CHAPTER 3

Polynomials and Polynomial Functions

Then

- You graphed quadratic functions and solved quadratic equations.

Now

- You will:
  - Add, subtract, multiply, divide, and factor polynomials.
  - Analyze and graph polynomial functions.
  - Evaluate polynomial functions and solve polynomial equations.
  - Find factors and zeros of polynomial functions.

Why? ▲

- **TRANSPORTATION** Polynomial functions can be used to determine bus schedules, highway capacity, traffic patterns, average fuel costs, and the prices of new and used cars.

Get Ready for the Chapter

1 Textbook Option Take the Quick Check below. Refer to the Quick Review for help.

QuickCheck

Rewrite each difference as a sum.

1. $-5 - 13$
2. $5 - 3y$
3. $5mr - 7mp$
4. $3x^2y - 14xy^2$
5. **PARTIES** Twenty people attended a going away party for Khamis. The guests left in groups of 2. By 9:00, x groups had left. Rewrite the number of guests remaining at 9:00 as a sum.

Use the Distributive Property to rewrite each expression without parentheses.

6. $-4(a + 5)$
7. $-1(3b^2 + 2b - 1)$
8. $-\frac{1}{2}(2m - 5)$
9. $-\frac{3}{4}(3z + 5)$
10. **MONEY** Ibrahim is buying pizza and soda for the members of the science club. A slice of pizza costs AED 2.25, and a soda costs AED 1.25. Write an expression to represent the amount that Mr. Ibrahim will spend on 15 students. Evaluate the expression by using the Distributive Property.

Solve each equation.

11. $x^2 + 2x - 8 = 0$
12. $2x^2 + 7x + 3 = 0$
13. $6x^2 + 5x - 4 = 0$
14. $4x^2 - 2x - 1 = 0$
15. **PHYSICS** If an object is dropped from a height of 50 feet above the ground, then its height after t seconds is given by $h = -16t^2 + 50$. Use the equation $0 = -16t^2 + 50$ to find how long it will take until the ball reaches the ground.

QuickReview

Example 1

Rewrite $2xy - 3 - z$ as a sum.

$$2xy - 3 - z$$
$$= 2xy + (-3) + (-z)$$

Example 2

Use the Distributive Property to rewrite $-3(a + b - c)$.

$$-3(a + b - c)$$
$$= -3(a) + (-3)(b) + (-3)(-c)$$
$$= -3a - 3b + 3c$$

Example 3

Solve $2x^2 + 8x + 1 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-8 \pm \sqrt{8^2 - 4(2)(1)}}{2(2)}$$
$$= \frac{-8 \pm \sqrt{56}}{4}$$
$$= -2 \pm \frac{\sqrt{14}}{2}$$

The exact solutions are $-2 + \frac{\sqrt{14}}{2}$ and $-2 - \frac{\sqrt{14}}{2}$.

The approximate solutions are -0.13 and -3.87.

Get Started on the Chapter

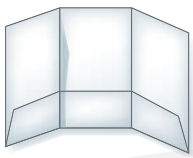
You will learn several new concepts, skills, and vocabulary terms as you study this chapter. To get ready, identify important terms and organize your resources. You may wish to refer to earlier chapters to review prerequisite skills.

FOLDABLES StudyOrganizer

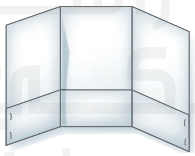
Polynomials and Polynomial Functions Make this Foldable to help you organize your notes about polynomials and polynomial functions. Begin with one sheet of $8\frac{1}{2}$" by 14" paper.

1 **Fold** a 2" tab along the bottom of a long side.

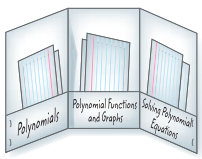
2 **Fold** along the width into thirds.



3 **Staple** the outer edges of the tab.



4 **Label** the tabs *Polynomials, Polynomial Functions and Graphs,* and *Solving Polynomial Equations.*



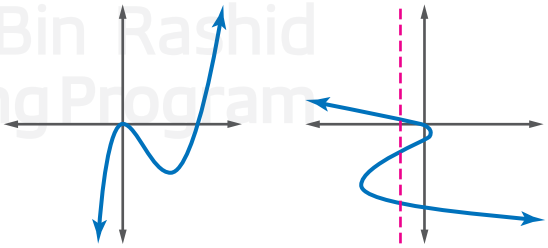
NewVocabulary

simplify
degree of a polynomial
synthetic division
polynomial in one variable
leading coefficient
polynomial function
power function
end behavior
relative maximum
relative minimum
extrema
turning points
prime polynomials
quadratic form
synthetic substitution
depressed polynomial

ReviewVocabulary

factoring to express a polynomial as the product of monomials and polynomials

function a relation in which each element of the domain is paired with exactly one element in the range



Function **Not a Function**

polynomial a monomial or sum of monomials

LESSON 3-1

Operations with Polynomials

Then

- You evaluated powers.

Now

1. Multiply, divide, and simplify monomials and expressions involving powers.
2. Add, subtract, and multiply polynomials.

Why?

- The light from the Sun takes approximately 8 minutes to reach Earth. So if you are outside right now you are basking in sunlight that the Sun emitted approximately 8 minutes ago.

Light travels very fast, at a speed of about 3×10^8 meters per second. How long would it take light to get here from the Andromeda galaxy, which is approximately 2.367×10^{21} meters away?

New Vocabulary

simplify

degree of a polynomial

Mathematical Practices

Reason abstractly and quantitatively.

1 Multiply and Divide Monomials

To **simplify** an expression containing powers means to rewrite the expression without parentheses or negative exponents. Negative exponents are a way of expressing the multiplicative inverse of a number. The following table summarizes the properties of exponents.

Concept Summary Properties of Exponents

For any real numbers x and y, integers a and b:

Property	Definition	Examples
Product of Powers	$x^a \cdot x^b = x^{a+b}$	$3^2 \cdot 3^4 = 3^{2+4}$ or 3^6 $p^2 \cdot p^9 = p^{2+9}$ or p^{11}
Quotient of Powers	$\frac{x^a}{x^b} = x^{a-b}, x \neq 0$	$\frac{9^5}{9^2} = 9^{5-2}$ or 9^3 $\frac{b^6}{b^4} = b^{6-4}$ or b^2
Negative Exponent	$x^{-a} = \frac{1}{x^a}$ and $\frac{1}{x^{-a}} = x^a, x \neq 0$	$3^{-5} = \frac{1}{3^5}$ $\frac{1}{b^{-7}} = b^7$
Power of a Power	$(x^a)^b = x^{ab}$	$(3^3)^2 = 3^{3 \cdot 2}$ or 3^6 $(d^2)^4 = d^{2 \cdot 4}$ or d^8
Power of a Product	$(xy)^a = x^a y^a$	$(2k)^4 = 2^4 k^4$ or $16k^4$ $(ab)^3 = a^3 b^3$
Power of a Quotient	$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}, y \neq 0$, and $\left(\frac{x}{y}\right)^{-a} = \left(\frac{y}{x}\right)^a$ or $\frac{y^a}{x^a}, x \neq 0, y \neq 0$	$\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$ $\left(\frac{a}{b}\right)^{-5} = \frac{b^5}{a^5}$
Zero Power	$x^0 = 1, x \neq 0$	$7^0 = 1$

Recall that a *monomial* is a number, a variable, or an expression that is the product of one or more variables with nonnegative integer exponents.

When simplifying a monomial, check to be sure that it has been simplified fully.

KeyConcept Simplifying Monomials

A monomial expression is in simplified form when:

- there are no powers of powers,
- each base appears exactly once,
- all fractions are in simplest form, and
- there are no negative exponents.

Example 1 Simplify Expressions

Simplify each expression. Assume that no variable equals 0.

a. $(2a^{-2})(3a^3b^2)(c^{-2})$

$$(2a^{-2})(3a^3b^2)(c^{-2})$$
$$= 2\left(\frac{1}{a^2}\right)(3a^3b^2)\left(\frac{1}{c^2}\right)$$
$$= \left(\frac{2}{a \cdot a}\right)(3 \cdot a \cdot a \cdot a \cdot b \cdot b)\left(\frac{1}{c \cdot c}\right)$$
$$= \left(\frac{2}{a \cdot a}\right)(3 \cdot a \cdot a \cdot a \cdot b \cdot b)\left(\frac{1}{c \cdot c}\right)$$
$$= \frac{6ab^2}{c^2}$$

b. $\dfrac{q^2r^4}{q^7r^3}$

$$\frac{q^2r^4}{q^7r^3} = q^{2-7} \cdot r^{4-3}$$
$$= q^{-5}r$$
$$= \frac{r}{q^5}$$

c. $\left(\dfrac{-2a^4}{b^2}\right)^3$

$$\left(\frac{-2a^4}{b^2}\right)^3 = \frac{(-2a^4)^3}{(b^2)^3}$$
$$= \frac{(-2)^3(a^4)^3}{(b^2)^3}$$
$$= \frac{-8a^{12}}{b^6}$$

Problem-Solving Tip

Check You can always check your answer using the definition of exponents.

$$\frac{q^2}{q^7} = \frac{q \cdot q}{q \cdot q \cdot q \cdot q \cdot q \cdot q \cdot q}$$
$$= \frac{1}{q^5}$$

GuidedPractice

1A. $(2x^{-3}y^3)(-7x^5y^{-6})$

1B. $\dfrac{15c^5d^3}{-3c^2d^7}$

1C. $\left(\dfrac{a}{4}\right)^{-3}$

1D. $(-2x^3y^2)^5$

StudyTip

Power of 1 Remember that a variable with no exponent indicated can be written as a power of 1.

2 Operations With Polynomials

The **degree of a polynomial** is the degree of the monomial with the greatest degree.

Example 2 Degree of a Polynomial

Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial.

a. $\frac{1}{4}x^4y^3 - 8x^5$

This expression is a polynomial because each term is a monomial. The degree of the first term is 4 + 3 or 7, and the degree of the second term is 5. The degree of the polynomial is 7.

b. $\sqrt{x} + x + 4$

This expression is not a polynomial because $\sqrt{x}$ is not a monomial.

c. $x^{-3} + 2x^{-2} + 6$

This expression is not a polynomial because x^{-3} and x^{-2} are not monomials: $x^{-3} = \frac{1}{x^3}$ and $x^{-2} = \frac{1}{x^2}$. Monomials cannot contain variables in the denominator.

GuidedPractice

2A. $\frac{x}{y} + 3x^2$ **2B.** $x^5y + 9x^4y^3 - 2xy$

You can simplify a polynomial just like you simplify a monomial. Perform the operations indicated, and combine like terms.

StudyTip

Alternative Methods Notice that Example 3a uses a horizontal method, and Example 3b uses a vertical method to simplify. Either method will yield a correct solution.

Example 3 Simplify Polynomial Expressions

Simplify each expression.

a. $(4x^2 - 5x + 6) - (2x^2 + 3x - 1)$

Remove parentheses, and group like terms together.

$$\begin{aligned}(4x^2 - 5x + 6) - (2x^2 + 3x - 1) &= 4x^2 - 5x + 6 - 2x^2 - 3x + 1 \\ &= (4x^2 - 2x^2) + (-5x - 3x) + (6 + 1) \\ &= 2x^2 - 8x + 7\end{aligned}$$

b. $(6x^2 - 7x + 8) + (-4x^2 + 9x - 5)$

Align like terms vertically and add.

$$\begin{array}{r} 6x^2 - 7x + 8 \\ (+)\ -4x^2 + 9x - 5 \\ \hline 2x^2 + 2x + 3 \end{array}$$

GuidedPractice

3A. $(-x^2 - 3x + 4) - (x^2 + 2x + 5)$ **3B.** $(3x^2 - 6) + (-x + 1)$

Adding or subtracting integers results in an integer, so the set of integers is closed under addition and subtraction. Similarly, because adding or subtracting polynomials results in a polynomial, the set of polynomials is closed under addition and subtraction.

You can use the Distributive Property to multiply polynomials.

Example 4 Simplify by Using the Distributive Property

Find $3x(2x^2 - 4x + 6)$.

$$\begin{aligned} 3x(2x^2 - 4x + 6) &= 3x(2x^2) + 3x(-4x) + 3x(6) \\ &= 6x^3 - 12x^2 + 18x \end{aligned}$$

GuidedPractice

Find each product.

4A. $\frac{4}{3}x^2(6x^2 + 9x - 12)$

4B. $-2a(-3a^2 - 11a + 20)$

Real-WorldCareer

Truck Driver Truck drivers are considered technical professionals because they are required to obtain specialized education and professional licensure. Although state motor vehicle departments administer the Commercial Driver's License program, federal law spells out the requirements to obtain one.

Real-World Example 5 Write a Polynomial Expression

DRIVING **The Department of Transportation limits the time a truck driver can work between periods of rest to ten hours. For the first part of his shift, Khalid drives at a speed of 60 kilometers per hour, and for the second part of the shift, he drives at a speed of 70 kilometers per hour. Write a polynomial to represent the distance driven.**

Words 60 kmph for some time and 70 kmph for the rest

Variable Let x = the number of hours he drives at 60 kilometers per hour.

Expression $60 \quad x \quad + \quad 70 \quad (10 - x)$

$$\begin{aligned} &60x + 70(10 - x) \\ &= 60x + 700 - 70x \\ &= 700 - 10x \end{aligned}$$

The polynomial is $700 - 10x$.

GuidedPractice

5. Rashid has AED 900 to invest in a savings account that has an annual interest rate of 1.8%, and a money market account that pays 4.2% per year. Write a polynomial for the interest he will earn in one year if he invests x dirhams in the savings account.

Like addition and subtraction, polynomials are closed under multiplication.

Example 6 Multiply Polynomials

Find $(n^2 + 4n - 6)(n + 2)$.

$$\begin{aligned} &(n^2 + 4n - 6)(n + 2) \\ &= n^2(n + 2) + 4n(n + 2) + (-6)(n + 2) \\ &= n^2 \cdot n + n^2 \cdot 2 + 4n \cdot n + 4n \cdot 2 + (-6) \cdot n + (-6) \cdot 2 \\ &= n^3 + 2n^2 + 4n^2 + 8n - 6n - 12 \\ &= n^3 + 6n^2 + 2n - 12 \end{aligned}$$

GuidedPractice

Find each product.

6A. $(x^2 + 4x + 16)(x - 4)$

6B. $(2x^2 - 4x + 5)(3x - 1)$

Check Your Understanding

Example 1 **Simplify. Assume that no variable equals 0.**

1. $(2a^3b^{-2})(-4a^2b^4)$
2. $\dfrac{12x^4y^2}{2xy^5}$
3. $\left(\dfrac{2a^2}{3b}\right)^3$
4. $(6g^5h^{-4})^3$

Example 2 **Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial.**

5. $3x + 4y$
6. $\frac{1}{2}x^2 - 7y$
7. $x^2 + \sqrt{x}$
8. $\dfrac{ab^3 - 1}{az^4 + 3}$

Examples 3–4, and 6 **Simplify.**

9. $(x^2 - 5x + 2) - (3x^2 + x - 1)$
10. $(3a + 4b) + (6a - 6b)$
11. $2a(4b + 5)$
12. $3x^2(2xy - 3xy^2 + 4x^2y^3)$
13. $(n - 9)(n + 7)$
14. $(a + 4)(a - 6)$

Example 5

15. **EXERCISE** Eiman exercises 75 minutes a day. She does cardio, which burns an average of 10 Calories per minute, and weight training, which burns an average of 7.5 Calories per minute. Write a polynomial to represent the amount of Calories Eiman burns in one day if she does x minutes of weight training.

Practice and Problem Solving

Example 1 **Simplify. Assume that no variable equals 0.**

16. $(5x^3y^{-5})(4xy^3)$
17. $(-2b^3c)(4b^2c^2)$
18. $\dfrac{a^3n^7}{an^4}$
19. $\dfrac{-y^3z^5}{y^2z^3}$
20. $\dfrac{-7x^5y^5z^4}{21x^7y^5z^2}$
21. $\dfrac{9a^7b^5c^5}{18a^5b^9c^3}$
22. $(n^5)^4$
23. $(z^3)^6$

Example 2 **Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial.**

24. $2x^2 - 3x + 5$
25. $a^3 - 11$
26. $\dfrac{5np}{n^2} - \dfrac{2g}{h}$
27. $\sqrt{m - 7}$

Examples 3–4, and 6 **REGULARITY** **Simplify.**

28. $(6a^2 + 5a + 10) - (4a^2 + 6a + 12)$
29. $(7b^2 + 6b - 7) - (4b^2 - 2)$
30. $3p(np - z)$
31. $4x(2x^2 + y)$
32. $(x - y)(x^2 + 2xy + y^2)$
33. $(a + b)(a^3 - 3ab - b^2)$
34. $4(a^2 + 5a - 6) - 3(2a^3 + 4a - 5)$
35. $5c(2c^2 - 3c + 4) + 2c(7c - 8)$
36. $5xy(2x - y) + 6y^2(x^2 + 6)$
37. $3ab(4a - 5b) + 4b^2(2a^2 + 1)$
38. $(x - y)(x + y)(2x + y)$
39. $(a + b)(2a + 3b)(2x - y)$

Example 5

40. **PAINTING** Jamal has hired two painters to paint his house. The first painter charges AED 12 per hour and the second painter charges AED 11 per hour. It will take 15 hours of labor to paint the house.

    a. Write a polynomial to represent the total cost of the job if the first painter does x hours of the labor.

    b. Write a polynomial to represent the total cost of the job if the second painter does y hours of the labor.

Simplify. Assume that no variable equals 0.

41. $\left(\dfrac{8x^2y^3}{24x^3y^2}\right)^4$ **42.** $\left(\dfrac{12a^3b^5}{4a^6b^3}\right)^3$ **43.** $\left(\dfrac{4x^{-2}y^3}{xy^{-4}}\right)^{-2}$ **44.** $\left(\dfrac{5a^{-7}b^2}{ab^{-6}}\right)^{-3}$

45. $(a^2b^3)(ab)^{-2}$ **46.** $(-3x^3y)^2(4xy^2)$ **47.** $\dfrac{3c^2d(2c^3d^5)}{15c^4d^2}$

48. $\dfrac{-10g^6h^9(g^2h^3)}{30g^3h^3}$ **49.** $\dfrac{5x^4y^2(2x^5y^6)}{20x^3y^5}$ **50.** $\dfrac{-12n^7p^5(n^2p^4)}{36n^6p^7}$

51. ASTRONOMY Refer to the beginning of the lesson.

a. How long does it take light from Andromeda to reach Earth?

b. The average distance from the Sun to Mars is approximately 2.28×10^{11} meters. How long does it take light from the Sun to reach Mars?

Simplify.

52. $\frac{1}{4}g^2(8g + 12h - 16gh^2)$ **53.** $\frac{1}{3}n^3(6n - 9p + 18np^4)$ **54.** $x^{-2}(x^4 - 3x^3 + x^{-1})$

55. $a^{-3}b^2(ba^3 + b^{-1}a^2 + b^{-2}a)$ **56.** $(g^3 - h)(g^3 + h)$ **57.** $(n^2 - 7)(2n^3 + 4)$

58. $(2x - 2y)^3$ **59.** $(4n - 5)^3$ **60.** $(3z - 2)^3$

61. MODELING The polynomials $0.108x^2 - 0.876x + 474.1$ and $0.047x^2 + 9.694x + 361.7$ approximate the number of bachelor's degrees, in thousands, earned by males and females, respectively, where x is the number of years after 1971.

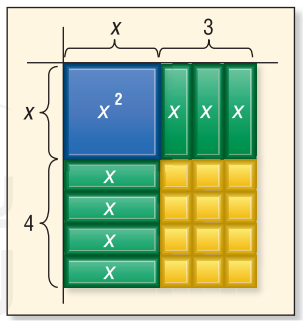
a. Find the polynomial that represents the total number of bachelor's degrees (in thousands) earned by both men and women.

b. Find the polynomial that represents the difference between bachelor's degrees earned by men and by women.

62. If $5^{k+7} = 5^{2k-3}$, what is the value of k?

63. What value of k makes $q^{41} = q^{4k} \cdot q^5$ true?

64. MULTIPLE REPRESENTATIONS Use the model at the right that represents the product of $x + 3$ and $x + 4$.



a. Geometric The area of each rectangle is the product of its length and width. Use the model to find the product of $x + 3$ and $x + 4$.

b. Algebraic Use FOIL to find the product of $x + 3$ and $x + 4$.

c. Verbal Explain how each term of the product is represented in the model.

H.O.T. Problems Use Higher-Order Thinking Skills

65. PROOF Show how the property of negative exponents can be proven using the Quotient of Powers Property and the Zero Power Property.

66. CHALLENGE What happens to the quantity of x^{-y} as y increases, for $y > 0$ and $x > 1$?

67. REASONING Explain why the expression 0^{-2} is undefined.

68. OPEN ENDED Write three different expressions that are equivalent to x^{12}.

69. WRITING IN MATH Explain why properties of exponents are useful in astronomy. Include an explanation of how to find the amount of time it takes for light from a source to reach a planet.

Standardized Test Practice

70. SHORT RESPONSE Simplify $\frac{(2x^2)^3}{12x^4}$.

71. STATISTICS For the numbers a, b, and c, the average (arithmetic mean) is twice the median. If $a = 0$ and $a < b < c$, what is the value of $\frac{c}{b}$?

A 2 **C** 4
B 3 **D** 5

72. Which is not a factor of $x^3 - x^2 - 2x$?

F x **H** $x - 1$
G $x + 1$ **J** $x - 2$

73. SAT/ACT The expression $(-6 + i)^2$ is equivalent to which of the following expressions?

A 35 **D** $35 - 12i$
B $-12i$ **E** $37 - 12i$
C $-12 + i$

Spiral Review

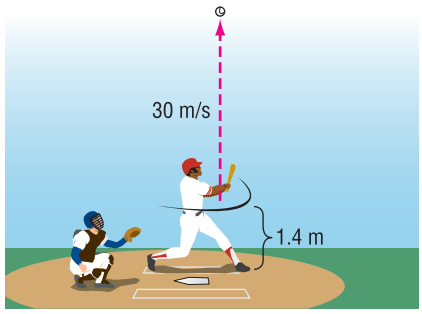
Evaluate each determinant.

74. $\begin{vmatrix} 6 & -3 \\ -1 & 8 \end{vmatrix}$ **75.** $\begin{vmatrix} -3 & -5 \\ -1 & -9 \end{vmatrix}$ **76.** $\begin{vmatrix} 8 & 6 \\ 4 & 3 \end{vmatrix}$

State whether each function is a linear function. Write *yes* or *no*. Explain.

77. $y = 4x^2 - 3x$ **78.** $y = -2x - 4$ **79.** $y = 4$

80. BASEBALL A baseball player hits a high pop-up with an initial upward velocity of 30 meters per second, 1.4 meters above the ground. The height $h(t)$ of the ball in meters t seconds after being hit is modeled by $h(t) = -4.9t^2 + 30t + 1.4$. How long does an opposing player have to get under the ball if he catches it 1.7 meters above the ground? Does your answer seem reasonable? Explain. (Lesson 3-3)



Evaluate each determinant.

81. $\begin{vmatrix} 3 & 0 & -2 \\ -1 & 4 & 3 \\ 5 & -2 & -1 \end{vmatrix}$ **82.** $\begin{vmatrix} -2 & -4 & -6 \\ 0 & 6 & -5 \\ -1 & 3 & -1 \end{vmatrix}$ **83.** $\begin{vmatrix} -3 & -1 & -2 \\ -2 & 3 & 4 \\ 6 & 1 & 0 \end{vmatrix}$

84. FINANCIAL LITERACY A couple is planning to invest AED 15,000 in certificates of deposit (CDs). For tax purposes, they want their total interest the first year to be AED 800. They want to put AED 1000 more in a 2-year CD than in a 1-year CD and then invest the rest in a 3-year CD. How much should they invest in each type of CD?

Years	1	2	3
Rate	3.4%	5.0%	6.0%

Find the slope of the line that passes through each pair of points.

85. (6, −2) and (−2, −9) **86.** (−4, −1) and (3, 8) **87.** (3, 0) and (−7, −5)

88. $\left(\frac{1}{2}, \frac{2}{3}\right)$ and $\left(\frac{1}{4}, \frac{1}{3}\right)$ **89.** $\left(\frac{2}{5}, \frac{1}{4}\right)$ and $\left(\frac{1}{10}, \frac{1}{12}\right)$ **90.** (−4.5, 2.5) and (−3, −1)

Skills Review

Factor each polynomial.

91. $12ax^3 + 20bx^2 + 32cx$ **92.** $x^2 + 2x + 6 + 3x$ **93.** $12y^2 + 9y + 8y + 6$

94. $2my + 7x + 7m + 2xy$ **95.** $8ax - 6x - 12a + 9$ **96.** $10x^2 - 14xy - 15x + 21y$

EXTEND 3-1

Algebra Lab
Dimensional Analysis

Real-world problems often involve units of measure. Performing operations with units is called **dimensional analysis** or **unit analysis**. You can use dimensional analysis to convert units or to perform calculations.

Example

A car is traveling at 65 miles per hour. How fast is the car traveling in meters per second?

You want to find the speed in meters per second, so you need to change the unit of distance from miles to meters and the unit of time from hours to seconds. To make the conversion, use fractions that you can multiply.

Step 1 Change the units of length from miles to meters.
Use the relationships of miles to feet and feet to meters.

$$\frac{65 \text{ miles}}{1 \text{ hour}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} \cdot \frac{1 \text{ meter}}{3.3 \text{ feet}}$$

Step 2 Change the units of time from hours to seconds.
Write fractions relating hours to minutes and minutes to seconds.

$$\frac{65 \text{ miles}}{1 \text{ hour}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} \cdot \frac{1 \text{ meter}}{3.3 \text{ feet}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}}$$

Step 3 Simplify and check by canceling the units.

$$\frac{65 \text{ miles}}{1 \text{ hour}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} \cdot \frac{1 \text{ meter}}{3.3 \text{ feet}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \cdot \frac{1 \text{ minute}}{60 \text{ seconds}}$$

$$= \frac{65 \cdot 5280}{3.3 \cdot 60 \cdot 60} \text{ m/s}$$

$$\approx 28.9 \text{ m/s}$$

So, 65 miles per hour is about 28.9 meters per second. This answer is reasonable because the final units are m/s, not m/hr, ft/s, or mi/hr.

Exercises

Solve each problem by using dimensional analysis. Include the appropriate units with your answer.

1. A horse can run 40 kilometers per hour. How far can a horse run in 3 minutes?
2. A cyclist traveled 43.2 kilometers at an average speed of 12 kilometers per hour. How long did the cyclist ride?
3. If you are driving 50 kilometers per hour, how many meters per second are you traveling?
4. The equation $d = \frac{1}{2}(9.8 \text{ m/s}^2)(3.5 \text{ s})^2$ represents the distance d that a ball falls 3.5 seconds after it is dropped from a tower. Find the distance.
5. **WRITING IN MATH** Explain how dimensional analysis can be useful in checking the reasonableness of your answer.

