



Teacher Edition

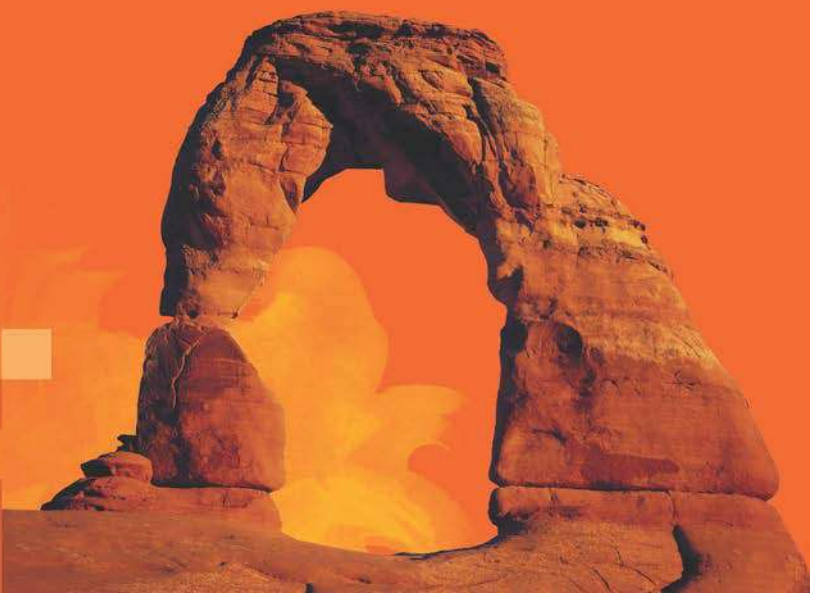
Volume 1

Reveal **MATH**TM

Integrated III



Mc
Graw
Hill





Teacher Edition
Volume 1

Reveal
MATH®
Integrated III



www.my.mheducation.com



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Send all inquiries to:
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Contents in Brief

Module 1 Quadratic Functions

2 Polynomials and Polynomial Functions

3 Polynomial Equations

4 Inverses and Radical Functions

5 Exponential Functions

6 Logarithmic Functions

7 Rational Functions

8 Inferential Statistics

9 Trigonometric Functions

Reveal Math Guiding Principles

Academic research and the science of learning provide the foundation for this powerful K–12 math program designed to help reveal the mathematician in every student.

Reveal Math is built on a solid foundation of **RESEARCH** that shaped the **PEDAGOGY** of the program.

Reveal Math Integrated I, Integrated II, and Integrated III (Reveal Math Integrated) used findings from research on teaching and learning mathematics to develop its instructional model. Based on analyses of research findings, these areas form the foundational structure of the program:

- Rigor
- Productive Struggle
- Formative Assessment
- Rich Tasks
- Mathematical Discourse
- Collaborative Learning

Instructional Model

1 Launch



WARM UP

During the **Warm Up**, students complete exercises to activate prior knowledge and review prerequisite concepts and skills.



INDIVIDUAL ACTIVITY



GROUP ACTIVITY



CLASS ACTIVITY



LAUNCH THE LESSON

In **Launch the Lesson**, students view a real-world scenario and image to pique their interest in the lesson content. They are introduced to questions that they will be able to answer at the end of the lesson.



EXPLORE

During the **Explore** activity, students work in partners or small groups to explore a rich mathematical problem related to the lesson content.



Reveal the full potential
in every student!

2 Explore and Develop



LEARN

In the **Learn** section, students gain the foundational knowledge needed to actively work through upcoming Examples.



EXAMPLES & CHECK

Students work through **Examples** related to the key concepts and engage in mathematical discourse.

Students complete a **Check** after several Examples as a quick formative assessment to help teachers adjust instruction as needed.

3 Reflect and Practice



EXIT TICKET

The **Exit Ticket** gives students an opportunity to convey their understanding of the lesson concepts.



PRACTICE

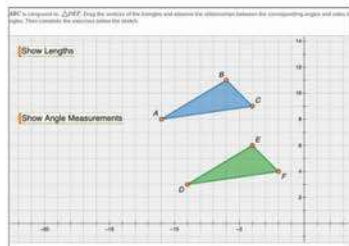
Students complete **Practice** exercises individually or collaboratively to solidify their understanding of lesson concepts and build proficiency with lesson skills.

Reveal Math Key Areas of Focus

Reveal Math Integrated I, II, III (Reveal Math Integrated) have a strong focus on rigor—especially the development of conceptual understanding—an emphasis on student mindset, and ongoing formative assessment feedback loops.

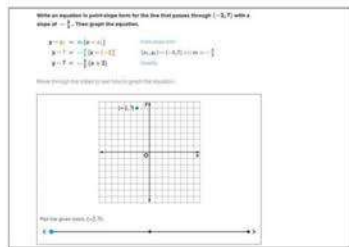
Rigor

Reveal Math Integrated has been thoughtfully designed to incorporate a balance of the three elements of rigor: conceptual understanding, procedural skills and fluency, and application.



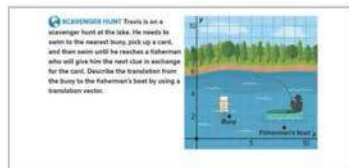
Conceptual Understanding

Explore activities give all students an opportunity to work collaboratively and discuss their thinking as they build conceptual understanding of new concepts. In the **Explore** activity to the left, students use **Web Sketchpad®** to build understanding of the relationships between corresponding sides and angles in congruent triangles.



Procedural Skills and Fluency

Students use different strategies and tools to build procedural fluency. In the **Example** shown, students build proficiency with writing equations in point-slope form.



Application

Real-world examples and practice problems are opportunities for students to apply their learning to new situations. In the real-world example shown, students apply their understanding by solving a multi-step problem with translations.

Student Mindset

Mindset Matters tips located in each module provide specific examples of how Reveal Math Integrated content can be used to promote a growth mindset in all students. Another feature focused on promoting a growth mindset is **Ignite!**

Activities developed by Dr. Raj Shah to spark student curiosity about why the math works. An **Ignite!** delivers problem sets that are flexible enough so that students with varying background knowledge can engage with the content and motivates them to ask questions, solve complex problems, and develop a can-do attitude toward math.

**Mindset Matters**

Growth Mindset vs. Fixed Mindset

Everyone has a core belief or mindset about how they learn. People with a *growth mindset* believe that they can grow their intelligence through hard work. Those with a *fixed mindset* believe that while they can learn new things, they cannot increase their intelligence. When a student approaches school, life, and the future workplace with a growth mindset, they are more likely to persevere through challenging problems, learn from their mistakes, and ultimately learn concepts in a deeper, more meaningful way.

How Can I Apply It?

Assign students rich tasks, such as the **Explore** activities, that can help them to develop their understanding. Encourage them with the thought that each time they learn a new idea, neurons fire electric currents that connect different parts of their brain!

Teacher Edition Mindset Tip

**IGNITE!**

Mind Your 9's and 10's

Suppose 9 is an expression, $9k + 1$, and 10 is an expression.

What do you notice?	What questions can you ask?

☐ Take about 10. Show you understand and connect with a partner. What do you notice about the relationship you each made?

With your class, discuss each of your questions and record 1 below. This should be a question that you can answer by generating your own strategies.

Have you ever written your question? What strategies did you use?	What connections, if any, did you make? What are you thinking about now?

☐ Take about 10. Why is it important to make connections in this context?

Student Ignite! Activity

Formative Assessment

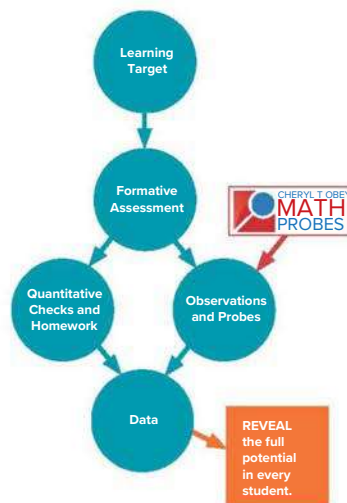
The key to reaching all learners is to adjust instruction based on each student's understanding. Reveal Math Integrated offers powerful formative assessment tools that help teachers to efficiently and effectively differentiate instruction for all students.

Math Probes

Each module includes a **Cheryl Tobey Formative Assessment Math Probe** that is focused on addressing student misconceptions about key math topics. Students can complete these probes at the beginning, middle, or end of a module. The teacher support includes a list of recommended differentiated resources that teachers assign based on students' responses.

Example Checks

After multiple examples, a formative assessment **Check** that students complete on their own allows teachers to gauge students' understanding of the concept or skill presented. When students complete the Check online, the teacher receives resource recommendations which can be assigned to students.



A Powerful Blended Learning Experience

The *Reveal Math Integrated I, II, III* (Reveal Math Integrated) blended learning experience was designed to include purposeful print and digital components focused on sparking student curiosity and providing teachers with flexible implementation options.

Reveal Math Integrated has been thoughtfully developed to provide a rich learning experience no matter where a district, school, or classroom falls on the digital spectrum. All of the instructional content can be projected or can be accessed via desktop, laptop, or tablet.

Lesson

1 Launch



WARM UP



The **Warm Up** exercise can be projected on an interactive whiteboard.



LAUNCH THE LESSON



Launch the Lesson can be projected or assigned to students to access on their own devices.



EXPLORE



The **Explore** activity can be projected while students record their observations in the Interactive Student Edition or can be assigned for students to complete on individual devices.



Launch the Lesson



Explore



INDIVIDUAL ACTIVITY



GROUP ACTIVITY



CLASS ACTIVITY



INTERACTIVE PRESENTATION



PRINT STUDENT EDITION

2 Explore and Develop

LEARN



As students are introduced to the key lesson concepts, they can progress through the **Learn** by recording notes in a notebook or on their own devices.

EXAMPLES & CHECK



Either in a notebook or on an individual device, students work through one or more **Examples** related to key lesson concepts.

A **Check** follows several Examples in either the Student Edition or on each student device.

3 Reflect and Practice

EXIT TICKET

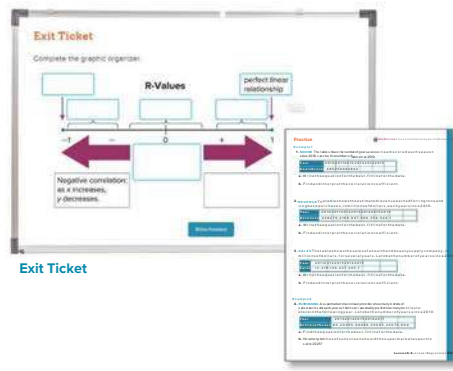
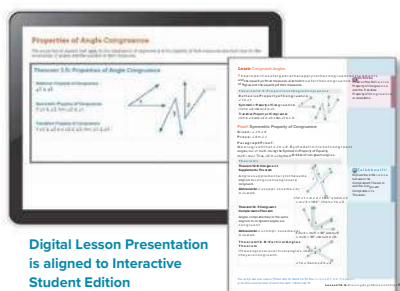


The **Exit Ticket** is projected or accessed via student devices to provide students with lesson closure and an opportunity to revisit the lesson concepts.

PRACTICE



Assign students **Practice** problems from their Student Edition or create a digital assignment for them to work on their device in class or at home to solidify lesson concepts.



Practice

Supporting All Learners

The *Reveal Math Integrated I, II, III* (Reveal Math Integrated) programs were designed so that all students have access to:

- rich tasks that promote productive struggle,
- opportunities to develop proficiency with the habits of mind and thinking strategies of mathematicians, and
- prompts to promote mathematical discourse and build academic language.

Resources for Differentiating Instruction

When needed, resources are available to differentiate math instruction for students who may need to see a concept in a different way, practice prerequisite skills, or are ready to extend their learning.

AL

Approaching Level Resources

- Remediation Activities
- Extra Examples

BL

Beyond Level Resources

- Beyond Level Differentiated Activities
- Extension Activities

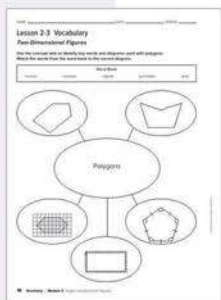
Resources for English Language Learners

Reveal Math Integrated also includes student and teacher resources to support students who are simultaneously learning grade-level math and building their English proficiency. Appropriate, research-based language scaffolds are also provided to support students as they engage in rigorous mathematical tasks and discussions.

ELL

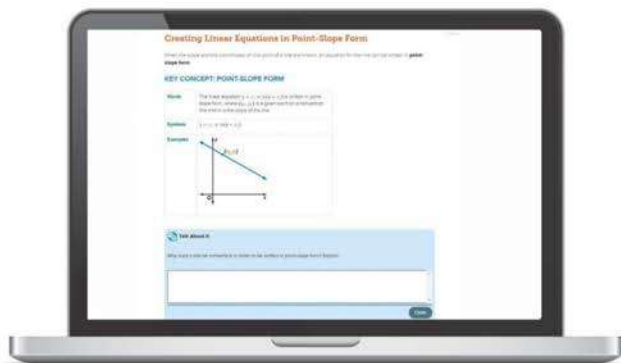
English Language Learners

- Spanish Personal Tutors
- Math Language-Building Activities
- Language Scaffolds
- *Think About It!* and *Talk About It!* Prompts
- Multilingual eGlossary
- Audio
- Graphic Organizers
- Web Sketchpad, Desmos, and eTools



Developing Mathematical Thinking and Strategic Questioning

Reveal Math Integrated I, II, III (Reveal Math Integrated) are comprised of high-quality math content designed to be accessible and relevant to each student. Throughout the program, students are presented with a variety of thoughtfully designed questioning strategies related to the content. Using these questions provides you with an additional, built-in type of formative assessment that can be used to modify instruction. They also strengthen students' ownership of mathematical content knowledge and daily use of the Standards for Mathematical Practice.



Key Concept Introduction followed by a **Talk About It** question to discuss with a classmate.

You will find these types of questioning strategies throughout Reveal Math Integrated. The related Standard for Mathematical Practice for each is also indicated.

- **Talk About It** questions encourage students to engage in mathematical discourse with classmates (SMP3)
- **Alternate Method** shows students another way to solve a problem and asks them to compare and contrast the methods and solutions (SMP1)
- **Avoid a Common Error** shows students a problem similar to an example but with a flaw in reasoning, and students have to find and explain the error (SMP3)
- **State Your Assumptions** requires that student state the assumptions they made to solve a problem (SMP4)
- **Use a Source** asks students to find information using an external source, such as the Internet, and use it to pose or solve a problem (SMP5)
- **Think About It** questions help students make sense of mathematical problems (SMP1)
- **Concept Checks** prompt students to analyze how the Key Concepts of the lesson apply to various use cases (SMP3)

Reveal Student Readiness with Individualized Learning Tools

Reveal Math Integrated I, II, III (Reveal Math Integrated) incorporate innovative, technology-based tools that are designed to extend the teacher's reach in the classroom to help address a wide range of knowledge gaps, set and align academic goals, and meet student individualized learning needs.

LEARNSMART®

Topic-Mastery

With embedded **LearnSmart**®, students have a built-in study partner for topic practice and review to prepare for multi-module, mid-year, or end-of-year testing.

LearnSmart's revolutionary adaptive technology measures students' awareness of their own learning, time on topic, answer accuracy, and suggests alternative resources to support student learning, confidence, and topic mastery.



ALEKS®

Individualized Learning Pathways

Learners of all levels benefit from the use of **ALEKS**' adaptive, online math technology designed to pinpoint what each student knows, does not know, and most importantly, what each student is ready to learn.

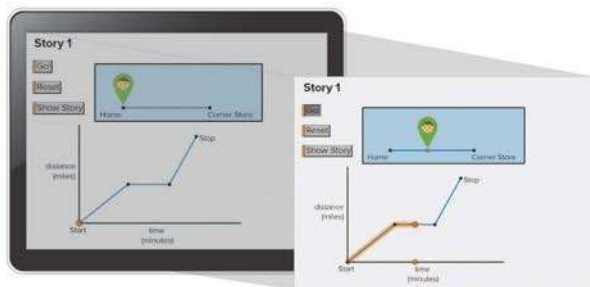
When paired with Reveal Math Integrated, **ALEKS** is a powerful tool designed to provide integrated instructionally actionable data enabling teachers to utilize Reveal Math Integrated resources for individual students, groups, or the entire classroom.



Activity Report

Powerful Tools for Modeling Mathematics

Reveal Math Integrated I, II, III (Reveal Math Integrated) have been designed with purposeful, embedded digital tools to increase student engagement and provide unique modeling opportunities.

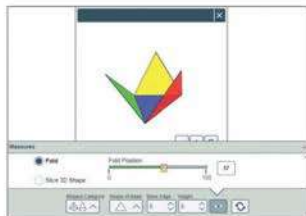


Web Sketchpad® Activities

The leading dynamic mathematics visualization software has now been integrated with **Web Sketchpad Activities** at point of use within Reveal Math Integrated. Student exploration (and practice) using **Web Sketchpad** encourages problem solving and visualization of abstract math concepts.



The powerful **Desmos** graphing calculator is available in Reveal Math Integrated for students to explore, model, and apply math to the real-world.



eTools

By using a wide variety of digital **eTools** embedded within Reveal Math Integrated, students gain additional hands-on experience while they learn and teachers have the option to create problem-based learning opportunities.

Technology-Enhanced Items

Embedded within the digital lesson, technology-enhanced items—such as drag-and-drop, flashcard flips, or diagram completion—are strategically placed to give students the practice with common computer functions needed to master computer-based testing.

TYPE



SWIPE



DRAW & DROP



FLASHCARDS



eTOOLS



MULTI-SELECT



WATCH



EXPAND



Assessment Tools to Reveal Student Progress and Success

Reveal Math Integrated I, II, III (Reveal Math Integrated) provide a comprehensive array of assessment tools, with both print and digital administration options, to measure student understanding and progress. The digital assessment tools include next-generation assessment items, such as multiple-response, selected-response, and technology-enhanced items.

Assessment Solutions

Reveal Math Integrated provides embedded, regular formative checkpoints to monitor student learning and provide feedback that can be used to modify instruction and help direct student learning using reports and recommendations based on resulting scores.

Summative assessments built in Reveal Math Integrated evaluate student learning at the module conclusion by comparing it against the state standards covered.

Formative Assessment Resources

- Cheryl Tobey Formative Assessment Math Probes
- Checks
- Exit Tickets
- Put It All Together

Summative Assessment Resources

- Module Tests
- Performance Tasks
- End-of-Course Tests
- LearnSmart

Or **Build Your Own** assessments focused on standards or objectives. Access to banks of thousands of questions, including those with tech-enhanced capabilities, enable a wide range of options to mirror high-stakes assessment formats.

Reporting

Clear, instructionally actionable data is a click away with the Reveal Math Integrated Reporting Dashboard.

Activity Report Real-time class and student reporting of activities completed by the class. Includes average score, submission rate, and skills covered for the class and each student.

- **Item Analysis Report** A detailed analysis of response rates and patterns, answers, and question types in a class snapshot or by student.

Standards Report Performance data by class or individual student are aggregated by standards, skills, or objectives linked to the related activities completed.



Activity Report

Professional Development Support for Continuous Learning

McGraw-Hill Education supports lifelong learning and demonstrates commitment to teachers with a built-in professional learning environment designed for support during planning or extended learning opportunities.

What You Will Find

- Best-practice resources
- Implementation support
- Teaching Strategies
- Classroom Videos
- Math Misconception Videos
- Content and Pedagogy Videos
- Content Progression Information

Why Professional Development Is so Important

- Research-based understanding of student learning
- Improved student performance
- Evidence-based instructional best practices
- Collaborative content strategy planning
- Extended knowledge of program how-to's



Reveal Math Expert Advisors



Cathy Seeley, Ed.D.

Austin, Texas

Mathematics educator, speaker, and writer, former Senior Fellow at the Charles A. Dana Center at the University of Texas at Austin, past President of NCTM, former Director of K-12 Mathematics for the State of Texas

Areas of expertise:

Mathematics Teaching, Equity, Assessment, STEM Learning, Informal Learning, Upside-Down Teaching, Productive Struggling, Mathematical Practices, Mathematical Habits of Mind, Family and Community Outreach, Mathematics Education Policy, Advocacy

"We want students to believe deeply that mathematics makes sense—in generating answers to problems, discussing their thinking and other students' thinking, and learning new material."

—Seeley, 2016, *Making Sense of Math*



Cheryl R. Tobey, M.Ed.

Gardiner, Maine

Senior Mathematics Associate at Education Development Center (EDC)

Areas of expertise:

Formative assessment and professional development for mathematics teachers; tools and strategies to uncovering misconceptions

"Misunderstandings and partial understandings develop as a normal part of learning mathematics. Our job as educators is to minimize the chances of students' harboring misconceptions by knowing the potential difficulties students are likely to encounter, using assessments to elicit misconceptions and implementing instruction designed to build new and accurate mathematical ideas."

—Tobey, et al 2007, 2009, 2010, 2013, 2104, *Uncovering Student Thinking Series*



Nevels Nevels, Ph.D.

Saint Louis, Missouri

PK-12 Mathematics Curriculum Coordinator for Hazelwood School District

Areas of expertise:

Mathematics Teacher Education; Student Agency & Identity; Socio-Cultural Perspective in Mathematics Learning

"A school building is one setting for learning mathematics. It is understood that all children should be expected to learn meaningful mathematics within its walls. Additionally, teachers should be expected to learn within the walls of this same building. More poignantly, I posit that if teachers are not learning mathematics in their school building, then it is not a school."

—Nevels, 2018



Raj Shah, Ph.D.

Columbus, Ohio

Founder of Math Plus Academy, a STEM enrichment program and founding member of The Global Math Project

Areas of expertise:

Sparking student curiosity, promoting productive struggle, and creating math experiences that kids love

"As teachers, it's imperative that we start every lesson by getting students to ask more questions because curiosity is the fuel that drives engagement, deeper learning and perseverance."

—Shah, 2017



Walter Secada, Ph.D.

Coral Gables, Florida

Professor of Teaching and Learning
at the University of Miami

Areas of expertise:

Improving education for English language learners, equity in education, mathematics education, bilingual education, school restructuring, professional development of teachers, student engagement, Hispanic dropout and prevention, and reform

“The best lessons take place when teachers have thought about how their individual English language learners will respond not just to the mathematical content of that lesson, but also to its language demands and mathematical practices.”

—Secada, 2018



Ryan Baker, Ph.D.

Philadelphia, Pennsylvania

Associate Professor and Director
of Penn Center for Learning Analytics
at the University of Pennsylvania

Areas of expertise:

Interactions between students and educational software; data mining and learning analytics to understand student learning

“The ultimate goal of the field of Artificial Intelligence in Education is not to promote artificial intelligence, but to promote education... systems that are designed intelligently, and that leverage teachers’ intelligence. Modern online learning systems used at scale are leveraging human intelligence to improve their design, and they’re bringing human beings into the decision-making loop and trying to inform them.”

—Baker, 2016



Chris Dede, Ph.D.

Cambridge, Massachusetts

Timothy E. Wirth Professor in
Learning Technologies at Harvard
Graduate School of Education

Areas of expertise:

Provides leadership in educational innovation; educational improvements using technology

“People are very diverse in how they prefer to learn. Good instruction is like an ecosystem that has many niches for alternative types of learning: lectures, games, engaging video-based animations, readings, etc. Learners then can navigate to the niche that best fulfills their current needs.”

—Dede, 2017



Dinah Zike, M.Ed.

Comfort, Texas

President of Dinah.com
in San Antonio, Texas and
Dinah Zike Academy

Areas of expertise:

Developing educational materials that include three-dimensional graphic organizers; interactive notebook activities for differentiation; and kinesthetic, cross-curricular manipulatives

“It is education’s responsibility to meet the unique needs of students, and not the students’ responsibility to meet education’s need for uniformity.”

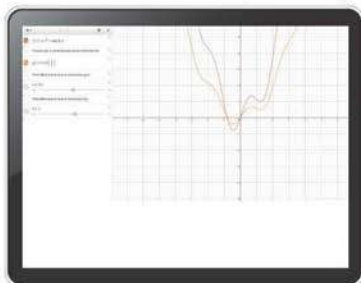
—Zike, 2017, InRIGORating Math Notebooks

Digital Classrooms

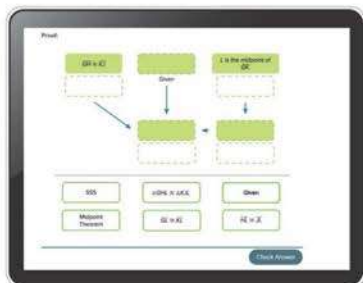
Projection is a focal point for key areas of the course with students interacting with the lesson using their own devices. Each student can access teacher-assigned sections of the lessons for **Explore** activities, **Learn** sections, and **Examples**. Point of use videos, animations, as well as interactive content enable students to experience math in interesting and impactful ways.



Web Sketchpad



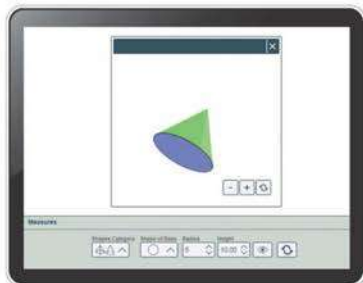
Desmos



Drag-and-Drop



Video



eTools

In Reveal AGA,
R is for—

- Research
- Rigor
- Relevant Connections

Are you...
READY to start?

Module 1

Quadratic Functions



	What Will You Learn?	1
1-1	Graphing Quadratic Functions	3
	Explore Transforming Quadratic Functions	
1-2	Solving Quadratic Equations by Graphing	13
	Explore Roots of Quadratic Equations	
1-3	Complex Numbers	21
	Explore Factoring the Sum of Two Squares	
1-4	Solving Quadratic Equations by Factoring	27
	Explore Finding the Solutions of Quadratic Equations by Factoring	
1-5	Solving Quadratic Equations by Completing the Square	33
	Explore Using Algebra Tiles to Complete the Square	
1-6	Using the Quadratic Formula and the Discriminant	43
	Explore The Discriminant	
1-7	Quadratic Inequalities	51
	Explore Graphing Quadratic Inequalities	
1-8	Solving Linear-Nonlinear Systems	57
	Explore Linear-Quadratic Systems	
	Module 1 Review	67



Module 2

Polynomials and Polynomial Functions

	What Will You Learn?	71
2-1	Polynomial Functions	73
	Explore Power Functions	
	Explore Polynomial Functions	
2-2	Analyzing Graphs of Polynomial Functions	83
2-3	Operations with Polynomials	93
	Explore Multiplying Polynomials	
2-4	Dividing Polynomials	101
	Explore Using Algebra Tiles to Divide Polynomials	
2-5	Powers of Binomials	109
	Explore Expanding Binomials	
	Module 2 Review	113

Module 3

Polynomial Equations



	What Will You Learn?	117
3-1	Solving Polynomial Equations by Graphing	119
	Explore Solutions of Polynomial Equations	
3-2	Solving Polynomial Equations Algebraically	123
3-3	Proving Polynomial Identities	131
	Explore Polynomial Identities	
3-4	The Remainder and Factor Theorems	135
	Explore Remainders	
3-5	Roots and Zeros	143
	Explore Roots of Quadratic Polynomials	
	Module 3 Review	153



Module 4

Inverses and Radical Functions

	What Will You Learn?	157
4-1	Operations on Functions	159
	Explore Adding Functions	
4-2	Inverse Relations and Functions	167
	Explore Inverse Functions	
4-3	nth Roots and Rational Exponents	175
	Explore Inverses of Power Functions	
4-4	Graphing Radical Functions	183
	Explore Using Technology to Analyze Square Root Functions	
4-5	Operations with Radical Expressions	193
4-6	Solving Radical Equations	203
	Explore Solutions of Radical Equations	
	Module 4 Review	211

Module 5

Exponential Functions

	What Will You Learn?	215
5-1	Graphing Exponential Functions	217
	Explore Using Technology to Analyze Graphs of Exponential Functions	
5-2	Solving Exponential Equations and Inequalities	225
	Explore Solving Exponential Equations	
5-3	Special Exponential Functions	233
	Explore Finding the Value of e	
5-4	Geometric Sequences and Series	239
	Explore Explicit and Recursive Formulas	
Expand 5-4	Sum of a Finite Geometric Series (Online Only)	
5-5	Modeling Data	249
	Explore Modeling Exponential Decay	
	Module 5 Review	253



Module 6

Logarithmic Functions

	What Will You Learn?	257
6-1	Logarithms and Logarithmic Functions	259
	Explore Transforming Logarithmic Functions	
6-2	Properties of Logarithms	269
	Explore Logarithmic Expressions and Equations	
6-3	Common Logarithms	277
6-4	Natural Logarithms.....	285
	Explore Using a Scatter Plot to Analyze Data	
6-5	Using Exponential and Logarithmic Functions	295
	Module 6 Review	305

Module 7

Rational Functions



	What Will You Learn?	309
7-1	Multiplying and Dividing Rational Expressions	311
	Explore Simplifying Complex Fractions	
7-2	Adding and Subtracting Rational Expressions	319
	Explore Closure of Rational Expressions	
7-3	Graphing Reciprocal Functions	327
	Explore Transforming Reciprocal Functions	
7-4	Graphing Rational Functions	337
	Explore Analyzing Rational Functions	
7-5	Variation	347
	Explore Variation	
7-6	Solving Rational Equations and Inequalities	355
	Explore Solving Rational Equations	
	Module 7 Review	365



Module 8

Inferential Statistics

	What Will You Learn?	369
8-1	Random Sampling	371
8-2	Using Statistical Experiments	379
	Explore Simulations and Experiments	
	Explore Fair Decisions	
8-3	Analyzing Population Data	387
8-4	Normal Distributions	393
	Explore Probability Distributions	
8-5	Estimating Population Parameters	403
	Module 8 Review	411

Module 9

Trigonometric Functions



What Will You Learn?	415
9-1 Angles and Angle Measure	417
Explore Arc Length	
9-2 Trigonometric Functions of General Angles	425
9-3 Circular and Periodic Functions	435
Explore Trigonometric Functions of Special Angles	
9-4 Graphing Sine and Cosine Functions.....	445
9-5 Graphing Other Trigonometric Functions	455
9-6 Translations of Trigonometric Graphs	463
Explore Analyzing Graphs of Trigonometric Functions by Using Technology	
9-7 Inverse Trigonometric Functions	473
Module 9 Review	477
Selected Answers	SA1
Glossary.....	G1
Index	IN1

Standards for Mathematical Content, Reveal Math Integrated III

This correlation shows the alignment of *Reveal Math Integrated III* to the Standards for Mathematical Content, from the Common Core State Standards for Mathematics.

Standard		Lesson(s)
Number and Quantity		
The Complex Number System N.CN		
N.CN.8	(+) Extend polynomial identities to the complex numbers. <i>For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.</i>	1-3, 1-4, 1-6
N.CN.9	(+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.	3-5
Algebra		
Seeing Structures in Expressions A.SSE		
A.SSE.1	Interpret expressions that represent a quantity in terms of its context.★ a. Interpret parts of an expression, such as terms, factors, and coefficients. b. Interpret complex expressions by viewing one or more of their parts as a single entity. <i>For example, interpret $P(1+r)^t$ as the product of P and a factor not depending on P.</i>	1-1, 1-6, 2-1, 2-2, 5-1, 9-4
A.SSE.2	Use the structure of an expression to identify ways to rewrite it. <i>For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$; thus recognizing it as a difference of squares that can be factored as $(x^2 + y^2)(x^2 - y^2)$.</i>	1-4, 1-5, 4-3, 4-5, 5-2, 6-1, 6-3, 6-4, 6-5
A.SSE.4	Derive the formula for the sum of a geometric series (when the common ratio is not 1), and use the formula to solve problems. <i>For example, calculate mortgage payments.</i> ★	5-4
Arithmetic With Polynomials and Rational Expressions A.APR		
A.APR.1	Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.	2-3
A.APR.2	Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a , the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.	3-4
A.APR.3	Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.	3-5
A.APR.4	Prove polynomial identities and use them to describe numerical relationships. <i>For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.</i>	1-4, 3-3
A.APR.5	(+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n , where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.	2-5
A.APR.6	Rewrite simple rational expressions in different forms; write $\frac{a(x)}{b(x)}$ in the form $q(x) + \frac{r(x)}{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division or, for the more complicated examples, a computer algebra system.	2-4
A.APR.7	(+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.	7-1, 7-2

Creating Equations A.CED		
A.CED.1	Create equations and inequalities in one variable and use them to solve problems. <i>Include equations arising from linear and quadratic functions, and simple rational and exponential functions.</i>	1-2, 1-4, 1-5, 1-6, 1-7, 3-1, 3-2, 3-4, 3-5, 5-2, 6-2, 6-5, 7-5, 7-6
A.CED.2	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.	1-1, 1-2, 2-1, 2-2, 3-1, 4-4, 4-6, 5-1, 5-3, 5-5, 6-1, 6-5, 7-3, 7-4, 7-5, 9-3, 9-4, 9-5, 9-6
A.CED.3	Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. <i>For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.</i>	1-7, 6-5, 7-6
A.CED.4	Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. <i>For example, rearrange Ohm's law $V = IR$ to highlight resistance R.</i>	1-6, 4-2, 6-1, 7-5
Reasoning With Equations and Inequalities A.REI		
A.REI.2	Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.	4-6, 7-6
A.REI.11	Explain why the x -coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★	1-8, 3-1, 5-2, 6-3, 7-6
FUNCTIONS		
Interpreting Functions F.IF		
F.IF.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. <i>Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.</i> ★	1-1, 1-2, 2-1, 2-2, 3-5, 4-2, 4-4, 5-1, 6-1, 6-4, 6-5, 7-3, 7-4, 9-4, 9-5, 9-6
F.IF.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. <i>For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.</i> ★	1-1, 2-1, 4-2, 4-4, 5-1, 6-1, 7-3, 7-4, 9-4, 9-5, 9-6
F.IF.6	Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★	1-1, 2-2, 5-3
F.IF.7	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★	2-1, 2-2, 3-5, 4-4, 5-1, 6-1, 9-4, 9-5, 9-6
	b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.	
	c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.	
	e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.	
F.IF.8	Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.	1-4, 1-5, 5-1, 6-5
F.IF.9	Compare properties of two functions, each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). <i>For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.</i>	1-1, 2-1, 4-4, 5-1, 6-1, 7-4

Building Functions F.BF		
F.BF.1	Write a function that describes a relationship between two quantities.★ b. Combine standard function types using arithmetic operations. <i>For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.</i>	4-1 function
F.BF.3	Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. <i>Include recognizing even and odd functions from their graphs and algebraic expressions for them.</i>	4-4, 5-1, 6-1, 7-3, 9-6
F.BF.4	Find inverse functions. a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. <i>For example, $f(x) = 2x^3$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$.</i>	4-2 inverse and write an
Linear, Quadratic, and Exponential Models F.LE		
F.LE.4	For exponential models, express as a logarithm the solution to $ab^t = d$ where a , c , and d are numbers and the base b is 2, 10, or e ; evaluate the logarithm using technology.	6-3, 6-4, 6-5
Trigonometric Functions F.TF		
F.TF.1	Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.	9-1
F.TF.2	Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.	9-3
F.TF.5	Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★	9-3, 9-4, 9-5, 9-6
GEOMETRY		
Similarity, Right Triangles, and Trigonometry G.SRT		
G.SRT.9	(+) Derive the formula $A = \frac{1}{2} ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.	Standard G.SRT.9 is taught in Integrated Math Course 2, 9-6 Solving Systems of Inequalities
G.SRT.10	(+) Prove the Laws of Sines and Cosines and use them to solve problems.	Standard G.SRT.10 is taught in Integrated Math Course 2, 4-7 The Law of Sines and 4-8 The Law of Cosines
G.SRT.11	(+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).	Standard G.SRT.11 is taught in Integrated Math Course 2, 4-7 The Law of Sines and 4-8 The Law of Cosines
Geometric Measurement and Dimension G.GMD		
G.GMD.4	Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.	Standard G.GMD.4 is taught in Integrated Math Course 2, 6-5 Cross Sections and Solids of Revolution

Modeling With Geometry G.MG		
G.MG.1	Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).★	2-1
G.MG.2	Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).★	6-9
G.MG.3	Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).★	Standard G.MG.3 is taught in Integrated Math Course 2, 6-4 Surface Area
STATISTICS		
Interpreting Categorical and Quantitative Data S.ID		
S.ID.4	Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.	8-4
Making Inferences and Justifying Conclusions S.IC		
S.IC.1	Understand that statistics allow inferences to be made about population parameters based on a random sample from that population.	8-1
S.IC.2	Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?	8-2
S.IC.3	Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.	8-1
S.IC.4	Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.	8-3, 8-5
S.IC.5	Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.	8-2
S.IC.6	Evaluate reports based on data.	8-1, 8-2, 8-4, 8-5
Using Probability to Make Decisions S.MD		
S.MD.6	(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).	8-2
S.MD.7	(+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).	8-2

Standards for Mathematical Practice

This correlation shows the alignment of *Reveal Math Integrated III* to the Standards for Mathematical Practice, from the Common Core State Standards.

Standard	Lesson(s)
1 Make sense of problems and persevere in solving them. Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.	<i>Reveal Math Integrated III</i> requires students to make sense of problems and persevere in solving them in Examples and Practice throughout the program. Some specific lessons for review are: Lessons 1-1, 1-3, 1-4, 1-5, 1-6, 1-7, 1-8, 2-1, 3-2, 3-5, 4-1, 4-4, 5-1, 6-1, 7-2, 7-6, 8-1, 9-3, 9-4
2 Reason abstractly and quantitatively. Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to <i>decontextualize</i>—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to <i>contextualize</i>, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.	<i>Reveal Math Integrated III</i> requires students to reason abstractly and quantitatively in Think About It features and Higher Order Thinking Skills throughout the program. Some specific lessons for review are: Lessons 1-1, 2-3, 3-1, 4-2, 4-3, 5-3, 6-3, 6-5, 7-5, 7-6, 8-3, 8-5, 9-1, 9-4, 9-5
3 Construct viable arguments and critique the reasoning of others. Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.	<i>Reveal Math Integrated III</i> requires students to construct viable arguments and critique the reasoning of others in Talk About It features and Practice throughout the program. Some specific lessons for review are: Lessons 2-4, 3-4, 4-4, 4-6, 5-4, 6-2, 7-1, 7-3, 8-2, 8-5, 9-2, 9-6
4 Model with mathematics. Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.	<i>Reveal Math Integrated III</i> requires students to model with mathematics, collaborate, and discuss mathematics in Examples and Practice throughout the program. Some specific lessons for review are: Lessons 1-2, 1-3, 1-6, 1-7, 1-8, 2-2, 2-5, 3-4, 4-1, 4-5, 5-5, 6-4, 7-4, 7-5, 8-3, 9-3, 9-7

Standard	Lesson(s)
5 Use appropriate tools strategically. Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.	<i>Reveal Math Integrated III</i> requires students to use appropriate tools strategically in Explore activities throughout the program. Some specific lessons for review are: Lessons 1-2, 2-2, 3-1, 4-4, 4-6, 5-1, 5-5, 6-1, 6-3, 6-4, 6-5, 7-4, 7-6, 8-4, 9-1, 9-6
6 Attend to precision. Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.	<i>Reveal Math Integrated III</i> requires students to attend to precision in Examples and Practice throughout the program. Some specific lessons for review are: Lessons 2-1, 3-3, 4-2, 4-3, 5-2, 6-5, 7-3, 7-5, 8-4, 9-2, 9-3, 9-7
7 Look for and make use of structure. Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.	<i>Reveal Math Integrated III</i> requires students to look for and make use of structure in Explore activities and Higher Order Thinking Skills throughout the program. Some specific lessons for review are: Lessons 1-4, 1-5, 2-5, 3-2, 3-5, 4-1, 4-3, 4-5, 5-2, 5-3, 6-3, 6-4, 7-2, 7-3, 8-1, 9-4, 9-6
8 Look for and express regularity in repeated reasoning. Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.	<i>Reveal Math Integrated III</i> requires students to look for and express regularity in repeated reasoning in Concept Check and Think About It features and Higher Order Thinking Skills throughout the program. Some specific lessons for review are: Lessons 2-3, 2-4, 3-3, 4-6, 5-4, 6-2, 7-1, 8-2, 9-5

Quadratic Functions

Module Goals

- Students graph quadratic functions and inequalities.
- Students solve quadratic equations and inequalities using a variety of methods.
- Students use quadratic functions to model real-world situations and solve problems.

Focus

Domain: Algebra, Functions, Number and Quantity

Standards for Mathematical Content:

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

F.IF.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

Also addresses *N.CN.1, N.CN.2, N.CN.7, N.CN.8, A.SSE.1b, A.CED.2, A.CED.3, A.REI.11, F.IF.4, and F.IF.6.*

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this module.

Coherence

Vertical Alignment

Now

Students solve quadratic equations and inequalities algebraically and by graphing, including those with complex solutions.

A.CED.3, F.IF.8a, N.CN.7

Next

Students will graph and identify key features of polynomial functions.

F.IF.4, F.IF.7c

Rigor

The Three Pillars of Rigor

To help students meet standards, they need to illustrate their ability to use the three pillars of rigor. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. Once they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Practice.

1 CONCEPTUAL UNDERSTANDING 2 FLUENCY 3 APPLICATION

EXPLORE

LEARN

EXAMPLE & PRACTICE

Suggested Pacing

Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
1-1 Graphing Quadratic Functions	F.IF.4, F.IF.6	2	1
1-2 Solving Quadratic Equations by Graphing	A.CED.2, F.IF.4	1	0.5
1-3 Complex Numbers	N.CN.1, N.CN.2	2	1
1-4 Solving Quadratic Equations by Factoring	N.CN.7, N.CN.8, F.IF.8a	2	1
1-5 Solving Quadratic Equations by Completing the Square	N.CN.7, F.IF.8a	3	1.5
1-6 Using the Quadratic Formula and the Discriminant	N.CN.7, N.CN.8, A.SSE.1b	1	0.5
Put It All Together: Lessons 1-1 through 1-6		1	0.5
1-7 Quadratic Inequalities	A.CED.1, A.CED.3	1	0.5
1-8 Solving Linear-Nonlinear Systems	A.REI.11	2	1
Module Review		1	0.5
Module Assessment		1	0.5
Total Days		18	9

NAME _____ DATE _____

Cheryl Tobey Math Probe
Factoring

Circle your choice.	Circle your choice.
1. Which is the completely factored form of $x^2 - 25$? A. $(x-5)$ B. $(x+5)$ C. $(x+5)(x-5)$ D. $(x+5)(x+5)$ E. Cannot be factored	
2. Which is the completely factored form of $x^2 - 16$? A. $(x-16)$ B. $(2x+16)(x-16)$ C. $(2x+8)(x-8)$ D. $(2x+4)(x-4)$ E. Cannot be factored	
3. Which is the completely factored form of $x^2 - 9$? A. (x^2-9) B. $(x^2+9)(x^2-9)$ C. $(x^2+3)(x^2-3)$ D. $(x+3)(x-3)$ E. Cannot be factored	

Source: Cheryl Tobey Math Probe - Factoring

Correct Answers:
1. D 2. B 3. D

Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will determine the correct ways to factor polynomial expressions.

Targeted Concepts Understand how polynomial expressions, such as the difference of squares, are factored.

Targeted Misconceptions

- Students may confuse the variable's exponent with a common factor.
- Students do not understand that the expanded form of a squared binomial is a trinomial.
- Students only take the square root of the variable and leave numbers as squares.
- Students do not understand that one cannot distribute an exponent over addition and subtraction.
- Students may not recognize the difference of squares.
- Students do not factor completely.

Use the Probe after Lesson 1-4.

Collect and Assess Student Answers

If the student selects these responses...	Then the student likely...
1. A	is using the square as a common factor with 16.
1. B 2. A 3. A	is incorrectly viewing factoring as taking the square root of each term and then squaring the expression. This error is related to a common error in the binomial expansion $(x - 4)^2 \neq x^2 - 16$.
1. C 2. C 3. C	is not considering how the middle term is eliminated by multiplying conjugates.
2. D	is only taking the square root of the variable and leaving the numbers as perfect squares. They are also not eliminating the linear term by considering a squared binomial.
3. B	does not recognize that factoring the original expression produces a difference of squares binomial $(x^2 - 9)$ that can also be factored.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

-  ALEKS® Quadratic Equations
- Lesson 1-4, all Learns, all Examples

Revisit the probe at the end of the module to be sure that your students no longer carry these misconceptions.



The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

What are important characteristics of a quadratic function, and what real-world situations can be modeled by quadratic functions and equations? Sample answer: Quadratic functions have an independent variable that is squared as the highest degree. These quadratic functions are a parabola shape when graphed, opening either up or down. Many physical situations follow a quadratic pattern, including the motion of a thrown object falling due to gravity.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

Focus Students write about the different ways quadratic functions and relations can be solved as these methods are presented in the lessons of this module.

Teach Have students make and label their Foldables as illustrated. Students should fill in the appropriate sections with their notes, diagrams, and examples as they cover each lesson in this module. Have students use the appropriate tabs as they cover each lesson in this module.

When to Use It Encourage students to add to their Foldables as they work through the module and to use them to review for the module test.

Launch the Module

For this module, the Launch the Module video uses a trip to Mars to explore quadratic functions and quadratic models. Students learn how to connect quadratic graphs to real-world situations.

Quadratic Functions

Essential Question

What are important characteristics of a quadratic function, and what real-world situations can be modeled by quadratic functions and equations?

What Will You Learn?

How much do you already know about each topic **before** starting this module?

KEY	Before	After
👉 I don't know.		
👉 I've heard of it.		
👉 I know it!		
find and interpret average rate of change of a quadratic function		
estimate solutions of quadratic equations by graphing		
perform operations with complex numbers		
solve quadratic equations by factoring		
solve quadratic equations by completing the square		
use the discriminant to determine the number and type of roots of a quadratic equation		
solve quadratic inequalities in two variables by graphing		
solve systems of two quadratic equations		
solve systems of nonlinear relations		

Foldables: Make this Foldable to help you organize your notes about quadratic functions. Begin with one sheet of 11" by 17" paper.

- 1. Fold** in half lengthwise.
- 2. Fold** in thirds crosswise.
- 3. Cut** along the middle fold from the edge to the last crease as shown.
- 4. Refold** along the lengthwise fold and tape the uncut section at the top. Label each section with a lesson number. Close to form a booklet.



Module 1 • Quadratic Functions 1

Interactive Student Presentation



What Vocabulary Will You Learn?

- average rate of change
- axis of symmetry
- completing the square
- complex conjugates
- complex number
- difference of squares
- discriminant
- factored form
- imaginary unit i
- maximum
- minimum
- perfect square binomials
- projectile motion problems
- pure imaginary number
- quadratic equation
- quadratic function
- quadratic inequality
- quadratic relations
- rate of change
- rationalizing the denominator
- standard form of a quadratic equation
- vertex
- vertex form

Are You Ready?

Complete the Quick Review to see if you are ready to start this module. Then complete the Quick Check.

Quick Review	
Example 1 Given $f(x) = -2x^2 + 3x - 1$ and $g(x) = 3x^2 - 5$, find each value.	
a. $f(2)$ $f(x) = -2x^2 + 3x - 1$ Original function $f(2) = -2(2)^2 + 3(2) - 1$ Substitute 2 for x . $= -8 + 6 - 1$ Simplify	Example 2 Factor $2x^2 - x - 3$ completely. If the polynomial is not factorable, write prime. To find the coefficients of the x term, you must find two numbers whose product is $2(-3) = -6$, and whose sum is -1 . The two coefficients must be 2 and -3 since $2(-3) = -6$ and $2 + (-3) = -1$. Rewrite the expression and factor by grouping. $2x^2 - x - 3$ $= 2x^2 + 2x - 3x - 3$ Substitute $2x$ for $-x$. $= (2x^2 + 2x) + (-3x - 3)$ Associative Property $= 2x(x + 1) - 3(x + 1)$ Factor out the GCF. $= (2x - 3)(x + 1)$ Distributive Property
b. $g(-2)$ $g(x) = 3x^2 - 5$ Original function $g(-2) = 3(-2)^2 - 5$ Substitute -2 for x . $= 12 - 5$ Simplify	
Quick Check Given $f(x) = 2x^2 + 4$ and $g(x) = -x^2 - 2x + 3$, find each value.	
1. $f(3)$ 22 2. $f(0)$ 4 3. $g(4)$ -21 4. $g(-3)$ 0	Factor completely. If the polynomial is not factorable, write prime. 5. $x^2 - 10x + 21$ $(x - 3)(x - 7)$ 6. $2x^2 + 7x - 4$ $(2x - 1)(x + 4)$ 7. $2x^2 - 7x - 15$ $(2x + 3)(x - 5)$ 8. $x^2 - 15x + 18$ prime
How Did You Do? Which exercises did you answer correctly in the Quick Check?	

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define A quadratic inequality is an inequality that includes a quadratic expression.

Example Consider $y < x^2 - 5x + 4$.

Ask Based on your knowledge of graphing inequalities and quadratic functions, what will the boundary of the graph of the quadratic inequality look like? **a dashed parabola**

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- finding function values
- finding x -intercepts
- simplifying expressions
- factoring
- factoring perfect squares
- evaluating methods for solving quadratic equations
- determining whether an ordered pair is a solution
- solving linear systems of equations
- solving linear-quadratic systems of equations



ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You may want to use the **Quadratic and Polynomial Functions** section to ensure student success in this module.



Mindset Matters

Mistakes = Learning

When a student makes a mistake and goes on to learn from it, he or she can actually build new connections in their brain while figuring out a new path or process.

How Can I Apply It?

ALEKS is a great tool to not only individualize learning for each student, but to also help students understand that making mistakes and trying new problems will help them to learn and grow over the long run. Have students keep track of their ALEKS Pie Chart to see their progress.

Graphing Quadratic Functions

LESSON GOAL

Students graph quadratic functions.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Explore:** Transforming Quadratic Functions

 **Develop:**

Graphing Quadratic Functions

- Graph a Quadratic Function by Using a Table
- Compare Quadratic Functions
- Use Quadratic Functions

Finding and Interpreting Average Rate of Change

- Find Average Rate of Change from an Equation
- Find Average Rate of Change from a Table
- Find Average Rate of Change from Graph

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources	AL	JE	ET
Remediation: Graphing Quadratic Functions	●	●	●
Extension: Parametric Equations	●	●	●

Language Development Handbook

Assign page 1 of the *Language Development Handbook* to help your students build mathematical language related to graphing quadratic functions.

ELL You can use the tips and suggestions on page T1 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

2 Reason abstractly and quantitatively.

Coherence

Vertical Alignment

Now

Students graph quadratic functions and identify key features of the graphs.
F.IF.4, F.IF.6

Next

Students will solve quadratic equations using graphs.
A.CED.2, F.IF.4

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

 **Conceptual Bridge** In this lesson, students develop an understanding of quadratic functions and use it to build fluency by graphing quadratic functions. They apply their understanding of quadratic functions by solving real-world problems.

Mathematical Background

Graphs of quadratic functions of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$, are called parabolas. All parabolas have an axis of symmetry, a vertex, and a y -intercept. Some parabolas open up, and others open down.



Interactive Presentation

Warm Up

If $f(x) = -4x + 2$ and $g(x) = x^2 - 2x + 5$, find each value.

- $f(3)$
- $g(-4)$
- $f(n - 2)$
- $g(-1) + 2$
- $f(-3) + g(2)$

Show Answers

Warm Up

Launch the Lesson

A football passed downhill, a long jumper leaping for a record distance, a softball hit for a home run... the action of a sporting event often centers on a parabola. A projectile is an object that through the air, and they all follow a path shaped like a **parabola**. The graphs of quadratic functions, parabolas all share certain characteristics.

What do footballs, long jumpers, and softballs have in common? They all follow the same rules of motion. Who makes the rules? The Parabolas.

PARABOLAS ARE GOOD SPORTS

Launch the Lesson

Vocabulary

Expand All Collapse All

- quadratic function**
A function with a graph that is a parabola.
- axis of symmetry**
The line about which a graph is symmetric.
- vertex**
Either the lowest point or the highest point on a parabola.
- rate of change**
How a quantity is changing with respect to a change in another quantity.
- average rate of change**
The change in the value of the dependent variable divided by the change in the value of the independent variable.

- Compare and contrast $y = x^2$ and $y = x^2 - 2$.
- What are differences between linear and quadratic functions?
- What types of real-world situations are modeled by quadratic relationships?
- How can we sketch a graph of a quadratic function?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- finding function values

Answers:

- 16
- 29
- $-6n + 14$
- 10
- 25

Launch the Lesson



Teaching the Mathematical Practices

4 Analyze Relationships Mathematically In the Launch the Lesson, encourage students to explain the relationships between the data about the athletes and the graph of a parabola. Guide students to see how the length of the long jump, height of a basket toss, and hang time of a slam dunk relate to the key features of a parabola described in the infographic.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Transforming Quadratic Functions

Objective

Students use a sketch to explore graphing quadratic functions.



Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to complete the Explore, students will need to use the sketch to draw conclusions. Work with students to explore and deepen their understanding of how the parameters of a quadratic function change the graph of the function.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

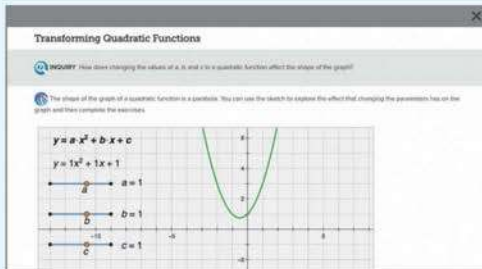
What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students interact with a sketch of a quadratic function by adjusting the constants and coefficients a , b , and c . They make observations about how changes to these values affect the graph of the quadratic function. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore

Explore

WEB SKETCHPAD



Students will use a sketch to explore the features of a quadratic graph.



Interactive Presentation

INQUIRY How can you use the values of a , b , and c in the equation of a quadratic function $y = ax^2 + bx + c$ to visualize its graph?

Submit

Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Transforming Quadratic Functions (continued)

Questions

Have students complete the Explore activity.

Ask:

- What happens to the standard form of the quadratic function when $x = 0$? How does this help you to describe c ? **Sample answer:** When $x = 0$, the equation becomes $y = 0$. This reminds me that c is the y -intercept for the function.
- How does knowing the value of a help you graph a quadratic function? **Sample answer:** If a is positive, I know that the graph will open up and the vertex will be the minimum value. If a is negative, I know that the graph will open down and the vertex will be the maximum value.

Inquiry

How can you use the values of a , b , and c in the equation of a quadratic function $y = ax^2 + bx + c$ to visualize its graph? **Sample answer:** The value of a tells you how wide the graph will be and whether it opens up or down. The value of b tells you whether the graph is shifted horizontally. The value of c tells you the y -intercept.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Graphing Quadratic Functions

Objective

Students graph quadratic functions by making a table of values.

MP Teaching the Mathematical Practices

- 1 Explain Correspondences** Encourage students to explain the relationships between the parameters, key features, and graph of a quadratic function.

Common Misconception

Students may think all nonlinear functions are quadratic. Consider discussing different types of nonlinear functions that are not represented by parabolas to help to prepare for future work with polynomial, radical, and exponential functions.

Common Error

Make sure students realize that $f(x)$ and y can be used interchangeably. They should also realize that the maximum or minimum value of the quadratic function is given by the y -coordinate of the vertex of the parabola.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Graphing Quadratic Functions

Lesson 1-1

Explore Transforming Quadratic Functions

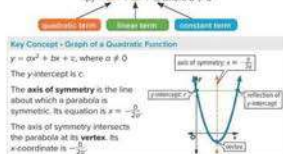
Online Activity Use graphing technology to complete the Explore.

INQUIRY How can you use the values of a , b , and c in the equation of a quadratic function $y = ax^2 + bx + c$ to visualize its graph?

Learn Graphing Quadratic Functions

A quadratic function has an equation of the form $y = ax^2 + bx + c$, where $a \neq 0$. A quadratic function has a graph that is a parabola.

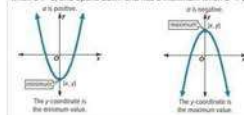
$$f(x) = ax^2 + bx + c, \text{ where } a \neq 0$$



You will derive the equation for the axis of symmetry in Lesson 1-5.

Key Concept • Maximum or Minimum Values of Quadratic Graphs

The vertex of a quadratic function is either a **maximum** or **minimum**. The graph of $y = ax^2 + bx + c$, where $a \neq 0$, opens up and has a minimum when $a > 0$, and opens down and has a maximum when $a < 0$.



Today's Goals

- Graph quadratic functions.

- Find and interpret the average rate of change of quadratic functions, given symbolically, in tables, and in graphs.

Today's Vocabulary

- quadratic function
- axis of symmetry
- vertex
- maximum
- minimum
- rate of change
- average rate of change

Think About It!

Why is it important to know whether the vertex is a maximum or minimum when graphing a quadratic function?

Sample answer: It will indicate which way the graph opens.

Watch Out!

Maxima and Minima

Just because a vertex is a maximum does not mean it will be located above the x -axis, and just because a vertex is a minimum does not mean it will be located below the x -axis.

Lesson 1-1 • Graphing Quadratic Functions 3

Interactive Presentation

Graphing Quadratic Functions

A **quadratic function** has a graph that is a parabola. It is a function of the form $y = ax^2 + bx + c$, where $a \neq 0$. Quadratic functions have a maximum or a minimum, and a constant term.

The general form of a quadratic function is:

$$f(x) = ax^2 + bx + c, \text{ where } a \neq 0$$

quadratic term linear term constant term

Learn

TYPE

a

Students answer a question about the importance of the vertex when graphing a quadratic function.



Example 1 Graph a Quadratic Function by Using a Table

MP Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. In Example 1, guide students to see what information they can gather about the graph just by looking at the quadratic function.

Questions for Mathematical Discourse

- AL** How is the axis of symmetry related to the vertex of a quadratic function? **The x -coordinate for all points on the axis of symmetry and the x -coordinate for the vertex are the same.**
- OL** How would the graph change if a were -1 ? **The axis of symmetry shifts to 1, the vertex becomes a maximum, and the y -intercept remains the same.**
- BL** Can a parabola ever exist in only one quadrant? Explain. **No; sample answer: All parabolas extend to infinity and negative infinity, so it will always exist in at least two quadrants, unless additional constraints are present.**

Example 2 Compare Quadratic Functions

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the key features of the verbal description and graph used in Example 2.

Questions for Mathematical Discourse

- AL** Is the vertex of $g(x)$ a maximum or a minimum? Explain your reasoning. **Maximum; sample answer: The y -coordinate of the vertex is greater than the y -intercept, which means the parabola opens down and has a maximum.**
- OL** If the y -intercept of a quadratic function is the same as the y -coordinate of the vertex, what is the x -coordinate of the vertex? **0**
- BL** Could you determine which function has the greater maximum without graphing $g(x)$? Explain. **Yes; sample answer: Because the y -coordinate of the vertex gives the value of the maximum or minimum, once I confirmed that the function had a maximum, I could just compare the y -coordinate of the vertex of $g(x)$ to the maximum of $f(x)$.**

Go Online
You can watch videos to see how to graph quadratic functions by using a table or its key features.

Think About It!
If you know that $f(-4) = 5$, find $f(2)$ without substituting 2 for x in the function. Justify your argument.

Sample answer: Because $f(x)$ is symmetric about $x = -1$, $f(-4) = f(2)$.

Think About It!
Compare the end behavior of $f(x)$ and $g(x)$.

Sample answer: Both functions have the same end behavior, as $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$ and as $x \rightarrow \infty$, $f(x) \rightarrow -\infty$.

Example 1 Graph a Quadratic Function by Using a Table

Graph $f(x) = x^2 + 2x - 3$. State the domain and range.

Step 1 Analyze the function.

For $f(x) = x^2 + 2x - 3$, $a = 1$, $b = 2$, and $c = -3$.
 c is the y -intercept, so the y -intercept is -3 .
 Find the axis of symmetry.

$$x = -\frac{b}{2a}$$

$$= -\frac{2}{2(1)}$$

$$= -1$$

Equation of the axis of symmetry

$$x = 1, b = 2$$

Simplify

The equation of the axis of symmetry is $x = -1$. Because $a > 0$, the vertex is a minimum.

Step 2 Graph the function.

x	$x^2 + 2x - 3$	$(x, f(x))$
-3	$(-3)^2 + 2(-3) - 3 = -3$	$(-3, -3)$
-2	$(-2)^2 + 2(-2) - 3 = -3$	$(-2, -3)$
-1	$(-1)^2 + 2(-1) - 3 = -3$	$(-1, -4)$
0	$(0)^2 + 2(0) - 3 = -3$	$(0, -3)$
1	$(1)^2 + 2(1) - 3 = -3$	$(1, -3)$

Step 3 Analyze the graph.

The parabola extends to positive and negative infinity, so the domain is all real numbers. The range is $\{y \mid y \geq -4\}$.

Example 2 Compare Quadratic Functions

Compare the graph of $f(x)$ to a quadratic function $g(x)$ with a y -intercept of -1 and a vertex at $(1, 2)$. Which function has a greater maximum?

From the graph, $f(x)$ appears to have a maximum of 5. Graph $g(x)$ using the given information. The vertex is at $(1, 2)$, so the axis of symmetry is $x = 1$.

