and then each result is doubled, what values will 20% of the resulting data lie between?

REASONING **The gas prices, in dollars per gallon, at four gas stations over a period of one month are shown.**



25. Which of the stations has the greatest variation in prices? the least variation? Explain your reasoning.

26. Which of the distributions is positively skewed? negatively skewed? symmetrical? Explain your reasoning.

27. **WRITING IN MATH** Why is the median less affected by outliers than the mean? Justify your answer.

Spiral Review

Write each complex number in exponential form.

28. $\sqrt{3} + \sqrt{3}i$

29. $\sqrt{5} - \sqrt{5}i$

30. $\sqrt{2} - \sqrt{6}i$

Use the Binomial Theorem to expand each binomial.

31. $(3a + 4b)^5$

32. $(5c - 2d)^4$

33. $(-2x + 4y)^6$

Find the sum of each geometric series described.

34. first five terms of $\frac{5}{3} + 5 + 15 + \cdots$

35. first six terms of $65 + 13 + 2.6 + \cdots$

36. first ten terms of $1 - \frac{3}{2} + \frac{9}{4} - \cdots$

Find the angle θ between vectors u and v.

37. $\mathbf{u} = 4\mathbf{i} - 2\mathbf{j} + 9\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 7\mathbf{j} - 10\mathbf{k}$

38. $\mathbf{u} = \langle -7, 4, 2 \rangle$, $\mathbf{v} = \langle 9, -5, 1 \rangle$

39. $\mathbf{u} = \langle 4, 4, -6 \rangle$, $\mathbf{v} = \langle 8, -5, 2 \rangle$

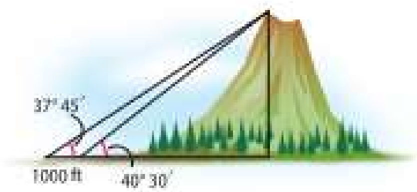
Graph the ellipse given by each equation.

40. $\frac{x^2}{4} + \frac{(y-3)^2}{25} = 1$

41. $\frac{(x+6)^2}{16} + \frac{(y-5)^2}{9} = 1$

42. $\frac{(x-2)^2}{28} + \frac{y^2}{8} = 1$

43. **SURVEYING** To determine the new height of a volcano after an eruption, a surveyor measured the angle of elevation to the top of the volcano to be 37° 45′. He then moved 1,000 feet closer to the volcano and measured the angle of elevation to be 40° 30′. Determine the new height of the volcano.



Skills Review for Standardized Tests

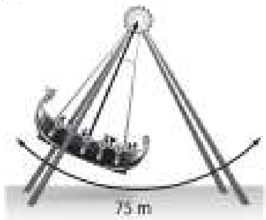
44. **REVIEW** An amusement park ride operates like the bob of a pendulum. On its longest swing, the ship travels through an arc 75 meters long. Each successive swing is two-fifths the length of the preceding swing. What is the total distance the ship will travel from the beginning of its longest swing if the ride is allowed to continue without intervention?

A 75 m

B 125 m

C 150 m

D 187.5 m



45. **REVIEW** The value of a certain car depreciated at a constant rate. If the initial value was $25,000 and the car was worth $8192 after five years, find the annual rate of depreciation.

F 10%

G 20%

H 30%

J 40%

46. **SAT/ACT** The values of each house in a city are collected and analyzed. Which descriptive statistic will best describe the data?

A mean

B median

C mode

D range

E standard deviation

47. The table shows the frequency distribution of scores on the state driving test at a particular center on a given day. Estimate the percentile rank of someone who scored a 72 that day.

F 27%

G 30%

H 34%

J 72%

Class Boundaries	Frequency f
0–65.5	12
65.5–70.5	3
70.5–75.5	4
75.5–80.5	1
80.5–85.5	9
85.5–90.5	13
90.5–95.5	8
95.5–100	6

LESSON 4-2 Probability Distributions

::Then	::Now	::Why?
• You found probabilities of events involving combinations.	• **1** Construct a probability distribution, and calculate its summary statistics. **2** Construct and use a binomial distribution, and calculate its summary statistics.	• Car insurance companies use statistics to measure the risk associated with particular events, such as collisions. Using data about what has happened in the past, they assign probabilities to all possible outcomes relating to the event and calculate statistics based on how these probabilities are distributed. With these statistics, they can predict the likelihood of certain outcomes and make decisions accordingly.

NewVocabulary
- random variable
- discrete random variable
- continuous random variable
- probability distribution
- expected value
- binomial experiment
- binomial distribution
- binomial probability distribution function

1 Probability Distributions

In the previous lesson, you used descriptive statistics to analyze a *variable*, a characteristic of a population. In that lesson, the values the variable could take on were determined by collecting data. In this lesson, you will consider variables with values that are determined by chance.

A **random variable** X represents a numerical value assigned to an outcome of a probability experiment. There are two types of random variables: discrete and continuous.

KeyConcept Discrete and Continuous Random Variables

A **discrete random variable** can take on a finite or countable number of possible values.

Example

Number of Car Accidents Per Year, X

A **continuous random variable** can take on an infinite number of possible values within a specified interval.

Example

Number of Miles Driven, X

Since different statistical techniques are used to analyze these two types of random variables, it is important to be able to distinguish between them. To correctly classify a random variable, consider whether X represents counted or measured data.

Example 1 Classify Random Variables as Discrete or Continuous

Classify each random variable X as *discrete* or *continuous*. Explain your reasoning.

a. X represents the weight of the cereal in a 450-gram box of cereal chosen at random from those on an assembly line.

The weight of the cereal could be any weight between 0 and 450 grams. Therefore, X is a continuous random variable.

b. X represents the number of cars in a school parking lot chosen at a random time during the school day.

The number of cars in the parking lot is countable. There could be 0, 1, 2, 3, or some other whole number of cars. Therefore, X is a discrete random variable.

GuidedPractice

1A. X represents the time it takes to serve a fast-food restaurant customer chosen at random.

1B. X represents the attendance at a randomly selected monthly school board meeting.

The sample space for the familiar theoretical probability experiment of tossing two coins is {TT, TH, HT, HH}. If X is the random variable for the number of heads, then X can assume the value 0, 1, or 2. From the sample space, you can find the theoretical probability of getting no, one, or two heads.

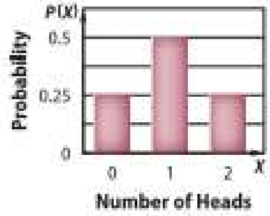
$P(0) = \frac{1}{4}$ $\quad P(1) = \frac{1}{2}$ $\quad P(2) = \frac{1}{4}$

ReadingMath

Probabilities of Random Variables The notation $P(1)$ is read *the probability that the random variable X is equal to 1.*

The table below and the graph at the right show the *probability distribution* of X.

Number of heads, X	0	1	2
Probability, $P(X)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$



KeyConcept Probability Distribution

A **probability distribution** of a random variable X is a table, equation, or graph that links each possible value of X with its probability of occurring. These probabilities are determined theoretically or by observation.

A probability distribution must satisfy the following conditions.

- The probability of each value of X must be between 0 and 1. That is, $0 \leq P(X) \leq 1$.
- The sum of all the probabilities for all the values of X must equal 1. That is, $\sum P(X) = 1$.

StudyTip

Continuous Distributions This lesson focuses on discrete random variables.

To construct a discrete probability distribution using observed instead of theoretical data, use the frequency of each observed value to compute its probability.

Example 2 Construct a Probability Distribution

TEACHER EVALUATION On a teacher evaluation form, students were asked to rate the teacher's explanations of the subject matter using a score from 1 to 5, where 1 was too simplified and 5 was too technical. Use the frequency distribution shown to construct and graph a probability distribution for the random variable X.

Score, X	Frequency
1	1
2	8
3	20
4	16
5	5

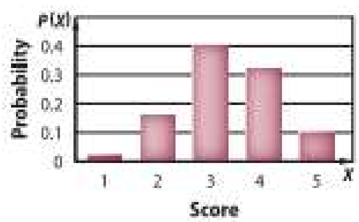
To find the probability that X takes on each value, divide the frequency of each value by the total number of students rating this teacher, which is $1 + 8 + 20 + 16 + 5$ or 50.

$P(1) = \frac{1}{50}$ or 0.02 $\qquad P(2) = \frac{8}{50}$ or 0.16 $\qquad P(3) = \frac{20}{50}$ or 0.40

$P(4) = \frac{16}{50}$ or 0.32 $\qquad P(5) = \frac{5}{50}$ or 0.10

The probability distribution of X is shown below, and its graph is shown at the right.

Score, X	1	2	3	4	5
$P(X)$	0.02	0.16	0.40	0.32	0.10



CHECK Note that all of the probabilities in the table are between 0 and 1 and that $\sum P(X) = 0.02 + 0.16 + 0.4 + 0.32 + 0.1$ or 1. ✔

GuidedPractice

2. **CAR SALES** A car salesperson tracked the number of cars he sold each day during a 30-day period. Use the frequency distribution of the results to construct and graph a probability distribution for the random variable X, rounding each probability to the nearest hundredth.

Cars Sold, X	0	1	2	3
Frequency	20	7	2	1

To compute the mean of a probability distribution, we must use a formula different from that used to compute the mean of a population. To understand why, consider computing the mean of the number of heads X resulting from an infinite number of two-coin tosses. We cannot compute the mean using $\mu = \frac{\sum X}{N}$, since N would be infinite. However, the probability distribution of X tells us what fraction of those tosses we would *expect* to have a value of 0, 1, or 2.

Number of Heads after Two Coin Tosses

{TT, TT, ..., TT, TT,	HT, HT, ..., HT, HT, TH, TH, ..., TH, TH,	HH, HH, ..., HH, HH}
{0, 0 ..., 0, 0,	1, 1, ..., 1, 1, 1, 1, ..., 1, 1,	2, 2, ..., 2, 2}
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Therefore, we would expect that on average the number of heads for many or an infinite number of tosses would be $\frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2$ or 1. This method for finding the mean of a probability distribution is summarized below.

KeyConcept Mean of a Probability Distribution

Words To find the mean of a probability distribution of X, multiply each value of X by its probability and find the sum of the products.

Symbols The mean of a random variable X is given by $\mu = \sum[X \cdot P(X)]$, where $X_1, X_2, \ldots, X_n$ are the values of X and $P(X_1), P(X_2), \ldots, P(X_n)$ are the corresponding probabilities.

Example 3 Mean of a Probability Distribution

TEACHER EVALUATION The table shows the probability distribution for the teacher evaluation question from Example 2. Find the mean score to the nearest hundredth, and interpret its meaning in the context of the problem situation.

Score, X	$P(X)$
1	0.02
2	0.16
3	0.40
4	0.32
5	0.10

Multiply each score by its probability, and find the sum of these products. Organize your calculations by extending the table.

Score, X	$P(X)$	$X \cdot P(X)$
1	0.02	$1 \cdot 0.02 = 0.02$
2	0.16	$2 \cdot 0.16 = 0.32$
3	0.40	$3 \cdot 0.40 = 1.20$
4	0.32	$4 \cdot 0.32 = 1.28$
5	0.10	$5 \cdot 0.10 = 0.50$
		$\sum[X \cdot P(X)]$ or 3.3

Therefore, the mean μ of this probability distribution is about 3.3.

Since a score of 3 indicates that the teacher's explanations were neither too simplified nor too technical, a mean of 3.3 indicates that on average, students felt that this teacher's explanations were appropriate but leaned slightly towards being too technical.

StudyTip

Rounding Rule The mean, as well as the variance and standard deviation, discussed on the next page, should be rounded to one decimal place more than that of an actual value that X can assume.

GuidedPractice

3. **CAR SALES** Find the mean of the probability distribution that you constructed in Guided Practice 2 and interpret its meaning in the context of the problem situation.

The variance formula used for population distributions can also not be used to calculate the variance or standard deviation of a probability distribution, because the value of N would be infinite. Instead, the following formulas are used to find the spread of a probability distribution.

KeyConcept Variance and Standard Deviation of Probability Distribution

Words To find the variance of a probability distribution of X, subtract the mean of the probability distribution from each value of X and square the difference. Then multiply each difference by its corresponding probability and find the sum of the products. The standard deviation is the square root of the variance.

Symbols The variance of a random variable X is given by $\sigma^2 = \sum[(X - \mu)^2 \cdot P(X)]$, and the standard deviation is given by $\sigma = \sqrt{\sigma^2}$.

StudyTip

Alternate Formula
A mathematically equivalent formula for the variance of a probability distribution that can significantly simplify the calculation of this statistic is $\sigma^2 = \sum[(X^2 \cdot P(X)] - \mu^2$.

Example 4 Variance and Standard Deviation of a Probability Distribution

TEACHER EVALUATION Find the variance and standard deviation of the probability distribution for the teacher evaluation question from Example 2 to the nearest hundredth.

Score, X	$P(X)$
1	0.02
2	0.16
3	0.40
4	0.32
5	0.10

Subtract each value of X from the mean found in Example 3, 3.32 and square the difference. Then multiply each difference by its corresponding probability and find the sum of the products.

Score, X	$P(X)$	$(X - \mu)^2$	$(X - \mu)^2 \cdot P(X)$
1	0.02	$(1 - 3.32)^2 = 5.38$	$5.38 \cdot 0.02 = 0.1076$
2	0.16	$(2 - 3.32)^2 = 1.74$	$1.74 \cdot 0.16 = 0.2788$
3	0.40	$(3 - 3.32)^2 = 0.10$	$0.10 \cdot 0.40 = 0.0410$
4	0.32	$(4 - 3.32)^2 = 0.46$	$0.46 \cdot 0.32 = 0.1480$
5	0.10	$(5 - 3.32)^2 = 2.82$	$2.82 \cdot 0.10 = 0.2822$
			$\sum[(X - \mu)^2 \cdot P(X)] = 0.8576$

The variance σ^2 is about 0.86, and the standard deviation is $\sqrt{0.8576}$ or about 0.93.

GuidedPractice

4. **CAR SALES** Find the variance and standard deviation of the probability distribution that you constructed in Guided Practice 2 to the nearest hundredth.

The **expected value** $E(X)$ of a random variable for a probability distribution is equal to the mean of the random variable. That is, $E(X) = \mu = \sum[X \cdot P(X)]$.

Example 5 Find an Expected Value

FUNDRAISERS At a fundraiser, 500 tickets are sold at $1 each for three prizes of $100, $50, and $10. What is the expected value of your net gain if you buy a ticket?

Construct a probability distribution for the possible net gains. Then find the expected value. The net gain for each prize is the value of the prize minus the cost of the tickets purchased.

Gain, X	$100 − 1 or $99	$50 − 1 or $49	$10 − 1 or $9	$0 − 1 or −$1
Probability, $P(X)$	$\frac{1}{500}$ or 0.002	$\frac{1}{500}$ or 0.002	$\frac{1}{500}$ or 0.002	$\frac{497}{500}$ or 0.994

$$E(X) = \sum[X \cdot P(X)]$$
$$= (99 \cdot 0.002) + (49 \cdot 0.002) + (9 \cdot 0.002) + (-1 \cdot 0.994) \text{ or about } -\$0.68$$

This expected value means that the average loss for someone purchasing a ticket is $0.68.

WatchOut!

Misinterpreting Expected Value
An expected value such as that calculated in Example 5 is *not* an indication of how much a person might win or lose. In Example 5, a person can lose only $1 for each ticket purchase and can win only $100, $50, or $10.

GuidedPractice

5. **WATER PARK** A water park makes \$350,000 when the weather is normal and loses \$80,000 per season when there are more bad weather days than normal. If the probability of having more bad weather days than normal this season is 35%, find the park's expected profit.

2 Binomial Distribution

Many probability experiments can be reduced to one involving only two outcomes: success or failure. For example, a multiple-choice question with five answer choices can be classified as simply correct or incorrect, or a medical treatment can be classified as effective or ineffective. Such experiments have been reduced to *binomial* experiments.

KeyConcept Binomial Experiment

A **binomial experiment** is a probability experiment that satisfies the following conditions.

- The experiment is repeated for a fixed number of independent trials n.
- Each trial has only two possible outcomes, success S or failure F.
- The probability of success $P(S)$ or p is the same in every trial. The probability of failure $P(F)$ or q is $1 - p$.
- The random variable X represents the number of successes in n trials.

Real-WorldLink

One in five American teens ages 12 years and older owns a portable MP3 player. More than one in twenty teens own more than one player.

Source: Digital Trends

Real-World Example 6 Identify a Binomial Experiment

Determine whether each experiment is a binomial experiment or can be reduced to a binomial experiment. If it can be presented as a binomial experiment, state the values of *n*, *p*, and *q*. Then list all possible values of the random variable. If it is not, explain why not.

a. The results of a school survey indicate that 68% of students own an MP3 player. Six students are randomly selected and asked if they own an MP3 player. The random variable represents the number of students who say that they do own an MP3 player.

The experiment satisfies the conditions of a binomial experiment.

- Each student selected represents one trial, and the selection of each of the six students is independent of the others.
- There are only two possible outcomes: either the student owns an MP3 player S or does not own an MP3 player F.
- The probability of success is the same for each student selected, $P(S) = 0.68$.

In this experiment, $n = 6$ and $p = P(S)$ or 0.68. The probability of failure is $q = 1 - p$, so $q = 1 - 0.68$ or 0.32. X represents the number of students who own an MP3 player out of those selected, so $X = 0, 1, 2, 3, 4, 5,$ or 6.

b. Five cards are drawn at random from a deck to make a hand for a card game. The random variable represents the number of spades.

In this experiment, each card selected represents one trial. The probability of drawing a spade for the first card is $\frac{13}{52}$ or $\frac{1}{4}$. However, since this card is kept to make a player's hand, the trials are not independent, and the probability of success for each draw will not be the same. Therefore, this experiment cannot be reduced to a binomial experiment.

GuidedPractice

6A. The results of a survey indicate that 61% of students like the new school uniforms and 24% do not. Twenty students are randomly selected and asked if they like the uniforms. The random variable represents the number who say that they do like the uniforms.

6B. You complete a test by randomly guessing the answers to 20 multiple-choice questions that each have 4 answer choices, only one of which is correct. The random variable represents the number of correct answers.

Study Tip

Look Back Refer to Lesson 3–5 to review binomial expansion and the Binomial Theorem.

The distribution of the outcomes of a binomial experiment and their corresponding probabilities is called a **binomial distribution.** The probabilities in this distribution can be calculated using the following formula, which represents the $p^x q^{n-x}$ term in the binomial expansion of $(p+q)^n$.

KeyConcept Binomial Probability Formula

The probability of X successes in n independent trials of a binomial experiment is

$$P(X) = {}_nC_x p^x q^{n-x} = \frac{n!}{(n-x)!\,x!} p^x q^{n-x},$$

where p is the probability of success and q is the probability of failure for an individual trial.

Notice that this formula represents a discrete function of the random variable X, known as the **binomial probability distribution function**.

Example 7 Binomial Distributions

EXERCISE In a recent poll, 35% of teenagers said they exercise regularly. Five teenagers chosen at random are asked if they exercise regularly. Construct and graph a binomial distribution for the random variable X, which represents the number of teenagers who said yes. Then find the probability that at least three of these teenagers said yes.

This is a binomial experiment in which $n = 5$, $p = 0.35$, $q = 1 - 0.35$ or 0.65. Use a calculator to compute the probability of each possible value for X using the Binomial Probability Formula.

$P(0) = {}_5C_0 \cdot 0.35^0 \cdot 0.65^5 \approx 0.116$

$P(1) = {}_5C_1 \cdot 0.35^1 \cdot 0.65^4 \approx 0.312$

$P(2) = {}_5C_2 \cdot 0.35^2 \cdot 0.65^3 \approx 0.336$

$P(3) = {}_5C_3 \cdot 0.35^3 \cdot 0.65^2 \approx 0.181$

$P(4) = {}_5C_4 \cdot 0.35^4 \cdot 0.65^1 \approx 0.049$

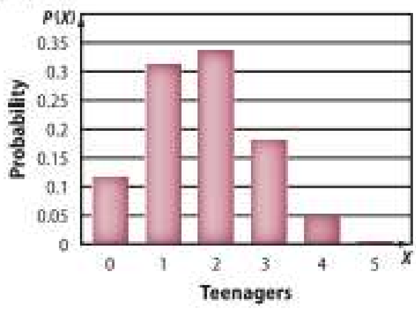
$P(5) = {}_5C_5 \cdot 0.35^5 \cdot 0.65^0 \approx 0.005$

Technology Tip

Binomial Probability To calculate each binomial probability on a graphing calculator, use binompdf(n, p, x) under the DISTR menu.

The probability distribution of X and its graph are shown below.

X	$P(X)$
0	0.116
1	0.312
2	0.336
3	0.181
4	0.049
5	0.005



To find the probability that *at least* three of the teenagers exercise regularly, find the sum of $P(3)$, $P(4)$, and $P(5)$.

$P(X \geq 3) = P(3) + P(4) + P(5)$ — P (at least three)

$= 0.181 + 0.049 + 0.005$ — $P(3) = 0.181$, $P(4) = 0.049$, and $P(5) = 0.005$

$= 0.235$ or 23.5% — Simplify.

Guided Practice

7. **CLASSES** In a certain high school graduating class, 48% of the students took a world language during their Grade 12 year. Seven students chosen at random are asked if they took a world language during their Grade 12 year. Construct and graph a probability distribution for the random variable X, which represents the number of students who said yes. Then find the probability that fewer than 4 of these students said yes.

Use the following formulas to find the mean, variance, and standard deviation of a binomial distribution.

KeyConcept Mean and Standard Deviation of a Binomial Distribution

The mean, variance, and standard deviation of a random variable X that has a binomial distribution are given by the following formulas.

Mean $\mu = np$

Variance $\sigma^2 = npq$

Standard Deviation $\sigma = \sqrt{\sigma^2}$ or $\sqrt{npq}$

These formulas are simpler than, but algebraically equivalent to, the formulas that you used to find the mean, variance, and standard deviation of probability distributions.

Real-World Example 8 Mean and Standard Deviation of a Binomial Distribution

EXERCISE The table shows the binomial distribution in Example 7. Find the mean, variance, and standard deviation of this distribution. Interpret the mean in the context of the problem situation.

X	0	1	2	3	4	5
$P(X)$	0.116	0.312	0.336	0.181	0.049	0.005

Real-WorldLink

According to a recent Gallup poll, approximately 58% of U.S. teens are classified as highly active. Activities in which the teens said that they regularly participate include basketball, running, jogging, biking, and swimming.

Source: The Gallup Poll

Method 1 Use the formulas for the mean, variance, and standard deviation of a probability distribution.

$$\mu = \sum[X \cdot P(X)]$$
$$= 0(0.116) + 1(0.312) + 2(0.336) + 3(0.181) + 4(0.049) + 5(0.005)$$
$$= 1.748$$

$$\sigma^2 = \sum[(X - \mu)^2 \cdot P(X)]$$
$$= (0 - 1.748)^2 \cdot 0.116 + (1 - 1.748)^2 \cdot 0.312 + (2 - 1.748)^2 \cdot 0.336 + (3 - 1.748)^2 \cdot 0.181 + (4 - 1.748)^2 \cdot 0.049 + (5 - 1.748)^2 \cdot 0.005$$
$$\approx 1.1354$$

$$\sigma = \sqrt{\sigma^2}$$
$$= \sqrt{1.1354} \text{ or about } 1.0656$$

Method 2 Use the formulas for the mean, variance, and standard deviation of a binomial probability distribution. In this binomial experiment, $n = 5$, $p = 0.35$, and $q = 0.65$.

$$\mu = np$$
$$= 5(0.35) \text{ or } 1.75$$

$$\sigma^2 = npq$$
$$= 5(0.35)(0.65) \text{ or } 1.1375$$

$$\sigma = \sqrt{\sigma^2}$$
$$= \sqrt{1.1375} \text{ or about } 1.0665$$

Both methods give approximately the same results. Therefore, the mean of the distribution is about 1.8 or 2, which means that on average about 2 out of the 5 students would say that they exercise regularly. The variance and standard deviation of the distribution are both about 1.1.

GuidedPractice

8. **CLASSES** Find the mean, variance, and standard deviation of the distribution that you constructed in Guided Practice 7. Interpret the mean in the context of the problem situation.

Exercises

Classify each random variable X as *discrete* or *continuous*. Explain your reasoning. (Example 1)

1. X represents the number of text messages sent by a randomly chosen student during a given day.
2. X represents the time it takes a randomly selected student to complete a physics test.
3. X represents the weight of a chocolate chip cookie selected at random in the school cafeteria.
4. X represents the number of CDs owned by a student chosen at random during a given day.
5. X represents the number of votes received by a candidate selected at random for a particular election.
6. X represents the weight of a wrestler selected at random on a given day.

Construct and graph a probability distribution for each random variable X. Find and interpret the mean in the context of the given situation. Then find the variance and standard deviation. (Examples 2–4)

7. **MUSIC** Students were asked how many MP3 players they own.

Players, X	Frequency
0	9
1	17
2	9
3	5
4	2

8. **AMUSEMENT** There were 20 participants in a pie eating contest at a county fair.

Pies Eaten, X	Frequency
1	1
2	5
3	9
4	3
5	2

9. **BREAKFAST** A sample of high school students was asked how many days they ate breakfast last week.

Days, X	Frequency
0	5
1	3
2	17
3	27
4	6
5	19
6	18
7	65

10. **HEALTH** Patients at a dentist's office were asked how many times a week they floss their teeth.

Flosses, X	Frequency
1	9
2	15
3	5
4	2
5	1
6	0
7	1

11. **CAR INSURANCE** A car insurance policy that costs $300 will pay $25,000 if the car is stolen and not recovered. If the probability of a car being stolen is $p = 0.0002$, what is the expected value of the profit or loss to the insurance company for this policy? (Example 5)

12. **FUNDRAISERS** A school hosts an annual fundraiser where tickets are sold for baked goods, the values of which are indicated below. Suppose 100 tickets were sold for a drawing for each of the four cakes.

What is the expected value of a participant's net gain or loss if she buys a ticket for AED 1? (Example 5)

Determine whether each experiment is a binomial experiment or can be reduced to a binomial experiment. If it can be presented as a binomial experiment, state the values of n, p, and q. Then list all possible values of the random variable. If it is not, explain why not. (Example 6)

13. You survey 25 students to find out how many are left-handed. The random variable represents the number of left-handed people.
14. You survey 200 people to see if they watch Monday Night Football. The random variable represents the number who watched Monday Night Football.
15. You roll a die 10 times to see if a 5 appears. The random variable represents the number of 5s.
16. You toss a coin 20 times to see how many tails occur. The random variable represents the number of tails.
17. You ask 15 people how old they are. The random variable represents their age.
18. You survey 40 students to find out whether they passed their driving test. The random variable represents the number that passed.
19. You select 10 cards from a deck without replacement. The random variable represents the number of hearts.

Construct and graph a binomial distribution for each random variable. Find and interpret the mean in the context of the given situation. Then find the variance and standard deviation.
(Examples 7 and 8)

20. In a recent poll, 89% of Americans order toppings on their pizza. Five teenagers chosen at random are asked if they order toppings.

21. In Eureka, California, 21% of the days are sunny. Consider the number of sunny days in February.

22. According to a survey, 26% of a company's employees have surfed the internet at work. Ten co-workers were selected at random and asked if they have surfed the internet at work.

23. A high school newspaper reported that 65% of students wear their seatbelts while driving. Eight students chosen at random are asked if they wear seatbelts.

24. According to a recent survey, 41% of high school students own a car. Seven students chosen at random are asked if they own a car.

25. **VOLUNTEERING** In a recent poll, 62% of Americans said that they had donated their time volunteering for a charity in the past year. If a random sample of 10 Americans is selected, find each of the following probabilities.
    a. Exactly 6 people donated their time to a charity.
    b. At least 5 people donated their time to a charity.
    c. At most 3 people donated their time to a charity.
    d. More than 8 people donated their time to a charity.

26. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the shape of a binomial distribution.
    a. **GRAPHICAL** Construct and graph the binomial distribution that corresponds to each of the following experiments.
       i. $n = 6$, $p = 0.5$
       ii. $n = 6$, $p = 0.3$
       iii. $n = 6$, $p = 0.7$
       iv. $n = 8$, $p = 0.5$
       v. $n = 10$, $p = 0.5$
    b. **VERBAL** Describe the shape of each of the distributions you found in part **a**.
    c. **ANALYTICAL** Make a conjecture regarding the shape of a distribution with each of the following probabilities of success: $p < 0.5$, $p = 0.5$, and $p > 0.5$.
    d. **ANALYTICAL** What happens to the spread of a binomial distribution as n increases?

H.O.T. Problems Use Higher-Order Thinking Skills

27. **PROOF** Use the distribution below to prove that $\mu = np$ and $\sigma^2 = npq$ for a binomial distribution, given $\mu = \sum[X \cdot P(X)]$ and $\sigma^2 = \sum[(X - \mu)^2 \cdot P(X)]$ for a probability distribution.

X	$P(X)$
0	$1 - p$
1	p

28. **REASONING** Suppose a coin is tossed ten times and lands on heads each time. Will the probability of the coin landing on tails increase during the next toss? Explain your reasoning.

29. **OPEN ENDED** A probability distribution in which all of the values of the random variable occur with equal probability is called a *uniform probability distribution*. Describe an example of an experiment that would produce a uniform distribution. Then find the theoretical probabilities that would result from this experiment. Include a table and graph of the distribution.

REASONING **Determine whether each of the following statements is *true* or *false*. Explain your reasoning.**

30. The probabilities associated with rolling two dice are determined theoretically.

31. The mean of a random variable is always a possible outcome of the experiment.

32. **CHALLENGE** Consider a binomial distribution in which $n = 50$ and $\sigma = 1.54$. What is the mean of the distribution? (*Hint*: p is closer to 0 than 1.)

33. **WRITING IN MATH** Describe another way you could find the probability that at least three of the teenagers exercise regularly or $P(X \geq 3)$ from Example 7. Give an example of when this method would be faster to use.

Spiral Review

34. **ART** The prices in dirhams of paintings sold at an art auction are shown.
   a. Construct a histogram, and use it to describe the shape of the distribution.
   b. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

Art Prices ($)					
1,800	600	750	600	600	1,800
1,350	450	300	1,200	750	600
750	450	2,700	600	750	300
750	2,300	600	450	2,100	1,200

Use the fifth partial sum of the exponential series to approximate each value to three decimal places.

35. $e^{0.2}$

36. $e^{-0.4}$

37. $e^{-0.75}$

Find the indicated geometric means for each pair of nonconsecutive terms.