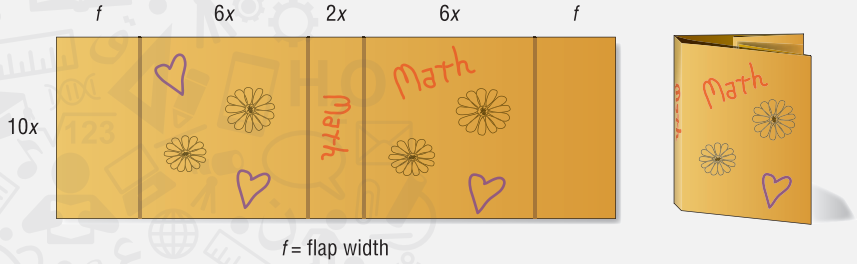
LESSON 3-2

Dividing Polynomials

Then
- You divided monomials.

Now
1. Divide polynomials using long division.
2. Divide polynomials using synthetic division.

Why?
- Suha needed $140x^2 + 60x$ square centimeters of paper to make a book jacket $10x$ centimeters tall. In figuring the area, she allowed for a front and back flap. If the spine is $2x$ centimeters wide, and the front and back are $6x$ centimeters wide, how wide are the front and back flaps? You can use a quotient of polynomials to help you find the answer.



New Vocabulary
synthetic division

Mathematical Practices
Attend to precision.

1 Long Division

In Lesson 3-1, you learned how to divide monomials. You can divide a polynomial by a monomial by using those same skills.

Example 1 Divide a Polynomial by a Monomial

Simplify $\dfrac{6x^4y^3 + 12x^3y^2 - 18x^2y}{3xy}$.

$$\frac{6x^4y^3 + 12x^3y^2 - 18x^2y}{3xy} = \frac{6x^4y^3}{3xy} + \frac{12x^3y^2}{3xy} - \frac{18x^2y}{3xy}$$
$$= \frac{6}{3} \cdot x^{4-1}y^{3-1} + \frac{12}{3} \cdot x^{3-1}y^{2-1} - \frac{18}{3} \cdot x^{2-1}y^{1-1}$$
$$= 2x^3y^2 + 4x^2y - 6x$$

Guided Practice **Simplify.**

1A. $(20c^4d^2f - 16cdf^2 + 4cdf) \div (4cdf)$

1B. $(18x^2y + 27x^3y^2z)(3xy)^{-1}$

You can use a process similar to long division to divide a polynomial by a polynomial with more than one term. The process is known as the *division algorithm*.

Example 2 Division Algorithm

Use long division to find $(x^2 + 3x - 40) \div (x - 5)$.

$$\begin{array}{r} x + 8 \\ x - 5 \overline{)\, x^2 + 3x - 40} \\ (-)\ x^2 - 5x \\ \hline 8x - 40 \\ (-)\ 8x - 40 \\ \hline 0 \end{array}$$

The quotient is $x + 8$. The remainder is 0.

Guided Practice **Use long division to find each quotient.**

2A. $(x^2 + 7x - 30) \div (x - 3)$

2B. $(x^2 - 13x + 12) \div (x - 1)$

Just as with the division of whole numbers, the division of two polynomials may result in a quotient with a remainder. Remember that $11 \div 3 = 3 + \text{R}2$, which is often written as $3\frac{2}{3}$. The result of a division of polynomials with a remainder can be written in a similar manner.

Standardized Test Example 3 Divide Polynomials

Which expression is equal to $(a^2 + 7a - 11)(3 - a)^{-1}$?

A $a + 10 - \frac{19}{3 - a}$

B $-a + 10$

C $-a - 10 + \frac{19}{3 - a}$

D $-a - 10 - \frac{19}{3 - a}$

Test-TakingTip

Multiple Choice You may be able to eliminate some of the answer choices by substituting the same value for a in the original expression and the answer choices, and then evaluate.

Read the Test Item

Since the second factor has an exponent of -1, this is a division problem.

$(a^2 + 7a - 11)(3 - a)^{-1} = \frac{a^2 + 7a - 11}{3 - a}$

Solve the Test Item

$$
\begin{array}{r}
-a - 10 \\
-a + 3 \overline{)\, a^2 + 7a - 11} \\
\underline{(-)\ a^2 - 3a} \\
10a - 11 \\
\underline{(-)\ 10a - 30} \\
19
\end{array}
$$

The quotient is $-a - 10$, and the remainder is 19.

Therefore, $(a^2 + 7a - 11)(3 - a)^{-1} = -a - 10 + \frac{19}{3 - a}$. The answer is **C**.

GuidedPractice

3. Which expression is equal to $(r^2 + 5r + 7)(1 - r)^{-1}$?

F $-r - 6 + \frac{13}{1 - r}$

G $r + 6$

H $r - 6 + \frac{13}{1 - r}$

J $r + 6 - \frac{13}{1 - r}$

2 Synthetic Division

Synthetic division is a simpler process for dividing a polynomial by a binomial. Suppose you want to divide $2x^3 - 13x^2 + 26x - 24$ by $x - 4$ using long division. Compare the coefficients in this division with those in Example 4.

$$
\begin{array}{r}
2x^2 - 5x + 6 \\
x - 4 \overline{)\, 2x^3 - 13x^2 + 26x - 24} \\
\underline{(-)\ 2x^3 - 8x^2} \\
-5x^2 + 26x \\
\underline{(-)\ -5x^2 + 20x} \\
6x - 24 \\
\underline{(-)\ 6x - 24} \\
0
\end{array}
$$

When the polynomial in the dividend is missing a term, a zero must be used to represent the missing term. So, with a dividend of $2x^3 - 4x^2 + 6$, a 0 will be used as a placeholder for the x-term.

$$\overline{)\, 2x^3 - 4x^2 + 0x + 6}$$

KeyConcept Synthetic Division

Step 1 Write the coefficients of the dividend so that the degrees of the terms are in descending order. Write the constant r of the divisor $x - r$ in the box. Bring the first coefficient down.

Step 2 Multiply the first coefficient by r, and write the product under the second coefficient.

Step 3 Add the product and the second coefficient.

Step 4 Repeat Steps 2 and 3 until you reach a sum in the last column. The numbers along the bottom row are the coefficients of the quotient. The power of the first term is one less than the degree of the dividend. The final number is the remainder.

Example 4 Synthetic Division

Use synthetic division to find $(2x^3 - 13x^2 + 26x - 24) \div (x - 4)$.

Step 1 Write the coefficients of the dividend. Write the constant r in the box. In this case, $r = 4$. Bring the first coefficient, 2, down.

4	2	−13	26	−24
	↓			
	2			|

Step 2 Multiply the first coefficient by r: $2 \cdot 4 = 8$. Write the product under the second coefficient.

4	2	−13	26	−24
		8		
	2 ↗			|

Step 3 Add the product and the second coefficient: $-13 + 8 = -5$.

4	2	−13	26	−24
		8		
	2	−5		|

Step 4 Multiply the sum, -5, by r: $-5 \times 4 = -20$. Write the product under the next coefficient, and add: $26 + (-20) = 6$. Multiply the sum, 6, by r: $6 \cdot 4 = 24$. Write the product under the next coefficient and add: $-24 + 24 = 0$.

4	2	−13	26	−24
		8	−20	24
	2	−5 ↗	6 ↗	| 0

CHECK Multiply the quotient by the divisor. The answer should be the dividend.

$$\begin{array}{rrrr} & 2x^2 - & 5x + & 6 \\ (\times) & & x - & 4 \\ \hline & -8x^2 + & 20x - & 24 \\ 2x^3 - & 5x^2 + & 6x & \\ \hline 2x^3 - & 13x^2 + & 26x - & 24 \end{array}$$

The quotient is $2x^2 - 5x + 6$. The remainder is 0.

WatchOut!
Synthetic Division
Remember to *add* terms when performing synthetic division.

GuidedPractice

Use synthetic division to find each quotient.

4A. $(2x^3 + 3x^2 - 4x + 15) \div (x + 3)$

4B. $(3x^3 - 8x^2 + 11x - 14) \div (x - 2)$

4C. $(4a^4 + 2a^2 - 4a + 12) \div (a + 2)$

4D. $(6b^4 - 8b^3 + 12b - 14) \div (b - 2)$

To use synthetic division, the divisor must be of the form $x - r$. If the coefficient of x in a divisor is not 1, you can rewrite the division expression so that you can use synthetic division.

WatchOut!

Precision Remember to divide *all* terms in the numerator and denominator.

Example 5 Divisor with First Coefficient Other than 1

Use synthetic division to find $(3x^4 - 5x^3 + x^2 + 7x) \div (3x + 1)$.

$$\frac{3x^4 - 5x^3 + x^2 + 7x}{3x + 1} = \frac{(3x^4 - 5x^3 + x^2 + 7x) \div 3}{(3x + 1) \div 3}$$

$$= \frac{x^4 - \frac{5}{3}x^3 + \frac{1}{3}x^2 + \frac{7}{3}x}{x + \frac{1}{3}}$$

Since the numerator does not have a constant term, use a coefficient of 0 for the constant term.

$x - r = x + \frac{1}{3}$, so $r = -\frac{1}{3}$. $\rightarrow$

$-\frac{1}{3}$	1	$-\frac{5}{3}$	$\frac{1}{3}$	$\frac{7}{3}$	0
		$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$
	1	-2	1	2	$-\frac{2}{3}$

The result is $x^3 - 2x^2 + x + 2 - \frac{\frac{2}{3}}{x + \frac{1}{3}}$. Now simplify the fraction.

$$\frac{\frac{2}{3}}{x + \frac{1}{3}} = \frac{2}{3} \div \left(x + \frac{1}{3}\right)$$

$$= \frac{2}{3} \div \frac{3x + 1}{3}$$

$$= \frac{2}{3} \cdot \frac{3}{3x + 1}$$

$$= \frac{2}{3x + 1}$$

The solution is $x^3 - 2x^2 + x + 2 - \frac{2}{3x + 1}$.

CHECK Divide using long division.

$$\begin{array}{r} x^3 - 2x^2 + x + 2 \\ 3x + 1 \overline{)\,3x^4 - 5x^3 + x^2 + 7x} \\ \underline{(-)\ 3x^4 + x^3} \\ -6x^3 + x^2 \\ \underline{(-)\ -6x^3 - 2x^2} \\ 3x^2 + 7x \\ \underline{(-)\ 3x^2 + x} \\ 6x + 0 \\ \underline{(-)\ 6x + 2} \\ -2 \end{array}$$

The result is $x^3 - 2x^2 + x + 2 - \frac{2}{3x + 1}$. ✓

GuidedPractice

Use synthetic division to find each quotient.

5A. $(8x^4 - 4x^2 + x + 4) \div (2x + 1)$

5B. $(8y^5 - 2y^4 - 16y^2 + 4) \div (4y - 1)$

5C. $(15b^3 + 8b^2 - 21b + 6) \div (5b - 4)$

5D. $(6c^3 - 17c^2 + 6c + 8) \div (3c - 4)$

Check Your Understanding

Examples 1, 2, and 4 Simplify.

1. $\frac{4xy^2 - 2xy + 2x^2y}{xy}$
2. $(3a^2b - 6ab + 5ab^2)(ab)^{-1}$
3. $(x^2 - 6x - 20) \div (x + 2)$
4. $(2a^2 - 4a - 8) \div (a + 1)$
5. $(3z^4 - 6z^3 - 9z^2 + 3z - 6) \div (z + 3)$
6. $(y^5 - 3y^2 - 20) \div (y - 2)$

Example 3

7. **MULTIPLE CHOICE** Which expression is equal to $(x^2 + 3x - 9)(4 - x)^{-1}$?

   A $-x - 7 + \frac{19}{4 - x}$ **B** $-x - 7$ **C** $x + 7 - \frac{19}{4 - x}$ **D** $-x - 7 - \frac{19}{4 - x}$

Example 5 Simplify.

8. $(10x^2 + 15x + 20) \div (5x + 5)$
9. $(18a^2 + 6a + 9) \div (3a - 2)$
10. $\frac{12b^2 + 23b + 15}{3b + 8}$
11. $\frac{27y^2 + 27y - 30}{9y - 6}$

Practice and Problem Solving

Example 1 Simplify.

12. $\frac{24a^3b^2 - 16a^2b^3}{8ab}$
13. $\frac{5x^2y - 10xy + 15xy^2}{5xy}$
14. $\frac{7g^3h^2 + 3g^2h - 2gh^3}{gh}$
15. $\frac{4a^3b - 6ab + 2ab^2}{2ab}$
16. $\frac{16c^4d^4 - 24c^2d^2}{4c^2d^2}$
17. $\frac{9n^3p^3 - 18n^2p^2 + 21n^2p^3}{3n^2p^2}$

18. **ENERGY** Compact fluorescent light (CFL) bulbs reduce energy waste. The amount of energy waste that is reduced each day in a certain community can be estimated by $-b^2 + 8b$, where b is the number of bulbs. Divide by b to find the average amount of energy saved per CFL bulb.

19. **BAKING** The number of cookies produced in a factory each day can be estimated by $-w^2 + 16w + 1000$, where w is the number of workers. Divide by w to find the average number of cookies produced per worker.

Examples 2, 4, and 5 Simplify.

20. $(a^2 - 8a - 26) \div (a + 2)$
21. $(b^3 - 4b^2 + b - 2) \div (b + 1)$
22. $(z^4 - 3z^3 + 2z^2 - 4z + 4)(z - 1)^{-1}$
23. $(x^5 - 4x^3 + 4x^2) \div (x - 4)$
24. $\frac{y^3 + 11y^2 - 10y + 6}{y + 2}$
25. $(g^4 - 3g^2 - 18) \div (g - 2)$
26. $(6a^2 - 3a + 9) \div (3a - 2)$
27. $\frac{6x^5 + 5x^4 + x^3 - 3x^2 + x}{3x + 1}$
28. $\frac{4g^4 - 6g^3 + 3g^2 - g + 12}{4g - 4}$
29. $(2b^3 - 6b^2 + 8b) \div (2b + 2)$
30. $(6z^6 + 3z^4 - 9z^2)(3z - 6)^{-1}$
31. $(10y^6 + 5y^5 + 10y^3 - 20y - 15)(5y + 5)^{-1}$

32. **REASONING** A rectangular box for a new product is designed in such a way that the three dimensions always have a particular relationship defined by the variable x. The volume of the box can be written as $6x^3 + 31x^2 + 53x + 30$, and the height is always $x + 2$. What are the width and length of the box?

33. **PHYSICS** The voltage V is related to current I and power P by the equation $V = \frac{P}{I}$. The power of a generator is modeled by $P(t) = t^3 + 9t^2 + 26t + 24$. If the current of the generator is $I = t + 4$, write an expression that represents the voltage.

34. ENTERTAINMENT A magician gives these instructions to a volunteer.

- Choose a number and multiply it by 4.
- Then add the sum of your number and 15 to the product you found.
- Now divide by the sum of your number and 3.

a. What number will the volunteer always have at the end?

b. Explain the process you used to discover the answer.

35. BUSINESS The number of magazine subscriptions sold can be estimated by $n = \frac{3500a^2}{a^2 + 100}$, where a is the amount of money the company spent on advertising in hundreds of dirhams and n is the number of subscriptions sold.

a. Perform the division indicated by $\frac{3500a^2}{a^2 + 100}$.

b. About how many subscriptions will be sold if AED 1500 is spent on advertising?

Simplify.

36. $(x^4 - y^4) \div (x - y)$

37. $(28c^3d^2 - 21cd^2) \div (14cd)$

38. $(a^3b^2 - a^2b + 2b)(-ab)^{-1}$

39. $\frac{n^3 + 3n^2 - 5n - 4}{n + 4}$

40. $\frac{p^3 + 2p^2 - 7p - 21}{p + 3}$

41. $\frac{3z^5 + 5z^4 + z + 5}{z + 2}$

42. MULTIPLE REPRESENTATIONS Consider a rectangle with area $2x^2 + 7x + 3$ and length $2x + 1$.

a. Concrete Use algebra tiles to represent this situation. Use the model to find the width.

b. Symbolic Write an expression to represent the model.

c. Numerical Solve this problem algebraically using synthetic or long division. Does your concrete model check with your algebraic model?

H.O.T. Problems Use Higher-Order Thinking Skills

43. ERROR ANALYSIS Shaikha and Jamal are dividing $2x^3 - 4x^2 + 3x - 1$ by $x - 3$. Shaikha claims that the remainder is 26. Jamal argues that the remainder is -100. Is either of them correct? Explain your reasoning.

44. CHALLENGE If a polynomial is divided by a binomial and the remainder is 0, what does this tell you about the relationship between the binomial and the polynomial?

45. REASONING Review any of the division problems in this lesson. What is the relationship between the degrees of the dividend, the divisor, and the quotient?

46. OPEN ENDED Write a quotient of two polynomials for which the remainder is 3.

47. ARGUMENTS Identify the expression that does not belong with the other three. Explain your reasoning.

$3xy + 6x^2$	$\frac{5}{x^2}$	$x + 5$	$5b + 11c - 9ad^2$

48. WRITING IN MATH Use the information at the beginning of the lesson to write assembly instructions using the division of polynomials to make a paper cover for your textbook.

Standardized Test Practice

49. An office employs x women and 3 men. What is the ratio of the total number of employees to the number of women?

A $\frac{x+3}{x}$ C $\frac{3}{x}$

B $\frac{x}{x+3}$ D $\frac{x}{3}$

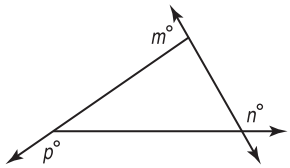
50. SAT/ACT Which polynomial has degree 3?

F $x^3 + x^2 - 2x^4$ J $x^2 + x + 12^3$

G $-2x^2 - 3x + 4$ K $1 + x + x^3$

H $3x - 3$

51. GRIDDED RESPONSE In the figure below, $m + n + p = ?$



52. $(-4x^2 + 2x + 3) - 3(2x^2 - 5x + 1) =$

A $2x^2$ C $-10x^2 + 17x$

B $-10x^2$ D $2x^2 + 17x$

Spiral Review

Simplify. (Lesson 3-1)

53. $(5x^3 + 2x^2 - 3x + 4) - (2x^3 - 4x)$

54. $(2y^3 - 3y + 8) + (3y^2 - 6y)$

55. $4a(2a - 3) + 3a(5a - 4)$

56. $(c + d)(c - d)(2c - 3d)$

57. $(xy)^2(2xy^2z)^3$

58. $(3ab^2)^{-2}(2a^2b)^2$

Determine whether each pair of matrices are inverses of each other.

59. $\begin{bmatrix} 4 & -3 \\ -1 & -6 \end{bmatrix}$ and $\begin{bmatrix} \frac{3}{13} & -\frac{1}{18} \\ -\frac{1}{26} & -\frac{2}{13} \end{bmatrix}$

60. $\begin{bmatrix} 6 & -3 \\ 4 & 8 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{10} & \frac{1}{20} \\ -\frac{1}{15} & \frac{2}{15} \end{bmatrix}$

61. $\begin{bmatrix} 2 & 4 \\ -3 & -2 \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ \frac{3}{8} & \frac{1}{4} \end{bmatrix}$

Solve each system of equations.

62. $4x - 7y = -9$
$5x + 2y = -22$

63. $8y - 2x = 38$
$5x - 3y = -27$

64. $3x + 8y = 24$
$-16y - 6x = 48$

Solve each inequality.

65. $3x - 6 \leq -14$

66. $6 - 4x \leq 2$

67. $-6x + 3 \geq 3x - 16$

68. BUSINESS A landscaper can mow a lawn in 30 minutes and perform a small landscape job in 90 minutes. He works at most 10 hours per day, 5 days per week. He earns AED 35 per lawn and AED 125 per landscape job. He cannot do more than 3 landscape jobs per day. Find the combination of lawns mowed and completed landscape jobs per week that will maximize income. Then find the maximum income.

Skills Review

Find each value if $f(x) = 4x + 3$, $g(x) = -x^2$, and $h(x) = -2x^2 - 2x + 4$.

69. $f(-6)$

70. $g(-8)$

71. $h(3)$

72. $f(c)$

73. $g(3d)$

74. $h(2b + 1)$

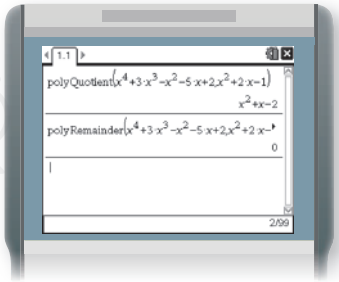
EXTEND

3-2 Graphing Technology Lab: Dividing Polynomials

Long division and synthetic division are two alternatives for dividing polynomials with linear divisors. You can use a graphing calculator with a computer algebra system (CAS) to divide polynomials with any divisor.

Activity 1 Divide Polynomials Without Remainders

Use CAS to find $(x^4 + 3x^3 - x^2 - 5x + 2) \div (x^2 + 2x - 1)$.



Step 1 Add a new **Calculator** page.

Step 2 From the menu, select **Algebra**, then **Polynomial Tools** and **Quotient of Polynomial**.

Step 3 Type the dividend, a comma, and the divisor.

The CAS indicates that $(x^4 + 3x^3 - x^2 - 5x + 2) \div (x^2 + 2x - 1)$ is $x^2 + x - 2$.

Step 4 To verify that there is no remainder, select **Remainder of a Polynomial** from the **Algebra, Polynomial Tools** menu then type the dividend, a comma, and the divisor.

In Activity 1, there was no remainder. But in many cases, there will be a remainder.

Activity 2 Divide Polynomials With Remainders

Use CAS to find $(4x^5 - 12x^4 - 7x^3 + 32x^2 + 3x + 20) \div (x^2 - 2x + 4)$.

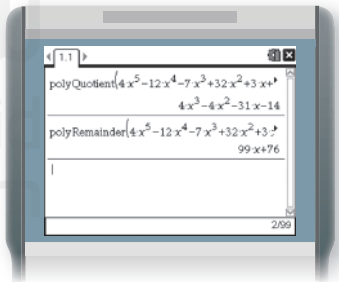
Step 1 Add a new **Calculator** page.

Step 2 From the menu, select **Algebra, Polynomial Tools,** and **Quotient of Polynomial**.

Step 3 Type the dividend, a comma, and the divisor.

The CAS indicates that $(4x^5 - 12x^4 - 7x^3 + 32x^2 + 3x + 20) \div (x^2 - 2x + 4)$ is $4x^3 - 4x^2 - 31x + 14$.

We need to determine whether there is a remainder.



Step 4 Use the **Remainder of a Polynomial** option from the **Algebra, Polynomial Tools** menu to determine the remainder. Then type the dividend, a comma, and the divisor.

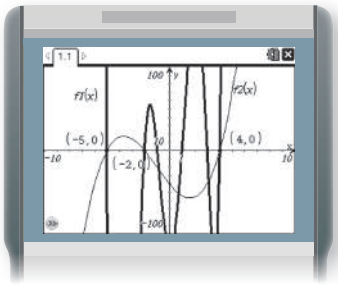
The remainder is $99x + 76$.
Therefore, $(4x^5 - 12x^4 - 7x^3 + 32x^2 + 3x + 20) \div (x^2 - 2x + 4)$ is $4x^3 - 4x^2 - 31x - 14 + \frac{99x + 76}{x^2 - 2x + 4}$.

You can also use a graphing calculator to determine roots of a polynomial so you can divide with synthetic division.

Activity 3 Divide with Synthetic Division

Use synthetic division to find $(x^6 - 28x^4 + 14x^3 + 147x^2 - 14x - 120) \div (x^3 + 3x^2 - 18x - 40)$.

Step 1 Graph the dividend as $f1(x)$ and the divisor as $f2(x)$ on the same calculator page. Use the **intersection points** tool from **Points & Lines** menu to find where the graphs have the same x-intercepts.

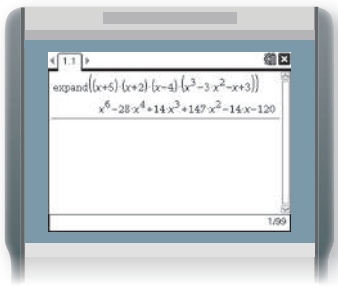


[−10, 10] scl: 1 by [−100, 100] scl: 10

Step 2 Use the roots from Step 1 as the divisors for synthetic division.

−5	1	0	−28	14	147	−14	−120
		−5	25	15	−145	−10	120
	1	−5	−3	29	2	−24	0
−2	1	−5	−3	29	2	−24	
		−2	14	−22	−14	24	
	1	−7	11	7	−12	0	
4	1	−7	11	7	−12		
		4	−12	−4	12		
	1	−3	−1	3	0		

Step 3 Use the **Expand** function to verify that $x^3 - 3x^2 - x + 3$ is the quotient when -5, -2, and 4 are roots.



Thus, $(x^6 - 28x^4 + 14x^3 + 147x^2 - 14x - 120) \div (x^3 + 3x^2 - 18x - 40)$ is $x^3 - 3x^2 - x + 3$.

Exercises

Find each quotient.

1. $(2x^4 + x^3 - 8x^2 + 17x - 12) \div (x^2 + 2x - 3)$
2. $(x^4 + 7x^3 + 8x^2 + x - 12) \div (x^2 + 3x - 4)$
3. $(9x^5 - 9x^3 - 5x^2 + 5) \div (9x^3 - 5)$
4. $(x^5 - 8x^4 + 10x^3 + 14x^2 + 61x - 30) \div (x^2 - 5x + 3)$
5. $(2x^6 + 2x^5 - 4x^4 - 18x^3 - 16x^2 + 8x + 16) \div (2x^3 + 2x^2 - 4x - 2)$
6. $(6x^6 - 2x^5 - 14x^4 + 10x^3 - 4x^2 - 28x - 5) \div (3x^3 - x^2 - 7x - 1)$
7. Use synthetic division to find $(x^6 - 7x^5 - 21x^4 + 175x^3 + 56x^2 - 924x + 720) \div (x^3 - 5x^2 - 12x + 36)$.

LESSON 3-3

Polynomial Functions

Then
- You analyzed graphs of quadratic functions.

Now
1. Evaluate polynomial functions.
2. Identify general shapes of graphs of polynomial functions.

Why?
- The volume of air in the lungs during a 5-second respiratory cycle can be modeled by $v(t) = -0.037t^3 + 0.152t^2 + 0.173t$, where v is the volume in liters and t is the time in seconds. This model is an example of a polynomial function.

New Vocabulary
- polynomial in one variable
- leading coefficient
- polynomial function
- power function
- quartic function
- quintic function
- end behavior

Mathematical Practices
Make sense of problems and persevere in solving them.

1 Polynomial Functions

A **polynomial in one variable** is an expression of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, where $a_n \neq 0$, a_{n-1}, a_2, a_1, and a_0 are real numbers, and n is a nonnegative integer.

The polynomial is written in standard form when the values of the exponents are in descending order. The degree of the polynomial is the value of the greatest exponent. The coefficient of the first term of a polynomial in standard form is called the **leading coefficient**.