The y -intercept is -1 , so $(0, -1)$ is on the graph.

Reflect $(0, -1)$ in the axis of symmetry. So, $(2, -1)$ is also on the graph.

Connect the points with a smooth curve.

2 is the maximum, so $f(x)$ has the greater maximum.

Interactive Presentation

Graph a Quadratic Function by Using a Table

Graph $f(x) = x^2 + 2x - 3$. State the domain and range.

Step 1 Analyze the function.

Step 2 Graph the function.

Step 3 Analyze the graph.

Example 1

EXPAND



Students step through the process of graphing a quadratic function using a table.

TYPE



Students answer a question about the graph of a quadratic function.

**Common Error**

Make sure students understand that a parabola that opens up is the graph of a function with a minimum value, and that a parabola that opens down is the graph of a function with a maximum value. Compare these parabolas to valleys (where the altitude of the valley floor is a minimum) and hills (where the peak of the hill is the maximum altitude).

Example 3 Use Quadratic Functions**MP Teaching the Mathematical Practices**

2 Represent Situations Symbolically Guide students to define variables to solve the problem in Example 3. Help students to identify the independent and dependent variables. Then work with them to find other relationships in the problem and represent them with an equation.

Questions for Mathematical Discourse

- AL** What about the equation indicates there will be a maximum amount of money generated? **The a value is negative.**
- OL** What does the y -intercept represent in this situation? **The money generated last year. This is their expected profit if the price stays the same.**
- BL** Assuming this model works for all values of x , how would you determine how many price increases would lead to a total of 0 tickets sold? **Replace $P(x)$ with 0 and solve for x or use the calculator to find the x -intercept.**

Example 3 Use Quadratic Functions

SKING A ski resort has extended hours on one holiday weekend per year. Last year, the resort sold 680 ski passes at \$120 per holiday weekend pass. This year, the resort is considering a price increase. They estimate that for each \$5 increase, they will sell 20 fewer holiday weekend passes.

Part A How much should they charge in order to maximize profit? **Step 1** Define the variables.

Let x represent the number of \$5 price increases, and let $P(x)$ represent the total amount of money generated. $P(x)$ is equal to the price of each pass ($120 + 5x$) times the total number of passes sold ($680 - 20x$).

Step 2 Write an equation.

$$P(x) = (120 + 5x)(680 - 20x)$$

$$= 81,600 - 2400x + 3400x - 100x^2$$

$$= -100x^2 + 1000x + 81,600$$

Original equation

Multiply

Simplify

Step 3 Find the axis of symmetry.

Because a is negative, the vertex is a maximum.

$$x = -\frac{b}{2a}$$

$$= -\frac{1000}{2(-100)}$$

$$= 5$$

Formula for the axis of symmetry

$$x = -100; b = 1000$$

Simplify

Step 4 Interpret the results.

The ski resort will make the most money with 5 price increases, so they should charge $120 + 5(5)$ or \$145 for each holiday weekend pass.

Part B Find the domain and range in the context of the situation.

The domain is $\{x \mid 0 \leq x \leq 34\}$ because the number of price increases, and the number of passes sold cannot be negative. The range is

$\{y \mid 0 \leq y \leq 84,100\}$ because the amount of money generated cannot be negative, and the maximum amount of money generated is $P(5) = -100(5)^2 + 1000(5) + 81,600$ or 84,100.

Check

CONCERTS Last year, a ticket provider sold 1350 lawn seats for a concert at \$70 per ticket. This year, the provider is considering increasing the price. They estimate that for each \$2 increase, they will sell 30 fewer tickets.

Part A How much should the ticket provider charge in order to maximize profit? **\$... 80**

Part B Find the domain and range in the context of the situation.

$$D = \{x \mid 0 \leq x \leq 9\}; R = \{y \mid 0 \leq y \leq 96,000\}$$

$$\{x \mid x \geq 0\}; \{y \mid 0 \leq y \leq 96,000\}$$

Go Online You can complete an Extra Example online.

Study Tip

Assumptions You assumed that the ski resort has the ability to increase the price indefinitely and that every price increase will be \$5 and will cause the resort to lose sales from exactly 20 holiday weekend passes.

Think About It!

Why is the maximum amount of money generated from holiday weekend passes $P(5)$?

Sample answer: Because the x -coordinate of the vertex is 5, the y -coordinate can be found by substituting 5 for x in the original equation.

Interactive Presentation

Example 3

EXPAND

Students step through the process of analyzing a quadratic function and its graph in a real-world context.

TYPE

Students answer a question about the maximum of a real-world quadratic model.

CHECK

Students complete the Check online to determine whether they are ready to move on.

**Learn** Finding and Interpreting Average Rate of Change

A function's **rate of change** is how a quantity is changing with respect to a change in another quantity. For nonlinear functions, the rate of change is not the same over the entire function. You can calculate the **average rate of change** of a nonlinear function over an interval.

Key Concept Average Rate of Change

The average rate of change of a function $f(x)$ is equal to the change in the value of the dependent variable $f(b) - f(a)$ divided by the change in the value of the independent variable $b - a$ over the interval $[a, b]$.

$$\frac{f(b) - f(a)}{b - a}$$
Example 4 Find Average Rate of Change from an Equation

Determine the average rate of change of $f(x) = -x^2 + 2x - 1$ over the interval $[-4, 4]$.

The average rate of change is equal to $\frac{f(b) - f(a)}{b - a}$.

First find $f(-4)$ and $f(4)$.

$$f(-4) = -(-4)^2 + 2(-4) - 1 = -16 - 8 - 1 = -25 \quad f(4) = -(4)^2 + 2(4) - 1 = -16 + 8 - 1 = -9$$

Then use $f(-4)$ and $f(4)$ to find the average rate of change.

$$\frac{f(4) - f(-4)}{4 - (-4)} = \frac{-9 - (-25)}{4 - (-4)} = \frac{-9 + 25}{4 + 4} = \frac{16}{8} = 2$$

The average rate of change of the function over the interval $[-4, 4]$ is 2.

Check

Find the average rate of change of $f(x) = -3x^2 - 5x + 7$ over the interval $[-5, 5]$.

average rate of change = -7 , -5

Example 5 Find Average Rate of Change from a Table

Determine the average rate of change of $f(x)$ over the interval $[-3, 3]$.

The average rate of change is equal to $\frac{f(b) - f(a)}{b - a}$.

First find $f(-3)$ and $f(3)$ from the table.

$$f(-3) = -24 \quad f(3) = 48$$

$$\frac{f(3) - f(-3)}{3 - (-3)} = \frac{48 - (-24)}{3 - (-3)} = \frac{48 + 24}{3 + 3} = \frac{72}{6} = 12$$

The average rate of change of the function over the interval $[-3, 3]$ is 12.

See Online You can complete an Extra Example online.

x	$f(x)$
-3	48
-2	21
-1	0
0	-15
1	-24
2	-27
3	-24

Think About It!

Find the average rate of change of the function over the interval $[-3, 5]$. Compare it to your results from the interval $[-4, 4]$.

Sample answer:

$f(-3) = -(-3)^2 + 2(-3) - 1 = -9 - 6 - 1 = -16$
 $f(5) = -(5)^2 + 2(5) - 1 = -25 + 10 - 1 = -16$
 or 4. The rate of change is greater over the interval $[-3, 5]$ than over the interval $[-4, 4]$.

Talk About It!

Without graphing, how can you tell that this function is nonlinear? Justify your argument.

Sample answer: The average rate of change is not the same over all intervals of the function. For example, the average rate of change over the interval $[-2, 1]$ is -15 , and the average rate of change over the interval $[-3, 3]$ is 12 .

Interactive Presentation



Learn

TYPE

a

Students answer a question to determine if they understand average rate of change of a linear function.

Learn Finding and Interpreting Average Rate of Change**MP Teaching the Mathematical Practices**

2 Make Sense of Quantities Mathematically proficient students need to be able to make sense of quantities and their relationships. Guide students to see how rate of change represents the relationship between problem variables.

Example 4 Find Average Rate of Change from an Equation**MP Teaching the Mathematical Practices**

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. In Example 4, guide students to first calculate the value of the function for -4 and for 4 before calculating the average rate of change.

Questions for Mathematical Discourse

- AL** Does the shape of the function over an interval matter when determining the average rate of change for that interval? Explain. **No; sample answer:** Only the function values at the interval endpoints are used to determine the average rate of change.
- OL** If the average rate of change over an interval is positive, what do you know about the end of the interval compared to the beginning of the interval? **The function will have a greater output at the end of the interval.**
- BL** If a function has a positive average rate of change over the interval $[a, b]$, but a negative average rate of change over the interval $[a, c]$, where $c > b$, what can you conclude about the extrema of the function? **The function has a maximum in the interval $[a, c]$.**

Example 5 Find Average Rate of Change from a Table**MP Teaching the Mathematical Practices**

3 Construct Arguments Point out that to answer the question in the Talk About It! feature, students will need to use definitions and previously established results to construct an argument.

Questions for Mathematical Discourse

- AL** Over which interval is the average rate of change positive? **[2, 3]**
- OL** What does it mean if the average rate of change over an interval is 0? **Sample answer:** The function has the same value at the beginning and end of the interval.
- BL** How could you use the average rate of change to estimate where an extrema occurs? **Find where the average rate of change switches from positive to negative (maximum) or from negative to positive (minimum).**



Example 6 Find Average Rate of Change from a Graph

MP Teaching the Mathematical Practices

5 Use a Source Guide students to find external information to answer the question posed in the Use a Source feature.

Questions for Mathematical Discourse

- AL** If the value of a function is greater at the end of an interval than it is at the beginning of the interval, what can you tell about the average rate of change over that interval? **The average rate of change will be positive over that interval.**
- OL** Is the amount spent on sports footwear from 2005 to 2015 linear or nonlinear? Explain. **Nonlinear; the average rate of change is not constant over each interval.**
- BL** How would the average rate of change from 2005 to 2015 differ if the spending each year between 2005 and 2015 increased? **The average rate of change would remain the same.**

Common Error

Students may subtract in the wrong direction, ending up with a rate of change that is the negative of what they expect. Encourage students to look at the interval and see if they expect a positive or negative average rate of change over the interval, and then use that to check their solution.

DIFFERENTIATE

Language Development **ELL**

Discuss how the average rate of change and slope are similar concepts. Average rate of change is the change in values of the dependent variable divided by the change in values of the independent variable. Remind students that slope is found by taking the difference of the y -coordinates divided by the difference of the x -coordinates. Students should understand that change in the values of the dependent variable is another, equivalent way of saying the difference of the y -coordinates, and that the change in the values of the independent variable is another, equivalent way of saying the difference of the x -coordinates.

Check

TESTING The table shows the number of students who took the ACT between 2011 and 2015.

Year	Number of Students
2011	1,623,112
2012	1,666,017
2013	1,799,243
2014	1,845,787
2015	1,924,436

Part A

Find the average rate of change in the number of students taking the ACT from 2011 to 2015.

average rate of change = $\frac{75,331}{4}$

Part B

Interpret your results in the context of the situation.

From 2011 to 2015, the number of students taking the ACT

increased by an average of $\frac{75,331}{4}$ students per year.

Example 6 Find Average Rate of Change from a Graph

FOOTWEAR The graph shows the amount of money the United States has spent on sports footwear since 2005.

Part A

Use the graph to estimate the average rate of change of spending on sports footwear from 2005 to 2015. Then check your results algebraically.

Estimate

From the graph, the change in the y -values is approximately 5.5, and the change in the x -values is 10.

So, the rate of change is approximately $\frac{5.5}{10}$ or 0.55.

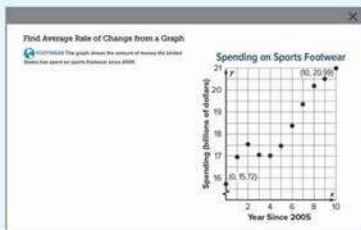
(continued on the next page)

Lesson 14 • Graphing Quadratic Functions 7

Use a Source
Research the sales of another industry over a ten-year period. Then find the average rate of change during that time.

See students' work.

Interactive Presentation



Example 6

TYPE



Students answer a question to determine whether they understand average rate of change.



Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Algebraically

The average rate of change is equal to $\frac{f(b) - f(a)}{b - a}$.
 $\frac{f(10) - f(0)}{10 - 0} = \frac{20.99 - 15.72}{10 - 0}$ or 0.527

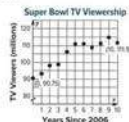
Part B

Interpret your results in the context of the situation.

From 2005 to 2015, the amount of money spent on sports footwear in the United States increased by an average of \$527 million per year.

Check

SUPER BOWL. The graph shows the number of television viewers of the Super Bowl since 2006.



Part A

Use the graph to estimate the average rate of change in Super Bowl viewers from 2006 to 2016 to the nearest hundredth of a million. Then check your results algebraically.

estimate = $\frac{7}{2.195}$ million

average rate of change = $\frac{7}{2.195}$ million per year

Part B

Interpret your results in the context of the situation.

From $\frac{2006}{2}$ to $\frac{2016}{2}$, the number of Super Bowl viewers $\frac{7}{2.195}$ increased by an average of $\frac{7}{2.195}$ million viewers per year.

Go Online You can complete an Extra Example online.

8 Module 1 • Quadratic Functions

CHECK



Students complete the Check online to determine whether they are ready to move on.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–26
2	exercises that use a variety of skills from this lesson	27–38
2	exercises that extend concepts learned in this lesson to new contexts	39–44
3	exercises that emphasize higher-order and critical-thinking skills	45–48

ASSESS AND DIFFERENTIATE

 Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, THEN assign:

- Practice, Exercises 1–43 odd, 45–48
- Extension: Parametric Equations
-  Quadratic Functions

IF students score 66%–89% on the Checks, THEN assign:

- Practice, Exercises 1–47 odd
- Remediation, Review Resources: Graphing Quadratic Functions
- Personal Tutors
- Extra Examples 1–5
-  Quadratic Functions

IF students score 65% or less on the Checks, THEN assign:

- Practice, Exercises 1–25 odd
- Remediation, Review Resources: Graphing Quadratic Functions
- Quick Review Math Handbook: Graphing Quadratic Functions
-  Quadratic Functions

Practice

Graph each function. Then state the domain and range. 1–6. See margin for graphs.

- $f(x) = x^2 + 6x + 8$
a. $x \in \{\text{all real numbers}\}$
b. $x \in \{x | x \geq -2\}$
- $f(x) = -x^2 - 2x + 2$
a. $x \in \{\text{all real numbers}\}$
b. $x \in \{x | x \leq 3\}$
- $f(x) = 2x^2 - 4x + 3$
a. $x \in \{\text{all real numbers}\}$
b. $x \in \{x | x \geq 1\}$
- $f(x) = -2x^2$
a. $x \in \{\text{all real numbers}\}$
b. $x \in \{x | x \geq 0\}$
- $f(x) = x^2 - 4x + 4$
a. $x \in \{\text{all real numbers}\}$
b. $x \in \{x | x \geq 0\}$
- $f(x) = x^2 - 6x + 8$
a. $x \in \{\text{all real numbers}\}$
b. $x \in \{x | x \geq -1\}$

Example 2

7. Compare the graph of $f(x)$ to a quadratic function $g(x)$ with a y-intercept of 1 and a vertex at $(1, 5)$. Which function has a greater maximum? Explain.



$g(x)$: Sample answer: Its vertex is a maximum point at $(1, 5)$, which is 4 units above the vertex of $f(x)$ which is $(1, -5)$.

8. Compare the graph of $f(x)$ to a quadratic function $g(x)$ with a y-intercept of 0.5 and a vertex at $(1, -8)$. Which function has a lesser minimum? Explain.



$g(x)$: Sample answer: Its vertex is a minimum point at $(1, -8)$, which is 4 units below the vertex of $f(x)$ which is $(1, -5)$.

9. Compare $f(x) = x^2 - 10x + 5$ to the quadratic function $g(x)$ shown in the table. Which function has the lesser minimum? Explain.

x	$g(x)$
-10	125
-5	75
0	19
5	-10
10	10

$g(x)$: Sample answer: Its vertex is a minimum point at $(5, -20)$, which is 10 units below the vertex of $f(x)$ which is $(5, -5)$.

$g(x)$: Sample answer: Its vertex is a maximum point at $(3, -6)$, which is 7 units above the vertex of $f(x)$ which is $(3, -11)$.

Example 3

10. Interpreting A county park sells annual permits to its fishing lake. Last year, the county sold 480 fishing permits for \$80 each. This year, the park is considering a price increase. They estimate that for each \$4 increase, they will sell 16 fewer annual fishing permits.

- Define variables and write a function to represent the situation. For x = the number of price increases and y = revenue, $y = (80 + 4x)(480 - 16x)$.
- How much would the park have to charge for each permit in order to maximize its revenue from fishing permits? \$300 per permit.
- Find the domain and range in the context of this situation. $D = \{x | 0 \leq x \leq 30\}$, $R = \{y | 0 \leq y \leq 40,000\}$.

Lesson 1-1 • Graphing Quadratic Functions 9

- SALES Last month, a retailer sold 120 jar candles at \$30 per candle. This month, the retailer is considering raising the candles on sale. They estimate that for each \$2 increase in price, they will sell 10 additional candles.
 - How much should the retailer charge for each candle in order to maximize its profit? \$37.
 - Find the domain and range in the context of this situation. $D = \{x | 0 \leq x \leq 30\}$, $R = \{y | 0 \leq y \leq 3645\}$.

Example 4

Determine the average rate of change of $f(x)$ over the specified interval.

- $f(x) = x^2 - 10x + 5$, interval $[-4, 4]$ 14. $f(x) = 2x^2 + 4x - 6$, interval $[-3, 3]$ 4
- $f(x) = 3x^2 - 3x + 1$, interval $[-5, 5]$ 15. $f(x) = 4x^2 + x + 3$, interval $[-2, 2]$ 1
- $f(x) = 2x^2 - 1$, interval $[-3, 3]$ 0 16. $f(x) = -2x^2 + 8x + 7$, interval $[-4, 4]$ 8

Example 5

Determine the average rate of change of $f(x)$ over the specified interval.

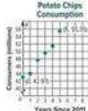
Interval	x	$f(x)$	Interval	x	$f(x)$
19. $[-3, 3]$	-3	0	20. $[-4, 4]$	-4	-27
	-1	6		-2	-3
	1	0		0	5
	3	0		2	-3
				4	-27

Interval	x	$f(x)$	Interval	x	$f(x)$
22. $[-5, 5]$	-5	-39	23. $[-3, 3]$	-3	27
	-3	-6		-1	3
	-1	1		1	3
	1	9		3	27
	3	9			
	5	1			

Example 6

25. FOOD The graph shows the number of people in the U.S. who consumed between 8 and 10 bags of potato chips in a year since 2011.

- Use the graph to estimate the average rate of change of consumption from 2011 to 2016. Then check your results algebraically: $\frac{10 - 0}{5} = 2$ or $\frac{10 - 0}{5} = 2$.
- Interpret your results in the context of the situation. See margin.



10 Module 1 • Quadratic Functions

3 REFLECT AND PRACTICE



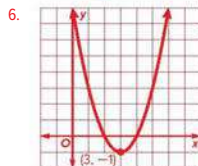
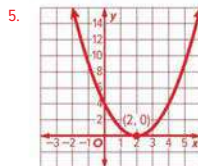
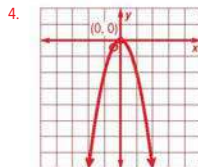
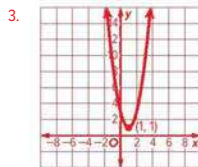
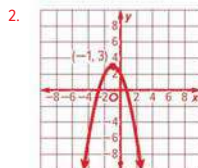
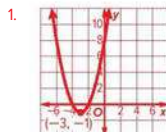
F.IF.4, F.IF.6

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Answers



25b. Sample answer: From 2011–2016, the number of people in the U.S. who consumed between 8 and 11 bags increased by an average of 2.476 million people per year.

26a. $\approx \frac{5}{10}, \approx \frac{2}{2} \approx \$5000; \frac{72-21}{10} = \frac{51}{10} = \$5100$

26b. Sample answer: From 2008–2018, the amount of money Sheila earned increased by an average of \$5100 per year.

26. **EARNINGS** The graph shows the amount of money Sheila earned each year since 2008. a. b. See margin.
- Use the graph to estimate the average rate of change of Sheila's earnings from 2008 to 2018. Then check your results algebraically.
 - Interpret your results in the context of the situation.



Mixed Exercises

Complete parts a–c for each quadratic function. 27–32. See Mod. 1 Answer Appendix.

- Find the y-intercept, the equation of the axis of symmetry, and the x-coordinate of the vertex.
- Make a table of values that includes the vertex.
- Use this information to graph the function.

27. $f(x) = 2x^2 - 5x + 9$ 28. $f(x) = -3x^2 - 9x + 2$ 29. $f(x) = -4x^2 + 5x$
 30. $f(x) = 2x^2 + 17x$ 31. $f(x) = 0.25x^2 + 3x + 4$ 32. $f(x) = -0.75x^2 + 4x + 6$
- Determine whether each function has a maximum or a minimum value. Then find and use the x-coordinate of the vertex to determine the maximum or minimum.
33. $f(x) = -3x^2 - 12x + 19$ max; -1 ; 23 34. $f(x) = 7x^2 - 25x + 8$ min; 15 ; -375
 35. $f(x) = -5x^2 + 34x - 6$ max; 14 ; 3.8 36. $f(x) = 2x^2 - 13x - 9$ min; 3.25 ; -30.125
 37. $f(x) = 9x - 1 - 18x^2$ max; 0.25 ; 0.125 38. $f(x) = -36 - 18x - 12x^2$ max; -0.75 ; -8.25

39. **HEALTH CLUBS** Last year, the Sports Time Athletic Club charged \$20 per month to participate in an aerobics class. Seventy people attended the classes. The club wants to increase the class price. They expect to lose one customer for each \$1 increase in the price.

- Define variables and write an equation for a function that represents the situation. Make a table and graph the function. See Mod. 1 Answer Appendix.
 - Find the vertex of the function and interpret it in the context of the situation. Does it seem reasonable? Explain. See Mod. 1 Answer Appendix.
40. **TICKETS** The manager of a community symphony estimates that the symphony will earn $-40P^2 + 1050P$ dollars per concert if they charge P dollars for tickets. What ticket price should the symphony charge in order to maximize its profits? $\$13.75$
41. **REASONING** On Friday nights, the local cinema typically sells 200 tickets at \$6.00 each. The manager estimates that for each \$0.50 increase in the ticket price, 10 fewer people will come to the cinema. a–c. See Mod. 1 Answer Appendix.
- Define the variables x and y . Then write and graph a function to represent the expected revenue from ticket sales, and determine the domain of the function for the situation.
 - What price should the manager set for a ticket in order to maximize the revenue? Justify your reasoning.
 - Explain why the graph decreases from $x = 4$ to $x = 20$, and interpret the meaning of the x-intercept of the graph.

Lesson 9.4 • Graphing Quadratic Functions 11

42. **USE A MODEL** From 4 feet above a swimming pool, Tomas throws a ball upward with a velocity of 32 feet per second. The height $h(t)$ of the ball t seconds after Tomas throws it is given by $h(t) = -16t^2 + 32t + 4$. For $t \geq 0$, find the maximum height reached by the ball and the time that this height is reached. **20 ft; 1 second**

43. **TRICK-OR-TREAT** At a special seasonal re-enactment, a cannonball is launched from a cannon on the wall of Fort Chambly, Quebec. If the path of the cannonball is traced on a graph so that the cannon is situated on the y-axis, the equation that describes the path is $y = -\frac{1}{1600}x^2 + \frac{1}{2}x + 20$, where x is the horizontal distance from the cliff and y is the vertical distance above the ground in feet. How high above the ground is the cannon? **20 ft**

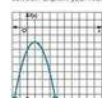


44. **CONSTRUCT ARGUMENTS** Which function has a greater maximum: $f(x) = -2x^2 + 6x - 7$ or the function shown in the graph at the right? Explain your reasoning by copying the graph of $g(x)$ and graphing $f(x)$. Graph $f(x)$ on the same coordinate plane and compare $g(x)$ with the greater maximum value.



Higher-Order Thinking Skills

45. **FIND THE ERROR** Lucas thinks that the functions $f(x)$ and $g(x)$ have the same maximum. Madison thinks that $g(x)$ has a greater maximum. Is either of them correct? Explain your reasoning.



$g(x)$ is a quadratic function with x-intercepts of 4 and 2 and a y-intercept of -8 .

Madison's sample answer: $f(x)$ has a maximum of -2 . $g(x)$ has a maximum of 1.

46. **PERSEVERE** The table at the right represents some points on the graph of a quadratic function.
- Find the values of a , b , c , and d . $a = 22$; $b = 26$; $c = -8$; $d = 2$
 - What is the x-coordinate of the vertex of the function? 0
 - Does the function have a maximum or a minimum? **maximum**

x	y
-20	-377
-1	-8
1	-8
2	-377

47. **WRITE** Describe how you determine whether a function is quadratic and, if it is, has a maximum or minimum value. See Mod. 1 Answer Appendix.
48. **CREATE** Give an example of a quadratic function with each characteristic.
- maximum of 8
 - minimum of -4
 - vertex at $(-2, 9)$
- Sample answer: $f(x) = x^2 + 8$ Sample answer: $f(x) = x^2 - 4$ Sample answer: $f(x) = x^2 + 4x + 10$

12 Module 1 • Quadratic Functions

Solving Quadratic Equations by Graphing

LESSON GOAL

Students solve quadratic equations by graphing.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Explore:** Roots of Quadratic Equations

 **Develop:**

Solving Quadratic Equations by Graphing

- One Real Solution
- Two Real Solutions
- Estimate Roots
- Solve by Using a Table
- Solve by Using a Calculator

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources

Remediation: Solving Quadratic Equations by Graphing

Extension: Solving Absolute Value Equations by Graphing

AL	LB	EL
●	●	●
	●	●

Language Development Handbook

Assign page 2 of the *Language Development Handbook* to help your students build mathematical language related to solving quadratic equations by factoring.

 You can use the tips and suggestions on page T2 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra, Functions

Standards for Mathematical Content:

A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

Standards for Mathematical Practice:

4 Model with mathematics.

5 Use appropriate tools strategically.

Coherence

Vertical Alignment

Previous

Students graphed quadratic functions.

F.IF.4, F.IF.6

Now

Students solve quadratic equations by graphing.

A.CED.2, F.IF.4

Next

Students will solve quadratic equations by factoring.

N.CN.7, N.CN.8, F.IF.8a

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

 **Conceptual Bridge** In this lesson, students expand on their understanding of graphing quadratic functions by using the graphs to solve related quadratic equations. They build fluency by finding solutions by graphing, and they apply their understanding by using graphs to solve real-world problems.



Interactive Presentation

Warm Up

Find the x-intercept(s) of each function.

Warm Up

Launch the Lesson

Uly is at a baseball game. Between innings, the mascot launches T-shirts into the stands. The path each T-shirt takes can be modeled by a parabola. Uly can write and solve a quadratic function to determine where each T-shirt will land.

Launch the Lesson

Vocabulary

Expand All Collapse All

quadratic equation

An equation that includes a quadratic expression.

standard form a quadratic equation

A quadratic equation can be written in this form: $ax^2 + bx + c = 0$, where $a \neq 0$ and a, b , and c are integers.

- How can you prove that any quadratic equation will have at most two roots?
- Tell a quadratic equation with real coefficients does not have real solutions? Why or why not?
- How does writing for x by graphing a quadratic function compare to solving for x by graphing a linear function?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- finding x-intercepts

Answers:

1. $-4, 2$
2. $-1, 4$
3. $-3, 0$
4. $-1, 1$

Launch the Lesson



Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will watch an animation about a real-world situation that can be modeled and solved by using a quadratic function.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can Statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Mathematical Background

If the graph of the related quadratic function has one x-intercept, then there is one real solution (double root). If the graph of the related quadratic function has two x-intercepts, then there are two real solutions. If the graph of the related quadratic function does not intersect the x-axis, then there are no real solutions, but two imaginary solutions.

Explore Roots of Quadratic Equations

Objective

Students use a sketch to explore solving quadratic equations by graphing.

MP Teaching the Mathematical Practices

1 Explain Correspondences Throughout the Explore, students will explain the relationships between the graphs and roots of quadratic functions and the solutions of the related quadratic equations.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. The students will use the sketch to graph various quadratic functions and identify their roots. They will observe the relationship between the zeros of the functions and the solutions of the related equations. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Roots of Quadratic Equations

INQUIRY How can you use the graph of a quadratic function to find the solutions of its related equation?

You can use a sketch to explore the roots of a quadratic equation and then complete the exercises.

$f(x) = x^2 + 2x - 3$

Graph

Roots

Explore

Exercise 1

Press Graph to graph $f(x) = x^2 + 2x - 3$. Before pressing Roots, estimate the zeros of the function. Justify your answer.

Done

Exercise 2

Exercise 3

Explore

WEB SKETCHPAD



Students use a sketch to explore the roots of quadratic equations.

EXPAND



Students tap to see guiding exercises about finding the roots of quadratic equations.



Interactive Presentation

Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Roots of Quadratic Equations

(continued)

Questions

Have students complete the Explore activity.

Ask:

- What does it mean to be the solution to a quadratic equation? **Sample answer:** The solution to any equation is a value or values of x that make the given equation true.
- How does changing the equation $6 = x^2 - x$ into $0 = x^2 - x - 6$ make it easier to find the solutions? **Sample answer:** By setting the equation equal to zero, you can graph $y = x^2 - x - 6$ and find where the graph is equal to zero, or where it crosses the x -axis.

Inquiry

How can you use the graph of a quadratic function to find the solution(s) of its related equation? **Sample answer:** I can find the x -intercepts of the graph of a quadratic function. Then I can substitute the coordinates of the x -intercepts into the related equation to see if they can make the equation true. If so, they are solutions of the related equation.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Solving Quadratic Equations by Graphing

Objective

Students solve quadratic equations by graphing.

MP Teaching the Mathematical Practices

3 Analyze Cases The Study Tip features introduces students to the cases where a quadratic equation has zero, one, or two solutions. Ask students to describe what the graph of the related quadratic function would look like for each case.

Essential Question Follow-Up

Students have explored solving quadratic equations by graphing.

Ask:

How can the graph of a quadratic function help you solve the corresponding quadratic equation? **Sample answer:** From the placement of the graph, you can see whether there are 0, 1, or 2 real solutions. The symmetry of the graph helps you identify a second solution when one solution is found.

Example 1 One Real Solution

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the equation, table, and graph used in Example 1.

Questions for Mathematical Discourse

- A1** If there is only one real solution, how is the vertex related to that solution? **The x-coordinate of the vertex is the solution.**
- OL** For a quadratic equation that has only one real solution, does the graph of the related function always have a minimum? Explain. **No; sample answer:** If you rearrange the terms in the equation such that the x^2 -term is negative, the parabola would open down and have a maximum.
- BL** Suppose you rearranged the terms so that the other side of the equation was 0. How would this affect your solution? **Sample answer:** Rearranging the terms so that the other side of the equation is 0 gives $-x^2 - 4x - 4 = 0$. The graph of this related function opens down, but has the same axis of symmetry and zero. Therefore, the solution is the same.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Solving Quadratic Equations by Graphing

Explore Roots of Quadratic Equations
 Online Activity Use graphing technology to complete the Explore.

INQUIRY How can you use the graph of a quadratic function to find the solutions of its related equation?

Learn Solving Quadratic Equations by Graphing
 A quadratic equation is an equation that includes a quadratic expression.
Key Concept - Standard Form of a Quadratic Equation
 The standard form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$ and a , b , and c are integers.
 One method for finding the roots of a quadratic equation is to find the zeros of a related quadratic function. You can identify the solutions or roots of an equation by finding the x -intercepts of the graph of a related function. Often, exact roots cannot be found by graphing. You can estimate the solutions by finding the integers between which the zeros are located on the graph of the related function.

Example 1 One Real Solution
 Solve $10 - x^2 = 4x + 14$ by graphing.
 Rearrange terms so that one side of the equation is 0.
 $0 = x^2 + 4x + 4$
 Find the axis of symmetry:
 $x = -\frac{b}{2a} = -\frac{4}{2(1)} = -2$
 Make a table of values, plot the points, and connect them with a curve.

x	y
-4	4
-3	1
-2	0
-1	1
0	4

The zero of the function is -2 . Therefore, the solution of the equation is -2 or $\{x | x = -2\}$.
Check:
 Solve $x^2 + 4x + 4 = 14$ by graphing: $x = -2, 12$

Today's Goals
 • Solve quadratic equations by graphing.

Today's Vocabulary
 quadratic equation
 standard form of a quadratic equation

Think About It!
 How can you determine the number of solutions of a quadratic equation?

Sample answer: Graph a related function and find the number of x -intercepts.

Study Tip
Solutions of Quadratic Equations A quadratic equation can have one real solution, two real solutions, or no real solutions.

Think About It!
 How can you find the solution of the equation from the table?

Sample answer: I can see from the table that -2 is the x -intercept of the related function, so -2 is a solution of the equation.

Lesson 1-2 • Solving Quadratic Equations by Graphing **13**

Interactive Presentation

Solving Quadratic Equations by Graphing

A quadratic equation is an equation that includes a quadratic expression.

KEY CONCEPT: STANDARD FORM OF A QUADRATIC EQUATION

Words	The standard form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$ and a , b , and c are integers.
Example	$x^2 + 2x - 3 = 0$

Learn

TYPE



Students will answer a question about determining the number of solutions of a quadratic equation.

**Example 2: Two Real Solutions**

Use a quadratic equation to find two real numbers with a sum of 24 and a product of 143.

Let x represent one of the numbers. Then $24 - x$ will represent the other number. So $x(24 - x) = 143$.

What do you need to find?
 x and $24 - x$

Step 1 Solve the equation for 0.

$$x(24 - x) = 143$$

Original equation

$$24x - x^2 = 143$$

Distributive Property

$$0 = x^2 - 24x + 143 \quad \text{Subtract } 24x - x^2 \text{ from each side.}$$

Step 2 Find the axis of symmetry.

$$x = -\frac{b}{2a}$$

Equation of the axis of symmetry

$$x = -\frac{-24}{2(1)}$$

$$x = 12$$

Simplify.

Step 3 Make a table of values and graph the function.

x	y
14	3
13	0
12	-1
11	0
10	3



Steps 4 and 5 Find the zero(s) and determine the solution.

The zeros of the function are 11 and 13.

$x = 11$ or $x = 13$, so $24 - x = 13$ or $24 - x = 11$. Thus, the two numbers with a sum of 24 and a product of 143 are 11 and 13.

Check

Use a quadratic equation to find two real numbers with a sum of -43 and a product of 306 . $\frac{-2}{-9}$ and $\frac{7}{-34}$

Go Online: You can complete an Extra Example online.

Example 2 Two Real Solutions**Teaching the Mathematical Practices**

1 Analyze Givens and Constraints In Example 2, guide students through the use of the four-step plan to identify the meaning of the problem and look for entry points to its solution.

Questions for Mathematical Discourse

A1 How can you tell from the graph that there will be two solutions?
The graph crosses the x -axis in two places.

O1 When the first number is defined as x , why is the other number $24 - x$? $x + (24 - x) = 24$

BL Given one root and the x -coordinate of the vertex, how can you find the other root? **Sample answer:** Because the line of symmetry passed through the vertex, I can determine how far the root is from the x -coordinate of the vertex and then go that far from the vertex in the other direction.

Common Error

Students may identify the y -intercept of the quadratic function as the a solution. Be sure to emphasize that it is only the x -intercepts that relate to the solutions. Emphasize the relationship of $f(x) = 0$ with the graph to reinforce the reason why these intercepts are the solutions.

Interactive Presentation

Two Real Solutions

NUMBER THEORY Use a quadratic equation to find two real numbers with a sum of 24 and a product of 143.

Tip: or apply steps of the four-step plan to solve the problem.

Example 2

DRAG & DROP

Students drag to sort the steps needed to solve the problem.

TYPE

Students explain why other possible pairs do not meet the criteria.

WATCH

Students watch a video to learn how to solve quadratic equations by graphing them on a calculator.

**Example 3** Estimate Roots**MP Teaching the Mathematical Practices**

2 Different Properties Mathematically proficient students look for different ways to solve problems. Use the Talk About It! feature to encourage students to consider how the table can be used to estimate the solutions.

Questions for Mathematical Discourse

- AL** Would this method work if there was only one real solution? Explain. **Yes; sample answer:** You would just look for the two consecutive integers the one solution is between.
- OL** How can you tell from the table that a root is between $x = 4$ and $x = 5$? **When $x = 4$, the value of y is positive and when $x = 5$, the value of y is negative, which means the graph crosses the x -axis between these two values.**
- BL** When making a table of values and graphing, why is it important to calculate and plot several points? **Sample answer:** You need to plot enough points over a large enough range to capture the zeros.

Example 4 Solve by Using a T table**MP Teaching the Mathematical Practices**

5 Use Estimation In Example 4, students estimate the solutions of the equation by using a table. Guide students to try smaller intervals to get more accurate estimates.

Questions for Mathematical Discourse

- AL** Why is $x = 0.209$ selected as the approximation of the zero rather than $x = 0.208$? **The y -value when $x = 0.209$ is closer to 0 than the y -value when $x = 0.208$.**
- OL** Between which consecutive integers is the second zero of the function located? **4 and 5**
- BL** How does each change in ΔTbl make the approximation more accurate? **With each change in ΔTbl , the approximation is accurate to another decimal place.**

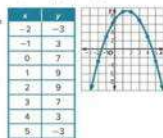
Example 3 Estimate Roots

Solve $-x^2 + 3x + 7 = 0$ by graphing. If the exact roots cannot be found, state the consecutive integers between which the roots are located.

Find the axis of symmetry: $x = -\frac{b}{2a} = -\frac{3}{2(-1)} = \frac{3}{2}$

Make a table of values, plot the points, and connect them with a curve.

The x -intercepts of the graph indicate that one solution is between -2 and -1 , and the other solution is between 4 and 5 .

**Check**

Use a graph to find all of the solutions of $x^2 + 5x - 1 = 0$. Select all of the pairs of consecutive integers between which the roots are located.
between -10 and -9 ; between 0 and 1

Example 4 Solve by Using a Table

Use a table to solve $-x^2 + 5x - 1 = 0$.

Steps 1 and 2 Enter the function and view the table.

Enter $-x^2 + 5x - 1$ in the Y_1 list. Use the **TABLE** window to find where the sign of Y_1 changes. The sign changes between $x = 0$ and $x = 1$.

Steps 3 and 4 Edit the table settings and find a more accurate location.

Use **TBLSET** to change ΔTbl to 0.1 and look again for the sign change. Repeat this for 0.01 and 0.001 to get a more accurate location of one zero.

One zero is located at approximately $x = 0.209$.

Go Online: You can complete an Extra Example online.

(continued on the next page)

Lesson 1-2 • Solving Quadratic Equations by Graphing 15

Interactive Presentation

Solve by Using a Table

Use a table to solve $-x^2 + 5x - 1 = 0$.

Select a calculator.

Example 4

TYPE

Students answer a question to determine whether they understand how to estimate roots from a table.

TAP

Students select a calculator to solve a quadratic equation by using a table.



Example 5 Solve by Using a Calculator



Teaching the Mathematical Practices

4 Interpret Mathematical Results In Example 5, point out that to solve the problem, students should interpret their mathematical results in the context of the problem.

Questions for Mathematical Discourse

- AL** What does $h(0)$ represent? the height of the ball when the punter's foot meets the ball
- OL** What does it mean when $h(t) = 0$? The ball has a height of 0 feet and is on the ground.
- BL** Describe two ways to check your solution. Sample answer: substitute 3.15 into the original function to check that $h(3.15) \approx 0$ or use the table function on my calculator with an interval of 0.01 or smaller to verify that $y = 0$ when $x = 3.15$ in the table.

Common Error

Remind students that when solving real-world problems, there may be solutions that do not make sense within the context.

DIFFERENTIATE

Language Development Activity

Tell students the meaning of *quadratic* comes from French and Latin words for *square*. A quadratic equation must have an x^2 -term. Help them recall that a square with side length x has an area of x^2 .

Think About It!
How can you check your solutions?

Sample answer: I can substitute the values for x in $-x^2 + 5x - 1 = 0$, and see if it is a true statement.

Watch Out!
Graphing Calculator: If you cannot see the graph of the function on your graphing calculator, you may need to adjust the viewing window. Having the proper viewing window will also make it easier to see the zeros.

Think About It!
Why did you only find the positive zero?

Sample answer: The negative zero is not a solution in the context of the situation because you cannot have a negative number of seconds.

Go Online: to see how to use a TI-Nspire graphing calculator with this example.

Steps 5 and 6: Find the other zero determine the solutions of the equation. Repeat the process to find the second zero of the function.

The zeros of the function are at approximately 0.209 and 4.791, so the solutions to the equation are approximately 0.209 and 4.791.

Check:
Use a table to find all of the solutions of $-x^2 - 3x + 8 = 0$.

-4.702	0
1.702	0

Example 5 Solve by Using a Calculator

FOOTBALL: A kicker punts a football. The height of the ball after t seconds is given by $h(t) = -16t^2 + 50t + h_0$, where h_0 is the initial height. If the ball is 15 feet above the ground when the punter's foot meets the ball, how long will it take the ball to hit the ground?

We know that h_0 is the initial height, so $h_0 = 15$. We need to find t when $h(t) = 0$. Use a graphing calculator to graph the related function $h(t) = -16t^2 + 50t + 15$.

Step 1: Enter the function in the $Y=$ list, and press graph.

Step 2: Use the zero feature in the **CALC** menu to find the positive zero.

Step 3: Find the left bound by placing the cursor to the left of the intercept.

Step 4: Find the right bound.

Step 5: Find and interpret the solution.

The zero is approximately 3.15. Thus, the ball hit the ground approximately 3.15 seconds after it was punted.

Check:

SOCCER: A goalie punts a soccer ball. If the ball is 1 foot above the ground when her foot meets the ball, find how long it will take, to the nearest hundredth of a second, for the ball to hit the ground. Use the formula $h(t) = -16t^2 + 35t + h_0$, where t is the time in seconds and h_0 is the initial height.

0	1
2.32	0

Go Online: You can complete an Extra Example online.

56 Module 1 • Quadratic Functions

Interactive Presentation

Solve by Using a Calculator

FOOTBALL: A kicker punts a football. If the ball is 15 feet above the ground when the punter's foot meets the ball, how long will it take the ball to hit the ground? Use the formula $h(t) = -16t^2 + 50t + h_0$, where t is the time in seconds and h_0 is the initial height.

Example 5

TYPE



Students answer a question about the viability of the zero of a quadratic equation in a real-world situation.

CHECK



Students complete the Check online to determine whether they are ready to move on.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–34
2	exercises that use a variety of skills from this lesson	35–49
2	exercises that extend concepts learned in this lesson to new contexts	50–58
3	exercises that emphasize higher-order and critical-thinking skills	59–63

ASSESS AND DIFFERENTIATE

 Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, THEN assign:

- Practice Exercises 1–57 odd, 59–63
- Extension: Solving Absolute Value Equations by Graphing
-  **ALEKS** Quadratic Equations

IF students score 66%–89% on the Checks, THEN assign:

- Practice Exercises 1–63 odd
- Remediation, Review Resources: Solving Quadratic Equations by Graphing
- Personal Tutors
- Extra Examples 1–3
-  **ALEKS** Quadratic Equations

IF students score 65% or less on the Checks, THEN assign:

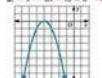
- Practice Exercises 1–33 odd
- Remediation, Review Resources: Solving Quadratic Equations by Graphing
- Quick Review Math Handbook: Solving Quadratic Equations by Graphing
-  **ALEKS** Quadratic Equations

Practice

Example 1

Use the related graph of each equation to determine its solutions.

1. $x^2 + 2x + 3 = 0$ no real solution 2. $x^2 - 3x + 10 = 0$ -2.5



Solve each equation by graphing. 4–12. See margin for graphs.

4. $x^2 - 10x + 21 = 0$ 3, 7

5. $4x^2 + 4x + 1 = 0$ $-\frac{1}{2}$

6. $x^2 + x - 6 = 0$ $-3, -2$

7. $x^2 + 2x - 3 = 0$ $-3, 1$

8. $-x^2 - 6x + 9 = 0$ -3

9. $x^2 - 6x + 5 = 0$ $1, 5$

10. $x^2 + 2x + 3 = 0$ no real solution

11. $x^2 - 3x + 10 = 0$ -2.5

12. $-x^2 - 8x + 16 = 0$ -4

Example 2

13. Use a quadratic equation to find two real numbers with a sum of 2 and a product of -24 . 8 and -4

14. Use a quadratic equation to find two real numbers with a sum of -15 and a product of -54 . 3 and -18

Example 3 15–23. See Mod. 1 Answer Appendix for graphs.

Solve each equation by graphing. If the exact roots cannot be found, state the consecutive integers between which the roots are located.

15. $x^2 - 4x + 2 = 0$

16. $x^2 + 6x + 6 = 0$

17. $x^2 + 4x + 2 = 0$

18. $-x^2 - 4x = 0$

19. $-x^2 + 36 = 0$

20. $x^2 - 6x + 4 = 0$

21. $x^2 + 5x + 3 = 0$

22. $x^2 - 7 = 0$

23. $x^2 - 4x + 6 = 0$

24. $x^2 - 4x + 2 = 0$

25. $x^2 + 6x + 6 = 0$

26. $x^2 + 4x + 2 = 0$

27. $x^2 - 4x = 0$

28. $-x^2 + 36 = 0$

29. $x^2 - 6x + 4 = 0$

30. $-x^2 - 4x = 0$

31. $-x^2 + 2x + 4 = 0$

32. $x^2 - 12x + 17 = 0$

33. $x^2 + 5x + 3 = 0$

34. $x^2 - 7 = 0$

35. $x^2 - 4x + 6 = 0$

Lesson 1.2 • Solving Quadratic Equations by Graphing 17

Example 4
Use the tables to determine the location of the zeros of each quadratic function. State the consecutive integers between which the roots are located.

x	-7	-6	-5	-4	-3	-2	-1	0
$f(x)$	-8	-1	4	4	-1	-8	-22	-48

x	-2	-1	0	1	2	3	4	5
$f(x)$	32	14	2	-3	-3	2	14	32

x	-6	-2	0	3	6	9	12	15
$f(x)$	-9	-1	2	5	3	-1	-6	-14

Use a table to solve each equation. If the exact roots cannot be found, approximate the roots to the nearest hundredth.

27. $-3x^2 + 3 = 0$ $-1, 1$

28. $x^2 - 3x + 2 = 0$ $1, 2$

29. $x^2 + x + \frac{1}{2} = 0$ $-1.618, 0.618$

30. $x^2 - 2x - 2 = 0$ $-0.73, 2.73$

31. $-x^2 + 2x + 4 = 0$ $-1.24, 3.24$

32. $x^2 - 12x + 17 = 0$ $2.29, 3.71$

Example 5

Investigate: Use the formula $A(t) = -16t^2 + v_0t + h_0$, where $A(t)$ is the height of an object in feet, v_0 is the object's initial velocity in feet per second, t is the time in seconds, and h_0 is the initial height in feet from which the object is launched.

Round to the nearest tenth, if necessary.

33. Melah throws a baseball with an initial upward velocity of 32 feet per second. The baseball is released from Melah's hand at a height of 4 feet. Use a graphing calculator to determine how long it will take the ball to hit the ground. 2.3 seconds

34. A punter kicks a football with an initial upward velocity of 40 feet per second. The ball is 2 feet above the ground when his foot meets the ball. Use a graphing calculator to determine how long will it take the ball to hit the ground. 3.8 seconds

Minced Exercises 35–43. See Mod. 1 Answer Appendix for graphs.

Solve each equation by graphing. If the exact roots cannot be found, state the consecutive integers between which the roots are located.

35. $4x^2 - 15 = -4x$

36. $-35 = -3x - 2x^2$

37. $-3x^2 + 10x + 9 = 1$

38. $-4x^2 = 12x + 8$

39. $-0.5x^2 + 18 = -6x + 33$

40. $0.5x^2 + 0.75 = 0.25x$

41. $3x^2 + 8x = 0$

42. $2x^2 + x = 11$

43. $-0.1x^2 + 0.5x + 10 = 0$

44. $-x^2 - 4x = 0$

45. $-x^2 + 36 = 0$

46. $x^2 - 6x + 4 = 0$

47. $x^2 + 5x + 3 = 0$

48. $x^2 - 7 = 0$

49. $x^2 - 4x + 6 = 0$

50. $-x^2 - 4x = 0$

51. $-x^2 + 2x + 4 = 0$

52. $x^2 - 12x + 17 = 0$

53. $x^2 + 5x + 3 = 0$

54. $x^2 - 7 = 0$

55. $x^2 - 4x + 6 = 0$

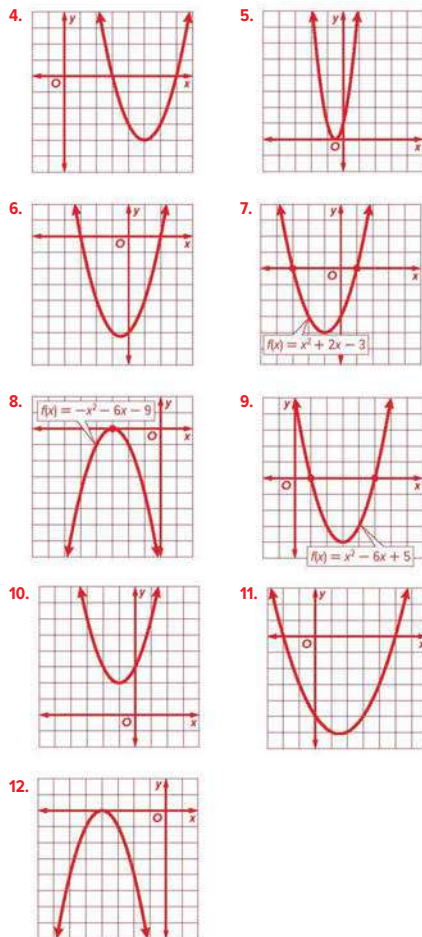


1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Answers



Use a graph or table to solve each equation. If exact roots cannot be found, state the consecutive integers between which the roots are located.

44. $x^2 + 4x = 0$
-4, 0

45. $-2x^2 - 4x - 5 = 0$
no real solution

46. $0.5x^2 - 2x + 2 = 0$
2

47. $-0.25x^2 - x - 1 = 0$
-2

48. $x^2 - 6x + 11 = 0$
no real solution

49. $-0.5x^2 + x + 6 = 0$
between -3 and -2, between 4 and 5

IRREGULAR Write a quadratic equation to find two real numbers that satisfy each situation, or show that no such numbers exist.

50. Their sum is 4, and their product is -17. 13 and -9

51. Their sum is 12, and their product is -85. -5 and 17

52. Their sum is -13, and their product is 42. -6 and -7

53. Their sum is -8, and their product is -209. 11 and -19

54. BRIDGES In 1895, a brick arch railway bridge was built on North Avenue in Baltimore, Maryland. The arch is described by the equation $y = 9 - \frac{1}{10}x^2$, where h is the height in yards and x is the distance in yards from the center of the bridge. Graph this equation and determine, to the nearest yard, where the bridge touches the ground. See Mod. 1 Answer Appendix for graph. The bridge touches the ground 27 yards from the center of the bridge on either side at (-27, 0) and (27, 0).

55. RADIO TELESCOPES The cross-section of a large radio telescope is a parabola. The equation that describes the cross-section is $y = \frac{1}{10}x^2 - \frac{3}{5}x + \frac{1}{2}$, where y is the depth of the dish in meters at a point x meters from the center of the dish. If the dish does not extend above the ground level, what approximation is the diameter of the dish? Solve by graphing. (44 m; See Mod. 1 Answer Appendix for graph.)



56. VOLCANOES A volcanic eruption blasts a boulder upward with an initial velocity of 240 feet per second. The height $h(t)$ of the boulder in feet, t seconds after the eruption can be modeled by the function $h(t) = -16t^2 + 240t + v_0$. How long will it take the boulder to hit the ground if it lands at the same elevation from which it was ejected? 15 seconds

Lesson 1.3 Solving Quadratic Equations by Graphing 19

57. TRAJECTORIES Daniela hits a golf ball from ground level. The function $h = 80t - 16t^2$ represents the height of the ball in feet, where t is the time in seconds after Daniela hits it. Use the graph of the function to determine how long it took for the ball to reach the ground. 5 seconds



58. HIKING Antonio is hiking and reaches a steep part of the trail that runs along the edge of a cliff in order to descend more safely; she drops her heavy backpack over the edge of the cliff so that it will land on a lower part of the trail, 38.75 feet below. The height $h(t)$ of an object t seconds after it is dropped straight down can also be modeled by the function $h(t) = -16t^2 + v_0t + h_0$, where v_0 is the initial velocity of the object, and h_0 is the initial height.

a. Write a quadratic function that can be used to determine the amount of time t that it takes for the backpack to land on the trail below the cliff after Antonio drops it. $h(t) = -16t^2 + 38.75$

b. Use a graphing calculator to determine how long until the backpack hits the ground. Round to the nearest tenth. 1.6 seconds

Higher-Order Thinking Skills

59. FIND THE ERROR Hakeem and Nami were asked to find the location of the roots of the quadratic function represented by the table. Is either of them correct? Explain.

x	-4	-2	0	2	4	6	8	10
f(x)	5.2	26	8	-2	-4	2	10	28

Hakeem

The roots are between x and $x + 4$ because $f(x)$ stops decreasing and begins to increase between $x = -4$ and $x = 4$.

Nami

The roots are between -3 and 0 because x changes sign at that location.

No; sample answer: No, Hakeem is right about the location of one of the roots, but his reason is not accurate. The roots are located where $f(x)$ changes signs.

60. INTERPRET Find the value of a positive integer k such that $f(x) = x^2 - 2kx + 55$ has roots at $k + 3$ and $k - 3$. $k = 8$

61. ANALYZE If a quadratic function has a minimum at $(-6, -14)$ and a root at $x = -15$, what is the other root? Explain your reasoning. **Sample answer:** The intercepts are equidistant from the axis of symmetry.

62. CREATE Write a quadratic function with a maximum at $(3, 125)$ and roots at -2 and 8 . $f(x) = -5x^2 + 33x + 80$

63. WRITE Explain how to solve a quadratic equation by graphing its related quadratic function. **Sample answer:** Graph the function using the axis of symmetry. Determine where the graph intersects the x -axis. The x -coordinates of these points are solutions of the quadratic equation.

Complex Numbers

LESSON GOAL

Students perform operations with pure imaginary and complex numbers.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Develop:**

Pure Imaginary Numbers

- Square Roots of Negative Numbers
- Products of Pure Imaginary Numbers
- Equation with Pure Imaginary Solutions

 **Explore:** Factoring the Sum of Two Squares

 **Develop:**

Complex Numbers

- Equate Complex Numbers
- Add or Subtract Complex Numbers
- Multiply Complex Numbers
- Divide Complex Numbers

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources

Remediation: Properties of Real Numbers

Extension: Conjugates and Absolute Value

AL	UB	ELL
●	●	●
●	●	●

Language Development Handbook

Assign page 3 of the *Language Development Handbook* to help your students build mathematical language related to performing operations with imaginary and complex numbers.

 You can use the tips and suggestions on page T3 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Number and Quantity

Standards for Mathematical Content:

N.CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

N.CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

7 Look for and make use of structure.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous

Students solved linear equations and inequalities in one variable.

A.CED.1, A.CED.2

Now

Students perform operations with pure imaginary and complex numbers.

N.CN.1, N.CN.2

Next

Students will solve quadratic equations by factoring.

N.CN.7, N.CN.8, F.IF.8a

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

 **Conceptual Bridge** In this lesson, students develop an understanding of imaginary and complex numbers and build fluency by performing operations with complex numbers and solving equations with complex solutions. They apply their understanding by solving real-world problems.

Mathematical Background

A complex number is any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit. If $b = 0$, the complex number is a real number. If $b \neq 0$, the complex number is imaginary. If $a = 0$, the complex number is a pure imaginary number. Real numbers and imaginary numbers are both subsets of the set of complex numbers.



Interactive Presentation

Warm Up

Simplify each expression.

- $(2 - 3i) + (5 + i)$
- $(7 - 3i) - (6 + 4i)$
- $(-1 + 5i) + (-2 - 8i)$
- $(2 - 3i)(5 + i)$
- $i(-2 - 8i)^2$

Show Answers

Warm Up

Launch the Lesson

Electrical engineers use algebra to describe voltage, current, and impedance (or resistance to current). In alternating current systems like those you use every day, different parts of the circuit are active at different times. However, the other parts of the circuit still carry current. Voltages, currents, and impedance even when they are not active. This cannot be measured by a real number, but can be measured with complex numbers, which contain both a real and imaginary part.

Launch the Lesson

Vocabulary

Expand All Collapse All

- imaginary unit**
The principal square root of -1 .
- pure imaginary number**
A number of the form bi , where a is a real number and $i = \sqrt{-1}$.
- complex number**
Any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit.
- complex conjugates**
Two complex numbers of the form $a + bi$ and $a - bi$.
- rationalizing the denominator**
A method used to eliminate radicals from the denominator of a fraction or fractions from a radicand.

What are imaginary numbers?
How are real numbers, imaginary numbers, and complex numbers related?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- simplifying expressions

Answers:

- $7 - 2i$
- $1 - 7i$
- $-3 - 3i$
- $10 - 13i - 3i^2$
- $4 + 32i + 64i^2$

Launch the Lesson



Teaching the Mathematical Practices

- 4 Apply Mathematics** In the Launch the Lesson, students will learn how complex numbers are used by engineers to measure circuits.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.



Explore Factoring the Sum of Two Squares

Objective

Students use a sketch to explore imaginary numbers by factoring the sum of two squares.



Teaching the Mathematical Practices

7 Use Structure Guide students to use the structure of prime polynomials to rewrite them in a form that can be factored using the imaginary number i .

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will explore how defining the imaginary number $i = \sqrt{-1}$ allows you to factor the sum of two squares. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore



Explore

EXPAND



Students tap to see the guiding exercises about factoring the sum of two squares.

TYPE



Students answer questions about the sum of two squares.



Interactive Presentation

INQUIRY Can you factor a prime polynomial?

Submit

Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Factoring the Sum of Two Squares
(continued)

Questions

Have students complete the Explore activity.

Ask:

- Why is it helpful to factor a polynomial? **Sample answer:** Factoring a polynomial can help find the solution to an equation or the zeros of a function.
- How does factoring a prime polynomial like the sum of two squares relate to the phrase "no real solution"? **Sample answer:** In order to factor a prime polynomial, you have to use imaginary numbers. So, there are no "real" solutions because they are imaginary.

Inquiry

Can you factor a polynomial of the form $a^2 + b$? **Sample answer:** Yes, if you use an imaginary number i defined such that $i = \sqrt{-1}$.

Go Online to find additional teaching notes and sample answers to the guiding exercises.



Learn Pure Imaginary Numbers

Objective

Students perform operations with pure imaginary numbers.

MP Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in the powers of i . Ask them to look for a general rule for determining the values of powers of i .

About the Key Concept

Draw a distinction between the square root of a negative and a negative square root. Explore with students what happens when you square examples of both types, like $-\sqrt{5}$ and $\sqrt{-5}$, to help reinforce this difference.

DIFFERENTIATE

Language Development Activity **AL** **BLELL**

IF students need help remembering the mathematical characteristics of i , **THEN** have students write poems about the imaginary number i and the repeating values of its powers, perhaps including wordplay with the terms *real* and *imaginary*. The content of the poems should be helpful for remembering the mathematical characteristics of i .

Example 1 Square Roots of Negative Numbers

MP Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of the radicand to rewrite it so that it can be simplified using imaginary numbers.

Questions for Mathematical Discourse

AL Why is there not a negative sign under the radical in the answer? The $\sqrt{-1}$ has been replaced with i .

OL How can you tell when there will be an i in your answer just by looking at a square root? when the radicand is negative

BL Does $\sqrt{-1} \cdot \sqrt{-1} = \sqrt{((-1)(-1))} = \sqrt{(-1)^2}$? Explain. **No**; sample answer: $\sqrt{-1} \cdot \sqrt{-1} = (\sqrt{-1})^2 = -1$, and $\sqrt{((-1)(-1))} = \sqrt{(-1)^2} = \sqrt{1} = 1$.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Complex Numbers

Learn Pure Imaginary Numbers

In your math studies so far, you have worked with real numbers. However, some equations such as $x^2 + x + 1 = 0$ do not have real solutions. This led mathematicians to define imaginary numbers. The imaginary unit i is the principal square root of -1 . Thus, $i = \sqrt{-1}$ and $i^2 = -1$.

Numbers of the form $6i$, $-2i$, and $i\sqrt{3}$ are called pure imaginary numbers. A pure imaginary number is a number of the form bi , where b is a real number and $i = \sqrt{-1}$. For any positive real number $\sqrt{-2} = \sqrt{2}i$, $\sqrt{-3} = \sqrt{3}i$, or b .