38. 8 and 312.5; 3 means

39. $\frac{2}{9}$ and 54; 4 means

40. $\frac{3}{4}$ and $\frac{24}{3125}$; 4 means

Find the next four terms of each sequence.

41. $a_1 = -12, a_n = a_{n-1} + 3, n \geq 2$

42. $a_1 = 19, a_n = a_{n-1} - 13, n \geq 2$

43. $a_1 = 81, a_n = a_{n-1} - 72, n \geq 2$

Find the dot product of u and v. Then determine if u and v are orthogonal.

44. $\mathbf{u} = \langle 2, 9, -2 \rangle, \mathbf{v} = \langle -4, 7, 6 \rangle$

45. $\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ and $\mathbf{v} = -7\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$

46. $\mathbf{u} = \langle 8, -2, -2 \rangle, \mathbf{v} = \langle -6, 6, -10 \rangle$

Graph the hyperbola given by each equation.

47. $\frac{(y+6)^2}{36} - \frac{(x-1)^2}{24} = 1$

48. $\frac{(y+5)^2}{49} - \frac{(x-6)^2}{20} = 1$

49. $\frac{(y+3)^2}{9} - \frac{(x+5)^2}{4} = 1$

Find *AB* and *BA*, if possible.

50. $A = [2\ \ -1], B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

51. $A = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 2 & 7 \end{bmatrix}$

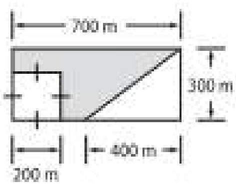
52. $A = \begin{bmatrix} 4 & -1 \\ 6 & 1 \\ 5 & -8 \end{bmatrix}, B = \begin{bmatrix} 2 & 9 & -3 \\ 4 & -1 & 0 \end{bmatrix}$

Skills Review for Standardized Tests

53. **REVIEW** Find the sum of 16 + 8 + 4 + ...
   A 28
   B 32
   C 48
   D 64

54. In a recent poll, 48% of Americans said that they shopped online for at least one holiday gift. If a random sample of 10 Americans is selected, what is the probability that at least 7 shopped online for a gift?
   F 3.4%
   G 4.8%
   H 10.0%
   J 14.1%

55. **SAT/ACT** Find the area of the shaded region.



   A $90{,}000\text{ m}^2$ C $130{,}000\text{ m}^2$ E $210{,}000\text{ m}^2$
   B $110{,}000\text{ m}^2$ D $150{,}000\text{ m}^2$

56. **REVIEW** Which of the following distributions best describes the data?

   {14, 15, 11, 13, 13, 14, 15, 14, 12, 13, 14, 15}

   F positively skewed   H normal
   G negatively skewed   J binomial

LESSON 4-3 The Normal Distribution

Then

- You analyzed probability distributions for discrete random variables.

Now

1. Find area under normal distribution curves.
2. Find probabilities for normal distributions, and find data values given probabilities.

Why?

- In a recent year, approximately 107 million Americans 20 years and older had a total blood cholesterol level of 200 milligrams per deciliter or higher. Physicians use variables of this type to compare patients' cholesterol levels to *normal* cholesterol ranges. In this lesson, you will determine the probability of a randomly selected person having a specific cholesterol level.

NewVocabulary
normal distribution
empirical rule
z-value
standard normal distribution

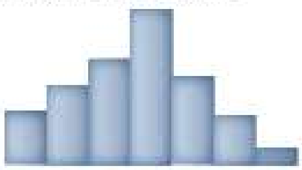
1 The Normal Distribution

The probability distribution for a continuous variable is called a *continuous probability distribution*. The most widely used continuous probability distribution is called the **normal distribution**. The characteristics of the normal distribution are as follows.

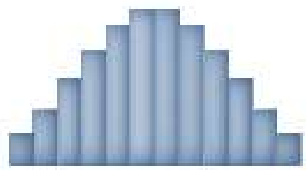
KeyConcept Characteristics of the Normal Distribution

- The graph of the curve is bell-shaped and symmetric with respect to the mean.
- The mean, median, and mode are equal and located at the center.
- The curve is continuous.
- The curve approaches, but never touches, the *x*-axis.
- The total area under the curve is equal to 1 or 100%.

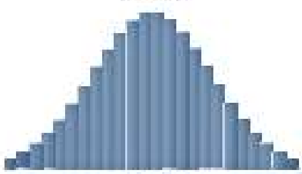
Consider a continuous probability distribution of times for a 400-meter run in a random sample of 100 athletes. By increasing sample size and decreasing class width, the distribution becomes more and more symmetrical. If it were possible to sample the entire population, the distribution would approach the normal distribution, as shown.



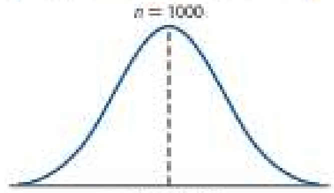
$n = 100$



$n = 1000$



$n = 10{,}000$



n = population

For every normally distributed random variable, the shape and position of the normal distribution curve are dependent on the mean and standard deviation. For example, in Figure 4.3.1, you can see that a larger standard deviation results in a flatter curve. A change in the mean, as shown in Figure 4.3.2, results in a horizontal translation of the curve.

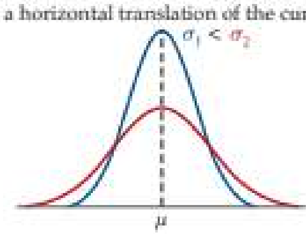


Figure 4.3.1

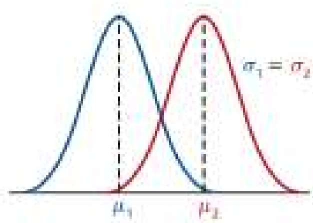


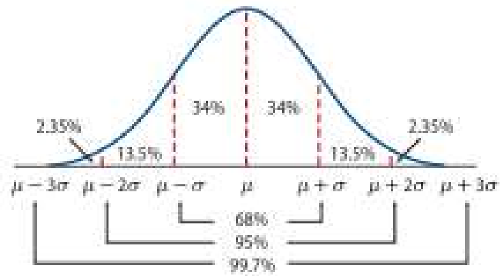
Figure 4.3.2

> **StudyTip**
> **Empirical Rule** The empirical rule is also known as the *68–95–99.7 rule*.

The area under the normal distribution curve between two data values represents the percent of data values that fall within that interval. The **empirical rule** can be used to describe areas under the normal curve over intervals that are one, two , or three standard deviations from the mean.

KeyConcept The Empirical Rule

In a normal distribution with mean μ and standard deviation σ:



- approximately 68% of the data values fall between $\mu - \sigma$ and $\mu + \sigma$.
- approximately 95% of the data values fall between $\mu - 2\sigma$ and $\mu + 2\sigma$.
- approximately 99.7% of the data values fall between $\mu - 3\sigma$ and $\mu + 3\sigma$.

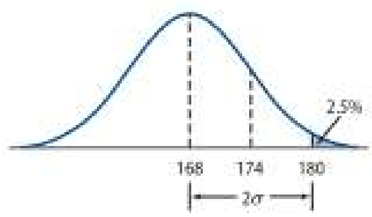
You can solve problems involving approximately normal distributions using the empirical rule.

Example 1 Use the Empirical Rule

HEIGHT The heights of the 880 students at East High School are normally distributed with a mean of 168 centimeters and a standard deviation of 6 centimeters.

a. Approximately how many students are more than 180 centimeters tall?

To determine the number of students that are more than 180 centimeters tall, find the corresponding area under the curve.

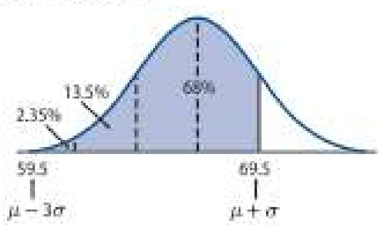


In the figure shown, you can see that 180 is 2σ from the mean. Because 95% of the data values fall within two standard deviations from the mean, each tail represents 2.5% of the data. The area to the right of 180 is 2.5% of 880 or 22.

Thus, about 22 students are more than 180 centimeters tall.

b. What percent of the students are between 159 and 174 centimeters tall?

The percent of students between 159 and 174 centimeters tall is represented by the shaded area in the figure at the right, which is between $\mu - 3\sigma$ and $\mu + \sigma$. The total area under the curve between 159 and 174 is equal to the sum of the areas of each region.



$2.35\% + 13.5\% + 68\% = 83.85\%$

Therefore, about 84% of the students are between 159 and 174 centimeters tall.

> **StudyTip**
> **Everything Under the Curve** Notice that in Example 1a, we used 2.5%, while in Example 1b, we used 2.35%. When you are asked for *greater than* or *less than*, you need to include everything under that side of the graph.

GuidedPractice

1. **MANUFACTURING** A machine used to fill water bottles dispenses slightly different amounts into each bottle. Suppose the volume of water in 120 bottles is normally distributed with a mean of 1.1 liters and a standard deviation of 0.02 liter.
   - **A.** Approximately how many bottles of water are filled with less than 1.06 liters?
   - **B.** What percent of the bottles have between 1.08 and 1.14 liters?

While the empirical rule can be used to analyze a normal distribution, it is only useful when evaluating specific values, such as $\mu + \sigma$. A normally distributed variable can be transformed into a standard value or z-value, which can be used to analyze any range of values in the normal distribution. This transformation is known as *standardizing*. The **z-value**, also known as the z-score and z test statistic, represents the number of standard deviations that a given data value is from the mean.

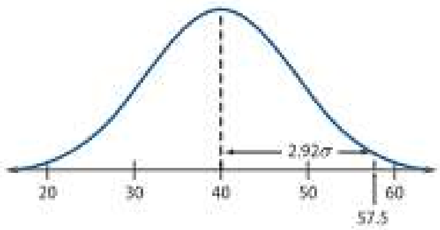
KeyConcept Formula for z-Values

The z-value for a data value in a set of data is given by $z = \frac{X - \mu}{\sigma}$, where X is the data value, μ is the mean, and σ is the standard deviation.

StudyTip

Positive and Negative z-Values If a data value is less than the mean, the corresponding z-value will be negative. Alternately, a data value that is greater than the mean will have a positive z-value.

You can use z-values to determine the position of *any* data value within a set of data. For example, consider a distribution with $\mu = 40$ and $\sigma = 6$. A data value of 57.5 is located about 2.92 standard deviations away from the mean, as shown. Therefore, in this distribution, $X = 57.5$ correlates to a z-value of 2.92.



StudyTip

Relative Position Like percentiles, z-values can be used to compare the relative positions of two values in two different sets of data.

Example 2 Find z-Values

Find each of the following.

a. z if $X = 24$, $\mu = 29$, and $\sigma = 4.2$

$z = \frac{X - \mu}{\sigma}$ — Formula for z-values

$= \frac{24 - 29}{4.2}$ — $X = 24$, $\mu = 29$, and $\sigma = 4.2$

≈ -1.19 — Simplify.

The z-value that corresponds to $X = 24$ is -1.19. Therefore, 24 is 1.19 standard deviations less than the mean in the distribution.

b. X if $z = -1.73$, $\mu = 48$, and $\sigma = 2.3$

$z = \frac{X - \mu}{\sigma}$ — Formula for z-values

$-1.73 = \frac{X - 48}{2.3}$ — $\mu = 48$, $\sigma = 2.3$, and $z = -1.73$

$-3.979 = X - 48$ — Multiply each side by 2.3.

$44.021 = X$ — Add 48 to each side.

A z-value of -1.73 corresponds to a data value of approximately 44 in the distribution.

GuidedPractice

2A. z if $X = 32$, $\mu = 28$, and $\sigma = 1.7$

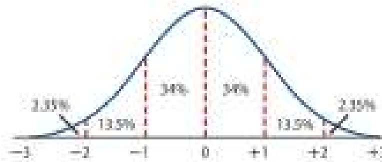
2B. X if $z = 2.15$, $\mu = 39$, and $\sigma = 0.4$

Every normally distributed random variable has a unique mean and standard deviation, which affect the position and shape of the curve. As a result, there are infinitely many normal probability distributions. Fortunately, they can all be related to one distribution known as the *standard* normal distribution. The **standard normal distribution** is a normal distribution of z-values with a mean of 0 and a standard deviation of 1.

The characteristics of the standard normal distribution are summarized below.

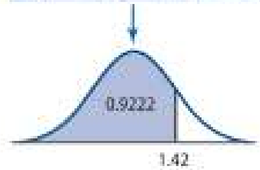
KeyConcept Characteristics of the Standard Normal Distribution

- The total area under the curve is equal to 1 or 100%.
- Almost all of the area is between $z = -3$ and $z = 3$.
- The distribution is symmetric.
- The mean is 0, and the standard deviation is 1.
- The curve approaches, but never touches, the x-axis.



You can solve normal distribution problems by finding the z-value that corresponds to a given X-value, and then finding the approximate area under the standard normal curve. The corresponding area can be found by using a table of z-values that shows the area *to the left* of a given z-value. For example, the area under the curve to the left of a z-value of 1.42 is 0.9222, as shown.

z	0.00	0.01	0.02
0.0	.5000	.5040	.5080
•	•	•	•
•	•	•	•
1.4	.9192	.9207	.9222



You can also find the area under the curve that corresponds to any z-value with a graphing calculator. This method will be used for the remainder of this chapter.

StudyTip
You can use the Normal distribution tables at the end of the book to determine the area of the region corresponding to a given z-value or to determine the z-value corresponding to a given area.

Using the Normal distribution tables to determine the area corresponding to $z = 1.42$

- Locate the table of positive z-values.
- Locate in the first column the value 1.4 and in the first row the value 0.02.
- The area corresponding to the z-value of 1.42 is located at the intersection of the row and column, which is 0.9222.

Example 3 Use the Standard Normal Distribution

COMMUNICATION The average number of phone calls received by a customer service representative each day during a 30-day month was 105 with a standard deviation of 12. Find the number of days with fewer than 110 phone calls. Assume that the number of calls is normally distributed.

$z = \frac{X - \mu}{\sigma}$ Formula for z-values

$= \frac{110 - 105}{12}$ or about 0.42 $X = 110$, $\mu = 105$, and $\sigma = 12$

Although the standard normal distribution extends to negative and positive infinity, when you are finding the area less than or greater than a given value, you can use a lower value of −4 and an upper value of 4.

In this case, enter a lower z-value of −4 and an upper z-value of 0.42. The resulting area is 0.66. Since there were 30 days in the month, there were fewer than 110 calls on 30 • 0.66 or 19.8 days.

```
normalcdf(-4,0.4
2)
      .6627255515
```

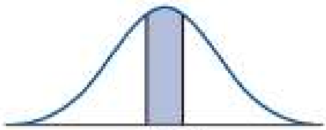
Therefore, there were approximately 20 days with fewer than 110 calls.

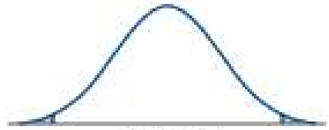
TechnologyTip
Area Under the Normal Curve You can use a graphing calculator to find the area under a standard normal curve that corresponds to any pair of z-values by selecting [2nd] [DISTR] and normalcdf (*lower z value, upper z value*).

GuidedPractice

3. **BASKETBALL** The average number of points that a basketball team scored during a single season was 63 with a standard deviation of 18. If there were 15 games during the season, find the percentage of games in which the team scored more than 70 points. Assume that the number of points is normally distributed.

In Example 3, you found the area under the normal curve that corresponds to a z-value. You can also find z-values that correspond to specific areas. For example, you can find the z-value that corresponds to a cumulative area of 1%, 20%, or 99%. You can also find intervals of z-values that contain or are between a certain percentage of data.



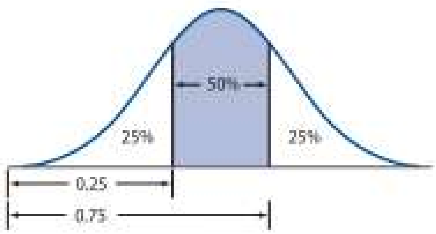


Example 4 Find z-Values Corresponding to a Given Area

Find the interval of z-values associated with each area.

a. middle 50% of the data

The middle 50% of the data corresponds to the data between 25% and 75% of the distribution, or 0.25 and 0.75, as shown.



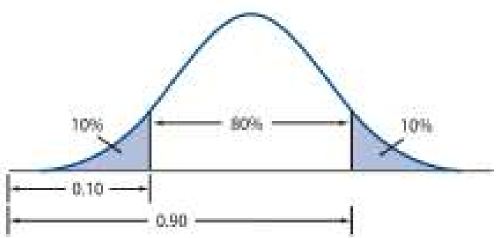
To find the z-scores that correspond to 0.25 and 0.75, select [2nd] [DISTR] to display the DISTR menu on a graphing calculator. Select invNorm(and enter 0.25. Repeat to find the value corresponding to 0.75. As shown at the right, the z-value corresponding to 0.25 is -0.67 and the z-value corresponding to 0.75 is 0.67.

```
invNorm(0.25)
   -.6744897495
invNorm(0.75)
    .6744897495
```

Therefore, the interval that represents the middle 50% of the data is $-0.67 < z < 0.67$.

b. the outside 20% of the data

The outside 20% of the data represents the top 10% and the bottom 10% of the distribution or 0.1 and 0.9, as shown.



To find the z-value corresponding to 0.10, enter 0.10 into a graphing calculator under invNorm(and repeat for 0.90. As shown, the z-value corresponding to 0.10 is -1.28 and the z-value corresponding to 0.90 is 1.28.

```
invNorm(0.10)
   -1.281551567
invNorm(0.90)
    1.281551567
```

Therefore, the interval that represents the outside 20% of the data is $-1.28 > z$ or $z > 1.28$.

GuidedPractice

4A. the middle 25% of the data

4B. the outside 60% of the data

> **StudyTip**
> **Symmetry** The normal distribution is symmetrical, so when you are asked for the middle or outside set of data, the z-values will be opposites.

StudyTip

Percentage, Proportion, Probability, and Area When a problem asks for a percentage, proportion, or probability, it is asking for the same value—the corresponding area under the normal curve.

2 Probability and the Normal Distribution

You have seen how the area under the normal curve corresponds to the proportion of data values in an interval. The area also corresponds to the probability of data values falling within a given interval. If a z-value is chosen randomly, the probability of choosing a value between 0 and 1 would be equivalent to the area under the curve between 0 and 1, which is 0.3413. Therefore, the probability of randomly choosing a value between 0 and 1 would be approximately 34%.

Example 5 Find Probabilities

METEOROLOGY **The temperatures for one month for a city in California are normally distributed with $\mu = 81°$ and $\sigma = 6°$. Find each probability, and use a graphing calculator to sketch the corresponding area under the curve.**

StudyTip

Continuity Factors In a continuous distribution, there is no difference between $P(x \geq c)$ and $P(x > c)$ because the probability that x is equal to c is zero.

a. $P(70° < X < 90°)$

The question is asking for the percentage of temperatures that were between 70° and 90°. First, find the corresponding z-values for $X = 70$ and $X = 90$.

$z = \frac{X - \mu}{\sigma}$ — Formula for z-values

$= \frac{70 - 81}{6}$ — $X = 70$, $\mu = 81$, and $\sigma = 6$

≈ -1.83 — Simplify.

Use 90 to find the other z-value.

$z = \frac{X - \mu}{\sigma}$ — Formula for z-values

$= \frac{90 - 81}{6}$ — $X = 90$, $\mu = 81$, and $\sigma = 6$

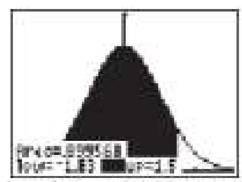
≈ 1.5 — Simplify.

Using the Standard Normal Distribution tables to find the area between $z = -1.83$ and $z = 1.5$

- In the table of positive z-values, we locate in the first column the value 1.5 and in the first row 0.00, then we locate the area corresponding to $z = 1.5$ at the intersection of the row and column. The area is 0.9332.
- In the table of negative z-values, we locate in the first column the value -1.8 and in the first row 0.03, then we locate the area corresponding to $z = -1.83$ at the intersection of the row and column. The area is 0.0336.
- The needed area is the area to the left of the greater z-value minus the area to the left of the smaller z-value; that is,

  $0.9332 - 0.0336 = 0.8996$

You can use a graphing calculator to display the area that corresponds to any z-value by selecting [2nd] [DISTR]. Then, under the DRAW menu, select ShadeNorm (*lower z value, upper z value*). The area between $z = -1.83$ and $z = 1.5$ is 0.899568, as shown.



$[-4, 4]$ scl: 1 by $[0, 0.5]$ scl: 0.125

Therefore, approximately 90% of the temperatures were between 70 and 90.

b. $P(X \geq 95°)$

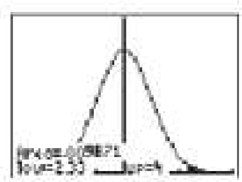
$z = \frac{X - \mu}{\sigma}$ — Formula for z-values

$= \frac{95 - 81}{6}$ — $X = 95$, $\mu = 81$, and $\sigma = 6$

≈ 2.33 — Simplify.

Using a graphing calculator, you can find the area between $z = 2.33$ and $z = 4$ to be 0.0099.

Therefore, the probability that a randomly selected temperature is at least 95° is about 1%.



$[-4, 4]$ scl: 1 by $[0, 0.5]$ scl: 0.125

GuidedPractice

5. **TESTING** The scores on a standardized test are normally distributed with $\mu = 72$ and $\sigma = 11$. Find each probability and use a graphing calculator to sketch the corresponding area under the curve.

A. $P(X < 89)$ **B.** $P(65 < X < 85)$

You can find specific intervals of data for given probabilities or percentages by using the standard normal distribution.

Real-WorldLink

In a recent year, the average national SAT scores were 502 in Critical Reading, 515 in Math, and 494 in Writing. The average national ACT score in that same year was 21.1.

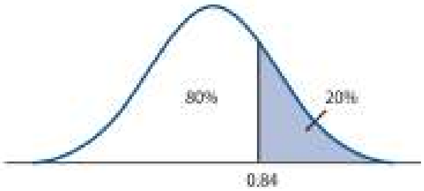
Source: *USA TODAY*

Real-World Example 6 Find Intervals of Data

COLLEGE The scores for the entrance exam for a college's mathematics department is normally distributed with $\mu = 65$ and $\sigma = 8$.

a. If Fatema wants to be in the top 20%, what score must she get?

To find the top 20% of the exam scores, you must find the exam score X that separates the upper 20% of the area under the normal curve, as shown. The top 20% correlates with $1 - 0.2$ or 0.8. Using a graphing calculator, you can find the corresponding z-value to be 0.84.



Now, use the formula for the z-value for a population to find the corresponding exam score.

$$z = \frac{X - \mu}{\sigma}$$ Formula for z-values

$$0.84 = \frac{X - 65}{8}$$ $\mu = 65$, $\sigma = 8$, and $z = 0.84$

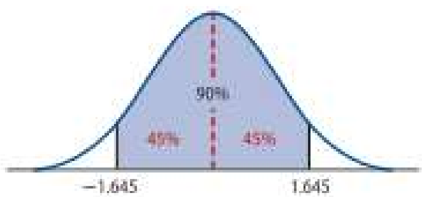
$$6.72 = X - 65$$ Multiply each side by 8.

$$71.72 = X$$ Add 65 to each side.

Fatema needs a score of at least 72 to be in the top 20%.

b. Fatema expects to earn a grade in the middle 90% of the distribution. What range of scores fall in this category?

The middle 90% of the exam scores represents 45% on each side of the mean and therefore corresponds to the interval of area from 0.05 to 0.95. Using a graphing calculator, the z-values that correspond to 0.05 and 0.95 are -1.645 and 1.645, respectively.



Use the z-values to find each value of X.

$z = \frac{X - \mu}{\sigma}$	Formula for z-values	$z = \frac{X - \mu}{\sigma}$
$-1.645 = \frac{X - 65}{8}$	$\mu = 65$ and $\sigma = 8$	$1.645 = \frac{X - 65}{8}$
$-13.16 = X - 65$	Multiply.	$13.16 = X - 65$
$51.84 = X$	Simplify.	$78.16 = X$

Therefore, Fatema expects to score between 52 and 78.

GuidedPractice

6. **RESEARCH** As part of a medical study, a researcher selects a study group with a mean weight of 86 kilograms and a standard deviation of 5.5 kilograms. Assume that the weights are normally distributed.

   A. If the study will mainly focus on participants whose weights are in the middle 80% of the data set, what range of weights will this include?

   B. If participants whose weights fall in the outside 5% of the distribution are contacted 2 weeks after the study, people in what weight range will be contacted?

Exercises

1. **NOISE POLLUTION** As part of a noise pollution study, researchers measured the sound level in decibels of a busy city street for 30 days. According to the study, the average noise was 82 decibels with a standard deviation of 6 decibels. Assume that the data are normally distributed. (Example 1)

   a. If a normal conversation is held at about 64 decibels, determine the number of hours during the study that the noise level was this low.

   b. Determine the percent of the study during which the noise was between 76 decibels and 88 decibels.

2. **GAS MILEAGE** Khamis commutes 290 miles each week for work. His car averages 29.6 miles per gallon with a standard deviation of 5.4 miles per gallon. Assume that the data are normally distributed. (Example 1)

   a. Approximate the number of miles that Khamis's car gets a gas mileage of 35 miles per gallon or better.

   b. For what percentage of Khamis's commute does his car have a gas mileage between 24.2 miles per gallon and 40.4 miles per gallon?

Find each of the following. (Example 2)

3. z if $X = 19$, $\mu = 22$, and $\sigma = 2.6$
4. X if $z = 2.3$, $\mu = 64$, and $\sigma = 1.3$
5. z if $X = 52$, $\mu = 43$, and $\sigma = 3.7$
6. X if $z = 2.5$, $\mu = 27$, and $\sigma = 0.4$
7. z if $X = 32$, $\mu = 38$, and $\sigma = 2.8$
8. X if $z = 1.7$, $\mu = 49$, and $\sigma = 4.1$

9. **ICHTHYOLOGY** As part of a science project, Mazen studied the growth rate of 797 green gold catfish and found the following information. Assume that the data are normally distributed. (Example 3)

   a. Determine the number of fish with a length less than 4.5 millimeters at birth.

   b. Determine the number of fish with a length greater than 5 millimeters at birth.

10. **ROLLER COASTER** The average wait in line for the 16,000 daily passengers of a roller coaster is 72 minutes with a standard deviation of 15 minutes. Assume that the data are normally distributed. (Example 3)

    a. Determine the number of passengers who wait less than 60 minutes to ride the roller coaster.

    b. Determine the number of passengers who wait more than 90 minutes to ride the roller coaster.

Find the interval of z-values associated with each area. (Example 4)

11. middle 30%
12. outside 15%
13. outside 40%
14. middle 10%
15. outside 25%
16. middle 84%

17. **BATTERY** The life of a certain brand of AA battery is normally distributed with $\mu = 8$ hours and $\sigma = 1.5$ hours. Find each probability. (Example 5)

    a. The battery will last less than 6 hours.

    b. The battery will last more than 12 hours.

    c. The battery will last between 8 and 9 hours.

18. **HEALTH** The average blood cholesterol level in adult Americans is 203 mg/dL (milligrams per deciliter) with a standard deviation of 38.8 mg/dL. Find each probability. Assume that the data are normally distributed. (Example 5)

    a. a blood cholesterol level below 160 mg/dL, which is considered low and can lead to a higher risk of stroke

    b. a blood cholesterol level above 240 mg/dL, which is considered high and can lead to higher risk of heart disease

    c. a blood cholesterol level between 180 and 200 mg/dL, which is considered normal

19. **SNOWFALL** The average annual snowfall in centimeters for the U.S. and Canada region from 45°N to 55°N is normally distributed with $\mu = 260$ and $\sigma = 27$. (Example 6)

    a. Determine the minimum amount of snowfall occurring in the top 15% of the distribution.

    b. Determine the maximum amount of snowfall occurring in the bottom 30%.

    c. What range of snowfall occurs in the middle 60%?

20. **TRAFFIC SPEED** The speed in kilometers per hour of traffic on North Street is normally distributed with $\mu = 60$ and $\sigma = 9$. (Example 6)

    a. Determine the maximum speed of the slowest 10% of cars driving on North Street.

    b. Determine the minimum speed of the fastest 5% of cars driving on North Street.

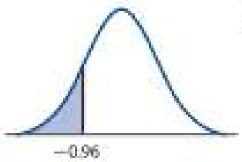
    c. At what range of speed do the middle 25% of cars on North Street drive?

21. **TESTS** Saleh took the ACT and SAT and earned the math scores shown. Which of the scores has a higher relative position? Explain your reasoning.

Test	Saleh's Score	National Average	Standard Deviation
ACT	27	21	4.7
SAT	620	508	111

22. **EXAMS** Asma scored 76 on a physics test that had a mean of 72 and a standard deviation of 10. She also scored 81 on a sociology test that had a mean of 78 and a standard deviation of 9. Compare her relative scores on each test. Assume that the data are normally distributed.

Find the area that corresponds to each shaded region.

23.


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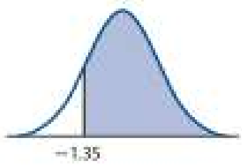

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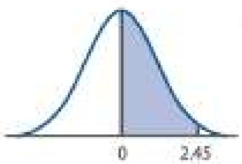

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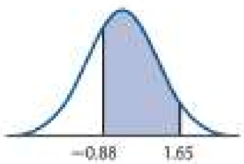


27. **FRACTILES** Quartiles, percentiles, and deciles are three types of fractiles, which divide an ordered set of data into equal groups. Find the z-values that correspond to each of the following fractiles.

a. D_{20}, D_{40}, and D_{80}

b. Q_1, Q_2, and Q_3

c. P_{10}, P_{40}, and P_{90}

28. **METEOROLOGY** The humidity observed in the morning during the same day in Chicago, Orlando, and Phoenix is shown. Assume that the data are normally distributed.

City	Humidity	Average Humidity	Standard Deviation
Chicago	85%	82%	12%
Orlando	94%	91%	15%
Phoenix	46%	43%	10%

a. Which city has the highest humidity? the lowest humidity? Explain your reasoning.

b. How would a fourth city compare that has a humidity of 81% and an average humidity of 8% with a standard deviation of 8%?

29. **JOBS** The salaries of employees in the sales department of an advertising agency are normally distributed with a standard deviation of $8000. During the holiday season, employees who earn less than $35,000 receive a gift basket.

a. Suppose 10% of the employees receive a gift basket. What is the mean salary of the sales department?

b. Suppose employees who make $10,000 greater than the mean salary receive an incentive bonus. If 200 employees work in the sales department, how many employees will receive a bonus?

30. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate the shape of a normal distribution. Consider a population of 4, 10, 6, 8.

a. **GRAPHICAL** Construct a bar graph, and use it to describe the shape of the distribution. Then find the mean and standard deviation of the data set.

b. **GRAPHICAL** Select eight random samples of size 2, with replacement, from the data set. Construct a bar graph, and use it to describe the shape of the distribution. Find the mean and standard deviation of the sample means.

c. **TABULAR** The table includes all samples of size 2 that can be taken, with replacement, from the data set. Find the mean of each sample and the mean and standard deviation of all of the sample means.

Sample	Mean	Sample	Mean
4, 4		8, 4	
4, 6		8, 6	
4, 8		8, 8	
4, 10		8, 10	
6, 4		10, 4	
6, 6		10, 6	
6, 8		10, 8	
6, 10		10, 10	

d. **GRAPHICAL** Construct a bar graph of the sample means from part **c** and use it to describe the shape of the distribution. What happens to the shape of a distribution of data as the sample size increases?

e. **ANALYTICAL** Divide the standard deviation of the population that you found in part **a** by the square root of the sample size. What do you think happens to the mean and standard deviation of a distribution of data as the sample size increases?

H.O.T. Problems Use Higher-Order Thinking Skills

31. **ERROR ANALYSIS** Husam and Salem are finding the z interval associated with the outside 35% of a distribution of data. Husam thinks it is the interval $z < -0.39$ or $z > 0.39$, while Salem thinks it is the interval $z < -0.93$ or $z > 0.93$. Is either of them correct? Explain your reasoning.

32. **REASONING** In real-life applications, z-values usually fall between −3 and +3 in the standard normal distribution. Why do you think this is the case? Explain your reasoning.

33. **CHALLENGE** Find two z-values, one positive and one negative, so that the combined area of the two equivalent tails is equal to each of the following.

a. 1%　　b. 5%　　c. 10%

34. **REASONING** Continuous variables *sometimes*, *always*, or *never* have normal distributions. Explain your reasoning.

35. **WRITING IN MATH** Compare and contrast the characteristics of a normal distribution and the standard normal distribution.

Spiral Review

36. **BASEBALL** The number of hits by each Wildcats player during a doubleheader is shown in the frequency distribution.

Hits, X	Frequency
0	3
1	1
2	8
3	2
4	3

a. Construct and graph a probability distribution for the random variable X.

b. Find and interpret the mean in the context of the situation.

c. Find the variance and standard deviation.

37. **AMERICAN FOOTBALL** The number of penalties a football team received for each game during two recent seasons is shown. Construct side-by-side box plots of the data sets. Then use this display to compare the distributions.

Season 1				Season 2			
8	11	6	13	9	1	3	5
9	18	16	11	8	3	6	4
15	14	14	9	10	6	3	1
8	5	10	5	5	5	3	2

Find the sum of each arithmetic series.

38. S_{51} of $-92 + (-88) + (-84) + \ldots$

39. 24th partial sum of $-13 + 2 + 17 + \ldots$

40. S_{46} of $295 + 281 + 267 + \ldots$

Find rectangular coordinates for each point with the given polar coordinates.

41. $\left(\frac{1}{4}, \frac{\pi}{2}\right)$

42. $\left(3, \frac{\pi}{3}\right)$

43. $(-2, \pi)$

Given v and u · v, find u. There may be more than one answer.

44. $\mathbf{v} = \langle -4, 2, -7 \rangle$, $\mathbf{u} \cdot \mathbf{v} = 17$

45. $\mathbf{v} = \langle 2, 8, 5 \rangle$, $\mathbf{u} \cdot \mathbf{v} = -6$

46. $\mathbf{v} = \left\langle \frac{2}{3}, -3, \frac{1}{3} \right\rangle$, $\mathbf{u} \cdot \mathbf{v} = 10$

Find the direction angle of each vector.

47. $6\mathbf{i} + 3\mathbf{j}$

48. $-3\mathbf{i} + 4\mathbf{j}$

49. $2\mathbf{i} - 8\mathbf{j}$

Write an equation of an ellipse with each set of characteristics.

50. vertices $(-3, 11)$, $(-3, -9)$; foci $(-3, 7)$, $(-3, -5)$

51. co-vertices $(-1, -6)$, $(-3, -6)$; length of major axis equals 10

52. vertices $(-4, 2)$, $(8, 2)$; length of minor axis equals 8

Skills Review for Standardized Tests

53. **SAT/ACT** If X is the sum of the first 1000 positive even integers and Y is the sum of the first 500 positive odd integers, about what percent greater is X than Y?

A 100% C 300% E 500%

B 200% D 400%

54. **REVIEW** In a recent year, the mean and standard deviation for scores on the ACT was 21.0 and 4.7. Assume that the scores were normally distributed. What is the approximate probability that a test taker scored higher than 30.2?

F 1% H 2%

G 1.5% J 2.5%

55. The length of each song in a music collection is normally distributed with $\mu = 4.12$ minutes and $\sigma = 0.68$ minutes. Find the probability that a song selected from the collection at random is longer than 5 minutes.

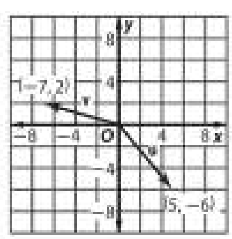
A 10% C 39%

B 19% D 89%

56. **REVIEW** Find $\mathbf{u} \cdot \mathbf{v}$.

F -47 H -6

G -24 J 47



Graphing Technology Lab
Transforming Skewed Data

Objective

- Use a graphing calculator to transform skewed data into data that resemble a normal distribution.

It is common for biological, medical, and other data to be positively skewed. It can sometimes be helpful to *transform* the original data so that it better resembles a normal distribution. This allows for the data to be spread out as opposed to being bunched at one end of a display.

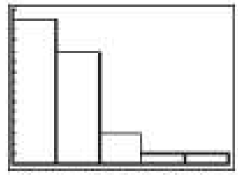
Activity Transform Data Using Natural Logarithms

Use the data to construct a histogram, and describe the shape of the distribution. Then transform the data by calculating the common logarithm of each entry. Graph the new data, and describe the shape of the distribution.

Data									
15	7	2	5	8	17	15	8	3	4
9	18	13	10	9	8	10	23	26	10
7	14	25	7	6	13	35	48	14	6

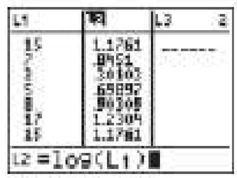
Step 1 Input the data into L1. Construct a histogram for the data using the intervals and scales shown.

The data appear to be positively skewed.



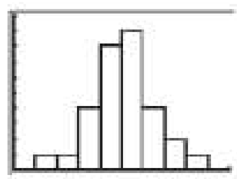
[0, 50] scl: 10 by [0, 15] scl: 1

Step 2 Input the common logarithm for each value into L2. Place the cursor on L2. Press LOG and enter L1. Press [ENTER].



Step 3 Construct a histogram for the new data using the intervals and scales shown.

The data appear to have a normal distribution.



[0, 2] scl: 0.2 by [0, 10] scl: 1

Data may also be transformed by calculating the square roots or powers of the entries. When data are transformed, the type of operation performed should always be specified. A transformation will not always result in the new data being normally distributed.

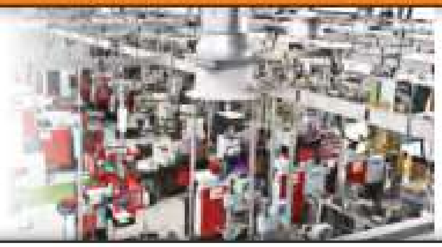
Exercise

Use the data to construct a histogram, and describe the shape of the distribution. Then transform the data by calculating the square root of each entry. Graph the new data, and describe the shape of the distribution. Explain how the transformation affected the summary statistics.

Data									
23	30	36	39	36	24	31	33	42	36
26	32	46	45	27	34	52	41	28	33
43	20	24	34	30	40	29	35	61	35

LESSON 4-4 The Central Limit Theorem

Then	Now	Why?
You used the normal distribution to find probabilities for intervals of data values in distributions.	1 Use the Central Limit Theorem to find probabilities. 2 Find normal approximations of binomial distributions	In manufacturing processes, quality control systems are used to determine when a process is outside of upper and lower control limits or "out of control." The mean of the process is controlled; therefore, successive sample means should be normally distributed around the actual mean.



NewVocabulary
- sampling distribution
- standard error of the mean
- sampling error
- continuity correction factor

1 The Central Limit Theorem Sampling is an important statistical tool in which subgroups of a population are selected so that inferences can be made about the entire population. The means of these subgroups, or sample means, can be compared to the mean of the population by using a sampling distribution. A **sampling distribution** is a distribution of the means of random samples of a certain size that are taken from a population.

Consider a population consisting of 16, 18, 20, 20, 22, and 24, with $\mu = 20$ and $\sigma = 2.582$. Suppose 12 random samples of size 2 are taken, with replacement. The mean $\bar{x}$ of each sample is shown.

Sample	$\bar{x}$	Sample	$\bar{x}$	Sample	$\bar{x}$
20, 22	21	20, 18	19	22, 22	22
22, 18	20	16, 22	19	18, 18	18
20, 24	22	24, 16	20	20, 16	18
20, 20	20	20, 24	22	24, 22	23

The distribution of the means of the 12 random samples, shown in Figure 4.4.1, does not appear to be normal. However, if all 36 samples of size of 2 from the population are found, the distribution of sample means will approach the normal distribution, as shown in Figure 4.4.2.

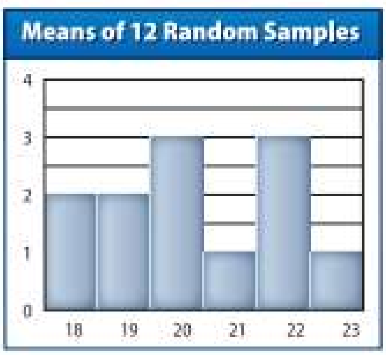


Figure 4.4.1

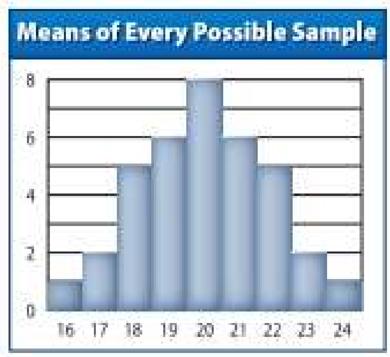


Figure 4.4.2

The mean of the means of every possible sample of size 2 from the population is

$$\mu_{\bar{x}} = \frac{16 + 17 + \ldots + 24}{36} = \frac{720}{36} \text{ or } 20.$$

Notice that this value is equal to the population mean $\mu = 20$. So, when the mean of the means of every possible sample of size 2 are found, $\mu_{\bar{x}} = \mu$. The standard deviation of the sample means $\sigma_{\bar{x}}$ and the standard deviation of the population σ when divided by the square root of the sample of size n are

$$\sigma_{\bar{x}} = \frac{\sqrt{(16 - 20)^2 + (17 - 20)^2 + \ldots + (24 - 20)^2}}{36} \approx 1.826 \quad \text{and} \quad \frac{\sigma}{\sqrt{n}} = \frac{2.582}{\sqrt{2}} \approx 1.826.$$

Since these two values are equal, the standard deviation of the sample means, also known as the **standard error of the mean**, can be found by using the formula $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$.

In general, randomly selected samples will have sample means that differ from the population mean. These differences are caused by **sampling error**, which occurs because the sample is not a complete representation of the population. However, if *all* possible samples of size n are taken from a population with mean μ and a standard deviation σ, the distribution of sample means will have:

- a mean $\mu_{\bar{x}}$ that is equal to μ and
- a standard deviation $\sigma_{\bar{x}}$ that is equal to $\frac{\sigma}{\sqrt{n}}$.

When the sample size n is large, regardless of the shape of the original distribution, the Central Limit Theorem states that the shape of the distribution of the sample means will approach a normal distribution.

StudyTip

Normally Distributed Variables If the original variable is not normally distributed, then n must be greater than 30 in order to use the standard normal distribution to approximate a distribution of sample means.

KeyConcept Central Limit Theorem

As the sampling size n increases:

- the shape of the distribution of the sample means of a population with mean μ and standard deviation σ will approach a normal distribution and
- the distribution will have a mean μ and standard deviation $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$.

The Central Limit Theorem can be used to answer questions about sample means in the same way that the normal distribution was used to answer questions about individual values. In this case, we can use a formula for the z-value of a sample mean.

KeyConcept z-Value of a Sample Mean

The z-value for a sample mean in a population is given by $z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$, where $\bar{x}$ is the sample mean, μ is the mean of the population, and $\sigma_{\bar{x}}$ is the standard error.

Example 1 Use the Central Limit Theorem

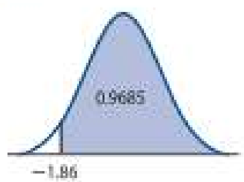
AGE According to a recent study, the average age that an American adult leaves home is 26 years old. Assume that this variable is normally distributed with a standard deviation of 2.4 years. If a random sample of 20 adults is selected, find the probability that the mean age the participants left home is greater than 25 years old.

Since the variable is normally distributed, the distribution of the sample means will be approximately normal with $\mu = 26$ and $\sigma_{\bar{x}} = \frac{2.4}{\sqrt{20}}$ or about 0.537. Find the z-value.

$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$ — z-value for a sample mean

$= \frac{25 - 26}{0.537}$ — $\bar{x} = 25$, $\mu = 26$, and $\sigma_{\bar{x}} = 0.537$

≈ -1.86 — Simplify.



The area to the right of a z-value of -1.86 is 0.9685. Therefore, the probability that the mean age of the sample is greater than 25 or $P(\bar{x} > 25)$ is about 96.85%.

GuidedPractice

1. **TORNADOES** The average number of tornadoes in Kansas is 47 per year, with a standard deviation of approximately 14.2 tornadoes. If a random sample of 15 years is selected, find the probability that the mean number of tornadoes is less than 50.

You can also determine the probability that a sample mean will fall within a given interval of the sampling distribution.

Real-WorldLink

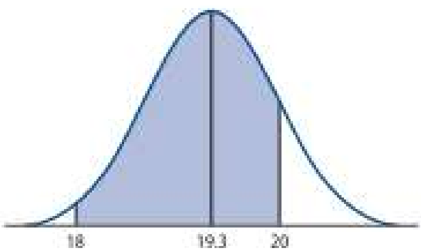
In 1994, a nonprofit organization called The Rechargeable Battery Recycling Corporation was formed to promote the recycling of rechargeable batteries in North America. It provides information for over 50,000 collection locations nationwide where rechargeable batteries can be recycled for free.

Source: Battery University

Real-World Example 2 Find the Area Between Two Sample Means

BATTERY LIFE A company that produces rechargeable batteries is designing a battery that will need to be recharged after an average of 19.3 hours of use. Assume that the distribution is normal with a standard deviation of 2.4 hours. If a random sample of 20 batteries is selected, find the probability that the mean life of the batteries before recharging is between 18 and 20 hours.

The area that corresponds to an interval of 18 to 20 hours is shown at the right.



First, find the standard deviation of the sample means.

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ Standard deviation of a sample mean

$= \frac{2.4}{\sqrt{20}}$ $\sigma = 2.4$ and $n = 20$

≈ 0.536 Simplify.

Use the z-value formula for a sample mean to find the corresponding z-values for 18 and 20.

z-value for $\bar{x} = 18$:

$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$ z-value formula for a sample mean

$= \frac{18 - 19.3}{0.536}$ $\bar{x} = 18$, $\mu = 19.3$, and $\sigma_{\bar{x}} = 0.536$

≈ -2.42 Simplify.

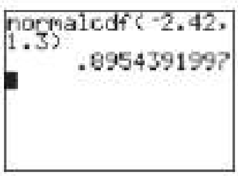
z-value for $\bar{x} = 20$:

$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$ z-value formula for a sample mean

$= \frac{20 - 19.3}{0.536}$ $\bar{x} = 20$, $\mu = 19.3$, and $\sigma_{\bar{x}} = 0.536$

≈ 1.30 Simplify.

Using a graphing calculator, select normalcdf(to find the area between $z = -2.42$ and $z = 1.30$.



The area between z-values of -2.42 and 1.30 is 0.8954. Therefore, $P(18 < \mu < 20)$ is 89.54%. So, the probability that the mean life of the batteries is between 18 and 20 hours is 89.54%.

GuidedPractice

2. **DAIRY** The average cost of a gallon of milk in a U.S. city is \$3.49 with a standard deviation of \$0.24. If a random sample of 40 1-gallon containers of milk is selected, find the probability that the mean of the sample will be between \$3.40 and \$3.60.

Example 3 Analyze Individual Values and Sample Means

CLASS SIZE According to a recent study, the average class size in high schools nationwide is 24.7 students per class. Assume that the distribution is normal with a standard deviation of 3.6 students.

a. Find the probability that a randomly selected class will have fewer than 23 students.

The question is asking for an individual value in which $P(x < 23)$. Use the z-value formula for an individual data value to find the corresponding z-value.

$z = \frac{X - \mu}{\sigma}$ z-value formula for an individual value

$= \frac{23 - 24.7}{3.6}$ or about -0.47 $X = 23$, $\mu = 24.7$, and $\sigma = 3.6$

The area associated with $z < -0.47$, or $P(z < -0.47)$, is 0.3192. Therefore, the probability that a randomly selected class has fewer than 23 students is 31.9%.

b. If a sample of 15 classes is selected, find the probability that the mean of the sample will be fewer than 23 students per class.

This question deals with a sample mean, so use the z-value formula for a sample mean to find the corresponding z-value. First, find the standard error of the mean.

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ Standard error of the mean

$= \frac{3.6}{\sqrt{15}}$ or about 0.93 $\sigma = 3.6$ and $n = 15$

Next, find the z-value using the z-value formula for a sample mean.

$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$ z-value formula for a sample mean

$= \frac{23 - 24.7}{0.93}$ or about -1.83 $\bar{x} = 23$, $\mu = 24.7$, and $\sigma_{\bar{x}} = 0.93$

The area associated with $z < -1.83$, or $P(z < -1.83)$, is 0.0336. Therefore, the probability that a sample of 15 classes will have a mean class size of fewer than 23 students is 3.36%.

StudyTip

z-Value Formulas Notice that the difference between the z-value formula for an individual data value and the z-value formula for a sample mean is that $\bar{x}$ is substituted for X and $\sigma_{\bar{x}}$ is substituted for σ in the formula for an individual value.

GuidedPractice

3. **APPLES** Consumers in the U.S. eat an average of 19 kilograms of apples per year. Assume that the standard deviation is 4 kilograms and the distribution is approximately normal.
   A. Find the probability that a randomly selected person consumes more than 21 kilograms of apples per year.
   B. If a sample of 30 people is selected, find the probability that the mean of the sample would be more than 21 kilograms of apples per year.

Notice in Figure 4.4.3 that the probability that an individual class has fewer than 23 students is much greater than the probability associated with the mean of a sample being fewer than 23 shown in Figure 4.4.4. This means that as the sample size increases, the distribution becomes narrower and the variability decreases.

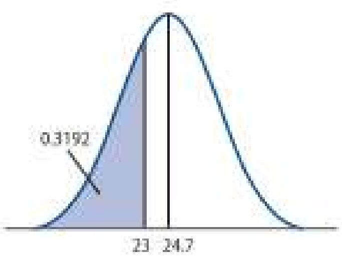


Figure 4.4.3

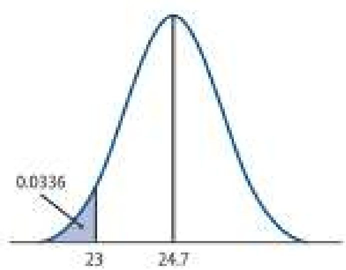


Figure 4.4.4

Math HistoryLink

Pierre-Simon Laplace
(1749–1827)
A French mathematician and astronomer, Pierre-Simon Laplace was born in Beaumont-en Auge, France. Laplace first approximated the binomial distribution with the normal distribution in his 1812 work *Théorie Analytique des Probabilités*.

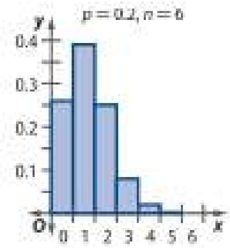
2 The Normal Approximation

According to the Central Limit Theorem, any sampling distribution can approach the normal distribution as n increases. As a result, other distributions such as the binomial distribution can be approximated with the normal distribution. The binomial distribution can be determined by using the equation

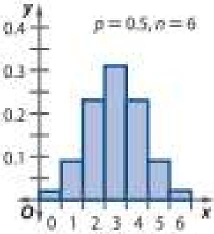
$$P(X) = {}_nC_x\, p^x q^{n-x},$$

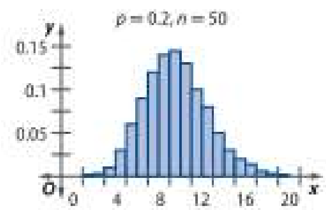
where n is the number of trials, p is the probability of success, and q is the probability of failure.

If the number of trials increases or the probability of success gets close to 0.5, the shape of the binomial distribution begins to resemble the normal distribution. For example, consider the binomial distribution at the right. When $p = 0.2$ and $n = 6$, the distribution is positively skewed.



However, when $p = 0.5$ and $n = 6$ or when $p = 0.2$ and $n = 50$, as shown below, the distribution is approximately normal.





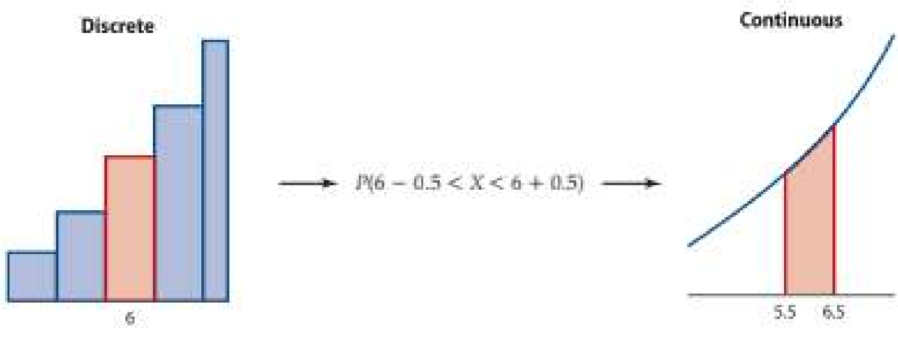
When the probability of success is close to 0 or 1 and the number of trials is relatively small, the normal approximation is not accurate. Therefore, as a rule, the normal approximation is typically used only when $np \geq 5$ and $nq \geq 5$.

KeyConcept Approximation Rule for Binomial Distributions

Words	The normal distribution can be used to approximate a binomial distribution when $np \geq 5$ and $nq \geq 5$.
Example	If p is 0.4 and n is 5, then $np = 5(0.4)$ or 2. Since $2 < 5$, the normal distribution should not be used to approximate the binomial distribution.

It also is important to remember that the normal distribution should only be used to approximate a binomial distribution if the original variable is normally distributed or $n \geq 30$.

Since binomial distributions are *discrete* and normal distributions are *continuous*, a correction for continuity called the **continuity correction factor** must be used when approximating a binomial distribution. To use the correction factor, 0.5 unit is added to or subtracted from a given discrete boundary. For example, to find $P(X = 6)$ in a discrete distribution, the correction would be to find $P(5.5 < X < 6.5)$ for a continuous distribution, as shown below.



Use the following steps to approximate a binomial distribution with the normal distribution.

StudyTip

Binomial Formulas The mean μ and standard deviation σ of a binomial distribution are found using $\mu = np$ and $\sigma = \sqrt{npq}$, respectively.

KeyConcept Normal Approximation of a Binomial Distribution

The procedure for the normal approximation of a binomial distribution is as follows.

Step 1 Find the mean μ and standard deviation σ.

Step 2 Write the problem in probability notation using X.

Step 3 Find the continuity correction factor, and rewrite the problem to show the corresponding area under the normal distribution.

Step 4 Find any corresponding z-values for X.

Step 5 Use a graphing calculator to find the corresponding area.

Example 4 Normal Approximation of a Binomial Distribution

COLLEGE A school newspaper reported that 20% of the current Grade 12 class would be attending an out-of-state college. If 35 Grade 12 students are selected at random, find the probability that fewer than 5 of the the students will be attending an out-of-state college.

In this binomial experiment, $n = 35$, $p = 0.2$, and $q = 0.8$.

Step 1 Find the mean μ and standard deviation σ.

$\mu = np$	Mean and standard deviation of a binomial distribution	$\sigma = \sqrt{npq}$
$= 35 \cdot 0.2$	$n = 35$, $p = 0.2$, and $q = 0.8$	$= \sqrt{35 \cdot 0.2 \cdot 0.8}$
$= 7$	Simplify.	≈ 2.37

Since $np = 35(0.2)$ or 7 and $nq = 35(0.8)$ or 28, which are both greater than 5, the normal distribution can be used to approximate the binomial distribution.

Step 2 Write the problem in probability notation using X.

The probability that fewer than 5 of the seniors will be attending an out-of-state college is $P(X < 5)$.

Step 3 Rewrite the problem with the continuity factor included.

Since the question is asking for the probability that *fewer than* 5 will be attending, subtract 0.5 unit from 5.

$P(X < 5) = P(X < 5 - 0.5)$ or $P(X < 4.5)$

WatchOut!

z-Value Formula When approximating the binomial distribution using the normal distribution, remember to use the z-value formula for an individual data value, not the formula for a sample mean.

Step 4 Find the corresponding z-value for X.

$$z = \frac{X - \mu}{\sigma} \quad \text{z-value formula}$$

$$= \frac{4.5 - 7}{2.37} \quad X = 4.5,\ \mu = 7, \text{ and } \sigma = 2.37$$

$$\approx -1.05 \quad \text{Simplify.}$$

Step 5 Use a graphing calculator to find the area to the left of z.

The approximate area to the left of $z = -1.05$ is 0.147, as shown at the right. Therefore, the probability that fewer than 5 seniors will be attending an out-of-state college in a random sample of 35 seniors is about 14.7%.

```
normalcdf(-4,-1.
05)
      .1468273946
```

GuidedPractice

4. **ADVERTISING** According to the results of an advertising survey sent to customers selected at random, 65% of the customers had not seen a recent television advertisement. Find the probability that from a sample of 50 customer responses, 15 or more had not seen the advertisement.

Real-WorldLink

Product recalls occur when a manufacturer sends out a request to the consumers to return a product after discovering a safety issue. Recalls are costly, but are done to limit the liability of the manufacturer.

Source: National Highway Traffic Safety Administration

Real-World Example 5 Normal Approximation of a Binomial Distribution

MANUFACTURING An automaker has discovered a defect in a new model. The defect is expected to affect 30% of the cars that were produced. What is the probability that there are at least 10 and at most 15 cars with the defect in a random sample of 40 cars?

In this binomial experiment, $n = 40$, $p = 0.3$, and $q = 0.7$.

Step 1 Begin by finding the mean μ and standard deviation σ.

$\mu = np$	Mean and standard deviation of a binomial distribution	$\sigma = \sqrt{npq}$
$= 40 \cdot 0.3$	$n = 40$, $p = 0.3$, and $q = 0.7$	$= \sqrt{40 \cdot 0.3 \cdot 0.7}$
$= 12$	Simplify.	≈ 2.9

Since $np = 40(0.3)$ or 12 and $nq = 40(0.7)$ or 28, which are both greater than 5, the normal distribution can be used to approximate the binomial distribution.

Step 2 Write the problem in probability notation: $P(10 \leq X \leq 15)$.

Step 3 Rewrite the problem with the continuity factor included.

$P(10 \leq X \leq 15) = P(10 - 0.5 < X < 15 + 0.5)$ or $P(9.5 \leq X \leq 15.5)$

Step 4 Find the corresponding z-values for $X = 9.5$ and $X = 15.5$.

$z = \frac{X - \mu}{\sigma}$	z-value formula	$z = \frac{X - \mu}{\sigma}$
$= \frac{9.5 - 12}{2.9}$	Substitute.	$= \frac{15.5 - 12}{2.9}$
≈ -0.86	Simplify.	≈ 1.21

Step 5 Use a graphing calculator to find the area between $z = -0.86$ and $z = 1.21$.

The approximate area that corresponds to $-0.86 < z < 1.21$ is 0.692, as shown at the right. Therefore, the probability of there being at least 10 and at most 15 cars with the defect in a random sample of 40 cars is about 69.2%.

```
normalcdf(-0.86,
1.21)
             .6919660179
```

GuidedPractice

5. **MANUFACTURING** Suppose a defect in a second model by the same automaker is expected to affect 20% of the cars that were produced. What is the probability that there are at least 8 and at most 10 defects in a random sample of 30 cars?

It may seem difficult to know whether to add or subtract 0.5 unit from a discrete data value to find the continuity correction factor. The table below shows each case.

WatchOut!

Writing Inequalities When a problem is asking for a probability *between* two values, write the inequality as $P(c_1 < X < c_2)$, not $P(c_1 \leq X \leq c_2)$. For instance, in Example 5, the probability that there are *between* 10 and 15 defects would be $P(10 < X < 15)$.

ConceptSummary Binomial Distribution Correction Factors

When using the normal distribution to approximate a binomial distribution, the following correction factors should be used, where c is a given data value in the binomial distribution.

Binomial	Normal
$P(X = c)$	$P(c - 0.5 < X < c + 0.5)$
$P(X > c)$	$P(X > c + 0.5)$
$P(X \geq c)$	$P(X > c - 0.5)$
$P(X < c)$	$P(X < c - 0.5)$
$P(X \leq c)$	$P(X < c + 0.5)$

Exercises

1. **VIDEO GAMES** The average prices for three video games at an online auction site are shown. Assume that the variable is normally distributed. (Examples 1 and 2)

Game	Average Price ($)
Column Craze	35
Dungeon Attack!	45
Space Race	52

   a. For a sample of 35 online prices for Column Craze, find the probability that the mean price is more than $38, if the standard deviation is $9.

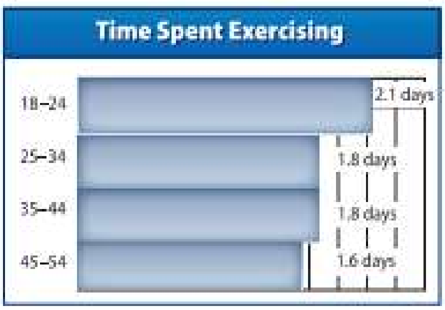
   b. For a random sample of 40 online prices for Space Race, find the probability that the mean price will be between $50 and $55 if the standard deviation is $12.