Polynomial	Expression	Degree	Leading Coefficient
Constant	12	0	12
Linear	$4x - 9$	1	4
Quadratic	$5x^2 - 6x - 9$	2	5
Cubic	$8x^3 + 12x^2 - 3x + 1$	3	8
General	$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$	n	a_n

Example 1 Degrees and Leading Coefficients

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

a. $8x^5 - 4x^3 + 2x^2 - x - 3$

This is a polynomial in one variable. The greatest exponent is 5, so the degree is 5 and the leading coefficient is 8.

b. $12x^2 - 3xy + 8x$

This is not a polynomial in one variable. There are two variables, x and y.

c. $3x^4 + 6x^3 - 4x^8 + 2x$

This is a polynomial in one variable. The greatest exponent is 8, so the degree is 8 and the leading coefficient is -4.

Guided Practice

1A. $5x^3 - 4x^2 - 8x + \frac{4}{x}$ **1B.** $5x^6 - 3x^4 + 12x^3 - 14$ **1C.** $8x^4 - 2x^3 - x^6 + 3$

A **polynomial function** is a continuous function that can be described by a polynomial equation in one variable. For example, $f(x) = 3x^3 - 4x + 6$ is a cubic polynomial function. The simplest polynomial functions of the form $f(x) = ax^b$ where a and b are non zero real numbers are called **power functions**.

If you know an element in the domain of any polynomial function, you can find the corresponding value in the range.

Real-World Example 2 Evaluate a Polynomial Function

RESPIRATION **Refer to the beginning of the lesson. Find the volume of air in the lungs 2 seconds into the respiratory cycle.**

By substituting 2 into the function we can find $v(2)$, the volume of air in the lungs 2 seconds into the respiratory cycle.

$$\begin{aligned} v(t) &= -0.037t^3 + 0.152t^2 + 0.173t \\ v(2) &= -0.037(2)^3 + 0.152(2)^2 + 0.173(2) \\ &= -0.296 + 0.608 + 0.346 \\ &= 0.658 \text{ L} \end{aligned}$$

GuidedPractice

2. Find the volume of air in the lungs 4 seconds into the respiratory cycle.

You can also evaluate functions for variables and algebraic expressions.

Example 3 Function Values of Variables

Find $f(3c - 4) - 5f(c)$ if $f(x) = x^2 + 2x - 3$.

To evaluate $f(3c - 4)$, replace the x in $f(x)$ with $3c - 4$.

$$\begin{aligned} f(x) &= x^2 + 2x - 3 \\ f(3c - 4) &= (3c - 4)^2 + 2(3c - 4) - 3 \\ &= 9c^2 - 24c + 16 + 6c - 8 - 3 \\ &= 9c^2 - 18c + 5 \end{aligned}$$

To evaluate $5f(c)$, replace x with c in $f(x)$, then multiply by 5.

$$\begin{aligned} f(x) &= x^2 + 2x - 3 \\ 5f(c) &= 5(c^2 + 2c - 3) \\ &= 5c^2 + 10c - 15 \end{aligned}$$

Now evaluate $f(3c - 4) - 5f(c)$.

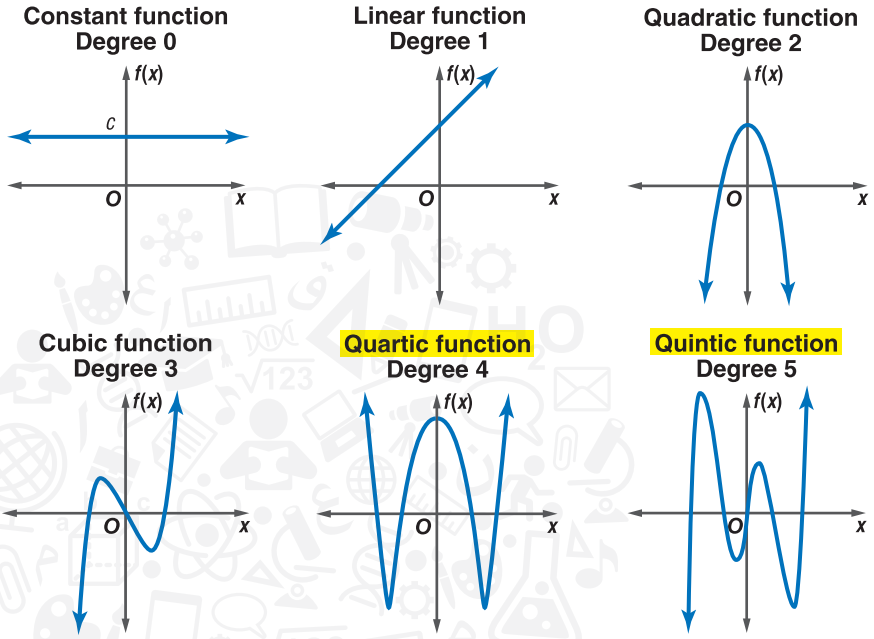
$$\begin{aligned} f(3c - 4) - 5f(c) &= (9c^2 - 18c + 5) - (5c^2 + 10c - 15) \\ &= 9c^2 - 18c + 5 - 5c^2 - 10c + 15 \\ &= 4c^2 - 28c + 20 \end{aligned}$$

GuidedPractice

3A. Find $g(5a - 2) + 3g(2a)$ if $g(x) = x^2 - 5x + 8$.

3B. Find $h(-4d + 3) - 0.5h(d)$ if $h(x) = 2x^2 + 5x + 3$.

2 Graphs of Polynomial Functions The general shapes of the graphs of several polynomial functions show the *maximum* number of times the graph of each function may intersect the x-axis. This is the same number as the degree of the polynomial.



StudyTip

Sense-Making The leading coefficient and degree are the sole determining factors for the end behavior of a polynomial function. With very large or very small numbers, the rest of the polynomial is insignificant in the appearance of the graph.

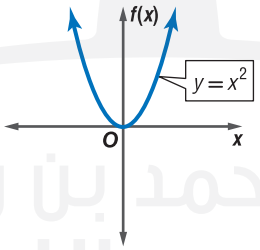
The domain of any polynomial function is all real numbers. The **end behavior** is the behavior of the graph of $f(x)$ as x approaches positive infinity ($x \to +\infty$) or negative infinity ($x \to -\infty$). The degree and leading coefficient of a polynomial function determine the end behavior of the graph and the range of the function.

KeyConcept End Behavior of a Polynomial Function

Degree: even
Leading Coefficient: positive
End Behavior:

$f(x) \to +\infty$ as $x \to -\infty$

$f(x) \to +\infty$ as $x \to +\infty$

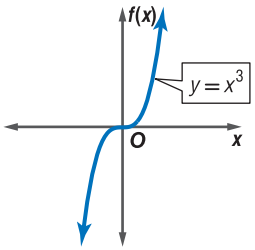


Domain: all real numbers
Range: all real numbers ≥ minimum

Degree: odd
Leading Coefficient: positive
End Behavior:

$f(x) \to -\infty$ as $x \to -\infty$

$f(x) \to +\infty$ as $x \to +\infty$

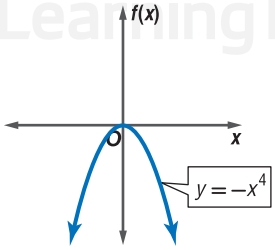


Domain: all real numbers
Range: all real numbers

Degree: even
Leading Coefficient: negative
End Behavior:

$f(x) \to -\infty$ as $x \to -\infty$

$f(x) \to -\infty$ as $x \to +\infty$

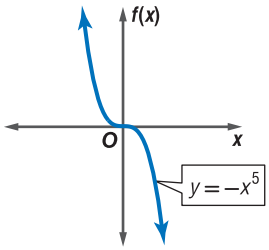


Domain: all real numbers
Range: all real numbers ≤ maximum

Degree: odd
Leading Coefficient: negative
End Behavior:

$f(x) \to +\infty$ as $x \to -\infty$

$f(x) \to -\infty$ as $x \to +\infty$



Domain: all real numbers
Range: all real numbers

ReviewVocabulary

infinity endless or boundless

ReviewVocabulary

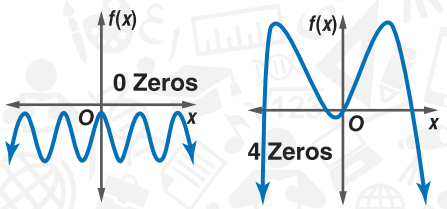
zero the *x*-coordinate of the point at which a graph intersects the *x*-axis

The number of real zeros of a polynomial function can be determined by examining its graph. Recall that real zeros occur at *x*-intercepts, so the number of times a graph crosses the *x*-axis equals the number of real zeros.

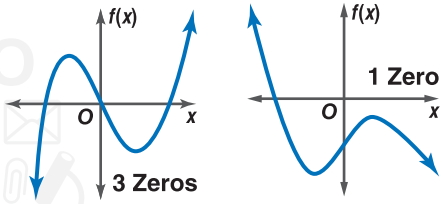
KeyConcept Zeros of Even- and Odd-Degree Functions

Odd-degree functions will always have an odd number of real zeros. Even-degree functions will always have an even number of real zeros or no real zeros at all.

Even-Degree Polynomials



Odd-Degree Polynomials



StudyTip

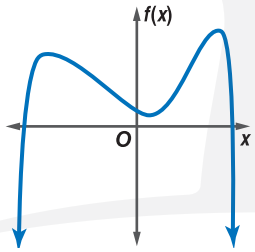
Double roots When a graph is tangent to the *x*-axis, there is a *double root*, which represents two of the same root.

Example 4 Graphs of Polynomial Functions

For each graph,

- **describe the end behavior,**
- **determine whether it represents an odd-degree or an even-degree polynomial function, and**
- **state the number of real zeros.**

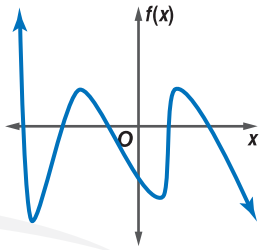
a.



$f(x) \to -\infty$ as $x \to -\infty$.
$f(x) \to -\infty$ as $x \to +\infty$.

Since the end behavior is in the same direction, it is an even-degree function. The graph intersects the *x*-axis at two points, so there are two real zeros.

b.

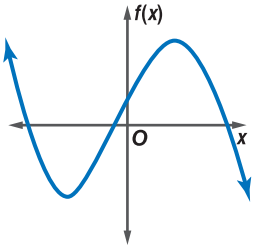


$f(x) \to +\infty$ as $x \to -\infty$.
$f(x) \to -\infty$ as $x \to +\infty$.

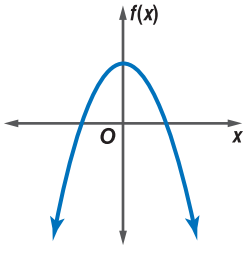
Since the end behavior is in opposite directions, it is an odd-degree function. The graph intersects the *x*-axis at five points, so there are five real zeros.

GuidedPractice

4A.



4B.



Check Your Understanding

Example 1 State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

1. $11x^6 - 5x^5 + 4x^2$

2. $-10x^7 - 5x^3 + 4x - 22$

3. $14x^4 - 9x^3 + 3x - 4y$

4. $8x^5 - 3x^2 + 4xy - 5$

Example 2 Find $w(5)$ and $w(-4)$ for each function.

5. $w(x) = -2x^3 + 3x - 12$

6. $w(x) = 2x^4 - 5x^3 + 3x^2 - 2x + 8$

Example 3 If $c(x) = 4x^3 - 5x^2 + 2$ and $d(x) = 3x^2 + 6x - 10$, find each value.

7. $c(y^3)$

8. $-4[d(3z)]$

9. $6c(4a) + 2d(3a - 5)$

10. $-3c(2b) + 6d(4b - 3)$

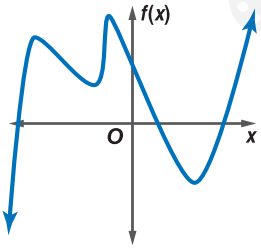
Example 4 For each graph,

a. describe the end behavior,

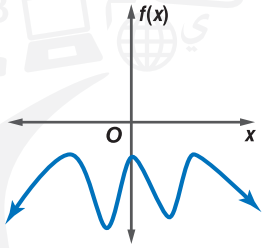
b. determine whether it represents an odd-degree or an even-degree function, and

c. state the number of real zeros.

11.



12.



Practice and Problem Solving

Example 1 **PERSEVERANCE** State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

13. $-6x^6 - 4x^5 + 13xy$

14. $3a^7 - 4a^4 + \frac{3}{a}$

15. $8x^5 - 12x^6 + 14x^3 - 9$

16. $-12 - 8x^2 + 5x - 21x^7$

17. $15x - 4x^3 + 3x^2 - 5x^4$

18. $13b^3 - 9b + 3b^5 - 18$

19. $(d + 5)(3d - 4)$

20. $(5 - 2y)(4 + 3y)$

21. $6x^5 - 5x^4 + 2x^9 - 3x^2$

22. $7x^4 + 3x^7 - 2x^8 + 7$

Example 2 Find $p(-6)$ and $p(3)$ for each function.

23. $p(x) = x^4 - 2x^2 + 3$

24. $p(x) = -3x^3 - 2x^2 + 4x - 6$

25. $p(x) = 2x^3 + 6x^2 - 10x$

26. $p(x) = x^4 - 4x^3 + 3x^2 - 5x + 24$

27. $p(x) = -x^3 + 3x^2 - 5$

28. $p(x) = 2x^4 + x^3 - 4x^2$

Example 3 If $c(x) = 2x^2 - 4x + 3$ and $d(x) = -x^3 + x + 1$, find each value.

29. $c(3a)$

30. $5d(2a)$

31. $c(b^2)$

32. $d(4a^2)$

33. $d(4y - 3)$

34. $c(y^2 - 1)$

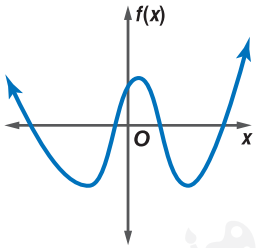
Example 4 **For each graph,**

a. describe the end behavior,

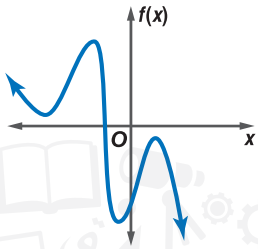
b. determine whether it represents an odd-degree or an even-degree function, and

c. state the number of real zeros.

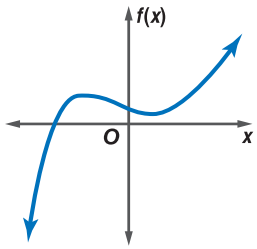
35.



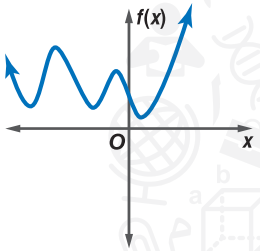
36.



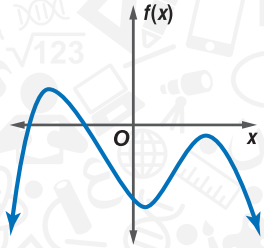
37.



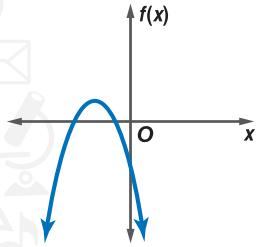
38.



39.



40.



41. PHYSICS For a moving object with mass m in kilograms, the kinetic energy KE in joules is given by the function $KE(v) = 0.5mv^2$, where v represents the speed of the object in meters per second. Find the kinetic energy of an all-terrain vehicle with a mass of 171 kilograms moving at a speed of 11 meters/second.

42. MODELING A microwave manufacturing firm has determined that their profit function is $P(x) = -0.0014x^3 + 0.3x^2 + 6x - 355$, where x is the number of microwaves sold annually.

a. Graph the profit function using a calculator.

b. Determine a reasonable viewing window for the function.

c. Approximate all of the zeros of the function using the **CALC** menu.

d. What must be the range of microwaves sold in order for the firm to have a profit?

Find $p(-2)$ and $p(8)$ for each function.

43. $p(x) = \frac{1}{4}x^4 + \frac{1}{2}x^3 - 4x^2$

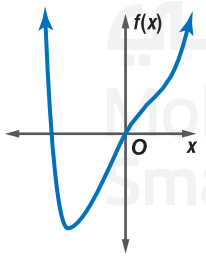
44. $p(x) = \frac{1}{8}x^4 - \frac{3}{2}x^3 + 12x - 18$

45. $p(x) = \frac{3}{4}x^4 - \frac{1}{8}x^2 + 6x$

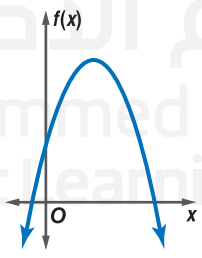
46. $p(x) = \frac{5}{8}x^3 - \frac{1}{2}x^2 + \frac{3}{4}x + 10$

Use the degree and end behavior to match each polynomial to its graph.

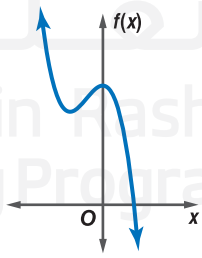
A



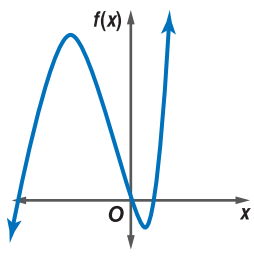
B



C



D



47. $f(x) = x^3 + 3x^2 - 4x$

48. $f(x) = -2x^2 + 8x + 5$

49. $f(x) = x^4 - 3x^2 + 6x$

50. $f(x) = -4x^3 - 4x^2 + 8$

If $c(x) = x^3 - 2x$ and $d(x) = 4x^2 - 6x + 8$, find each value.

51. $3c(a - 4) + 3d(a + 5)$

52. $-2d(2a + 3) - 4c(a^2 + 1)$

53. $5c(a^2) - 8d(6 - 3a)$

54. $-7d(a^3) + 6c(a^4 + 1)$

55. BUSINESS A clothing manufacturer's profitability can be modeled by $p(x) = -x^4 + 40x^2 - 144$, where x is the number of items sold in thousands and $p(x)$ is the company's profit in thousands of dirhams.

a. Use a table of values to sketch the function.

b. Determine the zeros of the function.

c. Between what two values should the company sell in order to be profitable?

d. Explain why only two of the zeros are considered in part **c**.

56. MULTIPLE REPRESENTATIONS Consider $g(x) = (x - 2)(x + 1)(x - 3)(x + 4)$.

a. Analytical Determine the x- and y-intercepts, roots, degree, and end behavior of $g(x)$.

b. Algebraic Write the function in standard form

c. Tabular Make a table of values for the function.

d. Graphical Sketch a graph of the function by plotting points and connecting them with a smooth curve.

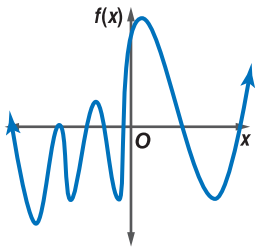
Describe the end behavior of the graph of each function.

57. $f(x) = -5x^4 + 3x^2$ **58.** $g(x) = 2x^5 + 6x^4$ **59.** $h(x) = -4x^7 + 8x^6 - 4x$

60. $f(x) = 6x - 7x^2$ **61.** $g(x) = 8x^4 + 5x^5$ **62.** $h(x) = 9x^6 - 5x^7 + 3x^2$

H.O.T. Problems Use Higher-Order Thinking Skills

63. CRITIQUE Abeer and Amani are determining the number of zeros of the graph at the right. Is either of them correct? Explain your reasoning.



Abeer

There are 7 zeros because the graph crosses the x-axis 7 times.

Amani

There are 8 zeros because the graph crosses the x-axis 7 times, and there is a double root.

64. CHALLENGE Of $f(x)$ and $g(x)$, which function has more potential real roots? What is the degree of that function?

x	−24	−18	−12	−6	0	6	12	18	24
$f(x)$	−8	−1	3	−2	4	7	−1	−8	5

$g(x) = x^4 + x^3 - 13x^2 + x + 4$

65. CHALLENGE If $f(x)$ has a degree of 5 and a positive leading coefficient and $g(x)$ has a degree of 3 and a positive leading coefficient, determine the end behavior of $\frac{f(x)}{g(x)}$. Explain your reasoning.

66. OPEN ENDED Sketch the graph of an even-degree polynomial with 8 real roots, one of them a double root.

67. REASONING Determine whether the following statement is *always*, *sometimes*, or *never* true. Explain.

A polynomial function that has four real roots is a fourth-degree polynomial.

68. WRITING IN MATH Describe what the end behavior of a polynomial function is and how to determine it.

Standardized Test Practice

69. SHORT RESPONSE Four students solved the same math problem. Each student's work is shown below. Who is correct?

Student A

$x^2x^{-5} = \frac{x^2}{x^5}$

$= \frac{1}{x^3}, x \neq 0$

Student C

$x^2x^{-5} = \frac{x^2}{x^{-5}}$

$= x^7, x \neq 0$

Student B

$x^2x^{-5} = \frac{x^2}{x^{-5}}$

$= x^{-7}, x \neq 0$

Student D

$x^2x^{-5} = \frac{x^2}{x^5}$

$= x^3, x \neq 0$

70. SAT/ACT What is the remainder when $x^3 - 7x + 5$ is divided by $x + 3$?

A -11
B -1
C 1
D 11
E 35

71. EXTENDED RESPONSE A company manufactures tables and chairs. It costs AED 40 to make each table and AED 25 to make each chair. There is AED 1440 available to spend on manufacturing each week. Let t = the number of tables produced and c = the number of chairs produced.

a. The manufacturing equation is $40t + 25c = 1500$. Construct a graph of this equation.

b. The company always produces two chairs with each table. Write and graph an equation to represent this situation on the same graph as the one in part **a**.

c. Determine the number of tables and chairs that the company can produce each week.

d. Explain how to determine this answer using the graph.

72. If $i = \sqrt{-1}$, then $5i(7i) =$

F 70
G 35
H -35
J -70

Spiral Review

Simplify. (Lesson 3-2)

73. $\frac{16x^4y^3 + 32x^6y^5z^2}{8x^2y}$

74. $\frac{18ab^4c^5 - 30a^4b^3c^2 + 12a^5bc^3}{6abc^2}$

75. $\frac{18c^5d^2 - 3c^2d^2 + 12a^5c^3d^4}{3c^2d^2}$

Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial. (Lesson 3-1)

76. $8x^2 + 5xy^3 - 6x + 4$

77. $9x^4 + 12x^6 - 16$

78. $3x^4 + 2x^2 - x^{-1}$

79. MANUFACTURING The Community Service Committee is making canvas tote bags and leather tote bags for a fundraiser. They will line both types of bags with canvas and use leather handles on both. For the canvas bags, they need 4 meters of canvas and 1 yard of leather. For the leather bags, they need 3 meters of leather and 2 meters of canvas. The committee leader purchased 56 meters of leather and 104 meters of canvas.

a. Let c represent the number of canvas bags, and let ℓ represent the number of leather bags. Write a system of inequalities for the number of bags that can be made.

b. Draw the graph showing the feasible region.

c. List the coordinates of the vertices of the feasible region.

d. If the club plans to sell the canvas bags at a profit of AED 20 each and the leather bags at a profit of AED 35 each, write a function for the total profit on the bags.

e. How can the club make the maximum profit?

f. What is the maximum profit?

Solve each inequality.

80. $|2x + 4| \leq 8$

81. $|-3x + 2| \geq 4$

82. $|2x - 8| - 4 \leq -6$

Skills Review

Determine whether each function has a maximum or minimum value, and find that value.

83. $f(x) = 3x^2 - 8x + 4$

84. $f(x) = -4x^2 + 2x - 10$

85. $f(x) = -0.25x^2 + 4x - 5$

LESSON 3-4

Analyzing Graphs of Polynomial Functions

Then
- You used maxima and minima and graphs of polynomials.

Now
1. Graph polynomial functions and locate their zeros.
2. Find the relative maxima and minima of polynomial functions.

Why?
- Annual attendance at the movies has fluctuated since the first movie theater, the Nickelodeon, opened in Pittsburgh in 1906. Overall attendance peaked during the 1920s, and it was at its lowest during the 1970s. A graph of the annual attendance to the movies can be represented by a polynomial function.

New Vocabulary
- Location Principle
- relative maximum
- relative minimum
- extrema
- turning points

Mathematical Practices
Construct viable arguments and critique the reasoning of others.

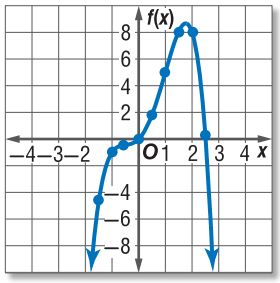
1 Graphs of Polynomial Functions

To graph a polynomial function, make a table of values to find several points and then connect them to make a smooth continuous curve. Knowing the end behavior of the graph will assist you in completing the graph.

Example 1 Graph of a Polynomial Function

Graph $f(x) = -x^4 + x^3 + 3x^2 + 2x$ by making a table of values.

x	$f(x)$	x	$f(x)$
−2.5	≈ −41	0.5	≈ 1.8
−2.0	−16	1.0	5.0
−1.5	≈ −4.7	1.5	≈ 8.1
−1.0	−1.0	2.0	8.0
−0.5	≈ −0.4	2.5	≈ 0.3
0.0	0.0	3.0	−21

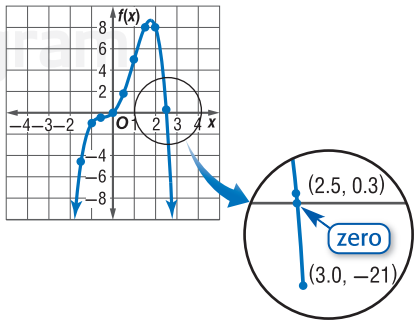


This is an even-degree polynomial with a negative leading coefficient, so $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$. Notice that the graph intersects the x-axis at two points, indicating there are two zeros for this function.

Guided Practice

1. Graph $f(x) = x^4 - x^3 - 2x^2 + 4x - 6$ by making a table of values.

In Example 1, one of the zeros occurred at $x = 0$. Another zero occurred between $x = 2.5$ and $x = 3.0$. Because $f(x)$ is positive for $x = 2.5$ and negative for $x = 3.0$ and all polynomial functions are continuous, we know there is a zero between these two values.



So, if the value of $f(x)$ *changes signs* from one value of x to the next, then there is a zero between those two x-values. This idea is called the **Location Principle**.

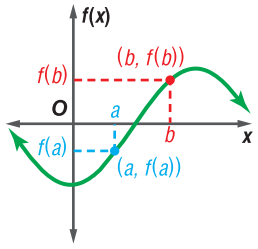
StudyTip

Degree Recall that the degree of the function is also the maximum number of zeros the function can have.

KeyConcept Location Principle

Words Suppose $y = f(x)$ represents a polynomial function and a and b are two real numbers such that $f(a) < 0$ and $f(b) > 0$. Then the function has at least one real zero between a and b.

Model

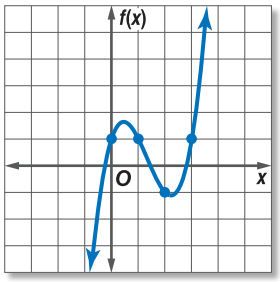


Example 2 Locate Zeros of a Function

Determine consecutive integer values of x between which each real zero of $f(x) = x^3 - 4x^2 + 3x + 1$ is located. Then draw the graph.