The Commutative and Associative Properties of Multiplication hold true for pure imaginary numbers. The first few powers of i are shown.

$$i^0 = 1 \quad i^1 = i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1$$

Example 1 Square Roots of Negative Numbers

Simplify $\sqrt{-256}$.

$$\begin{aligned} \sqrt{-256} &= \sqrt{-1 \cdot 256} && \text{Factor the radicand.} \\ &= \sqrt{-1} \cdot \sqrt{256} && \text{Factor out the imaginary unit.} \\ &= i \cdot 16 && \text{Simplify.} \end{aligned}$$

Check:

Simplify $\sqrt{-256}$.

- A. $i\sqrt{256}$
 B. $3i\sqrt{5}$
 C. $5i\sqrt{3}$
 D. $-3i\sqrt{5}$

Go Online: You can complete an Extra Example online.

Today's Goals

- Perform operations with pure imaginary numbers.
- Perform operations with complex numbers.

Today's Vocabulary
imaginary unit i
pure imaginary number
complex number
complex conjugates
rationalizing the denominator

Go Online
You may want to complete the Concept Check to check your understanding.

Study Tip

Square Factors: When factoring an expression under a radical, look for perfect square factors.

Interactive Presentation



Example 1



Talk About It

How can an expression with two imaginary expressions, $\sqrt{-10}$ and $\sqrt{-15}$, have a product that is real?

Sample answer: Because $\sqrt{-10}$ and $\sqrt{-15}$ are both imaginary, you can factor out the imaginary unit i from each. The product of $i \cdot i = i^2 = (\sqrt{-1})^2 = -1$. So, the product of two imaginary numbers is a negative real number.

Example 2 Products of Pure Imaginary Numbers

Simplify $\sqrt{-10} \cdot \sqrt{-15}$.

$$\begin{aligned}\sqrt{-10} \cdot \sqrt{-15} &= i\sqrt{10} \cdot i\sqrt{15} & i &= \sqrt{-1} \\ &= i^2 \cdot \sqrt{150} & \text{Multiply.} \\ &= -1 \cdot \sqrt{25} \cdot \sqrt{6} & \text{Simplify.} \\ &= -5\sqrt{6} & \text{Multiply.}\end{aligned}$$

Check

$$\begin{aligned}\text{Simplify } \sqrt{-10} \cdot \sqrt{-15} \\ = -20\end{aligned}$$

Example 3 Equation with Pure Imaginary Solutions

Solve $x^2 + 81 = 0$.

$$\begin{aligned}x^2 + 81 &= 0 & \text{Original equation} \\ x^2 &= -81 & \text{Subtract 81 from each side.} \\ x &= \pm\sqrt{-81} & \text{Take the square root of each side.} \\ x &= \pm 9i & \text{Simplify.}\end{aligned}$$

Check

$$\begin{aligned}\text{Solve } 3x^2 + 27 &= 0 \\ x &= 3i, x = -3i\end{aligned}$$

Explore Factoring the Sum of Two Squares

Online Activity Use guiding exercises to complete the Explore.

ENQUIRY Can you factor a polynomial of the form $a^2 + b^2$?

Learn Complex Numbers

Key Concept Complex Numbers

A complex number is any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit. a is called the real part, and b is called the imaginary part.

Go Online You can complete an Explore Example online.

22 Module 1 • Quadratic Functions

Interactive Presentation

Products of Pure Imaginary Numbers

Simplify $\sqrt{-10} \cdot \sqrt{-15}$.

$$\sqrt{-10} \cdot \sqrt{-15} = i\sqrt{10} \cdot i\sqrt{15} \quad i = \sqrt{-1}$$

Example 2

SWIPE



Students move through the steps to see how to multiply two imaginary numbers.

TYPE



Students answer a question about the product of two imaginary numbers.

CHECK



Students complete the Check online to determine whether they are ready to move on.

Example 2 Products of Pure Imaginary Numbers

MP Teaching the Mathematical Practices

3 Construct Arguments In the Talk About It! feature, students will use definitions and previously established results to construct an argument.

Questions for Mathematical Discourse

AL What is the prime factorization of 150? $5 \cdot 5 \cdot 2 \cdot 3$

OL List the possible values for all powers of i when simplified.
 $i, -1, -i, 1$

BL Determine a rule you can use to simplify all powers of i ? **Sample answer:** After you divide the exponent by 4, if the remainder is 0 it is equal to 1, if the remainder is 1 it is equal to i , if the remainder is 2 it is equal to -1 , and if the remainder is 3 it is equal to $-i$.

Example 3 Equation with Pure Imaginary Solutions

MP Teaching the Mathematical Practices

2 Attend to Quantities Remind students a quadratic equation has two solutions, which is indicated by the $\pm$ symbol.

Questions for Mathematical Discourse

AL How can you check your solution? **Sample answer:** Substitute $-9i$ and then $9i$ into the original equation and check that the equation is true.

OL How are the roots of $x^2 - 81$ and $x^2 + 81$ different? $x^2 - 81$ has two real roots and $x^2 + 81$ has two imaginary roots.

BL Would the graph of $f(x) = x^2 + 81$ ever cross the x -axis? How do you know? **No; sample answer:** Because the equation $x^2 + 81 = 0$ does not have any real solutions, the graph of the related function does not have any x -intercepts.

Common Error

Make sure students understand that when they take the square root of each side of an equation, they must use the $\pm$ symbol in front of the radical sign.



Learn Complex Numbers

Objective

Students perform operations with complex numbers.

MP Teaching the Mathematical Practices

3 Construct Arguments In the Think About It! feature, students will use the Venn diagram to construct arguments about the complex number system.

About the Key Concept

Be sure that students recognize that complex conjugates are not inverses of each other, but must have the same real components. Confirm that the product of complex conjugates is always a real number by working through examples. Keep in mind that these problems parallel the difference of squares that will be introduced in Lesson 1-4.

Essential Question Follow-Up

Students have explored complex numbers.

Ask:

How do complex numbers relate to solving quadratic equations?

Sample answer: When the graph of the related quadratic function does not cross the x -axis, the solutions to a quadratic equation are complex numbers.

Example 4 Equate Complex Numbers

MP Teaching the Mathematical Practices

7 Interpret Complicated Expressions Guide students to see the equation as being composed of real and imaginary parts.

Questions for Mathematical Discourse

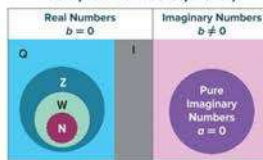
- A1** Why is $5x - 7$ set equal to 13 ? **They are real parts of each side of the equation.**
- OL** Why do you relate the real and imaginary parts separately? **Sample answer:** Because two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal, you need to find the value that makes the real parts of each side equal and the value that makes the imaginary parts of each side equal.
- BL** Could you have an x -term in both the real and imaginary parts of the equation? **Yes.** How would this affect the solution? **The x -value of the solution would have to satisfy both parts to be valid.**

Common Error

Emphasize that two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

The Venn diagram shows the set of complex numbers. Notice that all of the real numbers are part of the set of complex numbers.

Complex Numbers ($a + bi$)



Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. The Commutative and Associative Properties of Multiplication and Addition and the Distributive Property hold true for complex numbers. To add or subtract complex numbers, combine like terms. That is, combine the real parts, and combine the imaginary parts.

Two complex numbers of the form $a + bi$ and $a - bi$ are called **complex conjugates**. The product of complex conjugates is always a real number.

A radical expression is in simplest form if no radicands contain fractions and no radicals appear in the denominator of a fraction. Similarly, a complex number is in simplest form if no imaginary numbers appear in the denominator of a fraction. You can use complex conjugates to simplify a fraction with a complex number in the denominator. This process is called **rationalizing the denominator**.

Example 4 Equate Complex Numbers

Find the values of x and y that make $5x - 7 + (y + 4)i = 13 + 11i$ true.

Use equations relating the real and imaginary parts to solve for x and y .

$5x - 7 = 13$	Real parts
$5x = 20$	Add 7 to each side.
$x = 4$	Divide each side by 5.
$y + 4 = 11$	Imaginary parts
$y = 7$	Subtract 4 from each side.

Check

Find the values of x and y that make $5x + 13 + (2y - 7)i = -2 + i$ true.

$$x = -3; y = 4$$

Go Online: You can complete an Extra Example online.

Lesson 1-3 • Complex Numbers 23

Study Tip:

These abbreviations represent the sets of real numbers.

Letter	Set
Q	rational numbers
Z	integers
N	natural numbers

Think About It! Compare and contrast the subsets of the complex number system using the Venn diagram.

Sample answer: For $a + bi$, if $b = 0$ then the number is real and can be irrational, rational, an integer, whole, and/or a natural number. If $b \neq 0$ then the number is imaginary. It is pure imaginary if $a = 0$.

Interactive Presentation

Complex Numbers

KEY CONCEPT: COMPLEX NUMBERS

Words

A complex number is any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit. a is called the real part, and b is called the imaginary part.

Learn

TYPE



Students answer a question about complex numbers.

**Go Online**

You can watch a video to see how to add or subtract complex numbers.

Study Tip:**Imaginary Unit**

Complex numbers are often used with electricity in these problems. j is usually used in place of i .

Example 5 Add or Subtract Complex NumbersSimplify $(8 + 3j) - (4 - 10j)$.

$$(8 + 3j) - (4 - 10j) = (8 - 4) + (3 - (-10j))$$

Commutative and Associative Properties:
Simplify.

$$= 4 + 13j$$

Check

$$\text{Simplify } (-5 + 5j) + (-3 + 8j)$$

$$\begin{array}{r} -5 + 5j \\ -3 + 8j \\ \hline \end{array}$$

Example 6 Multiply Complex Numbers

ELECTRICITY The voltage V of an AC circuit can be found using the formula $V = CI$, where C is current and I is impedance. If $C = 3 + 2j$ amps and $I = 7 - 5j$ ohms, determine the voltage.

$$V = CI$$

$$= (3 + 2j)(7 - 5j)$$

$$C = 3 + 2j \text{ and } I = 7 - 5j$$

$$= 3(7) + 3(-5j) + 2(7) + 2(-5j)$$

FOIL Method

$$= 21 - 15j + 14 - 10j^2$$

Multiply.

$$= 21 - j - 10(-1)$$

Add.

$$= 31 - j$$

The voltage is $31 - j$ volts.**Example 7** Divide Complex NumbersSimplify $\frac{3}{3 + 2j}$.

Rationalize the denominator to simplify the fraction.

$$\frac{3}{3 + 2j} = \frac{3}{3 + 2j} \cdot \frac{3 - 2j}{3 - 2j}$$

 $3 + 2j$ and $3 - 2j$ are complex conjugates.

$$= \frac{3(3 - 2j)}{(3 + 2j)(3 - 2j)}$$

Multiply the numerator and denominator.

$$= \frac{9 - 6j}{9 - 4j^2}$$

$$j^2 = -1$$

$$= \frac{9 - 6j}{9 - 4(-1)}$$

$$= \frac{9 - 6j}{13}$$

Simplify.

$$= \frac{9}{13} - \frac{6j}{13}$$

 $(a + bi)$ form.
Check

$$\text{Simplify } -2j \div \frac{6 - 8j}{25} \text{ or } \frac{6 - 8j}{25} \div -2j$$

Go Online You can complete an Extra Example online.

Interactive Presentation

Example 5

SWIPE

Students move through the steps to see how to subtract two complex numbers.

WATCH

Students watch a video to learn how to add and subtract complex numbers.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 5 Add or Subtract Complex Numbers**MP Teaching the Mathematical Practices**

1 Seek Information Mathematically proficient students must be able to transform algebraic expression to reach solutions. In Example 5, students must rearrange and combine terms to simplify the expression.

Questions for Mathematical Discourse

- AL** When adding or subtracting two complex numbers, how many operations will you need to perform? **At least 2.**
- OL** How would removing the parentheses change the expression? **Sample answer:** Removing the first set does not change the expression. Removing the second set would make the imaginary term $-7i$.
- BL** Can adding or subtracting complex numbers result in a real number? Explain. **Yes, when the imaginary parts cancel.**

Example 6 Multiply Complex Numbers**MP Teaching the Mathematical Practices**

4 Apply Mathematics In Example 6, students apply what they have learned about complex numbers to solving a real-world problem.

Questions for Mathematical Discourse

- AL** Why do you use the FOIL method to multiply C and I ?
Each expression has two terms.
- OL** What is the product of a non-zero real number and an imaginary number? **An imaginary number**
- BL** Why can you combine real and imaginary parts with multiplication but not with addition? **Sample answer:** A real term plus an imaginary term is already in its simplest form, but the like factors of real and imaginary terms can be simplified when multiplying. (Similarly, $7 + 3x$ is in its simplest form, but $7(3x)$ can be simplified to $21x$.)

Example 7 Divide Complex Numbers**MP Teaching the Mathematical Practices**

7 Use Structure Help students to use the structure of the expression to rationalize the denominator and simplify the expression.

Questions for Mathematical Discourse

- AL** What is the complex conjugate of $a + bi$? **$a - bi$**
- OL** How do you rewrite $\frac{a + bi}{c + di}$ in the form $a + bi$? **$\frac{a}{c} + \frac{b}{c}i$**
- BL** Why is the product of two complex conjugates a real number?
Sample answer: When you distribute the multiplication by using the FOIL method, the outer and inner terms with i cancel one another.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using their whiteboards so that you can see all student responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2 exercises that mirror the examples		1–37
2	exercises that use a variety of skills from this lesson	38–42
2	exercises that extend concepts learned in this lesson to new contexts	43–46
3	exercises that emphasize higher-order and critical-thinking skills	47–51

ASSESS AND DIFFERENTIATE

 Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks,
THEN assign:

- Practice Exercises 1–45 odd, 47–51
- Extension: Conjugates and Absolute Value
-  **ALEKS** Complex Numbers and Complex Zeros of Polynomial Functions

BI

IF students score 66%–89% on the Checks,
THEN assign:

- Practice Exercises 1–51 odd
- Remediation, Review Resources: Properties of Real Numbers
- Personal Tutors
- Extra Examples 1–7
-  **ALEKS** Properties of Real Numbers

OL

IF students score 65% or less on the Checks,
THEN assign:

- Practice Exercises 1–37 odd
- Remediation, Review Resources: Properties of Real Numbers
- Quick Review Math Handbook*: Complex Numbers
-  **ALEKS** Properties of Real Numbers

AL

Practice

Examples 1 and 2:

Simplify.

1. $\sqrt{-48}$ $4\sqrt{3}$

2. $\sqrt{-83}$ $3\sqrt{7}$

3. $\sqrt{-72}$ $6\sqrt{2}$

4. $\sqrt{-24}$ $2\sqrt{6}$

5. $\sqrt{-84}$ $2\sqrt{21}$

6. $\sqrt{-99}$ $3\sqrt{11}$

7. $\sqrt{-23} \cdot \sqrt{-46}$ $-23\sqrt{2}$

8. $\sqrt{-6} \cdot \sqrt{-3}$ $-3\sqrt{2}$

9. $\sqrt{-5} \cdot \sqrt{-10}$ $-5\sqrt{2}$

10. $(3i)(-2i)(5i)$ $30i$

11. i^2 -1

12. $4i(-6i)$ $-24i^2$

Example 3:

Solve each equation.

13. $5x^2 + 45 = 0$ $\pm 3i$

14. $4x^2 + 24 = 0$ $\pm 2\sqrt{6}i$

15. $-9x^2 = 9$ $\pm i$

16. $7x^2 + 64 = 0$ $\pm 2\sqrt{7}i$

17. $5x^2 + 125 = 0$ $\pm 5i$

18. $8x^2 + 96 = 0$ $\pm 2\sqrt{3}i$

Example 4:

Find the values of x and y that make each equation true.

19. $9 + 12i = 3x + 4yi$ $3, 3$

20. $x + 1 + 2yi = 3 - 6i$ $2, -3$

21. $2x + 7 + (3 - yi) = -4 + 6i$ $-\frac{11}{2}, -3$

22. $5 + y + (3x - 7i) = 9 - 3i$ $\frac{6}{5}, 4$

23. $20 - 12i = 5x + (4y)i$ $4, -3$

24. $x - 16i = 3 - (2y)i$ $3, 8$

Examples 5 and 6:

Simplify.

25. $(5 + 4i) + (4 - 5i)$ $10 - 4i$

26. $(8 + 3i) - (5 - 2i)$ $3 + 5i$

27. $(5 - 4i) - (3 - 2i)$ $2 + i$

28. $(-4 + 2i) + (5 - 3i)$ $1 - i$

29. $(5 - 3i) + (4 - 2i)$ $10 - 5i$

30. $(-11 + 4i) - (1 - 5i)$ $-12 + 9i$

31. $(2 + 8i) - 3i$ $2 + 5i$

32. $(5 - 2i)(4 - i)$ $18 - 13i$

33. $(4 - 2i)(1 - 2i)$ $-10i$

34. **ELECTRICITY** Using the formula $V = CI$, find the voltage V in a circuit when the current $C = 3 - i$ amperes and the impedance $I = 3 + 2i$ ohms. $11 + \frac{1}{2}i$ volts

Example 7:

Simplify.

35. $\frac{3}{2} + \frac{1}{2}i$ $\frac{3}{2} + \frac{1}{2}i$

36. $\frac{7 - 13i}{2}$ $\frac{7}{2} - \frac{13}{2}i$

37. $\frac{8 - 5i}{3}$ $\frac{8}{3} - \frac{5}{3}i$

Lesson 1-3 • Complex Numbers 25



Mixed Exercises

STRUCTURE Simplify.

$$38. (1 + i)(2 + 3i)(4 - 3i)(1 + 2i) \quad 39. \frac{4 - i\sqrt{2}}{2 + i\sqrt{2}} \cdot \frac{1 - 5i\sqrt{2}}{9} \quad 40. \frac{2 - i\sqrt{2}}{2 + i\sqrt{2}} \cdot \frac{1 - 5i\sqrt{2}}{1}$$

$$41. \text{Find the sum of } ix^2 - (4 + 5i)x + 7 \text{ and } 3x^2 + (2 + 6i)x - 8i. \quad (3 + 9i)x^2 + (-2 + 9i - 8i)x + 7$$

$$42. \text{Simplify } (2 + ix + 5 + i)(-3 + 4i)x^2 + (5 - 5i)(x - 6). \quad (5 - 3i)x^2 + (-5 + 4i)x + 11$$

ELECTRICITY Use the formula $V = CI$, where V is the voltage, C is the current, and I is the impedance.

$$43. \text{The current in a circuit is } 2 + 4i \text{ amps, and the impedance is } 3 - i \text{ ohms. What is the voltage? } 10 + 10i \text{ volts.}$$

$$44. \text{The voltage in a circuit is } 24 - 8i \text{ volts, and the impedance is } 4 - 2i \text{ ohms. What is the current? } \frac{8}{5} + \frac{4}{5}i$$

$$45. \text{CIRCUITS The impedance in one part of a series circuit is } 1 + 3i \text{ ohms and the impedance in another part of the circuit is } 7 - 5i \text{ ohms. Add these complex numbers to find the total impedance in the circuit. } 8 - 2i \text{ ohms.}$$

$$46. \text{ELECTRICAL ENGINEERING The standard electrical voltage in Europe is 220 volts.}$$

$$a. \text{Find the impedance in a standard European circuit if the current is } 22 - 11i \text{ amps. } 8 + 4i \text{ ohms}$$

$$b. \text{Find the current in a standard European circuit if the impedance is } 10 - 5i \text{ ohms. } 17.6 + 8.8i \text{ amps}$$

$$c. \text{Find the impedance in a standard European circuit if the current is } 20i \text{ amps. } -11i \text{ ohms}$$

Higher-Order Thinking Skills

$$47. \text{FIND THE ERROR Jose and Zoe are simplifying } 2i(3i/4). \text{ Is either of them correct? Explain your reasoning. } \text{Zoe, } i^2 = -1, \text{ not } -i.$$

Jose

$$14i^3 = 14i$$

Zoe

$$24i^3 = -24i$$

$$48. \text{PERSISTENCE Simplify } (1 + 2i)^2. \quad -11 - 2i$$

$$49. \text{ANALYZE Determine whether the following statement is always, sometimes, or never true. Explain your reasoning. See margin.}$$

Every complex number has both a real part and an imaginary part.

$$50. \text{CREATE Write two complex numbers with a product of 20. Sample answer: } (4 + 2i)(4 - 2i)$$

$$51. \text{WRITE Explain how complex numbers are related to quadratic equations. See margin.}$$

Answers

49. Always; sample answer: The value of 5 can be represented by $5 + 0i$, and the value of $3i$ can be represented by $0 + 3i$.

51. Some quadratic equations have complex solutions and cannot be solved using only the real numbers.

Solving Quadratic Equations by Factoring

LESSON GOAL

Students solve quadratic equations by factoring.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Explore:** Finding the Solutions of Quadratic Equations by Factoring

 **Develop:**

Solving Quadratic Equations by Factoring

- Factor by Using the Distributive Property
- Factor a Trinomial
- Solve an Equation by Factoring
- Factor a Trinomial Where a is Not 1

Solving Quadratic Equations by Factoring Special Products

- Factor a Difference of Squares
- Factor a Perfect Square Trinomial
- Complex Solutions

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

 Formative Assessment Math Probe

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources

Remediation: Using the Distributive Property

Extension: Using Identities to Factor

Language Development Handbook

Assign page 4 of the *Language Development Handbook* to help your students build mathematical language related to solving quadratic equations by factoring.

ELL You can use the tips and suggestions on page T4 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Number and Quantity, Functions

Standards for Mathematical Content:

N.CN.7 Solve quadratic equations with real coefficients that have complex solutions.

N.CN.8 Extend polynomial identities to the complex numbers.

F.IF.8a Use the properties of exponents to interpret expressions for exponential functions.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous

Students performed operations with pure imaginary and complex numbers.
N.CN.1, N.CN.2

Now

Students solve quadratic equations by factoring.
N.CN.7, N.CN.8, F.IF.8a

Next

Students will simplify quadratic equations by using the Square Root Property and completing the square.
N.CN.7, F.IF.8a

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

 **Conceptual Bridge** In this lesson, students draw on their understanding of factoring to solve quadratic equations. They build fluency by factoring to find solutions, and they apply their understanding of factoring by solving real-world problems.



Interactive Presentation

Warm Up

Factor:

- $mn + mnw$
- $4x^2 - 25$
- $x^2 + 4x - 5$
- $25x^2 - 30x + 9$
- $2x^2 + 9x + 4$

Warm Up

Launch the Lesson

The equation $d = vt + \frac{1}{2}at^2$ can be used to approximate the stopping distance d of a car where v is the initial velocity t is the time, and a is the acceleration. If you know all of the above information, you can factor to solve a quadratic equation that can tell you how long it will take a car to stop once the driver applies the brakes.



Launch the Lesson

Vocabulary

Exponent **BE** Coefficient **AB**

- factored form**
A form of quadratic equation, $0 = a(x - p)(x - q)$, where $a \neq 0$, in which p and q are the x -intercepts of the graph of the related function.
- differences of squares**
A binomial in which the first and last terms are perfect squares.
- perfect square binomials**
Squares of binomials.

• If a polynomial can be factored, is the factored form unique?

• When a quadratic equation is rewritten in factored form, what do you know about the quadratic function that you didn't know before?

• What is the difference between factoring the sum of two squares and factoring the difference of two squares?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- factoring

Answers:

- $mn(1 + w)$
- $(2x + 5)(2x - 5)$
- $(x + 5)(x - 1)$
- $(5x - 3)^2$
- $(2x + 1)(x + 4)$

Launch the Lesson



Teaching the Mathematical Practices

2 Attend to Quantities Point out that it is important to note the meaning of the variables used in the equation for the stopping distance of a car.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Mathematical Background

Quadratic equations can be solved using several different methods. Factoring can be a quick method. Once a polynomial has been factored, the Zero Product Property may be used to find the roots of the equation. If the polynomial is difficult to factor or is not factorable, then other methods must be used.



Explore Finding the Solutions of Quadratic Equations by Factoring

Objective

Students use a sketch to explore solving quadratic equations by factoring.



Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the factors and solutions of quadratic equations and the graphs of the related quadratic functions.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

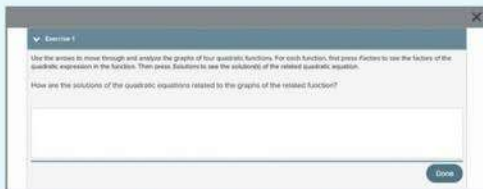
Students will complete guiding exercises throughout the Explore activity. Students review a variety of equations and graphs, connecting the solutions to the factors of the quadratic equation. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore



Explore

WEB SKETCHPAD



Students will use a sketch to explore how to solve quadratic equations by factoring.

TYPE



Students will answer questions about the solutions and factors of quadratic equations.



Interactive Presentation

INQUIRY How can you use factoring to solve a quadratic equation?

Submit

Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Finding the Solutions of Quadratic Equations by Factoring (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Which function is the difference of two squares? **Sample answer:** $h(x)$ is the difference of two squares, 49 and x^2 .
- When might you want to solve by factoring instead of by graphing?
Sample answer: When I am looking for an exact answer rather than an estimate.

Inquiry

How can you use factoring to solve a quadratic equation? **Sample answer:** Rewrite the equation so that one side is 0, factor the expression in the equation, set each factor equal to 0, and solve.



Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Solving Quadratic Equations by Factoring

Objective

Students solve quadratic equations by factoring.

MP Teaching the Mathematical Practices

7 Use the Distributive Property Point out that the Distributive Property is one of the most-used properties in algebra. Students should know that whenever they see a number outside of a sum or difference within parentheses, they should apply the Distributive Property.

Example 1 Factor by Using the Distributive Property

MP Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. In Example 1, students must transform the quadratic equation to reach a solution.

Questions for Mathematical Discourse

A1 What is a greatest common factor? the greatest single factor that can be factored out of multiple terms

O1 How is the Zero Product Property used to identify the solutions to this equation? Sample answer: Because the equation can be rewritten as $3x(4x - 1) = 0$, the Zero Product Property means the solutions come from $3x = 0$ and $4x - 1 = 0$.

B1 Can you tell one of the roots just by looking at the original equation? Explain. Yes; sample answer: Zero is a root because all the terms have a factor of x .

DIFFERENTIATE

Language Development Activity ELL

Beginning Ask questions about the lesson content to elicit yes/no answers: "Consider the trinomial $x^2 - 2x - 3$ and its factored form $(x - 3)(x + 1)$ given in the Learn section. Is x the product of the first terms of the two binomials?" **yes**

Intermediate/Advanced Ask questions about the lesson content to elicit short answers: "Consider the trinomial $x^2 - 2x - 3$ and its factored form $(x - 3)(x + 1)$ given in the Learn section. What is the product of the inner terms of the two binomials?" $(-3)(x)$ "What two terms can be combined in the product?" x and $-3x$

Advanced High Ask questions about lesson content to elicit complete sentences: "What is the Zero Product Property?" **If the product of two numbers is 0, one or both of the numbers is 0.** "How does the Zero Product Property help you solve equations?" **If you know that the product of two expressions is 0, one or both expressions equals 0. So finding the values where either expression is 0 solves the equation.**



Solve Quadratic Equations by Factoring

Lesson 1-4

Explore: Finding the Solutions of Quadratic Equations by Factoring

Online Activity Use graphing technology to complete the Explore.

INQUIRY How can you use factoring to solve a quadratic equation?

Learn: Solving Quadratic Equations by Factoring

The factored form of a quadratic equation is $0 = (x - p)(x - q)$, where $p \neq 0$ and $q \neq 0$. In this equation, p and q represent the x -intercepts of the graph of the related function. For example, $0 = x^2 - 2x - 3$ can be written in the factored form $0 = (x - 3)(x + 1)$ and its related graph has x -intercepts of -1 and 3 .

Key Concepts - Factoring

Using the Distributive Property: $ax + bx = a(x + b)$

Factoring Trinomials: $x^2 + bx + c = (x + m)(x + p)$ when $m + p = b$ and $mp = c$

Key Concept - Zero-Product Property

Words: For any real numbers a and b , if $ab = 0$, then either $a = 0$, $b = 0$, or both a and $b = 0$.

Example: $15x - 20x + 4 = 0$, then $x - 2 = 0$, $x + 4 = 0$, or both: $x - 2 = 0$ and $x + 4 = 0$.

To solve a quadratic equation by factoring, first make sure that one side of the equation is 0, and factor the binomial. Use the Zero Product Property to write separate equations. Then use the properties of equality to isolate the variable.

Example 1 Factor by Using the Distributive Property

Solve $12x^2 - 2x = x$ by factoring. Check your solution.

$$12x^2 - 2x = x$$

Original equation

$$12x^2 - 3x = 0$$

Subtract x from each side.

$$3x(4x - 1) = 0$$

Factor the GCF.

$$3x(4x - 1) = 0$$

Distributive Property

$$3x = 0 \text{ or } 4x - 1 = 0$$

Zero Product Property

$$x = 0 \text{ or } x = \frac{1}{4}$$

Solve.

Go Online You can complete an Extra Example online.

Today's Goals

- Solve quadratic equations by factoring.
- Solve quadratic equations by factoring special products.

Today's Vocabulary
factored form
difference of squares
perfect square
trinomials

Think About It!

The equation $x^2 - 2x - 3 = 0$ could be solved by factoring, where $x^2 - 2x - 3 = (x - 3)(x + 1)$. How are the factors of the equation related to the roots, or zeros, of the related function $f(x) = x^2 - 2x - 3$?

Sample answer: setting each factor equal to 0 and solving the resulting equations results in the zeros of the related function. So, $f(x) = 0$ and $(x - 3) = 0$.

Go Online You can watch a video to see how to use algebra tiles to factor a polynomial using the Distributive Property.

Lesson 1-4 • Solve Quadratic Equations by Factoring 27

Interactive Presentation



Learn

TAP



Students tap to see the Study Tip.



Example 2 Factor a Trinomial

MP Teaching the Mathematical Practices

7 Use the Distributive Property In Example 2, students must use the Distributive Property in reverse to factor the trinomial and solve the equation.

Questions for Mathematical Discourse

- AL** What is the first step before factoring any quadratic equation?
Solve the equation so that one side is 0.
- OL** How do you find the factors of the trinomial? **Sample answer:** Look for two numbers that have a sum of -6 and a product of -27 .
- BL** What would you change in the original equation so that the solution were 3 and -9 ? **Change** $-6x$ to $6x$.

Apply Example 3 Solve an Equation by Factoring

MP Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them, 4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- The car's speed is measured in mph and the distance is measured in feet. What first step must you take to solve the exercise?
- What type of equation is obtained and how can this type of equation be solved?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.

Example 2 Factor a Trinomial

Solve $x^2 - 6x - 9 = 18$ by factoring. Check your solution.

$$x^2 - 6x - 9 = 18$$

$$x^2 - 6x - 27 = 0$$

$$(x + 3)(x - 9) = 0$$

$$x + 3 = 0 \text{ or } x - 9 = 0$$

$$x = -3 \quad x = 9$$

Original equation

Subtract 18 from each side.

Factor the trinomial.

Zero Product Property

Solve

Apply Example 3 Solve an Equation by Factoring

ACCELERATION The equation $d = vt + \frac{1}{2}at^2$ represents the displacement d of a car traveling at an initial velocity v where the acceleration a is constant over a given time t . Find how long it takes a car to accelerate from 30 mph to 45 mph if the car moved 605 feet and accelerated slowly at a rate of 2 feet per second squared.

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: Solve the equation to find the time for the car to accelerate. The acceleration is given in feet per second squared and the velocity is given in miles per hour. How do I address the difference in units?

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: Convert the velocity to feet per second. Then substitute the distance, velocity, and acceleration into the formula and solve for time.

3 What is your solution?

Use your strategy to solve the problem.

What is the velocity in feet per second? 44 fps.

How long it takes the car to accelerate from 30 mph to 45 mph? 11 s.

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: The solutions of the equation are -55 and 11 . Because time cannot be negative, $t = 11$ is the only viable solution in the context of the situation.

Go Online You can complete an Extra Example online.

Think About It! Choose two integers and write an equation in standard form with these roots. How would the equation change if the signs of the two roots were switched?

Sample answer: 2 and 5; $x^2 - 7x + 10 = 0$; -2 and -5 ; $x^2 + 7x + 10 = 0$; the linear term is the only term that changes sign.

Think About It! Why did you not use 45 mph to solve this problem?

Sample answer: The equation uses the initial velocity and 45 mph is the final velocity.

Interactive Presentation

Actions may print by pressing:

ACCELERATION The equation $d = vt + \frac{1}{2}at^2$ represents the displacement d of a car traveling at an initial velocity v where the acceleration a is constant over a given time t . Find how long it takes a car to accelerate from 30 mph to 45 mph if the car moved 605 feet and accelerated slowly at a rate of 2 feet per second squared.



Apply Example 3

DRAG & DROP



Students drag to sort the steps needed to solve the problem.

Example 4 Factor a Trinomial Where a Is Not 1

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. The Think About It! feature asks students to explain how they determined the values of m and p when factoring the equation.

Questions for Mathematical Discourse

- AL** How can you check that your factored expression is correct?
 Sample answer: Use the FOIL method or the Distributive Property to check that the product of the factors is equal to the original quadratic expression.
- OL** What must be true about both b and c for $\frac{b}{a}$ and $\frac{c}{a}$ to both be integers? Both b and c would have to be multiples of a .
- BL** If you first divided the equation by 3, would you find the same solutions? Explain. Yes; sample answer: The solutions would be the same, but the factoring process may be more complicated.

Learn Factoring Special Products

Objective

Students solve quadratic equations by factoring special products.

MP Teaching the Mathematical Practices

7 Look for a Pattern Help students to see how to apply the patterns for factoring special products.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Check

SALES A clothing store is analyzing their market to determine the profitability of their new dress design. If $P(x) = -16x^2 + 172x - 44,640$ represents the store's profit when x is the price of each dress, find the prices at which the store makes no profit on the design. **B**

- A. \$11.25 and \$15.50
 B. \$45 and \$62
 C. \$50 and \$54
 D. \$180 and \$248

Example 4 Factor a Trinomial Where a Is Not 1

Solve $3x^2 + 5x + 15 = 17$ by factoring. Check your solution.

$3x^2 + 5x + 15 = 17$	Original equation
$3x^2 + 5x - 2 = 0$	Subtract 17 from each side.
$(3x - 1)(x + 2) = 0$	Factor the trinomial.
$3x - 1 = 0$ or $x + 2 = 0$	Zero Product Property
$x = \frac{1}{3}$ $x = -2$	Solve.

Check

Solve $4x^2 + 12x - 27 = 13$ by factoring. Check your solution.

$$x = \frac{3}{4}, \frac{9}{2}$$

Learn Solving Quadratic Equations by Factoring Special Products

Key Concept • Factoring Differences of Squares

Words: To factor $a^2 - b^2$, find the square roots of a^2 and b^2 . Then apply the pattern.

$$\text{Symbols: } a^2 - b^2 = (a + b)(a - b)$$

Key Concept • Factoring Perfect Square Trinomials

Words: To factor $a^2 + 2ab + b^2$, find the square roots of a^2 and b^2 . Then apply the pattern.

$$\text{Symbols: } a^2 + 2ab + b^2 = (a + b)^2$$

Not all quadratic equations have solutions that are real numbers. In some cases, the solutions are complex numbers of the form $a + bi$, where $b \neq 0$. For example, you know that the solution of $x^2 = -4$ must be complex because there is no real number for which its square is -4 . If you take the square root of each side, $x = 2i$ or $-2i$.

Talk About It!

Explain how to determine which values should be chosen for m and p when factoring a polynomial of the form $ax^2 + bx + c$.

Sample answer: Find two numbers, m and p , with a product of ac and a sum of b .



Math History Minute

English mathematician and astronomer **Thomas Harriot (1560–1621)** was one of the first, if not the first, to consider the imaginary roots of equations. Harriot advanced the notation system for algebra and studied negative and imaginary numbers.

Lesson 1-4 • Solving Quadratic Equations by Factoring 29

Interactive Presentation

Factor a Trinomial Where a Is Not 1

Solve $3x^2 + 5x + 15 = 17$ by factoring. Check your solution.

$3x^2 + 5x + 15 = 17$
Original equation

Example 4

SWIPE



Students move through the steps to see how to factor a trinomial.

CHECK



Students complete the Check online to determine whether they are ready to move on.

**Example 5** Factor a Difference of SquaresSolve $81 = x^2$ by factoring. Check your solution.

$$\begin{aligned} 81 &= x^2 \\ 81 - x^2 &= 0 \\ 9^2 - x^2 &= 0 \\ (9 + x)(9 - x) &= 0 \\ 9 + x &= 0 \text{ or } 9 - x = 0 \\ x &= -9 \quad x = 9 \end{aligned}$$

Original equation

Subtract x^2 from each side.Write in the form $a^2 - b^2$.

Factor the difference of squares.

Zero Product Property

Solve.

CheckSolve $x^2 = 529$ by factoring. Check your solution.

$$x = \frac{23}{-23}$$

Example 6 Factor a Perfect Square TrinomialSolve $16y^2 - 22y + 9 = 26y - 13$ by factoring. Check your solution.

$$\begin{aligned} 16y^2 - 22y + 23 &= 26y - 13 && \text{Original equation} \\ 16y^2 - 48y + 36 &= -13 && \text{Subtract } 26y \text{ from each side.} \\ 16y^2 - 48y + 36 &= 0 && \text{Add } 13 \text{ to each side.} \\ (4y)^2 - 2(4y)(3) + (-3)^2 &= 0 && \text{Factor the perfect square trinomial.} \\ (4y - 3)^2 &= 0 && \text{Simplify.} \\ 4y - 3 &= 0 && \text{Take the square root of each side and solve.} \\ y &= \frac{3}{4} \end{aligned}$$

CheckSolve $16x^2 - 22x + 15 = 50x - 1$ by factoring. Check your solution.

$$x = \frac{2}{-2}, 1$$

Example 7 Complex SolutionsSolve $x^2 = -144$ by factoring. Check your solution.

$$\begin{aligned} x^2 &= -144 && \text{Original equation} \\ x^2 + 144 &= 0 && \text{Add } 144 \text{ to each side.} \\ x^2 - (-144) &= 0 && \text{What is a difference of squares?} \\ x^2 - (12i)^2 &= 0 && \sqrt{-144} = 12i \\ (x + 12i)(x - 12i) &= 0 && \text{Factor the difference of squares.} \\ x + 12i &= 0 \text{ or } x - 12i = 0 && \text{Zero Product Property} \\ x &= -12i \quad x = 12i && \text{Solve.} \end{aligned}$$

Go Online! You can complete an Extra Example online.

Go Online! You can watch a video to see how to use algebra tiles to factor a difference of squares.

Think About It! Why does this equation have one solution instead of two?

Sample answer: The factor $4y - 3$ is repeated, so the equation only has one solution.

Think About It! Explain why both $(-12)^2$ and $(12)^2$ equal -144 .

Sample answer: $(-12)(-12) = 144$ or -144 , and $(12)(12) = 144$ or -144 .

Watch Out! Complex Numbers Remember i^2 equals -1 , not i .

30 Module 1 • Quadratic Functions

Interactive Presentation

Example 6

Solve $16y^2 - 22y + 9 = 26y - 13$ by factoring. Check your solution.

$$\begin{aligned} 16y^2 - 22y + 23 &= 26y - 13 \\ 16y^2 - 48y + 36 &= 0 \\ (4y)^2 - 2(4y)(3) + (-3)^2 &= 0 \\ (4y - 3)^2 &= 0 \\ \sqrt{(4y - 3)^2} &= \sqrt{0} \\ 4y - 3 &= 0 \\ y &= \frac{3}{4} \end{aligned}$$

Original equation
Subtract $26y$ from each side.
Add 13 to each side.
Factor the perfect square trinomial.
Simplify.
Take the square root of each side.
Solve.

Example 6

TYPE

Students answer a question to determine whether they understand how to factor perfect square trinomials.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 5 Factor a Difference of Squares**MP Teaching the Mathematical Practices**

7 Use Structure Help students to use the structure of the equation to apply the pattern for factoring a difference of squares.

Questions for Mathematical Discourse

- AL** What does it mean for a number to be a perfect square? The square root is an integer.
- OL** When multiplying $(x - a)(x + a)$, what always happens to the two resulting x -terms? They add to 0 and cancel out.
- BL** How would you use this method to solve $(x + 2)^2 = 81$? **Sample answer:** Let $u = (x + 2)$, solve for u , then substitute $x + 2$ back in for u .

Example 6 Factor a Perfect Square Trinomial**MP Teaching the Mathematical Practices**

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. In Example 6, students must transform the quadratic equation to apply a pattern for factoring and reach a solution.

Questions for Mathematical Discourse

- AL** How do you check for a perfect square trinomial? **Sample answer:** Once all the terms are simplified on one side of the equation, check to see if the constant and the coefficient of the x^2 -term are perfect squares. If they are, check that the coefficient of the x -term is -2 times the square roots of the perfect squares.
- OL** What is the coefficient of the x -term when you simplify $(x - a)^2 - 2a$?
- BL** Why would the root of a perfect square trinomial be called a double root? **Sample answer:** The trinomial has two roots that are the same value.

Example 7 Complex Solutions**MP Teaching the Mathematical Practices**

6 Use Definitions In the Think About It! feature, students will use the definition of i to justify their solutions in Example 7.

Questions for Mathematical Discourse

- AL** If $x^2 = m$ has a pure imaginary number as a solution, what do you know about m ? **m is a negative number.**
- OL** What is the relationship between $(x + 12i)$ and $(x - 12i)$? They are complex conjugates.
- BL** Why can you write -144 as $(12i)^2$? $12^2 = 144$ and $i^2 = -1$, so $(12i)^2 = 144(-1) = -144$.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION



N.CN.7, N.CN.8, F.IF.8a

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–32
2	exercises that use a variety of skills from this lesson	33–40
2	exercises that extend concepts learned in this lesson to new contexts	41, 42
3	exercises that emphasize higher-order and critical-thinking skills	43–47

ASSESS AND DIFFERENTIATE

Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks,
THEN assign:

BL

- Practice Exercises 1–41 odd, 43–47
- Extension: Using Identities to Factor
- ALEKS Solving Quadratic Equations by Factoring

IF students score 66%–89% on the Checks,
THEN assign:

OL

- Practice Exercises 1–47 odd
- Remediation, Review Resources: Using the Distributive Property
- Personal Tutors
- Extra Examples 1–7
- ALEKS Factoring Using the GCF; Factoring by Grouping

IF students score 65% or less on the Checks,
THEN assign:

AL

- Practice Exercises 1–31 odd
- Remediation, Review Resources: Using the Distributive Property
- Quick Review Math Handbook: Solving Quadratic Equations by Factoring
- ALEKS Factoring Using the GCF; Factoring by Grouping

Practice

Go Online You can complete your homework online.

Examples 1 and 2

Solve each equation by factoring. Check your solution.

- $6x^2 - 2x = 0$ $\frac{1}{6}$
- $x^2 = 7x$ $0, 7$
- $20x^2 = -25x$ $0, -\frac{5}{4}$
- $x^2 + x - 30 = 0$ $5, -6$
- $x^2 + 14x + 33 = 0$ $-11, -3$
- $x^2 - 3x = 10$ $5, -2$

Example 3

7. **GEOMETRY** The length of a rectangle is 2 feet more than its width. Find the dimensions of the rectangle if its area is 63 square feet. **7 ft by 9 ft**

8. **PHOTOGRAPHY** The length and width of a 6-inch by 8-inch photograph are reduced by the same amount to make a new photograph with area that is half that of the original. By how many inches will the dimensions of the photograph have to be reduced? **2 in.**

Example 4

Solve each equation by factoring. Check your solution.

- $2x^2 - x - 3 = 0$ $\frac{3}{2}, -1$
- $6x^2 - 5x - 4 = 0$ $-\frac{1}{2}, \frac{4}{3}$
- $5x^2 + 28x - 12 = 0$ $\frac{2}{5}, -\frac{6}{5}$
- $12x^2 - 8x + 1 = 0$ $\frac{1}{6}, \frac{1}{3}$
- $2x^2 - 11x - 40 = 0$ $8, -\frac{5}{2}$
- $3x^2 + 2x = 21$ $\frac{7}{3}, -3$

Examples 5–7

Solve each equation by factoring. Check your solution.

- $x^2 = 64$ $-8, 8$
- $x^2 - 100 = 0$ $-10, 10$
- $7x^2 = x^2 - 12$ 17
- $x^2 + 14 = 50$ $-6, 6$
- $x^2 - 169 = 0$ $-13, 13$
- $124 = x^2 + 3$ $-11, 11$
- $4x^2 - 28x + 49 = 0$ $\frac{7}{2}$
- $9x^2 + 6x = -1$ $-\frac{1}{3}$
- $16x^2 - 24x + 9 = 4$ $\frac{3}{2}$
- $81x^2 + 36x = -4$ $-\frac{2}{3}$
- $25x^2 + 80x + 64 = 0$ $-\frac{8}{5}$
- $9x^2 + 60x + 95 = -5$ $-\frac{10}{3}$
- $x^2 + 12 = -13$ $5i, -5i$
- $x^2 + 100 = 0$ $10i, -10i$
- $x^2 = -225$ $15i, -15i$
- $36x^2 = -25$ $\frac{5}{6}i, -\frac{5}{6}i$
- $64x^2 = -49$ $\frac{7}{8}i, -\frac{7}{8}i$

Lesson 1-4 • Solve Quadratic Equations by Factoring 31



Mixed Exercises

STRUCTURE Solve each equation by factoring. Check your solution.

33. $10x^2 + 25x + 15 = -3$ 34. $27x^2 + 5 = 48x$ 35. $x^2 + |t| = 0$ 36. $45x^2 - 3x = 2x + 0$ 37. $80x^2 = -36$ 38. $16x^2 + 8x = -1$

39. **USE A MODEL** The drawing *Maisons près de la mer* by Claude Monet is approximately 10 inches by 15 inches. Jennifer wants to make an art piece inspired by the drawing, and with an area 60% greater. If she chooses the size of her artwork by increasing both the width and height of Monet's work by same amount, what will be the dimensions of Jennifer's artwork? **13 inches by 16 inches**

40. **ANIMATION** A computer graphics animator would like to make a realistic simulation of a tossed ball. The animator wants the ball to follow the parabolic trajectory represented by $f(x) = -0.2x^2 + 50x - 5$.

- What are the solutions of $f(x) = 0$? **-5, 5**
 - If the animator changes the equation to $f(x) = -0.2x^2 + 20$, what are the solutions of $f(x) = 0$? **-10, 10**
41. Find two consecutive even positive integers that have a product of 624. **24, 26**
42. Find two consecutive odd positive integers that have a product of 323. **17, 19**

Higher-Order Thinking Skills

43. **FIND THE ERROR** Jade and Matteo are solving $-12x^2 + 5x + 2 = 0$. Is either of them correct? Explain your reasoning. **See margin.**

Jade	Matteo
$-12x^2 + 5x + 2 = 0$	$-12x^2 + 5x + 2 = 0$
$-12x^2 + 6x - 5x + 2 = 0$	$-12x^2 + 6x - 5x + 2 = 0$
$4x(-3x + 2) - 5x + 2 = 0$	$4x(-3x + 2) - (5x + 2) = 0$
$4x(-3x + 2) = 0$	$(4x + 1)(-3x + 2) = 0$
$x = \frac{1}{4}$ or $x = \frac{2}{3}$	$x = -\frac{1}{4}$ or $x = \frac{2}{3}$

44. **PERSEVERE** The rule for factoring a difference of cubes is shown below. Use this rule to factor $49x^3 - 125y^3$. **$5x(7x^2 - 35y^2 + 49y + 5y^2)$**

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

45. **CREATE** Choose two integers. Then write an equation in standard form with those roots. How would the equation change if the signs of the two roots were switched? **See margin.**

46. **ANALYZE** Determine whether the following statement is sometimes, always, or never true. Explain your reasoning. **See margin.**

If a quadratic equation in standard form where a , b , and c are integers, if b is odd, then the quadratic cannot be a perfect square trinomial.

47. **WRITE** Explain how to factor a trinomial in standard form with $a > 1$. **See margin.**

Answers

43. Neither; both students made mistakes in Step 3.

45. Sample answer: 3 and $6 \rightarrow x^2 - 9x + 18 = 0$. -3 and $-6 \rightarrow x^2 + 9x + 18 = 0$. The linear term changes sign.

46. Always; Sample answer: To factor using perfect square trinomials, the coefficient of the linear term, bx must be a multiple of 2, or even.

47. Sample answer: Standard form is $ax^2 + bx + c$. Multiply a and c . Then find a pair of integers, g and h , that multiply to equal ac and add to equal b . Then write the quadratic expression, substituting the middle term, bx , with $gx + hx$. The expression is now $ax^2 + gx + hx + c$. Then factor the GCF from the first two terms and factor the GCF from the second two terms. So, the expression becomes $GCF(x - q) + GCF_2(x - q)$. Simplify to get $(GCF + GCF_2)(x - q)$ or $(x - p)(x - q)$.

Solving Quadratic Equations by Completing the Square

LESSON GOAL

Students simplify quadratic expressions by using the Square Root Property and completing the square.

1 LAUNCH



Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP



Develop:

Solving Quadratic Equations by Using the Square Root Property

- Solve a Quadratic Equation with Rational Roots
- Solve a Quadratic Equation with Irrational Roots
- Solve a Quadratic Equation with Complex Solutions



Explore: Using Algebra Tiles to Complete the Square



Develop:

Solving Quadratic Equations by Completing the Square

- Complete the Square
- Solve by Completing the Square
- Solve a Problem by Completing the Square (online)
- Solve When a Is Not 1
- Solve Equations with Imaginary Solutions

Quadratic Functions in Vertex Form

- Write Functions in Vertex Form
- Determine the Vertex and Axis of Symmetry
- Model with a Quadratic Function



You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE



Exit Ticket



Practice

DIFFERENTIATE



View reports of student progress on the **Checks** after each example.

Resources	AL	DE	ET	
Remediation: Factoring Special Products	●	●		●
Extension: The Golden Quadratic Equations		●	●	●

Suggested Pacing

90 min	1.5 days
45 min	3 days

Focus

Domain: Number and Quantity, Functions

Standards for Mathematical Content:

N.CN.7 Solve quadratic equations with real coefficients that have complex solutions.

F.IF.8a Use the properties of exponents to interpret expressions for exponential functions.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous

Students solved quadratic equations by factoring.

N.CN.7, N.CN.8, F.IF.8a

Now

Students use completing the square to solve quadratic equations.

N.CN.7, F.IF.8a

Next

Students will solve quadratic equations with the quadratic formula and use the discriminant to determine the number of real roots.

N.CN.7, A.SSE.1b

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION



Conceptual Bridge In this lesson, students develop an understanding of the process of completing the square. They build fluency by using this process to solve equations, and they apply their understanding by solving real-world problems.

Language Development Handbook

Assign page 5 of the *Language Development Handbook* to help your students build mathematical language related to simplifying quadratic formula and discriminant to solve quadratic equations.

ETP You can use the tips and suggestions on page T5 of the handbook to support students who are building English proficiency.





Interactive Presentation

Warm Up

Factor:

- $x^2 + 2x + 1$
- $x^2 - 12x + 36$
- $x^2 + 18x + 81$
- $x^2 - 22x + 121$
- $4x^2 - 28x + 49$

Show Answers

Warm Up

Launch the Lesson

Seattle's Pacific Science Center contains five white parabolic arches in the middle of the center's six exhibition halls. The 100-foot-tall arches are lit with various colors to correspond to special events throughout the year. You can write quadratic equations to model the arches and solve to find where each one meets the ground.



Launch the Lesson

Vocabulary

Expand All Collapse All

- completing the square**
A process used to turn a quadratic expression into a perfect square trinomial.
- vertex form**
A form of quadratic equation, $y = a(x - h)^2 + k$, where the vertex is (h, k) .
- projectile motion problems**
Problems that involve objects being thrown or dropped.
 - What are the advantages of writing a quadratic function in vertex form?
 - What are the effects of a , h , and k on the graph of $y = a(x - h)^2 + k$?
 - How do you know what the initial value is?
 - How can you determine the length of time it takes for a projectile to hit the ground?
 - How can you determine the maximum height of the projectile?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- factoring perfect squares

Answers:

- $(x + 1)^2$
- $(x - 6)^2$
- $(x + 9)^2$
- $(x - 11)^2$
- $(2x - 7)^2$

Launch the Lesson



Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will learn how quadratic equations can be used to model parabolic arches in a real-world situation.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Mathematical Background

To complete the square for an expression of the form $x^2 + bx$, add the square of one half of the coefficient b to $x^2 + bx$. Written in symbols:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$



2 EXPLORE AND DEVELOP

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Using Algebra Tiles to Complete the Square

Objective

Students use algebra tiles to explore simplifying quadratic expressions by completing the square.

MP Teaching the Mathematical Practices

5 Use Mathematical Tools To complete the exercises in the Explore, students will need to use algebra tiles. Work with students to explore and deepen their understanding of completing the square.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. The students will watch a video describing the method of using algebra tiles to complete the square. Students will use algebra tiles and the method shown in the video to complete the square for quadratic expressions. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore



Explore

WATCH



Students can watch a video that explains how to complete the square to make a perfect square trinomial.



Interactive Presentation



Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Using Algebra Tiles to Complete the Square (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- How does the shape of the algebra tiles help to complete the square?

Sample answer: It is easy to evenly distribute the x -tiles around the x^2 -tile so that the sides of the product-rectangle are equal, making it a square. The rest can be filled in with 1-tiles to complete the fourth corner of the square.

- Can you model completing the square with algebra tiles for $x^2 + 5x$?

Why or why not? No; **sample answer:** There is an odd number of x -tiles. The fifth x -tile cannot be broken in half, so this would need to be done algebraically.

Inquiry

How does forming a square to create a perfect square trinomial help you solve quadratic equations? **Sample answer:** You can use the side length to solve for the variable.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Solving Quadratic Equations by Using the Square Root Property

Objective

Students solve quadratic equations by using the Square Root Property.

MP Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of equations that can be solved by using the Square Root Property.

Common Misconception

Students often forget that taking the square root of a number results in both positive and negative values. When applying the Square Root Property, you will want to reinforce this. If confusion remains, verify this by showing that both the positive and negative square roots yield the same answer when they are each squared.

Example 1 Solve a Quadratic Equation with Rational Roots

MP Teaching the Mathematical Practices

1 Understand the Approaches of Others In Example 1, encourage students to consider another way to solve the equation. Work with students to look at the Alternate Method. Ask students to compare and contrast the original method and the alternate method.

Questions for Mathematical Discourse

- A** What type of trinomial is the left side of the equation? **A perfect square trinomial**
- OL** Why is the $\pm$ sign placed before the radical? **Sample answer:** The radical refers to the principal square root of 25. Without including $\pm$ before the radical, you are only finding the positive root.
- BL** If you set each side of the equation equal to y and graphed, what would be the coordinates of the vertex of the quadratic function and the points of intersection of the graphs? **vertex: (2, 0); points of intersection: (-3, 25) and (7, 25)**

Important to Know

Point out that both constants in the Example 1 equation, 4 and 25, are perfect squares. Establishing this will help solidify the difference between rational and irrational roots when it is addressed in Example 2.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example online.

Solving Quadratic Equations by Completing the Square

Lesson 1-5

Learn Solving Quadratic Equations by Using the Square Root Property

You can use square roots to solve equations like $x^2 - 49 = 0$. Remember that 7 and -7 are both square roots of 49 because $7^2 = 49$ and $(-7)^2 = 49$. Therefore, the solution set of $x^2 - 49 = 0$ is $\{-7, 7\}$. This can be written as $\{\pm 7\}$.

Key Concept • Square Root Property

Words: To solve a quadratic equation in the form $x^2 = n$, take the square root of each side.

Symbols: For any number $n \geq 0$, if $x^2 = n$, then $x = \pm \sqrt{n}$.

Example: $x^2 = 121$, $x = \pm \sqrt{121}$ or ± 11 .

Not all quadratic equations have solutions that are whole numbers. Roots that are irrational numbers may be written as exact answers in radical form or as approximate answers in decimal form when a calculator is used. Sometimes solutions of quadratic equations are not real numbers. Solutions that are complex numbers can be written in the form $a + bi$, where $b \neq 0$.

Example 1 Solve a Quadratic Equation with Rational Roots

Solve $x^2 - 4x + 4 = 25$ by using the Square Root Property.

$$x^2 - 4x + 4 = 25$$

$$(x - 2)^2 = 25$$

$$x - 2 = \pm \sqrt{25}$$

$$x - 2 = \pm 5$$

$$x = 2 \pm 5$$

$$x = 2 + 5 \text{ or } x = 2 - 5$$

$$x = 7 \quad x = -3$$

$$\text{The solution set is } \{x | x = -3, 7\}.$$

Solving Quadratic Equations by Completing the Square

Today's Goal

- Solve quadratic equations by using the Square Root Property.
- Complete the square in quadratic expressions to solve quadratic equations.
- Complete the square in a quadratic function to interpret key features of its graph.

Today's Vocabulary
completing the square
vertex form
projectile motion problems

Think About It!
How can you determine whether an equation of the form $x^2 = n$ will have an answer that is a whole number?

Sample answer: If n is a perfect square, $x^2 = n$ will have an answer that is a whole number.

Study Tip
Square Roots: When using the Square Root Property, remember to include the $\pm$ before the radical.

Go Online: An alternate method is available for this example.

Lesson 1-5 • Solving Quadratic Equations by Completing the Square 33

Interactive Presentation

Solving Quadratic Equations by Using the Square Root Property

You can use square roots to solve equations like $x^2 - 49 = 0$. Remember that 7 and -7 are both square roots of 49 because $7^2 = 49$ and $(-7)^2 = 49$. Therefore, the solution set of $x^2 - 49 = 0$ is $\{-7, 7\}$. This can be written as $\{\pm 7\}$.

KEY CONCEPT: SQUARE ROOT PROPERTY

Words: To solve a quadratic equation in the form $x^2 = n$, take the square root of each side.

Symbols: For any number $n \geq 0$, if $x^2 = n$, then $x = \pm \sqrt{n}$.

Example: $x^2 = 121$, $x = \pm \sqrt{121}$ or ± 11 .

Learn

TYPE



Students answer a question to determine whether they understand solving quadratic equations by using the Square Root Property.


Example 2 Solve a Quadratic Equation with Irrational Roots

 Solve $x^2 + 24x + 144 = 192$ by using the Square Root Property.

$$x^2 + 24x + 144 = 192$$

$$(x + 12)^2 = 192$$

$$x + 12 = \pm\sqrt{192}$$

$$x + 12 = \pm 8\sqrt{3}$$

$$x = -12 \pm 8\sqrt{3}$$

$$x = -12 + 8\sqrt{3} \text{ or } -12 - 8\sqrt{3}$$

$$x = 1.86, -25.86$$

 The exact solutions are $-12 - 8\sqrt{3}$ and $-12 + 8\sqrt{3}$. The approximate solutions are -25.86 and 1.86 .

Example 3 Solve a Quadratic Equation with Complex Solutions

 Solve $2x^2 - 92x + 1058 = -72$ by using the Square Root Property.

$$2x^2 - 92x + 1058 = -72$$

$$x^2 - 46x + 529 = -36$$

$$(x - 23)^2 = -36$$

$$x - 23 = \pm\sqrt{-36}$$

$$x - 23 = \pm 6i$$

$$x = 23 \pm 6i$$

$$x = 23 + 6i \text{ or } 23 - 6i$$

 The solutions are $23 + 6i$ and $23 - 6i$.

Explore Use Algebra Tiles to Complete the Square

Online Activity Use algebra tiles to complete the Explore.

INQUIRY How does forming a square to create a perfect square trinomial help you solve quadratic equations?

Go Online You can complete an Extra Example online.

Watch Out!

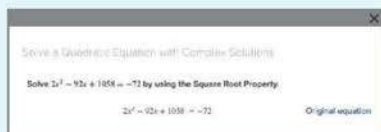
Perfect Squares. The constant, 192, on the right side of the equation is not a perfect square. This means that the roots will be irrational numbers.

Think About It!

 Can you solve a quadratic equation by completing the square if the coefficient of the x^2 term is not 1? Justify your argument.

Yes, sample answer: First divide the equation by the coefficient.

Interactive Presentation



Example 3

SWIPE


Students move through the steps to solve a quadratic equation with complex solutions.

TYPE


Students answer a question to show they understand complex solutions of quadratic equations.

CHECK


Students complete the Check online to determine whether they are ready to move on.

Example 2 Solve a Quadratic Equation with Irrational Roots

Teaching the Mathematical Practices
6 Use Precision In Example 2, students calculate the solutions of the quadratic equation accurately and efficiently. Point out that the exact answers are in radical form, but a calculator may be used to find the approximate solutions.

Questions for Mathematical Discourse
AL Is there an exact decimal answer to this example? **No; sample answer:** Irrational numbers cannot be written as a terminating or repeating decimal.

OL Are irrational numbers real numbers? **Yes.**
BL When would you want to use the exact solutions versus using the approximate solutions? **Sample answer:** If you will be doing further calculations with the solutions, you should use the exact solutions. Approximate solutions are useful for the final step but can introduce error in intermediate steps.