2. **CHEWING GUM** Americans chew an average of 182 sticks of gum per year. Assume a standard deviation of 13 sticks for each question. Assume that the variable is normally distributed. (Examples 1 and 2)

   a. Find the probability that 50 randomly selected people chew an average of 175 sticks or more per year.

   b. If a random sample of 45 people is selected, find the probability that the mean number of sticks of gum they chew per year is between 180 and 185.

3. **EXERCISE** The average number of days per week that Americans from four different age groups spent exercising during a recent year is shown. Assume that the variable is normally distributed. (Examples 1 and 2)



   a. Find the probability that a random sample of 30 Americans ages 45 to 54 spent more than 1.5 days a week exercising, if the standard deviation is 0.5 day.

   b. Assuming a standard deviation of 1.2 days, in a random sample of 30 Americans ages 18 to 24, find the probability that the average time spent exercising is between 2 and 2.5 days per week.

4. **MEDICINE** The mean recovery time for patients with a certain virus is 4.5 days with a standard deviation of 2 days. Assume that the variable is normally distributed. (Examples 1 and 2)

   a. Find the probability of an average recovery time of less than 4 days for a random sample of 75 people.

   b. In a random sample of 80 people, find the probability that average recovery time is between 4.4 and 4.8 days.

5. **TOURISM** The average number of tourists that visit a national monument every month is 55,000, with a standard deviation of 8,000. Assume that the variable is normally distributed. (Example 3)

   a. If a random month is selected, find the probability that there would be fewer than 50,000 tourists visiting the monument.

   b. If a sample of 10 months is selected, find the probability that there would be fewer than 50,000 visiting tourists.

6. **NUTRITION** The average protein content of a certain brand of energy bar is 12 grams with a standard deviation of 2 grams. Assume that the variable is normally distributed. (Example 3)

   a. Find the probability that a randomly selected bar will have more than 10 grams of protein.

   b. In a sample of 15 bars, find the probability that the average protein content will be greater than 10 grams.

7. **WORLD CUP** In a recent year, 33% of Americans said that they were planning to watch the World Cup football tournament. What is the probability that in a random sample of 45 people, fewer than 14 people plan to watch the World Cup? Assume that the variable is normally distributed. (Example 4)

8. **MOVIES** According to a national poll, in a recent year, 27% of Americans saw 5 or more movies in theaters. What is the probability that in a random sample of 40 people, between 6 and 11 people saw more than 5 movies in a movie theater that year? Assume that the variable is normally distributed. (Example 5)

9. **LIBRARY** A poll was conducted at a library to approximate the percent of books, CDs, magazines, and movies that were checked out during one month. The results are shown. Assume that the variable is normally distributed. (Examples 4 and 5)

Resources	Percent
books	45
CDs	20
magazines	3
movies	32

   a. What is the probability that of 65 randomly selected resources, exactly 35 were books?

   b. Find the probability that of 85 randomly selected resources, at least 15 and at most 18 were CDs.

10. **DRIVING** A driving instructor has found that 12% of students cancel or forget about lessons. Assume that the variable is normally distributed. (Examples 4 and 5)

   a. If the instructor has 60 students, what is the probability that more than 10 of the students will miss a lesson?

   b. What is the probability that of 80 students, exactly 7 students will miss a lesson?

11. **TESTS** A multiple-choice test consists of 50 questions, with possible answers A, B, C, and D. Find the probability that, with random guessing, the number of correct answers will be each of the following.

a. less than 18
b. exactly 12
c. at least 14
d. between 10 and 15

Find the minimum sample size needed for each probability so that the normal distribution can be used to approximate the binomial distribution.

12. $p = 0.1$
13. $p = 0.4$
14. $p = 0.5$
15. $p = 0.8$

16. **BASKETBALL** The average points per game scored by four different basketball players are shown.

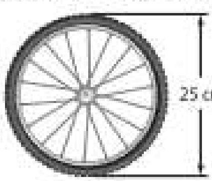
Player	A	B	C	D
Average	8.1	6.3	4.9	10.3

a. Find the mean and standard deviation of the averages.

b. Identify each possible combination of 3 players' averages, and find the mean of each combination.

c. Find the mean of each of the means that you found in part **b.** How does this compare to the mean that you found in part **a**?

17. **BICYCLES** Consider the bicycle rim shown, where $\mu = 25$ centimeters and $\sigma = 0.125$ centimeter.



The diameters for 10 random samples of 3 bicycle rims from a company's assembly line are shown.

Sample	Diameter	Sample	Diameter
1	25.2, 24.9, 25	6	24.9, 25.1, 24.8
2	25.1, 25, 24.8	7	25.3, 24.9, 25.1
3	25.3, 24.9, 24.8	8	25.4, 24.8, 25.3
4	24.9, 25.3, 25.2	9	24.8, 24.9, 25.2
5	25, 25.2, 24.7	10	25, 25.3, 24.7

a. Find $\bar{x}$ and s for each sample.

b. Construct a scatter plot with the sample number on the x-axis and the sample means on the y-axis.

c. In this process, the upper control limit is $\bar{x} + \frac{3\sigma}{\sqrt{n}}$ and the lower control limit is $\bar{x} - \frac{3\sigma}{\sqrt{n}}$. If the process is in control, all values should fall within the control limits. Use the graph from part **b** to determine whether the process is in control. Explain your reasoning.

18. **BLOOD TYPES** The distributions of blood types of U.S. and Canadian citizens are shown.

U.S.		Canada	
Type	Distribution	Type	Distribution
O	44%	O	46%
A	42%	A	42%
B	10%	B	9%
AB	4%	AB	3%

a. If 50 U.S. citizens are selected at random, find the probability that fewer than 20 of those chosen will have type O blood.

b. Find the probability that out of 100 randomly selected Canadian citizens, between 80 and 90 of those chosen will have types O or A blood.

c. What is the probability that two randomly chosen people from the U.S. or Canada will have the same blood type?

H.O.T. Problems Use Higher-Order Thinking Skills

19. **ERROR ANALYSIS** Halima and Hana are calculating results for a survey that they are taking as part of a summer internship. They found that of the residents they surveyed, 65% do not recycle. Halima found the probability that fewer than 30 out of 50 random residents do not recycle is 18.7%, while Hana found that it would be 27.7%. Is either of them correct? Explain your reasoning.

20. **WRITING IN MATH** Explain how the Central Limit Theorem can be used to describe the shape, center, and spread of a distribution of sample means.

21. **CHALLENGE** In the United States, 7% of the male population and 0.4% of the female population are color-blind. Suppose random samples of 100 men and 1500 women are selected. Is there a greater probability that the men's or women's sample will include at least 10 people who are color-blind? Explain your reasoning.

22. **OPEN ENDED** Give an example of a population and a sample of the population. Explain what is meant by the corresponding sampling distribution.

REASONING Determine whether each statement is *true* or *false*. Explain your reasoning.

23. As the number of samples increases, a sampling distribution of sample means will approach the normal distribution.

24. In a binomial distribution, $P(X \geq c) \neq P(X > c)$.

25. **WRITING IN MATH** Explain why the normal distribution can be used to approximate a binomial distribution, what conditions are necessary to do so, and why a correction for continuity is needed.

Spiral Review

26. **COMMUNITY SERVICE** A recent study of 1,286 high school seniors revealed that the students completed an average of 38 hours of volunteer work over the summer with a standard deviation of 6.7 hours. Determine the number of seniors who completed more than 42 hours of community service. Assume that the variable is normally distributed.

27. **GAMES** Managers of a fitness club randomly surveyed 56 members and recorded the number of days that each member attended the club in a given week. Use the frequency distribution shown to construct a probability distribution for the random variable X. Then find the mean, variation, and standard deviation of the probability distribution.

Days, X	Frequency	Days, X	Frequency
0	3	4	11
1	5	5	9
2	10	6	3
3	14	7	1

Find each sum.

28. $\sum_{n=1}^{19} -50 + 5n$

29. $\sum_{n=12}^{68} 5 - \frac{n}{4}$

30. $\sum_{n=10}^{16} 24n - 90$

Find the specified term of each sequence.

31. 7th term, $a_n = (a_{n-1} - 6)^2$, $a_1 = 4$

32. 6th term, $a_n = 3n^2 - 4n$

33. 4th term, $a_n = (a_{n-1})^2 - 11$, $a_1 = 3$

Find rectangular coordinates for each point with the given polar coordinates.

34. $\left(2, \frac{\pi}{2}\right)$

35. $\left(\frac{1}{4}, \frac{\pi}{4}\right)$

36. $(6, 210^\circ)$

Find each of the following for $\mathbf{p} = \langle 4, 0\rangle$, $\mathbf{q} = \langle -2, -3\rangle$, and $\mathbf{t} = \langle -4, 2\rangle$.

37. $\mathbf{p} - \mathbf{t} - 2\mathbf{q}$

38. $\mathbf{q} - 4\mathbf{p} + 3\mathbf{t}$

39. $4\mathbf{p} + 3\mathbf{q} - 6\mathbf{t}$

Write an equation for and graph each parabola with focus F and the given characteristics.

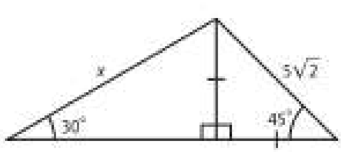
40. $F(-6, 8)$; opens up; contains $(0, 16)$

41. $F(2, -5)$; opens down; contains $(10, -11)$

42. **YOGURT** The Frozen Yogurt Shack sells cones in three sizes: small, $2.89; medium, $3.19; and large, $3.39. On Friday, 78 cones were sold totaling $246.42. The Shack sold six more medium cones than small cones that day. Use Cramer's Rule to determine the number of each type of cone sold on Friday.

Skills Review for Standardized Tests

43. **SAT/ACT** What is the value of x?



A $2\sqrt{2}$
B 5
C $5\sqrt{3}$
D 10
E $5\sqrt{6}$

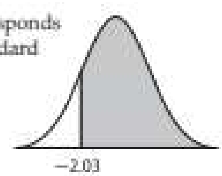
44. **REVIEW** In a study, 62% of registered voters said they voted in the 2008 presidential election. If 6 registered voters are chosen at random, what is the probability that at least 4 of them voted?

F 32%
G 41.2%
H 58.6%
J 73.2%

45. The average number of patients who are seen every week at a certain hospital is normally distributed. The average per week is 12,423, with a standard deviation of 3269. If a week is selected at random, find the probability that there would be fewer than 4000 patients.

A 0.50%
B 2.37%
C 32.20%
D 36.73%

46. **REVIEW** Find the area that corresponds to the shaded region of this standard normal distribution.



F 0.02
G 0.04
H 0.96
J 0.98

CHAPTER 4 Mid-Chapter Quiz

Lessons 4-1 through 4-4

1. **AUDITIONS** The ages of 20 students who auditioned for roles in a high school production of *Gone with the Wind* are shown. (Lesson 4-1)

Ages of Students				
14	15	17	16	14
16	17	16	18	16
15	16	18	15	17
14	18	15	17	16

a. Construct a histogram, and use it to describe the shape of the distribution.

b. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

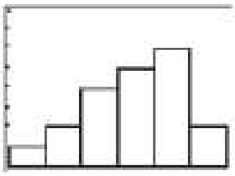
2. **VACATION** Suhaila is planning a trip for spring vacation. She has narrowed her choices down to two locations. The temperatures during twelve days around the time of spring vacation are shown below for each location. (Lesson 4-1)

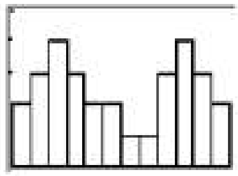
Cape Hatteras, North Carolina					
52	60	62	57	55	63
64	59	54	52	54	60
Orlando, Florida					
77	77	76	76	72	71
72	74	74	72	73	73

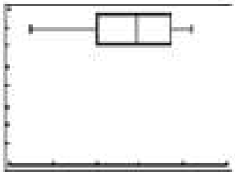
a. Construct side-by-side box plots of the data sets, and use this display to compare the center and spread of the distributions.

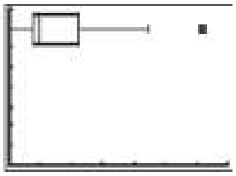
b. Which location has the greater variation in temperature?

3. **MULTIPLE CHOICE** Which of the following displays a data set that is positively skewed? (Lesson 4-1)

A


C


B


D


Classify each random variable *X* as *discrete* or *continuous*. Explain your reasoning. (Lesson 4-2)

4. X represents the number of times a coin lands on heads if flipped a random number of times.

5. X represents the time it takes for a randomly chosen marathon runner to complete a race.

6. **TRAVEL** In a recent poll, 20% of American teenagers said that they have visited Washington, D.C. Find the probability that out of 6 randomly chosen teenagers, at least 3 have visited their nation's capital. (Lesson 4-2)

7. **SHAMPOO** The amount of water in milliliters in a particular shampoo is normally distributed with $\mu = 125$ and $\sigma = 7$. Find each of the following. (Lesson 4-3)

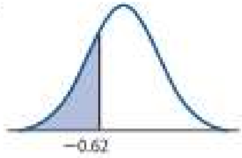
a. $P(X < 105)$

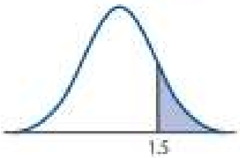
b. $P(X > 140)$

c. $P(115 < X < 130)$

8. **GOLF** A random sample of 130 golfers resulted in an average score of 78 with a standard deviation of 6.3. Find the number of golfers with an average of 70 or lower. (Lesson 4-3)

Find the area that corresponds with the shaded region. (Lesson 4-3)

9.


10.


11. **PROJECTS** The scores on a science project for one class are normally distributed with $\mu = 78$ and $\sigma = 8$. Find each probability. (Lesson 4-3)

a. $P(X \geq 96)$

b. $P(60 < X < 85)$

Find the probability of each sample mean. (Lesson 4-4)

12. $P(\bar{x} < 38)$; $\mu = 40$, $\sigma = 5.5$, $n = 25$

13. $P(\bar{x} > 82.2)$; $\mu = 82.5$, $\sigma = 4.1$, $n = 50$

14. **EMPLOYMENT** According to a recent study, the average age that a person starts his or her first job is 16.8 years old. Assume that this variable is normally distributed with a standard deviation of 1.7 years. If a random sample of 25 people is selected, find the probability that the mean age the participants started their first jobs is greater than 17 years old. (Lesson 4-4)

LESSON 4-5 Confidence Intervals

Then

- You analyzed sample means and the effect of the Central Limit Theorem on a sampling distribution.

Now

- **1** Use the normal distributions to find confidence intervals for the mean.
- **2** Use *t*-distributions to find confidence intervals for the mean.

Why?

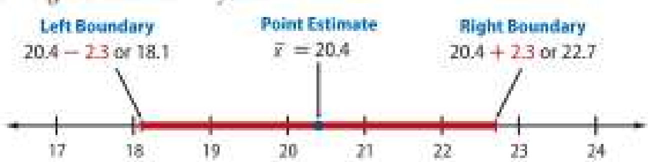
- Executives at a film studio want to know the average age of people seeing a movie. A survey of 200 people who saw the movie finds that the average age was 20.4 years. The studio executives decide to estimate the mean age μ for all customers as between 18.1 and 22.7 or $18.1 < \mu < 22.7$.

New Vocabulary

inferential statistics
parameter
point estimate
interval estimate
confidence level
maximum error of estimate
critical value
confidence interval
t-distribution
degrees of freedom

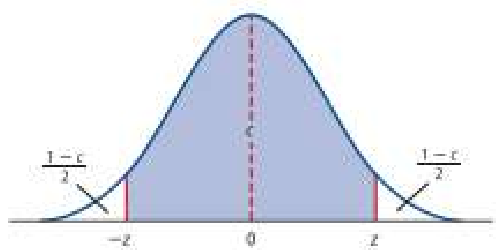
1 Normal Distribution In **inferential statistics**, a sample of data is analyzed and conclusions are made about the entire population. This procedure is used because it is usually too challenging to get information from every member of a population. A measure that describes a characteristic of a population, such as the mean or standard deviation, is called a **parameter**. Many different parameters may be used to analyze data; but in this lesson, we will concentrate on the mean.

The average age of 20.4 years is an example of a **point estimate**, a single value estimate of an unknown population parameter. Due to sampling error and relatively small sampling size, the point estimate will most likely not match the population mean. For this reason, the studio executives used an **interval estimate** of $18.1 < \mu < 22.7$. An interval estimate is a range of values used to estimate an unknown population parameter. To form an interval estimate, a point estimate is used as the center of the interval and a margin of error is added to and subtracted from the point estimate. For this study, the studio executives let the margin of error be 2.3 years.



Before an interval estimate is created, it is helpful to know just how reliable you want it to be. The probability that the interval estimate will include the actual population parameter is known as the **confidence level**, and is denoted as c. We can illustrate a confidence level using the normal distribution if the standard deviation of the population σ is known and the population is normally distributed or if $n \geq 30$. Recall that the Central Limit Theorem states that when $n \geq 30$, the sampling distribution of sample means resembles a normal distribution.

The confidence level for a normal distribution is equal to the area under the standard normal curve between $-z$ and z, as shown. The remaining area in the two tails is then $1 - c$, or $\frac{1}{2}(1 - c)$ for each tail.



Suppose you are conducting an experiment in which you want a confidence level of 95%. When $c = 95\%$, 2.5% of the area lies to the left of $-z$ and 2.5% lies to the right of z. Using a graphing calculator, you can find the corresponding value for $-z$ to be -1.96 and z to be 1.96. By calculating the product of the z-values and the standard deviation of the sample means $\sigma_{\bar{x}}$, the **maximum error of estimate** E, the maximum difference between the point estimate and the actual value of the parameter, can be determined.

StudyTip

Maximum Error of Estimate The maximum error of estimate E will be a positive value since it is the maximum difference between the point estimate and the actual value of the parameter.

KeyConcept Maximum Error of Estimate

The maximum error of estimate E for a population mean μ is given by

$$E = z \cdot \sigma_{\bar{x}} \text{ or } z \cdot \frac{\sigma}{\sqrt{n}},$$

where z is a critical value that corresponds to a particular confidence level, and $\sigma_{\bar{x}}$ or $\frac{\sigma}{\sqrt{n}}$ is the standard deviation of the sample means. When $n \geq 30$, s, the sample standard deviation, may be substituted for σ.

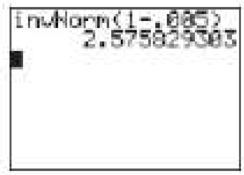
The z-values that correspond to a particular confidence level are known as **critical values**. The three most widely used confidence levels and their corresponding critical values are shown below.

Confidence Level	z-Value
90%	1.645
95%	1.960
99%	2.576

Example 1 Find Maximum Error of Estimate

TEXTBOOKS A poll of 75 randomly selected college students showed that the students spent an average of $230 on textbooks per session. Assume from past studies that the standard deviation is $55. Use a 99% confidence level to find the maximum error of estimate for the amount of money spent by students on textbooks.

In a 99% confidence interval, 0.5% of the area lies in each tail. You can find the corresponding z-value to be 2.576 by using a graphing calculator or the table above.



$E = z \cdot \frac{\sigma}{\sqrt{n}}$ Maximum Error of Estimate

$= 2.576 \cdot \frac{55}{\sqrt{75}}$ $z = 2.576$, $\sigma = 55$, and $n = 75$

≈ 16.36 Simplify.

This means that you can be 99% confident that the population mean of money spent on textbooks will be no more than $16.36 from the sample mean of $230.

GuidedPractice

1. **MUSIC** Executives at a music label surveyed 125 people who actively download music and found that the listeners have an average of 740 songs downloaded to their computers. Assume a standard deviation of 86 songs. Use a 94% confidence level to find the maximum error of estimate for the number of songs on the computer of someone who actively downloads music.

Once a confidence level is established and a maximum error of estimate is calculated, it can be used to determine a **confidence interval**. A confidence interval, denoted CI, is a specific interval estimate of a parameter and can be found when the maximum error of estimate is added to and subtracted from the sample mean.

KeyConcept Confidence Interval for the Mean

A confidence interval CI for a population mean μ is given by

$$CI = \bar{x} \pm E \text{ or } \bar{x} \pm z \cdot \frac{\sigma}{\sqrt{n}},$$

where $\bar{x}$ is the sample mean and E is the maximum error of estimate.

Example 2 Find Confidence Intervals When σ is Known

HOMEWORK A poll of 20 randomly selected high school students showed that the students spent a mean time of 35 minutes per weeknight on homework. Assume a normal distribution with a standard deviation of 12 minutes. Find a 90% confidence interval for the mean of all of the students.

Substitute 1.645, the corresponding z-value for a 90% confidence interval, into the confidence interval formula.

$CI = \bar{x} \pm z \cdot \frac{\sigma}{\sqrt{n}}$ — Confidence Interval for the Mean

$= 35 \pm 1.645 \cdot \frac{12}{\sqrt{20}}$ — $\bar{x} = 35$, $z = 1.645$, $\sigma = 12$, and $n = 20$

$\approx 35 \pm 4.41$ — Simplify.

Add and subtract the margin of error.

Left Boundary	Right Boundary
$35 - 4.41 = 30.59$	$35 + 4.41 = 39.41$

A 90% confidence interval is $30.59 < \mu < 39.41$. Therefore, with 90% confidence, the mean time spent on homework by students is between 30.6 and 39.4 minutes.

GuidedPractice

2. **SHOPPING** A sample of 65 randomly selected mall patrons showed that they spent an average of \$70 that day. Assume a standard deviation of \$12. Find a 95% confidence interval for the average amount spent by a mall patron that day.

Real-WorldCareer

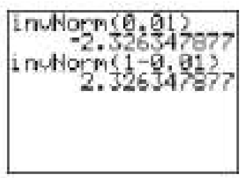
Engineering Engineers use science and math to find economical solutions to technical problems. A bachelor's degree in engineering is usually required for most entry-level jobs.

In many real-world situations, the population standard deviation is unknown. When this is the case, the standard deviation s of the sample can be used in place of the population standard deviation, as long as the variable is normally distributed and $n \geq 30$.

Real-World Example 3 Find Confidence Intervals When σ is Unknown

ENGINEERING Tensile strength is the stress at which a material breaks or deforms. A company wants to estimate the mean tensile strength of a new material. A random sample of 40 units is normally distributed with an average tensile strength of 2,552 pounds per square inch (psi) and a standard deviation of 203 pounds per square inch. Find a 98% confidence interval for the mean tensile strength of the material.

In a 98% confidence interval, 1% of the area lies in each tail. You can find the corresponding z-value to be 2.33 by using a graphing calculator.



Since the distribution is normal and $n \geq 30$, the sample standard deviation can be used to find a confidence interval.

$CI = \bar{x} \pm z \cdot \frac{s}{\sqrt{n}}$ — Confidence Interval for the Mean

$= 2{,}552 \pm 2.33 \cdot \frac{203}{\sqrt{40}}$ — $\bar{x} = 2{,}552$, $z = 2.33$, $s = 203$, and $n = 40$

$\approx 2{,}552 \pm 74$ — Simplify.

Therefore, a 98% confidence interval is $2{,}478 < \mu < 2{,}626$.

TechnologyTip

Calculate Confidence Intervals You can check your answer by using a graphing calculator. Press STAT and select ZInterval under the TESTS menu. For Input: select Stats and then enter each of the values. Then select Calculate.

GuidedPractice

3. **ENGINEERING** Suppose a random sample of 50 units of the same material is normally distributed with an average tensile strength of 39.2 psi and a standard deviation of 3.1 psi. Estimate the mean tensile strength with 99% confidence.

2 ***t*-Distribution** In many cases, the population standard deviation is not known and, due to constraints such as time and cost, sample sizes exceeding 30 are not realistic. In these cases, another distribution called the *t*-distribution can be used, as long as the variable is approximately normally distributed.

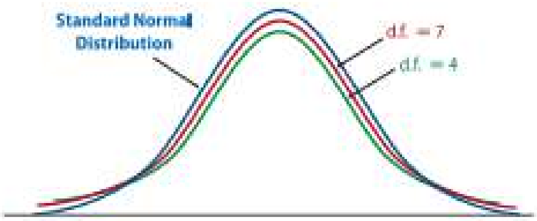
Study Tip

Distributions That Are Not Normal You cannot use a normal distribution or *t*-distribution to construct a confidence interval if the population is not normally or approximately normally distributed.

The ***t*-distribution** is a family of curves that are dependent on a parameter known as the *degrees of freedom*. The **degrees of freedom (d.f.)** are equal to $n - 1$ and represent the number of values that are free to vary after a sample statistic is determined.

For example, if $\bar{x} = 4$ in a sample of 10 values, 9 of the 10 values are free to vary. Once the first 9 values are selected, the tenth value must be a specific number in order for $\bar{x} = \frac{40}{10}$. So, the degrees of freedom are 10 − 1 or 9, which corresponds to a specific curve.

Notice in the figure below that as the degrees of freedom increase, or as d.f. approaches 30, the *t*-distribution approaches the standard normal distribution.



The characteristics of the *t*-distribution are summarized below.

Key Concept Characteristics of the *t*-Distribution

- The distribution is bell-shaped and symmetric about the mean.
- The mean, median, and mode equal 0 and are at the center of the distribution.
- The curve never touches the *x*-axis.
- The standard deviation is greater than 1.
- The distribution is a family of curves based on the sample size *n*.
- As *n* increases, the distribution approaches the standard normal distribution.

Similar to the normal distribution, the *t*-distribution can be used to construct a confidence interval by using a *t*-value rather than a *z*-value to calculate the maximum error of the estimate *E*. A *t*-value can be found by

$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} \text{ or } \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}, \text{ where } \mu \text{ is the population mean.}$$

You will use a graphing calculator to find values for *t* since the population mean μ is the parameter that you are attempting to estimate. You can find a confidence interval when using the *t*-distribution by using the formula shown.

Study Tip

Distributions When $n \geq 30$, it is standard procedure to use the normal distribution. However, *t*-distributions can still be used.

Key Concept Confidence Interval Using *t*-Distribution

A confidence interval *CI* for the *t*-distribution is given by

$$CI = \bar{x} \pm t \cdot \frac{s}{\sqrt{n}},$$

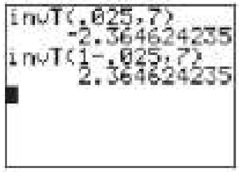
where $\bar{x}$ is the sample mean, *t* is a critical value with $n - 1$ degrees of freedom, *s* is the sample standard deviation, and *n* is the sample size.

Example 4 Find Confidence Intervals with the t-Distribution

CAPACITY The capacities of eight randomly selected tanks are measured. The mean capacity is 143 liters and the standard deviation is 3.0. Find the 95% confidence interval of the mean capacity of the tanks. Assume that the variable is normally distributed.

The population standard deviation is unknown and $n < 30$, so the t-distribution must be used. Since $n = 8$, there are $8 - 1$ or 7 degrees of freedom. You can use a graphing calculator to find the corresponding t-value.

In the DISTR Menu, select InvT(a, df). The a value represents the area of one tail of the distribution and df represents degrees of freedom. So, for a 95% confidence interval, the area in either tail of the t-distribution is half of 5% or 0.025.



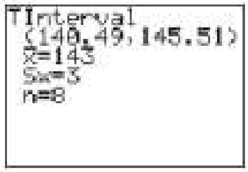
$CI = \bar{x} \pm t \cdot \frac{s}{\sqrt{n}}$ — Confidence Interval Using t-Distribution

$= 143 \pm 2.365 \cdot \frac{3}{\sqrt{8}}$ — $\bar{x} = 143$, $t = 2.365$, $s = 3$, and $n = 8$

$= 143 \pm 2.5$ — Simplify.

Therefore, the 95% confidence interval is $140.5 < \mu < 145.5$.

CHECK You can check your answer by using a graphing calculator. In the STAT menu, select TESTS and TInterval. Under the TInterval menu, select Stats and enter each of the values. Then select Calculate.



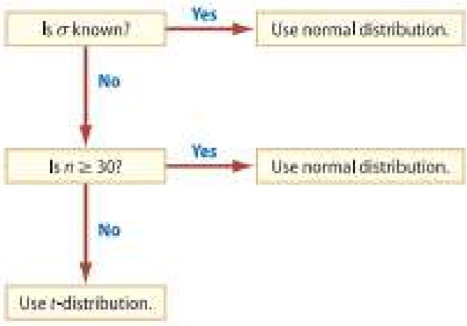
Guided Practice

4. **RESTAURANTS** The waiting time of ten randomly selected customers at a restaurant was measured with a mean of 25 minutes and a standard deviation of 4 minutes. Find the 99% confidence interval of the mean waiting time for all customers, assuming that the variable is normally distributed.

StudyTip

Using the t-Distribution Most real-world inferences about the population mean will be completed using the t-value because σ is rarely known.

It may be difficult to determine whether to use a normal distribution or a t-distribution for a given problem. The chart shown below summarizes when to use each, assuming that the population is normally or approximately normally distributed.



In all statistical experiments, the user determines the confidence level, which directly affects the confidence interval. With all other variables held constant, increasing the confidence level will expand the confidence interval. Expanding a confidence interval reduces the accuracy of the estimate. For example, observe the confidence interval from Example 2 when the confidence level is raised to 99%.

	90% Confidence Level	99% Confidence Level
z-value	1.645	2.576
E	4.41	6.91
CI	$30.59 < \mu < 39.41$	$28.09 < \mu < 41.91$

Generally, a high confidence level and a small confidence interval are desired. This can be achieved by increasing the sample size n. You can find the minimum sample size needed for a specific maximum error of estimate by starting with the formula for E and solving for n.

$$E = z \cdot \frac{\sigma}{\sqrt{n}} \quad \text{Maximum Error of Estimate}$$

$$\sqrt{n} \cdot E = z \cdot \sigma \quad \text{Multiply each side by } \sqrt{n}.$$

$$\sqrt{n} = \frac{z \cdot \sigma}{E} \quad \text{Divide each side by } E.$$

$$n = \left(\frac{z\sigma}{E}\right)^2 \quad \text{Square each side.}$$

KeyConcept Minimum Sample Size Formula

The minimum sample size needed when finding a confidence interval for the mean is given by $n = \left(\frac{z\sigma}{E}\right)^2$, where n is the sample size and E is the maximum error of estimate.

Example 5 Find a Minimum Sample Size

PRODUCT DEVELOPMENT You are testing the reliability of a thermometer. Your manager asks you to conduct an experiment with results that are accurate to ±0.05 degree with 95% confidence. If $\sigma = 0.8$, how many measurements are needed?

$$n = \left(\frac{z\sigma}{E}\right)^2 \quad \text{Minimum Sample Size Formula}$$

$$= \left(\frac{1.96(0.8)}{0.05}\right)^2 \quad z = 1.96,\ \sigma = 0.8, \text{ and } E = 0.05$$

$$= 983.45 \quad \text{Simplify.}$$

At least 984 observations are needed to have a margin of error of ±0.05 with 95% confidence.