Make a table of values. Since $f(x)$ is a third-degree polynomial function, it will have either 3 or 1 real zeros. Look at the values of $f(x)$ to locate the zeros. Then use the points to sketch a graph of the function.

x	$f(x)$	
−2	−29	
−1	−7	
0	1	change in sign
1	1	
2	−1	change in sign
3	1	change in sign
4	13	



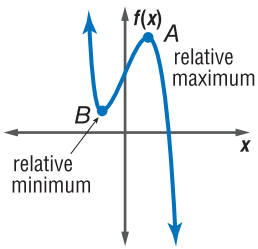
The changes in sign indicate that there are zeros between $x = -1$ and $x = 0$, between $x = 1$ and $x = 2$, and between $x = 2$ and $x = 3$.

GuidedPractice

2. Determine consecutive integer values of x between which each real zero of the function $f(x) = x\,4 - 3x\,3 - 2x\,2 + x + 1$ is located. Then draw the graph.

2 Maximum and Minimum Points

The graph below shows the general shape of a third-degree polynomial function.



Point A on the graph is a **relative maximum** of the function since no other nearby points have a greater y-coordinate. The graph is increasing as it approaches A and decreasing as it moves from A.

Likewise, point B is a **relative minimum** since no other nearby points have a lesser y-coordinate. The graph is decreasing as it approaches B and increasing as it moves from B. The maximum and minimum values of a function are called the **extrema**.

StudyTip

Odd Functions Some odd functions, like $f(x) = x^3$, have no turning points.

These points are often referred to as **turning points**. The graph of a polynomial function of degree n has at most $n - 1$ turning points.

StudyTip

Maximum and Minimum
A polynomial with a degree greater than 3 may have more than one relative maximum or relative minimum.

Example 3 Maximum and Minimum Points

Graph $f(x) = x^3 - 4x^2 - 2x + 3$. Estimate the x-coordinates at which the relative maxima and relative minima occur.

Make a table of values and graph the function.

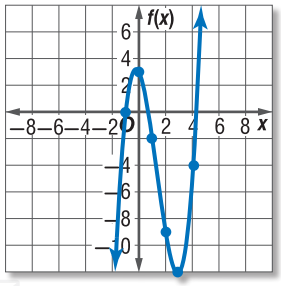
x	$f(x)$
−2	−17
−1	0
0	3
1	−2
2	−9
3	−12
4	−4
5	18

zero

indicates a relative maximum

indicates a relative minimum

zero between 4 and 5



Look at the table of values and the graph.

The value of $f(x)$ changes signs between $x = 4$ and $x = 5$, indicating a zero of the function.

StudyTip

Rational Values Zeros and turning points will not always occur at integral values of x.

The value of $f(x)$ at $x = 0$ is greater than the surrounding points, so there must be a relative maximum *near* $x = 0$.

The value of $f(x)$ at $x = 3$ is less than the surrounding points, so there must be a relative minimum *near* $x = 3$.

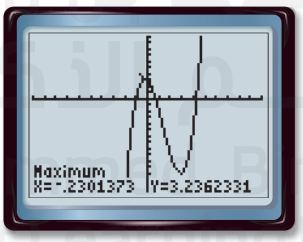
CHECK You can use a graphing calculator to find the relative maximum and relative minimum of a function and confirm your estimates.

Enter $y = x^3 - 4x^2 = 2x + 3$ in the **Y=** list and graph the function.

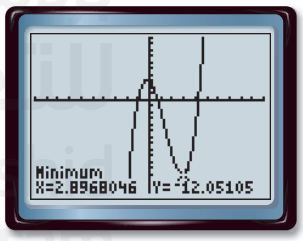
Use the **CALC** menu to find each maximum and minimum.

When selecting the left bound, move the cursor to the left of the maximum or minimum. When selecting the right bound, move the cursor to the right of the maximum or minimum.

Press ENTER twice.



[−10, 10] scl: 1 by [−15, 10] scl: 1



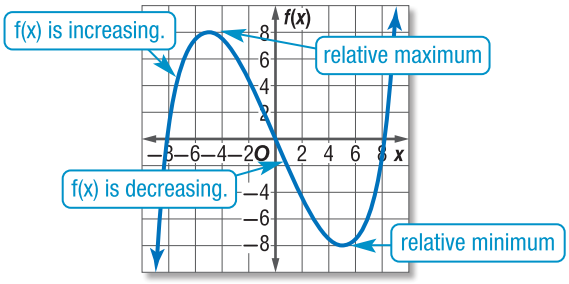
[−10, 10] scl: 1 by [−15, 10] scl: 1

The estimates for a relative maximum near x = 0 and a relative minimum near x = 3 are accurate.

GuidedPractice

3. Graph $f(x) = 2x^3 + x^2 - 4x - 2$. Estimate the x-coordinates at which the relative maxima and relative minima occur.

The graph of a polynomial function can reveal trends in real-world data. It is often helpful to note when the graph is increasing or decreasing.



Real-WorldLink

Over 1.4 billion movie tickets were sold in the United States in 2006.

Source: CNN

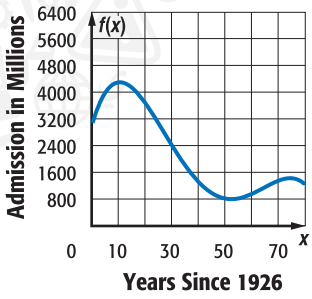
Real-World Example 4 Graph a Polynomial Model

MOVIES **Refer to the beginning of the lesson. Annual admissions to movies in the United States can be modeled by the function $f(x) = -0.0017x^4 + 0.31x^3 - 17.66x^2 + 277x + 3005$, where x is the number of years since 1926 and $f(x)$ is the annual admissions in millions.**

a. Graph the function.

Make a table of values for the years 1926–2006. Plot the points and connect with a smooth curve. Finding and plotting the points for every tenth year gives a good approximation of the graph.

x	$f(x)$
0	3005
10	4302
20	3689
30	2414
40	1317
50	830
60	977
70	1374
80	1229



b. Describe the turning points of the graph and its end behavior.

There are relative maxima near 1936 and 2000 and a relative minimum between 1976 and 1981. $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$.

c. What trends in movie admissions does the graph suggest? Is it reasonable that the trend will continue indefinitely?

Movie attendance peaked around 1936 and declined until about 1978. It then increased until 2000 and began a decline.

d. Is it reasonable that the trend will continue indefinitely?

This trend may continue for a couple of years, but the graph will soon become unreasonable as it predicts negative attendance for the future.

GuidedPractice

4. FAX MACHINES The annual sales of fax machines for home use can be modeled by $f(x) = -0.17x^4 + 6.29x^3 - 77.65x^2 + 251x + 1100$, where x is the number of years after 1990 and $f(x)$ is the annual sales in millions of dirhams.

A. Graph the function.

B. Describe the turning points of the graph and its end behavior.

C. What trends in fax machine sales does the graph suggest?

D. Is it reasonable that the trend will continue indefinitely?

Check Your Understanding

Example 1 **Graph each polynomial equation by making a table of values.**

1. $f(x) = 2x^4 - 5x^3 + x^2 - 2x + 4$

2. $f(x) = -2x^4 + 4x^3 + 2x^2 + x - 3$

3. $f(x) = 3x^4 - 4x^3 - 2x^2 + x - 4$

4. $f(x) = -4x^4 + 5x^3 + 2x^2 + 3x + 1$

Example 2 **Determine the consecutive integer values of x between which each real zero of each function is located. Then draw the graph.**

5. $f(x) = x^3 - 2x^2 + 5$

6. $f(x) = -x^4 + x^3 + 2x^2 + x + 1$

7. $f(x) = -3x^4 + 5x^3 + 4x^2 + 4x - 8$

8. $f(x) = 2x^4 - x^3 - 3x^2 + 2x - 4$

Example 3 **Graph each polynomial function. Estimate the x-coordinates at which the relative maxima and relative minima occur. State the domain and range for each function.**

9. $f(x) = x^3 + x^2 - 6x - 3$

10. $f(x) = 3x^3 - 6x^2 - 2x + 2$

11. $f(x) = -x^3 + 4x^2 - 2x - 1$

12. $f(x) = -x^3 + 2x^2 - 3x + 4$

Example 4

13. **SENSE-MAKING** Annual compact disc sales can be modeled by the quartic function $f(x) = 0.48x^4 - 9.6x^3 + 53x^2 - 49x + 599$, where x is the number of years after 1995 and $f(x)$ is annual sales in millions.

a. Graph the function for $0 \leq x \leq 10$.

b. Describe the turning points of the graph, its end behavior, and the intervals on which the graph is increasing or decreasing.

c. Continue the graph for $x = 11$ and $x = 12$. What trends in compact disc sales does the graph suggest?

d. Is it reasonable that the trend will continue indefinitely? Explain.

Practice and Problem Solving

Examples 1–3 **Complete each of the following.**

a. Graph each function by making a table of values.

b. Determine the consecutive integer values of x between which each real zero is located.

c. Estimate the x-coordinates at which the relative maxima and minima occur.

14. $f(x) = x^3 + 3x^2$

15. $f(x) = -x^3 + 2x^2 - 4$

16. $f(x) = x^3 + 4x^2 - 5x$

17. $f(x) = x^3 - 5x^2 + 3x + 1$

18. $f(x) = -2x^3 + 12x^2 - 8x$

19. $f(x) = 2x^3 - 4x^2 - 3x + 4$

20. $f(x) = x^4 + 2x - 1$

21. $f(x) = x^4 + 8x^2 - 12$

Example 4

22. **FINANCIAL LITERACY** The average annual price of gasoline can be modeled by the cubic function $f(x) = 0.0007x^3 - 0.014x^2 + 0.08x + 0.96$, where x is the number of years after 1987 and $f(x)$ is the price in dirhams.

a. Graph the function for $0 \leq x \leq 30$.

b. Describe the turning points of the graph and its end behavior.

c. What trends in gasoline prices does the graph suggest?

d. Is it reasonable that the trend will continue indefinitely? Explain.

Use a graphing calculator to estimate the x-coordinates at which the maxima and minima of each function occur. Round to the nearest hundredth.

23. $f(x) = x^3 + 3x^2 - 6x - 6$

24. $f(x) = -2x^3 + 4x^2 - 5x + 8$

25. $f(x) = -2x^4 + 5x^3 - 4x^2 + 3x - 7$

26. $f(x) = x^5 - 4x^3 + 3x^2 - 8x - 6$

Sketch the graph of polynomial functions with the following characteristics.

27. an odd function with zeros at -5, -3, 0, 2 and 4

28. an even function with zeros at -2, 1, 3, and 5

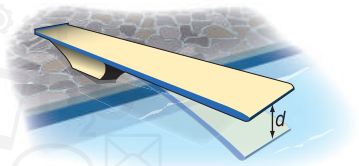
29. a 4-degree function with a zero at -3, maximum at $x = 2$, and minimum at $x = -1$

30. a 5-degree function with zeros at -4, -1, and 3, maximum at $x = -2$

31. an odd function with zeros at -1, 2, and 5 and a negative leading coefficient

32. an even function with a minimum at $x = 3$ and a positive leading coefficient

33. **DIVING** The deflection d of a 10-foot-long diving board can be calculated using the function $d(x) = 0.015x^2 - 0.0005x^3$, where x is the distance between the diver and the stationary end of the board in feet.



a. Make a table of values of the function for $0 \le x \le 10$.

b. Graph the function.

c. What does the end behavior of the graph suggest as x increases?

d. Will this trend continue indefinitely? Explain your reasoning.

Complete each of the following.

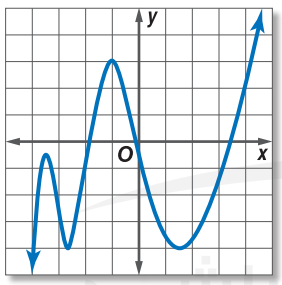
a. Estimate the x-coordinate of every turning point and determine if those coordinates are relative maxima or relative minima.

b. Estimate the x-coordinate of every zero.

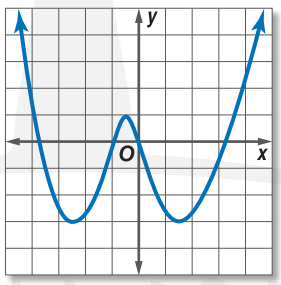
c. Determine the smallest possible degree of the function.

d. Determine the domain and range of the function.

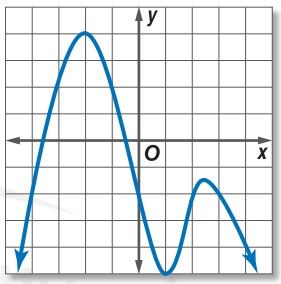
34.



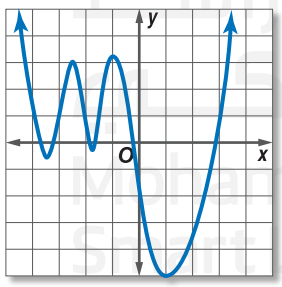
35.



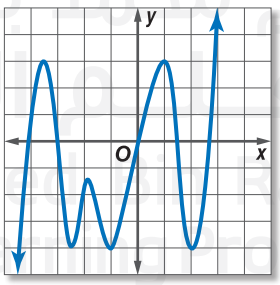
36.



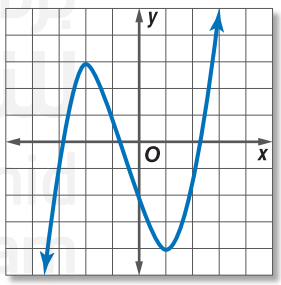
37.



38.



39.



40. **REASONING** The number of subscribers using pagers can be modeled by $f(x) = 0.015x^4 - 0.44x^3 + 3.46x^2 - 2.7x + 9.68$, where x is the number of years after 1990 and $f(x)$ is the number of subscribers in millions.

a. Graph the function.

b. Describe the end behavior of the graph.

c. What does the end behavior suggest about the number of pager subscribers?

d. Will this trend continue indefinitely? Explain your reasoning.

41. PRICING Khalid's vending machines currently sell an average of 3500 beverages per week at a rate of AED 0.75 per can. He is considering increasing the price. His weekly earnings can be represented by $f(x) = -5x^2 + 100x + 2625$, where x is the number of AED 0.05 increases. Graph the function and determine the most profitable price for Khalid.

For each function,

a. determine the zeros, x- and y-intercepts, and turning points,

b. determine the axis of symmetry, and

c. determine the intervals for which it is increasing, decreasing, or constant.

42. $y = x^4 - 8x^2 + 16$

43. $y = x^5 - 3x^3 + 2x - 4$

44. $y = -2x^4 + 4x^3 - 5x$

45. $y = \begin{cases} x^2 \text{ if } x \leq -4 \\ 5 \text{ if } -4 < x \leq 0 \\ x^3 \text{ if } x > 0 \end{cases}$

46. MULTIPLE REPRESENTATIONS Consider the following function.

$$f(x) = x^4 - 8.65x^3 + 27.34x^2 - 37.2285x + 18.27$$

a. Analytical What are the degree, leading coefficient, and end behavior?

b. Tabular Make a table of integer values $f(x)$ if $-4 \leq x \leq 4$. How many zeros does the function appear to have from the table?

c. Graphical Graph the function by using a graphing calculator.

d. Graphical Change the viewing window to [0, 4] scl: 1 by [−0.4, 0.4] scl: 0.2. What conclusions can you make from this new view of the graph?

H.O.T. Problems Use Higher-Order Thinking Skills

47. REASONING Explain why the leading coefficient and the degree are the only determining factors in the end behavior of a polynomial function.

48. REASONING The table below shows the values of $g(x)$, a cubic function. Could there be a zero between $x = 2$ and $x = 3$? Explain your reasoning.

x	−2	−1	0	1	2	3
$g(x)$	4	−2	−1	1	−2	−2

49. OPEN ENDED Sketch the graph of an odd polynomial function with 6 turning points and 2 double roots.

50. ARGUMENTS Determine whether the following statement is *sometimes, always,* or *never* true. Explain your reasoning.

For any continuous polynomial function, the y-coordinate of a turning point is also either a relative maximum or relative minimum.

51. REASONING A function is said to be even if for every x in the domain of f, $f(x) = f(-x)$. Is every even-degree polynomial function also an even function? Explain.

52. REASONING A function is said to be *odd* if for every x in the domain, $-f(x) = f(-x)$. Is every odd-degree polynomial function also an odd function? Explain.

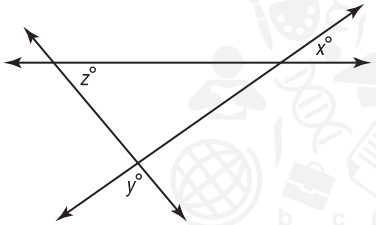
53. WRITING IN MATH How can you use the characteristics of a polynomial function to sketch its graph?

Standardized Test Practice

54. Which of the following is the factorization of $2x - 15 + x^2$?

A $(x - 3)(x - 5)$
B $(x - 3)(x + 5)$
C $(x + 3)(x - 5)$
D $(x + 3)(x + 5)$

55. SHORT RESPONSE In the figure below, if $x = 35$ and $z = 50$, what is the value of y?

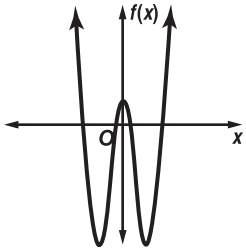


56. Which polynomial represents $(4x^2 + 5x - 3)(2x - 7)$?

F $8x^3 - 18x^2 - 41x - 21$
G $8x^3 + 18x^2 + 29x - 21$
H $8x^3 - 18x^2 - 41x + 21$
J $8x^3 + 18x^2 - 29x + 21$

57. SAT/ACT The figure at the right shows the graph of a polynomial function $f(x)$. Which of the following could be the degree of $f(x)$?

A 2
B 3
C 4
D 5
E 6



Spiral Review

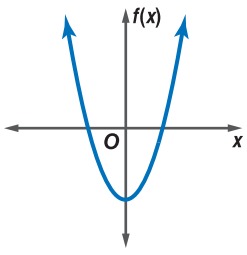
For each graph,

a. describe the end behavior,

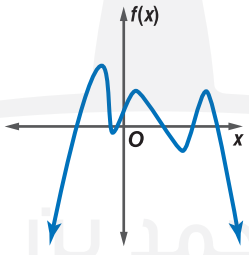
b. determine whether it represents an odd-degree or an even-degree function, and

c. state the number of real zeros. (Lesson 3-3)

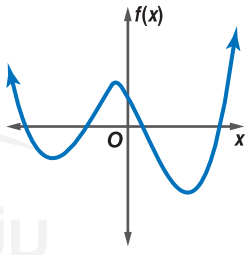
58.



59.



60.



Simplify. (Lesson 3-2)

61. $(x^3 + 2x^2 - 5x - 6) \div (x + 1)$

62. $(4y^3 + 18y^2 + 5y - 12) \div (y + 4)$

63. $(2a^3 - a^2 - 4a) \div (a - 1)$

64. CHEMISTRY Hessa needs 200 milliliters of a 48% concentration acid solution. She has 60% and 40% concentration solutions in her lab. How many milliliters of 40% acid solution should be mixed with 60% acid solution to make the required amount of 48% acid solution?

Skills Review

Factor.

65. $x^2 + 6x + 3x + 18$

66. $y^2 - 5y - 8y + 40$

67. $a^2 + 6a - 16$

68. $b^2 - 4b - 21$

69. $6x^2 - 5x - 4$

70. $4x^2 - 7x - 15$

EXTEND

3-4 Graphing Technology Lab: Modeling Data Using Polynomial Functions

You can use a graphing calculator to model data points when a curve of best fit is a polynomial function.

Mathematical Practices
Use appropriate tools strategically.

Example

The table shows the distance a seismic wave produced by an earthquake travels from the epicenter. Draw a scatter plot and a curve of best fit to show how the distance is related to time. Then determine approximately how far away from the epicenter a seismic wave will be felt 8.5 minutes after an earthquake occurs.

Travel Time (min)	1	2	5	7	10	12	13
Distance (km)	400	800	2500	3900	6250	8400	10,000

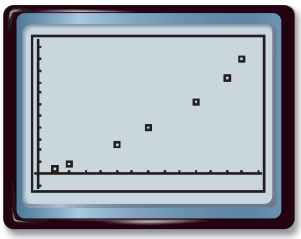
Source: University of Arizona

Step 1 Enter time in **L1** and distance in **L2**.

KEYSTROKES: STAT 1 1 ENTER 2 ENTER 5 ENTER 7 ENTER 10 ENTER 12 ENTER 13 ENTER ▶
400 ENTER 800 ENTER 2500 ENTER 3900 ENTER 6250 ENTER 8400 ENTER 10000 ENTER

Step 2 Graph the scatter plot.

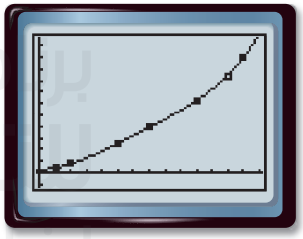
KEYSTROKES: 2nd [STAT PLOT] 1 ENTER ▼ ENTER ZOOM 9



[−0.2, 14.2] scl: 1
by [−1232, 11632] scl: 1000

Step 3 Determine and graph the equation for a curve of best fit. Use a quartic regression for the data.

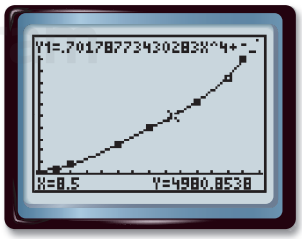
KEYSTROKES: STAT ▶ 7 ENTER Y= VARS 5 ▶ ▶ 1 GRAPH



The equation is shown in the Y= screen. If rounded, the regression equation shown on the calculator can be written as the algebraic equation $y = 0.7x^4 - 17x^3 + 161x^2 - 21x + 293$.

Step 4 Use the [CALC] feature to find the value of the function for $x = 8.5$.

KEYSTROKES: 2nd [CALC] 1 8.5 ENTER



After 8.5 minutes, the wave could be expected to be felt approximately 4980 kilometers from the epicenter.

MENTAL CHECK The table gives the distance for 7 minutes as 3900 and the distance for 10 minutes as 6250. Since 8.5 is halfway between 7 and 10 a reasonable estimate for the distance is halfway between 3900 and 6250. ✓

Exercises

The table shows how many minutes out of each eight-hour work day are used to pay one day's worth of taxes.

Year	Minutes
1930	56
1940	86
1950	119
1960	134
1970	144
1980	147
1990	148
2000	163
2005	151

Source: Tax Foundation

1. Draw a scatter plot of the data. Then graph several curves of best fit that relate the number of minutes to the number of years. Try LinReg, QuadReg, and CubicReg.
2. Write the equation for the curve that best fits the data.
3. Based on this equation, how many minutes should you expect to work each day in the year 2020 to pay one day's taxes? Use mental math to check the reasonableness of your estimate.

The table shows the estimated number of alternative-fueled vehicles in use in the United States per year from 1998 to 2007.

Year	Number of Vehicles	Year	Number of Vehicles
1998	295,030	2003	533,999
1999	322,302	2004	565,492
2000	394,664	2005	592,122
2001	425,457	2006	634,559
2002	471,098	2007	695,763

Source: U.S. Department of Energy

4. Draw a scatter plot of the data. Then graph several curves of best fit that relate the distance to the month.
5. Which curve best fits the data? Is that curve best for predicting future values?
6. Use the best-fit equation you think will give the most accurate prediction for how many alternative-fuel vehicles will be in use in 2018. Use mental math to check the reasonableness of your estimate.

The table shows the average distance from the Sun to Earth during each month of the year.

Month	Distance (astronomical units)
January	0.9840
February	0.9888
March	0.9962
April	1.0050
May	1.0122
June	1.0163
July	1.0161
August	1.0116
September	1.0039
October	0.9954
November	0.9878
December	0.9837

Source: The Astronomy Cafe

7. Draw a scatter plot of the data. Then graph several curves of best fit that relate the distance to the month.
8. Write the equation for the curve that best fits the data.
9. Based on your regression equation, what is the distance from the Sun to Earth halfway through September?
10. Would you use this model to find the distance from the Sun to Earth in subsequent years? Explain your reasoning.

Extension

11. Write a question that could be answered by examining data. For example, you might estimate the number of people living in your town 5 years from now or predict the future cost of a car.
12. Collect and organize the data you need to answer the question you wrote. You may need to research your topic on the Internet or conduct a survey to collect the data you need.
13. Make a scatter plot and find a regression equation for your data. Then use the regression equation to answer the question.

CHAPTER 3

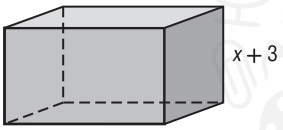
Mid-Chapter Quiz

Lessons 3-1 through 3-4

Simplify. Assume that no variable equals 0. (Lesson 3-1)

1. $(3x^2y^{-3})(-2x^3y^5)$
2. $4t(3rt - r)$
3. $\dfrac{3a^4b^3c}{6a^2b^5c^3}$
4. $\left(\dfrac{p^2r^3}{pr^4}\right)^2$
5. $(4m^2 - 6m + 5) - (6m^2 + 3m - 1)$
6. $(x + y)(x^2 + 2xy - y^2)$

7. **MULTIPLE CHOICE** The volume of the rectangular prism is $6x^3 + 19x^2 + 2x - 3$. Which polynomial expression represents the area of the base? (Lesson 3-1)



A $6x^4 + 37x^3 + 59x^2 + 3x - 9$

B $6x^2 + x + 1$

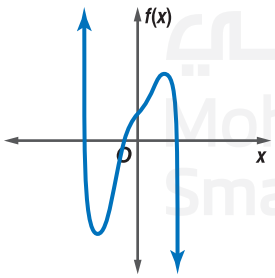
C $6x^2 + x - 1$

D $6x + 1$

Simplify. (Lesson 3-2)

8. $(4r^3 - 8r^2 - 13r + 20) \div (2r - 5)$
9. $\dfrac{3x^3 - 16x^2 + 9x - 24}{x - 5}$

10. Describe the end behavior of the graph. Then determine whether it represents an odd-degree or an even-degree polynomial function and state the number of real zeros. (Lesson 3-3)



11. **MULTIPLE CHOICE** Find $p(-3)$ if $p(x) = \frac{2}{3}x^3 + \frac{1}{3}x^2 - 5x$. (Lesson 3-3)

F 0 **H** 30

G 11 **J** 36

12. **PENDULUMS** The formula $L(t) = \dfrac{8t^2}{\pi^2}$ can be used to find the length of a pendulum in meters when it swings back and forth in t seconds. Find the length of a pendulum that makes one complete swing in 4 seconds. (Lesson 3-3)

13. **MULTIPLE CHOICE** Find $3f(a - 4) - 2h(a)$ if $f(x) = x^2 + 3x$ and $h(x) = 2x^2 - 3x + 5$. (Lesson 3-3)

A $-a^2 + 15a - 74$

B $-a^2 - 2a - 1$

C $a^2 + 9a - 2$

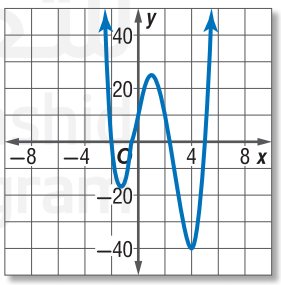
D $-a^2 - 9a + 2$

14. **ENERGY** The power generated by a windmill is a function of the speed of the wind. The approximate power is given by the function $P(s) = \dfrac{s^3}{1000}$, where s represents the speed of the wind in kilometers per hour. Find the units of power $P(s)$ generated by a windmill when the wind speed is 18 kilometers per hour. (Lesson 3-3)

Use $f(x) = x^3 - 2x^2 - 3x$ for Exercises 15–17. (Lesson 3-4)

15. Graph the function.
16. Estimate the x-coordinates at which the relative maxima and relative minima occur.
17. State the domain and range of the function.
18. Determine the consecutive integer values of x between which each real zero is located for $f(x) = 3x^2 - 3x - 1$. (Lesson 3-4)

Refer to the graph. (Lesson 3-4)



19. Estimate the x-coordinate of every turning point, and determine if those coordinates are relative maxima or relative minima.
20. Estimate the x-coordinate of every zero.
21. What is the least possible degree of the function?