Example 3 Solve a Quadratic Equation with Complex Solutions

Teaching the Mathematical Practices
1 Seek Information Students must transform the quadratic equation to reach solutions in Example 3.

Questions for Mathematical Discourse
AL Why is there an imaginary term in each solution? **Sample answer:** There is a negative under the square root.

OL How can you tell the solutions will be complex rather than purely imaginary? **Sample answer:** The quadratic expression factors to a square term with a constant subtracted from x . This constant will have to be added to the imaginary part.

BL When substituting the solutions for x in the original equation, how is it possible that the left side simplifies to a real number? **Sample answer:** The squared x -term will have an imaginary component that will cancel the imaginary component of the linear x -term.

Learn Solving Quadratic Equations by Completing the Square

Objective

Students complete the square in quadratic expressions to solve quadratic equations.


Teaching the Mathematical Practices
7 Use Structure Work with students to examine the structure of the completing the square method.

Important to Know

When discussing the steps for completing the square, emphasize that the coefficient of the quadratic term must be 1.

**Example 4** Complete the Square**MP Teaching the Mathematical Practices**

1 Explain Correspondences Encourage students to explain the relationships between the equations used in this example.

Questions for Mathematical Discourse

- AL** What formula is used to find c ? $\left(\frac{b}{2}\right)^2$
- OL** How does $\left(\frac{b}{2}\right)^2$ change if b is negative? **It does not change.**
- BL** What would be the denominator of c if b is odd? **4**

Common Error

If $\frac{b}{2}$ does not simplify to a whole number, remind students that to square a fraction, you must square the numerator and denominator.

Example 5 Solve by Completing the Square**MP Teaching the Mathematical Practices**

7 Use Structure Help students to use the structure of the equation in Example 5 to identify the ways to rewrite the equation to solve by completing the square.

Questions for Mathematical Discourse

- AL** Why do you want to find $\left(\frac{b}{2}\right)^2$? **to make the quadratic a perfect square trinomial**
- OL** Why is $\left(\frac{b}{2}\right)^2$ added to each side of the equation? **to keep both sides equal to each other**
- BL** How could you change the original equation so that the solution set contains only integers? **Sample answer: Change the value of c , -4 , so that when you add 81 to each side, the result is a perfect square. For example, change c to -19 .**

DIFFERENTIATE**Language Development Activity AL IL**

Have students solve the equation $x^2 + 6x - 40 = 0$ by completing the square. Then have them discuss with a partner as many ways as they can to check their solutions.

Learn Solving Quadratic Equations by Completing the Square

All quadratic equations can be solved by using the properties of equality to manipulate the equation until one side is a perfect square. This process is called **completing the square**.

Key Concept • Completing the Square

Words: To complete the square for any quadratic expression of the form $x^2 + bx$, follow the steps below.

Step 1 Find one half of b , the coefficient of x .

Step 2 Square the result in Step 1.

Step 3 Add the result of Step 2 to $x^2 + bx$.

Symbolic: $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

To solve an equation of the form $x^2 + bx + c = 0$ by completing the square, first subtract c from each side of the equation. Then add $\left(\frac{b}{2}\right)^2$ to each side of the equation and solve for x .

Example 4 Complete the Square

Find the value of c that makes $x^2 - 7x + c$ a perfect square. Then write the expression as a perfect square trinomial.

Step 1 Find one half of b . $-\frac{7}{2} = -3.5$

Step 2 Square the result from Step 1. $(-3.5)^2 = 12.25$

Step 3 Add the result from Step 2 to $x^2 - 7x$. $x^2 - 7x + 12.25$

The expression $x^2 - 7x + 12.25$ can be written as $(x - 3.5)^2$.

Example 5 Solve by Completing the Square

Solve $x^2 + 18x - 4 = 0$ by completing the square.

$$x^2 + 18x - 4 = 0$$

$$x^2 + 18x = 4$$

$$x^2 + 18x + 81 = 4 + 81$$

$$(x + 9)^2 = 85$$

$$x + 9 = \pm\sqrt{85}$$

$$x = -9 \pm \sqrt{85}$$

$$x = -9 + \sqrt{85} \text{ or}$$

$$x = -9 - \sqrt{85}$$

$$x = 0.22 \text{ or } -18.22$$

The solution set is $\{x | x = -9 - \sqrt{85}, -9 + \sqrt{85}\}$.

Go Online to see Example 6.

Original equation

Add 4 to each side.

Add $\left(\frac{18}{2}\right)^2$ to each side.

Factor.

Square Root Property

Subtract 9 from each side

Write as two equations.

Simplify.

Talk About It

If a and b are real numbers, can the value of c ever be negative? Explain your reasoning.

No; sample answer: Because this value is equal to half of b squared, and the square of a real number is never negative.

Think About It!

Why do we first add 4 to each side?

Sample answer: In order to complete the square, the expression on one side of the equation must be of the form $x^2 + bx$.

Go Online You can complete an Extra Example online.

Lesson 1-5 • Solving Quadratic Equations by Completing the Square **35**

Interactive Presentation

Solve by Completing the Square

Solve by completing the square.

$$x^2 + 18x - 4 = 0$$

Original equation

Example 5

SWIPE



Students move through the steps to solve a quadratic equation by completing the square.



Go Online
for Example 8.

Example 7 Solve When a Is Not 1Solve $4x^2 - 12x - 27 = 0$ by completing the square.

$$4x^2 - 12x - 27 = 0$$

$$x^2 - 3x - \frac{27}{4} = 0$$

$$x^2 - 3x = \frac{27}{4}$$

$$x^2 - 3x + \frac{9}{4} = \frac{27}{4} + \frac{9}{4}$$

$$\left(x - \frac{3}{2}\right)^2 = 9$$

$$x - \frac{3}{2} = \pm 3$$

$$x = \frac{3}{2} \pm 3$$

$$x = \frac{3}{2} + 3 \text{ or } x = \frac{3}{2} - 3$$

$$x = \frac{9}{2}$$

$$x = -\frac{3}{2}$$

The solution set is $\left\{\frac{9}{2}, -\frac{3}{2}\right\}$.**Check**Solve $6x^2 - 21x + 9 = 0$ by completing the square.

$$x = -\frac{7}{3}, -\frac{1}{3}$$

Original equation

Divide each side by 4.

Add $\frac{27}{4}$ to each side.Add $\left(\frac{3}{2}\right)^2$ or $\frac{9}{4}$ to each side.

Factor.

Square Root Property

Add $\frac{3}{2}$ to each side.

Write as two equations.

Simplify.

Think About It!
Compare and contrast the solutions of this equation and the ones in the previous example. Explain.

Sample answer: Both equations have two solutions. This equation has imaginary solutions because I took the square root of a negative number, and the previous equation had real solutions because I took the square root of a positive number.

Example 8 Solve Equations with Imaginary SolutionsSolve $3x^2 - 72x + 465 = 0$ by completing the square.

$$3x^2 - 72x + 465 = 0$$

$$x^2 - 24x + 155 = 0$$

$$x^2 - 24x = -155$$

$$x^2 - 24x + 144 = -155 + 144$$

$$(x - 12)^2 = -11$$

$$x - 12 = \pm\sqrt{-11}$$

$$x - 12 = \pm i\sqrt{11}$$

$$x = 12 \pm i\sqrt{11}$$

$$x = 12 + i\sqrt{11}$$

$$\text{or}$$

$$x = 12 - i\sqrt{11}$$

The solution set is $\left\{12 + i\sqrt{11}, 12 - i\sqrt{11}\right\}$.

Original equation

Divide each side by 3.

Subtract 155 from each side.

Add $\left(\frac{24}{2}\right)^2$ to each side.

Factor.

Square Root Property

 $\sqrt{-1} = i$

Add 12 to each side.

Write as two equations.

Go Online You can complete an Extra Example online.

Interactive Presentation

Solve When a Is Not 1

Solve $4x^2 - 12x - 27 = 0$ by completing the square

Example 7

SWIPE



Students move through the steps to solve a problem by completing the square.

CHECK



Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Example 7 Solve When a Is Not 1**MP** Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expression to reach solutions. In Example 7, students must transform the quadratic equation to apply the method of completing the square.

Questions for Mathematical Discourse

AL Why doesn't the right side of the equation change when dividing each side by 4? $\frac{0}{4} = 0$

OL How do you take the square root of a fraction? **Take the square root of the numerator and divide by the square root of the denominator.**

BL What formula could you use to find c just by looking at the original equation? $\left(\frac{b}{2a}\right)^2$

Example 8 Solve Equations with Imaginary Solutions**MP** Teaching the Mathematical Practices

3 Construct Arguments In the Think About! feature, students will use previously established results to construct arguments about the solutions in Examples 7 and 8.

Questions for Mathematical Discourse

AL Can a be negative? **Yes.** How would the first step change if $a = -3$? **The result would be $x^2 - 24x - 155 = 0$.**

OL Does the positive constant in the original equation guarantee a complex solution? **Explain. No; sample answer: It becomes a negative number when moved to the other side of the equation, it is also affected by the term added when completing the square.**

The solution will only be complex if the right side is negative after $\left(\frac{b}{2}\right)^2$ is added to each side.

BL What is an alternate method that can be used in step 2 to get a constant term of 144 on the left side of the equation? **Subtract 11 from each side.**

DIFFERENTIATE

Language Development **ELL**

Point out that when a quadratic equation has imaginary roots, the graph of its related function does not appear to cross the x -axis, but the word *imaginary* implies the possibility that the graph crosses the x -axis in an imaginary location that can't be seen.



Learn Quadratic Functions in Vertex Form

Objective

Students complete the square in a quadratic function to interpret key features of its graph.

MP Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. Guide students to see what information they can gather about quadratic functions just by looking at the vertex form of the function.

Common Misconception

The squared expression in the vertex form has the structure $(x - h)$, so it is easy for students to get confused about the sign on the value of h . Go over a series of examples of vertex forms where values of h are positive and negative, so students become familiar with the idea of transforming an addition expression into subtracting a negative h .

Example 9 Write Functions in Vertex Form

MP Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. In Example 9, students must transform the function to rewrite it in vertex form.

Questions for Mathematical Discourse

- AL** Why does -1 need to be factored out in the third step? **Vertex form** requires the coefficient of x^2 to be 1.
- OL** Why is it useful to put the function in vertex form? **Sample answer:** It makes it easier to graph because you can quickly see the vertex and the axis of symmetry of the graph.
- BL** When completing the square, why do you add and subtract $\left(\frac{b}{2}\right)^2$?

Sample answer: We cannot change the value of y , so we must add and subtract $\left(\frac{b}{2}\right)^2$ so that we are really adding 0.

Important to Know

As $|a|$ increases, the graph gets narrower. Because the quantity $(x - h)^2$ is multiplied by a , as $|a|$ increases, the corresponding y -value has a greater positive value or lesser negative value. This results in a steeper (and thus narrower) graph.

Learn Quadratic Functions in Vertex Form

When a function is given in standard form, $y = ax^2 + bx + c$, you can complete the square to write it in vertex form.

Key Concept • Vertex Form of a Quadratic Function

Words: The vertex form of a quadratic function is $y = a(x - h)^2 + k$.

Symbolic: Standard Form	Vertex Form
$y = ax^2 + bx + c$	$y = a(x - h)^2 + k$
	The vertex is (h, k) .
Example: Standard Form	Vertex Form
$y = 2x^2 + 12x + 16$	$y = 2(x + 3)^2 - 2$
	The vertex is $(-3, -2)$.

After completing the square and writing a quadratic function in vertex form, you can analyze key features of the function. The vertex is (h, k) and $x = h$ is the equation of the axis of symmetry. The shape of the parabola and the direction that it opens are determined by a . The value of k is a minimum value if $a > 0$ or a maximum value if $a < 0$.

The path that an object travels when influenced by gravity is called a trajectory, and trajectories can be modeled by quadratic functions. The formula below relates the height of the object $h(t)$ and time t , where g is acceleration due to gravity, v is the initial velocity of the object, and h_0 is the initial height of the object.

$$h(t) = -\frac{1}{2}gt^2 + vt + h_0$$

The acceleration due to gravity g is 9.8 meters per second squared or 32 feet per second squared. Problems that involve objects being thrown or dropped are called **projectile motion problems**.

Example 9 Write Functions in Vertex Form

Write $y = -x^2 - 12x - 9$ in vertex form.

$y = -x^2 - 12x - 9$	Original function
$y = [-(x^2 + 12x) - 9]$	Group $ax^2 + bx$.
$y = -(x^2 + 12x) - 9$	Factor out -1 .
$y = -(x^2 + 12x + 36) - 9 + (-9)(-36)$	Complete the square.
$y = -(x + 6)^2 + 27$	Simplify.

Check

Write each function in vertex form.

$$\begin{aligned} \text{a. } y &= x^2 + 8x - 3 & \text{b. } y &= -3x^2 - 6x - 5 \\ y &= (x + 4)^2 - 7 & y &= -3(x + 1)^2 - 2 \end{aligned}$$

Go Online You can complete an Extra Example online.

Lesson 1-5 • Solving Quadratic Equations by Completing the Square 37

Think About It!
What is the minimum value of $y = 2x^2 - 3x^2 - 17$? How do you know that this value is a minimum?

-1. Sample answer: Because a is positive, the graph has a minimum, and the vertex is $(3, -1)$.

Watch Out!
The coefficient of the x^2 -term must be 1 before you can complete the square.

Interactive Presentation



Example 9

TYPE



Students answer a question to determine whether they understand how to write functions in vertex form.

**Think About It!**

How would your equation for the axis of symmetry change if the vertex form of the equation was $y = 3x + 2x^2 - 7$? Justify your argument.

Sample answer: The axis of symmetry would be $x = -2$ because the vertex form of the function can be rewritten as $y = 3(x - (-2))^2 - 2$.

Think About It!

If the firework reaches a height of 241 feet after 3 seconds, what is the height of the firework after 5 seconds? Justify your answer.

241 feet. Sample answer: Because the axis of symmetry is $t = 4$, the height of the firework will be the same after 3 seconds and after 5 seconds.

Study Tip

Vertex: When you interpret the vertex of a function, it is important to also consider the value of a when the function is in vertex or standard form. The value of a will tell you whether the vertex is a maximum or minimum.

Example 10 Determine the Vertex and Axis of Symmetry

Consider $y = 3x^2 - 12x + 5$.

Part A Write the function in vertex form.

$$y = 3x^2 - 12x + 5$$

$$y = (3x^2 - 12x) + 5$$

$$y = 3(x^2 - 4x) + 5$$

$$y = 3(x^2 - 4x + 4) + 5 - 3(4)$$

$$y = 3(x - 2)^2 - 7$$

Original equation

Factor $a(x^2 + bx)$

Complete the square

Simplify

Part B Find the axis of symmetry.

The axis of symmetry is $x = h$ or $x = 2$.

Part C Find the vertex, and determine if it is a maximum or minimum.

The vertex is (h, k) or $(2, -7)$. Because $a > 0$, this is a minimum.

Example 11 Model with a Quadratic Function

PREWRITING If a firework is launched 1 foot off the ground at a velocity of 128 feet per second, write a function for the situation. Then find and interpret the axis of symmetry and vertex.

Step 1 Write the function.

$$h(t) = -\frac{1}{2}gt^2 + v_0t + h_0$$

$$h(t) = -\frac{1}{2}(32)t^2 + 128t + 1$$

$$h(t) = -16t^2 + 128t + 1$$

Function for projectile motion

$$g = 32 \frac{\text{ft}}{\text{s}^2}, v_0 = 128 \frac{\text{ft}}{\text{s}}, h_0 = 1 \text{ ft}$$

Simplify

Step 2 Rewrite the function in vertex form.

$$h(t) = -(16t^2 + 128t) + 1$$

$$h(t) = -(16t^2 + 128t + 64) + 1$$

$$h(t) = -(16t^2 + 128t + 64) + 1$$

$$h(t) = -(16t^2 + 128t + 64) + 1$$

Group $ax^2 + bx$

Factor

Complete the square

Simplify

Step 3 Find and interpret the axis of symmetry.

Because the axis of symmetry divides the function into two equal halves, the firework will be at the same height after 2 seconds as it is after 6 seconds.

Step 4 Find and interpret the vertex.

The vertex is the maximum of the function because $a < 1$. So the firework reached a maximum height of 257 feet after 4 seconds.

Go Online You can complete an Extra Example online.

38 Module 1 • Quadratic Functions

Interactive Presentation

Example 11

TAP

Students tap to see the steps to modeling with a quadratic function.

TYPE

Students answer a question to determine whether they understand modeling with a quadratic function.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 10 Determine the Vertex and Axis of Symmetry**Teaching the Mathematical Practices**

7 Use Structure Help students to use the structure of the function in vertex form to identify key features.

Questions for Mathematical Discourse

- AL** What are two features of the graph that can be found from the vertex form of the equation? **Sample answer:** the vertex and axis of symmetry
- OL** What does a negative value of a indicate about the graph? **The parabola opens down.**
- BL** As the absolute value of a increases, what happens to the parabola? **It becomes narrower.**

Common Error

When a quadratic expression is multiplied by a common factor, remind students to take that factor into account when adding a constant term to complete the square.

Example 11 Model with a Quadratic Function**Teaching the Mathematical Practices**

2 Create Representations Guide students to write a function that models the projectile motion in Example 11. Then use the function to analyze the situation.

Questions for Mathematical Discourse

- AL** What does the vertex of the parabola represent in this situation? **The maximum height of the firework.**
- OL** In **Step 2**, why is the coefficient of the t^2 term negative and the coefficient of the t -term positive? **Sample answer:** The coefficient of the t^2 -term represents the acceleration due to gravity, which pulls in the negative direction (down). The coefficient of the t -term represents the change in height due to the initial velocity, which is in the positive direction (up).
- BL** Why is transforming the standard form of the quadratic equation into the vertex form helpful? **Sample answer:** The vertex form can be used to identify the vertex of the function, which gives the time and height of the highest point in the path of the firework.

Exit Ticket**Recommended Use**

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.



Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–51
2	exercises that use a variety of skills from this lesson	52–66
2	exercises that extend concepts learned in this lesson to new contexts	67–71
3	exercises that emphasize higher-order and critical-thinking skills	72–77

ASSESS AND DIFFERENTIATE

Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, THEN assign:

- Practice Exercises 1–71 odd, 72–77
- Extension: The Golden Quadratic Equations
- Completing the Square and the Quadratic Formula

IF students score 66%–89% on the Checks, THEN assign:

- Practice Exercises 1–75 odd
- Remediation, Review Resources: Factoring Special Products
- Personal Tutors
- Extra Examples 1–10
- Factoring Special Products

IF students score 65% or less on the Checks, THEN assign:

- Practice, Exercises 1–51 odd
- Remediation, Review Resources: Factoring Special Products
- Quick Review Math Handbook: Completing the Square
- Factoring Special Products

Answers

50c. axis of symmetry is $t = 3$; because the axis of symmetry divides the function into two equal halves, the firework will be at the same height after 1 second as it is after 5 seconds. The vertex is (3, 146). This is the maximum value of the function because $a < 0$. So the firework reached a maximum height of 146 feet after 3 seconds.

51a. $h(t) = -4.9(t - 0.427)^2 + 8.391$

51b. $t = 0.427$; points equidistant from the axis of symmetry represent the times when the diver will be at the same height during his dive. Vertex = (0.427, 8.391); Malik reaches a maximum height of about 8.391 meters approximately 0.427 second after he begins his dive.

Practice

Examples 1–5

Solve each equation by using the Square Root Property.

- $x^2 - 18x + 81 = 49$
 $(x-9)^2 = 49$
 $x-9 = \pm 7$
 $x = 2, 16$
- $x^2 + 20x + 100 = 64$
 $(x+10)^2 = 64$
 $x+10 = \pm 8$
 $x = -2, -18$
- $9x^2 - 12x + 4 = 8$
 $(3x-2)^2 = 8$
 $3x-2 = \pm 2\sqrt{2}$
 $x = \frac{2 \pm 2\sqrt{2}}{3}$
- $4x^2 + 4x + 1 = 16$
 $(2x+1)^2 = 16$
 $2x+1 = \pm 4$
 $x = -\frac{5}{2}, -\frac{3}{2}$
- $4x^2 - 28x + 49 = 64$
 $(2x-7)^2 = 64$
 $2x-7 = \pm 8$
 $x = \frac{15}{2}, -\frac{1}{2}$
- $16x^2 + 24x + 9 = 81$
 $(4x+3)^2 = 81$
 $4x+3 = \pm 9$
 $x = \frac{3}{2}, -\frac{3}{2}$
- $25x^2 + 20x + 4 = 75$
 $(5x+2)^2 = 75$
 $5x+2 = \pm 5\sqrt{3}$
 $x = \frac{-2 \pm 5\sqrt{3}}{5}$
- $25x^2 + 40x + 16 = 28$
 $(5x+4)^2 = 28$
 $5x+4 = \pm 2\sqrt{7}$
 $x = \frac{-4 \pm 2\sqrt{7}}{5}$
- $36x^2 + 48x + 16 = 12$
 $(6x+4)^2 = 12$
 $6x+4 = \pm 2\sqrt{3}$
 $x = \frac{-2 \pm \sqrt{3}}{3}$
- $25x^2 - 30x + 9 = 96$
 $(5x-3)^2 = 96$
 $5x-3 = \pm 4\sqrt{6}$
 $x = \frac{3 \pm 4\sqrt{6}}{5}$
- $2x^2 - 20x + 50 = -328$
 $(x-5)^2 = -100$
 $x-5 = \pm 10i$
 $x = 5 \pm 10i$
- $2x^2 - 24x + 72 = -862$
 $(x-6)^2 = -200$
 $x-6 = \pm 10\sqrt{2}i$
 $x = 6 \pm 10\sqrt{2}i$
- $2x^2 + 20x + 50 = -200$
 $(x+5)^2 = -60$
 $x+5 = \pm \sqrt{60}i$
 $x = -5 \pm \sqrt{60}i$
- $x^2 - 2x + 1 = -36$
 $(x-1)^2 = -37$
 $x-1 = \pm \sqrt{37}i$
 $x = 1 \pm \sqrt{37}i$
- $3x^2 - 24x + 48 = -108$
 $(x-4)^2 = -32$
 $x-4 = \pm 4\sqrt{2}i$
 $x = 4 \pm 4\sqrt{2}i$
- $3x^2 - 24x + 48 = -362$
 $(x-4)^2 = -100$
 $x-4 = \pm 10i$
 $x = 4 \pm 10i$

Example 6

Find the value of c that makes each trinomial a perfect square. Then write the trinomial as a perfect square trinomial.

- $x^2 + 30x + c$ 25; $(x+15)^2$
- $x^2 - 14x + c$ 49; $(x-7)^2$
- $x^2 + 24x + c$ 144; $(x+12)^2$
- $x^2 + 5x + c$ $\frac{25}{4}$; $(x+\frac{5}{2})^2$
- $x^2 - 9x + c$ $\frac{81}{4}$; $(x-\frac{9}{2})^2$
- $x^2 - x + c$ $\frac{1}{4}$; $(x-\frac{1}{2})^2$

Examples 5 and 6

Solve each equation by completing the square.

- $2x^2 - 13x + 36 = 0$ 4, 9
- $x^2 + x - 6 = 0$ 2, -3
- $x^2 + 3x - 6 = 0$ $\frac{-3 \pm \sqrt{33}}{2}$
- $x^2 - x - 3 = 0$ $\frac{1 \pm \sqrt{13}}{2}$
- $x^2 - 8x - 65 = 0$ -5, 13

31. When the dimensions of a cube are reduced by 4 inches on each side, the surface area of the new cube is 864 square inches. What were the dimensions of the original cube? 16 in. by 16 in. by 16 in.

Lesson 15 • Solving Quadratic Equations by Completing the Square 39

Examples 7 and 8

Solve each equation by completing the square.

- $2x^2 - 8x - 24 = 0$ 6, -2
- $2x^2 - 3x + 1 = 0$ $\frac{1}{2}, 1$
- $3x^2 - 13x - 7 = 0$ $\frac{13 \pm \sqrt{205}}{6}$
- $25x^2 + 40x - 9 = 0$ $-\frac{4}{5}, -\frac{9}{5}$
- $3x^2 + 7x - 4 = 0$ $-\frac{4}{3}, 1$
- $3x^2 + 2x - 1 = 0$ $-\frac{1}{3}, 1$
- $x^2 - 4x + 12 = 0$ 2, 2
- $2x^2 - 3x + 5 = 0$ $\frac{3 \pm \sqrt{31}}{4}$
- $2x^2 + 5x + 7 = 0$ $-\frac{5 \pm \sqrt{33}}{4}$
- $x^2 - 2x + 3 = 0$ $1 \pm i\sqrt{2}$
- $x^2 - 24 = 0$ $\pm 2\sqrt{6}$
- $x^2 - 2x + 4 = 0$ $1 \pm i\sqrt{3}$

Examples 9 and 10

Write each function in vertex form. Find the axis of symmetry. Then find the vertex, and determine if it is a maximum or minimum.

- $y = x^2 + 2x + 5$ $y = (x+1)^2 + 4$; axis of symmetry: $x = -1$; vertex: $(-1, 4)$; minimum
- $y = x^2 + 6x + 2$ $y = (x+3)^2 - 7$; axis of symmetry: $x = -3$; vertex: $(-3, -7)$; minimum
- $y = -x^2 - 4x + 2$ $y = -(x+2)^2 + 6$; axis of symmetry: $x = -2$; vertex: $(-2, 6)$; maximum
- $y = x^2 - 8x + 5$ $y = (x-4)^2 - 11$; axis of symmetry: $x = 4$; vertex: $(4, -11)$; minimum
- $y = 2x^2 + 4x + 3$ $y = 2(x+1)^2 + 1$; axis of symmetry: $x = -1$; vertex: $(-1, 1)$; minimum
- $y = 3x^2 + 6x + 1$ $y = 3(x+1)^2 - 2$; axis of symmetry: $x = -1$; vertex: $(-1, -2)$; minimum

Example 11

50. FIREWORKS The height of a firework at an amusement park celebration can be modeled by a quadratic function. Suppose the firework is launched from a platform 2 feet off the ground at a velocity of 96 feet per second.

- Write a function to represent this situation. $h(t) = -16t^2 + 96t + 2$
- Rewrite the function in vertex form. $h(t) = -16(t - 3)^2 + 146$
- Find the axis of symmetry and the vertex and interpret their meaning in the context of the situation. See margin.

51. DIVING Malik is participating in a diving championship. For each of his dives, his height above the water can be modeled by a quadratic function. The diving board is 7.5 meters above the water and Malik jumps with a velocity of 4.8 meters per second. Use $h(t) = -\frac{1}{2}gt^2 + vt + h_0$, where $g = 9.8 \frac{m}{s^2}$. a. See margin.

- Write a function in vertex form to represent this situation.
- Find the axis of symmetry and the vertex and interpret their meaning in the context of the situation.

40 Module 1 • Quadratic Functions



Mixed Exercises

PRECISION Solve each equation. Round to the nearest hundredth, if necessary.

42. $4x^2 - 28x + 49 = 5$ 53. $9x^2 + 30x + 25 = 11$ 54. $x^2 + x + \frac{1}{4} = \frac{3}{4}$
 2.38, 4.62 -2.71, -0.56 -1.36, 0.26
55. $x^2 + 12x + 0.56 = 0.91$ 56. $x^2 + 0.7x + 4.025 = 0$ 57. $x^2 - 3.2x = -3.46$
 -1.44, 0.24 -0.35 ± 2i 1.6 ± 0.9i
58. $x^2 - 18x + 11.24 = 2.43$ 59. $0.3x^2 - 0.78x - 5.59 = 0$ 60. $15x^2 - 8.8x + 22 = 2.2$
 0.9 ± 2.83i -1.3 ± 4.09i 4 ± 1.47i

61. FREE FALL A rock falls from the top of a cliff that is 25.8 meters high. Use the formula $h(t) = -\frac{1}{2}gt^2 + v_0t + h_0$, where $g = 9.8 \frac{m}{s^2}$, to write a quadratic function that models the situation. Determine to the nearest tenth of a second the amount of time it takes the rock to strike the ground. Explain your reasoning. **See margin.**

62. REACTION TIME Teta was eating lunch when she saw her friend Joni approach. The room was crowded and Joni had to lift his tray to avoid obstacles. Suddenly, a glass on Joni's lunch tray tipped and fell off the tray. Teta lunged forward and managed to catch the glass just before it hit the ground. The height h in feet of the glass t seconds after it was dropped is given by $h = -16t^2 + 4.5$. If Teta caught the glass when it was six inches off the ground, how long was the glass in the air before she caught it? **0.5 second**

63. INVESTMENTS The amount of money A in an account in which P dollars are invested for 2 years is given by the formula $A = P(1 + r)^2$, where r is the interest rate compounded annually. If an investment of \$800 in the account grows to \$862 two years, at what interest rate was it invested, to the nearest percent? **5%**

64. INVESTMENTS Nyali invested \$1000 in a savings account with interest compounded annually. After two years the balance in the account is \$1210. Use the compound interest formula $A = P(1 + r)^t$ to find the annual interest rate, to the nearest percent. **10%**

Write each function in vertex form. Then find the vertex.

65. $y = x^2 - 8x + 28$ 66. $y = x^2 + 6x + 65$
 $y = (x - 4)^2 + 12$ $y = (x + 3)^2 + 11$
67. $y = x^2 - 20x + 104$ 68. $y = x^2 - 8x + 17$
 $y = (x - 10)^2 + 4(0.4)$ $y = (x - 4)^2 + 1(4.9)$

69. AUDITORIUM SEATING The seats in an auditorium are arranged in a square grid pattern. There are 45 rows and 45 columns of chairs. For a special concert, organizers decide to increase seating by adding n rows and n columns to make a square pattern of seating $45 + n$ seats on a side.
 a. How many seats are added in the expansion? **$n^2 + 90n$**
 b. What is n if organizers wish to add 1000 seats? **10**

Lesson 1.8: Solving Quadratic Equations by Completing the Square 41

70. DECK DESIGN The Rayburns current deck is 12 feet by 12 feet. They decide they would like to expand their deck and maintain its square shape. How much longer will each side need to be for the deck to have an area of 200 square feet? **About 2.94 feet**

71. VOLUME A piece of sheet metal has a length that is three times its width. It is used to make a box with an open top by cutting out 2-inch by 2-inch squares from each corner, then folding up the sides.



- a. Define variables and write a quadratic function that represents the volume of the box in cubic inches.
 $w = \text{width}$, $V(w) = \text{volume}$, $V(w) = 2(3w - 4)(3w - 4) = 6w^2 - 32w + 32$

b. What are the dimensions of the metal sheet that results in a box with a volume of 112 cubic feet? **18 by 54 ft.**

Higher-Order Thinking Skills

72. CONSTRUCTIVE ARGUMENTS Explain why the equation for the axis of symmetry for a quadratic function $y = ax^2 + bx + c$ is $x = -\frac{b}{2a}$. **See margin.**

73. FIND THE ERROR Alonso and Alka are solving $x^2 + 8x - 20 = 0$ by completing the square. Is either of them correct? Explain your reasoning. **Alonso: Alka did not add 16 to each side; she added it only to the left side.**

Alonso	Alka
$x^2 + 8x + 16 = 0$	$x^2 + 8x - 20 = 0$
$(x + 4)^2 = 0$	$x^2 + 8x + 16 = 20$
$x + 4 = 0$	$x^2 + 8x + 16 + 16 = 20 + 16$
$x = -4$	$x^2 + 8x + 32 = 36$
	$(x + 4)^2 = 36$
	$x + 4 = \pm 6$
	$x = -4 \pm 6$

74. REVERSE ENGINEER Solve $x^2 + bx + c = 0$ by completing the square. Your answer will be an expression for x in terms of b and c .
 $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

75. ANALYZE Without solving, determine how many unique solutions there are for each equation. Are they rational, real, or complex? Justify your reasoning. **See margin.**

- a. $(x + 2)^2 = 16$ b. $(x - 2)^2 = 16$ c. $(x - 2)^2 = 16$
 d. $36 - (x - 2)^2 = 16$ e. $36(x + 2)^2 = 0$ f. $(x + 4)^2 = (x + 6)^2$

76. CREATE Write a perfect square trinomial equation in which the linear coefficient is negative and the constant term is a fraction. Then solve the equation.
 Sample answer: $x^2 - \frac{3}{2}x + \frac{9}{16} = \frac{9}{16} - \frac{9}{16}$

77. WRITE Explain what it means to complete the square. Describe each step. Completing the square is rewriting one side of a quadratic equation in the form of a perfect square. Once in this form, the equation can be solved by using the Square Root Property.

Answers

61. $h(t) = -4.9t^2 + 25.8$; when the rock hits the ground, its height above the ground will be zero, so the time can be determined by setting $-4.9t^2 + 25.8$ equal to zero and solving for t : $-4.9t^2 + 25.8 = 0 \rightarrow 25.8 = 4.9t^2 \rightarrow t^2 = \frac{25.8}{4.9} \rightarrow t = 2.3$ seconds.

72. Sample answer: You can complete the square to write $y = ax^2 + bx + c$ in vertex form, $y = a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}$. So, $h = -\frac{b}{2a}$ and the equation of the axis of symmetry is $x = -\frac{b}{2a}$.

75a. 2; rational; 16 is a perfect square, so $x + 2$ and x are rational.

75b. 2; rational; 16 is a perfect square, so $x - 2$ and x are rational.

75c. 2; complex; if the opposite of a square is positive, the square is negative. The square root of a negative number is complex.

75d. 2; real; the square must equal 20. Since that is positive but not a perfect square, the solutions will be real but not rational.

75e. 1; rational; the expression must be equal to 0 and only -2 makes the expression equal to 0.

75f. 1; rational; the expressions $(x + 4)$ and $(x + 6)$ must either be equal or opposites. No value makes them equal, -5 makes them opposites. The only solution is -5 .

Using the Quadratic Formula and the Discriminant

LESSON GOAL

Students use the Quadratic Formula and discriminant to solve quadratic equations and determine the number of real roots.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Develop:**

Using the Quadratic Formula

- Real Roots, c Is Positive
- Real Roots, c Is Negative
- Complex Roots

 **Explore:** The Discriminant

 **Develop:**

Using the Discriminant

- The Discriminant, Real Roots
- The Discriminant, Complex Roots

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources

Remediation: Solving Quadratic Equations by Using the Quadratic Formula

Extension: Sum and Product of Roots



Language Development Handbook

Assign page 6 of the *Language Development Handbook* to help your students build mathematical language related to using the quadratic formula and discriminant to solve quadratic equations.

 You can use the tips and suggestions on page T6 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra

Standards for Mathematical Content:

N.CN.7 Solve quadratic equations with real coefficients that have complex solutions.

N.CN.8 Extend polynomial identities to the complex numbers.

A.SSE.1b Interpret complicated expressions by viewing one or more of their parts as a single entity.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

4 Model with mathematics.

Coherence

Vertical Alignment

Previous

Students solved quadratic equations by completing the square.

N.CN.7, F.IF.8a

Now

Students solve quadratic equations with the Quadratic Formula and use the discriminant to determine the number of real roots.

N.CN.7, A.SSE.1b

Next

Students will graph and solve quadratic inequalities.

A.CED.1, A.CED.3

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

 **Conceptual Bridge** In this lesson, students develop an understanding of the Quadratic Formula. They build fluency by using this formula to solve equations, and they apply their understanding by using the formula by solving real-world problems.

Mathematical Background

Any quadratic equation written in the form $ax^2 + bx + c = 0$, where $a \neq 0$, can be solved using the Quadratic Formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

where $a \neq 0$.



Interactive Presentation

Warm Up

Match each quadratic equation with the best method for solving it. Be prepared to justify your answers.

Equation	Method
1. $x^2 - 6x + 9 = 16$	A. Completing the square
2. $x^2 + 2x = 7$	B. factoring
3. $x^2 - 7x + 12 = 0$	C. graphing
4. $x^2 - 1 = 0$	D. Square Root Property

Show Answers

Warm Up

Launch the Lesson

When people go bungee jumping, the relationship between the time from their jump to their height can be modeled by a quadratic function. When designing a bungee jump, an organization uses this model to ensure safety. If 0 feet is the height where a jumper would hit the ground, then the organization can use the discriminant to determine whether a bungee jumper's arc would drop put them in jeopardy.



Launch the Lesson

Vocabulary

Expand All Collapse All

discriminant

In the Quadratic Formula, the expression under the radical sign.

- How do you decide the best method to solve a quadratic equation?
- How can you solve a quadratic equation with the square root?
- How do you find negative solutions for quadratic equations?
- What does the discriminant tell you about the solutions of the quadratic equation?
- What is the relationship between factors, roots, zeros, solutions, and intercepts?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- evaluating methods for solving quadratic equations

Answers:

- D
- A
- B
- C

Launch the Lesson



Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will learn how the discriminant can be used to determine whether a bungee jumper is in danger.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statement that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then discuss the questions below with the class.



Explore The Discriminant

Objective

Students use a graphing calculator to explore the discriminant.



Teaching the Mathematical Practices

3 Construct Arguments In the Explore, students will use stated assumptions, definitions, and previously established results to construct an argument about the discriminant.

5 Analyze Graphs Help students analyze the graphs they have generated using graphing calculators to draw conclusions about the number of real zeros of each function.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

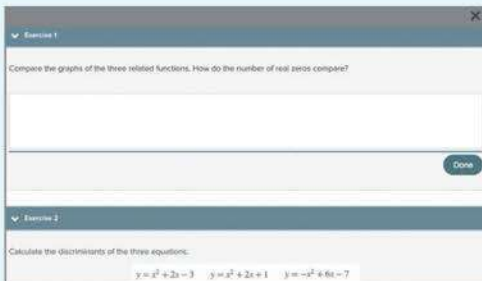
Students will complete guiding exercises throughout the Explore activity. The students calculate the discriminants of quadratic equations and observe the relationship between the values of the discriminant and roots of the equation. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore



Explore

TAP



Students tap to select a calculator to help them calculate the discriminant.

TYPE



Students complete the calculations to compare the discriminants of the three equations.



Interactive Presentation

INQUIRY How does the discriminant of a quadratic equation relate to its roots?

Submit

Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore The Discriminant (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What part of the quadratic formula is the discriminant? **Sample answer:** The discriminant is the expression inside the radical.
- How could you use the discriminant to help you sketch a graph?
Sample answer: You can use the discriminant to tell you how many rational zeros the graph will have. Then you know how many times the graph should cross the x -axis, including having a graph that does not cross the x -axis.

Inquiry

How does the discriminant of a quadratic equation relate to its roots?

Sample answer: If the discriminant is positive, then the roots are real. If the discriminant is a perfect square, then the roots are rational, otherwise they are irrational. If the discriminant is 0 then the equation has one real root. If the discriminant is negative, then the roots are complex.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Using the Quadratic Formula

Objective

Students solve quadratic equations by using the Quadratic Formula.

MP Teaching the Mathematical Practices

6 Use Precision In this lesson, students learn how to calculate solutions of quadratic equations accurately and efficiently and to express solutions with a degree of precision appropriate to the problem.

DIFFERENTIATE

Reteaching Activity AL ELL

IF students substitute values into the Quadratic Formula incorrectly, THEN encourage students to write down the values of a , b , and c from the standard form of the quadratic equation before they begin substituting them into the formula.

Example 1 Real Roots, c Is Positive

MP Teaching the Mathematical Practices

4 Interpret Mathematical Results In Example 1, point out that to solve the problem, students should interpret their mathematical results in the context of the problem.

Questions for Mathematical Discourse

AL How do you identify a , b , and c ? a is the coefficient of t^2 , b is the coefficient of t , and c is the constant term.

OL Why is -2 seconds not a viable solution? **Sample answer:** Time cannot be negative.

BL If you complete the square to solve $ax^2 + bx + c$, what is the result? **The Quadratic Formula**

Go Online

- F Ind additional teaching notes.
- V View performance reports of the Checks.
- A Assign or present an Extra Example.

Lesson 1-6

Using the Quadratic Formula and the Discriminant

Learn Using the Quadratic Formula

To solve any quadratic equation, you can use the Quadratic Formula.

Key Concept • Quadratic Formula

The solutions of a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the following formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Go Online You can see how the Quadratic Formula is derived.

Example 1 Real Roots, c Is Positive

CONTEST At the World Championship Pumpkin Chunkin contest in Bridgeville, Delaware, pumpkins are launched hundreds of yards. The path of a pumpkin can be modeled by $h = -4.9t^2 + 11.7t + 42$, where h is the height and t is the number of seconds after launch.

Part A Use the Quadratic Formula to solve $0 = -4.9t^2 + 11.7t + 42$.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-11.7 \pm \sqrt{(11.7)^2 - 4(-4.9)(42)}}{2(-4.9)}$$

$$= \frac{-11.7 \pm \sqrt{136.89 + 823.2}}{-9.8}$$

$$= \frac{-11.7 \pm \sqrt{960.09}}{-9.8}$$

$$= \frac{-11.7 \pm \sqrt{960.09}}{-9.8} \text{ or } \frac{-11.7 - \sqrt{960.09}}{-9.8}$$

Quadratic Formula

$$a = -4.9, b = 11.7, c = 42$$

Square and multiply.

Add.

Multiply by $-\frac{1}{2}$.

The approximate solutions are 4.4 seconds and -2.0 seconds.

Part B Interpret the roots.

The negative root does not make sense in this context because the pumpkin launches at 0 seconds. The pumpkin lands after 4.4 seconds.

Check

DIVING A diver jumps from a diving board that is 10 feet high, and she wants to figure out how far from the board she is before she enters the water. Her arc can be modeled by $y = -4.9x^2 + 2.5x + 10$, where y is her height in meters and x is time in seconds.

Part A Solve $0 = -4.9x^2 + 2.5x + 10$. $\frac{2.5 \pm \sqrt{250.25}}{9.8}$, $\frac{2.5 \pm \sqrt{250.25}}{9.8}$

Part B Interpret the roots.

The diver enters the water after approximately $\frac{7}{9}$ seconds. 1.7

Today's Goals

- Solving quadratic equations by using the Quadratic Formula.
- Determining the number and type of roots of a quadratic equation.

Today's Vocabulary

discriminant

Think About It!

Why are the roots not at 0 and 4.4 seconds, when the pumpkin is launched and when it lands?

Sample answer: The pumpkin was launched from a starting height of 42 feet, so it was not at 0 feet when $t = 0$.

Go Online

You can complete an Extra Example online.

Lesson 1-6 • Using the Quadratic Formula and the Discriminant 43

Interactive Presentation

The screenshot shows a step-by-step derivation of the Quadratic Formula. It starts with a general quadratic equation $ax^2 + bx + c = 0$ and shows how to complete the square to arrive at the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The steps include dividing by a , moving the constant term, adding the square of half the coefficient of x , and then taking the square root of both sides.

Learn

TYPE



Students answer questions to determine whether they understand using the Quadratic Formula.

**Example 2** Real Roots, c Is NegativeSolve $x^2 + 4x - 17 = 0$ by using the Quadratic Formula.

$$\begin{aligned}
 x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Quadratic Formula} \\
 &= \frac{-4 \pm \sqrt{4^2 - 4(1)(-17)}}{2(1)} && (a = 1, b = 4, c = -17) \\
 &= \frac{-4 \pm \sqrt{84}}{2} && \text{Simplify.} \\
 &= \frac{-4 \pm \sqrt{4 \cdot 21}}{2} && \text{Product Property of Square Roots} \\
 &= \frac{-4 \pm 2\sqrt{21}}{2} && \sqrt{4} = 2 \\
 &= -2 \pm \sqrt{21} && \text{Divide the numerator and denominator by 2.}
 \end{aligned}$$

CheckSolve $3x^2 - 5x - 1 = 0$ by using the Quadratic Formula.

$$x = \frac{5 \pm \sqrt{37}}{6}$$

Example 3 Complex RootsSolve $5x^2 + 8x + 11 = 0$ by using the Quadratic Formula.

$$\begin{aligned}
 x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Quadratic Formula} \\
 &= \frac{-8 \pm \sqrt{8^2 - 4(5)(11)}}{2(5)} && (a = 5, b = 8, c = 11) \\
 &= \frac{-8 \pm \sqrt{-356}}{10} && \text{Simplify.} \\
 &= \frac{-8 \pm \sqrt{4 \cdot (-89)}}{10} && \text{Product Property of Square Roots} \\
 &= \frac{-8 \pm 2i\sqrt{89}}{10} && \text{Write as a complex number.} \\
 &= \frac{-4 \pm i\sqrt{89}}{5} && \text{Divide the numerator and denominator by 2.}
 \end{aligned}$$

CheckSolve $9x^2 - 3x + 18 = 0$ by using the Quadratic Formula.

$$x = \frac{1 \pm i\sqrt{71}}{6}$$

Go Online You can complete an Extra Example online.

44 Module 1 • Quadratic Functions

Interactive Presentation

Real Roots, c Is Negative

Solve $x^2 + 4x - 17 = 0$ by using the Quadratic Formula.

Example 2

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Real Roots, c Is Negative**MP Teaching the Mathematical Practices**

1 Check Answers Mathematically proficient students continually ask themselves, “Does this make sense?” Encourage students to check their answers in Example 2 by using a different method.

Questions for Mathematical Discourse

- AL** Why can you divide the numerator and denominator by 2 in the last step? **2** is the common factor of the numerator and the denominator.
- OL** How does a negative value of c and a positive value of a affect the roots? **Sample answer:** The radicand will be positive; thus, the roots will be real.
- BL** How would a negative value of b affect the substitution in step 2? **Sample answer:** $-b$ would result in a positive value. The value of b^2 would remain the same.

Important to KnowStudents may forget to include the sign when c is negative. As students work through Example 2, stress the importance maintaining signs.**Example 3** Complex Roots**MP Teaching the Mathematical Practices**

6 Use Precision In Example 3, students will calculate accurately and efficiently using the quadratic formula. They will express their numerical answers exactly using radicals and complex numbers.

Questions for Mathematical Discourse

- AL** Why are the roots of the equation complex? **The radicand in the Quadratic Formula is negative.**
- OL** How can you write the solution as a complex number in $a + bi$ form? **Write each fraction separately.**
- BL** Why is it not possible to have one complex root and one real root? **Sample answer:** If one root has real and imaginary terms, the other will also have real and imaginary terms because they occur in complex conjugate pairs.

Important to KnowRemind students that they can also write their final answer in the standard form of a complex number, $a + bi$, where both a and b are real numbers.



Essential Question Follow-Up

Students have explored solving quadratic equations.

Ask:

How do you know what method to use when solving a quadratic equation? **Sample answer:** If the equation has terms that you know are easily factorable, you could solve by factoring. If the equation has more complex terms, you could solve by using the Quadratic Formula, completing the square, or by graphing. You could also use one method to solve and a second method to check your answer.

Learn Using the Discriminant

Objective

Students determine the number and type of roots of a quadratic equation by using the discriminant.

MP Teaching the Mathematical Practices

3 Analyze Cases Work with students to analyze at the cases of the discriminant. Encourage students to familiarize themselves with all of the cases.

Common Error

Because there both $-b$ and b^2 are used in the Quadratic Formula, it is easy for beginning students to confuse the locations of these two uses of the b coefficient.

DIFFERENTIATE

Language Development Activity AL EL

Have students research the root words that form the words quadratic and discriminant. Discuss how the root words relate to the mathematical meanings of quadratic and discriminant.

Explore The Discriminant

Online Activity Use graphing technology to complete the Explore.

INQUIRY How does the discriminant of a quadratic equation relate to its roots?

Learn Using the Discriminant

In the Quadratic Formula, the **discriminant** is the expression under the radical sign, $b^2 - 4ac$. The value of the discriminant can be used to determine the number and type of roots of a quadratic equation.

Key Concept • Discriminant

Consider $ax^2 + bx + c = 0$, where a , b , and c are rational numbers and $a \neq 0$.

Value of Discriminant	Type and Number of Roots	Example of Graph of Related Function
$b^2 - 4ac > 0$ $b^2 - 4ac$ is a perfect square	2 real, rational roots	
$b^2 - 4ac < 0$ $b^2 - 4ac$ is not a perfect square	2 real, irrational roots	
$b^2 - 4ac = 0$	1 real rational root	
$b^2 - 4ac < 0$	2 complex roots	

Go Online You can complete an Extra Example online.

Lesson 1-6 • Using the Quadratic Formula and the Discriminant 45

Talk About It
Why are the roots of a quadratic equation complex if the discriminant is negative?

Sample answer: If the discriminant is negative, then the radical expression in the Quadratic Formula will include the imaginary unit $i = \sqrt{-1}$. Thus, the roots will be complex numbers because they contain i .

Interactive Presentation

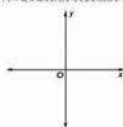
Using the Discriminant

ACT In Quadratic Formula, the discriminant is the expression under the radical sign, $b^2 - 4ac$. The value of the discriminant can be used to determine the number and type of roots of a quadratic equation.

Key Concept • Discriminant

Tip of each button to learn more about the types of roots associated with different values of the discriminant.

Consider $ax^2 + bx + c = 0$, where a , b , and c are rational numbers and $a \neq 0$.



Learn

TAP



Students tap on each button to see different cases of the discriminant.



Think About It!
Is it possible for a quadratic equation to have zero real or complex roots?

No; sample answer: If the discriminant is 0, then there is exactly one solution. If it is not zero, there are two solutions.

Example 4 The Discriminant, Real RootsExamine $2x^2 - 10x + 7 = 0$.**Part A** Find the value of the discriminant for $2x^2 - 10x + 7 = 0$.

$$\begin{aligned} a &= 2 & b &= -10 & c &= 7 \\ b^2 - 4ac &= (-10)^2 - 4(2)(7) \\ &= 100 - 56 \\ &= 44 \end{aligned}$$

Part B Describe the number and type of roots for the equation.

The discriminant is nonzero, so there are two roots. The discriminant is positive and not a perfect square, so the roots are irrational.

CheckExamine $2x^2 + 8x + 8 = 0$.**Part A** Find the value of the discriminant for $2x^2 + 8x + 8 = 0$.

$$b^2 - 4ac = 7, 0$$

Part B Describe the number and type of roots for the equation.There is/are $\frac{7}{2}$ root(s).
rational**Example 5** The Discriminant, Complex RootsExamine $-5x^2 + 10x - 15 = 0$.**Part A** Find the value of the discriminant for $-5x^2 + 10x - 15 = 0$.

$$\begin{aligned} a &= -5 & b &= 10 & c &= -15 \\ b^2 - 4ac &= (10)^2 - 4(-5)(-15) \\ &= 100 - 300 \\ &= -200 \end{aligned}$$

Part B Describe the number and type of roots for the equation.

The discriminant is nonzero, so there are two roots. The discriminant is negative, so the roots are complex.

CheckExamine $10x^2 - 4x + 7 = 0$.**Part A** Find the value of the discriminant for $10x^2 - 4x + 7 = 0$.

$$b^2 - 4ac = -264$$

Part B Describe the number and type of roots for the equation.There is/are $\frac{1}{2}$ root(s).
complex**Go Online** You can complete an Extra Example online.

Interactive Presentation

The Discriminant, Real Roots

Examine $2x^2 - 10x + 7 = 0$.

Part A
Find the value of the discriminant for $2x^2 - 10x + 7 = 0$.

$a = \square$ $b = \square$ $c = \square$

$b^2 - 4ac = (\square)^2 - 4(\square)(\square)$

$= 100 - 56$

$= 44$

Example 4

CHECK



Students complete the Check online to determine whether they are ready to move on.

Example 4 The Discriminant, Real Roots**MP Teaching the Mathematical Practices**

2 Attend to Quantities In Example 4, students must note the meaning of the discriminant. They use the meaning of the discriminant to determine the number and type of roots of the equation.

Questions for Mathematical Discourse

AL What is the value of the discriminant if the quadratic equation has exactly one real rational root? **0**

OL What type(s) of roots exist if the discriminant is a nonzero perfect square? **rational roots**

BL What does the discriminant tell you about the solutions of a quadratic equation that looking at the graph cannot always tell you? **Sample answer: It can distinguish between having rational and irrational roots.**

Example 5 The Discriminant, Complex Roots**MP Teaching the Mathematical Practices**

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. In Example 5, students use the discriminant from the Quadratic Formula to determine the number and type of roots of the equation.

Questions for Mathematical Discourse

AL What about the value of the discriminant tells you the solution must be complex? **It has a negative value.**

OL When will the discriminant have a negative value? **when $4ac > b^2$**

BL If the discriminant is the negative of a perfect square, what do you know about the solution? **The coefficient of the complex term will be a rational number.**

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–35
2	exercises that use a variety of skills from this lesson	36–52
3	exercises that emphasize higher-order and critical-thinking skills	53–58

ASSESS AND DIFFERENTIATE

 Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, THEN assign:

- Practice Exercises 1–59 odd, 60–66
- Extension: Sum and Product of Roots
-  Completing the Square and the Quadratic Formula

BL

IF students score 66%–89% on the Checks, THEN assign:

- Practice Exercises 1–65 odd
- Remediation, Review Resources: Solving Quadratic Equations by Using the Quadratic Formula
- Personal Tutors
- Extra Examples 1–5
-  Quadratic Equations

OL

IF students score 65% or less on the Checks, THEN assign:

- Practice, Exercises 1–35 odd
- Remediation, Review Resources: Solving Quadratic Equations by Using the Quadratic Formula
- Quick Review Math Handbook: The Quadratic Formula and the Discriminant
-  Quadratic Equations

AL

Practice

Example 1

Solve each equation by using the Quadratic Formula.

1. $x^2 + 8x + 15 = 0$ $-3, -5$

2. $x^2 - 18x + 72 = 0$ $6, 12$

3. $12x^2 - 22x + 6 = 0$ $\frac{1}{3}, \frac{1}{2}$

4. $4x^2 - 6x = -2$ $\frac{1}{2}, \frac{1}{4}$

5. $x^2 + 8x + 5 = 0$ $-4 \pm \sqrt{11}$

6. $-8x^2 + 4x = -5$ $\frac{1 \pm \sqrt{17}}{16}$

7. FOOTBALL A quarterback throws a football to a receiver. The path of a football can be modeled by the quadratic function $h = -16t^2 + 45t + 4$, where h is the height in feet and t is the number of seconds after the football is thrown. If the ball is overthrown and the receiver does not touch the ball, how long will it take the football to hit the ground? 2.9 s

Examples 2 and 3

Solve each equation by using the Quadratic Formula.

8. $x^2 + 2x - 35 = 0$ $5, -7$

9. $4x^2 + 18x - 5 = 0$ $-\frac{3}{4}, -\frac{5}{2}$

10. $2x^2 - x - 15 = 0$ $3, -\frac{5}{2}$

11. $3x^2 + 5x = 2$ $-\frac{5}{6}, -\frac{2}{3}$

12. $x^2 + x - 8 = 0$ $\frac{-1 \pm \sqrt{33}}{2}$

13. $8x^2 + 5x - 1 = 0$ $\frac{-5 \pm \sqrt{101}}{16}$

14. $x^2 - x - 5 = 0$ $\frac{1 \pm \sqrt{21}}{2}$

15. $16x^2 - 24x - 25 = 0$ $\frac{3 \pm \sqrt{149}}{8}$

16. $x^2 - 6x + 21 = 0$ $3 \pm 2\sqrt{3}$

17. $x^2 + 25 = 0$ $\pm 5i$

18. $3x^2 + 36 = 0$ $\pm 2\sqrt{3}i$

19. $8x^2 - 4x + 1 = 0$ $\frac{1 \pm \sqrt{5}}{4}$

20. $2x^2 + 2x + 3 = 0$ $\frac{-1 \pm \sqrt{17}}{2}$

21. $x^2 - 14x + 53 = 0$ $7 \pm 2i$

22. $4x^2 + 2x + 9 = 0$ $\frac{-1 \pm \sqrt{145}}{8}$

23. $3x^2 - 6x + 11 = 0$ $\frac{1 \pm 2\sqrt{2}}{3}$

Lesson 14 • Using the Quadratic Formula and the Discriminant 47

Examples 4 and 5

Find the value of the discriminant for each quadratic equation. Then describe the number and type of roots for the equation.

24. $x^2 - 8x + 16 = 0$

0; 1 rational root

26. $3x^2 - 2x = 0$

4; 2 rational roots

28. $15x^2 - 6 = 0$

120; 2 irrational roots

30. $x^2 + 8x + 13 = 0$

12; 2 irrational roots

32. $x^2 - 2x - 17 = 0$

72; 2 irrational roots

34. $x^2 - x + 1 = 0$

-3; 2 complex roots

25. $x^2 - 16 = 0$

225; 2 rational roots

27. $20x^2 + 7x - 3 = 0$

289; 2 rational roots

29. $x^2 - 6 = 0$

24; 2 irrational roots

31. $15x^2 - x - 1 = 0$

21; 2 irrational roots

33. $x^2 + 49 = 0$

-196; 2 complex roots

35. $2x^2 - 3x = -2$

-7; 2 complex roots

Mixed Exercises

REGULARITY Describe the discriminant of the related equation of each graph. Then determine the type and number of roots.



$b^2 - 4ac < 0$
2 complex roots



$b^2 - 4ac = 0$
2 real rational or irrational roots



$b^2 - 4ac > 0$
1 real rational root

Use the discriminant to describe the number and type of roots for each equation. Then solve each equation by using the Quadratic Formula.

39. $4x^2 - 4x + 17 = 0$

2 complex roots; $\frac{1 \pm \sqrt{65}}{4}$

40. $8x - 1 = 4x^2$

2 irrational roots; $\frac{2 \pm \sqrt{17}}{4}$

41. $7x^2 - 5x = 0$

2 rational roots; $0, \frac{5}{7}$

42. $x^2 + 10x + 24 = 0$

2 rational roots; $-4, -6$

43. $x^2 - 11x + 24 = 0$

2 rational roots; $3, 8$

44. $12x^2 + 9x + 1 = 0$

2 irrational roots; $-\frac{3 \pm \sqrt{17}}{24}$

45. $3x^2 - 15x + 10 = 0$

2 rational roots; $4, \frac{5}{3}$

46. $x^2 - \frac{8}{5}x + \frac{2}{5} = 0$

2 complex roots; $\frac{4 \pm \sqrt{10}}{5}$

47. $2x^2 + 10x + 11 = 0$

2 irrational roots; $-\frac{5 \pm \sqrt{23}}{2}$

48. Module 1 • Quadratic Functions



Answers

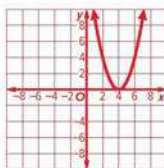
52. Set $h(t) = 20$; use the Quadratic Formula: $t = \frac{v \pm \sqrt{v^2 - 4(16)(20)}}{32}$. The ball fails to hit the target if the solutions are complex, when $v - 1280 < 0$. If the initial velocity is less than approximately 35.8 feet per second, the ball will not hit the target.

55a. Always; sample answer: When a and c are opposite signs, then ac will always be negative and $-4ac$ will always be positive. Because b^2 will always be positive, then $b^2 - 4ac$ represents the addition of two positive values, which will never be negative. Hence, the discriminant can never be negative and the solutions can never be imaginary.

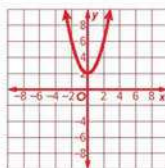
55b. Sometimes; sample answers: The roots will only be irrational if $b^2 - 4ac$ is not a perfect square.

56. Sample graphs shown.

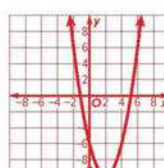
56a. 1 rational root



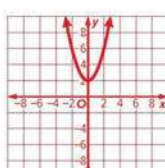
56b. 2 complex roots



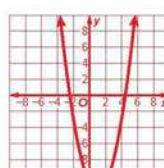
56c. 2 real roots



56d. 2 complex roots



56e. 2 rational roots



48. USE A MODEL The height $h(t)$ in feet of an object t seconds after it is propelled up from the ground with an initial velocity of 60 feet per second is modeled by the equation $h(t) = -16t^2 + 60t$, where will the object be at a height of 56 feet?
125; 2.4

49. SPORTS Natasha Litovskaya set the women's shot put world record of 22.63 meters. Her throw can be modeled by $h = -4.9t^2 + 13.7t + 1.6$, where t is time in seconds and h is the height in meters. About how long was the shot in the air?
2.9 s

50. STOPPING DISTANCE A car's stopping distance d is the sum of the distance traveled during the time it takes the driver to react and the distance traveled while braking. This is represented as $d = vt + \frac{v^2}{2g}$, where v is the initial velocity in feet per second, t is the driver's reaction time in seconds, g is the coefficient of friction, and d is acceleration due to gravity. Use $g = 32 \text{ ft/s}^2$.

a. Assume $\mu = 0.8$ for rubber tires on dry pavement and the average reaction time of 1.5 seconds. Copy and complete the table. Round to the nearest tenth.

Velocity, v (ft/s)	15	40	55	70	80
Stopping Distance, d (ft)	26.9	91.25	145.6	200.7	245

b. Make different assumptions. Copy and complete the table. Round to the nearest tenth. coefficient of friction $\mu =$ _____ Sample answer: 0.9 for rubber

Velocity, v (ft/s)	15	40	55	70	80
Stopping Distance, d (ft)	29.5	91.25	177.0	200.7	245

c. How did your different assumptions affect the data you found? Interpret the information in the context of the situation.
Sample answer: Using the coefficient of friction for wet pavement made the stopping distance for a given velocity longer. At a given velocity, a car will take a longer distance to stop on wet pavement than it does on dry pavement.