StudyTip

Rounding It is not possible to have a sample size that is a fraction. Therefore, when finding a minimum sample size, always round answers in the form of a fraction or a decimal to the next greater whole number.

Guided Practice

5. **MARKETING** Executives at a car dealership want to estimate the average age of their customers before making a television commercial. They want to be 90% confident that the mean age is ±2 years of the sample mean. If the standard deviation from a previous study is 12 years, how large should the sample be?

Exercises

1. **TRANSPORTATION** A random sample of 85 New York City residents showed that the average commuting time to work was 36.5 minutes. Assume that the standard deviation from previous studies was 11.3 minutes. Find the maximum error of estimate for a 99% confidence level. Then create a confidence interval for the mean commuting time of all New York City residents. (Examples 1–3)

2. **ORANGES** The owner of an orange grove randomly selects 50 oranges of the same type and weighs them with a resulting mean weight of 208.6 grams and a standard deviation of 22.5 grams. Find the maximum error of estimate for a 98% confidence level. Then estimate the mean weight of the oranges using a confidence interval. (Examples 1–3)

3. **TEMPERATURE** The average body temperature for 15 randomly selected polar bears was 36.4°C. Assume that the standard deviation from a recent study was 1.6°C. Find the maximum error of estimate for a 95% confidence level. Then estimate the mean body temperature for all polar bears in that region using a confidence interval. (Examples 1–3)

4. **TYPING SPEED** In a random sample of 20 students in a computer class, the average keyboarding speed was 40 words per minute (WPM) with a standard deviation of 8 WPM. Estimate the mean keyboarding speed for all students taking the class using a 90% confidence level. (Example 4)

5. **TEXT MESSAGES** A random sample of 25 students with cell phones found on average, the students send or receive 68 text messages a day with a standard deviation of 13 messages. Estimate the mean number of text messages for all students with cell phones using a 96% confidence level. (Example 4)

6. **COLLEGE VISITS** A random sample of 20 college-bound Grade 11 students found on average they visited 6.4 colleges with a standard deviation of 1.9. Estimate the mean number of college visits for all college-bound juniors using a 95% confidence level. (Example 4)

Determine whether the normal distribution or *t*-distribution should be used for each question. Then find each confidence interval given the following information. (Examples 2–4)

7. 90%; $\bar{x} = 128$, $s = 7$, $n = 20$

8. 95%; $\bar{x} = 65$, $s = 15.9$, $n = 300$

9. 95%; $\bar{x} = 39.4$, $s = 1.2$, $n = 15$

10. 98%; $\bar{x} = 122.3$, $\sigma = 2.2$, $n = 2000$

11. 99%; $\bar{x} = 28.3$, $\sigma = 4.5$, $n = 75$

12. 99%; $\bar{x} = 2489$, $\sigma = 18.3$, $n = 160$

13. **COFFEE** The owner of a coffee shop wants to determine the average price for a small cup of coffee in his city. How large should the sample be if he wishes to be accurate to within $0.015 at 90% confidence? A previous study showed that the standard deviation of the price was $0.10. (Example 5)

14. **TESTS** A teacher wants to estimate the average amount of time it takes students to finish a 25-question test. How large should the sample be if the teacher wishes to be 99% accurate within 8 minutes? A previous study showed that the standard deviation of the time was 11.3 minutes. (Example 5)

15. **SCHOOL** A survey was taken by 26 randomly selected students, recording the amount of time each student participated in after-school activities for a given week. Assume that the time is normally distributed.

Time (hours)						
11	7	2	7	6	12	9
10	8	6	4	8	8	7
4	7	8	8	6	5	
9	9	10	15	12	13	

a. Decide the type of distribution that can be used to estimate the population mean. Explain your reasoning.

b. Calculate the mean and the standard deviation to the nearest tenth.

c. Construct a 95% confidence interval for the average amount of time students participate in after-school activities.

d. Interpret the confidence interval in the context of the problem.

16. **WAGES** In a previous study, the standard deviation for starting wages among employed high school students was $0.50. A survey of 20 randomly selected employed high school students was conducted and their starting wages were recorded. Assume that the wages are normally distributed.

Wages ($)				
6.75	6.50	6.50	5.50	6.75
5.75	6.50	7.50	7.25	6.00
6.50	7.25	6.75	6.00	5.75
6.00	6.50	6.75	7.00	6.25

a. Decide the type of distribution that can be used to estimate the population mean. Explain your reasoning.

b. Calculate the mean to the nearest hundredth.

c. Construct a 90% confidence interval for the average starting wage for an employed high school student.

d. Interpret the confidence interval in the context of the problem.

17. **AGE** Yousif wants to estimate the average age of teachers with a 95% confidence level. He knows that the standard deviation from past studies is 9 years. If Yousif has only 50 teachers at his school to survey, how accurate can he make his estimate?

18. **TELEVISION** Nasser and Ayoub want to compare the average amount of time per day in minutes that boys and girls watch television. They surveyed 16 female students and 16 male students chosen at random and recorded the viewing times.

Female	
115	120
125	130
120	120
125	105
110	115
105	110
120	125
110	115

Male	
90	140
120	110
105	115
125	120
105	130
150	125
120	110
115	90

a. Calculate the mean and sample standard deviation for each data set.

b. Construct two 99% confidence intervals for the average amount of time spent watching television for both boys and girls.

c. Make a statement comparing the effectiveness of the two intervals.

19. **RESTAURANT** The owner of a restaurant wants a mean preparation time of 20 minutes for each order that is placed. To help ensure that the goal is achieved, the owner timed 24 randomly selected orders and found an average preparation time of 22 minutes with a standard deviation of 4 minutes. The owner will be satisfied if the goal preparation time falls within the 99% confidence interval at which the restaurant is currently operating. Is the owner satisfied? Explain your reasoning.

20. **INCOME** Khalifa is being transferred by his employer and has his choice of three cities. Before making a decision, he wants to compare the average incomes of his fellow employees in the three cities. With help from the human resource department, he records the following information. The sample size for each city was 35 employees.

City	$\bar{x}$ (\$)	σ (\$)
1	46,700	6,300
2	47,800	3,000
3	45,000	8,000

a. Construct a 95% confidence interval for the average income of employees in each city.

b. If salary is all that is considered, to what city should Khalifa choose to be transferred? Explain your reasoning.

21. **CELL PHONES** A cell phone manufacturer wants the mean talk time, the time a phone is engaged in sending a message or transmitting a conversation, for its long-life batteries to be 62 hours. To ensure the quality of its batteries, the manufacturer randomly samples 14 phones and records the talk times in hours.

Talk Time (hours)						
61.0	63.1	63.3	59.1	63.4	61.5	60.0
62.6	62.3	60.3	62.9	61.3	62.4	63.6

The manufacturer will be satisfied if the mean talk time falls within the 99% confidence interval at which the batteries are currently operating. Is the manufacturer satisfied? Explain your reasoning.

22. **READING** Rana is conducting a study on the average amount of time that people between the ages of 17–25 spend reading each day. She surveys 20 people chosen at random. She currently has a confidence interval with a maximum error of estimate of 10 minutes. What will Rana's sample size need to be if she wants to reduce the error to 5 minutes? 2.5 minutes?

H.O.T. Problems Use Higher-Order Thinking Skills

23. **CHALLENGE** A confidence interval of $40.872 < \mu < 49.128$ is created by a random study. If the sample standard deviation was 10 and the t-value used was 2.064, find the degrees of freedom for the t-distribution.

24. **WRITING IN MATH** Most studies desire results with high confidence levels. Explain why a 99% confidence level is not used for every study.

REASONING Determine whether each statement is *true* or *false*. Explain your reasoning.

25. Increasing the sample size will expand a confidence interval.

26. Increasing the confidence level will expand a confidence interval.

27. Increasing the standard deviation will expand a confidence interval.

28. Increasing the mean will expand a confidence interval.

29. **REASONING** If a person conducting a study wants to reduce the maximum error of estimate by $\frac{1}{3}$, what would the person have to do to the sample size?

30. **CHALLENGE** A study conducted with a random sample of size $n = 64$ results in a confidence interval of $3.19 < \mu < 4.01$. If the interval was created using a 90% confidence level, find the sample standard deviation.

31. **WRITING IN MATH** Explain why parameters are needed in statistics.

Spiral Review

32. **EDUCATION** According to a recent survey, 35% of adult Americans had obtained a bachelor's degree. What is the probability that in a random sample of 50 people, between 12 and 16 people had earned a bachelor's degree?

33. **CAR BATTERY** The useful life of a certain car battery is normally distributed with a mean of 100,000 kilometers and a standard deviation of 10,000 kilometers. The company makes 20,000 batteries a month. What is the probability that if you select a car battery at random, it will last between 80,000 and 110,000 kilometers?

Use the fifth partial sum of the trigonometric series for cosine or sine to approximate each value to three decimal places.

34. $\sin \frac{\pi}{7}$

35. $\cos \frac{2\pi}{11}$

36. $\sin \frac{4\pi}{17}$

Determine the eccentricity, type of conic, and equation of the directrix given by each polar equation.

37. $r = \frac{8}{\cos \theta + 5}$

38. $r = \frac{4}{7 \cos \theta + 4}$

39. $r = \frac{2}{\sin \theta + 3}$

Use the dot product to find the magnitude of the given vector.

40. $\mathbf{u} = \langle -8, 0 \rangle$

41. $\mathbf{v} = \langle 7, 2 \rangle$

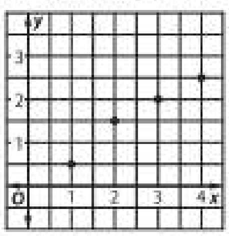
42. $\mathbf{u} = \langle 4, 8 \rangle$

43. **THRILL RIDES** At an amusement park, there is an additional cost per person to ride the Cloud Coaster as well as the Danger Coaster as a 1-person, 2-person, and 3-person ride. The table shows how many people paid for the rides during the first four hours that the park was open. Write and solve a system of equations to determine the cost of each ride per person. Interpret your solution.

Hour	Cloud Coaster	Danger Coaster			Total Paid (AED)
		1 Person	2 Person	3 Person	
1	8	5	10	3	575
2	10	8	2	6	574
3	16	4	8	3	661
4	13	11	6	0	722

Skills Review for Standardized Tests

44. **SAT/ACT** Which line best fits the data in the graph?



A $y = x$

B $y = -0.5x + 4$

C $y = -0.5x - 4$

D $y = 0.5 + 0.5x$

E $y = 0.75x$

45. **REVIEW** People were chosen at random and asked how many times they went out to eat per week. If $\sigma = 0.6$, the results had 95% confidence, and they were accurate to ± 0.05, about how many people were asked?

F 6 G 23 H 144 J 554

46. In a random sample of 28 college-educated adults aged 25 to 35, the average amount of student-loan debt was \$5566 with a standard deviation of \$1831. Estimate the mean student-loan debt for all college-educated adults ages 25 to 35 using a 90% confidence interval.

A $4188 < \mu < 6944$

B $4319 < \mu < 6813$

C $4507 < \mu < 6625$

D $4997 < \mu < 6135$

47. **REVIEW** A school has two independent backup generators having probabilities of 0.9 and 0.95, respectively, of successful operation in case of a power outage. What is the probability that at least one backup generator operates during a power outage?

F 0.855

G 0.89

H 0.95

J 0.995

LESSON

4-6 Hypothesis Testing

Then	Now	Why?
You found confidence intervals for the means of distributions.	1 Write null and alternative hypotheses, and identify which represents the claims. 2 Perform hypothesis testing using test statistics and *p*-values.	Saeed and Fahd are shooting baskets when Fahd proudly boasts, "I can make at least 90% of my free throws." Saeed is curious about Fahd's remark and wants to test the accuracy of his claim.

NewVocabulary

hypothesis test
null hypothesis
alternative hypothesis
level of significance
left-tailed test
two-tailed test
right-tailed test
***p*-value**

1 Hypotheses

A **hypothesis test** assesses evidence provided by data about a claim concerning a population parameter. A claim of this type is called a *statistical hypothesis* and may or may not be true. Fahd's claim at the beginning of the lesson is an example of a statistical hypothesis.

To test the validity of a claim, write it as a mathematical statement. Fahd's claim can be written as $\mu \geq 90\%$, where μ is his average shooting percentage. The statement $\mu < 90\%$ is the complement of the original statement, which represents Fahd not meeting his claim. These two statements represent the pair of hypotheses that need to be stated to test a claim.

- The **null hypothesis** H_0: There *is not* a significant difference between the sample value and the population parameter. It will contain a statement of *equality*, such as $\geq$, $=$, or $\leq$. In this example, $\mu \geq 90\%$ is the null hypothesis.
- The **alternative hypothesis** H_a: There is a difference between the sample value and the population parameter. It will contain a statement of *inequality*, such as $>$, $\neq$, or $<$. In this example, $\mu < 90\%$ is the alternative hypothesis.

If a claim k is made about a population mean μ, the possible combinations for the hypotheses are:

H_0: $\mu = k$ and H_a: $\mu \neq k$ H_0: $\mu \geq k$ and H_a: $\mu < k$ H_0: $\mu \leq k$ and H_a: $\mu > k$

Example 1 Null and Alternative Hypotheses

For each statement, write the null and alternative hypotheses. State which hypothesis represents the claim.

a. Makers of a gum brand claim that their gum will keep its flavor for at least 5 hours.

This claim becomes $\mu \geq 5$ and is the null hypothesis since it includes an equality symbol. The complement is $\mu < 5$.

H_0: $\mu \geq 5$ (Claim) H_a: $\mu < 5$

b. Technicians of an automotive company claim that they will perform an oil change on a car in less than 15 minutes.

This claim becomes $\mu < 15$ and is the alternative hypothesis since it includes an inequality symbol. The complement is $\mu \geq 15$.

H_0: $\mu \geq 15$ H_a: $\mu < 15$ (Claim)

c. A teacher claims that the average amount of time that his students are spending on homework every night is 35 minutes.

This claim becomes $\mu = 35$ and is the null hypothesis since it includes an equality symbol. The complement is $\mu \neq 35$.

H_0: $\mu = 35$ (Claim) H_a: $\mu \neq 35$

GuidedPractice

1A. A football player claims that he can achieve more than 100 rushing yards per game.

1B. A cross country star claims that it will take him no more than 4 minutes to run a mile.

1C. A salesperson claims that she averages 12 sales per month.

WatchOut!

Reject or Not Reject The null hypothesis is the hypothesis tested but it may not represent the claim. For example, if the alternative hypothesis represents the claim and the null hypothesis is rejected, then the claim is actually being supported.

2 Significance and Tests

To validate a claim, the null hypothesis is always tested. In the example at the beginning of the lesson, $\mu \geq 90\%$ would be tested. After a sample of data is analyzed, one of two decisions is made.

- Reject the null hypothesis.
- Do not reject the null hypothesis.

Every shot that Fahd could ever take cannot be recorded. Saeed can only analyze a sample of data, such as having Fahd take 100 shots. Thus, there is always a chance that Saeed can make the wrong decision. When the decision is incorrect, it is either a *Type I* or a *Type II* error.

	H_0 is True	H_0 is False
H_0 is rejected	**Type I Error** The null hypothesis is rejected, when it is actually true. Saeed rejects the statement $\mu \geq 90\%$ when Fahd actually shoots 90% or better.	**Correct decision** Saeed rejects the statement $\mu \geq 90\%$ when Fahd actually shoots less than 90%.
H_0 is not rejected	**Correct decision** Saeed does not reject the statement $\mu \geq 90\%$ when Fahd actually shoots 90% or better.	**Type II Error** The null hypothesis is not rejected, when it is actually false. Saeed does not reject the statement $\mu \geq 90\%$ when Fahd actually shoots less than 90%.

This suggests that there are actually four possible outcomes when a decision about the null hypothesis is made. The only way to guarantee complete accuracy is to test the entire population.

StudyTip

Errors The risk of making a Type I error is identical to the significance level. Reducing the risk of a Type I error increases the risk of a Type II error by widening the acceptance interval.

The **level of significance**, denoted α, is the maximum allowable probability of committing a Type I error. For example, if $\alpha = 0.10$, there is a 10% chance that H_0 was rejected when it was actually true, or there is a 90% chance that a correct decision was made. Any level of significance can be chosen. The three most commonly used levels are $\alpha = 0.10$, $\alpha = 0.05$, and $\alpha = 0.01$.

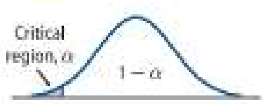
After a level of significance is chosen, a critical value can be found using either a z- or t-value. Similar to confidence intervals, the decision to use a z- or t-value is based on the characteristics of the study.

- If σ is known or $n \geq 30$, use a z-value.
- If σ is unknown and $n < 30$, use a t-value.

The z- or t-value and the alternative hypothesis will determine the *critical region*, the range of values that suggest a significant enough difference to reject the null hypothesis. The location of a critical region is determined by the inequality sign of the alternate hypothesis, which indicates whether the test is left-tailed, right-tailed, or two-tailed.

ConceptSummary Tests of Significance

If H_a: $\mu < k$, the hypothesis test is a **left-tailed test**.



If H_a: $\mu \neq k$, the hypothesis test is a **two-tailed test**.

If H_a: $\mu > k$, the hypothesis test is a **right-tailed test**.

Once the area corresponding to the level of significance is determined, a *test statistic* for the sample mean is calculated. The test statistic is the z- or t-value for the sample and will be referred to as the *z statistic* or *t statistic*. If the z or t statistic for the sample:

- is in the critical region, H_0 should be rejected.
- is *not* in the critical region, H_0 should not be rejected.

StudyTip

z Statistic and t Statistic Calculation To calculate a z statistic, use $z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$. To calculate a t statistic, use $t = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$.

The steps to conduct a hypothesis test are summarized below.

KeyConcept Steps for Hypothesis Testing

Step 1 State the hypotheses, and identify the claim.

Step 2 Determine the critical value(s) and region.

Step 3 Calculate the test statistic.

Step 4 Reject or fail to reject the null hypothesis.

Real-World Example 2 One-Sided Hypothesis Test

NUTRITION Representatives of a company report that their product contains no more than 5 grams of fat. A researcher tests a random sample of 50 products and finds that $\bar{x} = 5.03$ grams. If the standard deviation of the population is 0.14 gram, use a 5% level of significance to determine whether there is sufficient evidence to reject the company's claim.

Real-WorldLink

The Nutrition Labeling and Education Act of 1990, or NLEA, required nutrition labeling for most foods, excluding meat and poultry.

Source: U.S. Food and Drug Administration

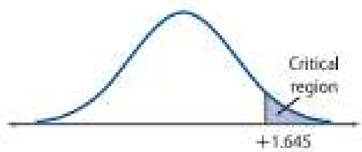
Step 1 State the null and alternative hypotheses, and identify the claim.

The claim written as a mathematical statement is $\mu \le 5$. This is the null hypothesis. The alternative hypothesis is $\mu > 5$.

$H_0: \mu \le 5$ (claim) $\qquad H_a: \mu > 5$

Step 2 Determine the critical value(s) and region.

The population standard deviation is known and $n \ge 30$, so you can use the z-value. The test is right-tailed since $\mu > 5$. Because a 5% level of significance is called for, $\alpha = 0.05$. Use a graphing calculator to find the z-value.



Step 3 Calculate the test statistic.

Find the z statistic. Since $\sigma = 0.14$ and $n = 50$, $\sigma_{\bar{x}} = \frac{0.14}{\sqrt{50}}$ or about 0.02.

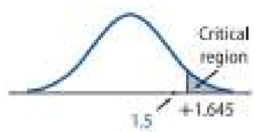
$$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}} \qquad \text{Formula for } z \text{ statistic}$$

$$= \frac{5.03 - 5}{0.02} \qquad \bar{x} = 5.03,\ \mu = 5,\ \text{and } \sigma_{\bar{x}} = 0.02$$

$$= 1.5 \qquad \text{Simplify.}$$

Step 4 Reject or fail to reject the null hypothesis.

H_0 is not rejected since the test statistic does not fall within the critical region.



Therefore, there is not enough evidence to reject the claim that there is no more than 5 grams of fat per product.

GuidedPractice

2. **JOBS** Employees at a bookstore claim that the mean wage per hour is less than the competitor's mean wage of $10.50. If a random sample of 20 employees shows a mean wage of $10.05 with a standard deviation of $0.75, test the employees' claim at $\alpha = 0.01$.

For a two-sided test, the level of significance α must be divided by 2 in order to determine the critical value at each tail.

Example 3 Two-Sided Hypothesis Test

FRUIT SNACKS Representatives of a company have stated that each box of fruit snacks contains 80 pieces. A researcher wants to determine if this is true. A random sample of 25 boxes is selected, with a sample mean of 84.1 pieces and a standard deviation of 7 pieces. Is this statistically significant at $\alpha = 0.01$?

Step 1 State the null and alternative hypotheses, and identify the claim.

The claim written as a mathematical statement is $\mu = 80$. This is the null hypothesis. The alternative hypothesis is $\mu \neq 80$.

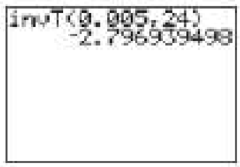
H_0: $\mu = 80$ (claim) H_a: $\mu \neq 80$

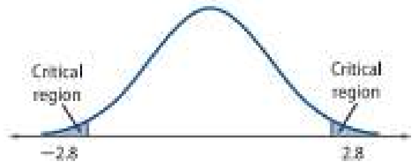
WatchOut!

Determining Critical Values
Remember to use the invT(function to find t-values for the t-distribution and invNorm(to find z-values for the normal distribution.

Step 2 Determine the critical value(s) and region.

The t-value should be used since $n < 30$ and σ is unknown. The test is two-tailed since $\mu \neq 80$, so the critical values are determined by $\frac{\alpha}{2}$ or 0.005. Using a graphing calculator, the critical values for $\alpha = 0.005$ with $25 - 1$ or 24 degrees of freedom are $t = -2.8$ and $t = 2.8$.





Step 3 Calculate the test statistic.

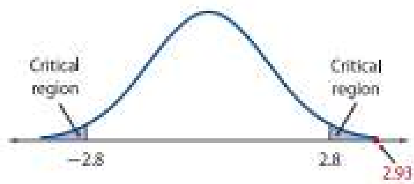
$t = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$ Formula for t statistic

$= \frac{84.1 - 80}{1.4}$ $\bar{x} = 84.1$, $\mu = 80$, and $\sigma_{\bar{x}} = \frac{7}{\sqrt{25}}$ or 1.4

≈ 2.93 Simplify.

Step 4 Reject or fail to reject the null hypothesis.

H_0 is rejected since the test statistic falls within the critical region.



There is enough evidence to reject the claim that there are 80 pieces in each box.

GuidedPractice

3. **TRAVEL** Representatives of a travel bureau in a U.S. city claim that in a recent year, an average of 110 people visited the city every day. In a sample of 90 days, there was an average of 115 visitors per day, with a standard deviation of 18 visitors. At $\alpha = 0.05$, is there enough evidence to reject the claim?

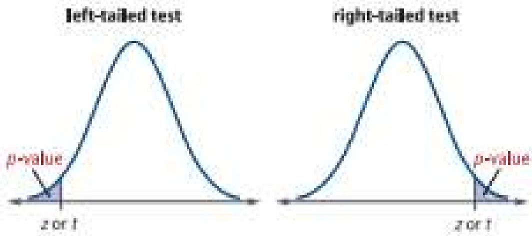
The **p-value** can also be used to determine whether H_0 should be rejected. The p-value is the lowest level of significance at which H_0 can be rejected for a given set of data. After calculating the z or t statistic for a hypothesis test, it can be converted into a p-value using a graphing calculator. To use the p-value to evaluate H_0, compare the p-value to α.

- If $p \leq \alpha$, then reject H_0.
- If $p > \alpha$, then do not reject H_0.

StudyTip

Law of Large Numbers The value of $\bar{x}$ is rarely identical to the true μ and often varies from sample to sample. However, due to the Law of Large Numbers, if we take larger and larger samples, $\bar{x}$ is guaranteed to get closer and closer to the true μ. This is true for *any* distribution.

The p-value corresponds to the area found under the normal curve to the left or right of the z or t statistic calculated for the sample data. The location of the area is determined by the type of test being preformed.



The α value is chosen by the researcher before the statistical test is performed, while the p-value is calculated after the sample mean is determined.

Example 4 Hypothesis Testing and p-Values

HORTICULTURE A biologist treated 40 plants with a chemical and then compared the amount of growth with untreated plants. For the untreated plants, the mean height is 21.6 centimeters. The treated plants have a mean height of 22.4 centimeters and $s = 1.8$ centimeters. The biologist claims that the chemical increased plant growth. Determine whether this result is significant at $\alpha = 0.01$.

The claim written as a mathematical statement is $\mu > 21.6$. This is the alternative hypothesis. The null hypothesis is $\mu \leq 21.6$.

H_0: $\mu \leq 21.6$ $\quad$ H_a: $\mu > 21.6$ (claim)

Since $n \geq 30$, the z statistic is used.

$z = \dfrac{\bar{x} - \mu}{\sigma_{\bar{x}}}$ $\quad$ Formula for z statistic

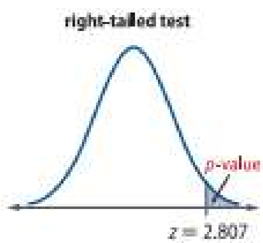
$\approx \dfrac{22.4 - 21.6}{0.285}$ $\quad$ $\bar{x} = 22.4$, $\mu = 21.6$, and $\sigma_{\bar{x}} = \frac{1.8}{\sqrt{40}}$ or about 0.285

≈ 2.807 $\quad$ Simplify.

TechnologyTip

Calculating the Area Under a t-Curve You can use a graphing calculator to find the area under the t-curve that corresponds to any t-value by selecting [2nd] [DISTR] and tcdf(*lower t value, upper t value, d.f.*).

This is a right-tailed test since the alternative hypothesis is $\mu > 21.6$. The area associated with $z = 2.807$ is 0.0025.



The p-value 0.0025 is less than 0.01. Therefore, the null hypothesis is rejected and there is significant evidence that the chemical increased plant growth.

GuidedPractice

4. **DRUGS** Makers of a sleep-aid claim that their product provides more than 8 hours of uninterrupted sleep. In a test of 50 patients, the mean amount of uninterrupted sleep was 8.07 hours with a standard deviation of 0.3 hour. Find the p-value and determine if there is enough evidence to reject the claim at $\alpha = 0.03$.

It is important to remember that statistical tests do not prove that a claim is true or false. These types of tests simply state that there is or is not enough evidence to say that a claim is likely to be true.

Exercises

Write the null and alternative hypotheses for each statement, and state which hypothesis represents the claim. (Example 1)

1. Makers of a cereal brand claim their product contains 4 grams of fiber.
2. A student claims that he has received at least an 85% on his math tests.
3. Khadija claims that she can drive to school from her house in less than 10 minutes.
4. Ayesha claims that her bowling average is 183.
5. Ian claims that he can recite the names of more than 38 former presidents of the United States.

Calculate the test statistic, and determine whether there is enough evidence to reject the null hypothesis. Then make a statement regarding the original claim.

6. **ADVERTISING** Company representatives claim that they will ship a product in less than four days. If a random selection of 60 delivery times has a sample mean of 3.9 days and a standard deviation of 0.6 day, is there enough evidence to reject the claim at $\alpha = 0.05$? Explain. (Examples 2 and 3)

7. **HEALTH** A researcher claims that a supplement does not increase bone density by at least 0.05 gram per square centimeter. If a study shows that the supplement increased bone density in a random sample of 35 people by 0.048 gram per square centimeter with a standard deviation of 0.004, is there enough evidence to reject the claim at $\alpha = 0.01$? Explain. (Examples 2 and 3)

8. **HOTELS** Owners of a hotel chain claim that the average cost of one of their hotel rooms in the U.S. is \$82. Sample data for 25 hotel rooms is collected. The average cost was found to be \$85 with a standard deviation of \$8. Is there enough evidence to reject the owners' claim at $\alpha = 0.02$? Explain. (Examples 2 and 3)

9. **CALCULATORS** A teacher claims that the average cost of a certain brand of graphing calculator is at least \$90. A random sample of 40 stores shows that the mean cost is \$89.25 with a standard deviation of \$4.95. Is there enough evidence to reject the teacher's claim at $\alpha = 0.05$? Explain. (Examples 2 and 3)

For each claim k, use the specified information to calculate the test statistic and determine whether there is enough evidence to reject the null hypothesis. Then make a statement regarding the original claim. (Examples 2 and 3)

10. $k: \mu = 1240,\ \alpha = 0.05,\ \bar{x} = 1245,\ s = 32,\ n = 50$
11. $k: \mu > 88,\ \alpha = 0.05,\ \bar{x} = 91.2,\ s = 3.9,\ n = 22$
12. $k: \mu < 500,\ \alpha = 0.01,\ \bar{x} = 490,\ s = 27,\ n = 35$
13. $k: \mu \neq 5500,\ \alpha = 0.01,\ \bar{x} = 5430,\ s = 236,\ n = 200$
14. $k: \mu \leq 10{,}000,\ \alpha = 0.01,\ \bar{x} = 10{,}015,\ s = 85,\ n = 18$

15. **REAL ESTATE** A researcher wants to test a claim that the average home sale price in the U.S. is less than \$260,000. She selects a sample of 40 homes and finds the mean sale price of the sample to be \$254,500 with a standard deviation of \$12,500. Find the p-value, and determine whether there is enough evidence to support the claim at $\alpha = 0.05$. (Example 4)

16. **MUSIC** Representatives of an electronics company claim that the average lifetime of an MP3 player is at least 5 years. A random sample of 100 MP3s shows a mean life span of 5.2 years with a standard deviation of 1.2 years. Find the p-value, and determine whether there is enough evidence to support the claim at $\alpha = 0.01$. (Example 4)

17. **BASEBALL** Faris believes that the cost of attending a baseball game for a family of two adults and two children is under \$125. He surveys 18 families selected at random and finds that the average cost is \$122.88 with a standard deviation of \$13.21. Find the p-value, and determine whether there is enough evidence to support the claim at $\alpha = 0.10$.

18. **CROSS COUNTRY** Mansour claims that the average 1600-meter time for the students in his school is less than 7 minutes. He records the 1600-meter times of 20 randomly selected students. Determine whether Mansour's claim is supported at $\alpha = 0.05$.