EXPLORE

3-5 Graphing Technology Lab: Solving Polynomial Equations by Graphing

You can use a graphing calculator to solve polynomial equations.

Activity

Solve $x^4 + 2x^3 = 7$.

Method 1

Step 1 Graph each side of the equation separately in a standard viewing window.

Let **Y1** $= x^4 + 2x^3$ and **Y2** $= 7$.

KEYSTROKES: Y= X,T,θ,*n* ^ 4 + 2
X,T,θ,*n* ^ 3 ENTER 7 ZOOM 6

Step 2 Find the points of intersection.

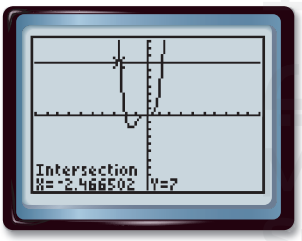
KEYSTROKES: 2nd [CALC] 5

Use ◀ or ▶ to position the cursor on **Y1** near the first point of intersection.
Press ENTER ENTER ENTER.
Then use ▶ to position the cursor near the second intersection point.
Press ENTER ENTER ENTER.

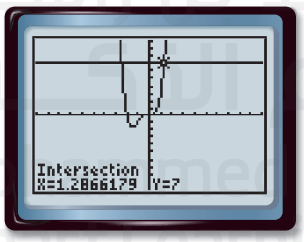
Step 3 Examine the graphs.

Determine where the graph of $y = x^4 + 2x^3$ intersects $y = 7$.

The intersections of the graphs of **Y1** and **Y2** are approximately −2.47 and 1.29, so the solution is approximately −2.47 and 1.29



[−10, 10] scl: 1 by [−10, 10] scl: 1



[−10, 10] scl: 1 by [−10, 10] scl: 1

Method 2

Step 1 Rewrite the related equation so it is equal to 0.

$$x^4 + 2x^3 = 7$$
$$x^4 + 2x^3 - 7 = 0$$

Let **Y1** $= x^4 + 2x^3 - 7$ and **Y2** $= 0$.

KEYSTROKES: Y= X,T,θ,*n* ^ 4 + 2
X,T,θ,*n* ^ 3 − 7 ENTER 0 ZOOM 6

Step 2 Because **Y2** $= 0$, to find the intersection points of **Y1** and **Y2**, find where **Y1** crosses the *x*-axis.

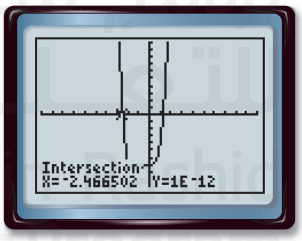
KEYSTROKES: 2nd [CALC] 5

Use ◀ or ▶ to position the cursor on **Y1** near the first point of intersection.
Press ENTER ENTER ENTER.
Then use ▶ to position the cursor near the second intersection point.
Press ENTER ENTER ENTER.

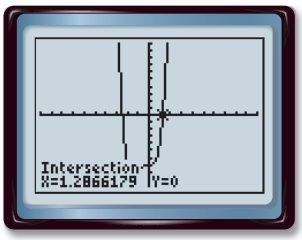
Step 3 Examine the graphs.

Determine where the graph of $y = x^4 + 2x^3 - 7$ crosses the *x*-axis.

The intersections of the graphs of **Y1** and **Y2** are approximately −2.47 and 1.29, so the solution is approximately −2.47 and 1.29



[−10, 10] scl: 1 by [−10, 10] scl: 1



[−10, 10] scl: 1 by [−10, 10] scl: 1

Exercises

Solve each equation. Round to the nearest hundredth.

1. $\frac{2}{3}x^3 + x^2 - 5x = -9$
2. $x^3 - 9x^2 + 27x = 20$
3. $x^3 + 1 = 4x^2$
4. $x^6 - 15 = 5x^4 - x^2$
5. $\frac{1}{2}x^5 = \frac{1}{5}x^2 - 2$
6. $x^8 = -x^7 + 3$
7. $x^4 - 15x^2 = -24$
8. $x^3 - 6x^2 + 4x = -6$
9. $x^4 - 15x^2 + x + 65 = 0$

LESSON 3-5

Solving Polynomial Equations

Then

- You solved quadratic functions by factoring.

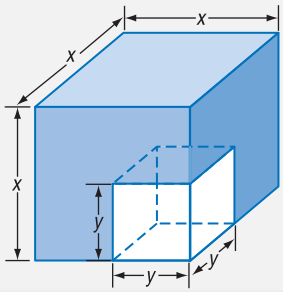
Now

1. Factor polynomials.
2. Solve polynomial equations by factoring.

Why?

- A small cube is cut out of a larger cube. The volume of the remaining figure is given and the dimensions of each cube need to be determined.

  This can be accomplished by factoring the cubic polynomial $x^3 - y^3$.



New Vocabulary

prime polynomials
quadratic form

Mathematical Practices
Model with mathematics.

1 Factor Polynomials

In Lesson 3-3, you learned that quadratics can be factored just like whole numbers. Their factors, however, are other polynomials. Like quadratics, some cubic polynomials can also be factored with special rules.

Key Concept Sum and Difference of Cubes

Factoring Technique	General Case
Sum of Two Cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of Two Cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Polynomials that cannot be factored are called **prime polynomials**.

Example 1 Sum and Difference of Cubes

Factor each polynomial. If the polynomial cannot be factored, write *prime*.

a. $16x^4 + 54xy^3$

$16x^4 + 54xy^3 = 2x(8x^3 + 27y^3)$

$8x^3$ and $27y^3$ are both perfect cubes, so we can factor the sum of two cubes.

$$\begin{aligned} 8x^3 + 27y^3 &= (2x)^3 + (3y)^3 \\ &= (2x + 3y)[(2x)^2 - (2x)(3y) + (3y)^2] \\ &= (2x + 3y)(4x^2 - 6xy + 9y^2) \end{aligned}$$

$16x^4 + 54xy^3 = 2x(2x + 3y)(4x^2 - 6xy + 9y^2)$

b. $9y^3 + 5x^3$

The first term is a perfect cube, but the second term is not. So, the polynomial cannot be factored using the sum of two cubes pattern. The polynomial also cannot be factored using quadratic methods or the GCF. Therefore, it is a prime polynomial.

Guided Practice

1A. $5y^4 - 320yz^3$

1B. $-54w^4 - 250wz^3$

Math HistoryLink

Sophie Germain
(1776–1831)
Sophie Germain taught herself mathematics with books from her father's library during the French Revolution when she was confined for safety. Germain discovered the identity $x^4 + 4y^4 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy)$, which is named for her.

The table below summarizes the most common factoring techniques used with polynomials. Whenever you factor a polynomial, always look for a common factor first. Then determine whether the resulting polynomial factors can be factored again using one or more of the methods below.

ConceptSummary Factoring Techniques

Number of Terms	Factoring Technique	General Case
any number	Greatest Common Factor (GCF)	$4a^3b^2 - 8ab = 4ab(a^2b - 2)$
two	Difference of Two Squares	$a^2 - b^2 = (a + b)(a - b)$
	Sum of Two Cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
	Difference of Two Cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
three	Perfect Square Trinomials	$a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$
	General Trinomials	$acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$
four or more	Grouping	$ax + bx + ay + by = x(a + b) + y(a + b) = (a + b)(x + y)$

Example 2 Factoring by Grouping

Factor each polynomial. If the polynomial cannot be factored, write *prime*.

a. $8ax + 4bx + 4cx + 6ay + 3by + 3cy$

$8ax + 4bx + 4cx + 6ay + 3by + 3cy$
$= (8ax + 4bx + 4cx) + (6ay + 3by + 3cy)$
$= 4x(2a + b + c) + 3y(2a + b + c)$
$= (4x + 3y)(2a + b + c)$

b. $20fy - 16fz + 15gy + 8hz - 10hy - 12gz$

$20fy - 16fz + 15gy + 8hz - 10hy - 12gz$
$= (20fy + 15gy - 10hy) + (-16fz - 12gz + 8hz)$
$= 5y(4f + 3g - 2h) - 4z(4f + 3g - 2h)$
$= (5y - 4z)(4f + 3g - 2h)$

StudyTip

Answer Checks
Multiply the factors to check your answer.

GuidedPractice

2A. $30ax - 24bx + 6cx - 5ay^2 + 4by^2 - cy^2$

2B. $13ax + 18bz - 15by - 14az - 32bx + 9ay$

Factoring by grouping is the only method that can be used to factor polynomials with four or more terms. For polynomials with two or three terms, it may be possible to factor according to one of the patterns listed above.

When factoring two terms in which the exponents are 6 or greater, look to factor perfect squares before factoring perfect cubes.

Example 3 Combine Cubes and Squares

Factor each polynomial. If the polynomial cannot be factored, write *prime*.

a. $x^6 - y^6$

This polynomial could be considered the difference of two squares or the difference of two cubes. The difference of two squares should always be done before the difference of two cubes for easier factoring.

$$x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$$
$$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$$

StudyTip
Grouping 6 or more terms
Group the terms that have the *most* common values.

b. $a^3x^2 - 6a^3x + 9a^3 - b^3x^2 + 6b^3x - 9b^3$

With six terms, factor by grouping first.

$$a^3x^2 - 6a^3x + 9a^3 - b^3x^2 + 6b^3x - 9b^3$$
$$= (a^3x^2 - 6a^3x + 9a^3) + (-b^3x^2 + 6b^3x - 9b^3)$$
$$= a^3(x^2 - 6x + 9) - b^3(x^2 - 6x + 9)$$
$$= (a^3 - b^3)(x^2 - 6x + 9)$$
$$= (a - b)(a^2 + ab + b^2)(x^2 - 6x + 9)$$
$$= (a - b)(a^2 + ab + b^2)(x - 3)^2$$

GuidedPractice

3A. $a^6 + b^6$

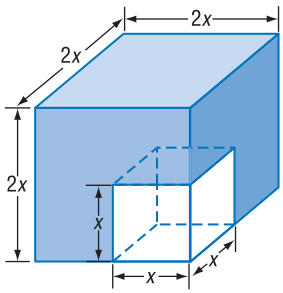
3B. $x^5 + 4x^4 + 4x^3 + x^2y^3 + 4xy^3 + 4y^3$

2 Solve Polynomial Equations

In Chapter 3, you learned to solve quadratic equations by factoring and using the Zero Product Property. You can extend these techniques to solve higher-degree polynomial equations.

Real-World Example 4 Solve Polynomial Functions by Factoring

GEOMETRY Refer to the beginning of the lesson. If the small cube is half the length of the larger cube and the figure is 7000 cubic centimeters, what should be the dimensions of the cubes?



Since the length of the smaller cube is half the length of the larger cube, then their lengths can be represented by x and $2x$, respectively. The volume of the object equals the volume of the larger cube minus the volume of the smaller cube.

$$(2x)^3 - x^3 = 7000$$
$$8x^3 - x^3 = 7000$$
$$7x^3 = 7000$$
$$x^3 = 1000$$
$$x^3 - 1000 = 0$$
$$(x - 10)(x^2 + 10x + 100) = 0$$
$$x - 10 = 0 \quad \text{or} \quad x^2 + 10x + 100 = 0$$
$$x = 10 \qquad x = -5 \pm 5i\sqrt{3}$$

Since 10 is the only real solution, the lengths of the cubes are 10 cm and 20 cm.

GuidedPractice

4. Determine the dimensions of the cubes if the length of the smaller cube is one third of the length of the larger cube, and the volume of the object is 3250 cubic centimeters.

In some cases, you can rewrite a polynomial in x in the form $au^2 + bu + c$. For example, by letting $u = x^2$, the expression $x^4 + 12x^2 + 32$ can be written as $(x^2)^2 + 12(x^2) + 32$ or $u^2 + 12u + 32$. This new, but equivalent, expression is said to be in **quadratic form**.

KeyConcept Quadratic Form

Words	An expression that is in quadratic form can be written as $au^2 + bu + c$ for any numbers a, b, and c, $a \neq 0$, where u is some expression in x. The expression $au^2 + bu + c$ is called the quadratic form of the original expression.
Example	$12x^6 + 8x^6 + 1 = 3(2x^3)^2 + 2(2x^3)^2 + 1$

StudyTip

Quadratic Form When writing a polynomial in quadratic form, choose the expression equal to u by examining the terms with variables. Pay special attention to the exponents in those terms. Not every polynomial can be written in quadratic form.

Example 5 Quadratic Form

Write each expression in quadratic form, if possible.

a. $\mathbf{150n^8 + 40n^4 - 15}$

$150n^8 + 40n^4 - 15 = 6(5n^4)^2 + 8(5n^4) - 15$

b. $\mathbf{y^8 + 12y^3 + 8}$

This cannot be written in quadratic form since $y^8 \neq (y^3)^2$.

GuidedPractice

5A. $x^4 + 5x + 6$

5B. $8x^4 + 12x^2 + 18$

You can use quadratic form to solve equations with larger degrees.

Example 6 Solve Equations in Quadratic Form

Solve $\mathbf{18x^4 - 21x^2 + 3 = 0}$.

$$18x^4 - 21x^2 + 3 = 0$$
$$2(3x^2)^2 - 7(3x^2) + 3 = 0$$
$$2u^2 - 7u + 3 = 0$$
$$(2u - 1)(u - 3) = 0$$
$$u = \frac{1}{2} \quad \text{or} \quad u = 3$$
$$3x^2 = \frac{1}{2} \qquad 3x^2 = 3$$
$$x^2 = \frac{1}{6} \qquad x^2 = 1$$
$$x = \pm\frac{\sqrt{6}}{6} \qquad x = \pm 1$$

The solutions of the equation are $1, -1, \frac{\sqrt{6}}{6}$, and $-\frac{\sqrt{6}}{6}$.

GuidedPractice

6A. $4x^4 - 8x^2 + 3 = 0$

6B. $8x^4 + 10x^2 - 12 = 0$

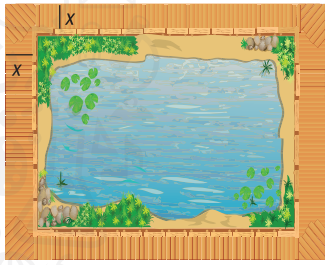
Check Your Understanding

Examples 1–3 **Factor completely. If the polynomial is not factorable, write *prime*.**

1. $3ax + 2ay - az + 3bx + 2by - bz$
2. $2kx + 4mx - 2nx - 3ky - 6my + 3ny$
3. $2x^3 + 5y^3$
4. $16g^3 + 2h^3$
5. $12qw^3 - 12q^4$
6. $3w^2 + 5x^2 - 6y^2 + 2z^2 + 7a^2 - 9b^2$
7. $a^6x^2 - b^6x^2$
8. $x^3y^2 - 8x^3y + 16x^3 + y^5 - 8y^4 + 16y^3$
9. $8c^3 - 125d^3$
10. $6bx + 12cx + 18dx - by - 2cy - 3dy$

Example 4 **Solve each equation.**

11. $x^4 - 19x^2 + 48 = 0$
12. $x^3 - 64 = 0$
13. $x^3 + 27 = 0$
14. $x^4 - 33x^2 + 200 = 0$
15. **PERSEVERANCE** A boardwalk that is x meters wide is built around a rectangular pond. The pond is 30 meters wide and 40 meters long. The combined area of the pond and the boardwalk is 2000 square meters. What is the width of the boardwalk?



Example 5 **Write each expression in quadratic form, if possible.**

16. $4x^6 - 2x^3 + 8$
17. $25y^6 - 5y^2 + 20$

Example 6 **Solve each equation.**

18. $x^4 - 6x^2 + 8 = 0$
19. $y^4 - 18y^2 + 72 = 0$

Practice and Problem Solving

Examples 1–3 **Factor completely. If the polynomial is not factorable, write *prime*.**

20. $8c^3 - 27d^3$
21. $64x^4 + xy^3$
22. $a^8 - a^2b^6$
23. $x^6y^3 + y^9$
24. $18x^6 + 5y^6$
25. $w^3 - 2y^3$
26. $gx^2 - 3hx^2 - 6fy^2 - gy^2 + 6fx^2 + 3hy^2$
27. $12ax^2 - 20cy^2 - 18bx^2 - 10ay^2 + 15by^2 + 24cx^2$
28. $a^3x^2 - 16a^3x + 64a^3 - b^3x^2 + 16b^3x - 64b^3$
29. $8x^5 - 25y^3 + 80x^4 - x^2y^3 + 200x^3 - 10xy^3$

Example 4 **Solve each equation.**

30. $x^4 + x^2 - 90 = 0$
31. $x^4 - 16x^2 - 720 = 0$
32. $x^4 - 7x^2 - 44 = 0$
33. $x^4 + 6x^2 - 91 = 0$
34. $x^3 + 216 = 0$
35. $64x^3 + 1 = 0$

Example 5 **Write each expression in quadratic form, if possible.**

36. $x^4 + 12x^2 - 8$
37. $-15x^4 + 18x^2 - 4$
38. $8x^6 + 6x^3 + 7$
39. $5x^6 - 2x^2 + 8$
40. $9x^8 - 21x^4 + 12$
41. $16x^{10} + 2x^5 + 6$

Example 6 **Solve each equation.**

42. $x^4 + 6x^2 + 5 = 0$
43. $x^4 - 3x^2 - 10 = 0$
44. $4x^4 - 14x^2 + 12 = 0$
45. $9x^4 - 27x^2 + 20 = 0$
46. $4x^4 - 5x^2 - 6 = 0$
47. $24x^4 + 14x^2 - 3 = 0$

48. ZOOLOGY A species of animal is introduced to a small island. Suppose the population of the species is represented by $P(t) = -t^4 + 9t^2 + 400$, where t is the time in years. Determine when the population becomes zero.

Factor completely. If the polynomial is not factorable, write *prime*.

49. $x^4 - 625$ **50.** $x^6 - 64$ **51.** $x^5 - 16x$ **52.** $8x^5y^2 - 27x^2y^5$

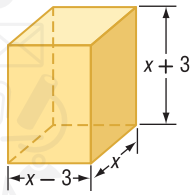
53. $15ax - 10bx + 5cx + 12ay - 8by + 4cy + 15az - 10bz + 5cz$

54. $6a^2x^2 - 24b^2x^2 + 18c^2x^2 - 5a^2y^3 + 20b^2y^3 - 15c^2y^3 + 2a^2z^2 - 8b^2z^2 + 6c^2z^2$

55. $6x^5 - 11x^4 - 10x^3 - 54x^3 + 99x^2 + 90x$

56. $20x^6 - 7x^5 - 6x^4 - 500x^4 + 175x^3 + 150x^2$

57. GEOMETRY The volume of the figure at the right is 440 cubic centimeters. Find the value of x and the length, height, and width.



Solve each equation.

58. $8x^4 + 10x^2 - 3 = 0$ **59.** $6x^4 - 5x^2 - 4 = 0$

60. $20x^4 - 53x^2 + 18 = 0$ **61.** $18x^4 + 43x^2 - 5 = 0$

62. $8x^4 - 18x^2 + 4 = 0$ **63.** $3x^4 - 22x^2 - 45 = 0$

64. $x^6 + 7x^3 - 8 = 0$ **65.** $x^6 - 26x^3 - 27 = 0$

66. $8x^6 + 999x^3 = 125$ **67.** $4x^4 - 4x^2 - x^2 + 1 = 0$

68. $x^6 - 9x^4 - x^2 + 9 = 0$ **69.** $x^4 + 8x^2 + 15 = 0$

70. SENSE-MAKING A rectangular prism with dimensions $x - 2$, $x - 4$, and $x - 6$ has a volume equal to $40x$ cubic units.

a. Write a polynomial equation using the formula for volume.

b. Use factoring to solve for x.

c. Are any values for x unreasonable? Explain.

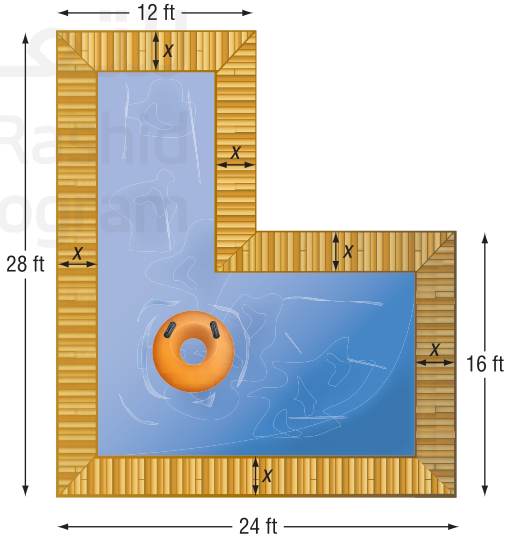
d. What are the dimensions of the prism?

71. POOL DESIGN Saeed wants to build a pool following the diagram at the right. The pool will be surrounded by a sidewalk of a constant width.

a. If the total area of the pool itself is to be 336 ft^2, what is x?

b. If the value of x were doubled, what would be the new area of the pool?

c. If the value of x were halved, what would be the new area of the pool?



72. **BIOLOGY** During an experiment, the number of cells of a virus can be modeled by $P(t) = -0.012t^3 - 0.24t^2 + 6.3t + 8000$, where t is the time in hours and P is the number of cells. Jamal wants to determine the times at which there are 8000 cells.

a. Solve for t by factoring.

b. What method did you use to factor?

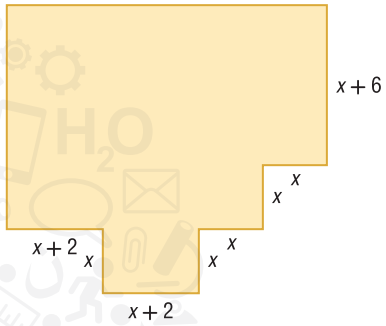
c. Which values for t are reasonable and which are unreasonable? Explain.

d. Graph the function for $0 \leq t \leq 20$ using your calculator.

73. **HOME BUILDING** Amna's parents want their basement home theater designed according to the diagram at the right.

a. Write a function in terms of x for the area of the basement.

b. If the basement is to be 1366 square meters, what is x?



74. **BIOLOGY** A population of parasites in an experiment can be modeled by $f(t) = t^3 + 5t^2 - 4t - 20$, where t is the time in days.

a. Use factoring by grouping to determine the values of t for which $f(t) = 0$.

b. At what times does the population reach zero?

c. Are any of the values of t unreasonable? Explain.

Factor completely. If the polynomial is not factorable, write *prime*.

75. $x^6 - 4x^4 - 8x^4 + 32x^2 + 16x^2 - 64$

76. $y^9 - y^6 - 2y^6 + 2y^3 + y^3 - 1$

77. $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$

78. **SENSE-MAKING** Ismail's corral, an enclosure for livestock, is currently 32 meters by 40 meters. He wants to enlarge the area to 4.5 times its current area by increasing the length and width by the same amount.

a. Draw a diagram to represent the situation.

b. Write a polynomial equation for the area of the new corral. Then solve the equation by factoring.

c. Graph the function.

d. Which solution is irrelevant? Explain.

H.O.T. Problems Use Higher-Order Thinking Skills

79. **CHALLENGE** Factor $36x^{2n} + 12x^n + 1$.

80. **CHALLENGE** Solve $6x - 11\sqrt{3x} + 12 = 0$.

81. **REASONING** Find a counterexample to the statement $a^2 + b^2 = (a + b)^2$.

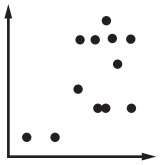
82. **OPEN ENDED** The cubic form of an equation is $ax^3 + bx^2 + cx + d = 0$. Write an equation with degree 6 that can be written in *cubic* form.

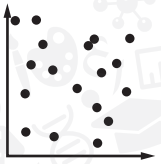
83. **WRITING IN MATH** Explain how the graph of a polynomial function can help you factor the polynomial.

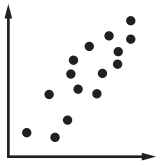
Standardized Test Practice

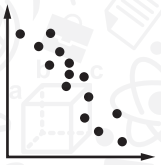
84. SHORT RESPONSE Tiles numbered from 1 to 6 are placed in a bag and are drawn to determine which of six tasks will be assigned to six people. What is the probability that the tiles numbered 5 and 6 are the last two drawn?

85. STATISTICS Which of the following represents a negative correlation?

A 

C 

B 

D 

86. Which of the following most accurately describes the translation of the graph $y = (x + 4)^2 - 3$ to the graph of $y = (x - 1)^2 + 3$?

F down 1 and to the right 3

G down 6 and to the left 5

H up 1 and to the left 3

J up 6 and to the right 5

87. SAT/ACT The positive difference between k and $\frac{1}{12}$ is the same as the positive difference between $\frac{1}{3}$ and $\frac{1}{5}$. Which of the following is the value of k?

A $\frac{1}{60}$

B $\frac{1}{20}$

C $\frac{1}{15}$

D $\frac{13}{60}$

E $\frac{37}{60}$

Spiral Review

Graph each polynomial function. Estimate the x-coordinates at which the relative maxima and relative minima occur. (Lesson 3-4)

88. $f(x) = 2x^3 - 4x^2 + x + 8$

89. $f(x) = -3x^3 + 6x^2 + 2x - 1$

90. $f(x) = -x^3 + 3x^2 + 4x - 6$

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why. (Lesson 3-3)

91. $f(x) = 4x^3 - 6x^2 + 5x^4 - 8x$

92. $f(x) = -2x^5 + 5x^4 + 3x^2 + 9$

93. $f(x) = -x^4 - 3x^3 + 2x^6 - x^7$

94. PART-TIME JOBS Thani makes AED 10 per hour cutting grass and AED 12 per hour for raking leaves. He cannot work more than 15 hours per week. Graph two inequalities that Thani can use to determine how many hours he needs to work at each job if he wants to earn at least AED 120 per week.

95. SKIING All 28 members of a ski club went on a trip. The club paid a total of AED 478 for the equipment. How many skis and snowboards did they rent?

96. GEOMETRY The sides of an angle are parts of two lines whose equations are $2y + 3x = -7$ and $3y - 2x = 9$. The angle's vertex is the point where the two sides meet. Find the coordinates of the vertex of the angle.