51. GEOMETRY A rectangular box has a square base and a height that is one more than 3 times the length of a side of the base. If the sides of the base are each increased by 2 inches and the height is increased by 3 inches, the volume of the box increases by 531 cubic inches. Define a variable and write an equation to represent the situation. Then find the dimensions of the original box.

$$V_{\text{new}} = V_{\text{original}} + 531 \text{ in}^3$$



$x =$ the length of a side of the base, $(x + 2)(3x + 4) = x^2(3x + 1) + 531$; by 5 in, by 16 in.

52. GAMES A carnival game has players hit a pad with a large rubber mallet. This fires a ball up a 20-foot vertical chute toward a target at the top. A prize is awarded if the ball hits the target. Explain how to find the initial velocity in feet per second for which the ball will hit the target. Assume the height of the ball can be modeled by the function $h(t) = -16t^2 + vt$, where v is the initial velocity.
See margin.

Higher-Order Thinking Skills

53. WHICH ONE DOESN'T BELONG? Use the discriminant to determine which of these equations is different from the others. Explain your reasoning.

$$x^2 - 3x - 40 = 0 \quad 12x^2 - x - 6 = 0 \quad 12x^2 + 2x - 4 = 0 \quad 7x^2 + 6x + 2 = 0$$

$7x^2 + 6x + 2 = 0$ is different from the other 3 equations because it has 2 complex roots, where the other 3 equations each have 2 rational roots.

54. FIND THE ERROR Tasha and Jonathan are determining the number of solutions of $3x^2 - 5x + 7 = 0$. Is either of them correct? Explain your reasoning.

Tasha

$$3x^2 - 5x + 7 = 0$$

$$b^2 - 4ac = (-5)^2 - 4(3)(7)$$

$$= -61$$

Since the discriminant is negative, there are no real solutions.

Jonathan

$$3x^2 - 5x + 7 = 0$$

$$b^2 - 4ac = 7 - 0 = 7$$

$$7^2 - 4(3)(-5) = 121$$

Since the discriminant is positive, there are two real roots.

Jonathan, you must first write the equation in the form $ax^2 + bx + c = 0$ to determine the values of a , b , and c . Therefore, the value of c is -7 , not 7.

55. ANALYZE Determine whether each statement is sometimes, always, or never true. Explain your reasoning. a, b. See margin.

- In a quadratic equation in standard form, if a and c have different signs, then the solutions will be real.
- If the discriminant of a quadratic equation is greater than 1, the two roots are real irrational numbers.

56. CREATE Sketch the corresponding graph and state the number and type of roots for each of the following. a–d. See margin.

- $x^2 - 40x = 0$
 - A quadratic function in which b never equals zero.
 - A quadratic function in which $h(x) = 0$ and $g(x) = 0$ if $x \neq 0$.
 - The discriminant is less than zero.
- e. a and d are both solutions and can be represented as fractions.

57. PERSIST Find the values of m in the quadratic equation $x^2 + x + m + 1 = 0$ such that it has one solution. -0.25

58. WRITE Describe three different ways to solve $x^2 - 2x - 15 = 0$. Which method do you prefer, and why? See margin.

Quadratic Inequalities

LESSON GOAL

Students graph and solve quadratic inequalities.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Explore:** Graphing Quadratic Inequalities

 **Develop:**

Graphing Quadratic Inequalities

- Graph a Quadratic Inequality ($<$ or $\leq$)
- Graph a Quadratic Inequality ($>$ or $\geq$)

Solving Quadratic Inequalities

- Solve a Quadratic Inequality ($<$ or $\leq$) by Graphing
- Solve a Quadratic Inequality ($>$ or $\geq$) by Graphing
- Solve a Quadratic Inequality Algebraically

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources	AL	DL	EL	EL
Remediation: Graphing Inequalities in Two Variables	●	●		●
Extension: Graphing Intersections of Quadratic Inequalities		●	●	●

Language Development Handbook

Assign page 7 of the *Language Development Handbook* to help your students build mathematical language related to graphing and solving quadratic inequalities.

ELL You can use the tips and suggestions on page 77 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra

Standards for Mathematical Content:

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

A.CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

4 Model with mathematics.

Coherence

Vertical Alignment

Previous

Students solved quadratic equations by graphing.
A.CED.2, F.IF.4

Now

Students graph and solve quadratic inequalities.
A.CED.1, A.CED.3

Next

Students will solve systems of linear and quadratic equations.
A.REI.11

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

 **Conceptual Bridge** In this lesson, students extend their understanding of quadratic equations to quadratic inequalities. They build fluency by solving quadratic inequalities, and they apply their understanding by solving real-world problems.

Mathematical Background

Graph quadratic inequalities in two variables using the same techniques as when graphing linear inequalities in two variables. Graph the related function. Use a dashed line if the symbol is $<$ or $>$. Test a point inside the parabola. If it is a solution, shade inside the parabola. If it is not a solution, shade outside the parabola.



Interactive Presentation

Warm Up

Determine whether each ordered pair is a solution of $y > -x^2 + x - 5$.

1. $(0,0)$
2. $(1, -7)$
3. $(1,2)$
4. $(3, -9)$
5. $(-2, -11)$

Show Answers

Warm Up

Launch the Lesson

Videos created with more pixels are higher quality. A digital video labeled 1080p has dimensions of 1920 × 1080 pixels, assuming that the ratio between width and height is 16:9. If a filmmaker wants to create a digital video that is of higher quality than 1080p while maintaining the same ratio, she can use a quadratic inequality to represent the possible heights and widths of the video that she could use.

area = 2,073,600 pixels

Launch the Lesson

Vocabulary

Expand All Collapse All

▼ quadratic inequality

An inequality of the form $y > ax^2 + bx + c$, $y \geq ax^2 + bx + c$, $y < ax^2 + bx + c$, or $y \leq ax^2 + bx + c$.

- How can you determine which region to shade when graphing a quadratic inequality in two variables?
- For which inequality symbol(s) would the roots of the related quadratic equation be included in the solution of a quadratic inequality in one variable?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- determining whether an ordered pair is a solution

Answers:

- yes
- no
- yes
- yes
- no

Launch the Lesson



Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will learn how a quadratic inequality can be used to determine the dimensions of a high quality video that maintains a given aspect ratio.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then discuss the questions below with the class.



Explore Graphing Quadratic Inequalities

Objective

Students explore the graphs of quadratic inequalities.



Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to complete the exercises in the Explore, students will need to use the sketch. Work with students to explore and deepen their understanding of quadratic inequalities.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

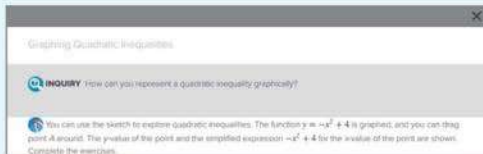
What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will use a sketch to examine the value of y at various points on the curve and coordinate plane to draw conclusions about graphing quadratic inequalities. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore



Explore

WEB SKETCHPAD



Students use a sketch to explore the graph of a quadratic inequality.



Interactive Presentation



Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Graphing Quadratic Inequalities
(continued)

Questions

Have students complete the Explore activity.

Ask:

- How do you represent the graph of a linear inequality? **Sample answer:** You use a solid or dashed line based on the inequality symbol, then shade one region, usually above or below the line.
- How would the graph be different for $y > -x + 4$? **Sample answer:** The curve would be dashed instead of solid, because the inequality symbol does not include "or equal to". Also, the shaded region would be above (outside) the curve instead of below (inside) it.



Inquiry

How can you represent a quadratic inequality graphically? **Sample answer:** I could graph the related function and shade to represent all the points for which that inequality is true.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Graphing Quadratic Inequalities

Objective

Students graph quadratic inequalities in two variables by using the related equations.



Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between graphing linear inequalities and graphing quadratic inequalities.

DIFFERENTIATE

Reteaching Activity **AL**

IF students are having trouble making connections between the graph of a quadratic inequality and the inequality itself,

THEN have students think about how the graph of a quadratic inequality helps them understand the meaning of the inequality. Ask them to explore whether the quadratic inequality itself or the graph of the inequality is more meaningful to them. Ask them to give explanations for their choices.

Example 1 Graph a Quadratic Inequality ($<$ or $\leq$)



Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the inequalities and graphs used in Example 1.

Questions for Mathematical Discourse

- AL** Why are the points on the parabola valid solutions to the inequality? **The inclusive symbol $\leq$ is used.**
- OL** How many points are solutions to this inequality? **infinitely many**
- BL** How does changing the shape of the parabola change the number of solutions? **Sample answer: The number of solutions does not change, it will still be infinite.**

Important to Know

When testing a point, ask the students if there is any point on the graph that would be easier to test than others. Guide them to realizing that testing the point $(0, 0)$ may be the easiest point to plot, unless that point is on the related function.



Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 1-7

Quadratic Inequalities

Explore: Graphing Quadratic Inequalities

Online Activity Use graphing technology to complete the Explore.

INQUIRY How can you represent a quadratic inequality graphically?

Learn: Graphing Quadratic Inequalities

You can graph quadratic inequalities in two variables by using the same techniques used to graph linear inequalities in two variables. A **quadratic inequality** is an inequality of the form $y > ax^2 + bx + c$, $y \geq ax^2 + bx + c$, $y < ax^2 + bx + c$, or $y \leq ax^2 + bx + c$.

Key Concept • Graphing Quadratic Inequalities

Step 1 Graph the related function.
Step 2 Test a point not on the parabola.
Step 3 Shade accordingly.

Example 1 Graph a Quadratic Inequality ($<$ or $\leq$)

Graph $y \leq x^2 - 2x + 8$.

Step 1 Graph the related function.
 Because the inequality is less than or equal to, the parabola should be solid.

Step 2 Test a point not on the parabola.

$y < x^2 - 2x + 8$
 $0 < (0)^2 - 2(0) + 8$
 $0 < 8$
 True

$(0, 0)$ is **not** on the parabola.

Shade the region that contains the point.

Step 3 Shade accordingly.

Go Online You can complete an Extra Example online.

Today's Goals

- Graph quadratic inequalities in two variables.
- Solve quadratic inequalities in two variables by graphing.

Today's Vocabulary

quadratic inequality

Think About It!

How do you know whether to make the parabola solid or dashed?

Sample answer: If the inequality contains $<$ or $>$, then it is dashed. If it contains $\leq$ or $\geq$, then it is solid.

Study Tip

(0, 0) If $(0, 0)$ is not a point on the parabola, then it is often the easiest point to test when determining which part of the graph to shade.

Lesson 1-7 • Quadratic Inequalities 51

Interactive Presentation

Graphing Quadratic Inequalities

You can graph quadratic inequalities in two variables by using the same techniques used to graph linear inequalities in two variables. A **quadratic inequality** is an inequality that includes a quadratic expression.

KEY CONCEPT: GRAPHING QUADRATIC INEQUALITIES

Learn

SWYPE



Students move through the step to see how to graph a quadratic inequality.

TYPE



Students answer a question to determine whether they understand how to graph a quadratic inequality.

**Example 2** Graph a Quadratic Inequality ($>$ or $\geq$)Graph $y > -5x^2 + 10x$.**Step 1** Graph the related function.

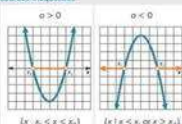
Because the inequality is greater than, the parabola should be dashed.

Step 2 Test a point not on the parabola.Because $(1, 0)$ is on the parabola, use $(1, 0)$ as a test point.

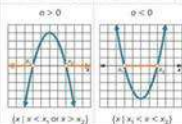
$y > -5x^2 + 10x$	Original Inequality
$0 > -5(1)^2 + 10(1)$	$(1, 0) = (1, 0)$
$0 > 5$	False

So, $(1, 0)$ is not a solution of the inequality.**Step 3** Shade accordingly.Because $(1, 0)$ is not a solution of the inequality, shade the region that does not contain the point.**Learn** Solving Quadratic Inequalities**Key Concept • Solving Quadratic Inequalities**

$$ax^2 + bx + c < 0$$

Graph $y = ax^2 + bx + c$ and identify the x -values for which the graph lies below the x -axis.For $<$, include the x -intercepts in the solution.

$$ax^2 + bx + c > 0$$

Graph $y = ax^2 + bx + c$ and identify the x -values for which the graph lies above the x -axis.For $>$, include the x -intercepts in the solution.

See Online! You can complete an Extra Example online.

52 Module 1 • Quadratic Functions

Interactive Presentation

Graph a Quadratic Inequality ($<$ or $\leq$)

Graph $y \leq x^2 - 2x + 6$.

Step 1

Graph the related function.

Step 2

Test a point.

Example 1

EXPAND

Students tap to see the step to graph a quadratic inequality.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 2 Graph a Quadratic Inequality ($>$ or $\geq$)**MP Teaching the Mathematical Practices****1 Explain Correspondences** Encourage students to explain the relationships between the inequality and graph in Example 2. Help students to see the relationship between the inequality symbol and the shaded region.**Questions for Mathematical Discourse****AL** Will the parabola of the related function will open up or down? How do you know? **down; $a < 0$.****OL** Can point $(0, 0)$ be used to test this inequality? Why or why not? **No; the parabola passes through the point $(0, 0)$.****BL** Is it necessary to test a point that you suspect will be in the shaded region of the inequality? **No, you can test any point not on the boundary line.** Why might you intentionally test a point not likely to be part of the solution set? **Sample answer: The calculations may be simpler when using a point outside the solution set. If the result is a false statement, you know the other region is the shaded region.****Learn** Solving Quadratic Inequalities**Objective**

Students solve quadratic inequalities in two variables by graphing.

MP Teaching the Mathematical Practices**3 Analyze Cases** Work with students to examine the cases of quadratic inequalities. Encourage students to familiarize themselves with all of the cases and how to solve each case by graphing.**DIFFERENTIATE****Reteaching Activity** **AL BL**

IF students are having trouble figuring out how the graphing of quadratic functions in two-dimensions relates to the solution of an inequality in one variable,

THEN review some examples of linear inequalities by plotting them on a coordinate plane in a similar way. Have students relate the two-dimensional graph to a linear inequality on a number line, then compare this to the solution set from graphing a quadratic inequality.

**Example 3** Solve a Quadratic Inequality ($\leq$ or $<$) by Graphing**MP** Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the inequality and graph used in Example 3.

Questions for Mathematical Discourse

AL How is this example different from Examples 1 and 2? **Sample answer:** the inequality has only the variable x .

OL Why do you shade a line on the x -axis rather than the whole area above the function? **Sample answer:** The original inequality only has x as a variable. The related function is used as a means of finding the x -values that are solutions to the original inequality.

BL What would the solution change if the inequality symbol was changed to $\leq$? **−3 and 2 would be included in the solution set.**

Common Error

The related equations show only a single variable, so the student may be confused about how to graph it on a 2-dimensional coordinate plane. Discuss the relationship between the related equation and the variable y .

Example 4 Solve a Quadratic Inequality ($>$ or $\geq$) by Graphing**MP** Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain conclusions drawn when solving a problem. The Talk About It feature asks students to justify their conclusion about the solution set of a quadratic inequality.

Questions for Mathematical Discourse

AL Why are the roots of the quadratic included in the solution? The inequality symbol for greater than or equal is used.

OL Do you need to determine the vertex to find the solution set? Explain. **No; sample answer:** Because the solution set is the x -values where the function is above the x -axis, so we only need to know the intercepts and which way the parabola opens.

BL Compare and contrast the solution sets for $x^2 - 3x - 4 \geq 0$ and $x^2 - 3x - 4 \leq 0$. **Sample** $x^2 - 3x - 4 \geq 0$; **sample answer:** The solution set for $x^2 - 3x - 4 \geq 0$ is $(-\infty, -1] \cup [4, \infty)$ and the solution set for $x^2 - 3x - 4 \leq 0$ is $[-1, 4]$.

The solution sets both contain -1 and 4 . The solution set for $x^2 - 3x - 4 \geq 0$ is two intervals. The solution set for $x^2 - 3x - 4 \leq 0$ is one interval.

Example 3 Solve a Quadratic Inequality ($<$ or $\leq$) by Graphing

Solve $x^2 + x - 6 < 0$ by graphing.

Because the quadratic expression is less than 0, the solution consists of x -values for which the graph of the related function lies below the x -axis. Begin by finding the zeros of the related function.

$$x^2 + x - 6 = 0$$

$$(x - 2)(x + 3) = 0$$

$$x = 2 \text{ or } x = -3$$

Related equation:

Factor:

Zero Product Property

Sketch the graph of a parabola that has x -intercepts at 2 and -3 . The graph should open up because $a > 0$.



The graph lies below the x -axis between -3 and 2 . Thus, the solution set is $\{x \mid -3 < x < 2\}$ or in interval notation $(-3, 2)$.

Think About It! How could you check your solution?

Sample answer: I could test a point on the segment, as well as one point to the left and one point to the right of the segment.

Example 4 Solve a Quadratic Inequality ($>$ or $\geq$) by Graphing

Solve $x^2 - 3x - 4 \geq 0$ by graphing.

Because the quadratic expression is greater than or equal to 0, the solution consists of x -values for which the graph of the related function lies on and above the x -axis. Begin by finding the zeros of the related function.

$$x^2 - 3x - 4 = 0$$

$$(x - 4)(x + 1) = 0$$

$$x = 4 \text{ or } x = -1$$

Related equation:

Factor:

Zero Product Property

Sketch the graph of a parabola that has x -intercepts at -1 and 4 . The graph should open up because $a > 0$.



The graph lies above and on the x -axis when $x \leq -1$ or $x \geq 4$. Thus, the solution set is $\{x \mid x \leq -1 \text{ or } x \geq 4\}$ or $(-\infty, -1] \cup [4, \infty)$.

Check:

Solve $-\frac{1}{2}x^2 + x + 1 > 0$ by graphing and write the solution set.

$$\{x \mid -\frac{7}{2} < x < \frac{7}{2}\}$$

$$(-3.5, 3.5)$$

Go Online You can complete an Extra Example online.

Lesson 1.7 • Quadratic Inequalities 53

Interactive Presentation

Solve a Quadratic Inequality ($<$ or $\leq$) by Graphing

Solve $x^2 + 2x - 8 < 0$ by graphing.

Because the quadratic expression is less than 0, the solution consists of x -values for which the graph of the related function lies below the x -axis. Begin by finding the zeros of the related function.

$$x^2 + 2x - 8 = 0$$

$$(x - 2)(x + 4) = 0$$

$$x = 2 \text{ or } x = -4$$

Sketch the graph of a parabola that has x -intercepts at -4 and 2 . The graph should open up because $a > 0$.

Example 3

TYPE

a

Students complete the calculations to solve a quadratic inequality by graphing.

TYPE

a

Students answer a question to determine whether they understand graphing quadratic inequalities.



Example 5 Solve a Quadratic Inequality Algebraically

Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to solve the problem in Exercise 5, students will need to analyze the mathematical relationships in the problem to draw a conclusion about the dimensions of the garden.

Questions for Mathematical Discourse

- A1** What does $A \leq 104$ mean in this context? The area of the garden that Marcus can make, without buying more soil, is 104 square feet or less.
- O1** How do you know from the context of the problem that the valid solutions are between the roots? **Sample answer:** We are looking for what length Marcus can add, which has an upper limit based on the amount of soil he has.
- E1** Consider the solution set if the direction of the inequality is flipped. How does the solution set compare to that of the original inequality? **Sample answer:** The original inequality has one solution interval that is bounded, and the flipped inequality has two solution intervals that are not bounded. Both solution sets contain an infinite number of points.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Example 5 Solve a Quadratic Inequality Algebraically

GARDENING Marcus is planning a garden. He has enough soil to cover 104 square feet, and wants the dimensions of the garden to be at least 5 feet by 10 feet. If he wants to increase the length and width by the same number of feet, by what value can he increase the dimensions of the garden without needing to buy more soil? Create a quadratic inequality and solve it algebraically.

Step 1 Determine the quadratic inequality.

$$\begin{aligned} A &= lw \\ &= (x+10)(x+5) \\ &= x^2 + 15x + 50 \end{aligned}$$

Area Formula
 $l = x + 10; w = x + 5$
FOIL and simplify.

The area must be less than or equal to 104 square feet, so $x^2 + 15x + 50 \leq 104$.

Step 2 Solve the related equation.

$$\begin{aligned} x^2 + 15x + 50 &= 104 \\ x^2 + 15x - 54 &= 0 \\ (x+18)(x-3) &= 0 \end{aligned}$$

Related equation
Subtract 104 from each side.
Factor.
 $x = -18$ or $x = 3$ Zero Product Property

Steps 3 and 4 Plot the solutions on a number line and test a value from each interval.

Use dots because -18 and 3 are solutions of the original inequality.



Test a value from each interval to see if it satisfies the original inequality.

Test $x = 20$, $x = 0$, and $x = 5$. The only value that satisfies the original inequality is $x = 0$, so the solution set is $[-18, 3]$. So, Marcus can't increase the length and width up to 3 feet without needing to buy more soil. The interval $-18 \leq x \leq 0$ is not relevant because Marcus does not want to decrease the length and width or leave it as is.

Check

MANUFACTURING An electronics manufacturer can model their profits in dollars P when they sell x video players by using the function $P(x) = -0.3x^2 + 75x - 1000$. How many video players can they sell so they make \$7500 or less?

The company will make \$7500 or less if they make $\frac{7}{3}$ video players or fewer and/or $\frac{7}{3}$ video players or more. **139**

511
Go Online You can complete an Extra Example online.

Interactive Presentation

Example 5

TAP



Students tap to see the steps to solve a quadratic inequality algebraically.

CHECK



Students complete the Check online to determine whether they are ready to move on.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–20
2	exercises that use a variety of skills from this lesson	21–39
2	exercises that extend concepts learned in this lesson to new contexts	40–41
3	exercises that emphasize higher-order and critical-thinking skills	42–46

ASSESS AND DIFFERENTIATE

 Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign:

- Practice Exercises 1–41 odd, 42–46
- Extension: Graphing Intersections of Quadratic Inequalities
-  **ALEKS** Quadratic Inequalities

IF students score 66%–89% on the Checks, **THEN** assign:

- Practice Exercises 1–45 odd
- Remediation, Review Resources: Graphing Inequalities in Two Variables
- Personal Tutors
- Extra Examples 1–5
-  **ALEKS** Graphing Linear Inequalities

IF students score 65% or less on the Checks, **THEN** assign:

- Practice Exercises 1–19 odd
- Remediation, Review Resources: Graphing Inequalities in Two Variables
-  **ALEKS** Graphing Linear Inequalities

Practice

Examples 1 and 2

Graph each inequality: 1–12. See margin.

- $y \leq x^2 + 6x + 4$
- $y < -x^2 + 4x - 6$
- $y \leq x^2 - 4$
- $y \leq x^2 + 4$
- $y < 2x^2 - 4x - 2$
- $-x^2 + 12x - 36 > y$
- $y > x^2 + 6x + 7$
- $y > x^2 - 8x + 17$
- $y \geq x^2 + 2x + 2$
- $y \geq 2x^2 + 4x$
- $y > -2x^2 - 4x + 2$
- $y \geq x^2 - 4x + 4$

Examples 3 and 4

Solve each inequality by graphing. 13–18. See Mod. 1 Answer Appendix for graphs.

- $x^2 - 6x + 9 \leq 0$
($x = 3$)
- $x^2 - 4x + 32 \geq 0$
($x = 2$)
- $x^2 - 8x + 16 \geq 0$
($x = 4$)
- $x^2 - 4x + 32 \geq 0$
($x = 2$)
- $x^2 - 8x + 16 \geq 0$
($x = 4$)
- $x^2 - 4x + 32 \geq 0$
($x = 2$)
- $x^2 - 8x + 16 \geq 0$
($x = 4$)
- $x^2 - 4x + 32 \geq 0$
($x = 2$)
- $x^2 - 8x + 16 \geq 0$
($x = 4$)

Example 5

19. FENCING Vanessa has 180 feet of fencing that she intends to use to build a rectangular play area for her dog. She wants the play area to enclose at least 1800 square feet. What are the possible widths of the play area? 30 ft to 60 ft

20. BUSINESS A bicycle maker sold 300 bicycles last year at a price of \$300 each. The maker wants to increase the profit margin this year, but predicts that each \$20 increase in price will reduce the number of bicycles sold by 10. How many \$20 price increases can the maker add and still expect to make a total profit of at least \$100,000? from 5 to 10

Mixed Exercises

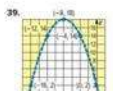
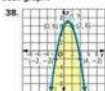
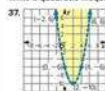
Solve each quadratic inequality by using a graph, a table, or algebraically.

- $-2x^2 + 12x < -15$
($x < -1.06$ or $x > 7.06$)
- $5x^2 + x + 3 \geq 0$
(all real numbers)
- $11 \leq 4x^2 + 7x$
($x \leq -2.75$ or $x \geq 1$)
- $x^2 - 4x \leq -7$
($x < 0.61$ or $x > 2.72$)
- $-3x^2 + 10x < 5$
($x < 0.61$ or $x > 2.72$)
- $-1 \geq -x^2 - 5x$
($x \leq -5.19$ or $x \geq 0.19$)
- $x^2 + 2x + 1 > 0$
(all real numbers)
- $x^2 - 3x + 2 \leq 0$
($1 \leq x \leq 2$)
- $x^2 + 10x + 7 \geq 0$
($x \leq -9.24$ or $x \geq -0.76$)
- $x^2 - 5x \geq 14$
($x \leq -2$ or $x \geq 7$)
- $-x^2 - 15 \leq 8x$
($x \leq -5$ or $x \geq -3$)
- $9x \leq 12x^2$
($x \leq 0$ or $x \geq \frac{4}{3}$)
- $4x^2 + 4x + 1 > 0$
($x \neq -\frac{1}{2}$)
- $5x^2 + 10 \geq 27x$
($x \leq \frac{3}{5}$ or $x \geq 5$)
- $3x^2 + 36x + 12 \leq 0$
($-3 \leq x \leq -4$)
- REASONING Consider the equation $ax^2 + bx + c = 0$. Assume that the discriminant is zero and that a is positive. What are the solutions of the inequality $ax^2 + bx + c \leq 0$? $x = -\frac{b}{2a}$

Lesson 17 • Quadratic Inequalities 55



Write a quadratic inequality for each graph.



$$y > x^2 - 4x - 6$$

$$y \leq -x^2 + 2x + 6$$

$$y > -0.25x^2 - 4x + 2$$

40. **BASEBALL** A baseball player hits a high pop-up with an initial velocity of 32 meters per second, 1.3 meters above the ground. The height $h(t)$ of the ball in meters t seconds after being hit is modeled by $h(t) = -4.9t^2 + 32t + 1.3$.

a. During what time interval is the ball higher than the camera located in the press box 43.4 meters above the ground? 1.83 to 6.50 s.

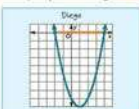
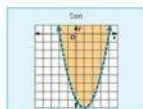
b. When is the ball within 2.4 meters of the ground where the catcher can attempt to catch it? 0 to 0.035 s and 6.50 to 6.57 s.

41. **CONSTRUCT ARGUMENTS** Are the boundaries of the solution set of $x^2 + 4x - 12 \leq 0$ twice the value of the boundaries of $x^2 + 2x - 6 \leq 0$? Explain.

See Mod. 1 Answer Appendix.

Higher-Order Thinking Skills

42. **FIND THE ERROR** Don and Diego used a graph to solve the quadratic inequality $x^2 - 2x - 8 > 0$. Is either of them correct? Explain your reasoning. See Mod. 1 Answer Appendix.



43. **CREATE** Write a quadratic inequality for each condition.

a. The solution set is all real numbers. Sample answer: $x^2 + 2x + 1 \geq 0$

b. The solution set is the empty set. Sample answer: $x^2 - 4x + 6 < 0$

44. **ANALYZE** Determine if the following statement is sometimes, always, or never true. Justify your reasoning. See Mod. 1 Answer Appendix.

The intersection of $y \leq -x^2 + c$ and $y \geq x^2 - c$ is the empty set.

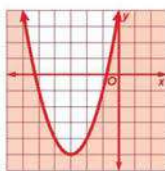
45. **PERSEVERE** Graph the intersection of the graphs of $y \leq -x^2 + 4$ and $y \geq x^2 - 4$. See Mod. 1 Answer Appendix.

46. **WRITE** How are the techniques used when solving quadratic inequalities and linear inequalities similar? How are they different? See Mod. 1 Answer Appendix.

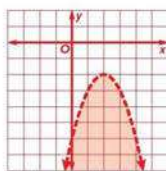
56 Module 1 • Quadratic Functions

Answers

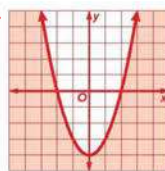
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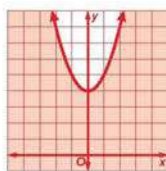
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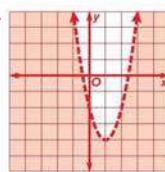
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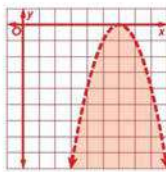
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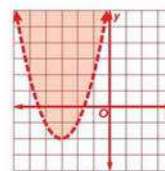
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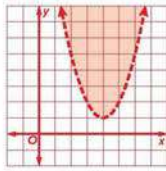
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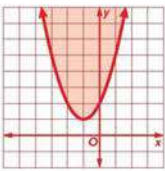
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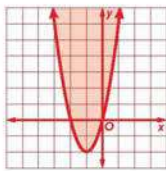
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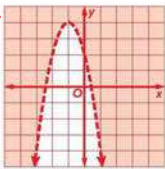
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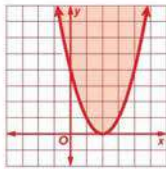
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11.



12.



Solving Linear-Nonlinear Systems

LESSON GOAL

Students solve systems of linear and quadratic equations.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Explore:** Linear-Quadratic Systems

 **Develop:**

Solving Linear-Quadratic Systems

- Solve a Linear-Quadratic System by Using Substitution
- Solve a Linear-Quadratic System by Using Elimination
- Use a System to Solve a Quadratic Equation
- Solve a Linear-Quadratic Systems by Graphing

Solving Quadratic-Quadratic Systems

- Solve a Quadratic-Quadratic System Graphically
- Solve a Quadratic-Quadratic System Algebraically
- Use a System to Solve a Quadratic Equation

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources

Remediation: Elimination Using Multiplication

Extension: Systems Involving Absolute Value Functions

AL	LE	EL	
●	●		●
	●	●	●

Language Development Handbook

Assign page 8 of the *Language Development Handbook* to help your students build mathematical language related to solving systems of linear and quadratic equations.

 You can use the tips and suggestions on page 18 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Algebra

Standards for Mathematical Content:

A.REI.11 Explain why the x -coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

4 Model with mathematics.

Coherence

Vertical Alignment

Previous

Students solved systems of equations by using substitution or elimination.
A.CED.3

Now

Students solve systems of linear and quadratic equations.
A.REI.11

Next

Students will solve polynomial equations by graphing.
A.CED.1, A.REI.11

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

 **Conceptual Bridge** In this lesson, students extend their understanding of solving systems of linear equations to graphing systems of linear and quadratic equations. They build fluency by solving systems algebraically and graphically, and they apply their understanding by solving real-world problems.

Mathematical Background

A system of a linear and a nonlinear equation can be solved algebraically or graphically. For a linear-quadratic system, a graph indicates whether the conic section and the line intersect in 0, 1, or 2 points. For a quadratic-quadratic system, a graph indicates the number of solutions (0, 1, 2, 3, or 4).



Interactive Presentation

Warm Up

Solve each system by using substitution.

- $3x + 2y = 12$
 $y = 2x - 1$
- $2x + y = 6$
 $x - 2y = 8$

Solve each system by using elimination.

- $5x + 2y = -13$
 $9x - 4y = -31$
- $7x + 3y = 2$
 $3x - y = -6$

Warm Up

Launch the Lesson

The European Space Agency has been running tests to see how nets can be used in orbit to collect space debris, such as detached rocket parts or satellites that are out of commission. To test the behavior of nets before trying the process in space, he and others at the ESA tested shooting nets in straight lines out of aircraft flying in parabolic arcs, which models the weightlessness experienced by objects in orbit. You can use a linear-quadratic system to predict



Launch the Lesson

Vocabulary

quadratic relations

Equations of parabolas with horizontal axes of symmetry that are not functions.

- How can you solve a system of two equations when one is linear and the other is quadratic?
- What are the different ways that the graphs of a linear function and a quadratic function can intersect?
- Does a system involving a quadratic function always have a solution?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- solving linear systems of equations

Answers:

- (2, 3)
- (4, -2)
- (-3, 1)
- (-1, 3)

Launch the Lesson

MP Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will learn how a linear-quadratic system of equations can be used to predict how nets should be deployed to collect debris in space.

Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then discuss the questions below with the class.

Explore Linear-Quadratic Systems

Objective

Students use a graphing calculator to explore linear-quadratic systems of equations.

MP Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to complete the exercises in the Explore, students will need to use a graphing calculator. Work with students to explore and deepen their understanding of linear-quadratic systems of equations.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

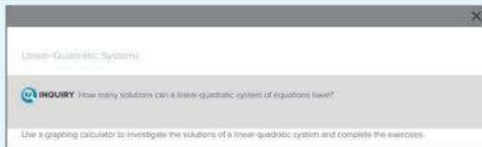
What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will use a graphing calculator to graph the equations in a linear-quadratic system and identify the solutions of the system. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore



Explore

TAP



Students tap to select a calculator to graph a linear-quadratic system of equations.

TYPE



Students answer questions about the solutions of various linear-quadratic systems.



Interactive Presentation

INQUIRY How many solutions can a linear-quadratic system of equations have?

Submit

Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Linear-Quadratic Systems

(continued)

Questions

Have students complete the Explore activity.

Ask:

- What does it mean if a linear-quadratic system does not have a solution? Does this only happen with horizontal lines? **Sample answer:** If the line and parabola do not intersect, then there is no solution. This could happen with a line that has a slope not equal to zero, like moving the given line up until it no longer intersects.
- Can a linear-quadratic system containing a vertical line and a parabola that opens up or down have no solution? Why or why not? **No; sample answer:** If a parabola opens up or down, then the domain is all real numbers. Because a vertical line has the form $x = k$, and k is in the domain of the quadratic function, the vertical line will intersect the parabola when $x = k$.

Inquiry

How many solutions can a linear-quadratic system of equations have?

A linear-quadratic system can have 0, 1, or 2 solutions.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Solving Linear-Quadratic Systems

Objective

Students solve systems of linear and quadratic equations.

MP Teaching the Mathematical Practices

3 Analyze Cases To answer the question in the Talk About It! feature, students must consider the cases of linear-quadratic systems to determine whether a system can have infinitely many solutions.

Example 1 Solve a Linear-Quadratic System by Using Substitution

MP Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of the linear and quadratic equations in the system to apply the substitution method.

Questions for Mathematical Discourse

- A1** How can you check your solutions? **Sample answer:** Substitute them into each equation and make sure they satisfy both equations.
- OL** What types of equations are in the system you are solving here? **a quadratic equation and a linear equation**
- BI** Describe another way to solve this system algebraically. **Sample answer:** Solve for y in the linear equation and substitute that expression into the quadratic equation.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Solving Linear-Nonlinear Systems

Lesson 1-8

Explore Linear-Quadratic Systems

Online Activity Use graphing technology to complete the Explore.

INQUIRY How many solutions can a linear-quadratic system of equations have?

Learn Solving Linear-Quadratic Systems

Like solving systems of linear equations, you can solve linear-quadratic systems by using graphical or algebraic methods. You can also use a system of equations to solve a quadratic equation by writing each side of the equation as a related function.

Example 1 Solve a Linear-Quadratic System by Using Substitution

Solve the system of equations.

$$x = 2y^2 + 3y + 1 \quad (1)$$

$$-x + y = -1 \quad (2)$$

Step 1 Solve Equation (2) for x :

$$-x + y = -1$$

$$\rightarrow x = y + 1$$

$$x = y + 1$$

Equation (2)

Subtract x from each side.Divide each side by -1 .

Step 2 Substitute for x in Equation (1). Then solve for y :

$$x = 2y^2 + 3y + 1 \quad \text{Equation (1)}$$

$$y + 1 = 2y^2 + 3y + 1$$

$$0 = 2y^2 + 2y$$

$$0 = 2y(y + 1)$$

$$y = 0 \text{ or } y = -1$$

Equation (1)

 $x = y + 1$

Simplify.

Factor out $2y$.

Zero Product Property

Step 3 Substitute the y -values and solve for x :

Case 1

$$x = y + 1$$

$$\rightarrow x = 0 + 1$$

Equation (2)

$$x = y + 1$$

$$\rightarrow -1 + 1 \text{ or } 0$$

The two solutions of the system are $(1, 0)$ and $(0, -1)$.

Go Online You can complete an Extra Example online.

Today's Goals

- Solve systems of linear and quadratic equations.
- Solve systems of two quadratic equations.

Today's Vocabulary

quadratic relations

Talk About It!

A system of linear equations can have infinitely many solutions. Can a linear-quadratic system have infinitely many solutions? Explain.

No, sample answer: A line cannot intersect a quadratic equation at more than 2 points. It either does not intersect it, intersects it at exactly one point, or goes through two points.

Study Tip

Algebra and Graphing Even when solving a system algebraically, it can be useful to graph the equations to ensure that you have the correct number of solutions.

Lesson 1-8 • Solving Linear-Nonlinear Systems 57

Interactive Presentation



Learn

WEB SKETCHPAD



Students use a sketch to see the solutions of a linear-quadratic system.

TYPE



Students answer a question to determine whether they understand the solutions of a linear-quadratic system.

**Example 2** Solve a Linear-Quadratic System by Using Elimination

Solve the system of equations.

$x^2 = y + 5$ (1)

$-x + y = 7$ (2)

Step 1 Solve so that the y s are on the same side of each equation.

$-x + y = 7$

$x = y + 7$

Equation (2)
Subtract y from each side

Step 2 Add the equations.

$x^2 = y + 5$

$(-x) - x = -y + 7$

$x^2 - x = 12$

Step 3 Solve for the remaining variable.

$x^2 - x = 12$

$x^2 - x - 12 = 0$

$(x - 4)(x + 3) = 0$

$x = 4$ or $x = -3$

Sub of Equations (1) and (2)
Subtract 12 from each side
Factor
Zero Product Property

Step 4 Solve for the other variable.

$-x + y = 7$

$-4 + y = 7$

$y = 11$

Equation (2)
Substitute x :
Simplify
 $-(-3) + y = 7$
 $y = 4$

The two solutions of the system are (4, 11) and (-3, 4).

Example 3 Use a System to Solve a Quadratic EquationUse a system of equations to solve $x^2 - 2x + 6 = 4x + 1$.

Step 1 Create a system of equations.

$y = x^2 - 2x + 6$ (1)

$y = 4x + 1$ (2)

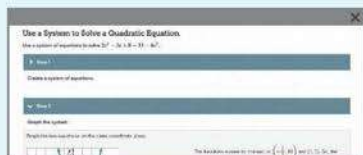
Step 2 Graph the system.

The functions appear to intersect at (1, 5) and (5, 26), so the solutions of $x^2 - 2x + 6 = 4x + 1$ are $x = 1$ and $x = 5$.

Go Online You can complete an Extra Example online.

58 Module 1 • Quadratic Functions

Interactive Presentation



Example 3

EXPAND



Students tap to see how to use a system to solve a quadratic equation.

Example 2 Solve a Linear-Quadratic System by Using Elimination**MP** Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. In Example 2, emphasize transforming the equations so you can eliminate the desired variable from the two equations.

Questions for Mathematical Discourse

- AL** What equation would you need to use to solve by substituting for y in the quadratic equation? $y = x + 7$
- OL** When using elimination, what has to be true about the variable you are eliminating? **Sample answer:** It must be on the same side of both equations and the absolute value of the coefficients must be equal.
- BL** If you rewrote the linear equation as $x + 7 = y$, what would be different about the elimination step? **You would have to subtract the linear equation instead of adding it.**

Example 3 Use a System to Solve a Quadratic Equation**MP** Teaching the Mathematical Practices

1 Explain Correspondences In Example 3, encourage students to explain the relationships between the system of equations, the graph, and the solutions.

Questions for Mathematical Discourse

- AL** What are you looking for in the graph to identify the solutions? **The points where the line and parabola intersect**
- OL** How would the graph and/or solution set change if you rearranged the equation before writing and graphing the system? **The graphs would change, but the x -values of the intersections would remain the same.**
- BL** How is solving a quadratic equation using a system related to solving by graphing the related function and finding the zeros of the function? **These methods are effectively the same. When you solve by graphing the related function, you are using a system to solve where one of the equations is $y = 0$.**



Example 4 Solve a Linear-Quadratic System by Graphing

MP Teaching the Mathematical Practices

4 Use Tools Point out that to solve the problem in Example 4, students will need to use a graph.

Questions for Mathematical Discourse

- AL** What is the maximum profit that can be obtained and for what price? **\$4 million; \$20**
- OL** Use the graph to determine the prices that would result in a profit of at least \$3 million. **\$10 to \$30**
- BL** What does the model suggest as a result if the price is above \$40? **Sample answer: The company would lose money at that price.**

Check

Use a system of equations to solve $2x + 11 = x^2 + x - 1$.
 $x = -3$ or -4

Example 4 Solve a Linear-Quadratic System by Graphing

PRODUCTION A software developer determines that her company can model their revenue R from a given product in hundreds of thousands of dollars given the unit price of the product x in dollars with the function $R = -0.5x^2 + 4x$. Create a linear-quadratic system and solve it graphically to determine the price for which the company will earn \$4.2 million.

Step 1 Create a linear-quadratic system.

The first equation is the given revenue model $R = -0.5x^2 + 4x$.

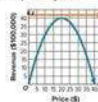
The line that represents revenue of \$4.2 million is $R = 42$.

So, the linear-quadratic system is:

$$R = -0.5x^2 + 4x \quad (1)$$

$$R = 42 \quad (2)$$

Step 2 Graph the system.



Step 3 Determine the solutions.

The graphs of the functions do not intersect at any point, so the system has 0 solutions.

Check

MOUNTAINS Engineers want to build a footbridge with steps across a steep valley. They can model the valley with the equation $y = 0.05x^2 - 4x + 80$, where y is the height above the lowest point in the valley in feet and x is the distance from where they plan to start the bridge. If they start the bridge 80 feet above ground and want it to go down an inch for every foot to the right, then at what points will the bridge start and end?

The bridge will start at $(0, 80)$ and end at approximately $(78, 73)$.

Go Online You can complete an Extra Example online.

Study Tip

R is measured in hundreds of thousands of dollars, so the value of 42 on the y -axis indicates revenue \$4.2 million.

Think About It!

What does the solution set of the system mean in the context of the situation?

Sample answer: Because the system has 0 solutions, there are no prices for which the developer could sell the software so they would earn \$4.2 million.

Go Online

You can watch videos to see how to solve systems of linear and nonlinear equations using a graphing calculator.

Interactive Presentation

Solve a Linear-Quadratic System by Graphing

PRODUCTION A software developer determines that they can model their profit P in hundreds of thousands of dollars given the price x in dollars with the function $P = -0.5x^2 + 4x$. Create a linear-quadratic system and solve it graphically to determine the price for which the company will earn \$4.2 million.



Example 4

EXPAND



Students tap to see the steps to solve a linear-quadratic system by graphing.

TYPE



Students answer a question to determine whether they understand how to solve a real-world problem by graphing a linear-quadratic system.

CHECK



Students complete the Check online to determine whether they are ready to move on.



Learn Solving Quadratic-Quadratic Systems

Objective

Students solve systems of two quadratic equations.

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between quadratic functions and quadratic relations. Guide students to look at how quadratic functions and relations are represented algebraically and graphically.

Common Misconception

Students may not have seen parabolas that open to the left or right before. They may confuse the x and y variable in these equations and try to graph the parabolas with a vertical axis of symmetry. Discuss the differences between a quadratic relation and a quadratic function.

Example 5 Solve a Quadratic-Quadratic System Graphically

MP Teaching the Mathematical Practices

1 Check Answers Mathematically proficient students continually ask themselves, “Does this make sense?” The Think About It! feature asks students to explain how they can check their solutions when solving a quadratic-quadratic system by graphing.

Questions for Mathematical Discourse

- A1** What can you tell about the second parabola from the fact that the squared expression contains a y ? **Sample answer:** It will have a horizontal axis of symmetry instead of a vertical one.
- A2** What does the positive value of a indicate in the second equation? **Sample answer:** The parabola opens in the positive direction, which is to the right.
- B1** Could a quadratic-quadratic system ever have infinitely many solutions? What would that mean? **Yes; sample answer:** It would mean that both equations in the system are equivalent.

Learn Solving Quadratic-Quadratic Systems

Equations of parabolas with vertical axes of symmetry have the parent function $y = x^2$ and can be written in the form $y = a(x - h)^2 + k$. Equations of parabolas with horizontal axes of symmetry are of the form $x = a(y - k)^2 + h$ and are not functions. These are often referred to as **quadratic relations**. The graph of $x = y^2$ is the parent graph for all quadratic relations that have a horizontal axis of symmetry.

If a system contains two quadratic relations, it may have zero to four solutions. Just as with linear-quadratic systems, you can solve quadratic-quadratic systems by using graphical or algebraic methods.

Example 5 Solve a Quadratic-Quadratic System Graphically

Solve the system of equations by graphing.

$$y = x^2 \quad (1)$$

$$x = \frac{1}{2}(y - 6)^2 + \frac{5}{2} \quad (2)$$

Step 1 Graph Equation (1).

Equation (1) has a vertex at (0, 0) and goes through the points (-2, 4), (-1, 1), (1, 1) and (2, 4).

Step 2 Graph Equation (2).

You can use a table of values to graph Equation (2). Because the expression on the right is set equal to x , find the value of y for several values of y .

y	$x = \frac{1}{2}(y - 6)^2 + \frac{5}{2}$	x
3	$x = \frac{1}{2}(3 - 6)^2 + \frac{5}{2}$	3
4	$x = \frac{1}{2}(4 - 6)^2 + \frac{5}{2}$	2
5	$x = \frac{1}{2}(5 - 6)^2 + \frac{5}{2}$	$\frac{7}{2}$
6	$x = \frac{1}{2}(6 - 6)^2 + \frac{5}{2}$	$\frac{5}{2}$
7	$x = \frac{1}{2}(7 - 6)^2 + \frac{5}{2}$	$\frac{9}{2}$
8	$x = \frac{1}{2}(8 - 6)^2 + \frac{5}{2}$	2
9	$x = \frac{1}{2}(9 - 6)^2 + \frac{5}{2}$	3

Step 3 Graph and solve the system.

To solve the system, graph both relations on the same coordinate plane and see where they intersect.

The relations intersect at (2, 4) and (3, 9), so those are the solutions of the system.



Think About It!
Is $x = \frac{1}{2}(y - 6)^2 + \frac{5}{2}$ a function? Explain your reasoning.

No; sample answer: For several values of x , there is more than one value of y .

Think About It!
How could you check your solutions?

Sample answer: I could enter the coordinates for each point into both equations to ensure that the solutions are correct.

Interactive Presentation

Solve a Quadratic-Quadratic System Graphically

Solve the system of equations by graphing.

$$y = x^2 \quad (1)$$

$$x = \frac{1}{2}(y - 6)^2 + \frac{5}{2} \quad (2)$$

Example 5

TAP



Students tap to see the steps to solve a quadratic-quadratic system by graphing.

TYPE



Students answer a question to determine whether they understand how to solve a quadratic-quadratic system by graphing.



Example 6 Solve a Quadratic-Quadratic System Algebraically

MP Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of the equations in the system to apply the substitution method.

Questions for Mathematical Discourse

AL How can you tell which quadratic equation, 1 or 2, opens down? Equation 2 has a coefficient a that is negative, so it will open down.

OL What would the graph of the system look like? **Sample answer:** The graph would have a parabola opening up and a parabola opening down, with two intersection points symmetric about the y -axis.

BL What inequality symbols would you use for each equation to get a bounded solution set with included boundaries? **Both of the original equations would use $\leq$.**

Common Error

Remind students to distribute the negative to all of the terms when performing the substitution.

Check

Solve the system of equations.

$$y = -x^2 + 5x - 6$$

$$3y = x^2 - 2x - 3$$

$$(17, -9) \quad (37, -97)$$

Example 6 Solve a Quadratic-Quadratic System Algebraically

Solve the system of equations.

$$2x^2 - y = 4 \quad (1)$$

$$y = -\frac{1}{2}x^2 + 6 \quad (2)$$

$$2x^2 - y = 4$$

$$2x^2 - (-\frac{1}{2}x^2 + 6) = 4$$

$$2x^2 + \frac{1}{2}x^2 - 6 = 4$$

$$\frac{5}{2}x^2 - 10 = 0$$

$$5x^2 - 20 = 0$$

$$x^2 - 4 = 0$$

$$(x + 2)(x - 2) = 0$$

$$x = \pm 2$$

Equation (1)

Substitution

Distributive Property

Simplify

Multiply each side by 2

Divide each side by 5

Difference of Two Squares

Zero Product Property

Substitute -2 and 2 into one of the original equations and solve for y .

Case 1

$$y = -\frac{1}{2}x^2 + 6$$

$$y = -\frac{1}{2}(-2)^2 + 6$$

$$y = 4$$

Equation (2)

Substitute for x

Simplify

Case 2

$$y = -\frac{1}{2}x^2 + 6$$

$$y = -\frac{1}{2}(2)^2 + 6$$

$$y = 4$$

The solutions are $(2, 4)$ and $(-2, 4)$.

Check

Solve the system of equations.

$$x = \frac{1}{9}y^2$$

$$y^2 = -8x + 72$$

$$(27, -67) \quad (27, -97)$$

Go Online You can complete an Extra Example online.

Interactive Presentation

Solve a Quadratic-Quadratic System Algebraically

Solve the system of equations.

$$2x^2 - y = 4 \quad (1)$$

$$y = -\frac{1}{2}x^2 + 6 \quad (2)$$

Example 6

SWIPE



Students move through the steps to see how to solve a quadratic-quadratic system algebraically.

**Example 7** Use a System to Solve a Quadratic EquationUse a system of equations to solve $2x^2 - 3x + 8 = 11 - 4x^2$.**Step 1** Create a system of equations.The related equations of each side of $2x^2 - 3x + 8 = 11 - 4x^2$ are:

$$y = 2x^2 - 3x + 8 \quad (1)$$

$$y = 11 - 4x^2 \quad (2)$$

Step 2 Graph and solve the system.

Graph the first equation.



Then graph the second equation on the same coordinate plane.

The functions appear to intersect at $(-\frac{1}{2}, 10)$ and $(1, 7)$, so the solutions of $2x^2 - 3x + 8 = 11 - 4x^2$ are $-\frac{1}{2}$ and 1.**Check**Use a system of equations to solve $-x^2 + 3x + 14 = -\frac{1}{2}x^2 + 5$.System Solutions: $(-7, -2.74)$, $(8.2, -8.7)$ Equation Solutions: $-7, -2, 7.6$

Go Online You can complete an Extra Example online.

62 Module 1 • Quadratic Functions

Interactive Presentation

Example 7

EXPAND

Students tap to see how to use a system to solve a quadratic equation.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 7 Use a System to Solve a Quadratic Equation**MP Teaching the Mathematical Practices****1 Explain Correspondences** Encourage students to explain the relationships between the quadratic equation, system of equations, and graph in Example 7.**Questions for Mathematical Discourse****AL** How do you know both equations will be graphed as parabolas?
Both can be written in quadratic form.**OL** How could you rewrite this equation to solve it as a linear-quadratic system? **Sample answer:** Adding $4x^2$ to each side would remove the x^2 term from the right side, turning it into a linear equation.**BL** Write an equation that would only intersect with the second related equation at one point. **Sample answer:** $y = 11 + 4x^2$ **Exit Ticket****Recommended Use**

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

3 REFLECT AND PRACTICE



A.REI.11

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Solve each system of equations by graphing, substitution, or elimination.

39. $y = x^2 + 5x + 8$ 40. $y = x^2 + 7x + 12$ 41. $y = x + 3$
 $y = 7x + 5$ (-1, -6), (0, 8) $y = x + 7$ (-5, 2), (-1, 6) $y = 2x^2 - x - 1$ (-1, 2), (2, 3)

42. $y = \left(\frac{1}{2}x\right)^2$ 43. $y = 2x^2$ 44. $y = -5x^2$
 $4x = y^2$ (0, 0) and (4, 4) $y = x^2 - 2y + 1$ (0, 4), (0.352) and (1, 2) $y = 4x^2 + 1$ no solution

45. $2x^2 - y = 2$ 46. $x^2 - y = 4$ 47. $y = x^2$
 $y = x^2 + 4$ (-1, 4), (2) and (4, 2) $y = y^2 - 2$ (-1, 4), (2) and (4, 6) $y = x^2 - 3x + 5$ (-1, 8), (2) and (5, 1)

48. $y = x - 3$ 49. $y = -x$ 50. $y = 3x - 4$ no solution
 $y = x^2 - 5$ (-2, -1), (1, -4) $y = 2x$ (2, -2), (0, 0) $y = x^2 + 9x + 20$

51. $x - 2 = y - 2^2$ 52. $3x - 1 = \frac{1}{2}y^2$ (4, 5.5) and (0.4, -1.5) 53. $x = y + 2^2$ no solution
 $y = x^2 + 5$ no solution $y = x^2 + 5x + 4$ (0, 4), (4, -15) $y = y + 2^2$

54. $x = \frac{1}{2}y^2$ 55. $4y + 3 = x^2$ (-1, 3), (1, 3) 56. $x = y + 2^2$ no solution
 $y^2 = -8x + 6$ $y = x^2 - 24$ and $(\sqrt{24}, 7)$ $y = 2x^2 + 3x + 2$ (1, 7) and $(\frac{1}{2}, -\sqrt{2})$

Use a system of equations to solve each quadratic equation.

57. $2x^2 + x - 1 = -10x + 12$ 58. $3x^2 - 5x - 2 = -36 - 8x$
 $x = \frac{1}{2}$ no solution

59. $x^2 + 3x + 2 = x + 5$ 60. $x^2 - 2x - 3 = -\frac{1}{2}x^2 + \frac{3}{2}x + \frac{3}{2}$
 $x = -3, 1$ $x = -2, 4$

61. $3x^2 + 4x + 1 = -x^2 - 2x - 1$ 62. $2x^2 - 6x + 4 = -x^2 + 3x - 2$
 $x = -\frac{1}{2}, -1$ $x = 1, 2$

63. $2x^2 + 6x + 5 = x^2 - 4$ 64. $4x^2 + 2x + 7 = 3x^2 + 6$
 $x = -\frac{1}{2}, -1$ $x = -1$

65. $5x^2 - 5x - 7 = 3x^2 - 4$ 66. $x^2 + 4x + 4 = x + 4$
 $x = -\frac{1}{2}, -1$ $x = -3, 0$

67. $x^2 + 5x + 12 = -4x - 6$ 68. $2x^2 - x - 6 = -3x - 2$
 $x = -6, -3$ $x = -2, 1$

69. **ROCKETS** Two model rockets are launched at the same time from different heights. The height in meters for one rocket after t seconds is modeled by $y = -4.9t^2 + 48.7t$. The height for the other rocket is modeled by $y = -4.9t^2 + 46.7t + 1.5$.

- After how many seconds are the rockets at the same height? Round to the nearest tenth, 0.7 s.
- From what height would the slower rocket have to be launched so that the rockets land at the same time? Justify your reasoning.

about 20.8 m. The faster rocket lands after about 9.96 seconds. Solving $y = -4.9t^2 + 46.7t + 1.5$ when $y = 0$ and $t = 9.96$, shows that $c \approx 20.8$.

Lesson 1.6 • Solving Linear-Quadratic Systems 65

Interactive Presentation

a. Create a quadratic equation that can be used to find the value of x , when the surface area of the packages is the same. $6x^2 = 8x + 6x^2 + 8x$.

b. Solve the quadratic equation using any method. $x = 0$.

c. Find and interpret the solution in the context of the solution. See margin.

70. **VOLLEYBALL** The function $h(t) = -16t^2 + vt + h_0$ models the height in feet of a volleyball, where v represents the initial velocity, h_0 represents the initial height, and t represents the time in seconds since the ball was hit. Suppose a player bounces a volleyball when it is 3 feet from the ground with an initial velocity of 25 feet per second.

- If the net is approximately 7 feet 8 inches, could the volleyball clear the net? Justify your reasoning. Yes, the maximum height of the ball is about 12.76 feet, which is higher than the net.
- The player on the other side of the net jumps 1 second after the ball is hit so that the path of her hands can be described by $h(t) = -16(t - 1)^2 + 120 = 9t + 7$. If the ball clears the net, is it possible that she blocks the ball? Justify your reasoning. Yes, the player's hands could strike the ball at a height of about 9 feet. Because that is above the height of the net, the ball may be blocked.
- What assumptions did you make? See margin.

Higher-Order Thinking Skills

71. **PERSEVERE** Describe three linear-quadratic systems of equations—one with no solution, one with 1 solution, and one with 2 solutions. See Mod. 1 Answer Appendix.

72. **CREATE** Write a system of two quadratic equations in which there is one solution. Sample answer: $y = x^2$ and $y = -x^2$.

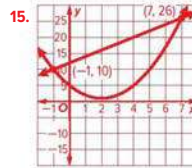
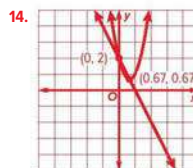
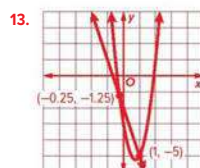
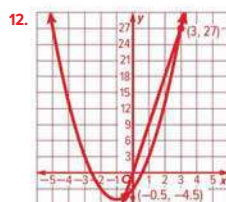
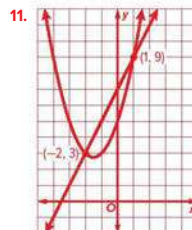
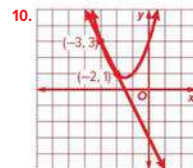
73. **WRITE** Describe a real-life situation that can be modeled by a system with a quadratic function and a linear function. See Mod. 1 Answer Appendix.

74. **FIND THE ERROR** Denny and Carol are solving the system $y = x^2 + 3x - 9$ and $-4x + y = 3$. Is either of them correct? Explain your reasoning.

Denny	Carol
$-4(x^2 + 3x - 9) = 3$	$x^2 + 3x - 9 = -4x + 3$
$-4x^2 - 12x + 36 = 3$	$x^2 + 7x - 12 = 6$
$-4x^2 - 12x = -33$	$x^2 + 7x - 18 = 0$
$(x + 3)(-4x + 11) = 0$	$x = -9, 2$
$x = -3, x = 11$	

Denny is correct. Carol incorrectly solved for y in the second equation of the system before using the substitution method.

Answers



17c. Sample answer: Because the selling price cannot be negative, (150.79, 50,000) is the only viable solution in the context of the situation. This means that the business can earn a \$50,000 profit when the selling price is about \$150.79 per item.

70c. Sample answer: Because the side length cannot be zero, there is no viable solution in the context of the situation. This means that it is impossible to make two boxes that meet these requirements.

71c. Sample answer: I assumed that the student bumping the ball was far enough away from the net that she wouldn't hit it.

I also assumed that the player attempting to block the ball is in the correct position for the path of the ball to intersect the path of her hands.

Review

Rate Yourself!

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have them respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- How can the graph of a quadratic function help you solve the corresponding quadratic equation?
- How do complex numbers relate to solving quadratic equations?
- How do you know what method to use when solving a quadratic equation?

Then have them write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

 A completed Foldable for this module should include the key concepts related to analyzing and graphing quadratic functions.

 **LearnSmart** Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Polynomial, Rational, and Radical Relationships**.

- Complex Numbers
- Solving Equations Graphically

Module 1 • Quadratic Functions
Review

 **Essential Question**

What characteristics of quadratic functions are important when analyzing real-world situations that are modeled by quadratic functions?

Quadratic functions have either a minimum or a maximum value. These values are often solutions when the functions model the path of a projectile or the profit made by a company.

Module Summary

Lessons 1.1 and 1.2

Graphs of Quadratic Functions

- When the coefficient on the x^2 -term is positive, the parabola opens up. When it is negative, the parabola opens down.
- The average rate of change for a parabola over the interval $[a, b]$ is $\frac{b^2 - a^2}{b - a}$.
- The solutions of an equation in one variable are the x -intercepts of the graph of a related function.

Lesson 1.3

Complex Numbers

- The imaginary unit i is the principal square root of -1 . Thus, $i = \sqrt{-1}$ and $i^2 = -1$.
- Two complex numbers of the form $a + bi$ and $c + di$ are called complex conjugates.