1600 Meter Times (minutes)									
5.25	7.27	5.46	7.63	7.75	5.42	6.00	8.17	9.45	6.20
6.63	7.38	6.97	7.85	7.03	6.53	6.87	7.22	7.16	6.92

a. Write the null and alternative hypotheses, and state which hypothesis represents the claim.

b. Determine whether there is enough evidence to reject the null hypothesis using critical values.

c. Make a statement regarding the original claim.

19. **HOMEWORK** Ms. Buthaina claims that her math students spend 25 minutes each night on homework. Muna asks her classmates to record the average amount of time that they spend on homework each night over the course of a week. Determine whether Ms. Buthaina's claim is supported at $\alpha = 0.10$.

Times (minutes)											
45	40	10	15	18	20	34	36	20	25	28	25
26	30	22	25	24	29	26	28	23	28	25	26
29	30	22	20	22	24	23	24	25	29	25	

a. Write the null and alternative hypotheses, and state which hypothesis represents the claim.

b. Determine whether there is enough evidence to reject the null hypothesis using critical values.

c. Make a statement regarding the original claim.

Describe the outcome if a type I or a type II error is committed when the null hypothesis is tested.

20. The accused person is not guilty.

21. The *X*-ray came back positive for an ankle sprain.

22. Students use study time efficiently.

23. The majority of students do not have jobs.

24. The average lifespan of a goldfish is 2 years.

25. The venom from the snake is not poisonous.

26. **SLEEP** Mr. Abdulkarim believes that college students get less than 6 hours of sleep each night. He randomly selected a group of students and recorded the average amount of sleep each student gets each night.

Average Sleep (hours/night)								
5.4	6.7	6.5	5.5	5.5	6.0	5.8	6.7	6.8
4.5	5.7	7.5	5.4	5.3	8.0	4.5	4.5	5.0

a. Write the null and alternative hypotheses, and state which hypothesis represents the claim.

b. Find the *p*-value.

c. Determine whether there is enough evidence to reject the null hypothesis at $\alpha = 0.05$.

d. Make a statement regarding the original claim.

27. **ACT** The average composite score on the ACT is a 21. Instructors of an ACT preparation class claim that they can raise test takers' scores. The scores of randomly selected attendees were recorded.

ACT Scores											
24	23	27	23	19	16	33	30	22	25	23	26
21	30	22	18	28	21	26	32	20	17	23	24
25	28	19	22	21	19	18	20	25	22	24	23

a. Write the null and alternative hypotheses, and state which hypothesis represents the claim.

b. Find the *p*-value.

c. Determine whether there is enough evidence to reject the null hypothesis at $\alpha = 0.01$.

d. Make a statement regarding the original claim.

28. **QUANTITY** A chocolatier claims that his candy averages at least 81 pieces per bag. Omar randomly selects bags of the candies and counts the pieces. Assume that $\sigma = 1.2$.

Number of Pieces								
81	80	82	82	83	82	84	81	81
80	83	83	82	81	80	84	81	81

a. Write the null and alternative hypotheses, and state which hypothesis represents the claim.

b. Determine whether there is enough evidence to reject the null hypothesis at $\alpha = 0.05$ using critical values.

c. Find the *p*-value. Determine whether there is enough evidence to reject the null hypothesis at $\alpha = 0.05$.

29. **GRADES** Mr. Eissa claims that the average grade for his students is an 85%. Two of his students, Adnan and Obaid, collect the following samples of grades from students in their classes.

Adnan's Scores								
64	84	86	99	76	90	79	94	85
84	85	88	91	80	85	76	86	96

Obaid's Scores							
95	86	95	83	86	85	84	88
88	86	87	88	95	86	85	95

a. Write the null and alternative hypotheses, and state which hypothesis represents the claim.

b. Suppose $\alpha = 0.10$. For each class, determine whether there is enough evidence to reject the null hypothesis and make a statement regarding the original claim.

c. Determine whether there is enough evidence to reject the null hypothesis if the two samples of data are combined. Use the result to make a statement regarding the original claim.

d. Make a conjecture in regard to the result found in part **c** and the Law of Large Numbers.

H.O.T. Problems Use Higher-Order Thinking Skills

30. **ERROR ANALYSIS** Abdalla and Abdulrahman are completing their statistics homework. Abdalla claims that it is always better to have a type I error rather than a type II error. Abdulrahman disagrees. Is either of them correct? Explain your reasoning.

31. **WRITING IN MATH** Describe the difference between conducting a hypothesis test using test statistics and critical values and conducting a hypothesis test using *p*-values.

REASONING Determine whether each statement is *true* or *false*. Explain your reasoning.

32. If the null hypothesis is rejected, then the claim is always rejected.

33. The alternative hypothesis can include an equality symbol if it represents the claim.

34. The *p*-value is always going to be a positive value.

35. **CHALLENGE** For a random set of data, $p = 0.0104$, $s = 0.3$, and $\bar{x} = 14.9$. If the study was conducted to test a claim of $\mu < 15$, find n.

36. **WRITING IN MATH** Explain why it may not always be in the researcher's best interest to have the lowest possible significance level in order to reduce the possibility of a type I error.

Spiral Review

37. **SHOES** A sample of 35 pairs of running shoes found the average cost to be \$45.25 with a standard deviation of \$7.60. Estimate the mean cost for running shoes using a confidence interval given a 90% confidence level.

38. **RARE BOOKS** The average prices for three antiquebooks are shown. The prices vary due to the age and condition of each book.

Book	Average Price (\$)
Don Quixote	155
Moby Dick	98
Oliver Twist	118

a. For a sample of 25 copies of *Don Quixote*, find the probability that the mean price is more than \$160, if the standard deviation is \$18.

b. For a random sample of 40 copies of *Oliver Twist*, find the probability that the mean price will be between \$115 and \$120, if the standard deviation is \$15.

Determine whether each sequence is *convergent* or *divergent*.

39. $a_n = (10 - n)^2$

40. $a_n = n^2 + 10n - 9$

41. $a_n = \frac{3n + 2}{n}$

Find the angle θ between vectors u and v.

42. $\mathbf{u} = 2\mathbf{i} + 8\mathbf{j} + 7\mathbf{k}, \mathbf{v} = -7\mathbf{i} - 3\mathbf{j} - 9\mathbf{k}$

43. $\mathbf{u} = -2\mathbf{i} - 8\mathbf{j} + 4\mathbf{k}, \mathbf{v} = 3\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}$

44. $\mathbf{u} = 5\mathbf{i} - 9\mathbf{k}, \mathbf{v} = 2\mathbf{i} - 3\mathbf{j} - 8\mathbf{k}$

Find a unit vector in the same direction as v.

45. $\mathbf{v} = \langle -9, 2 \rangle$

46. $\mathbf{v} = \langle -5, -1 \rangle$

47. $\mathbf{v} = \langle 4, 3 \rangle$

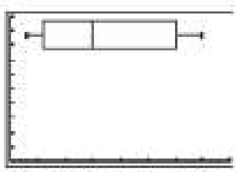
Write an equation for the hyperbola with the given characteristics.

48. vertices $(-6, 3)$, $(4, 3)$; conjugate axis is 8 units

49. vertices $(-2, 6)$, $(-2, -4)$; conjugate axis is 6 units

Skills Review for Standardized Tests

50. **REVIEW** Estimate the median and spread of the data represented by the box plot.



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A median ≈ 30, spread ≈ 50

B median ≈ 30, spread ≈ 65

C median ≈ 50, spread ≈ 50

D median ≈ 50; spread ≈ 65

51. **REVIEW** Find the solutions of $x^2 = i$.

F i and $-i$

G $\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$ and $\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}$

H 1 and −1

J $\cos \frac{\pi}{8} + i \sin \frac{\pi}{8}$ and $\cos \frac{9\pi}{8} + i \sin \frac{9\pi}{8}$

52. **SAT/ACT** Which of the following statements are true if n is an integer?

I. $3n + 6$ is divisible by 3.

II. $10n + 8$ is divisible by 2.

III. $4n - 2$ is divisible by 4.

A I only

B II only

C I and II only

D I and III only.

E I, II, and III are true.

53. Rashid believes that the average price of gasoline is still under \$2.50 per gallon. He randomly calls 40 different gas stations and finds that the average price is \$2.51 with a standard deviation of \$0.06. Find the p-value, and determine whether there is enough evidence to support the claim at $\alpha = 0.10$.

F 0.85; not enough evidence

G 0.95; enough evidence

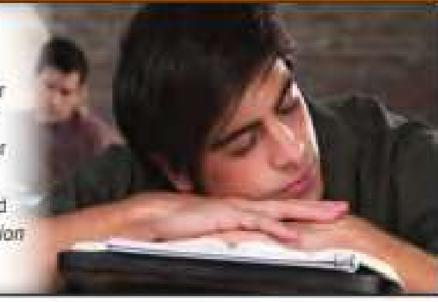
H 0.15; not enough evidence

J 0.05; enough evidence

LESSON 4-7

Correlation and Linear Regression

Then	Now	Why?
● You analyzed univariate data.	● **1** Measure the linear correlations for sets of bivariate data using the correlation coefficient, and determine if the correlations are significant. **2** Generate least-squares regression lines for sets of bivariate data, and use the lines to make predictions.	● A feature writer for a school newspaper is interested in determining whether the number of hours of sleep a student gets each night is related to his or her overall grade point average. In statistical terms, the writer would like to know if there is a *correlation* between sleep and grades.

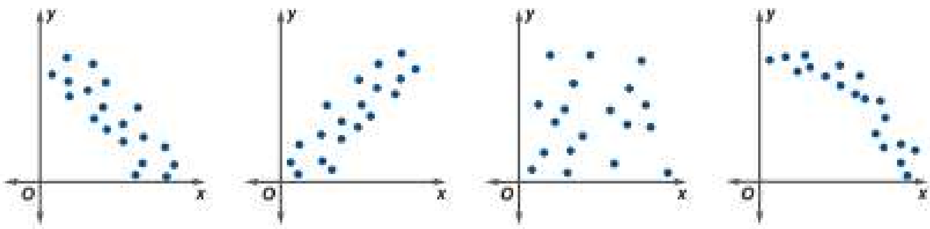


NewVocabulary
- correlation
- bivariate
- explanatory variable
- response variable
- correlation coefficient
- regression line
- line of best fit
- residual
- least-squares regression line
- residual plot
- influential
- interpolation
- extrapolation

1 Correlation

Thus far in this chapter, you have graphed, characterized, and used summary statistics to describe distributions of one-variable data sets. You have used sample statistics of such univariate data to make inferences about a population by developing confidence intervals and performing hypothesis tests. **Correlation** is another area of inferential statistics that involves determining whether a relationship exists between two variables in a set of **bivariate** data.

Bivariate data can be represented as ordered pairs (x, y), where x is the independent or **explanatory variable** and y is the dependent or **response variable**. To determine whether there may be a linear, a nonlinear, or no correlation between the variables, you can use a scatter plot.



Negative Linear Correlation | Positive Linear Correlation | No Correlation | Nonlinear Correlation

We say that the data have a strong linear relationship if the points lie close to a straight line and weak if they are widely scattered about the line, but interpreting correlation using a scatter plot tends to be subjective. A more precise way to determine the type and strength of the linear relationship between two variables is to calculate the **correlation coefficient**. A formula for this measure is given below.

KeyConcept Correlation Coefficient

For n pairs of sample data for the variables x and y, the correlation coefficient r between x and y is given by

$$r = \frac{1}{n-1}\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right),$$

where x_i and y_i represent the values for the ith pair of data, $\bar{x}$ and $\bar{y}$ represent the means of the two variables, and s_x and s_y represent the standard deviations of the two variables.

The correlation coefficient can take on values from -1 to 1. This value indicates the strength and type of linear correlation between x and y as shown in the diagram below.

Notice from the formula that the correlation coefficient is the average of the products of the standardized values for x and the standardized values for y. The correlation coefficient can be tedious to calculate by hand, so we most often rely on computer software or a graphing calculator.

StudyTip

Resistance of Correlation Coefficient Like the mean and standard deviation, r is a nonresistant statistic. It can be affected by outliers.

Example 1 Calculate and Interpret the Correlation Coefficient

SLEEP/GPA STUDY A feature writer for a student newspaper conducts a study to determine whether there is a linear relationship between the average number of hours a student sleeps each night and his or her overall grade point average. The table shows the data that the writer collected. Make a scatter plot of the data, and identify the relationship. Then calculate and interpret the correlation coefficient.

Hours of Sleep	GPA	Hours of Sleep	GPA
6.6	2.2	8.0	2.9
6.6	2.4	8.0	3.1
6.7	2.3	8.1	3.3
6.8	2.3	8.2	3.3
6.8	2.2	8.2	3.2
7.0	2.6	8.3	2.8
7.0	2.7	8.4	3.1
7.2	2.8	8.6	3.3
7.4	2.6	8.7	3.4
7.4	3.0	8.8	3.1
7.4	2.9	8.8	3.2
7.5	2.7	8.8	3.4
7.7	2.8	9.1	3.3
7.9	2.9	9.2	3.8
7.9	3.0	9.2	3.5

Step 1 Graph a scatter plot of the data.

Enter the data into L1 and L2 on your calculator. Then turn on Plot1 under the STAT PLOT menu and choose [scatter plot icon] using L1 for the Xlist and L2 for the Ylist. Graph the scatterplot by pressing ZoomStat or by pressing GRAPH and adjusting the window manually (Figure 4.7.1). From the graph, it appears that the data have a positive linear correlation.

Step 2 Calculate and interpret the correlation coefficient.

Press STAT and select LinReg(ax+b) under the CALC menu (Figure 4.7.2). The correlation coefficient r is about 0.9148. Because r is close to 1, this suggests that the data may have a strong positive linear correlation. This numerical assessment of the data is consistent with our graphical assessment.

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Figure 4.7.1

Figure 4.7.2

GuidedPractice

1. **METEOROLOGY** A weather program is featuring a special on a city where a study was conducted to determine whether there is a linear relationship between the average monthly rainfall and temperature. The table in Figure 4.7.3 shows the data collected. Make a scatter plot of the data. Then calculate and interpret the correlation coefficient for the data.

Rain (cm)	Temp. (°F)
5.35	41.3
4.03	44.3
3.77	46.6
2.51	50.4
1.84	56.1
1.59	61.4
0.85	65.3
1.22	65.7
1.94	60.8
3.25	53.5
5.65	46.3
6.00	41.6

Figure 4.7.3

In Example 1, the data collected represent just a sample of the entire school population; therefore, r represents a *sample correlation coefficient*. In order for r to be a valid estimate of the *population correlation coefficient* ρ, the following assumptions must be valid.

- The variables x and y are *linearly* related.
- The variables are *random* variables.
- The two variables have a *bivariate normal distribution*. That is, x and y each come from a normally distributed population.

We would like to use the value of r to make an inference about the relationship between the variables x and y for the entire population. In order to do that, we need to determine whether the value of $|r|$ is great enough to conclude that there is a significant relationship between x and y.

ReadingMath

Population Correlation Coefficient The Greek letter ρ used to represent the population correlation coefficient is pronounced *rho*.

To make this determination, you can perform a hypothesis test. The null and alternative hypotheses for a two-tailed test of the population correlation coefficient ρ are as follows.

H_0: $\rho = 0$ — There is no correlation between the x and y variables in the population.
H_a: $\rho \neq 0$ — There is a correlation between the x and y variables in the population.

We can use a t-test as described below to test the significance of the correlation coefficient.

KeyConcept Formula for the *t*-Test for the Correlation Coefficient

For a t-test of the correlation between two variables, the test statistic for ρ is the sample correlation coefficient r and the standardized test statistic t is given by

$$t = r\sqrt{\frac{n-2}{1-r^2}}, \text{ where } n - 2 \text{ is the degrees of freedom.}$$

Real-World Example 2 Test for Significance

SLEEP/GPA STUDY In Example 1, you calculated the correlation coefficient r for the 30 pairs of student sleep and GPA data to be about 0.9148. Test the significance of this correlation coefficient at the 5% level.

Step 1 State the hypotheses.

H_0: $\rho = 0$ $\quad$ H_a: $\rho \neq 0$

Step 2 Determine the critical values.

Testing for significance at the 5% level means that $\alpha = 0.05$. Since this is a two-tailed test, the critical values are determined by $\frac{\alpha}{2}$ or 0.025. Using a graphing calculator, the critical values for $\alpha = 0.025$ with $30 - 2$ or 28 degrees of freedom are $t = \pm 2.0$.

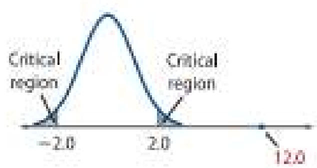
Step 3 Calculate the test statistic.

$$t = r\sqrt{\frac{n-2}{1-r^2}}$$ Calculate the test statistic for ρ.

$$= 0.9148\sqrt{\frac{30-2}{1-(0.9148)^2}} \text{ or about } 12.0$$ $r = 0.9148$ and $n = 30$

Step 4 Reject or fail to reject the null hypothesis.

Since $12.0 > 2.0$, the statistic falls within the critical region and the null hypothesis is rejected.



At the 5% level, there is enough evidence to conclude that there is a significant correlation between the average amount of sleep a student gets each night and his or her overall GPA.

Real-WorldLink

About 40 million people in the U.S. suffer from sleep problems every year. Lack of sleep can have many serious medical consequences.

Source: The National Center on Sleep Disorders Research

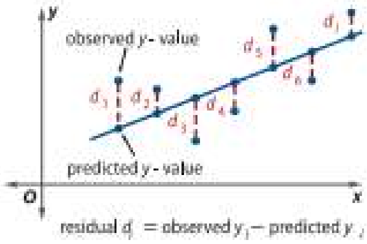
GuidedPractice

2. **METEOROLOGY** In the Guided Practice for Example 1, you calculated the correlation coefficient r for the 12 pairs of rainfall and temperature data. Test the significance of this correlation coefficient at the 10% level.

2 Linear Regression

Linear Regression Once the correlation between two variables is determined to be significant, the next step is to determine the equation of the **regression line**, also called a **line of best fit**. The regression line describes how the response variable y changes as the explanatory variable x changes.

While many lines of best fit can be drawn through a set of points, the one used most often is determined by specific criteria. Consider the scatter plot and regression line shown. The difference d between an observed y-value and its predicted y-value on the regression line is called a **residual**.



Residuals are positive when the observed value is above the line, negative when the observed value is below the line, and zero when it is on the line. The **least-squares regression line** is the line for which the sum of the squares of these residuals is a minimum.

ReadingMath

Regression Equation Notation The symbol $\hat{y}$ is read *y hat* and is used to emphasize that the equation gives the predicted and not the actual response y for any x.

KeyConcept Equation of the Least-Squares Regression Line

The equation of the least-squares regression line for an explanatory variable x and response variable y is $\hat{y} = ax + b$.

The slope a and y-intercept b in this equation are given by

$$a = r\frac{s_y}{s_x} \text{ and } b = \bar{y} - a\bar{x},$$

where r represents the correlation coefficient between the two variables, $\bar{x}$ and $\bar{y}$ represent their means, and s_x and s_y represent their standard deviations.

As with the correlation coefficient, it is not necessary to calculate the least-squares regression equation by hand. Computer software or a graphing calculator will provide the slope a and the y-intercept b of the least-squares regression line for keyed-in values of the variables.

Example 3 Find the Least-Squares Regression Line

SLEEP/GPA STUDY **Find the equation of the regression line for the data used in Example 1. Interpret the slope and intercept in context. Then assess the fit of the modeling equation by graphing it, along with the scatter plot of the data, in the same window.**

Using the same screen you used to obtain the correlation coefficient (Figure 4.7.4), the least-squares regression equation is approximately $\hat{y} = 0.457x - 0.667$. The slope $a = 0.457$ indicates that for each additional hour of sleep, a student will raise his or her GPA by 0.457 point. The y-intercept $b = -0.667$ indicates that when a student averages no sleep, his or her GPA will be less than 0, which is not possible.

Since the data appear to be randomly scattered about the line $\hat{y} = 0.457x - 0.667$, this regression line appears to be a good fit for the data (Figure 4.7.5).

Figure 4.7.4

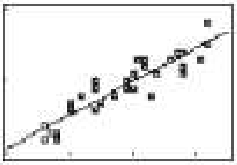


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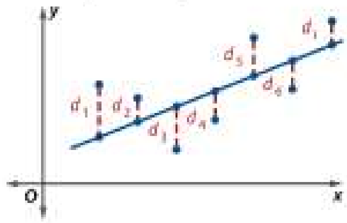
Figure 4.7.5

GuidedPractice

3. METEOROLOGY Find the equation of the regression line for the rainfall and temperature data used in the Guided Practice for Example 1. Interpret the slope and intercept in context. Then assess the fit of the modeling equation by graphing it and the scatter plot of the data in the same window.

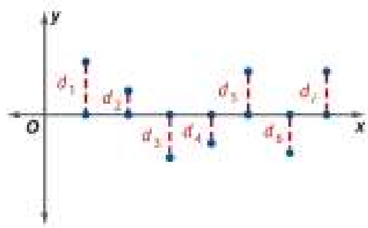
A least-squares regression line describes the overall pattern in a set of bivariate data. As with univariate data analysis, you should always be on the lookout for striking deviations, or outliers, from this pattern. Remember that the residuals measure how much the data deviate from the regression line.

Scatterplot with Regression Line



Examining a scatter plot of the residuals, called a **residual plot**, can help you assess how well the regression line describes the data. In a residual plot, the horizontal line at zero corresponds to the regression line. You can create a residual plot using your graphing calculator. If the plot of the residuals appears to be randomly scattered and centered about $y = 0$, the use of a linear model for the data is supported. If the plot displays a curved pattern, the use of a linear model would not be supported.

Residual Plot



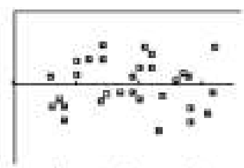
StudyTip

Residuals While residuals can be calculated from any regression line fitted to the data, the residuals from the least-squares regression line have a special property. The mean of the least-squares residuals will always be zero.

Example 4 Graph and Analyze a Residual Plot

SLEEP/GPA STUDY Graph and analyze the residual plot for the average sleep hours and GPA data in Example 1 to determine whether the linear model found in Example 3 is appropriate.

After calculating the least-squares regression line in Example 3, you can obtain the residual plot of the data by turning on Plot2 under the STAT PLOT menu and choosing , using L1 for the Xlist and RESID for the Ylist. You can obtain RESID by pressing [2nd] [STAT] and selecting RESID from the list of names. Graph the scatter plot of the residuals by pressing ZoomStat.



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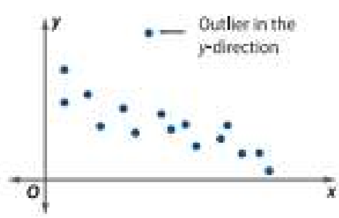
The residuals appear to be randomly scattered and centered about the regression line at $y = 0$. This supports the claim that the use of a linear model is appropriate.

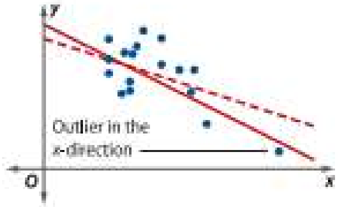
GuidedPractice

4. **METEOROLOGY** Graph and analyze the residual plot for the rainfall and temperature data to determine if the linear model found in the Guided Practice for Example 3 is appropriate.

The residual plot magnifies deviations of the data points from the regression line, making it easier to see outliers in the data that lie in the y-direction. Outliers in the y-direction can indicate errors in data recording or unique cases, especially when describing societal trends or behavioral traits.



Outliers in the x-direction can have a strong influence on the position of a regression line. In the figure, two least-squares regression lines are shown. The solid line is calculated using all the data, while the dashed line is calculated leaving out the outlier in the x-direction. Notice that leaving out this point noticeably moves the regression line.



StudyTip

Influence The influence of an outlier is not a yes or no question. It is a matter of degree and is therefore subjective.

An individual data point that substantially changes a regression line is said to be **influential**. Outliers in the x-direction are often influential to the least-squares regression line. To determine if a point is an influential outlier, calculate and graph regression lines with and without this point. The point is influential if there is a substantial difference in the positions of the regression lines when the point is removed.

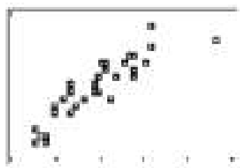
Example 5 Identify an Influential Outlier

SLEEP/GPA STUDY Suppose the feature writer in Example 1 conducting the sleep/GPA study later received the additional piece of data listed in the table, which is an outlier.

Hours of Sleep	GPA
10.7	3.6

a. Make a new scatter plot of the sleep/GPA data that includes the additional data point.

Add the data point to the end of L1 and L2 and then graph the data, adjusting your window as necessary. From the graph you can see that this point is an outlier in the x-direction.



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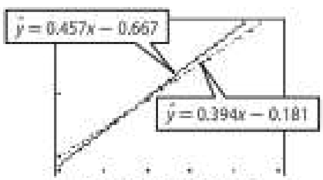
b. Calculate the correlation coefficient and least squares regression line with this outlier. Describe the effect this outlier has on the strength of the correlation and on the slope and intercept of the regression line.

Original data:	$r \approx 0.9148$	$\hat{y} = 0.457x - 0.667$
Data with outlier:	$r \approx 0.8934$	$\hat{y} = 0.394x - 0.181$

The outlier has reduced the strength of the correlation. The change in the slope of the regression equation has caused the rate at which a student's GPA is raised due to additional sleep to drop from 0.457 points per hour to 0.394 per hour. At the same time, this outlier has raised the y-intercept, indicating that a student who gets no sleep will have a GPA close to 0.

c. Plot both regression lines in the same window. Then state whether the outlier is influential. Explain your reasoning.

The graph of the regression lines shows that the regression line moves more than a small amount when the outlier is added. Therefore, the outlier (10.7, 3.6) is influential.



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GuidedPractice

5. **METEOROLOGY** Suppose the value (2.51, 50.4) for the rainfall and temperature data from Guided Practice 1 was replaced with (0.5, 50.4).

   A. Make a scatter plot of the original temperature/rainfall data that includes this outlier.

   B. Calculate the correlation coefficient and least squares regression line with this outlier. Describe the effect this outlier has on the strength of the correlation and on the slope and intercept of the regression line.

   C. Plot both regression lines in the same window. Then state whether the outlier is influential. Explain your reasoning.

WatchOut!

Making Predictions Do not use a least-squares regression line to make predictions unless a linear model is appropriate and the correlation coefficient is significant. Otherwise, these predictions would be meaningless.

Once you determine that the linear correlation coefficient for a set of data is significant and you find the least-squares regression line, you can then use the equation to make predictions over the range of the data. Making such predictions is called **interpolation**. Using the equation to make predictions far outside the range of the x-values you used to obtain the regression line is called **extrapolation**. Extrapolation should be avoided, since few real-world relationships are linear for all values of the explanatory variable.

Example 6 Predictions with Regression

SLEEP/GPA STUDY The regression equation for the average hours of sleep x and GPA y from Example 3 was $\hat{y} = 0.457x - 0.667$. Use this equation to predict the expected GPA (to the nearest tenth) for a student who averages the following hours of sleep and state whether this prediction is reasonable. Explain.

a. 8 hours

Evaluate the regression equation for $x = 8$ to calculate $\hat{y}$.

$\hat{y} = 0.457x - 0.667$ — Regression equation

$= 0.457(8) - 0.667$ — $x = 8$

$= 3.656 - 0.667$ — Multiply.

$= 2.989$ — Subtract.

Using this model, we would expect that a student averaging 8 hours of sleep would have GPA of about 3.0. This GPA is reasonable since 8 is an x-value in the range of the original data.

b. 10.5 hours

$\hat{y} = 0.457x - 0.667$ — Regression equation

$= 0.457(10.5) - 0.667$ — $x = 10.5$

$= 4.7985 - 0.667$ — Multiply.

$= 4.1315$ — Subtract.

Using this model, we would expect that a student averaging 10.5 hours of sleep would have a GPA of about 4.1. This value is not reasonable, since we are extrapolating a y-value for an x-value that falls far outside the range of the original data. It is also not meaningful, since a student cannot earn a GPA higher than a 4.0 in this model.

WatchOut!

Correlation vs. Causation Just because two variables are strongly correlated does not necessarily imply a cause-and-effect relationship. A significant correlation indicates only that y is in some way associated with x.

GuidedPractice

6. **METEOROLOGY** Use the regression equation for the rainfall and temperature data from Guided Practice 3 to predict the expected temperature (to the nearest tenth of a degree) for months with each average rainfall. State whether this prediction is reasonable. Explain.

A. 3 cm **B.** 8 cm

When analyzing bivariate data, follow the steps summarized below.

ConceptSummary Analyzing Bivariate Data

Step 1 Make a scatter plot, and decide whether the variables appear to be linearly related.

Step 2 If they appear to be linearly related, calculate the strength of the relationship by calculating the correlation coefficient.

Step 3 Use a t-test to determine if the correlation is significant.

Step 4 If significant, find the least-squares regression equation that models the data.

Exercises

For Exercises 1–6, analyze the bivariate data. (Examples 1-6)

a. Make a scatter plot of the data, and identify the relationship. Then calculate and interpret the correlation coefficient.

b. Determine if the correlation coefficient is significant at the 1%, 5%, and 10% levels. Explain your reasoning.

c. If the correlation is significant at the 10% level, state the least-squares regression equation and interpret the slope and intercept in context.

d. Graph and analyze the residual plot.

e. Identify any influential outliers. Describe the effect the outlier has on the strength of the original correlation and on the slope and intercept of the original regression line.

f. If any data were removed, reassess the significance of the correlation at the 10% level and, if still appropriate, recalculate the regression equation.

g. Use the regression equation to make the specified predictions. Interpret your results, and state whether the prediction is reasonable. Explain your reasoning.

1. **FAT GRAMS AND PROTEIN** An athlete wondered if there is a significant linear correlation between grams of fat and grams of protein in various foods. If appropriate, use the data below to predict the amount of protein (per serving) of an item with 1, 5, or 13 grams of fat.

Fat (g)	Protein (g)	Fat (g)	Protein (g)
12	14	9	13
57	30	18	24
9	15	30	25
20	25	18	25
12	15	32	24
39	28		

2. **FIBER AND CALORIES** The following data shows the caloric counts and amount of fiber in a variety of breakfast cereals. Use the data to predict the calories in a serving of cereal that has 4.5, 5.5, or 7 grams of fiber.

Fiber (g)	Calories	Fiber (g)	Calories
1.5	133.5	1	149
0.5	115.5	1.5	114.5
1	143	0.5	85.5
2.5	109.5	1	116
0	119	1.5	110
0.5	113.5	0	53.5
0.5	102	8	196.5
0.5	117.5	0.5	99.5
6	186.5	6.5	114.5
1	154	3.5	140.5
11	389	0.5	122.5
4	114.5	2	110

3. **EDUCATION AND HEALTH CARE** The following data lists the performance rankings of education and health care in 14 states. If appropriate, use the data to predict the health care ranking if the education ranking is 15, 28, or 42.