Skills Review

Divide.

97. $(x^2 + 6x - 2) \div (x + 4)$

98. $(2x^2 + 8x - 10) \div (2x + 1)$

99. $(8x^3 + 4x^2 + 6) \div (x + 2)$

EXTEND

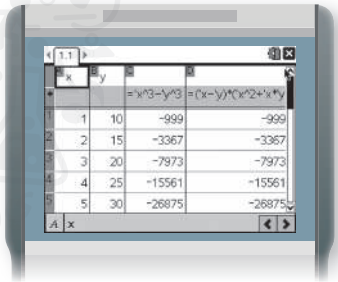
3-5 Graphing Technology Lab: Polynomial Identities

An **identity** is an equation that is satisfied by any numbers that replace the variables. Thus, a **polynomial identity** is a polynomial equation that is true for any values that are substituted for the variables.

You can use a table or a spreadsheet on your graphing calculator to determine whether a polynomial equation may be an identity.

Activity 1 Use a Table

Determine whether $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ may be an identity.



Step 1 Add a new **Lists & Spreadsheet** page. Label column A x and column B y. Type any values in columns A and B.

Step 2 Move the cursor to the formula row in column C and type $= x^3 - y^3$. In column D, type $= (x - y)(x^2 + xy + y^2)$.

No matter what values are entered for x and y in columns A and B, the values in columns C and D are the same. Thus, the equation may be an identity.

If you want to prove that an equation is an identity, you need to show that it is true for all values of the variables.

KeyConcept Verifying Identities by Transforming One Side

Step 1 Simplify one side of an equation until the two sides of the equation are the same. It is often easier to work with the more complicated side of the equation.

Step 2 Transform that expression into the form of the simpler side.

Activity 2 Transform One Side

Prove that $(x + y)^2 = x^2 + 2xy + y^2$ is an identity.

$$(x + y)^2 \stackrel{?}{=} x^2 + 2xy + y^2$$
$$(x + y)(x + y) \stackrel{?}{=} x^2 + 2xy + y^2$$
$$x^2 + xy + xy + y^2 \stackrel{?}{=} x^2 + 2xy + y^2$$
$$x^2 + 2xy + y^2 = x^2 + 2xy + y^2 \checkmark$$

Thus, the identity $(x + y)^2 = x^2 + 2xy + y^2$ is verified.

You can also use a graphing calculator with a computer algebra system (CAS) to prove an identity.

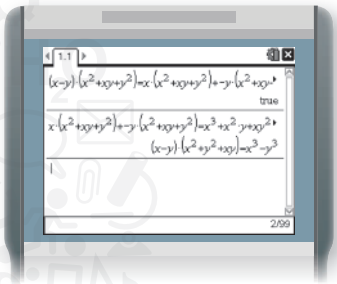
Activity 3 Use CAS

Prove that $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ is an identity.

Step 1 Add a new **Calculator** page on the graphing calculator CAS. Simplify the right side of the equation one step at a time.

Step 2 Enter the right side of the equation and then distribute.

Step 3 Multiply next. The CAS system will do the final simplification step.



The final step shown on the CAS screen is the results in $x^3 - y^3$. Thus, the identity has been proved.

You can also prove identities by transforming each side of the equation.

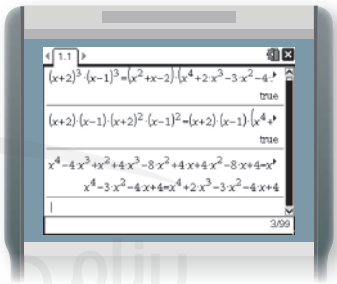
Activity 4 Use CAS to Transform Each Side

Prove that $(x + 2)^3(x - 1)^3 = (x^2 + x - 2)(x^4 + 2x^3 - 3x^2 - 4x + 4)$ is an identity.

Add a new **Calculator** page. Simplify the left and the right sides of the equation simultaneously.

The CAS will indicate if the changes are true, otherwise it will simplify for you.

The CAS system will do the final simplification step. The identity $(x + 2)^3 (x - 1)^3 = (x^2 + x - 2)(x^4 + 2x^3 - 3x^2 - 4x + 4)$ has been proved.



Analyze

1. Determine whether $a^2 - b^2 = (a + b)(a - b)$ may be an identity.

Exercises

Use CAS to prove each identity.

2. $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
3. $p^4 - q^4 = (p - q)(p + q)(p^2 + q^2)$
4. $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$
5. $g^6 + h^6 = (g^2 + h^2)(g^4 - g^2h^2 + h^4)$
6. $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$
7. $u^6 - w^6 = (u + w)(u - w)(u^2 + vw + w^2)(u^2 - vw + w^2)$
8. $(x + 1)^2(x - 4)^3 = (x^2 - 3x - 4)(x^3 - 7x^2 + 8x + 16)$

LESSON 3-6

The Remainder and Factor Theorems

Then

- You used the Distributive Property and factoring to simplify algebraic expressions.

Now

1. Evaluate functions by using synthetic substitution.
2. Determine whether a binomial is a factor of a polynomial by using synthetic substitution.

Why?

- The number of college students from the United States who study abroad can be modeled by the function $S(x) = 0.02x^4 - 0.52x^3 + 4.03x^2 + 0.09x + 77.54$, where x is the number of years since 1993 and $S(x)$ is the number of students in thousands.

You can use this function to estimate the number of U.S. college students studying abroad in 2018 by evaluating the function for $x = 25$. Another method you can use is *synthetic substitution*.

New Vocabulary
synthetic substitution
depressed polynomial

Mathematical Practices
Look for and make use of structure.

1 Synthetic Substitution

Synthetic division can be used to find the value of a function. Consider the polynomial function $f(x) = -3x^2 + 5x + 4$. Divide the polynomial by $x - 3$.

Method 1 Long Division

$$\begin{array}{r} -3x - 4 \\ x - 3 \overline{)\, -3x^2 + 5x + 4} \\ \underline{-3x^2 + 9x} \\ -4x + 4 \\ \underline{-4x + 12} \\ -8 \end{array}$$

Method 2 Synthetic Division

$$\begin{array}{r|rrr} 3 & -3 & 5 & 4 \\ & & -9 & -12 \\ \hline & -3 & -4 & -8 \end{array}$$

Compare the remainder of -8 to $f(3)$.

$$\begin{aligned} f(3) &= -3(3)^2 + 5(3) + 4 \\ &= -27 + 15 + 4 \\ &= -8 \end{aligned}$$

Notice that the value of $f(3)$ is the same as the remainder when the polynomial is divided by $x - 3$. This illustrates the **Remainder Theorem**.

KeyConcept Remainder Theorem

Words If a polynomial $P(x)$ is divided by $x - r$, the remainder is a constant $P(r)$, and

$$P(x) = Q(x) \cdot (x - r) + P(r),$$

where $Q(x)$ is a polynomial with degree one less than $P(x)$.

Example $x^2 + 6x + 2 = (x - 4) \cdot (x + 10) + 42$

Applying the Remainder Theorem using synthetic division to evaluate a function is called **synthetic substitution**. It is a convenient way to find the value of a function, especially when the degree of the polynomial is greater than 2.

Example 1 Synthetic Substitution

If $f(x) = 3x^4 - 2x^3 + 5x + 2$, find $f(4)$.

Method 1 **Synthetic Substitution**

By the Remainder Theorem, $f(4)$ should be the remainder when the polynomial is divided by $x - 4$.

4	3	−2	0	5	2
		12	40	160	660
	3	10	40	165	662

The remainder is 662. Therefore, by using synthetic substitution, $f(4) = 662$.

Method 2 **Direct Substitution**

Replace x with 4.

$f(x) = 3x^4 - 2x^3 + 5x + 2$

$f(4) = 3(4)^4 - 2(4)^3 + 5(4) + 2$

$= 768 - 128 + 20 + 2$ or 662

By using direct substitution, $f(4) = 662$. Both methods give the same result.

GuidedPractice

1A. If $f(x) = 3x^3 - 6x^2 + x - 11$, find $f(3)$.

1B. If $g(x) = 4x^5 + 2x^3 + x^2 - 1$, find $f(-1)$.

Synthetic substitution can be used in situations in which direct substitution would involve cumbersome calculations.

Real-World Example 2 Find Function Values

COLLEGE Refer to the beginning of the lesson. How many U.S. college students will study abroad in 2018?

Use synthetic substitution to divide $0.02x^4 - 0.52x^3 + 4.03x^2 + 0.09x + 77.54$ by $x - 20$.

25	0.02	−0.52	4.03	0.09	77.54
		0.5	−0.5	88.25	2208.5
	0.02	−0.02	3.53	88.34	2286.04

In 2018, there will be about 2,286,040 U.S. college students studying abroad.

GuidedPractice

2. COLLEGE The function $C(x) = 2.46x^3 - 22.37x^2 + 53.81x + 548.24$ can be used to approximate the number, in thousands, of international college students studying in the United States x years since 2000. How many international college students can be expected to study in the U.S. in 2015?

WatchOut!
Synthetic Substitution
Remember that synthetic substitution is used to divide a polynomial by $(x - a)$. If the binomial is $(x - a)$, use a. If the binomial is $(x + a)$, use $-a$.

2 Factors of Polynomials

The synthetic division below shows that the quotient of $2x^3 - 3x^2 - 17x + 30$ and $x + 3$ is $2x^2 - 9x + 10$.

$$\begin{array}{r|rrrr} -3 & 2 & -3 & -17 & 30 \\ & & -6 & 27 & -30 \\ \hline & 2 & -9 & 10 & 0 \end{array}$$

When you divide a polynomial by one of its binomial factors, the quotient is called a depressed polynomial. A **depressed polynomial** has a degree that is one less than the original polynomial. From the results of the division, and by using the Remainder Theorem, we can make the following statement.

$$2x^3 - 3x^2 - 17x + 30 = (2x^2 - 9x + 10) \cdot (x + 3) + 0$$

Since the remainder is 0, $f(-3) = 0$. This means that $x + 3$ is a factor of $2x^3 - 3x^2 - 17x + 30$. This illustrates the **Factor Theorem**, which is a special case of the Remainder Theorem.

KeyConcept Factor Theorem

The binomial $x - r$ is a factor of the polynomial $P(x)$ if and only if $P(r) = 0$.

The Factor Theorem can be used to determine whether a binomial is a factor of a polynomial. It can also be used to determine all of the factors of a polynomial.

Example 3 Use the Factor Theorem

Determine whether $x - 5$ is a factor of $x^3 - 7x^2 + 7x + 15$. Then find the remaining factors of the polynomial.

The binomial $x - 5$ is a factor of the polynomial if 5 is a zero of the related polynomial function. Use the Factor Theorem and synthetic division.

$$\begin{array}{r|rrrr} 5 & 1 & -7 & 7 & 15 \\ & & 5 & -10 & -15 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

StudyTip
Factoring The factors of a polynomial do not have to be binomials. For example, the factors of $x^3 + x^2 - x + 15$ are $x + 3$ and $x^2 - 2x + 5$.

Because the remainder is 0, $x - 5$ is a factor of the polynomial. The polynomial $x^3 - 7x^2 + 7x + 15$ can be factored as $(x - 5)(x^2 - 2x - 3)$. The polynomial $x^2 - 2x - 3$ is the depressed polynomial. Check to see if this polynomial can be factored.

$x^2 - 2x - 3 = (x + 1)(x - 3)$

So, $x^3 - 7x^2 + 7x + 15 = (x - 5)(x + 1)(x - 3)$.

You can check your answer by multiplying out the factors and seeing if you come up with the initial polynomial.

GuidedPractice

3. Show that $x - 2$ is a factor of $x^3 - 7x^2 + 4x + 12$. Then find the remaining factors of the polynomial.

Check Your Understanding

Example 1 **Use synthetic substitution to find $f(4)$ and $f(-2)$ for each function.**

1. $f(x) = 2x^3 - 5x^2 - x + 14$

2. $f(x) = x^4 + 8x^3 + x^2 - 4x - 10$

Example 2

3. NATURE The approximate number of bald eagle nesting pairs in the United States can be modeled by the function $P(x) = -0.16x^3 + 15.83x^2 - 154.15x + 1147.97$, where x is the number of years since 1970. About how many nesting pairs of bald eagles can be expected in 2018?

Example 3 **Given a polynomial and one of its factors, find the remaining factors of the polynomial.**

4. $x^3 - 6x^2 + 11x - 6; x - 1$

5. $x^3 + x^2 - 16x - 16; x + 1$

6. $3x^3 + 10x^2 - x - 12; x - 1$

7. $2x^3 - 5x^2 - 28x + 15; x + 3$

Practice and Problem Solving

Example 1 **Use synthetic substitution to find $f(-5)$ and $f(2)$ for each function.**

8. $f(x) = x^3 + 2x^2 - 3x + 1$

9. $f(x) = x^2 - 8x + 6$

10. $f(x) = 3x^4 + x^3 - 2x^2 + x + 12$

11. $f(x) = 2x^3 - 8x^2 - 2x + 5$

12. $f(x) = x^3 - 5x + 2$

13. $f(x) = x^5 + 8x^3 + 2x - 15$

14. $f(x) = x^6 - 4x^4 + 3x^2 - 10$

15. $f(x) = x^4 - 6x - 8$

Example 2

16. FINANCIAL LITERACY A specific car's fuel economy in miles per gallon can be approximated by $f(x) = 0.00000056x^4 - 0.000018x^3 - 0.016x^2 + 1.38x - 0.38$, where x represents the car's speed in miles per hour. Determine the fuel economy when the car is traveling 40, 50 and 60 miles per hour.

Example 3 **Given a polynomial and one of its factors, find the remaining factors of the polynomial.**

17. $x^3 - 3x + 2; x + 2$

18. $x^4 + 2x^3 - 8x - 16; x + 2$

19. $x^3 - x^2 - 10x - 8; x + 2$

20. $x^3 - x^2 - 5x - 3; x - 3$

21. $2x^3 + 17x^2 + 23x - 42; x - 1$

22. $2x^3 + 7x^2 - 53x - 28; x - 4$

23. $x^4 + 2x^3 + 2x^2 - 2x - 3; x - 1$

24. $x^3 + 2x^2 - x - 2; x + 2$

25. $6x^3 - 25x^2 + 2x + 8; 2x + 1$

26. $16x^5 - 32x^4 - 81x + 162; 2x - 3$

27. BOATING A motor boat traveling against waves accelerates from a resting position. Suppose the speed of the boat in meters per second is given by the function $f(t) = -0.04t^4 + 0.8t^3 + 0.5t^2 - t$, where t is the time in seconds.

a. Find the speed of the boat at 1, 2, and 3 seconds.

b. It takes 6 seconds for the boat to travel between two buoys while it is accelerating. Use synthetic substitution to find $f(6)$ and explain what this means.

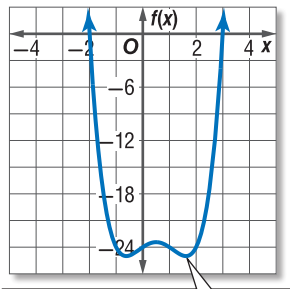
28. REASONING A company's sales, in millions of dirhams, of consumer electronics can be modeled by $S(x) = -1.2x^3 + 18x^2 + 26.4x + 678$, where x is the number of years since 2005.

a. Use synthetic substitution to estimate the sales for 2017 and 2020.

b. Do you think this model is useful in estimating future sales? Explain.

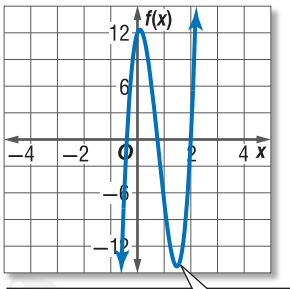
Use the graph to find all of the factors for each polynomial function.

29.



$f(x) = x^4 - 2x^3 - x^2 + 2x - 24$

30.



$f(x) = 20x^3 - 47x^2 + 8x + 12$

31. **MULTIPLE REPRESENTATIONS** In this problem, you will consider the function $f(x) = -9x^5 + 104x^4 - 249x^3 - 456x^2 + 828x + 432$.

a. Algebraic If $x - 6$ is a factor of the function, find the depressed polynomial.

b. Tabular Make a table of values for $-5 \leq x \leq 6$ for the depressed polynomial.

c. Analytical What conclusions can you make about the locations of the other zeros based on the table? Explain your reasoning.

d. Graphical Graph the original function to confirm your conclusions.

PERSEVERANCE **Find values of k so that each remainder is 3.**

32. $(x^2 - x + k) \div (x - 1)$

33. $(x^2 + kx - 17) \div (x - 2)$

34. $(x^2 + 5x + 7) \div (x + k)$

35. $(x^3 + 4x^2 + x + k) \div (x + 2)$

H.O.T. Problems Use Higher-Order Thinking Skills

36. **OPEN ENDED** Write a polynomial function that has a double root of 1 and a double root of −5. Graph the function.

CHALLENGE **Find the solutions of each polynomial function.**

37. $(x^2 - 4)^2 - (x^2 - 4) - 2 = 0$

38. $(x^2 + 3)^2 - 7(x^2 + 3) + 12 = 0$

39. **REASONING** Polynomial $f(x)$ is divided by $x - c$. What can you conclude if:

a. the remainder is 0?

b. the remainder is 1?

c. the quotient is 1, and the remainder is 0?

40. **CHALLENGE** Review the definition for the Factor Theorem. Provide a proof of the theorem.

41. **OPEN ENDED** Write a cubic function that has a remainder of 8 for $f(2)$ and a remainder of −5 for $f(3)$.

42. **CHALLENGE** Show that the quartic function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ will always have a rational zero when the numbers 1, −2, 3, 4, and −6 are randomly assigned to replace a through e, and all of the numbers are used.

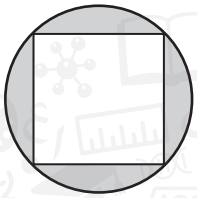
43. **WRITING IN MATH** Explain how the zeros of a function can be located by using the Remainder Theorem and making a table of values for different input values and then comparing the remainders.

Standardized Test Practice

44. $27x^3 + y^3 =$

A $(3x + y)(3x + y)(3x + y)$
B $(3x + y)(9x^2 - 3xy + y^2)$
C $(3x - y)(9x^2 + 3xy + y^2)$
D $(3x - y)(9x^2 + 9xy + y^2)$

45. GRIDDED RESPONSE In the figure, a square with side length $2\sqrt{2}$ is inscribed in a circle. The area of the circle is $k\pi$. What is the exact value of k?



46. What is the product of the complex numbers $(4 + i)(4 - i)$?

F 15
G $16 - i$
H 17
J $17 - 8i$

47. SAT/ACT The measure of the largest angle of a triangle is 14 less than twice the measure of the smallest angle. The third angle measure is 2 more than the measure of the smallest angle. What is the measure of the smallest angle?

A 46
B 48
C 50
D 52
E 82

Spiral Review

Solve each equation. (Lesson 3-5)

48. $x^4 - 4x^2 - 21 = 0$

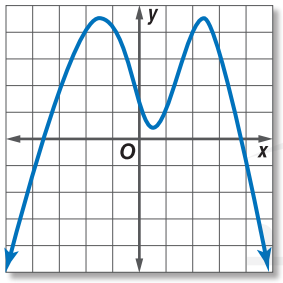
49. $x^4 - 6x^2 = 27$

50. $4x^4 - 8x^2 - 96 = 0$

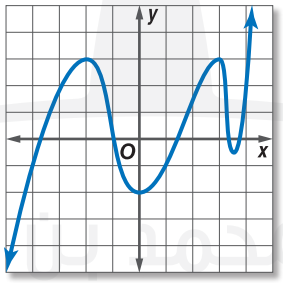
Complete each of the following. (Lesson 3-4)

a. Estimate the x-coordinate of every turning point and determine if those coordinates are relative maxima or relative minima.
b. Estimate the x-coordinate of every zero.
c. Determine the smallest possible degree of the function.
d. Determine the domain and range of the function.

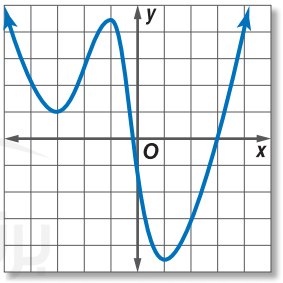
51.



52.



53.



54. HIGHWAY SAFETY Engineers can use the formula $d = 0.05v^2 + 1.1v$ to estimate the minimum stopping distance d in feet for a vehicle traveling v miles per hour. If a car is able to stop after 125 feet, what is the fastest it could have been traveling when the driver first applied the brakes? (Lesson 3-6)

Solve by graphing.

55. $y = 3x - 1$
$y = -2x + 4$

56. $3x + 2y = 8$
$-4x + 6y = 11$

57. $5x - 2y = 6$
$3x - 2y = 2$

Skills Review

If $c(x) = x^2 - 2x$ and $d(x) = 3x^2 - 6x + 4$, find each value.

58. $c(a + 2) - d(a - 4)$

59. $c(a - 3) + d(a + 1)$

60. $c(-3a) + d(a + 4)$

61. $3d(3a) - 2c(-a)$

62. $c(a) + 5d(2a)$

63. $-2d(2a + 3) - 4c(a^2 + 1)$

LESSON 3-7

Roots and Zeros

Then

- You used complex numbers to describe solutions of quadratic equations.

Now

1. Determine the number and type of roots for a polynomial equation.
2. Find the zeros of a polynomial function.

Why?

- The function $g(x) = 1.384x^4 - 0.003x^3 + 0.28x^2 - 0.078x + 1.365$ can be used to model the average price of a gallon of gasoline in a given year if x is the number of years since 1990. To find the average price of gasoline in a specific year, you can use the roots of the related polynomial equation.

Mathematical Practices
Attend to precision.

1 Synthetic Types of Roots

Previously, you learned that a zero of a function $f(x)$ is any value c such that $f(c) = 0$. When the function is graphed, the real zeros of the function are the x-intercepts of the graph.

ConceptSummary Zeros, Factors, Roots, and Intercepts

Words Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial function. Then the following statements are equivalent.

- c is a zero of $P(x)$.
- c is a root or solution of $P(x) = 0$.
- $x - c$ is a factor of $a_n x^n + \cdots + a_1 x + a_0$.
- If c is a real number, then $(c, 0)$ is an x-intercept of the graph of $P(x)$.

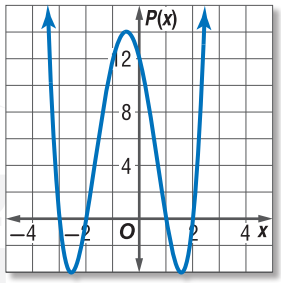
Example Consider the polynomial function $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$.

The zeros of $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$ are -3, -2, 1, and 2.

The roots of $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$ are -3, -2, 1, and 2.

The factors of $x^4 + 2x^3 - 7x^2 - 8x + 12$ are $(x + 3)$, $(x + 2)$, $(x - 1)$, and $(x - 2)$.

The x-intercepts of the graph of $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$ are $(-3, 0)$, $(-2, 0)$, $(1, 0)$, and $(2, 0)$.



When solving a polynomial equation with degree greater than zero, there may be one or more real roots or no real roots (the roots are imaginary numbers). Since real numbers and imaginary numbers both belong to the set of complex numbers, all polynomial equations with degree greater than zero will have at least one root in the set of complex numbers. This is the **Fundamental Theorem of Algebra**.

KeyConcept Fundamental Theorem of Algebra

Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.

Example 1 Determine Number and Type of Roots

Solve each equation. State the number and type of roots.

a. $x^2 + 6x + 9 = 0$

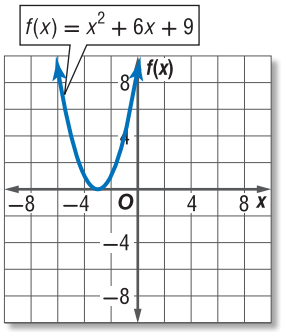
$x^2 + 6x + 9 = 0$

$(x + 3)^2 = 0$

$x + 3 = 0$

$x = -3$

Because $(x + 3)$ is twice a factor of $x^2 + 6x + 9$, -3 is a double root. Thus, the equation has one real repeated root, -3.



CHECK The graph of the equation touches the x-axis at $x = -3$. Since -3 is a double root, the graph does not cross the axis. ✓

b. $x^3 + 25x = 0$

$x^3 + 25x = 0$

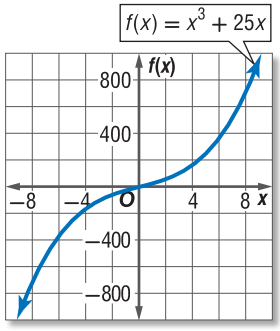
$x(x^2 + 25) = 0$

$x = 0$ or $x^2 + 25 = 0$

$x^2 = -25$

$x = \pm\sqrt{-25}$ or $\pm 5i$

This equation has one real root, 0, and two imaginary roots, $5i$ and $-5i$.



CHECK The graph of this equation crosses the x-axis at only one place, $x = 0$. ✓

ReadingMath

Repeated Roots Polynomial equations can have double roots, triple roots, quadruple roots, and so on. In general, these are referred to as *multiple roots*.

GuidedPractice

1A. $x^3 + 2x = 0$

1B. $x^4 - 16 = 0$

1C. $x^3 + 4x^2 - 7x - 10 = 0$

1D. $3x^3 - x^2 + 9x - 3 = 0$

Examine the solutions for each equation in Example 1. Notice that the number of solutions for each equation is the same as the degree of each polynomial. The following corollary to the Fundamental Theorem of Algebra describes this relationship between the degree and the number of roots of a polynomial equation.

KeyConcept Corollary to the Fundamental Theorem of Algebra

Words	A polynomial equation of degree n has exactly n roots in the set of complex numbers, including repeated roots.		
Example	$x^3 + 2x^2 + 6$	$4x^4 - 3x^3 + 5x - 6$	$-2x^5 - 3x^2 + 8$
	3 roots	4 roots	5 roots

Additionally, French mathematician René Descartes discovered a relationship between the signs of the coefficients of a polynomial function and the number of positive and negative real zeros.

StudyTip

Zero at the Origin If a zero of a function is at the origin, the sum of the number of positive real zeros, negative real zeros, and imaginary zeros is reduced by how many times 0 is a zero of the function.

KeyConcept Descartes' Rule of Signs

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial function with real coefficients. Then

- the number of positive real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and
- the number of negative real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this by an even number.

Example 2 Find Numbers of Positive and Negative Zeros

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $f(x) = x^6 + 3x^5 - 4x^4 - 6x^3 + x^2 - 8x + 5$.

Because $f(x)$ has degree 6, it has six zeros, either real or imaginary. Use Descartes' Rule of Signs to determine the possible number and type of *real* zeros.

Count the number of changes in sign for the coefficients of $f(x)$.

$$f(x) = x^6 + 3x^5 - 4x^4 - 6x^3 + x^2 - 8x + 5$$

There are 4 sign changes, so there are 4, 2, or 0 positive real zeros.