Lesson 1.4 through 1.6

Solving Quadratic Equations

- For any real numbers a and b , if $ab = 0$, then either $a = 0$, $b = 0$, or both a and $b = 0$.
- You can solve a quadratic equation by graphing, by factoring, by completing the square, or by using the Quadratic Formula.
- To solve a quadratic equation of the form $ax^2 + bx + c = 0$, take the square root of each side.
- The solutions of a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Lesson 1.7

Quadratic Inequalities

- To graph a quadratic inequality, graph the related function, test a point not on the parabola and shade accordingly.
- For $ax^2 + bx + c < 0$, graph $y = ax^2 + bx + c$ and identify the x -values for which the graph lies below the x -axis. For $\leq$, include the x -intercepts in the solution.
- For $ax^2 + bx + c > 0$, graph $y = ax^2 + bx + c$ and identify the x -values for which the graph lies above the x -axis. For $\geq$, include the x -intercepts in the solution.

Lesson 1.8

Systems Involving Quadratic Equations

- You can use the substitution method or the elimination method to solve a system that includes a quadratic equation.
- If a system contains two quadratic relations, it may have anywhere from zero to four solutions.

Study Organizer

 **Foldables**

Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.

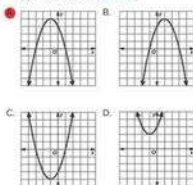


Module 1 Review • Quadratic Functions 67

Module 1 • Quadratic Functions 67

Test Practice

1. **MULTIPLE CHOICE** Which is the graph of $f(x) = -x^2 - 2x + 3$? (Lesson 1-1)



2. **OPEN RESPONSE** At a concert, a T-shirt cannon launches a T-shirt upward. The height of the T-shirt in feet at a given number of seconds after the launch is shown in the table.

Time (s)	Height (ft)
1	74
2	76
3	126
4	104
5	50

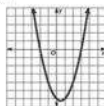
Find and interpret the average rate of change in the height between 1 and 3 seconds after launch. (Lesson 1-1)

The height of the T-shirt is increasing at an average rate of 26 feet per second between 1 and 3 seconds after the launch.

3. **OPEN RESPONSE** Use a quadratic equation to find two real numbers with a sum of 35 and a product of 292. (Lesson 1-2)

10 and 21

4. **MULTI-SELECT** The graph of $f(x) = x^2 - x - 6$ is shown.



Find the solutions of $x^2 - x - 6 = 0$. Select all that apply. (Lesson 1-2)

- A. -6
B. -3
☒ C. -2
D. 2
☒ E. 3
F. 6

5. **MULTIPLE CHOICE** Simplify $\sqrt{-9} \cdot \sqrt{-49}$. (Lesson 1-3)

- A. $-2i$
B. $2i$
☒ C. -21
D. 21

6. **OPEN RESPONSE** Every complex number can be written in the form $a + bi$. For the complex number $8 - 4i$, identify the values of a and b . (Lesson 1-3)

$a = 8; b = -4$

Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Put It All Together: Lessons 1-1 through 1-6

Vocabulary Activity

Module Review

Assessment Resources

Vocabulary Test

AL Module Test Form B

OL Module Test Form A

BL Module Test Form C

Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

Test Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1–17 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Multiple Choice	Students select one correct answer.	1, 5, 8, 9, 13, 14
Multi-Select	Multiple answers may be correct. Students must select all correct answers.	4, 11, 16
Table Item	Students complete a table by entering in the correct values.	18
Open Response	Students construct their own response.	2, 3, 6, 7, 10, 13, 15, 17

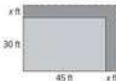
To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
A.SSE.1b	1-1, 1-6	3, 13
A.SSE.2	1-4, 1-5	7, 10, 11
A.REI.11	1-8	16, 17, 18
A.CED.1	1-4, 1-6, 1-7	8, 12, 15
A.CED.2	1-1	1
A.CED.3	1-7	14, 15
A.CED.4	1-6	12, 13
F.IF.4	1-1, 1-2	2, 4
F.IF.6	1-1	2
F.IF.8a	1-4, 1-5	7, 8, 9
N.CN.1	1-3	5, 6
N.CN.2	1-3	5
N.CN.7	1-5	9
N.CN.8	1-5	9

7. OPEN RESPONSE Solve $4x^2 - 64 = 8x - 4$ by factoring. (Lesson 1-4)

$$x = -3, x = 5$$

8. MULTIPLE CHOICE A rectangular lawn has a width of 30 feet and a length of 45 feet. A diagram of the lawn is shown.



A landscape designer wants to increase the length and width of the lawn by the same amount so that the total area will be 2200 square feet. By how many feet should the designer increase the length and width of the lawn? (Lesson 1-4)

- ☒ A. 10 feet
☐ B. 19 feet
☐ C. 28 feet
☐ D. 29 feet

9. MULTIPLE CHOICE Solve $x^2 + 24x + 150 = 0$ by completing the square. (Lesson 1-5)

- ☐ A. $12 \pm \sqrt{6}$
☐ B. $12 \pm \sqrt{16}$
☐ C. $-12 \pm \sqrt{6}$
☒ D. $-12 \pm \sqrt{16}$

10. OPEN RESPONSE Use the square root property to solve $x^2 - 16 = 0$. (Lesson 1-5)

$$x = -4, x = 4$$

11. MULTI-SELECT Use the Square Root Property to find the solutions of $x^2 + 10x + 25 = 81$. Select all that apply. (Lesson 1-5)

- ☒ A. -14
☐ B. -9
☐ C. -4
☒ D. 4
☐ E. 9
☐ F. 14

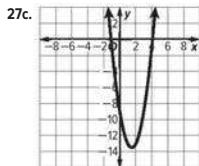
12. OPEN RESPONSE The height of a fireworks shell, in meters, t seconds after launch can be modeled by the function $h(t) = -4.9t^2 + 90t$. To the nearest tenth of a second, how many seconds does it take to first reach a height of 300 meters? Explain your reasoning. (Lesson 1-6)

Sample answer: The height of the fireworks shell is represented by $h(t)$, so substitute 300 for $h(t)$. Write the function in standard form. Use the quadratic formula, where $a = -4.9$, $b = 90$, and $c = -300$, to find the values of t . Using the quadratic formula given approximate values of t of 5.8 and 10.5. Because 5.8 is less than 10.5, the first time the fireworks shell reaches a height of 300 meters is at about 5.8 seconds.

Lesson 1-1

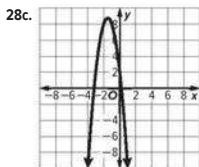
27a. y -int = -9 ; axis of symmetry: $x = 1.5$; x -coordinate of vertex = 1.5

x	$f(x)$
0	-9
1	-13
1.5	-3.5
2	-13
3	-9



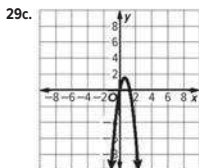
28a. y -int = 2 ; axis of symmetry: $x = -1.5$; x -coordinate of vertex = -1.5

x	$f(x)$
-3	2
-2	8
-1.5	8.75
-1	8
0	2



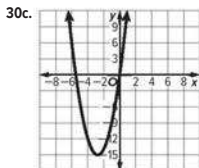
29a. y -int = 0 ; axis of symmetry: $x = \frac{5}{8}$; x -coordinate of vertex = $\frac{5}{8}$

x	$f(x)$
$-\frac{3}{4}$	-6
$\frac{1}{4}$	1
$\frac{5}{8}$	1.5625
1	1
2	-6



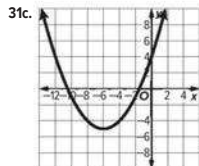
30a. y -int = 0 ; axis of symmetry: $x = -2.75$; x -coordinate of vertex = -2.75

x	$f(x)$
-4	-12
-3	-15
-2.75	-15.125
-2.5	-15
-1.5	-12



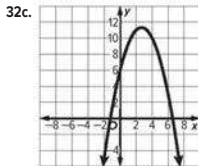
31a. y -int = 4 ; axis of symmetry: $x = -6$; x -coordinate of vertex = -6

x	$f(x)$
-10	-1
-8	-4
-6	-5
-4	-4
-2	-1



32a. y -int = 6 ; axis of symmetry: $x = \frac{8}{3}$; x -coordinate of vertex = $\frac{8}{3}$

x	$f(x)$
$\frac{4}{3}$	10
$\frac{7}{3}$	11.25
$\frac{8}{3}$	$11\frac{1}{3}$
3	11.25
4	10



39a. Let x represent the number of \$1 price increases. Let y represent the income. Then $y = (70 - x)(20 + x)$.

x	y
0	1400
5	1625
10	1800
15	1925
20	2000
25	2025
30	2000
35	1925
40	1800
45	1625
50	1400



39b. (25, 2025); The club should increase the price by \$25 to make a maximum profit of \$2025. Sample answer: Increasing the price by \$25 is more than twice the current price, so this may be unreasonable. The club may want to revisit their assumptions.

41a. If x = number of price increases, the revenue is $R(x) = -5x^2 + 40x + 1200$. Because the price changes and the revenue will not be negative, the domain is $\{x \mid 0 \leq x \leq 20\}$.

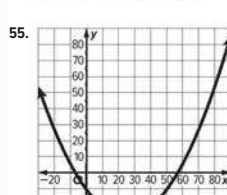
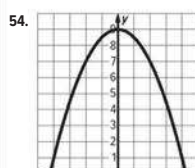
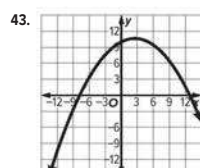
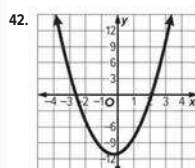
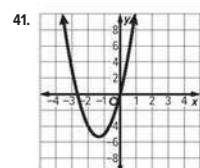
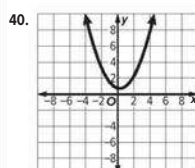
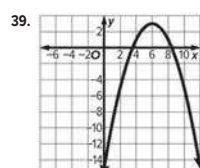
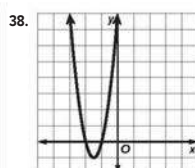
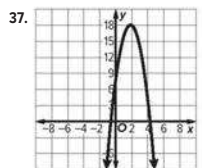
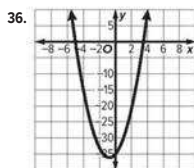
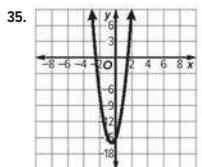
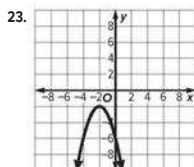
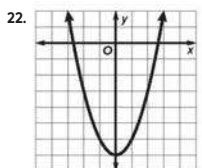
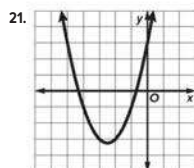
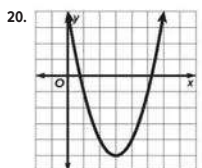
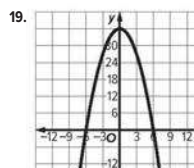
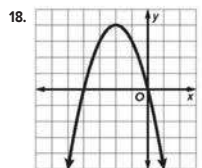
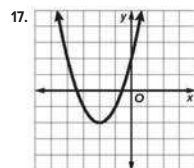
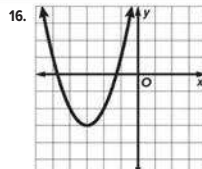
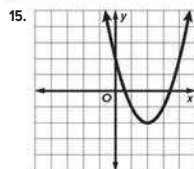


41b. The graph shows that the maximum revenue value occurs at $x = 4$, which corresponds to a ticket price of \$6.00 + 4(\$0.50) or \$8.00. The maximum revenue is $P(4) = \$1280.00$.

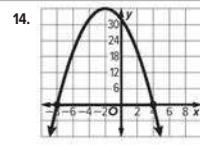
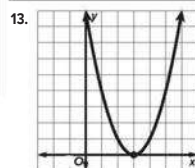
41c. As price increases continue, the demand for tickets will decrease. The x -intercept indicates a price that is too high, and no one is estimated to buy tickets the cinema.

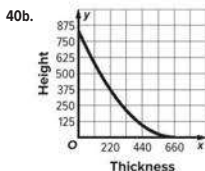
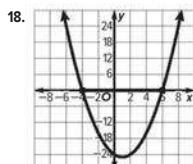
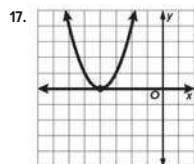
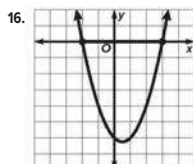
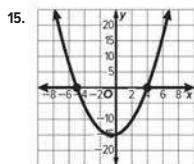
47. Sample answer: A function is quadratic if it has no other terms than a quadratic term, linear term, and constant term. The function has a maximum if the coefficient of the quadratic term is negative and has a minimum if the coefficient of the quadratic term is positive.

Lesson 1-2



Lesson 1-7

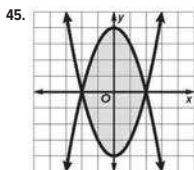




41. No; the graphs of the inequalities intersect the x -axis at the same points.

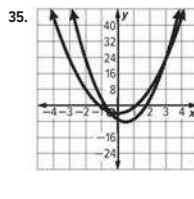
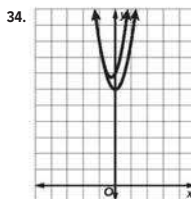
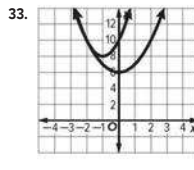
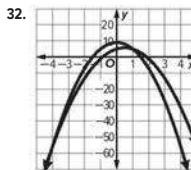
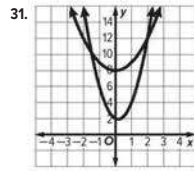
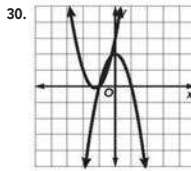
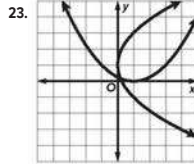
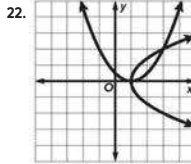
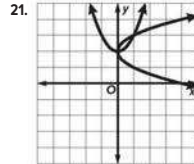
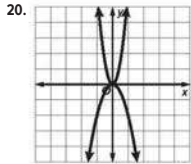
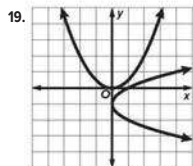
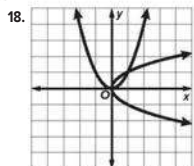
42. Neither; Don graphed the inequality in two variables, and Diego graphed the wrong interval.

44. Sometimes; sample answer: When a is positive and c is negative, there is no solution and when a is negative and c is positive there is a solution set.



46. Sample answer: For both quadratic and linear inequalities, you must first graph the related function. You use the inequality symbol to determine if the line is dashed or solid. Then you use test points to determine where to shade. One difference is that one related function is a straight line while the other related function is a curve.

Lesson 1-8



72. Sample answer: A linear-quadratic system of equations will have no points in common if the parabola opens up and the line is below the parabola with a slope that does not cause it to intersect. There will only be one solution if the line intersects the parabola at the vertex, either horizontally or vertically. There will be two solutions if the line intersects the parabola in a point other than the vertex and is not parallel to the line of symmetry of the parabola.

74. Sample answer: Company profits can be modeled by a quadratic equation and a horizontal line can be added to the model to determine specific price points or quantities of items that maximize profit for the company.

13. **MULTIPLE CHOICE** Solve $4x^2 - 6x - 5 = 0$ by using the Quadratic Formula. (Lesson 1.6)

A. $\frac{3 \pm \sqrt{39}}{4}$
 B. $\frac{-3 \pm \sqrt{39}}{4}$
☒ C. $\frac{3 \pm \sqrt{29}}{4}$
 D. $\frac{-3 \pm \sqrt{29}}{4}$

14. **MULTIPLE CHOICE** The graph of $y = x^2 - 4x + 3$ is shown. Select the values for which $x^2 - 4x + 3 < 0$. (Lesson 1.7)



A. $x < 1, x < 3$
 B. $x < 1, x > 3$
☒ C. $x > 1, x < 3$
 D. $x > 1, x > 3$

15. **OPEN ENDED** Colee wants to add an L-shaped deck along two sides of his garden. The garden is a rectangle 15 feet wide and 20 feet long. The deck width will be the same on both sides and the total area of the garden and deck cannot exceed 500 square feet. How wide can the deck be? (Lesson 1.7)

between 0 and 5 feet

16. **MULTI-SELECT** Solve the system of equations. Which ordered pair is part of the solution set? Select all that apply. (Lesson 1.6)

$$\begin{cases} y + 3 = x \\ y = x^2 - 5 \end{cases}$$

 A. $(-2, -1)$
 B. $(-2, 1)$
☒ C. $(-1, -4)$
 D. $(-1, 1)$
 E. $(1, -4)$
 F. $(1, 4)$
☒ G. $(2, -1)$
 H. $(2, -4)$

17. **OPEN RESPONSE** What are the solutions of this system of equations? (Lesson 1.6)

$$\begin{cases} y = (x - 4)^2 - 3 \\ y = x - 1 \end{cases}$$

 (2, 1), (7, 6)

18. **MULTI-SELECT** What is the solution to the system of equations? Select all that apply. (Lesson 1.6)

(1) $y = x^2 - 2x - 1$
 (2) $y = x + 3$
 A. $(-4, 1)$
 B. $(-3, 0)$
☒ C. $(-1, 2)$
 D. $(1, 0)$
 E. $(1, 4)$
☒ F. $(4, 7)$

Polynomials and Polynomial Functions

Module Goals

- Students analyze polynomial functions by examining key features and graphing.
- Students analyze the graphs of polynomial functions by identifying key features.
- Students add, subtract, and multiply polynomials.
- Students divide polynomials by using long division and synthetic division.
- Students expand powers of binomials.

Focus

Domain: Algebra, Functions

Standards for Mathematical Content:

F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

A.APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. *Also addresses A.APR.5, A.APR.6, and F.IF.7c.*

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this module.

Coherence

Vertical Alignment

Previous

Students graphed and performed operations with polynomials of degree one and degree two.

A.CED.2, F.IF.8a

Now

Students graph and analyze polynomial functions and perform operations with polynomial expressions.

A.APR.1, F.IF.7c

Next

Students will write and solve polynomial equations and identify zeros of polynomial functions.

A.CED.1, A.APR.3

Rigor

The Three Pillars of Rigor

To help students meet standards, they need to illustrate their ability to use the three pillars of rigor. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. Once they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Independent Practice.

1 CONCEPTUAL UNDERSTANDING 2 FLUENCY 3 APPLICATION

EXPLORE

LEARN

EXAMPLE & PRACTICE

Suggested Pacing

Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
2-1 Polynomial Functions	F.IF.4; F.IF.7c	3	1.5
2-2 Analyzing Graphs of Polynomial Functions	F.IF.4; F.IF.7c	1	0.5
2-3 Operations with Polynomials	A.APR.1	2	1
2-4 Dividing Polynomials	A.APR.6	1	0.5
2-5 Powers of Binomials	A.APR.5	1	0.5
Module Review		1	0.5
Module Assessment		1	0.5
Total Days		11	5.5

NAME _____ DATE _____

Cheryl Tobey Math Probe
Subtracting Polynomials

Indicate which of the responses, if any, is a simplified version of $(3x^2 - 5x + 6) - (2x^2 - 3x + 5)$.

Circle your answer.	Explain your choice.
1. $3x^2 - 11x + 11$ YES NO	
2. $5x^2 - 11x + 11$ YES NO	
3. $5x^2 + 11$ YES NO	
4. $-5x^2 - 4$ YES NO	
5. $5x^2 + x - 4$ YES NO	
6. $5x^2 - 4$ YES NO	
7. $5x^2 + 5$ YES NO	

Based on a math probe by Cheryl Tobey. © 2008. All rights reserved.

Correct Answers: 1. no 2. no
3. no 4. no 5. yes 6. no
7. no

Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will determine which of several expressions is a simplified version of a polynomial expression involving subtraction.

Targeted Concepts Understand that polynomials can be subtracted by taking the additive inverse of terms being subtracted and then combining like terms to simplify.

Targeted Misconceptions

- Students may incorrectly distribute the subtraction across the second polynomial by:
 - completely ignoring the subtraction and treating it like an addition problem, or
 - only subtracting the first term of the second polynomial.
- Students may incorrectly combine like terms, including adding exponents.

Use the Probe after Lesson 2-3.

Collect and Assess Student Answers

If the student selects these responses...

1. yes
2. yes

Then the student likely...

combines like terms without correctly distributing the negative across the second polynomial by taking the additive inverse of each term.

Examples: For Item 1, they did not take the additive inverse of any of the terms, and for Item 2, they only took the additive inverse of the first term.

3. yes
4. yes

does not distribute the negative across the second polynomial by taking the additive inverse of each term and does not correctly combine like terms.

Example: For Item 3, they did not take the additive inverse of any of the terms in the second polynomial and incorrectly combined all terms with a variable by adding coefficients and adding exponents.

6. yes
7. yes

distributes the negative across the second polynomial correctly but has errors with combining like terms.

Examples: For Item 7, they distributed the negative correctly and started to accurately combine like terms, but instead of leaving it as $6x^2 + x - 4$, they inaccurately combined terms with variables.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

- Lesson 2-3, Learn, Example 3

Revisit the Probe at the end of the module to be sure that your students no longer carry these misconceptions.



The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

How does an understanding of polynomials and polynomial functions help us understand and interpret real-world events? Sample answer: Polynomials and polynomial functions can be used to model situations where quantities increase or decrease in a nonlinear pattern.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

Focus Students read about polynomial equations.

Teach Throughout the module, have students take notes under the tabs of their Foldables while working through each lesson. They should include definitions of terms and key concepts. Encourage students to record new ways of graphing, analyzing, and solving polynomial equations in their Foldable.

When to Use It Use the appropriate tabs as students cover each lesson in this module. Students should add vocabulary to the back of their Foldable during each lesson.

Launch the Module

For this module, the Launch the Module video uses energy production and usage to model polynomial functions. Exploring wind turbines, hybrid cars, and solar panels, students learn how polynomials and polynomial functions can be applied in real-world contexts.

Module 2 Polynomials and Polynomial Functions

Essential Question

How does an understanding of polynomials and polynomial functions help us understand and interpret real-world events?

What Will You Learn?

How much do you already know about each topic **before** starting this module?

KEY:	Before	After
👉 I don't know	👉 I don't know	👉 I don't know
👉 I've heard of it	👉 I've heard of it	👉 I've heard of it
👉 I know it	👉 I know it	👉 I know it
graph power functions		
graph polynomial functions		
use the location principle to find zeros of a function		
identify extrema of graphs of functions		
add and subtract polynomials		
multiply polynomials		
divide polynomials using long division		
divide polynomials using synthetic division		
expand powers of binomials		

Foldables Make this Foldable to help you organize your notes about polynomials and polynomial functions. Begin with one sheet of 8½ by 11-inch paper.

1. Fold a 2-inch tab along the bottom of a long side.

2. Fold along the width into thirds.

3. Staple the outer edges of the tab.

4. Label the tabs Polynomials, Polynomial Functions, and Polynomial Graphs.



Module 2 • Polynomials and Polynomial Functions 71

Interactive Presentation



What Vocabulary Will You Learn?

- binomial
- closed
- degree
- degree of a polynomial
- FOIL method
- leading coefficient
- monomial function
- Pascal's triangle
- polynomial in one variable
- polynomial function
- power function
- quartic function
- quintic function
- standard form of a polynomial
- synthetic division
- trinomial

Are You Ready?

Complete the Quick Review to see if you are ready to start this module. Then complete the Quick Check.

Quick Review Example 1 Rewrite $2xy - 3 - z$ as a sum. $2xy - 3 - z$ $= 2xy + (-3) + (-z)$		Example 2 Use the Distributive Property to rewrite $-3(p + b - c)$. $-3(p + b - c)$ $= -3(p) + (-3)(b) + (-3)(-c)$ $= -3p - 3b + 3c$
Original expression	Rewrite using addition	Original expression
		Distributive Property
		Simplify

Quick Check Rewrite each difference as a sum. 1. $-5 - 13$ $-5 + (-13)$ 2. $5 - 3y$ $5 + (-3y)$ 3. $5uv - 7np$ $5uv + (-7np)$ 4. $3c^2y - 54xy^2$ $3c^2y + (-54xy^2)$		Use the Distributive Property to rewrite each expression without parentheses. 5. $-4(p + 5)$ $-4p - 20$ 6. $-5(3d^2 + 2b - 9)$ $-15d^2 - 10b + 45$ 7. $-\frac{1}{2}(2m - 5)$ $-m + \frac{5}{2}$ 8. $-\frac{3}{4}(3x + 5)$ $-\frac{9}{4}x - \frac{15}{4}$

How Did You Do?
Which exercises did you answer correctly in the Quick Check?

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define A polynomial function is a continuous function that can be described by a polynomial equation in one variable.

Example $f(x) = 3x^2 - 4x + 6$

Ask Is $g(x) = -4x^2 + 2$ a polynomial function? **yes**

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- identifying values of a , b , and c
- selecting graphs by attributes
- performing simple operations with polynomials
- multiplying polynomials
- expanding perfect squares



ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You may want to use the **Quadratic and Polynomial Functions** section to ensure student success in this module.



Mindset Matters

Promote Process Over Results

The process that a student takes as they encounter a new problem is just as important—if not more important—than the result.

How Can I Apply It?

Encourage students to consider the **Think About It!** prompts in their Interactive Student Edition. Have students discuss their problem-solving strategies with a partner. Be sure to support the process and reward effort as students explore and work through problems.

Polynomial Functions

LESSON GOAL

Students analyze polynomial functions by examining key features and graphing.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 INQUIRY AND DEVELOP

 **Explore:** Power Functions

 **Develop:**

Graphing Power Functions

- End Behavior and Degree of a Monomial Function
- Graph a Power Function by Using a Table

 **Explore:** Polynomial Functions

 **Develop:**

Graphing Polynomial Functions

- Degrees and Leading Coefficients
- Evaluate and Graph a Polynomial Function
- Zeros of a Polynomial Function
- Compare Polynomial Functions

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources

Remediation: Adding and Subtracting Polynomials

Extension: Approximation by Means of Polynomials

AL	OE	EL	
●	●		●
	●	●	●

Language Development Handbook

Assign page 9 of the *Language Development Handbook* to help your students build mathematical language related to analyzing polynomial functions.

ELL You can use the tips and suggestions on page T9 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	1.5 days
45 min	3 days

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

F.IF.7c Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

6 Attend to precision.

Coherence

Vertical Alignment

Previous

Students graphed quadratic functions. **F.IF.4, F.IF.6**

Now

Students analyze polynomial functions by examining key features and graphing. **F.IF.4, F.IF.7c**

Next

Students will analyze the graphs of polynomial functions by identifying key features. **F.IF.4, F.IF.7c**

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

 **Conceptual Bridge** In this lesson, students extend their understanding of polynomials to include polynomial functions and build fluency by graphing them. They apply their understanding of graphing polynomial functions by solving real-world problems.

Mathematical Background

An expression made up of a sum of monomials that contains one variable is called a polynomial in one variable. The degree of a polynomial in one variable is the greatest exponent of its variable. The leading coefficient is the coefficient of the term with the highest degree. A polynomial function can be described by a polynomial equation in one variable.



Interactive Presentation

Warm Up

If $f(x) = ax + b$, what can you tell about a and b from the graphs?

-
-

If $f(x) = ax^2 + bx + c$, what can you tell about a and c from the graphs?

-
-

Warm Up

Launch the Lesson

One of the oldest continuously running roller coasters in the world is the Jack Rabbit at Six Flags Great Escape Park near Rochester, New York. The roller coaster opened in 1920 and took enthusiasts with joy for the ride today. With a maximum height of 75 feet and 2330 feet of track, a ride on the wooden roller coaster takes passengers nearly 2 minutes to complete. The tracks of many roller coasters can be modeled by a polynomial function. You could even use the graphs of different roller coasters to see how they compare.

Launch the Lesson

Vocabulary

Expand All Collapse All

- > power function
- > monomial function
- > polynomial
- > standard form of a polynomial
- > degree of a polynomial
- > polynomial function

• How do the operations on polynomial expressions follow from the operations on real numbers?
• What characteristics of polynomial functions can be seen on their graphs?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- evaluating values of a , b , and c

Answers:

- $a > 0$ and $b < 0$
- $a < 0$ and $b > 0$
- $a > 0$ and $c < 0$
- $a < 0$ and $c > 0$
- 3: Since the axis of symmetry is left of the y -axis, $-\frac{b}{2a} < 0$. Since $a > 0$, $b > 0$. 4: Since the axis of symmetry is right of the y -axis, $-\frac{b}{2a} > 0$. Since $a < 0$, $b > 0$.

Launch the Lesson



Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students relate polynomial functions to the tracks of roller coasters.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.



Explore Power Functions

Objective

Students use a sketch to explore key features of power functions.



Teaching the Mathematical Practices

1 Explain Correspondences Throughout the Explore, encourage students to explain the relationships between the values of the coefficient and degree of monomial function and the end behavior of the function.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will use a sketch to explore end behaviors based on different values of a and n in monomial function $f(x) = ax^n$. Then, they will make conjectures about their findings. Then, students will answer the Inquiry Question.

(continued on next page)

Interactive Presentation

Power Functions

INQUIRY How do the coefficient and degree of a function of the form $f(x) = ax^n$ affect its end behavior?

A function of the form $f(x) = ax^n$, where a and n are nonzero real numbers, is called a **power function**. Power functions for which n is a positive integer are called **monomial functions**. Use the sketch to explore the end behavior of the graphs of monomial functions. Drag the points to change the values of a and n .

Explore

Exercise 1

Set $a = 2$ and $n = -4$. Describe the end behavior.

As $x \rightarrow -\infty$, $f(x) \rightarrow$ and as $x \rightarrow \infty$, $f(x) \rightarrow$.

Check Answer

Drag the point across the first slider to see the end behavior for various values of a . State the end behavior of a function of the form $f(x) = ax^3$ where $a \neq 0$.

a	x	$f(x)$
$a < 0$	$x \rightarrow -\infty$	$f(x) \rightarrow$
	$x \rightarrow \infty$	$f(x) \rightarrow$
$a > 0$	$x \rightarrow -\infty$	$f(x) \rightarrow$

Explore

WEB SKETCHPAD



Students use a sketch to model power functions.



Interactive Presentation

Exercise 2

Set $n = 3$ and $a = -8$. Describe the end behavior.

As $x \rightarrow -\infty$, $f(x) \rightarrow$ and as $x \rightarrow \infty$, $f(x) \rightarrow$.

Check Answer

Drag the point across the first slider to see the end behavior for various values of n . Make a conjecture about the end behavior of a function of the form $f(x) = ax^n$ where $a \neq 0$.

Submit

Explore

INQUIRY How do the coefficient and degree of a function of the form $f(x) = ax^n$ affect its end behavior?

Submit

Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Power Functions (*continued*)

Teaching the Mathematical Practices

5 Use Mathematical Tools To complete the Explore, students will need to use a sketch to analyze the end behavior of monomial functions for various coefficients and degrees. Work with students to explore and deepen their understanding of power functions.

Questions

Have students complete the Explore activity.

Ask:

- If the square of a real number is always positive, why does a negative value for a make the graph open down? **Sample answer:** The values for $y = x^2$ will always be positive but multiplying by a negative coefficient makes all values negative.
- What do you think is the end behavior for $y = x^n$? $y = x^n$? **Sample answer:** The degree is even for $y = ax^n$, so the end behavior is the same as x approaches both positive and negative infinity. The degree is odd for $y = ax^n$, so the end behavior is opposite as x approaches positive and negative infinity.



Inquiry

How do the coefficient and degree of a function of the form $f(x) = ax^n$ affect its end behavior? **Sample answer:** The end behavior of a function of the form $f(x) = ax^n$ changes based on whether the coefficient is positive or negative and whether the degree is even or odd. If n is even, then the end behavior is the same as x approaches both $-\infty$ and ∞ . If n is odd, the end behavior is different as x approaches $-\infty$ and ∞ .



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Explore Polynomial Functions

Objective

Students use a sketch to explore key features of polynomial functions.



Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between cubic, quartic, and quintic functions and key features of the graphs of the functions.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will use a sketch to explore the number of x-intercepts a polynomial function has based on coefficient values and the degree of the function. Then, students will answer the Inquiry Question.

(continued on next page)

Interactive Presentation

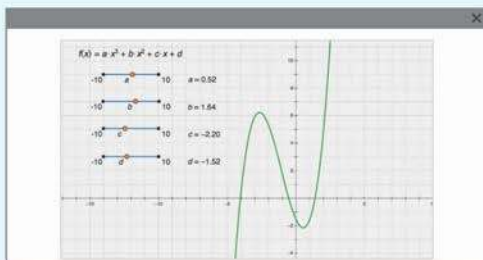
Polynomial Functions

INQUIRY How is the degree of a function related to the number of times its graph intersects the x-axis?

Cubic Functions

Use the sketch to explore key features of the graphs of cubic functions. Then complete the exercises.

Explore



Explore

WEB SKETCHPAD



Students use a sketch to model power functions.



Interactive Presentation



Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Polynomial Functions (*continued*)

Teaching the Mathematical Practices

3 Construct Arguments Throughout the Explore, students will use established results from the sketch to construct arguments about the key features of polynomial functions.

Questions

Have students complete the Explore activity.

Ask:

- What is the maximum number of times the graph of $y = ax^2$ can cross the x-axis? **Sample answer:** The graph of $y = ax^2$ can cross the x-axis 0, 1 or 2 times. The graph cannot cross more than twice.
- What is the maximum number of times the graph of $y = x^n$ can cross the x-axis? **Sample answer:** The graph of $y = x^n$ can cross a maximum of n times, so $y = x^n$ can cross at most 12 times.



Inquiry

How is the degree of a function related to the number of times its graph intersects the x-axis? **Sample answer:** The degree of a function is the maximum number of times its graph will intersect the x-axis.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Graphing Power Functions

Objective

Students graph and analyze power functions.

MP Teaching the Mathematical Practices

3 Construct Arguments In the Talk About It! feature, students will use definitions to determine whether $f(x) = \sqrt{x}$ is a power function and/or monomial function.

About the Key Concept

Help students understand how the graph of a monomial function changes for different values of the leading coefficient and degree of the function. Find patterns in different values, and use them to make conjectures that will help students understand the examples.

Common Misconception

Some students may not remember the vocabulary terms used to describe a function. Remind them that in $f(x) = ax^n$, a is the coefficient, and n is the degree of the function.

Essential Question Follow-Up

Students have explored power functions with different degrees.

Ask:

How does a power function's graph change as the degree of the function increases? **Sample answer:** As the degree increases, the ends of the graphs will alternate between going in the same and opposite directions.

DIFFERENTIATE

Language Development Activity

Beginning Reinforce the use of visual context to derive meaning through examples of environmental print such as graphs of functions in news articles you find online. Pantomime or elicit one-word responses to the meaning derived from such images.

Intermediate Provide graphs from real-world articles to illustrate problems that students can work in pairs to solve. Move around the room to monitor progress.

Advanced High Provide a graph and have students write an opinion or explanation of what the graph represents. Have volunteers share their observations with their group.

Go Online

- F ind additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 2-1

Polynomial Functions

Explore Power Functions

Online Activity Use graphing technology to complete the Explore.

INQUIRY How do the coefficient and degree of a function of the form $f(x) = ax^n$ affect its end behavior?

Learn: Graphing Power Functions

A **power function** is any function of the form $f(x) = ax^n$ where a and n are nonzero real numbers. For a power function, a is the **leading coefficient** and n is the **degree**, which is the value of the exponent. A power function with positive integer n is called a **monomial function**.

Key Concept: End Behavior of a Monomial Function

Degree: even	Degree: odd
Leading Coefficient: positive End Behavior: As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$. As $x \rightarrow \infty$, $f(x) \rightarrow \infty$. 	Leading Coefficient: positive End Behavior: As $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$. As $x \rightarrow \infty$, $f(x) \rightarrow \infty$. 
Leading Coefficient: negative End Behavior: As $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$. As $x \rightarrow \infty$, $f(x) \rightarrow \infty$. 	Leading Coefficient: negative End Behavior: As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$. As $x \rightarrow \infty$, $f(x) \rightarrow -\infty$. 

Today's Goals

- Graph and analyze power functions.
- Graph and analyze polynomial functions.

Today's Vocabulary

power function
leading coefficient
degree
monomial function
polynomial in one variable
standard form of a polynomial
degree of a polynomial
polynomial function
quartic function
quintic function

Talk About It!

Is $f(x) = \sqrt{x}$ a power function? Is it a monomial function? Explain your reasoning.

Sample answer: $f(x) = \sqrt{x}$ is a power function because it is equivalent to $f(x) = x^{\frac{1}{2}}$, and $\frac{1}{2}$ is a nonzero constant real number. It is not a monomial function because $\frac{1}{2}$ is not a positive integer.

Lesson 2-1 • Polynomial Functions 73

Interactive Presentation

Graphing Power Functions

A **power function** is any function of the form $f(x) = ax^n$ where a and n are nonzero real numbers. For a power function, a is the **leading coefficient** and n is the **degree**, which is the value of the exponent. A power function with positive integer n is called a **monomial function**.

Key Concept: End Behavior of a Monomial Function

Degree: even	Degree: odd
Leading Coefficient: positive End Behavior: As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$. As $x \rightarrow \infty$, $f(x) \rightarrow \infty$. 	Leading Coefficient: positive End Behavior: As $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$. As $x \rightarrow \infty$, $f(x) \rightarrow \infty$. 
Leading Coefficient: negative End Behavior: As $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$. As $x \rightarrow \infty$, $f(x) \rightarrow \infty$. 	Leading Coefficient: negative End Behavior: As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$. As $x \rightarrow \infty$, $f(x) \rightarrow -\infty$. 

Learn

TYPE



Students answer a question to identify whether a function is a power function or a monomial function.

**Key Concept • Zeros of Even and Odd Degree Functions**

Odd-degree functions will always have at least one real zero. Even-degree functions may have any number of real zeros or no real zeros at all.

Example 1 End Behavior and Degree of a Monomial Function

Describe the end behavior of $f(x) = -2x^3$ using the leading coefficient and degree, and state the domain and range.

The leading coefficient of $f(x)$ is -2 , which is negative.

The degree is 3, which is odd.

Because the leading coefficient is negative and the degree is odd, the end behavior is that as $x \rightarrow -\infty$, $f(x) \rightarrow \infty$ and as $x \rightarrow \infty$, $f(x) \rightarrow -\infty$.

Because the leading coefficient is negative and the degree is odd, the domain and range are all real numbers.

Check

Describe the end behavior, domain, and range of $f(x) = -10x^5$.
end behavior: As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$, and as $x \rightarrow \infty$, $f(x) \rightarrow -\infty$.
domain: all real numbers range: all real numbers

Example 2 Graph a Power Function by Using a Table

PRESSURE For water to flow through a garden hose at a certain rate in gallons per minute (gpm), it needs to have a specific pressure in pounds per square inch (psi). Through testing and measurement, a company that produces garden hoses determines that the pressure P given the flow rate F is defined by $P(F) = \frac{3}{2}F^2$. Graph the function $P(F)$, and state the domain and range.

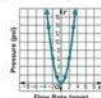
Steps 1 and 2 Find a and n . Then state the domain and range.

For $P(F) = \frac{3}{2}F^2$, $a = \frac{3}{2}$, and $n = 2$.

Because a is positive and n is even, the domain is all real numbers and the range is all real numbers ≥ 0 .

Steps 3–5 Create a table of values and graph the ordered pairs.

F	$P(F)$	$(F, P(F))$
-2	$\frac{3}{2}(-2)^2$	6
-1	$\frac{3}{2}(-1)^2$	1.5
0	$\frac{3}{2}(0)^2$	0
1	$\frac{3}{2}(1)^2$	1.5
2	$\frac{3}{2}(2)^2$	6



Go Online You can complete an Extra Example online.

Go Online

You can watch a video to see how to graph power functions on a TI-84.

Think About It!

Interpret the domain and range given the context of the situation.

Sample answer: Because it is unreasonable to have negative flow rate or negative pressure from a garden hose, both the domain and range are restricted to values greater than or equal to zero.

74 Module 2 • Polynomials and Polynomial Functions

Example 1 End Behavior and Degree of a Monomial Function**MP Teaching the Mathematical Practices**

1 Explain Correspondences Encourage students to explain the relationships between the leading coefficient and degree of a monomial function and the key features of the function.

Questions for Mathematical Discourse

- AL** Why does the negative leading coefficient cause this end behavior for the odd function? **The sign of x and y will always be opposite, so if $x \rightarrow \infty$, y must go to $-\infty$.**
- OL** Through how many quadrants will this monomial function pass? **two**
- BL** What happens to the graph of ax^n as a increases? **The graph gets narrower. What happens to the graph of x^n , where n is even positive integers, as n increases? The part of the graph below $y = 1$ widens, and the part of the graph above $y = 1$ narrows.**

Example 2 Graph a Power Function by Using a Table**MP Teaching the Mathematical Practices**

4 Interpret Mathematical Results Use the Think About It! feature to encourage students to interpret their mathematical results in the context of the problem.

Questions for Mathematical Discourse

- AL** How does the graph of $P(F) = \frac{3}{2}F^2$ compare to the graph of $P(F) = F^2$? **The coefficient $\frac{3}{2}$ makes the graph narrower.**
- OL** Does the function make sense for the context of the situation? **Yes, within the valid domain, an increase in flow yields an increase in pressure.**
- BL** What could a negative flow rate represent? **The flow is in the opposite direction.**

Interactive Presentation

Example 1

TAP

Student tap on each button to determine the end behavior of the function.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Learn Graphing Polynomial Functions

Objective

Students graph and analyze polynomial functions.

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Students should use clear definitions when they answer the question in the Think About It! feature.

Example 3 Degrees and Leading Coefficients

MP Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. In Example 3, guide students to see what information they can gather about the expression just from looking at it.

Questions for Mathematical Discourse

- A** What should you check first when determining the degree and leading coefficient of a polynomial? **Check to see if the terms are in order from greatest to least degree.**
- OL** What is the end behavior of a polynomial of degree $2n - 1$, where n is a non-negative integer? **This would be an odd degree function, which results in the end behavior as $x \rightarrow \infty$, $f(x) \rightarrow -\infty$ and as $x \rightarrow -\infty$, $f(x) \rightarrow \infty$.**
- BI** Why does the degree of the function determine the end behavior? **The greatest degree term has the more effect on the function for greater values of x .**
- Graph $y = 0.005x^4 - x$ using a graphing calculator or online graphing software. Does the end behavior match the degree of the polynomial? **Yes, but it does not appear to until you zoom out.**

Explore Polynomial Functions

Online Activity Use graphing technology to complete the Explore.

INQUIRY How is the degree of a function related to the number of times its graph intersects the x -axis?

Learn Graphing Polynomial Functions

A **polynomial in one variable** is an expression of the form $a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$, where $a_n \neq 0$, $a_0, a_1, \dots, a_{n-1}$ are real numbers, and n is a nonnegative integer. Because the terms are written in order from greatest to least degree, this polynomial is written in **standard form**. The **degree of a polynomial** is n and the leading coefficient is a_n .

A **polynomial function** is a continuous function that can be described by a polynomial equation in one variable. You have learned about constant, linear, quadratic, and cubic functions. A **quartic function** is a fourth-degree function. A **quintic function** is a fifth-degree function. The degree tells you the maximum number of times that the graph of a polynomial function intersects the x -axis.

Example 3 Degrees and Leading Coefficients

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

a. $2x^4 - 3x^2 - 4x^3 - 5x + 6x$ degree: 4 leading coefficient: 2

b. $7x^4 - 2$ degree: 3 leading coefficient: 7

c. $4x^2 - 2xy + 8y^2$ This is not a polynomial in one variable. There are two variables, x and y .

d. $x^3 + 12x^2 - 3x^2 + 2x^2 + 8x + 4$ degree: 5 leading coefficient: 1

Check

Select the degree and leading coefficient of $11x^3 + 5x^2 - 7x - \frac{5}{2}$.

A. degree: 3, leading coefficient: 11

B. degree: 11, leading coefficient: 3

C. This is not a polynomial in one variable. There are two variables, x and y .

D. This is not a polynomial in one variable. The term $\frac{5}{2}$ has the variable with an exponent less than 0.

Go Online You can complete an Extra Example online.

Go Online
You can learn how to graph a polynomial function by watching the video online.

Think About It!
If a polynomial function has a leading coefficient of a , can you determine its end behavior? Explain your reasoning.

Yes; sample answer: I would need to know the degree of the function to determine its end behavior.

Think About It!
Jamison says the leading coefficient of $4x^2 - 3x + 2x^3 - x$ is 4. Do you agree or disagree? Justify your reasoning.

Disagree; sample answer: The term with the greatest degree is $2x^3$, and the coefficient of that term is 2.

Watch Out!
Leading Coefficients
If the term with the greatest degree has no coefficient shown, as in part d, the leading coefficient is 1.

Lesson 2-1 • Polynomial Functions 75

Interactive Presentation

Complete the table to determine the degree and leading coefficient of different polynomial expressions.

Polynomial	Determine	Degree	Leading Coefficient
Constant	10	0	10
Linear	$5x - 4$	1	5
Quadratic	$4x^2 + 3x - 6$	2	4
Cubic	$3x^3 - 4x^2 + 5x - 1$	3	3
Quartic	$5x^4 + 4x^3 - 3x^2 + 2x - 1$	4	5
Quintic	$6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x - 1$	5	6
General	$a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$	n	a_n

Click Me! Click Again

Learn

SELECT



Students select the values to complete the table.

**Think About It!**

What values of x make sense in the context of the situation? Justify your reasoning.

Sample answer: Because x is a percent, only values between 0 and 1, inclusively, make sense in the context of the situation.

Study Tip

Axis Labels Notice that the x -axis is measuring the percent of the radius, not the actual length of the radius.

Example 4 Evaluate and Graph a Polynomial Function

SLU The density of the Sun, in grams per centimeter cubed, expressed as a percent of the distance from the core of the Sun to its surface can be modeled by the function $f(x) = 519x^4 - 1630x^3 + 1844x^2 - 889x + 155$, where x represents the percent as a decimal. At the core $x = 0$, and at the surface $x = 1$.

Part A Evaluate the function.

Find the core density of the Sun at a radius 60% of the way to the surface.

Because we need to find the core density at a radius 60% of the way to the surface, $x = 0.6$. So, replace x with 0.6 and simplify.

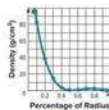
$$\begin{aligned} f(0.6) &= 519(0.6)^4 - 1630(0.6)^3 + 1844(0.6)^2 - 889(0.6) + 155 \\ &= 519(0.1296) - 1630(0.216) + 1844(0.36) - 889(0.6) + 155 \\ &= 67.2624 - 352.08 + 663.84 - 533.4 + 155 \\ &= 0.6224 \text{ g/cm}^3 \end{aligned}$$

Part B Graph the function.

Sketch a graph of the function.

Substitute values of x to create a table of values. Then plot the points, and connect them with a smooth curve.

x	$f(x)$
0.1	82.9619
0.2	38.7504
0.3	14.4539
0.4	3.4264
0.5	0.1875
0.6	0.6224
0.7	1.7891
0.8	1.9824
0.9	0.7859

**Check**

CARDIOLOGY To help predict heart attacks, doctors can inject a concentration of dye in a vein near the heart to measure the cardiac output in patients. In a normal heart, the change in the concentration of dye can be modeled by $f(x) = -0.006x^4 + 0.040x^3 - 0.0093x^2 + 1.78x$, where x is the time in seconds.

Part A Find the concentration of dye after 5 seconds.

$$f(5) = \underline{\underline{-7.21375}}$$

Go Online You can complete an Extra Example online.

Example 4 Evaluate and Graph a Polynomial Function**MP Teaching the Mathematical Practices**

6 Use Quantities In Example 4, guide students to clarify their use of quantities. Ensure that they specify the units of measure they are using in the problem and label axes appropriately.

Questions for Mathematical Discourse

AL What does the decreasing trend of the graph mean in the context of the example? **As you move farther from the center of the Sun, the density decreases.**

OL What do you expect the density to be at $x = 1$? Explain. **The density should be zero because $x = 1$ represents the surface. Evaluate $f(1)$. Does the result match your expectation? Explain. -1 ; Sample answer: No; because the density is estimated by a model, the value at $f(1)$ is close to the expected value but not exact.**

BL What is the density of the Sun at its core? **155 g/cm³. The density of air at sea level on Earth is about 1.225×10^{-3} g/cm³. What is the difference in order of magnitude between these values? The density at the Sun's core is 5 orders of magnitude greater than the density of air on Earth at sea level.**

Common Error

Students may assume that the x -axis is measuring the length of the radius, instead of a percent of the radius. Explain what the x - and y -axes represent on the coordinate plane in the context of the situation.

Interactive Presentation

Example 4

TYPE**a**

Students answer a question about the possible domain of the function in the context of the situation.

**Example 5** Zeros of a Polynomial Function**MP** Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationship between the graph of a polynomial function and the number of real zeros of the function.

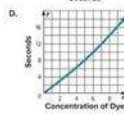
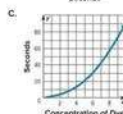
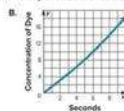
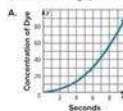
Questions for Mathematical Discourse

- AL** Is the degree of the function even or odd? Explain. **Odd** because the end behavior in the y -direction for $x \rightarrow -\infty$ is the opposite direction of the end behavior as $x \rightarrow \infty$.
- OL** Can you tell the degree of the function from this graph? **No**, you can only tell that the degree is odd and greater than 1.
- BL** How would you transform this graph to only have 2 unique zeros? **Sample answer:** Translate the graph up until the relative minimum is on the x -axis.

Common Error

Students may confuse zeros with y -intercept(s). Remind them that zeros are the values of x when $f(x) = 0$.

Part B Select the graph of the concentration of dye over 10 seconds. A

**Example 5** Zeros of a Polynomial Function

Use the graph to state the number of real zeros of the function.

The real zeros occur at $x = -2$, 1 , and 4 , so there are three real zeros.

**Check**

Use the graph to state the number of real zeros of the function.



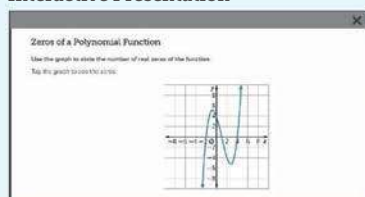
The function has 4 real zero(s).

Go Online You can complete an Extra Example online.

Study Tip

Zeros The real zeros occur at values of x where $f(x) = 0$, or where the polynomial intersects the x -axis. Recall that odd-degree polynomial functions have at least one real zero and even-degree polynomial functions have any number of real zeros. So, the minimum number of times that an odd-degree polynomial intersects the x -axis is 1, and the minimum number of times that an even-degree polynomial intersects the x -axis is 0.

Lesson 2.1 • Polynomial Functions **77**

Interactive Presentation

Example 5

TAP

Students tap on the graph to see the real zeros of a function.



Example 6 Compare Polynomial Functions



Teaching the Mathematical Practices

1 Explain Correspondences In Example 6, students will explain the relationships between the key features of the graphs of the given functions.

Think About It!
Find the domain and range of $f(x)$. Does $g(x)$ have the same domain and range? Explain.

The domain and range of $f(x)$ are both all real numbers. No; sample answer: The domain of $g(x)$ is also all real numbers, but the range is all real numbers greater than or equal to the minimum, which is between $y = -2$ and $y = -3$.

Study Tip:

Zeros. The zeros of a polynomial function are the x -coordinates of the points at which the graph intersects the x -axis.

Example 6 Compare Polynomial Functions

Examine $f(x) = x^3 + 2x^2 - 3x$ and $g(x)$ shown in the graph.

Part A Graph $f(x)$.

Substitute values for x to create a table of values. Then plot the points, and connect them with a smooth curve.

x	$f(x)$
-3	0
-2	6
-1	4
0	0
1	0
2	10
3	36



Part B Analyze the extrema.

Which function has the greater relative maximum?

$f(x)$ has a relative maximum at approximately $y = 6$, and $g(x)$ has a relative maximum between $y = 2$ and $y = 3$. So, $f(x)$ has the greater relative maximum.

Part C Analyze the key features.

Compare the zeros, x - and y -intercepts, and end behavior of $f(x)$ and $g(x)$ zeros.

$f(x)$: $-3, 0, 1$

$g(x)$: The graph appears to intersect the x -axis at $-1, -0.5, 1, 2$

Intercepts

$f(x)$: x -intercepts: $-3, 0, 1$; y -intercept: 0

$g(x)$: x -intercepts: $-1, -0.5, 1, 2$; y -intercept: 2

End behavior

$f(x)$: As $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$, and as $x \rightarrow \infty$, $f(x) \rightarrow \infty$.

$g(x)$: As $x \rightarrow -\infty$, $g(x) \rightarrow -\infty$, and as $x \rightarrow \infty$, $g(x) \rightarrow \infty$.

Pause and Reflect

Did you struggle with anything in this lesson? If so, how did you deal with it?

See students' observations.

Go Online You can complete an Exit Ticket online.

78 Module 2 • Polynomials and Polynomial Functions

Interactive Presentation

Compare Polynomial Functions
Examine $f(x) = x^3 + 2x^2 - 3x$ and $g(x)$ shown in the graph.

Example 6

CHECK



Students complete the Check online to determine whether they are ready to move on.

Questions for Mathematical Discourse

- AL** How many zeros and relative extrema does each function have?
 $f(x)$: 3 zeros, 2 relative extrema; $g(x)$: 4 zeros; 3 relative extrema
- OL** Based on the end behavior of the graph of $g(x)$, what type of degree and leading coefficient would you expect the function to have? **even degree, positive leading coefficient**
- BL** Can you tell whether the degree of a function is even or odd from the number of zeros? Explain. **No; sample answer: A function could intersect the x -axis at a relative extreme point, or it could continue through, and you cannot tell which is the case just by the number of unique real zeros.**

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2 exercises that mirror the examples		1–22
2	exercises that use a variety of skills from this lesson	23–30
2	exercises that extend concepts learned in this lesson to new contexts	31–35
3	exercises that emphasize higher-order and critical-thinking skills	36–39

ASSESS AND DIFFERENTIATE

 Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, THEN assign:

- Practice Exercises 1–35 odd, 36–39
- Extension: Approximation by Means of Polynomials
-  **ALEKS** Real Zeros of Polynomial Functions, Graphs of Polynomial Functions

BL

IF students score 66%–89% on the Checks, THEN assign:

- Practice, Exercises 1–39 odd
- Remediation, Review Resources: Adding and Subtracting Polynomials
- Personal Tutors
- Extra Examples 1–6
-  **ALEKS** Values of a , b , and c

OL

IF students score less than 65% on the Checks, THEN assign:

- Practice, Exercises 1–21 odd
- Remediation, Review Resources: Adding and Subtracting Polynomials
- Quick Review Math Handbook:** Polynomial Functions
-  **ALEKS** Values of a , b , and c

AL

Practice

 Go Online You can complete your homework online.

Example 1

Describe the end behavior of each function using the leading coefficient and degree, and state the domain and range.

- $f(x) = 3x^4$
As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$ and as $x \rightarrow \infty$, $f(x) \rightarrow \infty$.
Domain: $D = (-\infty, \infty)$, Range: $R = [0, \infty)$
- $f(x) = -2x^3$
As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$ and as $x \rightarrow \infty$, $f(x) \rightarrow -\infty$.
Domain: $D = (-\infty, \infty)$, Range: $R = (-\infty, \infty)$
- $f(x) = \frac{1}{2}x^5$
As $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$ and as $x \rightarrow \infty$, $f(x) \rightarrow \infty$.
Domain: $D = (-\infty, \infty)$, Range: $R = (-\infty, \infty)$
- $f(x) = \frac{1}{3}x^6$
As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$ and as $x \rightarrow \infty$, $f(x) \rightarrow \infty$.
Domain: $D = (-\infty, \infty)$, Range: $R = [0, \infty)$

Example 2

5. **USE A MODEL** The shape of a parabolic reflector inside a flashlight can be modeled by the function $f(x) = \frac{1}{2}x^2$. Graph the function $f(x)$, and state the domain and range.
See margin for graph. $D = (-\infty, \infty)$, $R = [0, \infty)$

6. **MACHINE EFFICIENCY** A company uses the function $f(x) = x^3 + 3x^2 - 18x - 40$ to model the change in efficiency of a machine based on its position x . Graph the function and state the domain and range. See margin for graph. $D = (-\infty, \infty)$, $R = (-\infty, \infty)$

Example 3

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

- $n + 8$
degree = 1, leading coefficient = 1
- $(2x - 1)(x^2 + 3)$
degree = 3, leading coefficient = 8
- $-5x^3 + 3x^2 - 8$
degree = 5, leading coefficient = -5
- $15 - 3x + 5x^2 - x^3 + 7x^6$
degree = 6, leading coefficient = 7

- $x^3 + 4x^2 + x^2$
11. not in one variable because there are two variables, x and t
12. not a polynomial because there is a negative exponent

Example 4

13. **TRIANGLES** Dylan drew n dots on a piece of paper making sure that no set of 3 points were collinear. The number of triangles that can be made using the dots as vertices is equal to $f(n) = \frac{1}{6}n^3 - \frac{1}{2}n^2 + 2n$, when $n \geq 0$.
a. If Dylan drew 15 dots, how many triangles can be made? 435
b. Sketch a graph of the function. See margin.

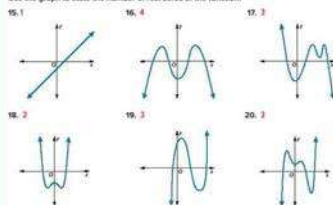


Lesson 2.4 • Polynomial Functions 79

14. **DISCING** The volume of a drill bit can be estimated by the formula for a cone, $V = \frac{1}{3}\pi r^2 h$, where h is the height of the bit and r is its radius. Substituting $\frac{1}{3}\pi$ for $\frac{1}{3}$, the volume of the drill bit can be estimated by $V = \frac{1}{3}\pi r^2 h$.
a. What is the volume of a drill bit with a radius of 3 centimeters? $3\pi\sqrt{3}$ cm³
b. Sketch a graph of the function. See margin.

Example 5

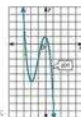
Use the graph to state the number of real zeros of the function.



Example 6

21. Examine $f(x) = x^3 - 2x^2 - 4x + 1$ and $g(x)$ shown in the graph.

- Which function has the greater relative maximum? $f(x)$
- Compare the zeros, x - and y -intercepts, and end behavior of $f(x)$ and $g(x)$.
zeros: $f(x)$: -1.39, 0.23, 3.19; $g(x)$: -2.75, -0.5, 0.25
 x -intercepts: $f(x)$: -1.39, 0.23, 3.19; $g(x)$: -2.75, -0.5, 0.25
 y -intercept: $f(x)$: 1; $g(x)$: 1
end behavior: $f(x)$: As $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$, and as $x \rightarrow \infty$, $f(x) \rightarrow \infty$.
 $g(x)$: As $x \rightarrow -\infty$, $g(x) \rightarrow -\infty$, and as $x \rightarrow \infty$, $g(x) \rightarrow \infty$.



3 REFLECT AND PRACTICE



F.IF.4, F.IF.7c

22. Examine the graph of $f(x)$ and $g(x)$ shown in the table.



x	-5	-3	0	1.5	3
$f(x)$	0	0	-3	-15	0

- a. Which function has the greater relative maximum? $f(x)$
 b. Compare the zeros, x - and y -intercepts, and end behavior of $f(x)$ and $g(x)$.
 zeros: $f(x) = -3, g(x) = -3, 3$
 x -intercepts: $f(x) = -3, 3, g(x) = -3, 3$
 y -intercept: $f(x) = -3$
 and behavior: $f(x)$: As $x \rightarrow -\infty, f(x) \rightarrow -\infty$, and as $x \rightarrow \infty, f(x) \rightarrow -\infty$; $g(x)$: As $x \rightarrow -\infty, g(x) \rightarrow \infty$, and as $x \rightarrow \infty, g(x) \rightarrow \infty$.

Mixed Exercises

Describe the end behavior, state the degree and leading coefficient of each polynomial. If the function is not a polynomial, explain why.

23. $f(x) = -5x^3 + 3x^2$
 As $x \rightarrow -\infty, f(x) \rightarrow \infty$ and as $x \rightarrow \infty, f(x) \rightarrow -\infty$; degree = 3; leading coefficient = -5.
 24. $g(x) = 2x^5 + 5x^4$
 As $x \rightarrow -\infty, g(x) \rightarrow -\infty$ and as $x \rightarrow \infty, g(x) \rightarrow \infty$; degree = 5; leading coefficient = 2.
 25. $h(x) = 0x^5 - 0x^4 + 3x^3$
 As $x \rightarrow -\infty, h(x) \rightarrow -\infty$ and as $x \rightarrow \infty, h(x) \rightarrow \infty$; degree = 3; leading coefficient = -5.
 26. $f(x) = 8x^4 + 5x^3$
 As $x \rightarrow -\infty, f(x) \rightarrow \infty$ and as $x \rightarrow \infty, f(x) \rightarrow \infty$; degree = 4; leading coefficient = 8.
 27. $g(x) = -6x^4 - 4x^3 + 13x^2$
 This is not a polynomial because there is a negative exponent.
 28. $h(x) = (x + 5)(3x - 4)$
 As $x \rightarrow -\infty, h(x) \rightarrow -\infty$ and as $x \rightarrow \infty, h(x) \rightarrow \infty$; degree = 2; leading coefficient = 3.
 29. $f(x) = (5 - 2x)(4 + 3x)$
 As $x \rightarrow -\infty, f(x) \rightarrow -\infty$ and as $x \rightarrow \infty, f(x) \rightarrow -\infty$; degree = 2; leading coefficient = -5.
 30. $g(x) = 3x^2 - 4x^3 + \frac{1}{x}$
 This is not a polynomial because there is a negative exponent.