Education	Health Care	Education	Health Care
1	45	8	35
2	48	9	18
3	50	10	13
4	37	11	20
5	39	12	28
6	26	13	15
7	21	14	29

4. **WEIGHT AND MPG** A shopper wants to determine if there is a significant linear correlation between the weight of cars and their highway fuel efficiency. If appropriate, use the data below to predict the gas mileage of automobiles that weigh 2,900, 3,300, and 4,000 pounds.

Weight (lb)	MPG	Weight (lb)	MPG
3,450	32	3,460	28
3,216	32	2,897	36
2,636	34	2,805	32
2,690	40	3,067	28
2,875	51	2,716	31
2,403	36	2,595	38
2,972	35	2,326	39
2,811	34	2,911	30

5. **GRADUATION AND UNEMPLOYMENT** An economist took a sample of the graduation rates and unemployment rates of various states in a given year. If appropriate, use the data below to predict the unemployment rate if the graduation rates are 70%, 80%, or 90%.

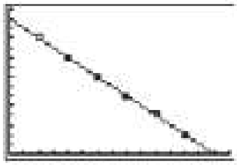
Graduated	73	85	64	79	68	82
Unemployed	6.9	4.1	3.2	2.9	4.3	5.1
Graduated	71	81	76	64	77	82
Unemployed	4.1	5.5	5	6.8	4.8	5.2

6. **POPULATION AND CRIME** The following data lists the performance rankings of population and crime in 14 states. If appropriate, use the data to predict the crime ranking if the population ranking is 15.

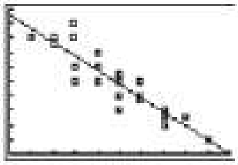
Population	1	2	3	4	5	6	7
Crime	14	15	13	4	5	9	7
Population	8	9	10	11	12	13	14
Crime	11	3	12	10	8	1	6

B **Match each graph to the corresponding correlation coefficient.**

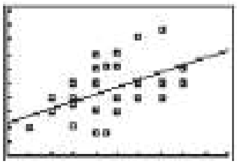
a.



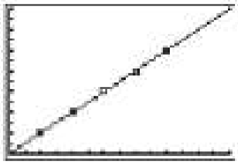
b.



c.



d.



7. $r = -0.90$ **b**

8. $r = 0.50$ **c**

9. $r = 1.00$ **d**

10. $r = -1.00$ **a**

11. **INCOME AND DINING OUT** A restaurant is conducting a study to determine the relationship between a person's monthly income and the number of times that person dines out each month.

Income ($)	500	1,125	300	750	1,250	950
Meals	4	10	3	6	12	8

a. Make a scatter plot of the data, and linearize the data by finding $(x, \ln y)$.

b. Make a scatter plot of the linearized data, and calculate and interpret r.

c. Determine if r is significant at the 5% level.

d. If r is significant, find the least-squares regression equation by using the model for the linearized data to find a model for the original data.

e. If appropriate, use the regression equation to predict the number of times that a person with a monthly income of $2000 will dine out. Is the prediction reasonable? Explain your reasoning.

12. **ADS AND SALES** An advertising firm wants to determine the strength of the relationship between the number of television ads aired each week and the amount of sales (in thousands of dollars) of the product.

Ads	2	3	5	7	7
Sales ($)	3	4	6	8	9
Ads	8	9	10	10	12
Sales ($)	10	12	12	13	15

a. Make a scatter plot of the data, and identify the relationship. Then find the correlation coefficient.

b. Determine if the correlation coefficient is significant at the 10% level. If so, find the least-squares regression equation.

c. Suppose the firm airs 15 ads during one week and 18 ads during the following week, and each ad spot costs $500. Make a prediction about the increase in profit from the first week to the second week.

13. **MULTIPLE REPRESENTATIONS** In this problem, you will **C** investigate the *coefficient of determination.*

a. **GRAPHICAL** Make a scatter plot of the data below. Then calculate the correlation coefficient r.

x	1	2	3	4	5	6
y	4	9	12	15	20	24

b. **NUMERICAL** Find the mean $\bar{y}$ of the y-values.

c. **NUMERICAL** Determine the least squares regression equation, and find the predicted $\hat{y}$-values by substituting each value of x into the equation.

d. **NUMERICAL** Use the following formulas to find the total variation $\Sigma(y - \bar{y})^2$, explained variation $\Sigma(\hat{y} - \bar{y})^2$ and unexplained variation $\Sigma(y - \hat{y})^2$.

e. **NUMERICAL** The coefficient of determination is given by $r^2 = \frac{\text{explained variation}}{\text{total variation}}$. Use the formula and your answers from part **d** to find r^2.

f. **ANALYTICAL** If the explained variation is the variation that can be explained by the relationship between x and y, what do you think the value of the coefficient of determination that you found means?

H.O.T. Problems Use Higher-Order Thinking Skills

REASONING **Determine whether each statement is *true* or *false*. Explain your reasoning.**

14. An r value of -0.85 indicates a stronger linear correlation than an r value of 0.75.

15. If the null hypothesis is rejected, it means that the value of ρ is not significantly different from 0.

16. **CHALLENGE** Consider two sets of bivariate data, C and D, which represent exponential relationships. With an exponential regression, the value of the base b in C is the reciprocal of the value of b in D. The correlation coefficients for each are equal to 0.99. What is the relationship of the linearized regression lines of C and D?

17. **REASONING** Consider the data set below where row A represents the explanatory variable and row B represents the response variable.

A	21	30	44	49	52	59
B	114	127	148	154	169	179

a. Make a scatter plot of the data. Then determine the equation for the least squares regression line and graph it in the same window as the scatter plot.

b. Interchange A and B and repeat part **a**.

c. What effect does switching the explanatory and response variables have on the regression line?

18. **WRITING IN MATH** Describe the strengths and weaknesses of the correlation coefficient as a measure of linear correlation for a set of bivariate data.

Spiral Review

19. **FOOTBALL** Jason claims that she can throw a football at least 55 yards. After 37 throws, his average distance is 57.7 yards with a standard deviation of 3.6 yards. Is there enough evidence to reject Jason's claim at $\alpha = 0.05$? Explain your reasoning.

20. **BOWLING** Suha and Shaikha want to compare their bowling scores. They recorded their scores for 16 games as shown.
   a. Calculate the mean and sample standard deviation for each data set.
   b. Construct two 99% confidence intervals for the average score for both Suha and Shaikha.
   c. Make a statement comparing the effectiveness of the two intervals.

Suha		Shaikha	
112	109	88	169
98	116	129	190
143	131	146	99
109	98	170	108
121	122	95	181
84	128	111	183
106	121	108	122
100	107	181	99

If possible, find the sum of each infinite geometric series.

21. $a_1 = 4, r = \frac{5}{7}$

22. $a_1 = 14, r = \frac{7}{3}$

23. $16 + 12 + 9 + \ldots$

Write an explicit formula and a recursive formula for finding the *n*th term of each arithmetic sequence.

24. 10, 26.5, 43, …

25. 15, −9, −33, …

26. $3, \frac{11}{3}, \frac{13}{3}, \ldots$

Express each complex number in polar form.

27. $6 - 8i$

28. $-4 + i$

29. $3 + 2i$

Determine whether each pair of vectors are parallel.

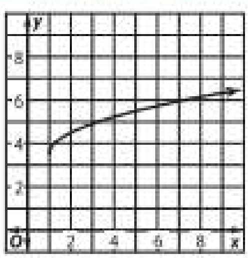
30. $\mathbf{g} = \langle 3, 4, -6 \rangle, \mathbf{h} = \langle 9, 12, -18 \rangle$

31. $\mathbf{j} = \langle 9, -15, 11 \rangle, \mathbf{k} = \langle -14, 10, 7 \rangle$

32. $\mathbf{n} = \langle -16, -8, -13 \rangle, \mathbf{p} = \langle -15, 9, 5 \rangle$

Skills Review for Standardized Tests

33. **SAT/ACT** Which of the following must be true about the graph?



   I. The domain is all real numbers.
   II. The function is $y = \sqrt{x} + 3.5$.
   III. The range is about $\{y \mid y \geq 3.5\}$.

   A I only
   B II and III
   C I, II, and III
   D II only
   E III only

34. The table shows the total attendance for minor league baseball in some recent years. Which of the following is a regression equation for the data?

Year	Attendance (millions)
1990	18.4
1995	25.2
2000	33.1
2005	37.6

   F $y = 1.31x - 2588.15$
   G $y = 1.46x - 2588.15$
   H $y = 1.31x - 18.4$
   J $y = 1.46x - 18.4$

35. **FREE RESPONSE** For the following problem, consider a real-life situation that exhibits the characteristics of exponential or logistic growth or decay.
   a. Identify the situation and the type of growth or decay that it represents.
   b. Pose a question or make a claim about the situation.
   c. Make a hypothesis to the answer of the question.
   d. Develop, justify, and implement a method to collect, organize, and analyze the related data.
   e. Extend the nature of collected, discrete data to that of a continuous function that describes the known data set.
   f. Generalize the results and make a conclusion.
   g. Compare the hypothesis and the conclusion.

EXTEND

4-7 Graphing Technology Lab
Median-Fit Lines

Objective

- Use a Graphing Calculator technology to find a median-fit line to model a relationship shown in a scatter plot.

In previous lessons, you have used regression equations to represent a set of data. Another type of regression used to model data is a *median-fit line*.

A median-fit line is found by dividing a set of data into three equal-sized groups and using the medians of those groups to determine a regression equation for the data.

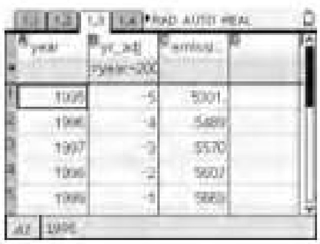
Activity 1 Draw a Median-Fit Line

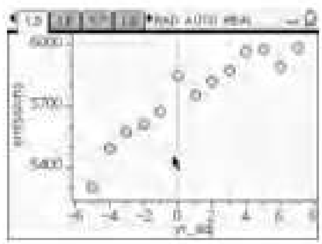
Use the data in the table to draw a median-fit line.

U.S. Energy Related Carbon Dioxide Emissions (million metric tons)			
Year	Emissions	Year	Emissions
1995	5,301	2002	5,820
1996	5,489	2003	5,872
1997	5,570	2004	5,966
1998	5,607	2005	5,974
1999	5,669	2006	5,888
2000	5,848	2007	5,984
2001	5,754		

Source: Energy Information Administration

Step 1 Enter the data in a spreadsheet. Then make a scatter plot of the data. Let the x-axis represent the number of years where 0 represents the year 2000 and the y-axis the metric tons of carbon dioxide.





Technology Tip

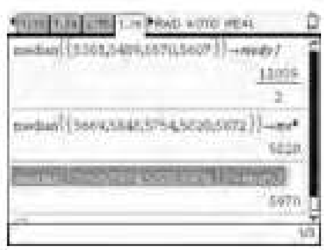
Icons Use different icons for the median points to easily distinguish them from the regular data points. Grab each point and select Attributes to change the icon type.

Step 2 Divide the data into three relatively equal and symmetric groups. The second group will have 5 data values, and the other groups will have 4. Then find the medians for the x- and y-values of each group.

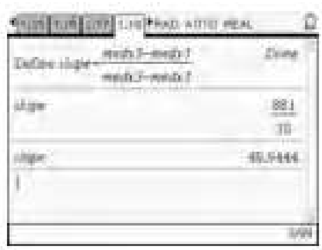
Group 1 median: $(-3.5, 5529.5)$

Group 2 median: $(1, 5820)$

Group 3 median: $(5.5, 5970)$



Step 3 The median-fit line uses the median points from the 1st and 3rd groups to determine slope and the average of the three median points as a point on the line. By using the point-slope form, $y = m(x - a) + b$, where m = slope and (a, b) is the average, you can form the median-fit line.



StudyTip

Geometric Interpretation Geometrically, the three median points determine a triangle and the average of these x- and y-values is the centroid of the triangle.

Step 4 Define the average of y as

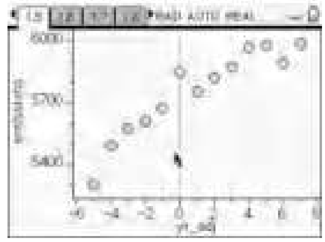
$$avey = \frac{medy1 + medy2 + medy3}{3}.$$

Define the median-fit line as $med\_med(x) = 48.944(x - 1) + 5773.17.$ Then graph the median-fit line.

Activity 2 Calculate a Median-Fit Line

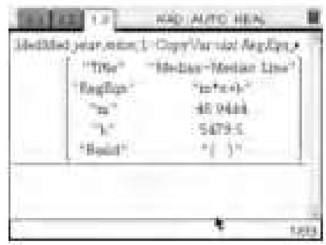
Use the data in Activity 1 to calculate the median-fit line.

Step 1 Remove the three ordered pairs that represent the medians. Then make a scatter plot of the data.

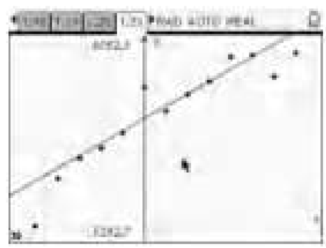


Step 2 Calculate the equation of the median-fit line. Then graph the line.

Open a new Calculator screen. Under the Statistics: Stat Calculations menu, select Median-Median Line. Enter the lists for the x- and y-values.



Notice that the equation of the median-fit line found in Activity 1 is identical to the calculator regression equation.



Analyze the Results

1. Explain the meaning of the slope of median-fit-line in this situation.
2. Is it reasonable to expect this line to represent the data indefinitely? Explain why or why not.
3. How many metric tons of carbon dioxide emissions were expected in 2015?

CHAPTER 4

Study Guide and Review

Chapter Summary

KeyConcepts

Descriptive Statistics (Lesson 4-1)

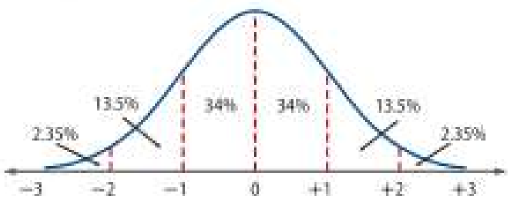
- The three most common shapes for distributions of data are negatively skewed, symmetrical, and positively skewed.

Probability Distributions (Lesson 4-2)

- The probability distribution of a random variable X links each possible value for X with its probability of occurring.

The Normal Distribution (Lesson 4-3)

- The z-value represents the number of standard deviations that a given data value is from the mean, and is given by $z = \frac{X - \mu}{\sigma}$.
- The standard normal distribution is a distribution of z-values with mean 0 and standard deviation 1.



The Central Limit Theorem (Lesson 4-4)

- As the sampling size n increases, the shape of the distribution of the sample means approaches a normal distribution.

Confidence Intervals (Lesson 4-5)

- When $n \geq 30$, $CI = \bar{x} \pm z \cdot \frac{\sigma}{\sqrt{n}}$; when $n < 30$ and σ is unknown, $CI = \sigma \pm t \cdot \frac{s}{\sqrt{n}}$.

Hypothesis Testing (Lesson 4-6)

- The steps to conduct a hypothesis test are as follows.
  - **Step 1** State the hypotheses, and identify the claim.
  - **Step 2** Determine the critical value(s) and region.
  - **Step 3** Calculate the test statistic.
  - **Step 4** Accept or reject the null hypothesis.

Correlation and Linear Regression (Lesson 4-7)

- To analyze bivariate data:
  - **Step 1** Make a scatter plot, and decide whether the variables appear to be linearly related.
  - **Step 2** Calculate the correlation coefficient.
  - **Step 3** Use a t-test to determine if the correlation is significant.
  - **Step 4** Find the least-squares regression equation.

KeyVocabulary

alternative hypothesis	negatively skewed distribution
binomial distribution	normal distribution
confidence interval	null hypothesis
continuous random variable	percentiles
correlation	positively skewed distribution
correlation coefficient	probability distribution
critical values	random variable
discrete random variable	regression line
empirical rule	response variable
explanatory variable	sampling distribution
extrapolation	sampling error
hypothesis test	standard normal distribution
inferential statistics	symmetrical distribution
interpolation	t-distribution
least squares regression line	z-value

VocabularyCheck

Identify the word or phrase that best completes each sentence.

1. The mean is less than the median and the majority of the data are on the right in a (negatively skewed, positively skewed) distribution.
2. A (continuous, discrete) random variable can take on an infinite number of possible values within a specified interval.
3. A distribution of z-values with a mean of 0 and a standard deviation of 1 is called a (binomial, standard normal) distribution.
4. The standard deviation of the sample means is called the (sampling error, standard error of the mean).
5. The (Central Limit Theorem, empirical rule) states that as n increases, the shape of the distribution of the sample means will approach a normal distribution.
6. A single value estimate of an unknown population parameter is called a(n) (point, interval) estimate.
7. The (alternative, null) hypothesis states that there is not a significant difference between a sample value and a population parameter.
8. Using an equation to make predictions far outside the range of the x-values you used to obtain the regression line is called (extrapolation, interpolation).

Lesson-by-Lesson Review

4-1 Descriptive Statistics

9. **SAT SCORES** The table gives the math SAT scores for 24 precalculus students.

SAT Math Scores					
373	437	477	491	503	516
392	454	479	491	508	519
405	463	485	498	508	522
417	470	485	499	513	533

a. Construct a histogram, and use it to describe the shape of the distribution.

b. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary.

10. **RADON GAS** The table shows the amount in picocuries per liter of radon gas in a sample of homes.

Amount of Radon (pCi/L)				
0.5	1.1	1.9	2.4	4.0
0.7	1.4	2.2	2.5	4.2
1.0	1.5	2.2	2.9	5.4
1.0	1.7	2.2	2.9	6.3
1.1	1.8	2.3	3.1	7.0

a. Construct a box plot, and use it to describe the shape of the distribution.

b. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

11. **MARATHON** The table gives the frequency distribution of the completion times for the Boston Marathon for the first 322 women finishers. Construct a percentile graph. Estimate the percentile rank for those finishing below 3 hours, and interpret its meaning.

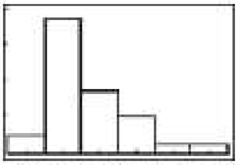
Time (hours)	Runners
2:45–2:49:59	3
2:50–2:54:59	4
2:55–2:59:59	28
3:00–3:04:59	35
3:05–3:09:59	54
3:10–3:14:59	80
3:15+	118

Example 1

BACKPACKS **The table shows the weight of school backpacks for a sample of high school students.**

Average Backpack Weight (lb)					
11.5	15.0	16.0	17.0	19.0	24.5
12.5	15.5	16.0	17.5	21.0	25.0
14.5	15.5	16.5	18.0	21.0	25.0
14.5	15.5	17.0	18.0	21.5	27.0
15.0	16.0	17.0	18.5	23.5	30.0

a. Construct a histogram, and use it to describe the shape of the distribution.



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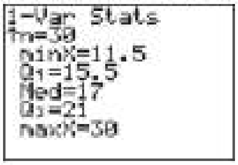
The graph is positively skewed. Most of the backpacks appear to weigh between 14 and 22 pounds, with a few that are heavier, so the tail of the distribution trails off to the right.

b. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary.



The distribution of data is skewed; therefore, the five-number summary can be used to describe the distribution. The five-number summary indicates that while the weights range from 11.5 to 30 pounds, the median weight is 17 pounds and half of the weights are between 15.5 and 21 pounds.

4-2 Probability Distributions

Classify each random variable X as *discrete* or *continuous*. Explain your reasoning.

12. X represents the number of people attending an opening show of a new movie on a given day.

13. X represents the amount of blood donated per person at a recent blood drive.

14. **FAMOUS PEOPLE** In a survey, 63% of adults said they recognized a certain famous athlete. Five adults are selected at random and asked if they recognize the athlete.

a. Construct and graph a binomial distribution for the random variable X representing the number of adults who recognized the athlete.

b. Find the probability that more than 2 adults recognized the athlete.

15. **DOGS** Find the variance and standard deviation of the probability distribution for the number of dogs per household in Greenville, South Carolina.

Dogs	Frequency
0	17,519
1	2720
2	1614
3	774
4	333

Example 2

GRAPHING In a school survey, 45% of the students said that they knew how to graph a conic. Five students chosen at random are asked if they can graph a conic.

a. Construct and graph a binomial distribution for the random variable X representing the number of students who said they could graph a conic.

Here $n = 5$, $p = 0.45$, and $q = 1 - 0.45$ or 0.55.

$P(0) = {}_5C_0 \cdot 0.45^0 \cdot 0.55^5 \approx 0.050$

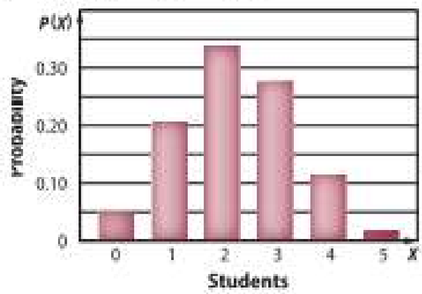
$P(1) = {}_5C_1 \cdot 0.45^1 \cdot 0.55^4 \approx 0.206$

$P(2) = {}_5C_2 \cdot 0.45^2 \cdot 0.55^3 \approx 0.337$

$P(3) = {}_5C_3 \cdot 0.45^3 \cdot 0.55^2 \approx 0.276$

$P(4) = {}_5C_4 \cdot 0.45^4 \cdot 0.55^1 \approx 0.113$

$P(5) = {}_5C_5 \cdot 0.45^5 \cdot 0.55^0 \approx 0.018$



b. Find the probability that fewer than three of the students interviewed could graph a conic.

$P(X < 3) = P(0) + P(1) + P(2)$

$= 0.05 + 0.21 + 0.34$ or 0.60 or 60%

4-3 The Normal Distribution

Find each of the following.

16. z if $X = 1.5$, $\mu = 1.1$, and $\sigma = 0.3$

17. X if $z = 2.34$, $\mu = 105$, and $\sigma = 18$

18. z if $X = 125$, $\mu = 100$, and $\sigma = 15$

19. X if $z = -1.12$, $\mu = 35$, and $\sigma = 3.4$

Find the interval of z-values associated with each area.

20. outside 55%

21. middle 24%

22. middle 96%

23. outside 49%

Example 3

Find z if $X = 36$, $\mu = 31$, and $\sigma = 1.3$.

$z = \frac{X - \mu}{\sigma}$ Formula for z-values

$= \frac{36 - 31}{1.3}$ $X = 36$, $\mu = 31$, and $\sigma = 1.3$

≈ 3.85 Simplify.

Study Guide and Review *Continued*

4-4 The Central Limit Theorem

24. **GRADES** The average grade-point average or GPA in a particular school is 2.88 with a standard deviation of approximately 0.67. Find each probability for a random sample of 50 students from that school.

a. the probability that the mean GPA will be less than 2.75

b. the probability that the mean GPA will be greater than 3.05

c. the probability that the mean GPA will be greater than 3.0 but less than 3.75

25. **PHOTOGRAPHY** A local photographer reported that 55% of Grade 12 students had their graduation photos taken outdoors. If 15 Grade 12s are selected at random, find the probability that fewer than 5 of them will get their pictures taken outdoors.

Find each of the following if z is the z-value, $\bar{x}$ is the sample mean, μ is the mean of the population, n is the sample size, and σ is the standard deviation.

26. z if $\bar{x} = 5.8$, $\mu = 5.5$, $n = 18$, and $\sigma = 0.2$

27. μ if $\bar{x} = 14.8$, $z = 4.49$, $n = 14$, and $\sigma = 1.5$

28. n if $z = 1.5$, $\bar{x} = 227$, $\mu = 224$, and $\sigma = 10$

29. σ if $z = -2.67$, $\bar{x} = 38.2$, $\mu = 40$, and $n = 16$

Example 4

WEATHER **The average annual snowfall for Albany, New York, is 62 centimeters with a standard deviation of approximately 20 centimeters. Find the probability that the mean snowfall will be between 60 and 70 centimeters using a random sample of data for 7 years.**

z-value for $\bar{x} = 60$:

$= \frac{60 - 62}{7.56}$ $\bar{x} = 60$, $\mu = 62$, and $\sigma_{\bar{x}} = \frac{20}{\sqrt{7}} = 7.56$

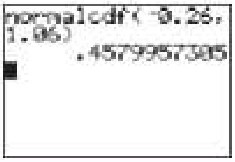
≈ -0.26 Simplify.

z-value for $\bar{x} = 70$:

$= \frac{70 - 62}{7.56}$ $\bar{x} = 70$, $\mu = 62$, and $\sigma_{\bar{x}} = \frac{20}{\sqrt{7}} = 7.56$

≈ 1.06 Simplify.

There is a 45.8% probability that the snowfall will be between 60 and 70 centimeters.



4-5 Confidence Intervals

Determine whether the normal distribution or t-distribution should be used for each question. Then find each confidence interval given the following information.

30. $c = 90\%$, $\bar{x} = 73$, $s = 4.8$, $n = 12$

31. $c = 96\%$, $\bar{x} = 34$, $\sigma = 2.3$, $n = 38$

32. $c = 99\%$, $\bar{x} = 16$, $s = 1.6$, $n = 55$

33. $c = 90\%$, $\bar{x} = 5.8$, $\sigma = 1.1$, $n = 47$

Determine the minimum sample size needed to conduct an experiment that has the given requirements.

34. $c = 90\%$, $\sigma = 3.9$, $E = 0.8$

35. $c = 95\%$, $\sigma = 1.6$, $E = 0.6$

36. $c = 98\%$, $\sigma = 6.8$, $E = 1.2$

37. $c = 92\%$, $\sigma = 10.2$, $E = 3.5$

Example 5

Determine whether the normal distribution or t-distribution should be used to find a 95% confidence interval in which $\bar{x} = 12.8$, $s = 3.8$, and $n = 50$. Then find the confidence interval.

Since $n \geq 30$, the normal distribution should be used.

In a 95% confidence interval, 2.5% of the area lies in each tail. Use a graphing calculator to find z.

$CI = \bar{x} \pm z \cdot \frac{s}{\sqrt{n}}$ Confidence Interval for the Mean

$= 12.8 \pm 1.96 \cdot \frac{3.8}{\sqrt{50}}$ $\bar{x} = 12.8$, $z = 1.96$, $s = 3.8$, and $n = 50$

$= 12.8 \pm 1.05$ Simplify.

The 95% confidence interval is $11.75 < \mu < 13.85$.

4-6 Hypothesis Testing

For each statement, write the null and alternative hypotheses, and state which hypothesis represents the claim.

38. Houriyya claims that she did not drive above 120 kilometers per hour during the entire trip.

39. Hamdah claims that she can type faster than 60 words per minute.

40. Reham claims that on average, it takes her less than 3 days to read a short novel.

41. Hessa claims that he can bake at least 6 dozen cookies per hour.

For each claim k, use the specified information to calculate the test statistic and determine whether there is enough evidence to reject the null hypothesis. Then make a statement regarding the original claim.

42. $k: \mu \leq 26.5, \alpha = 0.10, \bar{x} = 27.8, s = 1.0, n = 46$

43. $k: \mu = 56, \alpha = 0.05, \bar{x} = 58.9, s = 6.7, n = 98$

44. $k: \mu < 18, \alpha = 0.01, \bar{x} = 17.6, s = 0.8, n = 26$

45. $k: \mu \geq 39, \alpha = 0.10, \bar{x} = 38.6, s = 2.6, n = 42$

Example 6

For claim k, use the specified information to calculate the test statistic and determine whether there is enough evidence to reject the null hypothesis. Then make a statement regarding the original claim.

$k: \mu \geq 62, \alpha = 0.05, \bar{x} = 61.5, s = 4.3, n = 70$

State the null and alternative hypotheses, and identify the claim.

$H_0: \mu \geq 62$ (claim) $H_a: \mu < 62$

Determine the critical value(s) and region.

Use the z-value since $n \geq 30$ and a left-tailed test since $\mu < 62$. Since $\alpha = 0.05$, you can use a graphing calculator to find $z = -1.645$.

Calculate the test statistic.

$z = \frac{61.5 - 62}{0.51} \approx -0.98$ $\quad \bar{x} = 61.5, \mu = 62, \text{ and } \sigma_{\bar{x}} = \frac{4.3}{\sqrt{70}}$

Reject or fail to reject the null hypothesis.

H_0 is not rejected since the test statistic does not fall within the critical region. Therefore, there is not enough evidence to reject the claim.

4-7 Correlation and Linear Regression

46. **GRADES** The table shows the pre-test and final grades for a high school college prep class. (x = pre-test, y = final)

Scores for a College Prep Class							
x	y	x	y	x	y	x	y
86	3.5	77	2.5	85	3.0	62	1.9
70	3.0	97	3.9	85	3.8	92	3.6
100	4.0	79	3.0	68	2.2	84	3.0
87	3.8	69	2.4	73	2.4	84	3.6
99	4.0	67	2.1	91	3.7	74	2.8

a. Make a scatter plot of the data, and identify the relationship. Then calculate and interpret the correlation coefficient.

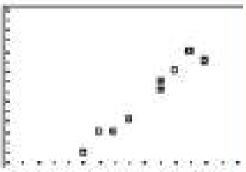
b. Test the significance of this correlation coefficient at the 10% level.

Example 7

HEIGHT **The table shows the heights of brothers and sisters. Make a scatter plot of the data and identify the relationship. Then calculate and interpret the correlation coefficient.**

Brother	71	68	66	67	70
Sister	69	64	63	63	68
Brother	71	70	73	72	65
Sister	69	67	70	71	61

The correlation coefficient r is about 0.9773. Since r is close to 1, this suggests that the data have a strong positive linear correlation. This numerical assessment of the data agrees with our graphical assessment.



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Study Guide and Review *Continued*

Applications and Problem Solving

47. **SPORTS** The body-fat levels of 20 professional basketball players are shown. (Lesson 4-1)

Body-Fat Levels (%)			
3.4	5.5	6.1	4.8
8.3	7.7	6.5	6.5
4.9	3.7	3.9	4.0
7.3	8.9	9.5	9.8
3.9	7.1	6.3	6.1

a. Construct a histogram, and use it to describe the shape of the distribution.

b. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

48. **EXERCISE** The number of hours that a sample of students exercises each week is shown. (Lesson 4-1)

Time Spent Exercising (hours)		
3	2.5	0
1.5	3	2
3.5	2	0
1.5	9.5	0
8	0.5	1.5
1	10	4

a. Construct a box plot, and use it to describe the shape of the distribution.

b. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary. Justify your choice.