Count the number of changes in sign for the coefficients of $f(-x)$.

$$\begin{aligned} f(-x) &= (-x)^6 + 3(-x)^5 - 4(-x)^4 - 6(-x)^3 + (-x)^2 - 8(-x) + 5 \\ &= x^6 - 3x^5 - 4x^4 + 6x^3 + x^2 + 8x + 5 \end{aligned}$$

There are 2 sign changes, so there are 2, or 0 negative real zeros.
Make a chart of the possible combinations of real and imaginary zeros.

Number of Positive Real Zeros	Number of Negative Real Zeros	Number of Imaginary Zeros	Total Number of Zeros
4	2	0	$4 + 2 + 0 = 6$
4	0	2	$4 + 0 + 2 = 6$
2	2	2	$2 + 2 + 2 = 6$
2	0	4	$2 + 0 + 4 = 6$
0	2	4	$0 + 2 + 4 = 6$
0	0	6	$0 + 0 + 6 = 6$

GuidedPractice

2. State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $h(x) = 2x^5 + x^4 + 3x^3 - 4x^2 - x + 9$.

2 Find Zeros You can use the various strategies and theorems you have learned to find all of the zeros of a function.

Example 3 Use Synthetic Substitution to Find Zeros

Find all of the zeros of $f(x) = x^4 - 18x^2 + 12x + 80$.

Step 1 Determine the total number of zeros.

Since $f(x)$ has degree 4, the function has 4 zeros.

Step 2 Determine the type of zeros.

Examine the number of sign changes for $f(x)$ and $f(-x)$.

$f(x) = x^4 - 18x^2 + 12x + 80$ $\qquad$ $f(-x) = x^4 - 18x^2 - 12x + 80$

Because there are 2 sign changes for the coefficients of $f(x)$, the function has 2 or 0 positive real zeros. Because there are 2 sign changes for the coefficients of $f(-x)$, $f(x)$ has 2 or 0 negative real zeros. Thus, $f(x)$ has 4 real zeros, 2 real zeros and 2 imaginary zeros, or 4 imaginary zeros.

StudyTip

Testing for Zeros If a value is not a zero for a polynomial, then it will not be a zero for the depressed polynomial either, so it does not need to be checked again.

Step 3 Determine the real zeros.

List some possible values, and then use synthetic substitution to evaluate $f(x)$ for real values of x.

x	1	0	−18	12	80
−3	1	−3	−9	39	−37
−2	1	**−2**	**−14**	**40**	**0**
−1	1	−1	−17	29	51
0	1	0	−18	12	80
1	1	1	−17	−5	75
2	1	2	−14	−2	76

Each row shows the coefficients of the depressed polynomial and the remainder.

From the table, we can see that one zero occurs at $x = -2$. Since there are 2 negative real zeros, use synthetic substitution with the depressed polynomial function $f(x) = x^3 - 2x^2 - 14x + 40$ to find a second negative zero.

x	1	−2	−14	40
−4	1	**−6**	**10**	**0**
−5	1	−7	21	−65
−6	1	−8	34	−164

A second negative zero is at $x = -4$. Since the depressed polynomial $x^2 - 6x + 10$ is quadratic, use the Quadratic Formula to find the remaining zeros of $f(x) = x^2 - 6x + 10$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)}$$

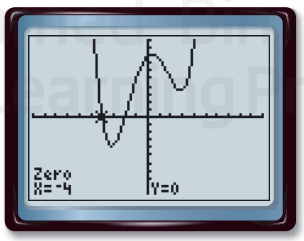
$$= 3 \pm i$$

The function has zeros at -4, -2, $3 + i$, and $3 - i$.

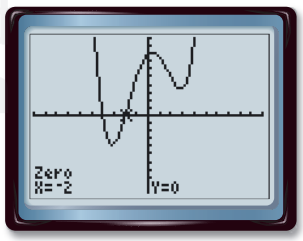
StudyTip

Locating Zeros Refer to Lesson 3-2 on how to use the CALC menu to locate a zero on your calculator.

CHECK Graph the function on a graphing calculator. The graph crosses the x-axis two times, so there are two real zeros. Use the zero function under the **CALC** menu to locate each zero. The two real zeros are -4 and -2.



[−10, 10] scl: 1 by [−100, 100] scl: 10



[−10, 10] scl: 1 by [−100, 100] scl: 10

GuidedPractice

3. Find all of the zeros of $h(x) = x^3 + 2x^2 + 9x + 18$.

ReviewVocabulary

complex conjugates two complex numbers of the form $a + bi$ and $a - bi$

In Chapter 3, you learned that the product of complex conjugates is always a real number and that complex roots always come in conjugate pairs. For example, if one root of $x^2 - 8x + 52 = 0$ is $4 + 6i$, then the other root is $4 - 6i$.

This applies to the zeros of polynomial functions as well. For any polynomial function with real coefficients, if an imaginary number is a zero of that function, its conjugate is also a zero. This is called the **Complex Conjugates Theorem**.

KeyConcept Complex Conjugates Theorem

Words	Let a and b be real numbers, and $b \neq 0$. If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.
Example	If $3 + 4i$ is a zero of $f(x) = x^3 - 4x^2 + 13x + 50$, then $3 - 4i$ is also a zero of the function.

When you are given all of the zeros of a polynomial function and are asked to determine the function, convert the zeros to factors and then multiply all of the factors together. The result is the polynomial function.

Example 4 Use Zeros to Write a Polynomial Function

Write a polynomial function of least degree with integral coefficients, the zeros of which include -1 and $5 - i$.

Understand If $5 - i$ is a zero, then $5 + i$ is also a zero according to the Complex Conjugates Theorem. So, $x + 1$, $x - (5 - i)$, and $x - (5 + i)$ are factors of the polynomial.

Plan Write the polynomial function as a product of its factors.

$$P(x) = (x + 1)[x - (5 - i)][x - (5 + i)]$$

Solve Multiply the factors to find the polynomial function.

$$\begin{aligned} P(x) &= (x + 1)\,[x - (5 - i)][x - (5 + i)] \\ &= (x + 1)\,[(x - 5) + i][(x - 5) - i] \\ &= (x + 1)\,[(x - 5)^2 - i^2] \\ &= (x + 1)\,[(x^2 - 10x + 25 - (-1)] \\ &= (x + 1)\,(x^2 - 10x + 26) \\ &= x^3 - 10x^2 + 26x + x^2 - 10x + 26 \\ &= x^3 - 9x^2 + 16x + 26 \end{aligned}$$

Check Because there are 3 zeros, the degree of the polynomial function must be 3, so $P(x) = x^3 - 9x^2 + 16x + 26$ is a polynomial function of least degree with integral coefficients and zeros of -1, $5 - i$, and $5 + i$.

GuidedPractice

4. Write a polynomial function of least degree with integral coefficients having zeros that include -1 and $1 + 2i$.

Check Your Understanding

Example 1 **Solve each equation. State the number and type of roots.**

1. $x^2 - 3x - 10 = 0$

2. $x^3 + 12x^2 + 32x = 0$

3. $16x^4 - 81 = 0$

4. $0 = x^3 - 8$

Example 2 **State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.**

5. $f(x) = x^3 - 2x^2 + 2x - 6$

6. $f(x) = 6x^4 + 4x^3 - x^2 - 5x - 7$

7. $f(x) = 3x^5 - 8x^3 + 2x - 4$

8. $f(x) = -2x^4 - 3x^3 - 2x - 5$

Example 3 **Find all of the zeros of each function.**

9. $f(x) = x^3 + 9x^2 + 6x - 16$

10. $f(x) = x^3 + 7x^2 + 4x + 28$

11. $f(x) = x^4 - 2x^3 - 8x^2 - 32x - 384$

12. $f(x) = x^4 - 6x^3 + 9x^2 + 6x - 10$

Example 4 **Write a polynomial function of least degree with integral coefficients that have the given zeros.**

13. $4, -1, 6$

14. $3, -1, 1, 2$

15. $-2, 5, -3i$

16. $-4, 4 + i$

Practice and Problem Solving

Example 1 **Solve each equation. State the number and type of roots.**

17. $2x^2 + x - 6 = 0$

18. $4x^2 + 1 = 0$

19. $x^3 + 1 = 0$

20. $2x^2 - 5x + 14 = 0$

21. $-3x^2 - 5x + 8 = 0$

22. $8x^3 - 27 = 0$

23. $16x^4 - 625 = 0$

24. $x^3 - 6x^2 + 7x = 0$

25. $x^5 - 8x^3 + 16x = 0$

26. $x^5 + 2x^3 + x = 0$

Example 2 **State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.**

27. $f(x) = x^4 - 5x^3 + 2x^2 + 5x + 7$

28. $f(x) = 2x^3 - 7x^2 - 2x + 12$

29. $f(x) = -3x^5 + 5x^4 + 4x^2 - 8$

30. $f(x) = x^4 - 2x^2 - 5x + 19$

31. $f(x) = 4x^6 - 5x^4 - x^2 + 24$

32. $f(x) = -x^5 + 14x^3 + 18x - 36$

Example 3 **Find all of the zeros of each function.**

33. $f(x) = x^3 + 7x^2 + 4x - 12$

34. $f(x) = x^3 + x^2 - 17x + 15$

35. $f(x) = x^4 - 3x^3 - 3x^2 - 75x - 700$

36. $f(x) = x^4 + 6x^3 + 73x^2 + 384x + 576$

37. $f(x) = x^4 - 8x^3 + 20x^2 - 32x + 64$

38. $f(x) = x^5 - 8x^3 - 9x$

Example 4 **Write a polynomial function of least degree with integral coefficients that have the given zeros.**

39. $5, -2, -1$

40. $-4, -3, 5$

41. $-1, -1, 2i$

42. $-3, 1, -3i$

43. $0, -5, 3 + i$

44. $-2, -3, 4 - 3i$

45. **REASONING** A computer manufacturer determines that the profit for producing x computers per day is $P(x) = -0.006x^4 + 0.15x^3 - 0.05x^2 - 1.8x$.

a. How many positive real zeros, negative real zeros, and imaginary zeros exist?

b. What is the meaning of the zeros in this situation?

Sketch the graph of each function using its zeros.

46. $f(x) = x^3 - 5x^2 - 2x + 24$

47. $f(x) = 4x^3 + 2x^2 - 4x - 2$

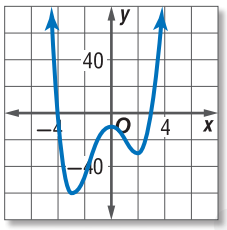
48. $f(x) = x^4 - 6x^3 + 7x^2 + 6x - 8$

49. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

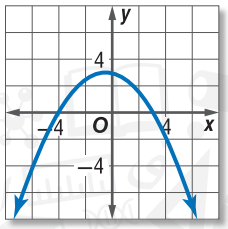
Match each graph to the given zeros.

a. $-3, 4, i, -i$ **b.** $-4, 3$ **c.** $-4, 3, i, -i$

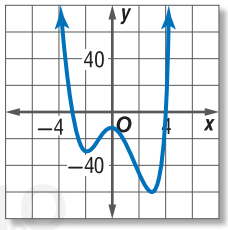
50.



51.



52.



53. CONCERTS The amount of money Hamdan's Music Hall took in from 2003 to 2010 can be modeled by $M(x) = -2.03x^3 + 50.1x^2 - 214x + 4020$, where x is the years since 2003.

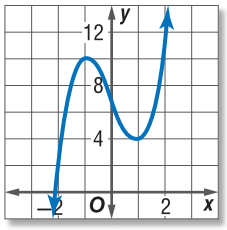
a. How many positive real zeros, negative real zeros, and imaginary zeros exist?

b. Graph the function using your calculator.

c. Approximate all real zeros to the nearest tenth. What is the significance of each zero in the context of the situation?

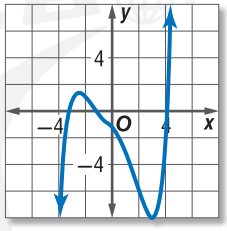
Determine the number of positive real zeros, negative real zeros, and imaginary zeros for each function. Explain your reasoning.

54.



degree: 3

55.



degree: 5

H.O.T. Problems Use Higher-Order Thinking Skills

56. OPEN ENDED Sketch the graph of a polynomial function with:

a. 3 real, 2 imaginary zeros **b.** 4 real zeros **c.** 2 imaginary zeros

57. CHALLENGE Write an equation in factored form of a polynomial function of degree 5 with 2 imaginary zeros, 1 nonintegral zero, and 2 irrational zeros. Explain.

58. ARGUMENTS Determine which equation is not like the others. Explain.

$r^4 + 1 = 0$	$r^3 + 1 = 0$	$r^2 - 1 = 0$	$r^3 - 8 = 0$

59. REASONING Provide a counterexample for each statement.

a. All polynomial functions of degree greater than 2 have at least 1 negative real root.

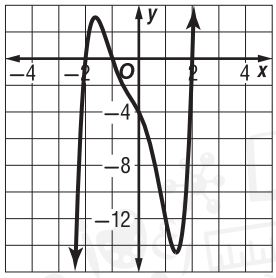
b. All polynomial functions of degree greater than 2 have at least 1 positive real root.

60. WRITING IN MATH Explain to a friend how you would use Descartes' Rule of Signs to determine the number of possible positive real roots and the number of possible negative roots of the polynomial function $f(x) = x^4 - 2x^3 + 6x^2 + 5x - 12$.

Standardized Test Practice

61. Use the graph of the polynomial function below. Which is not a factor of the polynomial $x^5 + x^4 - 3x^3 - 3x^2 - 4x - 4$?

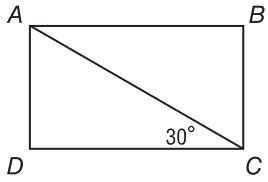
A $x - 2$
B $x + 2$
C $x - 1$
D $x + 1$



62. SHORT RESPONSE A window is in the shape of an equilateral triangle. Each side of the triangle is 8 feet long. The window is divided in half by a support from one vertex to the midpoint of the side of the triangle opposite the vertex. Approximately how long is the support?

63. GEOMETRY In rectangle $ABCD$, $\overline{AD}$ is 8 units long. What is the length of $\overline{AB}$?

F 4 units
G 8 units
H $8\sqrt{3}$ units
J 16 units



64. SAT/ACT The total area of a rectangle is $25a^4 - 16b^2$ square units. Which factors could represent the length and width?

A $(5a^2 + 4b)$ units and $(5a^2 + 4b)$ units
B $(5a^2 + 4b)$ units and $(5a^2 - 4b)$ units
C $(5a^2 - 4b)$ units and $(5a^2 - 4b)$ units
D $(5a - 4b)$ units and $(5a - 4b)$ units
E $(5a + 4b)$ units and $(5a - 4b)$ units

Spiral Review

Use synthetic substitution to find $f(-8)$ and $f(4)$ for each function. (Lesson 3-6)

65. $f(x) = 4x^3 + 6x^2 - 3x + 2$

66. $f(x) = 5x^4 - 2x^3 + 4x^2 - 6x$

67. $f(x) = 2x^5 - 3x^3 + x^2 - 4$

Factor completely. If the polynomial is not factorable, write *prime*. (Lesson 3-5)

68. $x^6 - y^6$

69. $a^6 + b^6$

70. $4x^2y + 8xy + 16y - 3x^2z - 6xz - 12z$

71. $5a^3 - 30a^2 + 40a + 2a^2b - 12ab + 16b$

Find the rate of change for each set of data.

72.

a.

Time (day)	3	6	9	12	15
Height (mm)	12	24	36	48	60

b.

Time (h)	2	4	6	8
Distance (mi)	35	70	105	140

c.

Time (s)	12	16	20	24	28
Volume (cm^3)	45	60	75	90	105

d.

Force (N)	32	40	48	56	64
Work (J)	48	60	72	84	96

73. RECREATION Ahmed estimates that he will need 50 tennis balls for every player that signs up for the tennis club and at least 150 more just in case. Write an inequality to express the situation.

Skills Review

Find all of the possible values of $\pm\frac{b}{a}$ for each replacement set.

74. $a = \{1, 2, 4\}$; $b = \{1, 2, 3, 6\}$

75. $a = \{1, 5\}$; $b = \{1, 2, 4, 8\}$

76. $a = \{1, 2, 3, 6\}$; $b = \{1, 7\}$

EXTEND

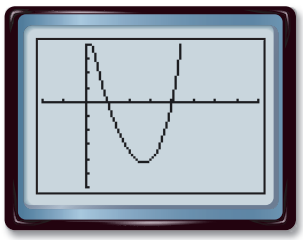
3-7 Graphing Technology Lab: Analyzing Polynomial Functions

You can use graphing technology to help you identify real zeros, maximum and minimum points, number and type of zeros, y-intercepts, and symmetry of polynomial functions.

Activity Identify Polynomial Characteristics

Graph each function. Identify the real zeros, maximum and minimum points, number and type of zeros, y-intercepts, and symmetry.

a. $g(x) = 3x^4 - 15x^3 + 87x^2 - 375x + 300$



[−2, 8] scl: 1 by [−300, 200] scl: 50

Step 1 Graph the equation.

Step 2 Use 2nd [CALC] **zero** to find the real zeros at $x = 1$ and $x = 4$.

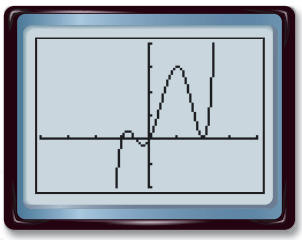
Step 3 Use 2nd [CALC] **minimum** to find the relative minimum at (2.68, −214.11). There is no relative maximum point.

Step 4 $g(x)$ has degree 4 and can have at most 4 zeros. Two real zeros were found through graphing. The other two zeros are either multiple zeros or imaginary zeros.

Step 5 Use 2nd [CALC] **value** 0 to find the y-intercept, 300.

Step 6 The line of symmetry passes through the vertex. Its equation is $x = 2.68$.

b. $f(x) = 2x^5 - 5x^4 - 3x^3 + 8x^2 + 4x$



[−4, 4] scl: 1 by [−4, 8] scl: 2

Step 1 Graph the equation.

Step 2 Locate the real zeros at $x = -1$, $x = -\frac{1}{2}$, $x = 0$ and $x = 2$.

Step 3 Find the relative maxima at (−0.81, 0.75) and (1.04, 6.02) and the relative minima at (−0.24, −0.48) and (2, 0).

Step 4 $f(x)$ has degree 5 and can have at most 5 zeros. Four real zeros were found through graphing. The other zero is either a multiple zero or an imaginary zero. In this case, there is a double zero at $x = 2$.

Step 5 The y-intercept is 0 because the graph goes through the origin.

Step 6 There is no symmetry.

Exercises

Graph each function. Identify the real zeros, maximum and minimum points, number and type of zeros, y-intercepts, and symmetry.

1. $f(x) = x^3 - 5x^2 + 6x$
2. $g(x) = x^4 - 3x^2 - 4$
3. $k(x) = -x^4 - x^3 + 2x^2$
4. $f(x) = -2x^3 - 4x^2 + 16x$
5. $g(x) = 3x^5 - 18x^4 + 27x^3$
6. $k(x) = x^4 - 8x^2 + 15$
7. $f(x) = -x^3 + 2x^2 + 8x$
8. $g(x) = x^5 + 3x^4 - 10x^2$

LESSON 3-8

Rational Zero Theorem

Then

- You found zeros of quadratic functions of the form $f(x) = ax^2 + bx + c$.

Now

1. Identify possible rational zeros of a polynomial function.
2. Find all of the rational zeros of a polynomial function.

Why?

- Annual sales of recorded music in the United States can be approximated by $d(t) = 30x^3 - 478x^2 + 1758x + 10{,}092$, where $d(t)$ is the total sales in millions of dirhams and t is the number of years since 2005. You can use this function to estimate when music sales will be AED 9 billion.

Mathematical Practices
Look for and express regularity in repeated reasoning.

1 Identify Rational Zeros

Usually it is not practical to test all possible zeros of a polynomial function using synthetic substitution. The **Rational Zero Theorem** can help you choose some possible zeros to test.

KeyConcept Rational Zero Theorem

Words If $P(x)$ is a polynomial function with integral coefficients, then every rational zero of $P(x) = 0$ is of the form $\frac{p}{q}$, a rational number in simplest form, where p is a factor of the constant term and q is a factor of the leading coefficient.

Example Let $f(x) = 6x^4 + 22x^3 + 11x^2 - 80x - 40$. If $\frac{4}{3}$ is a zero of $f(x)$, then 4 is a factor of -40, and 3 is a factor of 6.

Corollary to the Rational Zero Theorem

If $P(x)$ is a polynomial function with integral coefficients, a leading coefficient of 1, and a nonzero constant term, then any rational zeros of $P(x)$ must be factors of the constant term.

Example 1 Identify Possible Zeros

List all of the possible rational zeros of each function.

a. $f(x) = 4x^5 + x^4 - 2x^3 - 5x^2 + 8x + 16$

If $\frac{p}{q}$ is a rational zero, then p is a factor of 16 and q is a factor of 4.

p: $\pm1, \pm2, \pm4, \pm8, \pm16$ q: $\pm1, \pm2, \pm4$

Write the possible values of $\frac{p}{q}$ in simplest form.

$\frac{p}{q} = \pm1, \pm2, \pm4, \pm8, \pm16, \pm\frac{1}{2}, \pm\frac{1}{4}$

b. $f(x) = x^3 - 2x^2 + 5x + 12$

If $\frac{p}{q}$ is a rational zero, then p is a factor of 12 and q is a factor of 1.

p: $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$ q: ±1

So, $\frac{p}{q} = \pm1, \pm2, \pm3, \pm4, \pm6$, and ±12

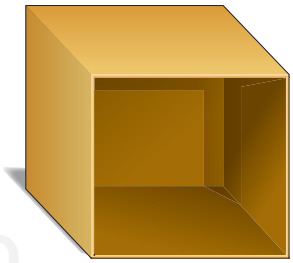
GuidedPractice

1A. $g(x) = 3x^3 - 4x + 10$ **1B.** $h(x) = x^3 + 11x^2 + 24$

2 Find Rational Zeros Once you have written the possible rational zeros, you can test each number using synthetic substitution and use the other tools you have learned to determine the zeros of a function.

Real-World Example 2 Find Rational Zeros

WOODWORKING **Ahmed is building a computer desk with a separate compartment for the computer. The compartment for the computer is a rectangular prism and will be 8019 cubic centimeters. The compartment will be 24 centimeters longer than it is wide and the height will be 18 centimeters greater than the width. Find the dimensions of the computer compartment.**



Let x = width, $x + 24$ = length, and $x + 18$ = height.

Write an equation for the volume.

$$\ell wh = V$$

$$(x + 24)(x)(x + 18) = 8019$$

$$x^3 + 42x^2 + 432x = 8019$$

$$x^3 + 42x^2 + 432x - 8019 = 0$$

The leading coefficient is 1, so the possible rational zeros are factors of 8019.

$\pm 1, \pm 3, \pm 9, \pm 11, \pm 27, \pm 33, \pm 81, \pm 99, \pm 243, \pm 297, \pm 729, \pm 891, \pm 2673$, and ± 8019

StudyTip

Structure Examine the signs of the coefficients of the equation. In this case, there is only one change of sign, so there is only one positive real zero.

Since length can only be positive, we only need to check positive values.

There is one change of sign of the coefficients, so by Descartes' Rule of Signs, there is only one positive real zero. Make a table for synthetic division and test possible values.

p	1	42	432	−8019
1	1	43	475	−7544
2	1	45	567	−6318
9	1	51	891	0

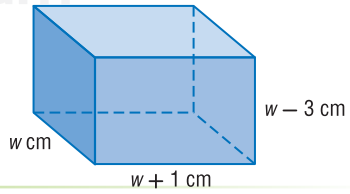
One zero is 9. Since there is only one positive real zero, we do not have to test the other numbers. The other dimensions are 9 + 24 or 33 inches, and 9 + 18 or 27 inches.

CHECK Multiply the dimensions and see if they equal the volume of 8019 cubic centimeters.

$9 \times 33 \times 27 = 8019$ ✓

GuidedPractice

2. GEOMETRY The volume of a rectangular prism is 1056 cubic centimeters. The length is 1 centimeter more than the width, and the height is 3 centimeters less than the width. Find the dimensions of the prism.



You usually do not need to test all of the possible zeros. Once you find a zero, you can try to factor the depressed polynomial to find any other zeros.

Example 3 Find All Zeros

Find all of the zeros of $f(x) = 5x^4 - 8x^3 + 41x^2 - 72x - 36$.

From the corollary to the Fundamental Theorem of Algebra, there are exactly 4 complex zeros. According to Descartes' Rule of Signs, there are 3 or 1 positive real zeros and exactly 1 negative real zero. The possible rational zeros are ±1, ±2, ±3, ±4, ±6, ±9, ±12, ±18, ±36, $\pm\frac{1}{5}$, $\pm\frac{2}{5}$, $\pm\frac{3}{5}$, $\pm\frac{4}{5}$, $\pm\frac{6}{5}$, $\pm\frac{9}{5}$, $\pm\frac{12}{5}$, $\pm\frac{18}{5}$, and $\pm\frac{36}{5}$.

Make a table and test some possible rational zeros.

$\frac{p}{q}$	5	−8	41	−72	−36
−1	5	−13	54	−126	90
1	5	−3	38	−34	−70
2	5	2	45	18	0

Because $f(2) = 0$, there is a zero at $x = 2$. Factor the depressed polynomial $5x^3 + 2x^2 + 45x + 18$.

$$5x^3 + 2x^2 + 45x + 18 = 0$$
$$(5x^3 + 2x^2) + (45x + 18) = 0$$
$$x^2(5x + 2) + 9(5x + 2) = 0$$
$$(x^2 + 9)(5x + 2) = 0$$
$$x^2 + 9 = 0 \quad \text{or} \quad 5x + 2 = 0$$
$$x^2 = -9 \qquad 5x = -2$$
$$x = \pm 3i \qquad x = -\frac{2}{5}$$

There is another real zero at $x = -\frac{2}{5}$ and two imaginary zeros at $x = 3i$ and $x = -3i$.

The zeros of the function are $-\frac{2}{5}$, 2, $3i$, and $-3i$.

GuidedPractice

Find all of the zeros of each function.

3A. $h(x) = 9x^4 + 5x^2 - 4$

3B. $k(x) = 2x^4 - 5x^3 + 20x^2 - 45x + 18$

Check Your Understanding

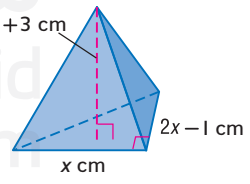
Example 1 **List all of the possible rational zeros of each function.**

1. $f(x) = x^3 - 6x^2 - 8x + 24$

2. $f(x) = 2x^4 + 3x^2 - x + 15$

Example 2

3. REASONING The volume of the triangular pyramid is 210 cubic centimeters. Find the dimensions of the solid.



Find all of the rational zeros of each function.