31. **REASONING** Describe the end behavior, and the possible degree and sign of the leading coefficient of the graph shown.
 As $x \rightarrow -\infty, y \rightarrow -\infty$, and as $x \rightarrow \infty, y \rightarrow \infty$; degree = 3; the leading coefficient is negative.



Lesson 2-4 • Polynomial Functions

32. **CONSTRUCT ARGUMENTS** Explain why a polynomial function with an odd degree must have at least one real zero. Sample answer: For a polynomial function with an odd degree, $f(x) \rightarrow \infty$ at one end of the graph and $f(x) \rightarrow -\infty$ at the other end. Connecting these two extremes with a continuous graph will mean crossing the x -axis at some point. That point represents a real zero.

33. **STRUCTURE** If $f(x) = ax^3 - bx^2 + c$, determine $f(-x)$. Express the result in standard form. How does the end behavior of $f(-x)$ compare to $f(x)$?
 See margin.

34. **COMPARING** Compare the end behavior of the functions $g(x) = -3x^4 + 15x^3 - 12x^2 + 3x + 20$ and $h(x) = -3x^4 - 18x - 1$. Explain your reasoning.
 See margin.

35. **USE A MODEL** A box has a square base with sides of 10 centimeters and a height of 4 centimeters. For a new box, the height is increased by twice a number x and the lengths of the sides of the base are decreased by x . Write and graph a function to represent the volume of the new box. What new dimensions will produce a box with the greatest volume? Describe your solution process.
 See margin.

Higher-Order Thinking Skills

36. **FIND THE ERROR** Shenequa and Virginia are determining the number of real zeros of the graph. Is either of them correct? Explain your reasoning.
 See margin.

Shenequa

There are 7 real zeros because the graph intersects the x -axis 7 times.

Virginia

There are 6 real zeros because the graph intersects the x -axis 7 times, and there is an invisible zero.



37. **ANALYZE** Compare the functions $g(x)$ and $f(x)$. Determine which function has the potential for more zeros and the degree of each function.

$g(x) = x^4 + x^3 - 13x^2 + x + 4$	24	-18	-10	-6	0	6	12	18	24
$f(x) = -x^4 + x^3 - 13x^2 + x + 4$	-6	-1	3	-2	4	7	-1	-8	5

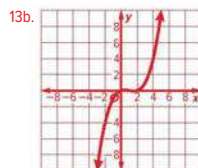
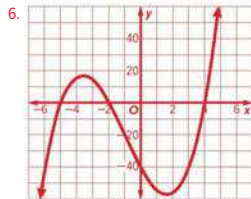
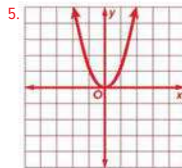
- $f(x)$, $g(x)$ has potential for 5 or more real zeros and a degree of 5 or more; $g(x)$ has potential for 4 real zeros and a degree of 4.

38. **PERSISTENCE** If $g(x)$ has a degree of 5 and a positive leading coefficient and $g(x)$ has a degree of 3 and a positive leading coefficient, determine the end behavior of $g(x)$. Explain your reasoning. Sample answer: As $x \rightarrow -\infty, g(x) \rightarrow \infty$, and as $x \rightarrow \infty, g(x) \rightarrow \infty$. The quotient is a 2nd-degree function with a positive leading coefficient.

39. **CREATE** Sketch the graph of an even-degree polynomial with 7 real zeros, one of which is a double zero, and the leading coefficient is negative.
 See margin.

1 CONCEPTUAL UNDERSTANDING

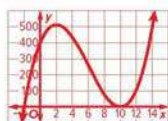
Answers



33. $f(1-x) = a(1-x)^3 - b(1-x) + (1-x)$. So, $f(1-x) = -ax^3 + 3a(1-x)^2 - (-3a + 2b - 1)x + (a - b + 1)$. The function $f(1-x)$ has the opposite leading coefficient, representing a reflection in the y -axis. So, it has the opposite end behavior.

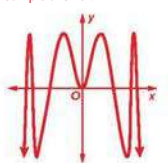
34. The end behavior is determined by the sign of the leading coefficient and the degree of the function. These are the same for $g(x)$ and $h(x)$. As $x \rightarrow \infty, f(x) \rightarrow \infty$ and as $x \rightarrow -\infty, f(x) \rightarrow -\infty$ for both functions.

35. Sample answer: The volume of the new box is modeled by the function $V(x) = (10 - x)^2(4 + 2x)$. The graph appears to have a relative maximum at $x = 2$ and $V(2) = 512$. So, the dimensions of the box with the greatest volume will be 8 centimeters by 8 centimeters by 8 centimeters.



36. Shenequa is correct; sample answer: The number of real zeros is equal to exactly the number of times the graph intersects the x -axis.

39. Sample answer:



Analyzing Graphs of Polynomial Functions

LESSON GOAL

Students analyze the graphs of polynomial functions by identifying key features.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Develop:**

The Location Principle

- Locate Zeros of a Function

Extrema of Polynomials

- Identify Extrema
- Analyze a Polynomial Function

Modeling with Polynomial Functions

- Use a Polynomial Function and Technology to Model
- Find Average Rate of Change

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources

Remediation: Polynomial Functions

Extension: The Bisection Method

AL	DL	EL	
●	●		●
	●	●	●

Language Development Handbook

Assign page 10 of the *Language Development Handbook* to help your students build mathematical language related to analyzing the graphs of polynomial functions.

 You can use the tips and suggestions on page T10 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

F.IF.7c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Standards for Mathematical Practice:

4 Model with mathematics.

5 Use appropriate tools strategically.

Coherence

Vertical Alignment

Previous

Students analyzed polynomial functions by examining key features and graphing

F.IF.4, F.IF.7c

Now

Students analyze the graphs of polynomial functions by identifying key features, including zeros and extrema

F.IF.4, F.IF.7c

Next

Students will solve polynomial equations by graphing.

A.CED.1, A.REI.11

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

 **Conceptual Bridge** In this lesson, students build on their understanding of the graphs of polynomial functions. They build fluency by using key features of the functions to graph them, and they apply their understanding by solving real-world problems.

Mathematical Background

Tables of values can be used to explore two types of changes in the values of a polynomial function. A change of signs in the value of $f(x)$ from one value of x to the next indicates that the graph of the function crosses the x -axis between the two x -values. A change between increasing values and decreasing values indicates that the graph is turning for that interval. Extrema occur at the relative maximum or minimum points of the function.



Interactive Presentation

Warm Up

Name the graph(s) that satisfy each requirement.

a. b. c. d.

- third-degree polynomial function
- negative leading coefficient
- domain and range of all real number
- the following end behavior: $x \rightarrow \infty, f(x) \rightarrow \infty$
- no real zeros

[Show Answers](#)

Warm Up

Launch the Lesson

The first payphone was installed in Hartford, Connecticut, in 1895. The number of payphones in the United States increased for more than 100 years until they reached their peak in 1999 with more than 2.1 million phones. Since then, the number of payphones quickly decreased and by 2013, there were fewer than 500,000. The number of payphones in the United States over time can be modeled by a polynomial function.



Launch the Lesson

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- identifying polynomial functions

Answers:

- c
- b, c
- a, c
- a, d
- b

Launch the Lesson



Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students apply what they have learned about polynomial functions to a real-world situation. Encourage students to identify key features of the function that models the number of payphones in the United States based on the description of the situation.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.



Learn The Location Principle

Objective

Students approximate zeros by graphing polynomial functions.

MP Teaching the Mathematical Practices

3 Analyze Cases The Think About It! feature guides students to examine a case where the Location Principle cannot be used to find all of the real zeros of a function. Encourage students to familiarize themselves with the cases in which the zeros of a function can and cannot be found by using the Location Principle.

Important to Know

A polynomial function must be continuous for the Location Principle to be true because if it was not continuous, then it could change signs but have discontinuity at 0.

Example 1 Locate Zeros of a Function

MP Teaching the Mathematical Practices

5 Use Estimation In Example 1, students must estimate the zeros of the function using the Location Principle and then find an exact answer.

Questions for Mathematical Discourse

AL Does a function of degree 4 have to have 4 zeros? Explain. **No**; 4 is the maximum number of possible zeros, but it can have fewer than 4.

OL If you substitute an estimated root into the equation, will you get zero? **Yes** may, or you may get a value close to zero because the root is an estimate.

BL Suppose $g(x)$ has two real zeros; but, when using the Location Principle with integers, no zeros are found. What are two possible descriptions of a function that would have this result?

Sample answer: The function could have two non-integer points that touch the axis without crossing, or the function could cross twice between two integers.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 2-2

Analyzing Graphs of Polynomial Functions

Learn The Location Principle

If the value of a polynomial function $f(x)$ changes signs from one value of x to the next, then there is a zero between those two x -values. This is called the Location Principle.

Key Concept: Location Principle

Suppose $y = f(x)$ represents a polynomial function, and a and b are two real numbers such that $f(a) < 0$ and $f(b) > 0$. Then the function has at least one real zero between a and b .



Example 1 Locate Zeros of a Function

Determine the consecutive integer values of x between which each real zero of $f(x) = x^4 - 2x^3 - x^2 + 1$ is located. Then draw the graph.

Step 1 Make a table.

Because $f(x)$ is a fourth-degree polynomial, it will have as many as 4 real zeros or none at all.

x	-2	-1	0	1	2	3	4
$f(x)$	29	3	1	-1	-3	19	113

Using the Location Principle, there are zeros between $x = 0$ and $x = 1$ and between $x = 2$ and $x = 3$.

Step 2 Sketch the graph.

Use the table to sketch the graph and find the locations of the zeros.



Check

Use technology to check the location of the zeros.

Input the function into a graphing calculator to confirm that the function crosses the x -axis between $x = 0$ and $x = 1$ and between $x = 2$ and $x = 3$.

You can find more accurate values of the zeros by using the **zero** feature in the **CALC** menu to find $x \approx 0.7233$ and $x \approx 2.3486$, which confirms the estimates.

Go Online You can complete an Extra Example online.

Lesson 2-2 • Analyzing Graphs of Polynomial Functions 83

Interactive Presentation

The Location Principle

You estimate the roots of a function by using a sign. The graph of a polynomial function of degree 4 crosses the x -axis three times, though there may not be four real roots.

For the graph, the zeros appear to be at $x = -2$, between $x = 0$ and $x = 1$, and between $x = 2$ and $x = 3$. Because the function is continuous, there must be one real zero between the two values. Finding the function's equation $f(x) = x^4 - 2x^3 - x^2 + 1$ and graphing it on a graphing calculator confirms the estimates.

A table of function values provides another way to analyze the function. Notice that the sign changes for the values of x between $x = 0$ and $x = 1$ and between $x = 2$ and $x = 3$, which confirms the estimates.

If the value of $f(x)$ changes signs from one value of x to the next, then there is at least one real zero between those two x -values. This is called the Location Principle.

x	$f(x)$
-2	29
-1	3
0	1
1	-1
2	-3
3	19
4	113

Learn

TYPE



Students answer a question to show they understand how to use the Location Principle to find zeros of a function.

CHECK



Students complete the Check online to determine whether they are ready to move on.



Learn Extrema of Polynomials

Objective

Students find the relative maxima and relative minima of polynomial functions.

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between extrema and degrees of polynomial functions.

Things to Remember

- Some odd functions, like $f(x) = x^3$ have no extrema.
- Zeros and extrema will not always occur at integral values of x .

Common Misconception

Students may believe that a function has only one relative maximum or minimum. A polynomial with a degree greater than 3 may have more than one relative maximum or minimum.

e Essential Question Follow-Up

Students will locate zeros of a function.

Ask:

Is it possible for a zero to be between two integers that both have positive $f(x)$ values? **Sample answer:** Yes, the function could become negative and come back up to positive between two integers.

Common Error

Students may assume that a function with degree 4 always has 4 zeros. Remind them that a function with degree 4 can have 0, 1, 2, 3, or 4 zeros.

Example 2 Identify Extrema

MP Teaching the Mathematical Practices

3 Construct Arguments In the Talk About It! feature, students will use stated assumptions and definitions to construct an argument.

Study Tip: **Turning Points** Relative maxima and relative minima of a function are sometimes called turning points.

Check

Determine the consecutive integer values of x between which each real zero of $f(x) = 2x^3 + x^2 - 3x - 2$ is located. Then draw the graph.

$$x = \frac{7}{2} \text{ and } x = \frac{7}{2} - 1$$

$$x = \frac{7}{2} - 1 \text{ and } x = \frac{7}{2} - 2$$



Learn Extrema of Polynomials

Extrema occur at relative maxima or minima of the function.

Point A is a relative minimum, and point B is a relative maximum. Both points A and B are extrema. The graph of a polynomial of degree n has at most $n - 1$ extrema.

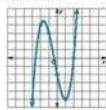


Example 2 Identify Extrema

Use a table to graph $f(x) = x^3 + x^2 - 5x - 2$. Estimate the x -coordinates at which the relative maxima and relative minima occur.

Step 1 Make a table of values and graph the function.

x	$f(x)$
-4	-30
-3	-5
-2	4
-1	3
0	-2
1	-5
2	0
3	19



Step 2 Estimate the locations of the extrema.

The value of $f(x)$ at $x = -2$ is greater than the surrounding points indicating a maximum near $x = -2$.

The value of $f(x)$ at $x = 1$ is less than the surrounding points indicating a minimum near $x = 1$.

You can use a graphing calculator to find the extrema of a function and confirm your estimates.

Go Online You can complete an Extra Example online.

Interactive Presentation

Identify Extrema and Turning Points

Use the table to graph $f(x) = x^3 + x^2 - 5x - 2$. Estimate the x -coordinates at which the relative maxima and relative minima occur.

Step 1 Make a table of values, and graph the function.

x	$f(x)$
-4	-30
-3	-5
-2	4
-1	3
0	-2
1	-5
2	0
3	19

Example 2

SWIPE



Students move through the steps to estimate the locations of the extrema of a function



Questions for Mathematical Discourse

AL Is there a relationship between the number of extrema and the number of zeros a function will have? Explain. **No; sample answer:** The number of extrema does not necessarily determine the number of times a graph crosses the x -axis.

OL What is another name for the extrema of a quadratic polynomial? **vertex**

BL Suppose $g(x)$ has no zeros and no extrema. What is a possible type of function $g(x)$ could be? **Sample answer:** $g(x)$ could be a linear function with $m = 0$ and $b \neq 0$.

Things to Remember

Remind students that while the degree of the function represents the maximum number of zeros a function can have, to find the maximum number of extrema, they should calculate one less than the degree of the function.

Example 3 Analyze a Polynomial Function

MP Teaching the Mathematical Practices

4 Analyze Relationships Mathematically Point out that to complete Example 3, students will need to analyze the mathematical relationships in the context of the problem.

Questions for Mathematical Discourse

AL What does $f(0)$ mean in this context? **the number of certified pilots, in thousands, in 1930**

OL $f(0) = 7.708$. Do you need to round this number when describing the number of pilots in 1930? **No, 7.708 represents 7708 pilots because x is scaled in thousands.**

BL Would this model make sense for years before 1930? Explain. **No; sample answer:** The graph of the function crosses the x -axis at -1.501 .

Check

Copy and complete the table for $f(x) = -x^4 - x^2 + 5x^2 + x - 3$ to estimate the x -coordinates at which the relative maxima and relative minima occur.

x	$f(x)$
-3	-55
-2	7
-1	1
0	-3
1	1
2	-5
3	-63

The relative maxima occur near $x = -2.7$ and $x = 1.1$.
The relative minimum occurs near $x = 0$.

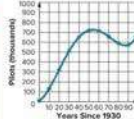
Example 3 Analyze a Polynomial Function

PLOTS The total number of certified pilots in the United States is approximated by $f(x) = 0.0000903x^4 - 0.0164x^3 + 0.762x^2 + 6.317x + 7708$, where x is the number of years after 1930 and $f(x)$ is the number of pilots in thousands. Graph the function and describe its key features over the relevant domain.

Step 1 Graph the function.

Make a table of values. Plot the points and connect them with a smooth curve.

x	$f(x)$
0	7708
10	131,381
20	320,496
30	507,961
40	648,356
50	717,933
60	754,636
70	658,001
80	589,356
90	571,621



Step 2 Describe the key features.

Domain and Range

The domain and range of the function is all real numbers. Because the function models years after 1930, the relevant domain and range are $x \geq 0$ and $f(x) \geq 7708$.

(continued on the next page)

Lesson 2-2 • Analyzing Graphs of Polynomial Functions 85

Go Online

You can learn how to graph and analyze a polynomial function on a graphing calculator by watching the video online.

Think About It

What trends in the number of pilots does the graph suggest?

Sample answer: The number of pilots peaked around 1955 and declined to a relative minimum around 2015. After 2015, the number of pilots will continue to increase.

Interactive Presentation



Example 3

TYPE



Students identify characteristics of a polynomial function in context.



Talk About It!

It is reasonable that the trend will continue indefinitely? Explain.

No, sample answer: It is unreasonable to assume that the number of pilots will continue to increase indefinitely. At some point, the number will stay relatively constant or begin to decline.

Study Tip:

Assumptions
Determining the end behavior for the graph of a polynomial that models data assumes that the trend continues and there are no other relative maxima or minima.

Extrema

There is a relative maximum between 1980 and 1990 and a relative minimum between 2010 and 2020 in the relevant domain.

End Behavior

As $x \rightarrow \infty$, $f(x) \rightarrow \infty$.

Intercepts

In the relevant domain, the y -intercept is at (0, 7708). There is no x -intercept, or zero, because the function begins at a value greater than 0 and as $x \rightarrow \infty$, $f(x) \rightarrow \infty$.

Symmetry The graph of the function does not have symmetry.

Check

COINS The number of quarters produced by the United States Mint can be approximated by the function $f(x) = 16.4x^2 - 140.3x + 148.9$, where x is the number of years since 2005 and $f(x)$ is the total number of quarters produced in millions. Use the graph of the function to complete the table and describe its key features.

Part A Copy and complete the table.

x , Years	$f(x)$, Quarters (millions)
0	3215
2	2451
4	1277
6	482
8	853
10	3176

Part B Describe the key features.

The relevant domain is $7 \leq x \leq 0$.

The relevant range is $7 \leq f(x) \leq 482$.

There is a relative minimum between 7201 and 7203 .

The y -intercept is 73215 .

The graph of the function does not have symmetry.

It is 7 unreasonable to assume that the trend will continue indefinitely.

See Online You can complete an Extra Example online.

86 Module 2 • Polynomials and Polynomial Functions

Things to Remember

Remind students how to find the domain, range, end behavior, and intercepts from the previous lesson.

DIFFERENTIATE

Language Development Activity

IF students ask how math functions can describe real-world situations, **THEN** have them discuss the appropriateness of describing real-world situations with mathematical functions. Help them to understand that a function is usually just an approximation of the real-world data, and is often only a reasonable model of a limited interval of the domain.

CHECK



Students complete the Check online to determine whether they are ready to move on.



Example 4 Use a Polynomial Function and Technology to Model

MP Teaching the Mathematical Practices

5 Compare Predictions with Data In Example 4, students should use a graphing calculator to visualize the data, determine a function of best fit, and compare their predictions with the data.

Questions for Mathematical Discourse

A1 What is the purpose of testing regressions for a set of data? The purpose is to find a polynomial function that models the data as closely as possible. The model can be used to make predictions of the dependent variable for values of the independent variable that are not part of the data.

OL Do you think testing the linear regression was necessary? Sample answer: No, the data does not appear to be well-modeled by a linear function.

BL How could you test the calculator's regression operation for accuracy? You could use points from a known function and see if the best fit is close to the original function.

Important to Know

Some students may need additional instructions to use the graphing calculator to graph polynomial functions and use it to solve problems.

Example 4 Use a Polynomial Function and Technology to Model

BACKGROUND The table shows U.S. backpack sales in millions of dollars, according to the Travel Goods Association. Make a scatter plot and a curve of best fit to show the trend over time. Then determine the backpack sales in 2015.

Year	Sales (million \$)	Year	Sales (million \$)
2000	1840	2008	1246
2001	1844	2009	1235
2002	1813	2010	1419
2003	1834	2011	1773
2004	1854	2012	1930
2005	1880	2013	2255
2006	1364	2014	2779
2007	1436		

Step 1 Enter the data.

Let the year 2000 be represented by 0. Enter the years since 2000 in List 1. Enter the backpack sales in List 2.

Step 2 Graph the scatter plot.

Choose the scatter plot feature in the **STAT PLOT** menu. Use List 1 for the Xlist and List 2 for the Ylist. Change the viewing window so that all the data are visible.



Step 3 Determine the polynomial function of best fit.

To determine the model that best fits the data, perform linear, quadratic, cubic, and quartic regressions, and compare the coefficients of determination, r^2 . The polynomial with a coefficient of determination closest to 1 will fit the data best. A quartic function fits the data best.



The regression equation with coefficients rounded to the nearest tenths is:

$$y = 0.2x^4 - 3.9x^3 + 26.4x^2 - 43.6x + 1139.9$$

Step 4 Graph and evaluate the regression function.

Assuming that the trend continues, the graph of the function can be used to predict backpack sales for a specific year. To determine the total sales in 2015, find the value of the function for $x = 15$.



In 2015, there were about \$3,523 million in backpack sales.



Math History Minute

By the age of 20, Italian mathematician **Maria Gaetana Agnesi (1718–1792)** had started working on her book *Analytical Institutions*, which was published in 1748. Early chapters included problems on maxima, minima, and turning points. Also described was a cubic curve called the “utility of Agnesi,” which was translated incorrectly from the original Italian.

Think About It!

Explain the approximation that is made when using the model to determine the backpack sales in a specific year.

Sample answer: The model is an approximation of the entire set of data, but it may not be accurate at a specific year. Also, the model may not be a good representation for specific years outside of the domain of the original data because it was not considered when creating the model.

Lesson 2-2 • Analyzing Graphs of Polynomial Functions 87

Interactive Presentation

BACKGROUND The table shows U.S. backpack sales in millions of dollars, according to the Travel Goods Association. Make a scatter plot and a curve of best fit to show the trend over time. Then determine the backpack sales in 2015.

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2003	1834	2011	1773
2004	1854	2012	1930
2005	1880	2013	2255
2006	1364	2014	2779
2007	1436		

Example 4

TAP



Students tap to move through the steps to graph and evaluate polynomial functions.



Example 5 Find Average Rate of Change



Teaching the Mathematical Practices

5 Use Estimation Point out that in Example 5, students need to estimate the rate of change from the graph and then find an exact answer algebraically.

Questions for Mathematical Discourse

AL For what interval of time might the average rate of change more accurately describe the acceleration of the rocket? Explain.

Sample answer: 80 seconds to 180 seconds because the acceleration appears to be increasing at a relatively consistent rate during this time.

OL If you draw a line connecting two points on the function, what is the relationship between the slope of the line and the average rate of change between the two points? **The slope of the line is the same as the average rate of change between the points.**

BL Consider a point P , which is a relative maximum on the curve. Now consider a point Q , which can vary. As Q approaches P , what value does the average rate of change approach? **zero** Using this, and that the average rate of change between two points is the slope of a line between the points, what would the instantaneous slope be at P ? **zero**

DIFFERENTIATE

Enrichment Activity BL

The nature of polynomial functions makes them perfect for inventing funny stories about how things change over time. Challenge students to create stories about something that is increasing or decreasing (or both) and display the growth graphically. Extend the problem by having them create stories for the graphs that match the context of their stories.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Check

TREES To estimate the amount of lumber that can be harvested from a tree, foresters measure the diameter of each tree. Determine the polynomial function of best fit, where x represents the diameter of a tree in inches and y is the estimated volume measured in board feet. Then estimate the volume of a tree with a diameter of 35 inches.

Diam. (in.)	17	19	20	23	25	28	32	38	39	41
Vol. (100s of board ft.)	19	25	32	57	71	103	123	252	259	294

Polynomial function of best fit:

$$y = -0.0006x^4 + 0.08x^3 - 3.55x^2 + 69.11x - 491.39$$

The estimated volume of a 35-inch diameter tree to the nearest board foot is $\frac{1}{100}$ of 100s of board feet.

Example 5 Find Average Rate of Change

ROCKETS The Ares-V rocket was designed to carry as much as 75 tons of supplies and 4 astronauts to the Moon and possibly even to Mars. The table shows the expected g-force on the rocket over the course of its 200-second launch.

Time (s)	Acceleration (Gs)	Time (s)	Acceleration (Gs)
0	1.34	120	1.46
20	1.26	140	1.93
40	1.12	160	2.47
60	1.01	180	2.84
80	1	200	2.2
100	1.15		

Part A Find the average rate of change.

Sketch the graph, and estimate the average rate of change of the acceleration. Then check your results algebraically.

Estimate: From the graph, the change in the y -values is about 0.5, and the change in the x -values is 200. So, the rate of change is about $\frac{0.5}{200}$, or 0.0043.

Check algebraically:

The average rate of change is $\frac{2.2 - 1}{200 - 0} = \frac{1.2}{200} = 0.006$.

Part B Interpret the results.

From 0 to 200 seconds, the average rate of change in acceleration was an increase of 0.0043 Gs per second.

Go Online: You can complete an Extra Example online.

Study Tip

g-force One G is the acceleration due to gravity at the Earth's surface. Defined as 9.80665 meters per second squared, this is the g-force you experience when you stand still on Earth. On a roller coaster, you experience 2 Gs and feel weightless at the top of the hills, and you can experience a g-force of 6 Gs or more as you are pushed into your seat at the bottom of the hills.

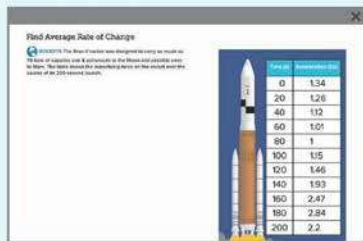
Think About It!

Does the average rate of change from 0 to 200 seconds accurately describe the acceleration of the launch? Justify your reasoning.

No; sample answer: For the first 80 seconds, the acceleration is relatively slow. For the next 100 seconds, it increases rapidly. Then, it decreases. The average rate of change does not describe the changes in acceleration over time.

88 Module 2 • Polynomials and Polynomial Functions

Interactive Presentation



Example 5

CHECK



Students complete the Check online to determine if they are ready to move on.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–15
2	exercises that use a variety of skills from this lesson	16–25
2	exercises that extend concepts learned in this lesson to new contexts	26–28
3	exercises that emphasize higher-order and critical-thinking skills	29–34

ASSESS AND DIFFERENTIATE

Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, THEN assign:

OL BL

- Practice Exercises 1–25 odd, 26–34
- Extension: The Bisection Method
- ALEKS® Graphs of Polynomial Functions

IF students score 66%–89% on the Checks, THEN assign:

AL OL

- Practice Exercises 1–33 odd
- Remediation, Review Resources: Analyzing Graphs of Polynomial Functions
- Personal Tutors
- Extra Examples 1–5
- ALEKS® Identifying Polynomial Functions

IF students score 65% or less on the Checks, THEN assign:

AL

- Practice Exercises 1–15 odd
- Remediation, Review Resources: Analyzing Graphs of Polynomial Functions
- Quick Review Math Handbook: Analyzing Graphs of Polynomial Functions
- ALEKS® Identifying Polynomial Functions

Practice

Example 1

Determine the consecutive integer values of x between which each real zero of each function is located by using a table. Then sketch the graph. See margin.

- $f(x) = x^2 + 3x - 1$
- $f(x) = -x^2 + 2x^2 - 4$
- $f(x) = x^2 + 4x^2 - 5x + 5$
- $f(x) = -x^2 - x^2 + 4$

Example 2

Use a table to graph each function. Then estimate the x -coordinates at which relative maxima and relative minima occur. See Mod. 2 Answer Appendix for tables and graphs.

- $f(x) = -2x^2 + 12x^2 - 8x$
rel. min between $x = 0$ and $x = 1$
rel. max near $x = 4$
- $f(x) = 2x^2 - 4x^2 - 3x + 4$
rel. max near $x = -0.2$, rel. min near $x = 1.6$
- $f(x) = x^2 + 2x - 1$
rel. min near $x = -1$, no rel. max
- $f(x) = x^2 + 8x^2 - 12$
min. near $x = 0$

Example 3

9. BUSINESS A banker models the expected value y of a company in millions of dollars by using the formula $y = ax^3 - 3x^2$, where x is the number of years in business. Graph the function and describe its key features over the relevant domain. See Mod. 2 Answer Appendix.

10. HEIGHT A plant's height is modeled by the function $f(x) = 1.5x^3 - 20x^2 + 85x - 84$, where x is the number of weeks since the seed was planted and $f(x)$ is the height of the plant. Graph the function and describe its key features over its relevant domain. See Mod. 2 Answer Appendix.

Example 4

11. USE ESTIMATION The table shows U.S. car sales in millions of cars. Use a graphing calculator to make a scatter plot and a curve of best fit to show the trend over time. Then use the equation to estimate the car sales in 2007. Let 2008 be represented by year 0. Round the coefficients of the regression equation to the thousandths place.

Year	Cars (millions)	Year	Cars (millions)	Year	Cars (millions)
2008	16.05	2011	16.765	2014	16.089
2009	17.81	2012	16.400	2015	12.413
2010	17.663	2013	16.635	2016	7.790

Curve of best fit: $y = -0.012x^3 + 0.217x^2 - 1.166x + 1.4886$; car sales in 2017: 5.941 million

Lesson 2.2 • Analyzing Graphs of Polynomial Functions 89

12. POPULATION The table shows the population in Cincinnati, Ohio, since 1960. Make a scatter plot and a curve of best fit to show the trend over the given time period. Then use the equation to estimate the population of Cincinnati in 2020.

Year	Population	Year	Population
1960	502,350	1990	364,553
1970	452,524	2000	331,258
1980	385,457	2010	296,943

Curve of best fit: $y = -0.58x^3 + 9.91x^2 - 22796x^2 - 4070.779x + 503,796.447$, population in 2020: 83,232

13. VOLUNTEERS The table shows average volunteer hours per month for a local non-profit. Make a scatter plot and a curve of best fit to show the trend over the given time period. Then use the equation to estimate the volunteer hours for September. Let January be represented by month 1.

Month	Volunteer Hours	Month	Volunteer Hours
Jan	48	May	100
Feb	60	June	110
Mar	72	July	105
Apr	75	Aug.	93

Curve of best fit: $y = -0.122x^3 + 1.585x^2 - 6.325x + 33.025$, where x is the number of months since January; average volunteer hours for September 44

Example 5

14. FARMS The table shows the number of farms in the U.S. at various years, according to the USDA, Census of Agriculture. Find the average rate of change from 1982 to 2012. Interpret the results in the context of the situation.

Year	Farms	Year	Farms
1982	2,480,000	2002	2,130,000
1987	2,340,000	2007	2,200,000
1992	2,180,000	2012	2,110,000
1997	2,220,000		

average rate of change: $-12,333$. From 1982 to 2012, there was an average of 12,333 fewer farms each year.

15. SALARY The table shows the annual salary of a salesperson over time. Find the average rate of change over the given time interval. Interpret the results in context of the situation.

Year	Salary	Year	Salary
2012	\$45,000	2015	\$55,500
2013	\$49,000	2016	\$72,000
2014	\$47,500	2017	\$67,500

average rate of change: 4500. From 2012 to 2017, the salesperson's average rate of change in salary was an increase of \$4500 per year.

90 Module 2 • Polynomials and Polynomial Functions

3 REFLECT AND PRACTICE



F.IF.4, F.IF.7c

Mixed Exercises

Graph each function by using a table of values. Then, estimate the x -coordinates at which each zero and relative extrema occur, and state the domain and range.

16. $f(x) = x^2 - 3x + 1$

17. $f(x) = 2x^2 + 9x^2 + 12x + 2$

18. $f(x) = 2x^3 - 3x^2 + 2$

19. $f(x) = x^3 - 2x^2 - 2$

20. Determine the key features for $y = \begin{cases} x^2 & \text{if } x \leq -4 \\ x - 4 & \text{if } -4 < x \leq 0 \\ x^2 + 1 & \text{if } x > 0 \end{cases}$. See Mod. 2 Answer Appendix.

USE TOOLS Use a graphing calculator to estimate the x -coordinates at which the maxima and minima of each function occur. Round to the nearest hundredth.

21. $f(x) = x^3 + 3x^2 - 6x - 6$
rel. max: $x \approx -2.72$; rel. min: $x \approx 0.73$

22. $f(x) = -2x^3 + 4x^2 - 5x + 8$
rel. max or min

23. $f(x) = -2x^3 + 5x^2 - 4x^2 + 3x - 7$
rel. max: $x \approx 1.34$; no rel. min

24. $f(x) = x^3 - 4x^2 + 3x^2 - 8x - 6$
rel. max: $x \approx -1.87$; rel. min: $x \approx 1.52$

25. **PREDICT** Sketch the graph of a third-degree polynomial function that has a relative minimum at $x = -3$, passes through the origin, and has a relative maximum at $x = 2$. Describe the end behavior of the graph. Based on the sketch, determine whether the leading coefficient is negative or positive. See Mod. 2 Answer Appendix.

26. **USE TOOLS** A canister has the shape of a cylinder with spherical caps on either end. The volume of the canister is cubic millimeters is modeled by the function $V(x) = \pi x^3 + 3\pi x^2 + \frac{3\pi}{2}x$, where x represents the radius in millimeters of a canister that is $x + 3$ millimeters wide.



- Use a graphing calculator to sketch the model that represents the volume of the canister. Include axis labels. See Mod. 2 Answer Appendix.
- What is the domain of the model? Explain any restrictions that apply. The domain is all x values greater than or equal to zero. Because x represents a radius, it cannot be negative.

Lesson 3.3 • Interpreting Graphs of Polynomial Functions 91

27. **CONSTRUCT ARGUMENTS** What type of polynomial function best models the data in the graph? Explain your reasoning.



The type of polynomial function that should be used to model the graph is a function with an even degree with a negative leading coefficient. This is based on the fact that the graph is reflected in the y -axis and as $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$ and as $x \rightarrow \infty$, $f(x) \rightarrow -\infty$.

28. **FORECASTING** The table shows the number of deliveries a grocery store has made since they began to offer the service. Use a scatter plot and a curve of best fit to determine the number of deliveries the grocery store can expect to deliver 10 years after they begin the service. 1037 deliveries

Year	Deliveries	Year	Deliveries
1	60	5	175
2	123	6	196
3	235	7	195
4	210	8	328

Higher-Order Thinking Skills

29. **ANALYZE** Explain why the leading coefficient and the degree are the only determining factors in the end behavior of a polynomial function. As the x values approach large positive or negative numbers, the term with the largest degree becomes more and more dominant in determining the value of $f(x)$.

30. **ANALYZE** The table below shows the values of a cubic function. Could there be a zero between $x = 2$ and $x = 3$? Justify your argument.

x	-2	-1	0	1	2	3
$f(x)$	-3	-2	-1	1	-2	-2

Yes, sample answer: The cubic function is of degree 3 and cannot have any more than three zeros. These zeros are located between -2 and -1 , 0 and 1 , and 2 and 3 .

31. **CREATE** Sketch the graph of an odd degree polynomial function with 6 extrema and a relative extreme at $y = 0$. See Mod. 2 Answer Appendix.

32. **ANALYZE** Determine whether the following statement is sometimes, always, or never true. Justify your argument.

For any continuous polynomial function, the y -coordinate of a point where a function changes from increasing to decreasing or from decreasing to increasing is either a relative maximum or relative minimum. **Always, sample answer:** a point at which the graph stops increasing and begins to decrease is a maximum, a point where a graph stops decreasing and begins to increase is a minimum.

33. **PERSERVE** A function is even if for every x in the domain of $f(x)$, $f(x) = f(-x)$ is every even-degree polynomial function also an even function? Explain.

No, sample answer: $f(x) = x^2 + x$ is an even degree, but $f(x) \neq f(-x)$.

34. **PERSERVE** A function is odd if for every x in the domain, $-f(x) = f(-x)$ is every odd-degree polynomial function also an odd function? Explain.

No, sample answer: $f(x) = x^3 + 2x^2$ is an odd degree, but $-f(x) \neq f(-x)$.

92 Module 3 • Polynomials and Polynomial Functions

1 CONCEPTUAL UNDERSTANDING

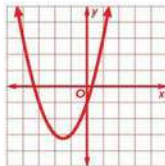
2 FLUENCY

3 APPLICATION

Answers

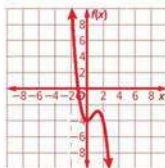
1. zeros between $x = -4$ and $x = -3$, and $x = 0$ and $x = 1$

x	$f(x)$
-4	3
-3	-1
-2	-3
-1	-3
0	-1
1	3



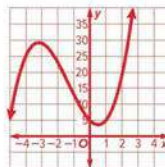
2. zero between $x = -2$ and $x = -1$

x	$f(x)$
-2	12
-1	-1
0	-4
1	-3
2	-4



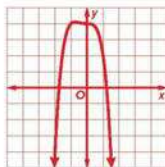
3. zero between $x = -6$ and $x = -5$

x	$f(x)$
-7	-107
-6	-37
-5	5
-4	25
-3	29



4. zeros between $x = -2$ and $x = -1$, and $x = 1$ and $x = 2$

x	$f(x)$
-3	-50
-2	-4
-1	4
0	4
1	2
2	-20
3	-104



Operations with Polynomials

LESSON GOAL

Students add, subtract, and multiply polynomials.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Develop:**

Adding and Subtracting Polynomials

- Identify Polynomials
- Add Polynomials
- Subtract Polynomials



Explore: Multiplying Polynomials



Develop:

Multiplying Polynomials

- Simplify by Using the Distributive Property
- Multiply Binomials
- Multiply Polynomials (online)
- Write and Simplify a Polynomial Expression



You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE



Exit Ticket



Practice



Formative Assessment Math Probe

DIFFERENTIATE



View reports of student progress on the **Checks** after each example.

Resources

Remediation: Adding and Subtracting Polynomials

Extension: Relationship Between GCF and LCM with Polynomials

AL	LE	EL
●	●	●
	●	●
	●	●

Language Development Handbook

Assign page 11 of the *Language Development Handbook* to help your students build mathematical language related to adding, subtracting, and multiplying polynomials.

EL You can use the tips and suggestions on page T11 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Algebra

Standard for Mathematical Content:

A.APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous

Students evaluated powers.

8.EE.1

Now

Students will add, subtract, and multiply polynomials.

A.APR.1

Next

Students will divide polynomials by using long division and synthetic division.

A.APR.6

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION



Conceptual Bridge In this lesson, students build on their understanding of polynomial operations. They build fluency by validating that polynomials are closed under arithmetic operations, and they apply their understanding by solving real-world problems.

Mathematical Background

When multiplying or dividing powers of variables, be sure that the base is the same. Add the exponents if multiplying powers of the same variable, and subtract them if dividing powers.



Interactive Presentation

Warm Up

Find the missing digits.

1. $(3x^2y^3)^2 = 9^{\square} x^{\square} y^{\square}$
 $= \square x^{\square} y^{\square}$

2. $(5 + 6i) + (-7 - 2i) = 5 + (-7) + 6i + (-2i)$
 $= \square + \square i$

3. $(x + 3)(x + 4) = x^2 + \square x + \square$
 $= x^2 + \square x + \square$

Show Answers

Warm Up

Launch the Lesson

In 1946, Russian architect Vladimir Shukhov described the Ten Books on Architecture in the book, *Vladimir Shukhov's Architectural Relationships*. One such relationship is that between length and width of an atrium. A common feature in some ancient Roman houses, the atrium was the main room where guests were greeted. It often had an open roof with a small draining pool to catch rainwater.



In *The Ten Books on Architecture*, Vitruvius stated that there were three ways to design an atrium. In one type, the width should be $\frac{2}{3}$ the length of the room. If an atrium was built using this relationship, you could use polynomials to represent its length and width. Then, you could add or multiply the polynomials to find its perimeter or area. You could also subtract a polynomial representing the area of the draining pool to find the total living area of the atrium.

Launch the Lesson

Vocabulary

Expand All Collapse All

Binomial
 The sum of two monomials.

Trinomial
 The sum of three monomials.

Are binomials and trinomials also polynomials? Explain.
 Could the sum of a binomial and trinomial be a binomial?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- performing simple operations with polynomials

Answers:

- first row: $3; 3 \cdot 3; 2 \cdot 3$; second row: $27; 9; 6$
- $-2; 4$
- first row: $4; 3; 12$; second row: $7; 12$

Launch the Lesson



Teaching the Mathematical Practices

2 Make Sense of Quantities Mathematically proficient students need to be able to make sense of quantities and their relationships. In the Launch the Lesson, notice the relationship between the length and width of the atrium described. Encourage students to write expressions to model the situation.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.



Explore Multiplying Polynomials

Objective

Students use a table to multiply polynomials.



Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. In the Explore, guide students to see the polynomial expressions as being composed of several terms that can be multiplied together to find the product of the two polynomials.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will be presented with two polynomials and instructed to find their product using a table. The terms of the polynomials are placed in the top row and left column of the table. Students will multiply each corresponding row and column to complete the table. Students will add all the products and combine like terms to find the product of the two polynomials. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Multiplying Polynomials

INQUIRY How is using a table to multiply polynomials related to the Distributive Property?

You can use a table to multiply two polynomials. A table allows you to find the product of two binomials, and just those with degrees less than or equal to 1.

The table below can be used to find $(x^2 + x + 1)(x^2 - 5x + 2)$. Complete the table by entering the product of the terms in the corresponding row and column of each cell. Then compute the expression.

	x^2	x	1
x^2	$4x^4$		
$-5x$			$-5x$
1			

Clear All Check Answer

Explore

TYPE



Students complete the calculations to multiply polynomials term by term.



Interactive Presentation



Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Multiplying Polynomials (*continued*)

Teaching the Mathematical Practices

2 Use the Distributive Property Point out that the Distributive Property is one of the most-used properties in algebra. Guide students to see how multiplying polynomials using a table relates to the Distributive Property.

Questions

Have students complete the Explore activity.

Ask:

- How is using the table similar to multiplying polynomials with algebra tiles? **Sample answer:** You are creating a rectangle with the polynomials as the length and width. The area inside each cell is equal to the product of each term, just like the different sections used with algebra tiles.
- Can the table only be used to multiply polynomials with degree 2? **Why or why not?** **Sample answer:** The table could be used to multiply polynomials of any degree by making enough rows or columns for each term in the given polynomial.



Inquiry

How is using a table to multiply polynomials related to the Distributive Property? **Sample answer:** The table organizes the results of multiplying each term in one polynomial by each term in the other. The result will be the same as using the Distributive Property.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Adding and Subtracting Polynomials

Objective

Students add and subtract polynomials by combining like terms.

MP Teaching the Mathematical Practices

7 Use Structure Guide students to use the structure of polynomials to see why polynomials are closed under addition and subtraction.

Important to Know

Terms and concepts students should be familiar with prior to this Learn include properties of exponents, factoring, simplifying, monomials, polynomials, degrees, and bases.

Common Misconception

Some students may believe that a common base or a common degree is enough for two terms to be like. Guide them to understand exponents and bases.

Essential Question Follow-Up

Students have begun to add and subtract polynomials.

Ask:

Are polynomials closed under addition and subtraction? **Sample answer:** Because adding or subtracting polynomials results in a polynomial, the set of polynomials is closed under addition and subtraction.

Example 1 Identify Polynomials

MP Teaching the Mathematical Practices

7 Interpret Complicated Expressions In Example 1, guide students to see each expression as composed of several terms to determine whether each expression is a polynomial.

Questions for Mathematical Discourse

- AL** Why is $\sqrt{x}$ not a monomial? **A monomial must have positive integer exponents.**
- OL** What would it mean if a polynomial had a degree of 0? **It is a constant polynomial.**
- BL** Can you have a term with exponents that are not positive integers and still have a polynomial? **This could only be true if the exponents simplified to positive integers.**

Things to Remember

Suggest that students return to the basic definitions for exponents rather than just memorizing rules. For example, they can derive the rule for multiplying quantities such as $x^{a^2}x$ by rewriting the problem as $x \cdot x \cdot x \cdot x \cdot x$.

Operations with Polynomials

Learn Adding and Subtracting Polynomials

A polynomial is a monomial or the sum of two or more monomials. A **binomial** is the sum of two monomials, and a **trinomial** is the sum of three monomials. The degree of a polynomial is the greatest degree of any term in the polynomial.

Polynomials can be added or subtracted by performing the operations indicated and combining like terms. You can subtract a polynomial by adding its additive inverse.

The sum or difference of polynomials will have the same variables and exponents as the original polynomials, but possibly different coefficients. Thus, the sum or difference of two polynomials is also a polynomial.

A set is **closed** if and only if an operation on any two elements of the set produces another element of the same set. Because adding or subtracting polynomials results in a polynomial, the set of polynomials is closed under the operations of addition and subtraction.

Example 1 Identify Polynomials

Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial.

a. $x^4 + \frac{1}{2}x - 4$

This expression is not a polynomial because $\frac{1}{2}x$ is not a monomial.

b. $5x^2y + 3x^2y^2 - 9$

This expression is a polynomial because each term is a monomial. The degree of the first term is $4 + 1 = 5$, the degree of the second term is $2 + 7 = 9$, and the degree of the third term is 0. So, the degree of the polynomial is 9.

c. $\frac{2}{3}x^5 - 6x^2 - x$

The expression is not a polynomial because x^{-1} and x^{-2} are not monomials.

Check:

State the degree of each polynomial.

A. $x^3 + 6x^2 - \frac{1}{3}x$

B. $3x^2y^2 + 5x^2y - 9$

C. $x^{10} - 10$

D. $25 - 0$

Go Online: You can complete an Extra Example online.

Today's Goals

- Add and subtract polynomials.
- Multiply polynomials.

Today's Vocabulary

- binomial
- closed
- FOIL method

Think About It!

Identify the like terms in $3x^2 + 3x^2 - 10$ and $4x^2 - 7x^2 + 5$.

$3x^2$ and $-7x^2$; -10 and 5

Study Tip

Degree 0 and 1 Remember that constant terms have a degree of 0 and variable terms with no exponent indicated have a degree of 1.

Go Online

You can learn how to add and subtract polynomials by watching the video online.

Interactive Presentation

Adding and Subtracting Polynomials

A **polynomial** is a sum of one or more monomials. A **monomial** is the product of a constant and a variable raised to a non-negative integer power. The **degree of a polynomial** is the greatest degree of any term in the polynomial.

Polynomials can be added or subtracted by performing the operations indicated and combining like terms. You can subtract a polynomial by adding its additive inverse. To find the additive inverse of a polynomial, write the opposite of each term in the polynomial. When finding the additive inverse of a polynomial, you are multiplying every term by -1 . The additive inverse of $3x^2 + 5x - 7$ is $-3x^2 - 5x + 7$.

$$-(3x^2 + 5x - 7) = -3x^2 - 5x + 7$$

Additive Inverse

Learn

TYPE



Students identify the like terms in two polynomials.

**Talk About It!**

Notice that the terms in the sum have the same variables and exponents as the terms of the original expressions x^2 , x , and a constant. Will this always be the case? Explain.

Sample answer: If the sum of any of the like terms is 0, then that degree term will not appear in the resulting polynomial.

Study Tip

Additive Inverse
Distributing -1 to each term in the polynomial being subtracted is the same as finding and adding its additive inverse.

Think About It!

Why is it helpful to insert placeholders for the $0x^2$ and $0x$ terms? Why were placeholders not included for the x^3 term in either polynomial?

Sample answer: Adding placeholders helps to align like terms of the expressions. Because neither polynomial has an x^3 term, the like terms can be aligned without adding a placeholder to each expression.

Example 2 Add Polynomials

Find $(8x^3 + 7x^2 - 2x + 5) + (4x^3 - 4x^2 - 8x + 1)$.

Method 1 Add horizontally.

Group and combine like terms.

$$(8x^3 + 7x^2 - 2x + 5) + (4x^3 - 4x^2 - 8x + 1)$$

$$= (8x^3 + 4x^3) + (7x^2 - 4x^2) + (-2x - 8x) + (5 + 1)$$

$$= 12x^3 + 3x^2 - 10x + 6$$

Group like terms.
Combine like terms.

Method 2 Add vertically.

Align like terms vertically and add.

$$\begin{array}{r} 8x^3 + 7x^2 - 2x + 5 \\ (+) 4x^3 - 4x^2 - 8x + 1 \\ \hline 12x^3 + 3x^2 - 10x + 6 \end{array}$$

Align like terms.
Combine like terms.

Check:

$$\text{Find } (2x^3 + 9x^2 + 6x - 3) - (4x^3 - 7x^2 + 5x). \quad 6x^3 + 2x^2 + 11x - 3$$

Example 3 Subtract Polynomials

Find $(2x^3 + 11x^2 + 7x - 8) - (5x^3 + 9x^2 - 3x + 4)$.

Method 1 Subtract horizontally.

Group and combine like terms.

$$(2x^3 + 11x^2 + 7x - 8) - (5x^3 + 9x^2 - 3x + 4)$$

$$= 2x^3 + 11x^2 + 7x - 8 - 5x^3 - 9x^2 + 3x - 4$$

$$= 2x^3 + (11x^2 - 9x^2) + (7x + 3x) + (-8 - 4)$$

$$= 2x^3 + 2x^2 + 10x - 12$$

Original equation.
Distribute -1 .

Group like terms.
Combine like terms.

Method 2 Subtract vertically.

Align like terms vertically and subtract.

$$\begin{array}{r} 2x^3 + 11x^2 + 7x - 8 \\ (-) 5x^3 + 9x^2 - 3x + 4 \\ \hline 2x^3 + 2x^2 + 10x - 12 \end{array}$$

Align like terms.
Combine like terms.

Check:

$$\text{Find } (8x^3 - 3x + 9) - (5x^3 + 2x^2 - 6x - 9). \quad -5x^3 + 6x^2 + 3x + 10$$

Explore: Multiplying Polynomials

Online Activity Use a table to complete the Explore.

INQUIRY How is using a table to multiply polynomials related to the Distributive Property?

Interactive Presentation

And Horizontal

Find $(4x^3 + 5x^2 - 2x + 3) + (x^3 - 4x^2 - 8x + 5)$.

Method 1 Add horizontally.

Group and combine like terms.

$$(4x^3 + 5x^2 - 2x + 3) + (x^3 - 4x^2 - 8x + 5)$$

$$= (4x^3 + x^3) + (5x^2 - 4x^2) + (-2x - 8x) + (3 + 5)$$

$$= 5x^3 + x^2 - 10x + 8$$

Group like terms.
Combine like terms.

Example 2

TAP



Students tap on each button to see how to add and subtract polynomials

CHECK



Students complete the Check online to determine whether they are ready to move on.

Example 2 Add Polynomials**Teaching the Mathematical Practices**

1 Special Cases Work with students to evaluate the two methods shown. Encourage students to familiarize themselves with both methods, and know the best time to use each.

Questions for Mathematical Discourse

AL Why should you group or align like terms when adding polynomials? **This makes it easier to combine the like terms without error.**

OL In what circumstances might each method be more useful? **Sample answer:** Horizontal may be more useful for combining polynomials with only a few terms, while vertical may be more useful for polynomials with several terms.

BL Can you think of another method to add two polynomials? **Sample answer:** You could combine from greatest to least degree while marking off each set of like terms as you record the resulting term.

Example 3 Subtract Polynomials**Teaching the Mathematical Practices**

7 Use Structure Help students to use the structure of polynomial expressions to combine like terms and find the difference of polynomials.

Questions for Mathematical Discourse

AL What must you remember to do if you are subtracting polynomials? **Distribute the -1 .**

OL What could you do in the vertical method to help you combine like terms with the correct operation? **Sample answer:** Change the sign of each term in the second polynomial and then add the polynomials.

BL Which method do you prefer for subtracting polynomials and why? **Sample answer:** I prefer the vertical method because I can use placeholders and line up the like terms without having to look back and forth between the polynomials.

Common Error

Students often only subtract the first term when subtracting one polynomial from another. Remind students to distribute -1 to all terms in the parentheses, not just the first term.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.



Learn Multiplying Polynomials

Objective

Students multiply polynomials by using the Distributive Property.

MP Teaching the Mathematical Practices

8 Notice Regularity Help students to see the regularity in the way that expressions are simplified when multiplying binomials using the FOIL method.

Essential Question Follow-Up

Students have begun to multiply polynomials.

Ask:

Are polynomials closed under multiplication? **Sample answer:** Because multiplying polynomials results in a polynomial, the set of polynomials is closed under multiplication.

DIFFERENTIATE

Reteaching Activity

IF students have difficulty describing or using properties of exponents, THEN have them write their own summary of the properties of powers, such as “to multiply expressions with exponents, you add the exponents; to divide, you subtract the exponents.”

Example 4 Simplify by Using the Distributive Property

MP Teaching the Mathematical Practices

7 Use the Distributive Property Point out that the Distributive Property is one of the most-used properties in algebra. Students should know that whenever they see a number outside of a sum or difference within parentheses, they should apply the Distributive Property.

Questions for Mathematical Discourse

A Why is $2x(4x^2)$ equal to $8x^3$? The coefficients 2 and 4 multiply to 8. The exponents for x are added because x to a power n is the same as multiplying x by x n times. $x(x \cdot x \cdot x) = x^4$

OL Can you ever end up with negative exponents when you multiply a monomial by a polynomial? **No, you are adding together two positive numbers.**

BL In general, if you are multiplying a monomial by a polynomial, how many pairs of monomials will you be multiplying together? **the number of terms the polynomial has**

Learn Multiplying Polynomials

Polynomials can be multiplied by using the Distributive Property to multiply each term in one polynomial by each term in the other. When polynomials are multiplied, the product is also a polynomial. Therefore, the set of polynomials is closed under the multiplication. This is similar to the system of integers, which is also closed under multiplication. To multiply two binomials, you can use a shortcut called the **FOIL method**.

Key Concept • FOIL Method

Words: Find the sum of the products of **F** the first terms, **O** the outer terms, **I** the inner terms, and **L** the last terms.

Symbolic:

$$\begin{array}{rcl}
 \text{Product of} & \text{Product of} & \text{Product of} \\
 \text{First Terms, Outer Terms} & \text{Inner Terms} & \text{Last Terms} \\
 (2x + 4y)(-3) & = & (2x)(-3) + (2x)(-3) + (4y)(-3) \\
 & = & 2x^2 - 6x + 4x - 12 \\
 & = & 2x^2 - 2x - 12
 \end{array}$$

Example 4 Simplify by Using the Distributive Property

Find $2x(4x^2 + 5x^2 - x - 7)$.

$$\begin{aligned}
 2x(4x^2 + 5x^2 - x - 7) &= 2x(4x^2) + 2x(5x^2) + 2x(-x) + 2x(-7) \\
 &= 8x^3 + 10x^3 - 2x^2 - 14x
 \end{aligned}$$

Example 5 Multiply Binomials

Find $(3x + 5)(x - 7)$.

Step 1 Multiply any two binomials.

$$\begin{aligned}
 (3x + 5)(x - 7) &= 3x(x) + 3x(-7) + 5(x) + 5(-7) && \text{FOIL Method} \\
 &= 3x^2 - 21x + 5x - 35 && \text{Multiply.} \\
 &= 3x^2 - 16x - 35 && \text{Combine like terms.}
 \end{aligned}$$

Step 2 Multiply the result by the remaining binomial.

$$\begin{aligned}
 (3x^2 - 16x - 35)(4x + 1) &= 3x^2(4x + 1) + (-16x)(4x + 1) + (-35)(4x + 1) \\
 &= 12x^3 + 3x^2 - 64x^2 - 16x - 140x - 35 \\
 &= 12x^3 - 61x^2 - 156x - 35
 \end{aligned}$$

Check

$$\text{Find } (-2x - 3)(5x^2 - 11x + 4). \quad -10x^3 - 53x^2 - 49x + 12$$

Go Online You can complete an Extra Example online.

Think About It!

Why are the exponents added when you multiply the monomials?

Sample answer: Because the monomials have the same base x , their exponents are added by using the Product of Powers Property.

Go Online for an example of how to multiply two binomials.

Lesson 2-3 • Operations with Polynomials 95

Interactive Presentation

Work with Polynomials

Polynomials can be multiplied by using the Distributive Property to multiply each term in one polynomial by each term in the other. When polynomials are multiplied, the product is also a polynomial. Therefore, the set of polynomials is closed under the multiplication. This is similar to the system of integers, which is also closed under multiplication.

The example below shows how to use the Distributive Property to multiply a monomial and a binomial.

$$\begin{aligned}
 5x(3x + 6) &= 5x \cdot 3x + 5x \cdot 6 \\
 &= 15x^2 + 30x
 \end{aligned}$$

Learn

TYPE



Students answer a question to show they understand how to multiply expressions with exponents.



Go Online
to see Example 6.

Think About It!
What does x represent
in the polynomial
expression for the
volume of the cake?

the height of tier 1

Problem-Solving Tip
Solve a Simpler Problem Some complicated problems can be more easily solved by breaking them into several simpler problems. In this case, finding the volume of each tier individually simplifies the situation and makes finding the total volume easier.

Apply Example 7 Write and Simplify a Polynomial Expression

BAKING Byron is baking a three-tier cake for a birthday party. Each tier will have $\frac{1}{2}$ the volume of the previous tier. The dimensions of the first tier are shown. Find the total volume of the cake.



1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to find the total volume of the cake, which is the sum of all 3 tiers. How can I represent the volume of each tier as a polynomial? Which properties will I need to know? I can find the answers to my questions by referencing other examples in the lesson.

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I will find and simplify the volume of each tier and then add them together. I will use the Distributive Property and FOIL method to complete the task.

3 What is your solution?

Use your strategy to solve the problem.

What is the volume of each tier?

Tier 1: $8x^3 - 2x^2 - 3x$; Tier 2: $4x^3 - x^2 - 1.5x$;
Tier 3: $2x^3 - 0.5x^2 - 0.75x$

What is the total volume of the cake?

$14x^3 - 3.5x^2 - 5.25x$

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: Because all the expressions are based on the expression for the volume of Tier 1, I can check that the expression for Tier 1 is correct. I can factor the expression for volume of Tier 1 to ensure that the factors are the same as the given dimensions.

$8x^3 - 2x^2 - 3x$ Expression for Tier 1
 $= (8x^2 - 2x - 3)x$ Factor x from each term.
 $= (4x - 3)(2x + 1)x$ Factor $8x^2 - 2x - 3$.

96 Module 2 • Polynomials and Polynomial Functions

Interactive Presentation



Example 6

CHECK



Students complete the Check online to determine whether they are ready to move on.

Example 5 Multiply Binomials

MP Teaching the Mathematical Practices

7 Interpret Complicated Expressions Guide students to see the original expression as being composed of several binomials. Encourage them to break the expression into simpler problems.

Questions for Mathematical Discourse

- AL** When multiplying a polynomial with three terms by a polynomial with two terms, how many pairs of monomials do you multiply? **6**
- OL** Can you tell the degree of the polynomial before multiplying? **Yes; you know the leading term will be the three a -terms multiplied together, so the degree will be 3.**
- BL** When multiplying more than two binomials, what features might you look for to decide which pair of binomials to multiply first? **Sample answer: Look for a pair that results in a difference of squares because the middle terms cancel. This would make the next step of multiplication simpler.**

Apply Example 7 (Ex. 6 is online only) Write and Simplify a Polynomial Expression

MP Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them, 4 Model with Mathematics Students will be presented with a task.

They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- How can you find the the volume of Tier 1?
- Do you have to find the dimensions of Tier 2 to find its volume?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using their whiteboards so that you can see all student responses. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2 exercises that mirror the examples		1–29
2	exercises that use a variety of skills from this lesson	30–45
2	exercises that extend concepts learned in this lesson to new contexts	46–47
3	exercises that emphasize higher-order and critical-thinking skills	48–51

ASSESS AND DIFFERENTIATE

 Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign:

- Practice Exercises 1–47 odd, 48–51
- Extension: Relationship Between GCF and LCM with Polynomials
-  **ALEKS**

IF students score 66%–89% on the Checks, **THEN** assign:

- Practice Exercises 1–51 odd
- Remediation, Review Resources: Adding and Subtracting Polynomials
- Personal Tutors
- Extra Examples 1–6
-  **ALEKS** Simple Operations with Polynomials

IF students score 65% or less on the Checks, **THEN** assign:

- Practice Exercises 1–29 odd
- Remediation, Review Resources: Adding and Subtracting Polynomials
- Quick Review Math Handbook*: Operations with Polynomials
-  **ALEKS** Simple Operations with Polynomials

Practice

Example 1

Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial.