49. **AP CLASSES** The table shows the number of AP classes per Grade 12 student. Find the mean, variance, and standard deviation of this distribution. (Lesson 4-2)

X	0	1	2	3	4
Frequency	12	18	25	19	11

50. **IQ** IQs for a group of people are normally distributed with a mean of 105 and a standard deviation of 22. Find the probability that a randomly chosen person will have an IQ that corresponds to each of the following. (Lesson 4-3)

a. above 101

b. below 94

c. between 110 and 120

51. **COOKIES** The number of chocolate chips in a cookie is normally distributed with $\mu = 25$ and $\sigma = 3$. Find each of the following. (Lesson 4-3)

a. $P(X < 35)$

b. $P(21 < X < 29)$

c. $P(X > 15)$

52. **WRESTLING** The average number of fans attending East High School's wrestling meets is normally distributed with $\mu = 88$ and $\sigma = 16$. If 6 random meets are selected, find the probability that the mean of the sample would be more than 90 fans. (Lesson 4-4)

53. **EXERCISE** A sample of 58 students found that on average, the students spend 185 minutes engaged in physical activity each week. Assume that the standard deviation from a recent study was 28 minutes. Estimate the mean time students spend engaged in physical activity each week using a confidence interval given a 95% confidence level. (Lesson 4-5)

54. **FLIGHT** An airline claims that its flights from Cleveland to Texas take less than 3.0 hours. A random sample of 30 flights found an average time of 2.9 hours and a standard deviation of 0.25 hour. Determine whether the airline's claim is supported at $\alpha = 0.05$. (Lesson 4-6)

a. Write the null and alternative hypotheses, and state which one represents the claim.

b. Calculate the test statistic.

c. Determine whether there is enough evidence to reject the null hypothesis.

d. Make a statement regarding the original claim.

55. **SCORES** **The table shows the aptitude and writing test scores for a class over the same material.** (Lesson 4-7)

Aptitude					Writing				
135	146	153	154	139	26	33	55	50	32
131	149	137	133	149	25	44	31	31	34
141	164	146	149	147	32	47	37	46	36
152	143	146	141	136	47	36	35	28	28
154	151	155	140	143	36	48	36	33	42
148	149	141	137	135	32	32	29	34	30

a. Make a scatter plot of the data, and identify the relationship. Then calculate and interpret the correlation coefficient.

b. Test the significance of this correlation coefficient at the 5% level.

c. Find the equation of the regression line.

d. Use this equation to predict the writing score for a student who scored a 142 on the aptitude test.

CHAPTER 4 Practice Test

1. **RACING** The ages of the last 20 winners of the Indianapolis 500 are shown.

Age (years)									
24	26	28	33	40	25	27	30	36	42
26	27	32	35	43	26	27	33	38	46

a. Construct a histogram, and use it to describe the shape of the distribution.

b. Summarize the center and spread of the data using either the mean and standard deviation or the five-number summary.

2. **TELEVISION** The number of televisions per household for 100 students is shown.

Televisions	0	1	2	3	4	5
Frequency	1	3	21	53	16	6

a. Use the frequency distribution to construct and graph a probability distribution for the random variable X.

b. Find the mean score, and interpret its meaning in the context of the problem situation.

c. Find the variance and standard deviation of the probability distribution.

3. **FOOD** Ms. Sumayya's math class took a poll to find how many drive-through trips students made in a week.

X	0	1	2	3	4	5
Frequency	10	16	12	22	8	2

a. Use the frequency distribution of the results to construct and graph a probability distribution for the random variable X, rounding each probability to the nearest hundredth.

b. Find the mean of the probability distribution.

c. Find the variance and standard deviation.

4. **VACATION** In the summer, the average temperature at a Caribbean vacation resort is 32°C with a standard deviation of 2.5°C. For a randomly selected day, find the probability that the temperature will be as follows.

a. above 22°C

b. below 20°C

c. between 29°C and 34°C

PACKAGING The mean weight of a box of cereal is 362 grams and standard deviation of 5. If a random selection of 5 boxes is sampled, find the following.

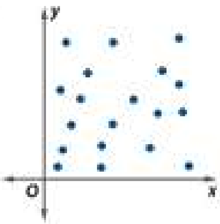
5. probability that the mean weight is less than 355

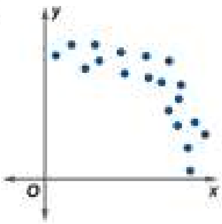
6. probability that the mean weight is greater than 370

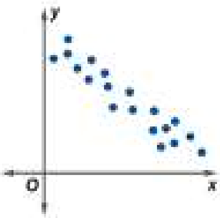
7. **CONCESSIONS** A survey of 97 movie patrons found that customers spent an average of $12.50 at the concessions counter. Assume that the standard deviation from a recent study was $2.25. Estimate the mean amount of money customers spend given a 95% confidence level.

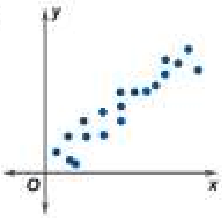
8. **RENT** Mohammad claims that the average college student spends less than $400 a month on rent. A sample of 48 students found that students spent an average of $385 on rent each month and a standard deviation of $30. At $\alpha = 0.10$, determine whether there is enough evidence to reject the null hypothesis, and make a statement regarding the original claim.

9. **MULTIPLE CHOICE** Identify the graph that could have a correlation coefficient of −0.96 in a linear regression.

A 

C 

B 

D 

10. **DRIVING** The table lists the average number of accidents per month for sections of highway with the given speed limits.

a. Make a scatter plot of the data, and identify the relationship.

b. Calculate and interpret the correlation coefficient.

c. Determine if the correlation coefficient is significant at the 5% level. Explain your reasoning.

Speed (kmph)	Accidents
25	2.6
30	3.5
35	6.9
40	10.3
45	15.2
50	18.3
55	22.3
60	24.8
65	26.0
70	29.2

CHAPTER 4

Connect to AP Calculus
Population Proportions

Objective

- Create confidence intervals for population proportions.

You learned that the probability of a success in a single trial of a binomial experiment is p and it can be expressed as a fraction, decimal, or percentage. For example, the probability of tossing a fair coin and recording a tail is $\frac{1}{2}$, 0.5, or 50%. This probability is a *population proportion* because for the fair coin, the entire population, both heads and tails, is accounted for.

It is not always plausible to calculate population proportions. For example, calculating the percentage of high school students that own their own car would require surveying every high school student. Thus, population proportions can be estimated using *sample proportions* in the same way that sample means were used to estimate population means in this chapter.

The sample proportion $\hat{p}$ is the proportion of successes in a sample and is given by $\hat{p} = \frac{x}{n}$, where x is the number of successes in the sample and n is the sample size. The probability of failure is then given by $\hat{q} = 1 - \hat{p}$.

Activity 1 Sample Proportion

A sample of 2,582 high school students found that 362 students own their own car. Estimate the population proportion of high school students that own their own car by calculating the sample proportion $\hat{p}$.

Step 1 Substitute $x = 362$ and $n = 2582$ into the formula for $\hat{p}$ and simplify.

Step 2 Interpret the result.
The percent of all high school students that own their own car is approximately 14%.

Analyze the Results

1. Is the sample proportion an accurate estimate for the population proportion? Explain your reasoning.
2. If the sample is conducted with a larger n, what can be said about the relationship between the sample proportion and the population proportion?
3. Will the sample proportion ever equal the population proportion? If not, what can be done to the sample proportion in addition to increasing n to give a better estimate for the population proportion? Explain your reasoning.

We know that the $\hat{p}$ found in Activity 1 is a *point estimate*. If we wanted to create a better estimate, we would want to construct an interval. The behavior of the distribution of sample proportions is similar to the distribution of sample means. As the sample size increases, the distribution becomes approximately normal and the average of the sample proportions approaches the population proportion p.

StudyTip

Normal Distribution and z-Values Recall that the normal distribution is used for binomial distribution when $np \geq 5$ and $nq \geq 5$. Thus we can find and use z-values to calculate E.

Just as a confidence interval can be calculated for a population mean by adding and subtracting a maximum error of estimate E to and from a sample mean $\bar{x}$ a maximum error of estimate can be added to and subtracted from a sample proportion $\hat{p}$ to create a confidence interval for a population proportion.

KeyConcept Confidence Interval For a Population Proportion

The confidence interval CI for a population proportion is given by

$$CI = \hat{p} \pm E,$$

where $\hat{p}$ is the sample proportion and E is the maximum error of estimate represented by $z\sqrt{\frac{\hat{p}\hat{q}}{n}}$.

Activity 2 Confidence Interval for a Proportion

A random survey of 825 college applicants recorded the students' high school grade point average a. Find the 90% confidence interval for the proportion of all college applicants with a grade point average of 3.0 or higher.

GPA a	Applicants
$4.0 \le a$	33
$3.0 \le a < 4.0$	600
$2.0 \le a < 3.0$	175
$a < 2.0$	17

Step 1 **Find $\hat{p}$ and $\hat{q}$.**

$\hat{p} = \frac{x}{n}$ — Sample Proportion Formula

$= \frac{633}{825}$ or about 0.77 — $x = 633$ and $n = 825$

Therefore, $\hat{q} = 1 - 0.77$ or about 0.23.

Step 2 **Verify that $n\hat{p} \ge 5$ and $n\hat{q} \ge 5$.**

$n\hat{p} \approx 825(0.77)$ or 635.25 $\quad n\hat{q} \approx (825)(0.23)$ or 189.75

Since $n\hat{p} \ge 5$ and $n\hat{q} \ge 5$, the sampling distribution of $\hat{p}$ can be approximated by the normal distribution.

Step 3 **Find the z-value.**

For a 90% confidence level, $z = 1.645$.

Step 4 **Find the maximum error of estimate.**

$E = z\sqrt{\frac{\hat{p}\hat{q}}{n}}$ — Maximum Error of Estimate Formula

$\approx 1.645\sqrt{\frac{0.77(0.23)}{825}}$ or about 0.0241 — $z = 1.645$, $\hat{p} = 0.77$, $\hat{q} = 0.23$, and $n = 825$

Step 5 **Find the left and right endpoints of the confidence interval.**

$CI = \hat{p} \pm E$ — Confidence Interval for a Proportion

$= 0.77 \pm 0.0241$ — $\hat{p} = 0.77$ and $E = 0.0241$

Left Boundary	Right Boundary
$0.77 - \mathbf{0.0241} = 0.7459$	$0.77 + \mathbf{0.0241} = 0.7941$

The 90% confidence interval is then $0.746 < p < 0.794$. Therefore, we are 90% confident that the proportion of applicants with a G.P.A. of 3.0 or higher is between 74.6% and 79.4%.

StudyTip

Finding z-Values Recall that the most common confidence levels and their corresponding z-values are as follows.

Confidence Level	z-value
90%	1.645
95%	1.960
99%	2.576

Remember that you can find the z-value for any confidence interval with a graphing calculator.

Analyze the Results

4. Describe two ways that the confidence interval found in Step 5 can be narrowed.
5. If the confidence level is held constant, what would n need to be to reduce the maximum error of estimate by $\frac{1}{2}$?

Model and Apply

6. In a 2006 Gallup Poll of 1,000 adults, 480 felt that the money the government spent on the space shuttle should have been spent on something else. Find the 95% confidence interval for the proportion of all adults who felt this way.
7. A random sample of 279 households found that 58% had at least one sport-utility vehicle (SUV). Find the 99% confidence interval for the proportion of all households that own an SUV.

Student Handbook

Symbols, Formulas, and Key Concepts

Symbols

Algebra

$\neq$	is not equal to
$\approx$	is approximately equal to
$\sim$	is similar to
$>, \geq$	is greater than, is greater than or equal to
$<, \leq$	is less than, is less than or equal to
$-a$	opposite or additive inverse of a
$\lvert a \rvert$	absolute value of a
$\sqrt{a}$	principal square root of a
$a : b$	ratio of a to b
(x, y)	ordered pair
(x, y, z)	ordered triple
i	the imaginary unit
$b^{\frac{1}{n}} = \sqrt[n]{b}$	nth root of b
$\mathbb{Q}$	rational numbers
$\mathbb{I}$	irrational numbers
$\mathbb{Z}$	integers
$\mathbb{W}$	whole numbers
$\mathbb{N}$	natural numbers
∞	infinity
$-\infty$	negative infinity
[ ]	endpoint included
( )	endpoints not included
$\log_b x$	logarithm base b of x
$\log x$	common logarithm of x
$\ln x$	natural logarithm of x
ω	omega, angular speed
α	alpha, angle measure
β	beta, angle measure
γ	gamma, angle measure
θ	theta, angle measure
λ	lambda, wavelength
ϕ	phi, angle measure
a	vector **a**
$\lvert \mathbf{a} \rvert$	magnitude of vector **a**

Sets and Logic

$\in$	is an element of
$\subset$	is a subset of
$\cap$	intersection
$\cup$	union
$\varnothing$	empty set
$\sim p$	negation of p, not p
$p \wedge q$	conjunction of p and q
$p \vee q$	disjunction of p and q
$p \longrightarrow q$	conditional statement, if p then q
$p \longleftrightarrow q$	biconditional statement, p if and only if q

Geometry

$\angle$	angle
$\triangle$	triangle
$^\circ$	degree
π	pi
$\measuredangle$	angles
$m\angle A$	degree measure of $\angle A$
$\overleftrightarrow{AB}$	line containing points A and B
$\overline{AB}$	segment with endpoints A and B
$\overrightarrow{AB}$	ray with endpoint A containing B
AB	measure of $\overline{AB}$, distance between points A and B
$\parallel$	is parallel to
$\nparallel$	is not parallel to
$\perp$	is perpendicular to
$\triangle$	triangle
▱	parallelogram
n-gon	polygon with n sides
$\vec{a}$	vector a
$\vec{AB}$	vector from A to B
$\lvert \vec{AB} \rvert$	magnitude of the vector from A to B
A'	the image of preimage A
$\rightarrow$	is mapped onto
$\odot A$	circle with center A
$\overset{\frown}{AB}$	minor arc with endpoints A and B
$\overset{\frown}{ABC}$	major arc with endpoints A and C
$m\overset{\frown}{AB}$	degree measure of arc AB

Trigonometry

$\sin x$	sine of x
$\cos x$	cosine of x
$\tan x$	tangent of x
$\sin^{-1} x$	Arcsin x
$\cos^{-1} x$	Arccos x
$\tan^{-1} x$	Arctan x

Symbols

Functions

$f(x)$	f of x, the value of f at x
$f(x) = \{$	piecewise-defined function
$f(x) = \lvert x \rvert$	absolute value function
$f(x) = [\![x]\!]$	function of greatest integer not greater than x
$f(x, y)$	f of x and y, a function with two variables, x and y
$[f \circ g](x)$	f of g of x, the composition of functions f and g
$f^{-1}(x)$	inverse of $f(x)$

Calculus

$\lim\limits_{x \to c}$	limit as x approaches c
m_{sec}	slope of a secant line
$f'(x)$	derivative of $f(x)$
Δ	delta, change
$\int$	indefinite integral
$\int_a^b$	definite integral
$F(x)$	antiderivative of $f(x)$

Probability and Statistics

$P(a)$	probability of a
$P(n, r)$ or ${}_nP_r$	permutation of n objects taken r at a time
$C(n, r)$ or ${}_nC_r$	combination of n objects taken r at a time
$P(A)$	probability of A
$P(A \mid B)$	the probability of A given that B has already occurred
$n!$	Factorial of n (n being a natural number)
Σ	sigma (uppercase), summation
μ	mu, population mean
σ	sigma (lowercase), population standard deviation
σ^2	population variance
s	sample standard deviation
s^2	sample variance
$\sum\limits_{n=1}^{k}$	summation from $n = 1$ to k
$\bar{x}$	x-bar, sample mean
H_0	null hypothesis
H_a	alternative hypothesis

Measures

Metric	Customary
Length	
1 kilometer (km) = 1000 meters (m) 1 meter = 100 centimeters (cm) 1 centimeter = 10 millimeters (mm)	1 mile (mi) = 1760 yards (yd) 1 mile = 5280 feet (ft) 1 yard = 3 feet 1 foot = 12 inches (in) 1 yard = 36 inches
Volume and Capacity	
1 liter (L) = 1000 milliliters (mL) 1 kiloliter (kL) = 1000 liters	1 gallon (gal) = 4 quarts (qt) 1 gallon = 128 fluid ounces (fl oz) 1 quart = 2 pints (pt) 1 pint = 2 cups (c) 1 cup = 8 fluid ounces
Weight and Mass	
1 kilogram (kg) = 1000 grams (g) 1 gram = 1000 milligrams (mg) 1 metric ton (t) = 1000 kilograms	1 ton (T) = 2000 pounds (lb) 1 pound = 16 ounces (oz)

Arithmetic Operations and Relations

Identity	For any number a, $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.
Substitution (=)	If $a = b$, then a may be replaced by b.
Reflexive (=)	$a = a$
Symmetric (=)	If $a = b$, then $b = a$.
Transitive (=)	If $a = b$ and $b = c$, then $a = c$.
Commutative	For any numbers a and b, $a + b = b + a$ and $a \cdot b = b \cdot a$.
Associative	For any numbers a, b, and c, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
Distributive	For any numbers a, b, and c, $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$.
Additive Inverse	For any number a, there is exactly one number $-a$ such that $a + (-a) = 0$.
Multiplicative Inverse	For any number $\frac{a}{b}$, where $a, b \neq 0$, there is exactly one number $\frac{b}{a}$ such that $\frac{a}{b} \cdot \frac{b}{a} = 1$.
Multiplicative (0)	For any number a, $a \cdot 0 = 0 \cdot a = 0$.
Addition (=)	For any numbers a, b, and c, if $a = b$, then $a + c = b + c$.
Subtraction (=)	For any numbers a, b, and c, if $a = b$, then $a - c = b - c$.
Multiplication and Division (=)	For any numbers a, b, and c, with $c \neq 0$, if $a = b$, then $ac = bc$ and $\frac{a}{c} = \frac{b}{c}$.
Addition (>)*	For any numbers a, b, and c, if $a > b$, then $a + c > b + c$.
Subtraction (>)*	For any numbers a, b, and c, if $a > b$, then $a - c > b - c$.
Multiplication and Division (>)*	For any numbers a, b, and c, 1. if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$. 2. if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
Zero Product	For any real numbers a and b, if $ab = 0$, then $a = 0$, $b = 0$, or both a and b equal 0.

* *These properties are also true for $<$, $\geq$, and $\leq$.*

Algebraic Formulas and Key Concepts

	Matrices		
Adding	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$	**Multiplying by a Scalar**	$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$
Subtracting	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$	**Multiplying**	$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$
	Polynomials		
Quadratic Formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, a \neq 0$	**Square of a Difference**	$(a - b)^2 = (a - b)(a - b)$ $= a^2 - 2ab + b^2$
Square of a Sum	$(a + b)^2 = (a + b)(a + b)$ $= a^2 + 2ab + b^2$	**Product of Sum and Difference**	$(a + b)(a - b) = (a - b)(a + b)$ $= a^2 - b^2$
	Logarithms		
Product Property	$\log_x ab = \log_x a + \log_x b$	**Power Property**	$\log_b m^p = p \log_b m$
Quotient Property	$\log_x \frac{a}{b} = \log_x a - \log_x b, b \neq 0$	**Change of Base**	$\log_a n = \frac{\log_b n}{\log_b a}$

Algebraic Formulas and Key Concepts

Exponential and Logarithmic Functions

Compound Interest	$A = P\left(1 + \frac{r}{n}\right)^{nt}$	**Exponential Growth or Decay**	$N = N_0(1 + r)^t$
Continuous Compound Interest	$A = Pe^{rt}$	**Continuous Exponential Growth or Decay**	$N = N_0e^{kt}$
Product Property	$\log_b xy = \log_b x + \log_b y$	**Power Property**	$\log_b x^p = p \log_b x$
Quotient Property	$\log_b \frac{x}{y} = \log_b x - \log_b y$	**Change of Base**	$\log_b x = \frac{\log_a x}{\log_a b}$
Logistic Growth	$f(t) = \frac{c}{1 + ae^{-bt}}$		

Sequences and Series

***n*th term, Arithmetic**	$a_n = a_1 + (n - 1)d$	***n*th term, Geometric**	$a_n = a_1r^{n-1}$
Sum of Arithmetic Series	$S_n = n\left(\frac{a_1 + a_n}{2}\right)$ or $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$	**Sum of Geometric Series**	$S_n = \frac{a_1 - a_1r^n}{1 - r}$ or $S_n = \frac{a_1 - a_nr}{1 - r}$, $r \neq 1$
Sum of Infinite Geometric Series	$S = \frac{a_1}{1 - r}$, $\lvert r \rvert < 1$	**Euler's Formula**	$e^{i\theta} = \cos\theta + i\sin\theta$
Power Series	$\sum_{n=0}^{\infty} a_nx^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$	**Exponential Series**	$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
Binomial Theorem	$(a + b)^n = {}_nC_0\,a^nb^0 + {}_nC_1\,a^{n-1}b^1 + {}_nC_2\,a^{n-2}b^2 + \cdots + {}_nC_r\,a^{n-r}b^r + \cdots + {}_nC_n\,a^0b^n$		
Cosine and Sine Power Series	$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^nx^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$		$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^nx^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Vectors

Addition in Plane	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2\rangle$	**Addition in Space**	$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$
Subtraction in Plane	$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2\rangle$	**Subtraction in Space**	$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3\rangle$
Scalar Multiplication in Plane	$k\mathbf{a} = \langle ka_1, ka_2\rangle$	**Scalar Multiplication in Space**	$k\mathbf{a} = \langle ka_1, ka_2, ka_3\rangle$
Dot Product in Plane	$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2$	**Dot Product in Space**	$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
Angle Between Two Vectors	$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$	**Projection of u onto v**	$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\lvert\mathbf{v}\rvert^2}\right)\mathbf{v}$
Magnitude of a Vector	$\lvert\mathbf{v}\rvert = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	**Triple Scalar Product**	$\mathbf{t} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} t_1 & t_2 & t_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

Equations of a Line on a Coordinate Plane

Slope-intercept form of a line	$y = mx + b$
Point-slope form of a line	$y - y_1 = m(x - x_1)$

Algebraic Formulas and Key Concepts

Conic Sections

Parabola	$(x-h)^2 = 4p(y-k)$ or $(y-k)^2 = 4p(x-h)$	**Circle**	$x^2 + y^2 = r^2$ or $(x-h)^2 + (y-k)^2 = r^2$
Ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$	**Hyperbola**	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Rotation of Conics	$x' = x\cos\theta + y\sin\theta$ and $y' = y\cos\theta - x\sin\theta$		

Parametric Equations

Vertical Position	$y = tv_0 \sin\theta - \frac{1}{2}gt^2 + h_0$	**Horizontal Distance**	$x = tv_0\cos\theta$

Complex Numbers

Product Formula	$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + \boldsymbol{i}\sin(\theta_1 + \theta_2)]$	**Quotient Formula**	$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + \boldsymbol{i}\sin(\theta_1 - \theta_2)]$
Distinct Roots Formula	$r^{\frac{1}{p}}\left(\cos\frac{\theta + 2n\pi}{p} + \boldsymbol{i}\sin\frac{\theta + 2n\pi}{p}\right)$	**De Moivre's Theorem**	$z^n = [r(\cos\theta + \boldsymbol{i}\sin\theta)]^n$ or $r^n(\cos n\theta + \boldsymbol{i}\sin n\theta)$

Geometric Formulas and Key Concepts

Coordinate Geometry

Slope	$m = \frac{y_2 - y_1}{x_2 - x_1}, x_2 \neq x_1$	**Distance on a number line**	$d = \lvert a - b \rvert$
Distance on a coordinate plane	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	**Arc length**	$\ell = \frac{x}{360} \cdot 2\pi r$
Midpoint on a coordinate plane	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$	**Midpoint on a number line**	$M = \frac{a + b}{2}$
Midpoint in space	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$	**Pythagorean Theorem**	$a^2 + b^2 = c^2$
Distance in space	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$		

Perimeter and Circumference

Square	$P = 4s$	**Rectangle**	$P = 2\ell + 2w$	**Circle**	$C = 2\pi r$ or $C = \pi d$

Lateral Surface Area

Prism	$L = Ph$	**Pyramid**	$L = \frac{1}{2}P\ell$
Cylinder	$L = 2\pi rh$	**Cone**	$L = \pi r\ell$

Total Surface Area

Prism	$S = Ph + 2B$	**Cone**	$S = \pi r\ell + \pi r^2$	**Cylinder**	$S = 2\pi rh + 2\pi r^2$
Pyramid	$S = \frac{1}{2}P\ell + B$	**Sphere**	$S = 4\pi r^2$	**Cube**	$S = 6s^2$

Volume

Prism	$V = Bh$	**Cone**	$V = \frac{1}{3}\pi r^2 h$	**Cylinder**	$V = \pi r^2 h$
Pyramid	$V = \frac{1}{3}Bh$	**Sphere**	$V = \frac{4}{3}\pi r^3$	**Cube**	$V = s^3$
Rectangular prism	$V = \ell wh$				

Trigonometric Functions and Identities

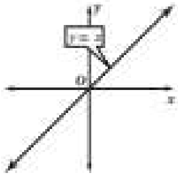
Trigonometric Functions			
Trigonometric Functions	$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin\theta}$	$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\cos\theta}$	$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{\cos\theta}{\sin\theta}$
Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos A$	$b^2 = a^2 + c^2 - 2ac\cos B$	$c^2 = a^2 + b^2 - 2ab\cos C$
Law of Sines	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	**Heron's Formula**	$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$
Linear Speed	$v = \frac{s}{t}$	**Angular Speed**	$\omega = \frac{\theta}{t}$

Trigonometric Identities			
Reciprocal	$\sin\theta = \frac{1}{\csc\theta}$ $\csc\theta = \frac{1}{\sin\theta}$	$\cos\theta = \frac{1}{\sec\theta}$ $\sec\theta = \frac{1}{\cos\theta}$	$\tan\theta = \frac{1}{\cot\theta}$ $\cot\theta = \frac{1}{\tan\theta}$
Pythagorean	$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$\cot^2\theta + 1 = \csc^2\theta$
Cofunction	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$	$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$	$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$
Odd-Even	$\sin(-\theta) = -\sin\theta$ $\csc(-\theta) = -\csc\theta$	$\cos(-\theta) = \cos\theta$ $\sec(-\theta) = \sec\theta$	$\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$
Sum & Difference	$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$	$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$	
Double-Angle	$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = 2\cos^2\theta - 1$ $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$	$\cos 2\theta = 1 - 2\sin^2\theta$
Power-Reducing	$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$	$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$	$\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$
Half-Angle	$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}$ $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}$	$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$ $\tan\frac{\theta}{2} = \frac{1 - \cos\theta}{\sin\theta}$	$\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta}$
Product-to-Sum	$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$ $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$	$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$ $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$	
Sum-to-Product	$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$ $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$	$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$ $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$	

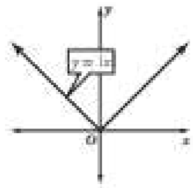
Parent Functions and Function Operations

Parent Functions

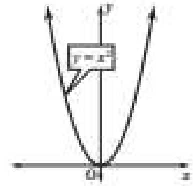
Linear Functions



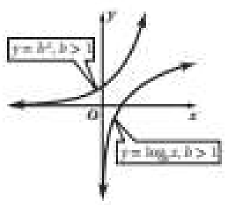
Absolute Value Functions



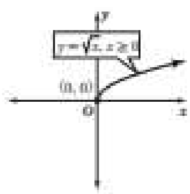
Quadratic Functions



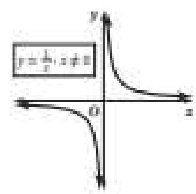
Exponential and Logarithmic Functions



Square Root Functions



Reciprocal and Rational Functions



Function Operations

Addition	$(f + g)(x) = f(x) + g(x)$	**Multiplication**	$(f \cdot g)(x) = f(x) \cdot g(x)$
Subtraction	$(f - g)(x) = f(x) - g(x)$	**Division**	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$

Calculus

Limits

Sum	$\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$	**Difference**	$\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
Scalar Multiple	$\lim_{x \to c} [k\, f(x)] = k \lim_{x \to c} f(x)$	**Product**	$\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
Quotient	$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, if $\lim_{x \to c} g(x) \neq 0$	**Power**	$\lim_{x \to c} [f(x)^n] = \left[\lim_{x \to c} f(x)\right]^n$
***n*th Root**	$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$, if $\lim_{x \to c} f(x) > 0$ when n is even	**Velocity**	**Average** $v_{avg} = \frac{f(b) - f(a)}{b - a}$ **Instantaneous** $v(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}$

Derivatives

Power Rule	If $f(x) = x^n$, $f'(x) = nx^{n-1}$	**Sum or Difference**	If $f(x) = g(x) \pm h(x)$, then $f'(x) = g'(x) \pm h'(x)$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$	**Quotient Rule**	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Integrals

Indefinite Integral	$\int f(x)\, dx = F(x) + C$	**Fundamental Theorem of Calculus**	$\int_a^b f(x)\, dx = F(b) - F(a)$

Statistics Formulas and Key Concepts

z-Values	$z = \frac{X - \mu}{\sigma}$	**z-Value of a Sample Mean**	$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}$
Binomial Probability	$P(X) = {}_nC_x\, p^x q^{n-x} = \frac{n!}{(n-x)!\,x!} p^x q^{n-x}$	**Maximum Error of Estimate**	$E = z \cdot \sigma_{\bar{x}}$ or $z \cdot \frac{\sigma}{\sqrt{n}}$
Confidence Interval, Normal Distribution	$CI = \bar{x} \pm E$ or $\bar{x} \pm z \cdot \frac{\sigma}{\sqrt{n}}$	**Confidence Interval, t-Distribution**	$CI = \bar{x} \pm t \cdot \frac{s}{\sqrt{n}}$
Correlation Coefficient	$r = \frac{1}{n-1}\sum\left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right)$	**t-Test for the Correlation Coefficient**	$t = r\sqrt{\frac{n-2}{1-r^2}}$, degrees of freedom: $n - 2$

Cover label guide

Cycle 03 Color

- Icon indicates the book type (**Example, Course book**)
- Number 3 has been shaded to represent the **volume of Course book**
- Number 9 Shows grade level
- Shows stream type

- Course book
- Activity book (workbook)
- Reading book (Arabic, social)
- Interactive Student Guide (Math)
- Activity Lab Manual (Science)
- Teacher Guide

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