4. $f(x) = x^3 - 6x^2 - 13x + 42$

5. $f(x) = 2x^4 + 11x^3 + 26x^2 + 29x + 12$

Example 3 **Find all of the zeros of each function.**

6. $f(x) = 3x^3 - 2x^2 - 8x + 5$

7. $f(x) = 8x^3 + 14x^2 + 11x + 3$

8. $f(x) = 4x^4 + 13x^3 - 8x^2 + 13x - 12$

9. $f(x) = 4x^4 - 12x^3 + 25x^2 - 14x - 15$

Practice and Problem Solving

Example 1 **List all of the possible rational zeros of each function.**

10. $f(x) = x^4 + 8x - 32$

11. $f(x) = x^3 + x^2 - x - 56$

12. $f(x) = 2x^3 + 5x^2 - 8x - 10$

13. $f(x) = 3x^6 - 4x^4 - x^2 - 35$

14. $f(x) = 6x^5 - x^4 + 2x^3 - 3x^2 + 2x - 18$

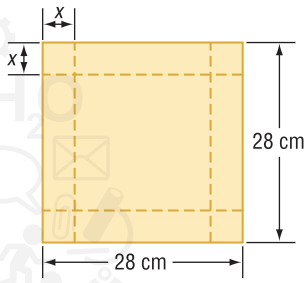
15. $f(x) = 8x^4 - 4x^3 - 4x^2 + x + 42$

16. $f(x) = 15x^3 + 6x^2 + x + 90$

17. $f(x) = 16x^4 - 5x^2 + 128$

Example 2

18. **MANUFACTURING** A box is to be constructed by cutting out equal squares from the corners of a square piece of cardboard and turning up the sides.



a. Write a function $V(x)$ for the volume of the box.

b. For what value of x will the volume of the box equal 1152 cubic centimeters?

c. What will be the volume of the box if $x = 6$ centimeters?

Find all of the rational zeros of each function.

19. $f(x) = x^3 + 10x^2 + 31x + 30$

20. $f(x) = x^3 - 2x^2 - 56x + 192$

21. $f(x) = 4x^3 - 3x^2 - 100x + 75$

22. $f(x) = 4x^4 + 12x^3 - 5x^2 - 21x + 10$

23. $f(x) = x^4 + x^3 - 8x - 8$

24. $f(x) = 2x^4 - 3x^3 - 24x^2 + 4x + 48$

25. $f(x) = 4x^3 + x^2 + 16x + 4$

26. $f(x) = 81x^4 - 256$

Example 3 **Find all of the zeros of each function.**

27. $f(x) = x^3 + 3x^2 - 25x + 21$

28. $f(x) = 6x^3 + 5x^2 - 9x + 2$

29. $f(x) = x^4 - x^3 - x^2 - x - 2$

30. $f(x) = 10x^3 - 17x^2 - 7x + 2$

31. $f(x) = x^4 - 3x^3 + x^2 - 3x$

32. $f(x) = 6x^3 + 11x^2 - 3x - 2$

33. $f(x) = 6x^4 + 22x^3 + 11x^2 - 38x - 40$

34. $f(x) = 2x^3 - 7x^2 - 8x + 28$

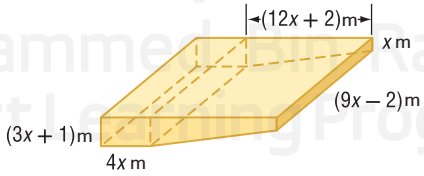
35. $f(x) = 9x^5 - 94x^3 + 27x^2 + 40x - 12$

36. $f(x) = x^5 - 2x^4 - 12x^3 - 12x^2 - 13x - 10$

37. $f(x) = 48x^4 - 52x^3 + 13x - 3$

38. $f(x) = 5x^4 - 29x^3 + 55x^2 - 28x$

39. **SWIMMING POOLS** A diagram of the swimming pool at the Midtown Community Center is shown below. The pool can hold 9175 cubic meters of water.



a. Write a polynomial function that represents the volume of the swimming pool.

b. What are the possible values of x? Which of these values are reasonable?

40. **MODELING** A portion of the path of a certain roller coaster can be modeled by $f(t) = t^4 - 31t^3 + 308t^2 - 1100t + 1200$ where t represents the time in seconds and $f(t)$ represents the height of the roller coaster. Use the Rational Zero Theorem to determine the four times at which the roller coaster is at ground level.

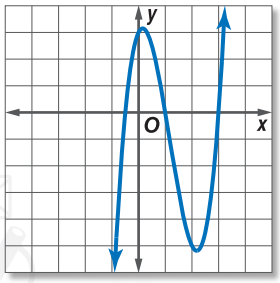
41. FOOD A restaurant orders spaghetti sauce in cylindrical metal cans. The volume of each can is about 160π cubic inches, and the height of the can is 6 inches more than the radius.

a. Write a polynomial equation that represents the volume of a can. Use the formula for the volume of a cylinder, $V = \pi r^2 h$.

b. What are the possible values of r? Which of these values are reasonable for this situation?

c. Find the dimensions of the can.

42. Refer to the graph at the right.



a. Find all of the zeros of $f(x) = 2x^3 + 7x^2 + 2x - 3$ and $g(x) = 2x^3 - 7x^2 + 2x + 3$.

b. Determine which function, f or g, is shown in the graph at the right.

43. MUSIC SALES Refer to the beginning of the lesson.

a. Write a polynomial equation that could be used to determine the year in which music sales would be about AED 9,000,000,000.

b. List the possible whole number solutions for your equation in part **a**.

c. Determine the approximate year in which music sales will be 9,000,000,000.

d. Does the model represent a realistic estimate for all future music sales? Explain your reasoning.

Find all of the zeros of each function.

44. $f(x) = x^5 + 3x^4 - 19x^3 - 43x^2 + 18x + 40$

45. $f(x) = x^5 - x^4 - 23x^3 + 33x^2 + 126x - 216$

H.O.T. Problems Use Higher-Order Thinking Skills

46. CRITIQUE Ayoub and Nasser are listing all of the possible rational zeros for $f(x) = 4x^4 + 8x^5 + 10x^2 + 3x + 16$. Is either of them correct? Explain your reasoning.

Ayoub	Nasser
$\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm\frac{1}{2}, \pm\frac{1}{4}$	$\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}$

47. CHALLENGE Give a polynomial function that has zeros at $1 + \sqrt{3}$ and $5 + 2i$.

48. REASONING Determine if the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

If all of the possible zeros of a polynomial function are integers, then the leading coefficient of the function is 1 or −1.

49. OPEN ENDED Write a function that has possible zeros of $\pm 18, \pm 9, \pm 6, \pm 3, \pm 2, \pm 1, \pm\frac{9}{4}, \pm\frac{9}{2}, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{1}{2}$, and $\pm\frac{1}{4}$.

50. CHALLENGE The roots of $x^2 + bx + c = 0$ are M and N. If $|M - N| = 1$, express c in terms of b.

51. WRITING IN MATH How can you find the zeros of a polynomial function?

Standardized Test Practice

52. ALGEBRA Which of the following is a zero of the function $f(x) = 12x^5 - 5x^3 + 2x - 9$?

A -6 C $\frac{3}{8}$

B $-\frac{2}{3}$ D 1

53. SAT/ACT How many negative real zeros does $f(x) = x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$ have?

F 5 J 1

G 3 K 0

H 2

54. ALGEBRA For all nonnegative numbers n, let $\boxed{n}$ be defined by $\boxed{n} = \frac{\sqrt{n}}{2}$. If $\boxed{n} = 4$, what is the value of n?

A 2 C 16

B 4 D 64

55. GRIDDED RESPONSE What is the y-intercept of a line that contains the point $(-1, 4)$ and has the same x-intercept as $x + 2y = -3$?

Spiral Review

Write a polynomial function of least degree with integral coefficients that has the given zeros. (Lesson 3-7)

56. $6, -3, \sqrt{2}$ **57.** $5, -1, 4i$ **58.** $-4, -2, i\sqrt{2}$

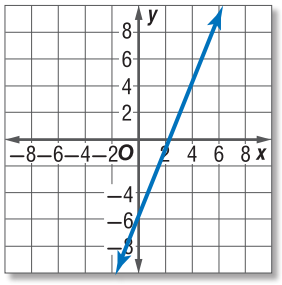
Given a polynomial and one of its factors, find the remaining factors of the polynomial. (Lesson 3-6)

59. $x^4 + 5x^3 + 5x^2 - 5x - 6; x + 3$ **60.** $a^4 - 2a^3 - 17a^2 + 18a + 72; a - 3$ **61.** $x^4 + x^3 - 11x^2 + x - 12; x + i$

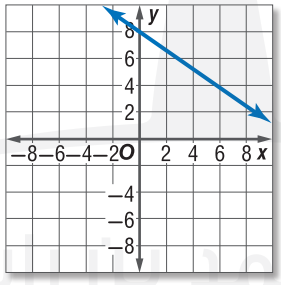
Write an equation in slope-intercept form for each graph.

62.

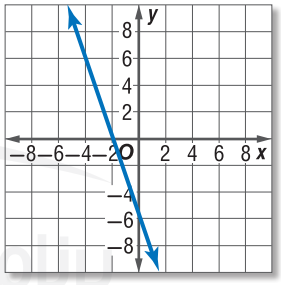
a.



b.



c.



63. DODGEBALL Six teams played in a dodgeball tournament. In how many ways can the top three teams finish?

Skills Review

Simplify.

64. $(x - 4)(x + 3)$ **65.** $3x(x^2 + 4)$ **66.** $x^2(x - 2)(x + 1)$

Find each value if $f(x) = 6x + 2$ and $g(x) = -4x^2$.

67. $f(5)$ **68.** $g(-3)$ **69.** $f(3c)$

CHAPTER 3

Study Guide and Review

Study Guide

KeyConcepts

Operations with Polynomials (Lessons 3-1 and 3-2)

- To add or subtract: Combine like terms.
- To multiply: Use the Distributive Property.
- To divide: Use long division or synthetic division.

Polynomial Functions and Graphs (Lessons 3-3 and 3-4)

- Turning points of a function are called *relative maxima* and *relative minima*.

Solving Polynomial Equations (Lesson 3-5)

- You can factor polynomials by using the GCF, grouping, or quadratic techniques.

The Remainder and Factor Theorems (Lesson 3-6)

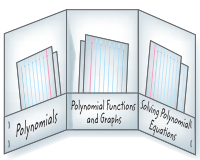
- Factor Theorem: The binomial $x - a$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$.

Roots, Zeros, and the Rational Zero Theorem (Lessons 3-7 and 3-8)

- Complex Conjugates Theorem: If $a + bi$ is a zero of a function, then $a - bi$ is also a zero.
- Integral Zero Theorem: If the coefficients of a polynomial function are integers such that $a_0 = 1$ and $a_n = 0$, any rational zeros of the function must be factors of a_n.
- Rational Zero Theorem: If $P(x)$ is a polynomial function with integral coefficients, then every rational zero of $P(x) = 0$ is of the form $\frac{p}{q}$, a rational number in simplest form, where p is a factor of the constant term and q is a factor of the leading coefficient.

FOLDABLES StudyOrganizer

Be sure the Key Concepts are noted in your Foldable.



KeyVocabulary

degree of a polynomial
depressed polynomial
end behavior
extrema
leading coefficient
Location principle
polynomial function
polynomial in one variable
power function
prime polynomials
quadratic form
relative maximum
relative minimum
simplify
synthetic division
synthetic substitution
turning points

VocabularyCheck

State whether each sentence is *true* or *false*. If *false*, replace the underlined term to make a true sentence.

1. The coefficient of the first term of a polynomial in standard form is called the leading coefficient.
2. Polynomials that cannot be factored are called polynomials in one variable.
3. A prime polynomial has a degree that is one less than the original polynomial.
4. A point on the graph of a function where no other nearby point has a greater y-coordinate is called a relative maximum.
5. A polynomial function is a continuous function that can be described by a polynomial equation in one variable.
6. To simplify an expression containing powers means to rewrite the expression without parentheses or negative exponents.
7. Synthetic division is a shortcut method for dividing a polynomial by a binomial.
8. The relative maximum and relative minimum of a function are often referred to as end behavior.
9. When a polynomial is divided by one of its binomial factors, the quotient is called a depressed polynomial.
10. $(x^3)^2 + 3x^3 - 8 = 0$ is a power function.

Lesson-by-Lesson Review

3-1 Operations with Polynomials

Simplify. Assume that no variable equals 0.

11. $\frac{14x^4y}{2x^3y^5}$

12. $3t(tn - 5)$

13. $(4r^2 + 3r - 1) - (3r^2 - 5r + 4)$

14. $(x^4)^3$

15. $(m + p)(m^2 - 2mp + p^2)$

16. $3b(2b - 1) + 2b(b + 3)$

Example 1

Simplify each expression.

a. $(-4a^3b^5)(5ab^3)$

$(-4a^3b^5)(5ab^3) = (-4)(5)(a^3 \cdot a)(b^5 \cdot b^3)$

$= -20a^4b^8$

b. $(2x^2 + 3x - 8) + (3x^2 - 5x - 7)$

$(2x^2 + 3x - 8) + (3x^2 - 5x - 7)$

$= (2x^2 + 3x^2) + (3x - 5x) + [-8 + (-7)]$

$= 5x^2 - 2x - 15$

3-2 Dividing Polynomials

Simplify.

17. $\frac{12x^4y^5 + 8x^3y^7 - 16x^2y^6}{4xy^5}$

18. $(6y^3 + 13y^2 - 10y - 24) \div (y + 2)$

19. $(a^4 + 5a^3 + 2a^2 - 6a + 4)(a + 2)^{-1}$

20. $(4a^6 - 5a^4 + 3a^2 - a) \div (2a + 1)$

21. **GEOMETRY** The volume of the rectangular prism is $3x^3 + 11x^2 - 114x - 80$ cubic units. What is the area of the base?

Example 2

Simplify $(6x^3 - 31x^2 - 34x + 22) \div (2x - 1)$.

$$\begin{array}{r} 3x^2 - 14x - 24 \\ 2x - 1\overline{)\,6x^3 - 31x^2 - 34x + 22} \\ \underline{(-)\ 6x^3 - \ 3x^2} \\ -28x^2 - 34x \\ \underline{(-)\ -28x^2 + 14x} \\ -48x + 22 \\ \underline{(-)\ -48x + 24} \\ -2 \end{array}$$

The result is $3x^2 - 14x - 24 - \frac{2}{2x - 1}$.

3-3 Polynomial Functions

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

22. $5x^6 - 3x^4 + x^3 - 9x^2 + 1$

23. $6xy^2 - xy + y^2$

24. $12x^3 - 5x^4 + 6x^8 - 3x - 3$

Find $p(-2)$ and $p(x + h)$ for each function.

25. $p(x) = x^2 + 2x - 3$

26. $p(x) = 3x^2 - x$

27. $p(x) = 3 - 5x^2 + x^3$

Example 3

What are the degree and leading coefficient of $4x^3 + 3x^2 - 7x^7 + 4x - 1$?

The greatest exponent is 7, so the degree is 7. The leading coefficient is -7.

Example 4

Find $p(a - 2)$ if $p(x) = 3x + 2x^2 - x^3$.

$p(a - 2) = 3(a - 2) + 2(a - 2)^2 - (a - 2)^3$

$= 3a - 6 + 2a^2 - 8a + 8 - (a^3 - 6a^2 + 12a - 8)$

$= -a^3 + 8a^2 - 17a + 10$

3-4 Analyzing Graphs of Polynomial Functions

Complete each of the following.

a. Graph each function by making a table of values.

b. Determine the consecutive integer values of x between which each real zero is located.

c. Estimate the x-coordinates at which the relative maxima and minima occur.

28. $h(x) = x^3 - 4x^2 - 7x + 10$

29. $g(x) = 4x^4 - 21x^2 + 5$

30. $f(x) = x^3 - 3x^2 - 4x + 12$

31. $h(x) = 4x^3 - 6x^2 + 1$

32. $p(x) = x^5 - x^4 + 1$

33. **BUSINESS** Humaid tracked the monthly profits for his sports store business for the first six months of the year. They can be modeled by using the following six points: (1, 675), (2, 950), (3, 550), (4, 250), (5, 600), and (6, 400). How many turning points would the graph of a polynomial function through these points have? Describe them.

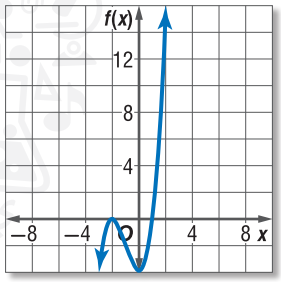
Example 5

Graph $f(x) = x^3 + 3x^2 - 4$ by making a table of values.

Make a table of values for several values of x.

x	−3	−2	−1	0	1	2
$f(x)$	−4	0	−2	−4	0	16

Plot the points and connect the points with a smooth curve.



3-5 Solving Polynomial Equations

Factor completely. If the polynomial is not factorable, write *prime*.

34. $a^4 - 16$

35. $x^3 + 6y^3$

36. $54x^3y - 16y^4$

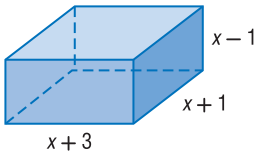
37. $6ay + 4by - 2cy + 3az + 2bz - cz$

Solve each equation.

38. $x^3 + 2x^2 - 35x = 0$

39. $8x^4 - 10x^2 + 3 = 0$

40. **GEOMETRY** The volume of the prism is 315 cubic centimeters. Find the value of x and the length, height, and width.



Example 6

Factor $r^7 + 64r$.

$$r^7 + 64r = r(r^6 + 64)$$
$$= r[(r^2)^3 + 4^3]$$
$$= r(r^2 + 4)(r^4 - 4r^2 + 16)$$

Example 7

Solve $4x^4 - 25x^2 + 36 = 0$.

$$(x^2 - 4)(4x^2 - 9) = 0$$

$x^2 - 4 = 0$ or $4x^2 - 9 = 0$

$x^2 = 4$ $\qquad$ $x^2 = \frac{9}{4}$

$x = \pm 2$ $\qquad$ $x = \pm\frac{3}{2}$

The solutions are -2, 2, $-\frac{3}{2}$, and $\frac{3}{2}$.

3-6 The Remainder and Factor Theorems

Use synthetic substitution to find $f(-2)$ and $f(4)$ for each function.

41. $f(x) = x^2 - 3$

42. $f(x) = x^2 - 5x + 4$

43. $f(x) = x^3 + 4x^2 - 3x + 2$

44. $f(x) = 2x^4 - 3x^3 + 1$

Given a polynomial and one of its factors, find the remaining factors of the polynomial.

45. $3x^3 + 20x^2 + 23x - 10; x + 5$

46. $2x^3 + 11x^2 + 17x + 5; 2x + 5$

47. $x^3 + 2x^2 - 23x - 60; x - 5$

Example 8

Determine whether $x - 6$ is a factor of $x^3 - 2x^2 - 21x - 18$.

6	1	−2	−21	−18
		6	24	18
	1	4	3	0

$x - 6$ is a factor because $r = 0$.

$x^3 - 2x^2 - 21x - 18 = (x - 6)(x^2 + 4x + 3)$

3-7 Roots and Zeros

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

48. $f(x) = -2x^3 + 11x^2 - 3x + 2$

49. $f(x) = -4x^4 - 2x^3 - 12x^2 - x - 23$

50. $f(x) = x^6 - 5x^3 + x^2 + x - 6$

51. $f(x) = -2x^5 + 4x^4 + x^2 - 3$

52. $f(x) = -2x^6 + 4x^4 + x^2 - 3x - 3$

Example 9

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $f(x) = 3x^4 + 2x^3 - 2x^2 - 26x - 48$.

$f(x)$ has one sign change, so there is 1 positive real zero.

$f(-x)$ has 3 sign changes, so there are 3 or 1 negative real zeros.

There are 0 or 2 imaginary zeros.

3-8 Rational Zero Theorem

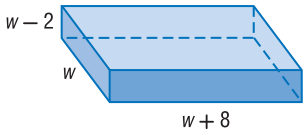
Find all of the zeros of each function.

53. $f(x) = x^3 + 4x^2 + 3x - 2$

54. $f(x) = 4x^3 + 4x^2 - x - 1$

55. $f(x) = x^3 + 2x^2 + 4x + 8$

56. **STORAGE** Hiyam is building a storage box that is shaped like a rectangular prism. It will have a volume of 96 cubic meters. Using the diagram below, find the dimensions of the box.



Example 10

Find all of the zeros of $f(x) = x^3 + 4x^2 - 11x - 30$.

There are exactly 3 zeros.

There are 1 positive real zero and 2 negative real zeros. The possible rational zeros are $\pm1, \pm2, \pm3, \pm5, \pm6, \pm10, \pm15, \pm30$.

3	1	4	−11	−30
		3	21	30
	1	7	10	0

$x^3 + 4x^2 - 11x - 30 = (x - 3)(x^2 + 7x + 10)$
$= (x - 3)(x + 2)(x + 5)$

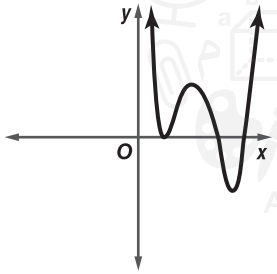
Thus, the zeros are 3, −2, and −5.

CHAPTER 3 Practice Test

Simplify.

1. $(3a)^2(7b)^4$
2. $(7x - 2)(2x + 5)$
3. $(2x^2 + 3x - 4) - (4x^2 - 7x + 1)$
4. $(4x^3 - x^2 + 5x - 4) + (5x - 10)$
5. $(x^4 + 5x^3 + 3x^2 - 8x + 3) \div (x + 3)$
6. $(3x^3 - 5x^2 - 23x + 24) \div (x - 3)$

7. **MULTIPLE CHOICE** How many unique real zeros does the graph have?



A 0 C 3

B 2 D 5

8. If $c(x) = 3x^3 + 5x^2 - 4$, what is the value of $4c(3b)$?

Complete each of the following.

a. Graph each function by making a table of values.

b. Determine consecutive integer values of x between which each real zero is located.

c. Estimate the x-coordinates at which the relative maxima and relative minima occur.

9. $g(x) = x^3 + 4x^2 - 3x + 1$
10. $h(x) = x^4 - 4x^3 - 3x^2 + 6x + 2$

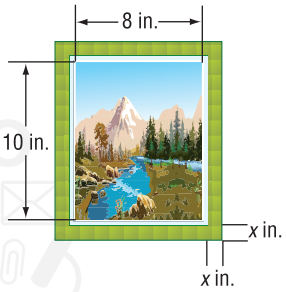
Factor completely. If the polynomial is not factorable, write *prime*.

11. $8y^4 + x^3y$
12. $2x^2 + 2x + 1$
13. $a^2x + 3ax + 2x - a^2y - 3ay - 2y$

Solve each equation.

14. $8x^3 + 1 = 0$
15. $x^4 - 11x^2 + 28 = 0$

16. **FRAMING** The area of the picture and frame shown below is 168 square inches. What is the width of the frame?



17. **MULTIPLE CHOICE** Let $f(x) = x^4 - 3x^3 + 5x - 3$. Use synthetic substitution to find $f(-2)$.

F 37 H −21

G 27 J −33

Given a polynomial and one of its factors, find the remaining factors of the polynomial.

18. $2x^3 + 15x^2 + 22x - 15;\ x + 5$
19. $x^3 - 4x^2 + 10x - 12;\ x - 2$

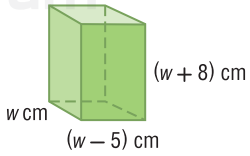
State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

20. $p(x) = x^3 - x^2 - x - 3$
21. $p(x) = 2x^6 + 5x^4 - x^3 - 5x - 1$

Find all zeros of each function.

22. $p(x) = x^3 - 4x^2 + x + 6$
23. $p(x) = x^3 + 2x^2 + 4x + 8$

24. **GEOMETRY** The volume of the rectangular prism shown is 612 cubic centimeters. Find the dimensions of the prism.



25. List all possible rational zeros of $f(x) = 2x^4 + 3x^2 - 12x + 8$.

CHAPTER 3

Preparing for Standardized Tests

Draw a Picture

Drawing a picture can be a helpful way for you to visualize how to solve a problem. Sketch your picture on scrap paper or in your test booklet (if allowed). Do not make any marks on your answer sheet other than your answers.

Strategies for Drawing a Picture

Step 1

Read the problem statement carefully.

Ask yourself:

- What am I being asked to solve?
- What information is given in the problem?
- What are the unknowns that I need to model and solve for?

Step 2

Sketch and label your picture.

- Draw your picture as clearly and accurately as possible.
- Label the picture carefully. Be sure to include all of the information given in the problem statement.

Step 3

Solve the problem.

- Use your picture to help you model the problem situation with an equation. Solve the equation.
- Check to be sure your answer makes sense.

Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

Mr. Khalifa has a rectangular swimming pool that measures 25 feet by 14 feet. He wants to have a cement walkway installed around the perimeter of the pool. The combined area of the pool and walkway will be 672 square feet. What will be the width of the walkway?

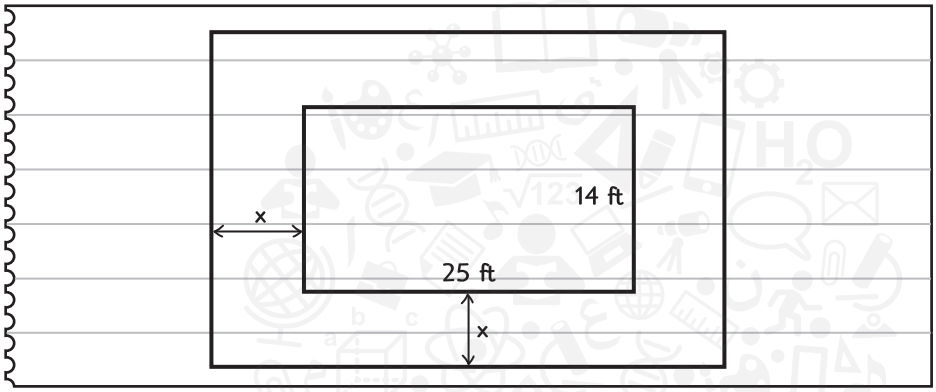
A 2.75 ft

B 3 ft

C 3.25 ft

D 3.5 ft

Draw a picture to help you visualize the problem situation. Let x represent the unknown width of the cement walkway.



The width of the pool and walkway is $14 + 2x$, and the length is $25 + 2x$. Multiply these polynomial expressions and set the result equal to the combined area, 672 square feet. Then solve for x.

$$(14 + 2x)(25 + 2x) = 672$$

$$350 + 78x + 4x^2 = 672$$
$$4x^2 + 78x - 322 = 0$$
$$x = -23 \text{ or } 3.5$$

Since the width cannot be negative, the width of the walkway will be 3.5 feet. The correct answer is D.

Exercises

Read each problem. Identify what you need to know. Then use the information in the problem to solve.

1. A farmer has 240 meters of fencing that he wants to use to enclose a rectangular area for his chickens. He plans to build the enclosure using the wall of his barn as one of the walls. What is the maximum amount of area he can enclose?

A 7200 m^2

B 4960 m^2

C 3600 m^2

D 3280 m^2

2. Metal washers are made by cutting a hole in a circular piece of metal. Suppose a washer is made by removing the center of a piece of metal with a 1.8-centimeters diameter. What is the radius of the hole if the washer has an area of 0.65π square centimeters?

F 0.35 cm.

G 0.38 cm.

H 0.40 cm.

J 0.42 cm.