1. $2x^2 - 3x + 5$

2. $x^3 - 1$

3. $\frac{5xy}{x^2 + 2}$

4. $\sqrt{x} - 7$

5. $3x^2 + 5$

6. $7x^2 + 6x - 7$

7. $2x^2 - 8 - 2$

8. $3x^2 + 6x - 5$

9. $3x + 5 + (2x + 7)$

10. $(3x + 5) - (x + 1)$

11. $5x^2 - 3x - 7 + (2x^2 + 7x - 2)$

12. $(-2x^2 - 3x + 5) + (-2x^2 - 3x + 8)$

13. $3x^2 + 4x - 5$

14. $(-2x^2 - 3x + 5) + (-2x^2 - 3x + 8)$

15. $(2x - 3) - (5x - 6)$

16. $(2x^2 + 3x - 5) - (2x^2 + 3x - 8)$

17. $3x^2 + 5$

18. $4x(2x^2 + 1)$

19. $3x^2 - 30x$

20. $8x^2 + 4xy$

21. $-50x^2 - x^3$

22. $x^2(2x + 5)$

23. $(x - 5)^2$

24. $(2x - 3)(3x - 5)$

25. $x^2 - 10x + 25$

26. $4x^2 - 12x + 9$

27. $x^2 + x^3 - 2xy + y^2$

28. $x^2 + x^3 - 2xy + y^2$

29. $x^2 + x^3 - 2xy + y^2$

30. $x^2 + x^3 - 2xy + y^2$

31. $x^2 + x^3 - 2xy + y^2$

32. $x^2 + x^3 - 2xy + y^2$

33. $x^2 + x^3 - 2xy + y^2$

34. $x^2 + x^3 - 2xy + y^2$

35. $x^2 + x^3 - 2xy + y^2$

36. $x^2 + x^3 - 2xy + y^2$

37. $x^2 + x^3 - 2xy + y^2$

38. $x^2 + x^3 - 2xy + y^2$

39. $x^2 + x^3 - 2xy + y^2$

40. $x^2 + x^3 - 2xy + y^2$

41. $x^2 + x^3 - 2xy + y^2$

42. $x^2 + x^3 - 2xy + y^2$

43. $x^2 + x^3 - 2xy + y^2$

44. $x^2 + x^3 - 2xy + y^2$

45. $x^2 + x^3 - 2xy + y^2$

46. $x^2 + x^3 - 2xy + y^2$

47. $x^2 + x^3 - 2xy + y^2$

48. $x^2 + x^3 - 2xy + y^2$

49. $x^2 + x^3 - 2xy + y^2$

50. $x^2 + x^3 - 2xy + y^2$

51. $x^2 + x^3 - 2xy + y^2$

52. $x^2 + x^3 - 2xy + y^2$

53. $x^2 + x^3 - 2xy + y^2$

54. $x^2 + x^3 - 2xy + y^2$

55. $x^2 + x^3 - 2xy + y^2$

56. $x^2 + x^3 - 2xy + y^2$

57. $x^2 + x^3 - 2xy + y^2$

58. $x^2 + x^3 - 2xy + y^2$

59. $x^2 + x^3 - 2xy + y^2$

60. $x^2 + x^3 - 2xy + y^2$

61. $x^2 + x^3 - 2xy + y^2$

62. $x^2 + x^3 - 2xy + y^2$

63. $x^2 + x^3 - 2xy + y^2$

64. $x^2 + x^3 - 2xy + y^2$

65. $x^2 + x^3 - 2xy + y^2$

66. $x^2 + x^3 - 2xy + y^2$

67. $x^2 + x^3 - 2xy + y^2$

68. $x^2 + x^3 - 2xy + y^2$

69. $x^2 + x^3 - 2xy + y^2$

70. $x^2 + x^3 - 2xy + y^2$

71. $x^2 + x^3 - 2xy + y^2$

72. $x^2 + x^3 - 2xy + y^2$

73. $x^2 + x^3 - 2xy + y^2$

74. $x^2 + x^3 - 2xy + y^2$

75. $x^2 + x^3 - 2xy + y^2$

76. $x^2 + x^3 - 2xy + y^2$

77. $x^2 + x^3 - 2xy + y^2$

78. $x^2 + x^3 - 2xy + y^2$

79. $x^2 + x^3 - 2xy + y^2$

80. $x^2 + x^3 - 2xy + y^2$

81. $x^2 + x^3 - 2xy + y^2$

82. $x^2 + x^3 - 2xy + y^2$

83. $x^2 + x^3 - 2xy + y^2$

84. $x^2 + x^3 - 2xy + y^2$

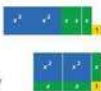
85. $x^2 + x^3 - 2xy + y^2$



43. **CONSTRUCT ARGUMENTS** Complete the table to show which sets are closed under the operations. Write yes if the set is closed under the operation. Write no and provide a counterexample if the set is not closed under the operation. Assume that zero division by zero is undefined. It does not affect closure.

	Addition and Subtraction	Multiplication	Division
Integers	yes	yes	No; $2 \div 2 = 1 \neq 2$
Rational Numbers	yes	yes	yes
Polynomials	yes	yes	No; $\frac{x}{x} = 1 \neq x^2$

44. **STRUCTURE** The polynomial $2x^2 + 3x + 1$ can be represented by the tiles shown in the figure at the right. These tiles can be arranged to form the rectangle shown. Notice that the area of the rectangle is $2x^2 + 3x + 1$ units².
- a. Find the length and width of the rectangle. See margin.
- b. Find the perimeter of the rectangle. Then explain your process for finding the perimeter. See margin.
- c. Select a value for x and substitute that value into each of the expressions above. For your value of x , state the length, width, perimeter, and area of the rectangle. Discuss any restrictions on the value of x . See margin.

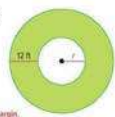


45. **BOOKING** Terry invests \$1500 in two mutual funds. In the first year, one fund grows 3.8% and the other grows 6%. Write a polynomial to represent the value of Terry's investment after the first year if he invested x dollars in the fund with the lesser growth rate. $-0.022x + 1590$

46. **GEOMETRY** Consider a trapezoid that has one base that measures five feet greater than its height. The other base is one foot less than twice its height. Let x represent the height.
- a. Write an expression for the area of the trapezoid.
 $A = 0.5(5x + 5) + (2x - 1)(5x) = 5x^2 + 5x$
- b. Write an expression for the area of the trapezoid if its height is increased by 4 feet. $A = 1.5x^2 + 14x + 32$

Lesson 3-3 • Operations with Polynomials 99

47. **URBAN DEVELOPMENT** The diagram represents an aerial view of a memorial in a town center. A sidewalk that is 12 feet wide with an area of 384 square feet surrounds a statue with a circular base of radius r .



- a. Find the radius of the smaller and larger circles. Show your work. See margin.
- b. A nearby town wants to use the same design concept, but use two squares rather than two circles. Draw and label a diagram with two squares to represent a sidewalk with the same uniform width and area as the circular sidewalk. See margin.
- c. If s represents the side length of the smaller square, write a polynomial expression for the area of the sidewalk that surrounds the smaller square. $48s + 576$

Higher-Order Thinking Skills

48. **ANALYZE** Given $f(x)$ and $g(x)$ are polynomials, is the product always a polynomial? Justify your argument. See margin.
49. **PERSEVERE** Use your result from Exercise 48 to make conjectures about the product of a polynomial with m terms and a polynomial with n terms. Justify your conjecture.
- a. How many times are two terms multiplied?
 $m \cdot n$ each term of one polynomial multiplies each term of the other one time.
- b. What is the least number of terms in the simplified product?
2; adding like terms may result in a sum of 0, but the first and last terms are unique.
50. **FIND THE ERROR** Isabelle found the product of $3x^2 - 4x + 1$ and $x^2 + 5x + 6$ using vertical alignment. Is her answer correct? Explain your reasoning.

Isabelle's Work

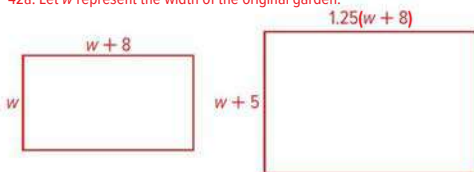
$$\begin{array}{r} 3x^2 - 4x + 1 \\ \times x^2 + 5x + 6 \\ \hline 18x^2 - 24x + 6 \\ x^4 - 20x^3 + 17x^2 - 4x + 6 \end{array}$$

Yes, Isabelle correctly aligned like terms after multiplying each term in the first polynomial by each term in the second polynomial. She then correctly added like terms.

51. **CREATE** Write an expression where two binomials are multiplied and have a product of $9 - 4b^2$. $(3 - 2b)(3 + 2b)$

Answers

- 42a. Let w represent the width of the original garden.



- 42b. $1.25(w+8)(w+5) - w(w+8) = 1.25w^2 + 16.25w + 50 - w^2 - 8w = 0.25w^2 + 8.25w + 50$; if $w = 10$ feet, then the garden's area increased by $0.25(10)^2 + 8.25(10) + 50$ or 157.5 square feet.

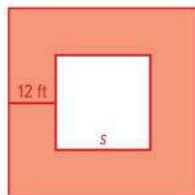
- 44a. The length and width are $2x + 1$ and $x + 1$ units.

- 44b. The perimeter can be found using the formula $P = 2(\ell + w)$. Substituting $2x + 1$ for length and $x + 1$ for width, $P = 2(2x + 1 + x + 1) = 2(3x + 2) = 6x + 4$.

- 44c. Sample answer: For $x = 3$, the length is 7 units, the width is 4 units, the perimeter is 22 units, and the area is 28 units². The value of x must be chosen so that the length, width, perimeter, and area are all positive. The expressions $2x + 1$, $x + 1$, $6x + 4$, and $2x^2 + 3x + 1$ will all be positive only if $x > -\frac{1}{2}$.

- 47a. Area of sidewalk = area of larger circle - area of smaller circle, so $384\pi = \pi(r + 12)^2 - \pi r^2 = \pi(r^2 + 24r + 144) - \pi r^2 = 24\pi r + 144\pi$. $24\pi r = 240\pi$; $r = 10$; the radius of the smaller circle is 10 feet, and the radius of the larger circle is $r + 12 = 10 + 12$ or 22 feet.

- 47b.



48. Yes; sample answer: For $f(x)$ and $g(x)$ to be polynomials, they must have real coefficients and whole-number exponents. Real numbers and whole numbers are closed under addition and multiplication. So, when $f(x)$ and $g(x)$ are multiplied, the coefficients of each term are multiplied, yielding new real coefficients, and the exponents of each term are added, producing new whole number exponents. By the definition of polynomials, the product of polynomials is a polynomial because the coefficients are real and the exponents are whole numbers.

Dividing Polynomials

LESSON GOAL

Students divide polynomials by using long division and synthetic division.

1 LAUNCH



Launch the lesson with a Warm Up and an introduction.

2 EXPLORE AND DEVELOP



Explore: Using Algebra Tiles to Divide Polynomials



Develop:

Dividing Polynomials by Using Long Division

- Divide a Polynomial by a Monomial
- Divide a Polynomial by a Binomial
- Find a Quotient with a Remainder

Dividing Polynomials by Using Synthetic Division

- Use Synthetic Division
- Divisor with a Coefficient Other Than 1



You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE



Exit Ticket



Practice

DIFFERENTIATE



View reports of student progress on the **Checks** after each example.

Resources	AL	IE	EL
Remediation: Multiplying a Polynomial by a Monomial	●	●	●
Extension: Oblique Asymptotes		●	●

Language Development Handbook

Assign page 12 of the *Language Development Handbook* to help your students build mathematical language related to dividing polynomials by using long division and synthetic division.

ELL You can use the tips and suggestions on page T12 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra

Standards for Mathematical Content:

A.APR.6 Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

8 Look for and expression regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous

Students added, subtracted, and multiplied polynomials.

A.APR.1

Now

Students divide polynomials by using long division and synthetic division.

A.APR.6

Next

Students will expand powers of binomials.

A.APR.5

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION



Conceptual Bridge In this lesson, students extend their understanding of multiplying polynomials to build fluency with dividing polynomials. They apply their understanding of dividing polynomials by solving real-world problems.

Mathematical Background

Synthetic division can be used to divide a polynomial by a binomial.

The terms in both the divisor and dividend must be in descending order of power, and a coefficient of 0 must be included for any term that is missing.



Interactive Presentation

Warm Up

Simplify each expression.

- $3(6x^2 + 3x - 8)$
- $7x(-3x^2 + 4x^2 + 18x)$
- $(x + 3)(2x^2 + 5x - 4)$
- $(-x^3 + 2x + 7)(2x^2 - 1)$

Show Answers

Warm Up

Launch the Lesson

GPS devices work by receiving data from the more than 30 satellites in the system orbiting Earth. A GPS needs to receive a signal from at least two satellites to calculate longitude and latitude, and it needs three or more satellites to be able to display altitude as well.



As the satellites transmit signals, they have to contend with atmospheric noise such as rain, fog, trees, tall buildings, or anything else that obstructs the communication between the satellite and the GPS device. In order to find the most accurate coordinates, the GPS performs thousands of error correction algorithms. The type of algorithm is a form of polynomial division that calculates the remainder of what the satellite signal is and what the signal should be. Based on the remainder, the GPS makes adjustments so that you can get to your destination.

Launch the Lesson

Vocabulary

synthetic division

An alternate method used to divide a polynomial by a binomial of degree 1.

- What does the remainder of polynomial division tell you about the division of the polynomial?
- Why is it necessary to have both synthetic division and long division?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- Multiplying polynomials

Answers:

- $18x^2 + 9x - 24$
- $-21x^4 + 28x^3 + 70x^2$
- $2x^3 + 11x^2 + 11x - 12$
- $-2x^5 + 5x^4 + 14x^3 - 2x^2 - 7x$

Launch the Lesson



Teaching the Mathematical Practices

4 Apply Mathematics Students see how division of polynomials is applicable to solving a real-world problem.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then, discuss the questions below with the class.



Explore Using Algebra Tiles to Divide Polynomials

Objective

Students use algebra tiles to explore dividing a trinomial by a binomial.



Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to complete the Explore, students will need to use the Algebra Tiles eTool. Work with students to explore and deepen their understanding of polynomial division.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

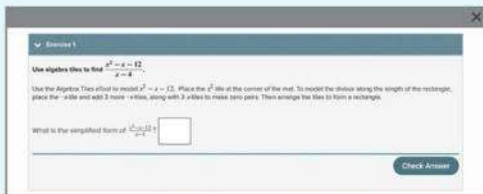
Students will complete guiding exercises throughout the Explore activity. Students will use algebra tiles to model polynomials of degree 0, 1, or 2. They will use these tiles to find quotients and remainders from division of polynomials. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore



Explore

TYPE



Students simplify an expression to show they understand how to use algebra tiles to divide polynomials.



Interactive Presentation



Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Using Algebra Tiles to Divide Polynomials (*continued*)

Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear definitions and mathematical language when they discuss their solutions to the Explore activity with others.

Questions

Have students complete the Explore activity.

Ask:

- Why do you need to add x -tiles to model $\frac{x^2 - x - 12}{x - 4}$? **Sample answer:** The length of the rectangle is $x - 4$, so the rectangle on the mat needs to have one x^2 -tile and 4 negative x -tiles. You have to add 3 positive and 3 negative x -tiles to make zero pairs and not change the value of the polynomial.
- Why did $\frac{x^2 + 2x - 9}{x - 3}$ have a remainder? What does this mean? **Sample answer:** The tiles couldn't be arranged into a single rectangle with length of $x - 3$. The pieces left created the remainder and shows that the polynomial does not have $x - 3$ as a factor.



Inquiry

How can you use a model to divide polynomials? **Sample answer:** Some polynomials can be modeled by the area of a rectangle. The length and width represent the divisor and quotient.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Dividing Polynomials by Using Long Division

Objective

Students divide polynomials by using long division.

MP Teaching the Mathematical Practices

3 Make Conjectures In the Talk About It! feature, students will make a conjecture about a quotient with a remainder of zero and then build a logical progression of statements to validate the conjecture. Once students have made their conjectures, guide students to validate them.

Important to Know

The degree of the quotient when a polynomial of degree n is divided by a polynomial of degree m is $n - m$.

Common Misconception

Students may think that they can treat each term in a polynomial as an entity that can 'cancel out' with other terms incorrectly. Review how to simplify polynomials using Example 1.

Essential Question Follow-Up

Students have begun to divide polynomials by using long division.

Ask:

How do you know you have your remainder and that you are done dividing? **Sample answer:** When the degree of the term you are dividing into is less than the degree of the divisor.

DIFFERENTIATE

Reteaching Activity **AL**

IF some students have trouble keeping their concentration throughout the sequence of steps required in long division, **THEN** encourage them to compare intermediate results with a partner so they can ask questions and catch errors before completing the entire problem.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 2-4

Dividing Polynomials

Explore Using Algebra Tiles to Divide Polynomials

Online Activity Use algebra tiles to complete the Explore.

INQUIRY How can you use a model to divide polynomials?

Learn Dividing Polynomials by Using Long Division

To divide a polynomial by a monomial, find the quotient of each term of the polynomial and the monomial.

$$\frac{6x^2 - 18x}{3x} = \frac{6x^2}{3x} - \frac{18x}{3x} = 2x - 6$$

You can divide a polynomial by a polynomial with more than one term by using a process similar to long division of real numbers. The resulting quotient may be a polynomial or a polynomial with a remainder.

Key Concept - Division Algorithm

Words: If $f(x)$ and $g(x)$ are two polynomials in which $g(x) \neq 0$ and the degree of $g(x)$ is less than the degree of $f(x)$, then there exists a unique quotient $q(x)$ and a unique remainder $r(x)$ such that $f(x) = q(x)g(x) + r(x)$, where the remainder is zero or the degree of $r(x)$ is less than the degree of $g(x)$.

Symbols: $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)} \quad \frac{f(x)}{g(x)} = q(x)g(x) + r(x)$$

$$\text{Example: } \frac{2x^3 - 5x - 3}{x + 1} = 2x^2 - 7x + 2$$

$$2x^3 - 5x - 3 = (x + 1)(2x^2 - 7x + 2)$$

$$= 2x^3 - 5x - 3$$

$$= 2x^3 - 7x^2 + 7x + 2x - 5x - 3$$

$$= 2x^3 - 7x^2 + 9x - 3$$

$$= 2x^3 - 7x^2 + 9x - 3$$

$$= 2x^3 - 7x^2 + 9x - 3$$

$$= 2x^3 - 7x^2 + 9x - 3$$

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$$= 2x^3 - 7x^2 + 9x - 3$$

Today's Goals

- Divide polynomials by using long division.
- Divide polynomials by using synthetic division.

Today's Vocabulary
synthetic division

Talk About It!

Is the following statement always, sometimes, or never true? Justify your argument.

If a quadratic polynomial is divided by a binomial with a remainder of 0, the binomial is a factor of the polynomial.

Always, sample answer: If the binomial divides the polynomial with a remainder of zero, the quotient is another binomial. The product of the two binomials is the original polynomial.

Study Tip

In algebra, unique means only one. So, there is only one quotient and remainder that will satisfy the Division Algorithm for each polynomial.

Lesson 2-4 • Dividing Polynomials 101

Interactive Presentation



Learn

**Example 1** Divide a Polynomial by a MonomialFind $(24x^4y^3 + 18x^2y^2 + 30xy)(6xy)^{-1}$.

$$(24x^4y^3 + 18x^2y^2 + 30xy)(6xy)^{-1}$$

$$= \frac{24x^4y^3}{6xy} + \frac{18x^2y^2}{6xy} + \frac{30xy}{6xy}$$

Write a fraction.

$$= \frac{24x^4y^3}{6xy} + \frac{18x^2y^2}{6xy} + \frac{30xy}{6xy}$$

Sum of quotients

$$= \frac{24}{6}x^{4-1}y^{3-1} + \frac{18}{6}x^{2-1}y^{2-1} + \frac{30}{6}x^{1-1}y^{1-1}$$

Divide

$$= 4x^3y^2 + 3xy + 5$$

Simplify

Check

$$\text{Find } (6xy^2 + 21x^2y + 12xy^2) \div (3xy^2): \quad 3x^3y^2 + 7x^2y^2 - 4x$$

Think About It!
How can you check the solution?**Sample answer:**
Multiply $x + 4$ and $x - 9$ and check that the product is $x^2 - 5x - 36$.**Watch Out!**
Signs Remember that you are subtracting throughout the process of long division. Carefully label the signs of the coefficients to avoid a sign error.**Go Online**
You can learn how to divide polynomials by watching the video online.**Example 2** Divide a Polynomial by a BinomialFind $(x^2 - 5x - 36) \div (x + 4)$.

$$\begin{array}{r} x - 9 \\ x + 4 \overline{) x^2 - 5x - 36} \\ \underline{-(x^2 + 4x)} \\ -9x - 36 \\ \underline{-(-9x - 36)} \\ 0 \end{array}$$

$$-9x - 36$$

$$\underline{-(-9x - 36)}$$

$$0$$

The quotient is $x - 9$ and the remainder is 0.

Check

$$\text{Find } \frac{x^2 - 5x - 36}{x + 4} = x - 9$$

Example 3 Find a Quotient with a RemainderFind $\frac{3x^3 - 14x^2 - 7x + 3}{x - 9}$.

$$\begin{array}{r} 3x^2 + x - 2 \\ x - 9 \overline{) 3x^3 - 14x^2 - 7x + 3} \\ \underline{-(3x^3 - 27x^2)} \\ 13x^2 - 7x + 3 \\ \underline{-(13x^2 - 117x)} \\ 110x + 3 \\ \underline{-(110x - 990)} \\ 993 \end{array}$$

$$13x^2 - 7x + 3$$

$$\underline{-(13x^2 - 117x)}$$

$$110x + 3$$

$$\underline{-(110x - 990)}$$

$$993$$

Go Online You can complete an Extra Example online.

Interactive Presentation



Example 1

TYPE



Students answer questions to show they understand how to divide polynomials.

CHECK



Students complete the Check online to determine whether they are ready to move on.

Example 1 Divide a Polynomial by a Monomial

Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. In Example 1, help students to see the expression as being composed of several simpler expressions.

Questions for Mathematical Discourse

- AL** Why is the fraction rewritten as three fractions? **We want to divide each term by $6ab$.**
- OL** What is an equivalent expression for $\frac{1}{x}$? x^{-1}
- BL** What is another method you could use to solve this example? **Sample answer:** Distribute the $(6ab)$ to each term as multiplication and then add the exponents. Adding the negative exponent is equivalent to subtracting the positive exponent.

Example 2 Divide a Polynomial by a Binomial

Teaching the Mathematical Practices

1 Check Answers Mathematically proficient students continually ask themselves, “Does this make sense?” In the Think About It! feature, students need to explain how to check their answer.

Questions for Mathematical Discourse

- AL** How do you determine each term of the quotient? **Look for the term that, when multiplied with the divisor, will result in canceling the first term of the remaining dividend.**
- OL** What does it mean for the remainder to be 0? $x + 4$ divides evenly into $x^2 - 5x - 36$ and $x + 4$ is a factor of $x^2 - 5x - 36$.
- BL** How could you tell the remainder would be zero before doing the division? $- \frac{36}{4} = -9$, and $-9 + 4 = -5$, so $x + 4$ is a factor of the dividend, along with $x - 9$.

Example 3 Find a Quotient with a Remainder

Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of the polynomials to find the quotient by using long division.

Questions for Mathematical Discourse

- AL** Why are there parentheses around the subtraction sign in the example? **The parentheses remind us that we are subtracting the entire expression, not just the first term.**
- OL** What statement can you make about the relationship between the divisor and the dividend for this example? **The divisor is not a factor of the dividend.**
- BL** Can you write $\frac{7}{(x-5)}$ as $\frac{7}{x} - \frac{7}{5}$? Explain. **No; sample answer:** Consider $\frac{7}{(3+5)} = \frac{7}{8}$, and $\frac{7}{3} + \frac{7}{5} = \frac{56}{15} \neq \frac{56}{8}$.



Learn Dividing Polynomials by Using Synthetic Division

Objective

Students divide polynomials by using synthetic division.

MP Teaching the Mathematical Practices

1 Understand Different Approaches Work with students to look at the alternate method for dividing polynomials. Ask students to compare and contrast long division and synthetic division.

Important to Know

You can use synthetic division to divide a polynomial by a binomial. The terms in both the divisor and dividend must be in descending order of power, and a coefficient of 0 must be included for any term that is missing.

Things to Remember

Remind students to *add* terms when performing synthetic division.

Essential Question Follow-Up

Students have begun to divide polynomials by using synthetic division.

Ask:

In what form must the divisor be to use synthetic division? What if it is not? **Sample answer:** The divisor needs to be in $x - a$ form. If the divisor is not in the form $x - a$, divide every term in both the divisor and dividend by the coefficient of the first term of the divisor and then use synthetic division.

Example 4 Use Synthetic Division

MP Teaching the Mathematical Practices

8 Look for a Pattern Help students to see how the calculations used in synthetic division are repeated in Example 4.

Questions for Mathematical Discourse

AL Why is $a = -3$? Synthetic division is defined for $(x - a)$, and $(x + 3) = (x - (-3))$.

OL How do you determine the degree of the terms of the quotient? The degree of the dividend minus the degree of the divisor yields the degree of the quotient, and subsequent terms go in descending order.

BL Why is this called synthetic division? **Sample answer:** It is a simplified process to achieve the same results as long division, derived from the pattern of operations performed in long division.

The quotient is $3x^2 + x - 2$ and the remainder is -7 .

Therefore, $\frac{3x^3 - 14x^2 + 27x + 3}{x - 3} = 3x^2 + x - 2 - \frac{7}{x - 3}$.

Check:

Find the quotient of $(-4x^3 + 5x^2 - 2x - 9)(x - 2)$.

$$-4x^3 - 3x - 8 - \frac{25}{x - 2}$$

Learn Dividing Polynomials by Using Synthetic Division

Synthetic division is an alternate method used to divide a polynomial by a binomial of degree 1. You may find this to be a quicker, simpler method.

Key Concept: Synthetic Division

Step 1 After writing a polynomial in standard form, write the coefficients of the dividend. If the dividend is missing a term, use a placeholder. Write the constant a of the divisor $x - a$ in the box. Bring the first coefficient down.

Step 2 Multiply the number just written in the bottom row by a , and write the product under the next coefficient.

Step 3 Add the product and the coefficient above it.

Step 4 Repeat Steps 2 and 3 until you reach a sum in the last column.

Step 5 Write the quotient. The numbers along the bottom row are the coefficients of the quotient. The power of the first term is one less than the degree of the dividend. The final number is the remainder.

Example 4 Use Synthetic Division

Find $(2x^3 - 2x^2 - 53x - 60) \div (x + 3)$.

Step 1 Write the coefficients of the dividend and write the constant a in the box. Because $x + 3 = x - (-3)$, $a = -3$. Then bring the first coefficient down.

Step 2 Multiply by a and write the product. The product of the coefficient and a is $3(-3) = -9$.

Step 3 Add the product and the coefficient. $-3(-3) = 9$.

Step 4 Repeat Steps 2 and 3 until you reach a sum in the last column.

(continued on the next page)

Think About It! Describe a method you could use to check your answer.

Sample answer: I can use long division to check that the quotient is the same.

Lesson 2-4 • Dividing Polynomials 103

Interactive Presentation

Dividing Polynomials by Using Synthetic Division

Synthetic division: A simpler method to divide a polynomial by a binomial of degree 1. You may find this to be a quicker, simpler method.

KEY CONCEPT: SYNTHETIC DIVISION

How do you divide a polynomial by a binomial of degree 1? Use synthetic division.

Step 1 After writing a polynomial in standard form, write the coefficients of the dividend. If the dividend is missing a term, use a placeholder. Write the constant a of the divisor $x - a$ in the box. Bring the first coefficient down.

Step 2 Multiply the number just written in the bottom row by a , and write the product under the next coefficient.

Step 3 Add the product and the coefficient above it.

Step 4 Repeat Steps 2 and 3 until you reach a sum in the last column.

Step 5 Write the quotient. The numbers along the bottom row are the coefficients of the quotient. The power of the first term is one less than the degree of the dividend. The final number is the remainder.

Learn

SWIPE



Students move through the steps to see how to divide polynomials using synthetic division.



Think About It!
Describe a method you could use to check your answer.

Sample answer: I can use long division to check that the quotient is the same.

Watch Out!
Missing terms: Add placeholders for terms that are missing from the polynomial. In this case, there are 0 x^2 terms.

Step 5 Write the quotient. Because the degree of the dividend is 3 and the degree of the divisor is 1, the degree of the quotient is 2. The final sum in the synthetic division is 0, so the remainder is 0.

The quotient is $3x^2 - 11x - 20$.

Example 5 Divisor with a Coefficient Other Than 1

Find $\frac{4x^3 - 37x^2 + 4x + 9}{2x - 1}$.

To use synthetic division, the lead coefficient of the divisor must be 1.

$$\begin{aligned} \frac{4x^3 - 37x^2 + 4x + 9}{2x - 1} &= \frac{2x^3 - 18x^2 + 2x + \frac{9}{2}}{x - \frac{1}{2}} \quad \begin{array}{l} \text{Divide the numerator and denominator by 2.} \\ \text{Simplify the numerator (and denominator).} \end{array} \end{aligned}$$

$$x - \frac{1}{2} \overline{) 2x^3 - 18x^2 + 2x + \frac{9}{2}}$$

Complete the synthetic division.

$\frac{1}{2}$	2	0	$-\frac{37}{2}$	-2	$\frac{9}{2}$
		1	$\frac{1}{2}$	-9	$-\frac{7}{2}$
	2	1	-18	-7	1

The resulting expression is $2x^2 + x^2 - 18x - 7 + \frac{1}{2} = \frac{3x^2 - 17x - \frac{13}{2}}{x - \frac{1}{2}}$.

Now simplify the fraction.

$$\frac{3x^2 - 17x - \frac{13}{2}}{x - \frac{1}{2}} = \frac{(3x^2 - 17x - \frac{13}{2}) \cdot 2}{(x - \frac{1}{2}) \cdot 2}$$

Multiply the numerator and denominator by 2.

$$= \frac{6x^2 - 34x - 13}{2x - 1}$$

Simplify.

The solution is $2x^2 + x^2 - 18x - 7 + \frac{1}{2} = \frac{3x^2 - 17x - \frac{13}{2}}{x - \frac{1}{2}}$.

You can check your answer by using long division.

Check:

$$\text{Find } (4x^3 + 3x^2 - 12x^2 - x + 6)(2x + 3)^{-1}.$$

$$x^3 - 3x + 2$$

Go Online: You can complete an Extra Example online.

104 Module 2 • Polynomials and Polynomial Functions

Example 5 Divisor with a Coefficient Other Than 1



Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expression to reach solutions. In Example 5, students must transform the numerator and denominator to apply synthetic division.

Questions for Mathematical Discourse

- AL** What number could you multiply each term by that would be the same as dividing each term by 2? $\frac{1}{2}$
- OL** What type of polynomial must the divisor be to use synthetic division? **linear**
- BL** Why do you divide both the numerator and the denominator by 2? **Sample answer:** You divide both so that the value of the expression does not change. Dividing the numerator and denominator by the same value is the same as dividing by 1.

Things to Remember

Remind students to divide all terms in the numerator and denominator when turning the divisor to $x - a$ form.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Interactive Presentation

Example 5

TYPE



Students enter numbers to complete a synthetic division.

CHECK



Students complete the Check online to determine whether they are ready to move on.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–16
2	exercises that use a variety of skills from this lesson	17–34
2	exercises that extend concepts learned in this lesson to new contexts	35–37
3	exercises that emphasize higher-order and critical-thinking skills	38–43

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign:

- Practice Exercises 1–37 odd, 38–43
- Extension: Oblique Asymptotes
- ALEKS**

OL EL

IF students score 66%–89% on the Checks, **THEN** assign:

- Practice Exercises 1–37 odd
- Remediation, Review Resources: Multiplying a Polynomial by a Monomial
- Personal Tutors
- Extra Examples 1–5
- ALEKS** Multiplying Polynomials

AL OL

IF students score 65% or less on the Checks, **THEN** assign:

- Practice Exercises 1–15 odd
- Remediation, Review Resources: Multiplying a Polynomial by a Monomial
- Quick Review Math Handbook: Dividing Polynomials
- ALEKS** Multiplying Polynomials

AL

Practice

Go Online You can complete your homework online.

Example 1
Simplify each expression.

$$1. \frac{15x^2 + 6x^2 - 3x}{5x^2 + 2x + 1}$$

$$2. 16x^2 - 6x^2 + 12x^2 - 8x^2 + 4x^2 - 1$$

$$3. \frac{6x^2 - 9x^2 + 3x^2}{5} + 3x^2$$

$$4. 14x^2 - 8x^2 + 3x^2 + 2x^2$$

Example 2 and 3

Simplify by using long division.

$$5. \frac{x^2 + 3x + 2}{x + 2} = (x + 1) + \frac{0}{x + 2}$$

$$6. \frac{x^2 + 4x + 3}{x + 3} = (x + 1) + \frac{0}{x + 3}$$

$$7. \frac{2x^2 + 13x + 15}{2x + 3} = (x + 2) + \frac{9}{2x + 3}$$

$$8. \frac{6x^2 + x - 2}{2x + 2} = (3x - 1) + \frac{3}{2x + 2}$$

$$9. \frac{4x^2 - 9}{2x - 3} = (2x + 3) + \frac{9}{2x - 3}$$

$$10. \frac{2x^2 - 5x - 4}{2x + 1} = (x - 3) + \frac{3}{2x + 1}$$

Example 4 and 5

Simplify using synthetic division.

$$11. \frac{3x^2 - 7x + 10}{x - 2} = 3x + 5 + \frac{20}{x - 2}$$

$$12. \frac{3x^2 - 6x^2 - 32x - 52}{x - 2} = -3x - 52 + \frac{4x^2}{x - 2}$$

$$13. \frac{x^2 + 5}{x - 2} = x + 7 + \frac{9}{x - 2}$$

$$14. \frac{3x^2 - x^2 - 9x + 33}{x^2 + 6x + 8} = \frac{2x^2 - 9x + 33}{x^2 + 6x + 8}$$

$$15. \frac{14x^2 - x^2 + 20x + 3}{x^2 + 6x + 8} = \frac{13x^2 + 20x + 3}{x^2 + 6x + 8}$$

$$16. \frac{3x^2 - 2x + 1}{x^2 + 6x + 8} = \frac{3x^2 - 2x + 1}{x^2 + 6x + 8}$$

Lesson 2.4 • Dividing Polynomials 105

Mixed Exercises

Simplify.

$$17. \frac{10x^2 + 3x - 4}{x - 2} = (10x + 23) + \frac{42}{x - 2}$$

$$18. \frac{x^2 - 6x^2 + 10x - 3}{x^2 - 3x + 1} = -5x + 13 + \frac{38}{x^2 - 3x + 1}$$

$$19. \frac{2x^2 - 7x^2 + 7x - 2}{2x^2 - 3x + 1} = -5x + 7 + \frac{9}{2x^2 - 3x + 1}$$

$$20. \frac{x^2 + 2x^2 - 34x + 9}{x^2 - 5x + 1} = 3x - 33 + \frac{10}{x^2 - 5x + 1}$$

$$21. \frac{x^2 + 8}{x^2 - 2x + 4} = 1 + \frac{8 + 2x}{x^2 - 2x + 4}$$

$$22. \frac{15x^2 + x^2 + x}{3x^2 - x + 1} = 5 + \frac{x + 1}{3x^2 - x + 1}$$

$$23. \frac{23x^2 - 23x^2}{3x^2 - 3x} = \frac{0}{3x^2 - 3x} = 0$$

$$24. \frac{x^4 - x^4}{x^2 + x^2} = \frac{0}{2x^2} = 0$$

$$25. \frac{x^2 + 3x^2 - 9x - 8}{x^2 - 4x + 1} = 4 + \frac{-9x - 8}{x^2 - 4x + 1}$$

$$26. \frac{10x^2 - 9x^2 - 20x - 100}{x^2 + 6x + 8} = \frac{x^2 - 20x - 100}{x^2 + 6x + 8}$$

$$27. \frac{14x^2 - x^2 + 20x + 3}{x^2 + 6x + 8} = \frac{13x^2 + 20x + 3}{x^2 + 6x + 8}$$

$$28. \frac{x^2 + 2x^2 - 9x - 8}{x^2 - 4x + 1} = 3 + \frac{-9x - 8}{x^2 - 4x + 1}$$

29. STRUCTURE Jada used long division to divide $x^3 + x^2 + x + 1$ by $x + 2$. Her work is shown below with three terms missing. What are A, B, and C? A is x^2 , B is $6x$, C is 15.

$$\begin{array}{r} x^2 - x^2 + 3x - 5 \\ x + 2 \overline{) x^3 + x^2 + x + 1} \\ \underline{(-) x^2 + 2x^2} \\ -x^2 + 3x - 5 \\ \underline{(-) 3x^2 - 6x} \\ 3x^2 + x - 5 \\ \underline{(-) 3x^2 - 6x} \\ -5x + 1 \\ \underline{(-) -5x + 10} \\ C \end{array}$$

106 Module 2 • Polynomials and Polynomial Functions



- 30. AVERAGES** Benia has a list of $n + 1$ numbers and she needs to find their average. Two of the numbers are n^2 and 2. Each of the other $n - 1$ numbers are all equal to 1. Find the average of these numbers.
- $$\frac{n^2 + 2 + 1 + \dots + 1}{n + 1}$$

- 31. VOLUME** The volume of a cylinder is $\pi x^2 + 32x^2 - 304x + 640$. If the height of the cylinder is $x + 40$ feet, find the area of its base (in terms of x and π).
- $$\pi(x^2 - 8x + 16)$$

- 32. REASONING** Rewrite $\frac{6x^2 + 3x - 1}{x^2 + 2} = \frac{6x^2 + 12x + 3}{x^2 + 2} - \frac{9x + 1}{x^2 + 2}$ using long division. What does the remainder indicate in this problem?
- $$\frac{6x^2 + 3x - 1}{x^2 + 2} = \frac{6x^2 + 12x + 3}{x^2 + 2} - \frac{9x + 1}{x^2 + 2}$$
- $2x + 4$ is a factor of $6x^2 + 3x - 1$. $2x^2 - 16x + 24 = 24x + 32$.

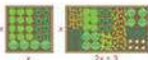
- 33. CONSTRUCT ARGUMENTS** Determine whether you have enough information to fill in the missing pieces of the long division exercise shown. If so, copy and complete the long division. Justify your response. See margin.
- $$\begin{array}{r} 3x - 11 \\ 9x^2 + 10x + 5 \\ \hline 27x^2 + 3x \\ \hline -3x + 5 \end{array}$$

- 34. REGULARITY** Rewrite $\frac{6x^3 - 5x^2 + 2x + 8}{2x + 3}$ as $q(x) + \frac{r(x)}{d(x)}$ using long division.

- a. Identify $q(x)$, $r(x)$, and $d(x)$. $q(x) = x^2 - 5x + x$; $r(x) = 8$; $d(x) = 2x + 3$
 b. How can you check your work using the expressions of $q(x)$, $r(x)$, and $d(x)$?
 Multiply $q(x)$ and $d(x)$ and then add $r(x)$. The result will equal the original dividend.

- 35. STRUCTURE** When a polynomial is divided by $4x - 6$, the quotient is $2x^2 + x + 1$ and the remainder is -4 . What is the dividend, $f(x)$? Explain.
 $8x^3 - 8x^2 - 2x - 10$. I multiplied the divisor and the quotient and added the remainder: $(2x^2 + x + 1)(4x - 6) + (-4) = 8x^3 - 8x^2 - 2x - 10$.

- 36. USE A MODEL** Luciano has a square garden. A new garden will have the same width and a length that is 3 feet more than twice the width of the original garden.



- a. Define a variable. Copy the diagrams. Label each side of the diagrams with an expression for its length.
 Let $x =$ a side of the original garden.
 b. Write a ratio to represent the percent increase in the area of the garden. Use polynomial division to simplify the expression. See margin.
 c. Use your expression from part b to determine the percent increase in area if the original garden was a 12-foot square. Check your answer. See margin.

Lesson 3-4 • Dividing Polynomials 107

- 37. REGULARITY** Marissa makes the following claims about the degrees of the polynomials in $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$. Do you agree with each claim? Justify your answers and provide examples.

- a. The degree of $r(x)$ must be less than the degree of $f(x)$.
 No, the degrees of $f(x)$ and $d(x)$ may be equal. For example, $\frac{x^2 + 8}{x^2 + 3} = 1 + \frac{5}{x^2 + 3}$.
 b. The degree of $q(x)$ must be at least 1 less than the degree of $d(x)$. See margin.
 c. The degree of $q(x)$ must be the degree of $f(x)$ minus the degree of $d(x)$. See margin.

Higher-Order Thinking Skills

- 38. FIND THE ERROR** Tiana and Jamal are dividing $2x^2 - 4x^2 + 3x - 1$ by $x - 3$. Tiana claims that the remainder is -100 . Jamal claims that the remainder is 25. Is either of them correct? Explain your reasoning.
 Sample answer: Jamal. Tiana divided by $x + 3$.

- 39. PERSEVERE** If a polynomial is divided by a binomial and the remainder is 0, what does this tell you about the relationship between the binomial and the polynomial?
 The binomial is a factor of the polynomial.

- 40. ANALYZE** What is the relationship between the degrees of the dividend, the divisor, and the quotient in any polynomial division exercise?
 Sample answer: The degree of the quotient plus the degree of the divisor equals the degree of the dividend.

- 41. CREATE** Write a quotient of two polynomials for which the remainder is 3.
 Sample answer: $\frac{x^2 + 5x + 8}{x + 2}$

- 42. WRITE** Compare and contrast dividing polynomials using long division and using synthetic division. See margin.

- 43. PERSEVERE** Mr. Collins has his class working with bases and polynomials. He wrote on the board that the number 1111 in base B has the value $B^3 + B^2 + B + 1$. The class was then given the following questions to answer.

- a. The number 11 in base B has the value $B + 1$. What is 1111 (in base B) divided by 11 (in base B)? $B^2 + 1$
 b. The number 111 in base B has the value $B^2 + B + 1$. What is 1111 (in base B) divided by 11 (in base B)? $B = \frac{1}{B^2 + B + 1}$

Answers

33. Yes; sample answer: Because $3x$ times the divisor is $9x^2 + 3x$, the divisor must be $3x + 1$. The second and third terms of the dividend must be $0x + 5$ because the first difference is $-3x + 5$.

- 36b. The area of the original garden is x^2 . The area of the new garden is

$2x^2 + 3x$. Percent of increase in the area of the new garden is $\left(\frac{2x^2 + 3x}{x^2}\right) \times 100$. Using long division and simplifying the ratio results in $200 + \frac{300}{x}$.

- 36c. Substitute 12 for x in the expression $200 + \frac{300}{x}$ and evaluate to determine that the percent of increase is $200 + \frac{300}{12} = 225$, or 225%.

Using 12 ft for a side of the original square, the area is 144 sq ft. The area of the new design would be $(2(12) + 3)(12) = 324$ sq ft. The percent of increase is $\frac{324 - 144}{144} = \frac{180}{144} = 1.25$ or 125%.

- 37b. Yes; if the degree of $r(x)$ is greater than or equal to the degree of $d(x)$,

then the expression $\frac{r(x)}{d(x)}$ may be simplified by division. For example,

if $\frac{r(x)}{d(x)} = \frac{8x + 1}{x}$, then $8x + 1$ may be divided by x to get $8 + \frac{1}{x}$.

- 37c. Yes; because $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$, the degree of $\frac{f(x)}{d(x)}$ must equal the degree

of $q(x) + \frac{r(x)}{d(x)}$. The degree of $r(x)$ is less than the degree of $d(x)$, so the degree of $q(x) + \frac{r(x)}{d(x)}$ equals the degree of $q(x)$. This means the degree of $\frac{f(x)}{d(x)}$ equals the degree of $q(x)$, and the degree of $\frac{f(x)}{d(x)}$ is the degree of $f(x)$ minus the degree of $d(x)$. For example, in $\frac{2x^2 - 1}{x^2 + 3} = 2x - \frac{6x - 1}{x^2 + 3}$, the degree of $q(x)$ is the degree of $f(x)$ minus the degree of $d(x)$.

42. Sample answer: A polynomial can be divided by a polynomial with more than one term using long division, which is a similar process to dividing real numbers. Synthetic division is an alternate method used to divide a polynomial by a binomial of degree 1. Synthetic division is typically a quicker, simpler method than long division.

Powers of Binomials

LESSON GOAL

Students expand powers of binomials.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Explore:** Expanding Binomials

 **Develop:**

Powers of Binomials

- Use Pascal's Triangle
- Use the Binomial Theorem
- Coefficients Other Than 1

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources

Remediation: Special Products

Extension: Patterns in Pascal's Triangle

AL	LE	ELL
●	●	●
●	●	●

Language Development Handbook

Assign page 12 of the *Language Development Handbook* to help your students build mathematical language related to powers of binomials.

 You can use the tips and suggestions on page T12 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra

Standards for Mathematical Content:

A.APR.5 Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n , where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.

Standards for Mathematical Practice:

4 Model with mathematics.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous

Students added, subtracted, and multiplied polynomials.

A.APR.1

Now

Students expand powers of binomials.

A.APR.5

Next

Students will solve polynomial equations by factoring and by writing them in quadratic form. **A.CED.1**

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

 **Conceptual Bridge** In this lesson, students develop an understanding of the Binomial Theorem, and they build fluency by using the Binomial Theorem to expand binomials. They apply their understanding by solving real-world problems.

Mathematical Background

For the expansion of $(a + b)^n$, the signs of the terms depend on the signs of a and b . The sign of each term is $+$ if both a and b are positive; the sign of each even-numbered term is $-$ if b alone is negative.



Interactive Presentation

Warm Up

Expand.

- $(x - 1)^2$
- $(x - 3y)^2$
- $(2x + 5y)^2$

Show Answers

Warm Up

Launch the Lesson

Watch this video to learn about Blaise Pascal.

Launch the Lesson

Vocabulary

Expand All Collapse All

Pascal's Triangle

A triangle of numbers in which a row represents the coefficients of an expanded binomial $(x + y)^n$.

What is the process to expand a binomial higher than degree four?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- expanding perfect squares

Answers:

1. $x^2 - 2x + 1$

2. $x^2 - 6xy + 9y^2$

3. $4x^2 + 20xy + 25y^2$

Launch the Lesson



Teaching the Mathematical Practices

7 Look for a Pattern Help students to see and use the patterns in Pascal's Triangle.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then, discuss the question below with the class.



Explore Expanding Binomials

Objective

Students use a sketch to explore using Pascal's Triangle to expand binomials.



Teaching the Mathematical Practices

3 Construct Arguments Students will use assumptions and previously established results to make a conjecture about how to use Pascal's Triangle to write expansions of binomials.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students use a sketch to construct Pascal's Triangle and study its relationship with the expansion of binomial $(a + b)^n$. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Expanding Binomials

INQUIRY How can you use Pascal's Triangle to write expansions of binomials?

You can use a triangular array of numbers known as Pascal's Triangle to expand binomials of the form $(a + b)^n$. Use the sketch to explore expanding binomials by using Pascal's Triangle. Then complete the exercises.

Explore

Expanding Binomials

$(a + b)^n = a^n + \dots + b^n$

row 0: 1

row 1: 1 1

Explore

WEB SKETCHPAD



Students use a sketch to explore expanding binomials.



Interactive Presentation

INQUIRY: How can you use Pascal's Triangle to write expansions of binomials?

Submit

Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Expanding Binomials (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- Does it make sense to use Pascal's Triangle for rows 0 and 1?
Sample answer: No, those are simple enough to find without using Pascal's Triangle.
- Which row of Pascal's Triangle do you need to use to expand $(a + b)^8$?
Sample answer: The degree is 8, so you need to find and use the 8th row.

Inquiry

How can you use Pascal's Triangle to write expansions of binomials? Sample answer: Find the n th row of Pascal's triangle by following the pattern of adding the numbers in the row above. Then, use the numbers in the n th row as the coefficients of the expanded binomial. Use descending powers of a and ascending powers of b in the terms.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Powers of Binomials

Objective

Students expand powers of binomials by using Pascal's triangle and the Binomial Theorem.

MP Teaching the Mathematical Practices

- 1 Explain Correspondences** Use the Think About It! feature to encourage students to explain how using Pascal's Triangle and the Binomial Theorem are related.

Essential Question Follow-Up

Students have begun to expand binomials using Pascal's triangle.

Ask:

How can you use Pascal's triangle to show that $C_{n,k} = C_{n,n-k}$? **Sample answer:** When Pascal's triangle is folded across a vertical center line, the numbers match. The corresponding numbers match k with $n - k$.

Example 1 Use Pascal's Triangle

MP Teaching the Mathematical Practices

- 7 Look for a Pattern** Help students to extend the pattern in Pascal's Triangle and use it to expand the binomial.

Questions for Mathematical Discourse

- A1** How can you describe the pattern of Pascal's triangle in words?
Sample answer: You add two terms to find the term below it in the triangle.
- OL** How do you determine the powers of the variables in the expansion? **Sample answer:** The first term has $a^n b^0$, and for subsequent terms the power of a goes in descending order while the power of b goes in ascending order.
- BL** Does the order of a and b matter? Explain. **No**, the expansion is symmetric.

Go Online

- F ind additional teaching notes.
- V iew performance reports of the Checks.
- A ssign or present an Extra Example.

Lesson 2-5

Powers of Binomials

Explore Expanding Binomials

- Online Activity** Use interactive tool to complete the Explore.

INQUIRY How can you use Pascal's triangle to write expansions of binomials?

Learn Powers of Binomials

You can expand binomials by following a set of rules and using patterns.

Key Concept - Binomial Expansion

In the binomial expansion of $(a + b)^n$,

- there are $n + 1$ terms,
- n is the exponent of a in the first term and b in the last term,
- in successive terms, the exponent of a decreases by 1, and the exponent of b increases by 1,
- the sum of the exponents in each term is n ,
- the coefficients are symmetric.

Pascal's triangle is a triangle of numbers in which a row represents the coefficients of an expanded binomial $(a + b)^n$. Each row begins and ends with 1. Each coefficient can be found by adding the two coefficients above it in the previous row.

Instead of writing out the rows of Pascal's triangle, you can use the Binomial Theorem to expand a binomial. The Binomial Theorem uses combinations to calculate the coefficients of the binomial expansion.

Key Concept - Binomial Theorem

If n is a natural number, then $(a + b)^n =$

$$C_n^0 a^n b^0 + C_n^1 a^{n-1} b^1 + C_n^2 a^{n-2} b^2 + C_n^3 a^{n-3} b^3 + \dots + C_n^{n-1} a^1 b^{n-1} + C_n^n a^0 b^n$$

$$1a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \dots + \frac{n!}{n!}a^0b^n$$

Example 1 Use Pascal's Triangle

Use Pascal's triangle to expand $(x + y)^5$.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 2 & & 1 & \\ & & 1 & & 3 & & 3 & & 1 \\ & 1 & & 4 & & 6 & & 4 & & 1 \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\ 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \end{array}$$

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

Lesson 2-5 • Powers of Binomials 109

Interactive Presentation



Learn

TYPE



Students explain how to write rows of Pascal's triangle.

**Talk About It!**

Describe a shortcut you could use to write out rows of Pascal's triangle instead of adding to find every number in a row. Explain your reasoning.

Sample answer: Since the coefficients in an expanded binomial are symmetric, you can just find the first or last half of the numbers in a row. Then, complete the row by using the same numbers.

Study Tip:

Assumptions: To use the Binomial Theorem, we assumed that the teams had an equal chance of winning and losing. Although teams are not always evenly matched and may benefit from a home field advantage, assuming there is an equal chance of either event occurring allows us to reasonably estimate the probability of an outcome.

Study Tip:

Coefficients: When the binomial is to be expanded, the coefficients other than 1, the coefficients will no longer be symmetric. In these cases, it may be easier to use the Binomial Theorem.

Check

Write the expansion of $(c + d)^5$. $c^5 + 5c^4d + 10c^3d^2 + 10c^2d^3 + 5cd^4 + d^5$

Example 2 Use the Binomial Theorem

BASEBALL In 2016, the Chicago Cubs won the World Series for the first time in 108 years. During the regular season, the Cubs played the Atlanta Braves 6 times, winning 3 games and losing 3 games. If the Cubs were as likely to win as to lose, find the probability of this outcome by expanding $(w + l)^6$.

$(w + l)^6$

$$\begin{aligned} &= {}_6C_0w^6 + {}_6C_1w^5l + {}_6C_2w^4l^2 + {}_6C_3w^3l^3 + {}_6C_4w^2l^4 + {}_6C_5wl^5 + {}_6C_6l^6 \\ &= w^6 + \frac{6}{1}w^5l + \frac{6 \cdot 5}{2}w^4l^2 + \frac{6 \cdot 5 \cdot 4}{3 \cdot 2}w^3l^3 + \frac{6 \cdot 5 \cdot 4 \cdot 3}{4 \cdot 3 \cdot 2}w^2l^4 + \frac{6 \cdot 5}{2}wl^5 + l^6 \\ &= w^6 + 6w^5l + 15w^4l^2 + 20w^3l^3 + 15w^2l^4 + 6wl^5 + l^6 \end{aligned}$$

By adding the coefficients, you can determine that there were 64 combinations of wins and losses that could have occurred.

$20w^3l^3$ represents the number of combinations of 3 wins and 3 losses. Therefore, there was a $\frac{20}{64}$ or about a 31% chance of the Cubs winning 3 games and losing 3 games against the Braves.

Check

GAME SHOW A group of 8 contestants are selected from the audience of a television game show. If there are an equal number of men and women in the audience, find the probability of the contestants being 5 women and 3 men by expanding $(w + w)^8$. Round to the nearest percent if necessary. $\frac{7}{22}$

Example 3 Coefficients Other Than 1

Expand $(2c - 6d)^4$.

$(2c - 6d)^4$

$$= {}_4C_0(2c)^4 + {}_4C_1(2c)^3(-6d) + {}_4C_2(2c)^2(-6d)^2 + {}_4C_3(2c)(-6d)^3 + {}_4C_4(-6d)^4$$

$$= 16c^4 + \frac{4}{1}(8c^3)(-6d) + \frac{6}{2}(4c^2)(36d^2) + \frac{4}{3}(2c)(-216d^3) + 1296d^4$$

$$= 16c^4 - 192c^3d + 864c^2d^2 - 1728cd^3 + 1296d^4$$

Go Online You can complete an Extra Example online.

Interactive Presentation

Interactive Presentation

You can use the Exit Ticket to find the probability of the Cubs winning 3 games and losing 3 games.

BASEBALL In 2016, the Chicago Cubs won the World Series for the first time in 108 years. During the regular season, they played the Atlanta Braves 6 times, winning 3 games and losing 3 games. If the Cubs were as likely to win as to lose, find the probability of this outcome by expanding $(w + l)^6$.



Example 2

TYPE

a

CHECK

Students learn to use the Binomial Theorem to find the probability of a combination of outcomes occurring.

Students complete the Check online to determine whether they are ready to move on.

Example 2 Use the Binomial Theorem**Teaching the Mathematical Practices**

4 Interpret Mathematical Results In Example 2, point out that to solve the problem, students should interpret their mathematical results in the context of the problem to determine the probability of the given outcome.

Questions for Mathematical Discourse

- AL** What do the coefficients of the expansion mean in the context of the example? **Each coefficient represents the number of combinations resulting in k number of losses.**
- OL** Why is an outcome of 3 wins and 3 losses more probable than an outcome of all wins? **There are more combinations of wins and losses that have the same result of 3 wins and 3 losses, whereas all wins can only occur as one combination.**
- BL** Was it necessary to use the binomial expansion rather than just solving the problem using combinations? **Explain. No; sample answer: The solution comes only from the coefficients, but using the binomial makes it easier to keep track of which values of ${}_nC_k$ go with which type of outcome.**

Example 3 Coefficients Other Than 1**Teaching the Mathematical Practices**

7 Use Structure Help students to use the structure of the Binomial Theorem to expand the binomial in Example 3.

Questions for Mathematical Discourse

- AL** Why is the first coefficient 16? **because ${}_4C_0 \times 2^4 = 16$**
- OL** How could you still use Pascal's triangle to solve this problem? **You can use Pascal's triangle to fill in the values for the combinations.**
- BL** How could you determine a formula to find the coefficients of $(Ca + Db)^n$, where C and D are constants? **Sample answer: We know from the Binomial Theorem that the coefficients for $(a + b)^n$ are $\frac{n!}{k!(n-k)!}$. We only need to determine the pattern for the coefficients C and D . The variable terms expand as $a^k b^{n-k}$, so the coefficients would also be to the same power. Thus $\frac{n!}{k!(n-k)!} C^k D^{n-k}$**

Exit Ticket**Recommended Use**

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–12
2	exercises that use a variety of skills from this lesson	13–25
3	exercises that emphasize higher-order and critical-thinking skills	26–30

ASSESS AND DIFFERENTIATE

 Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, THEN assign:

- Practice Exercises 1–25 odd, 26–30
- Extension: Patterns in Pascal's Triangle
- 

OL BL

IF students score 66%–89% on the Checks, THEN assign:

- Practice Exercises 1–29 odd
- Remediation, Review Resources: Special Products
- Personal Tutors
- Extra Examples 1–3
-  Expanding Perfect Squares

AL OL

IF students score 65% or less on the Checks, THEN assign:

- Practice Exercises 1–11 odd
- Remediation, Review Resources: Special Products
-  Expanding Perfect Squares

AL

Practice

 Go Online You can complete your homework online.

Example 1

Use Pascal's triangle to expand each binomial.

- $(x - y)^3$
 $x^3 - 3x^2y + 3xy^2 - y^3$
- $(a + b)^4$
 $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
- $(2 - 3y)^4$
 $16 - 48y + 60y^2 - 40y^3 + 8y^4$
- $(m + n)^4$
 $m^4 + 4m^3n + 6m^2n^2 + 4mn^3 + n^4$
- $(x - y)^5$
 $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$
- $(2 + 3t)^5$
 $32 + 96t + 720t^2 + 480t^3 + 1920t^4 + 243t^5$

Example 2

7. **BAND** A school band went to 4 competitions during the year and received a superior rating 2 times. If the band is as likely to receive a superior rating as to not receive a superior rating, find the probability of this outcome by expanding $(x + y)^4$. Round to the nearest percent if necessary. **38%**

8. **BASKETBALL** Oliver shot 8 free throws at practice, making 5 free throws and missing 2 free throws. If Oliver is equally likely to make a free throw as he is to miss a free throw, find the probability of this outcome by expanding $(x + y)^8$. Round to the nearest percent if necessary. **11%**

Example 3

Expand each binomial.

- $(3x + 4y)^3$
- $(2x - 2y)^3$
- $243x^3 + 960x^2y + 4320xy^2 + 5760y^3 + 3840x^3 + 9216y^3$
- $128x^4 - 996x^3y + 2688x^2y^2 - 4480xy^3 + 4480y^4 - 2688x^2y^2 + 896xy^3 - 128y^4$
- $(3b - 3c)^3$
- $(4x + 3y)^3$
- $4096x^4 - 6144x^3y + 3456x^2y^2 - 864xy^3 + 81y^4$
- $5096x^4 + 18,432x^3y + 34,560x^2y^2 + 34,560xy^3 + 18,440y^4 + 5832x^3y + 729y^4$

Mixed Exercises

Expand each binomial.

- $(x + \frac{1}{2})^3$
- $(x - \frac{1}{2})^4$
- $x^4 + \frac{5}{2}x^3 + \frac{5}{2}x^2 + \frac{5}{2}x + \frac{1}{2}$
- $x^4 - \frac{5}{2}x^3 + \frac{5}{2}x^2 - \frac{5}{2}x + \frac{1}{2}$
- $(2x + \frac{1}{2})^3$
- $(3x + \frac{1}{2})^3$
- $32x^3 + 288x^2 + 50x^2 + \frac{5}{2}x^2 + \frac{5}{2}x^2 + \frac{5}{2}x^2$
- $27x^3 + 9x^2d + d^3 + \frac{27}{2}$
- STRUCTURE** Out of 12 hanes, Vince tossed 5 strikes. If Vince is as likely bowl a strike as to not bowl a strike in one hane, find the probability of this outcome. Round to the nearest percent if necessary. **23%**
- REGULARITY** A group of 10 choir members are selected at random to perform solos. If there are an equal number of boys and girls in the choir, find the probability of the choir members selected being 7 boys and 3 girls. Round to the nearest percent if necessary. **12%**

Lesson 2-6 • Powers of Binomials 111



Answers

19. **USE A MODEL:** A company is developing a robotic welder that produces circuit boards. At this stage in its development, the robotic welder only produces 50% of the circuit boards correctly. Use the Binomial Theorem to find the probability that 5 of 7 circuit boards chosen at random are correct. **See margin.**
20. **USE A MODEL:** Diego flips a fair coin 12 times. What is the probability that the coin lands on tails 3 times? 5 times? 9 times? 5.4%, 10.3%, 5.4%.
21. **REASONING:** A test consists of 10 true-false questions. Matthew forgets to study and must guess on every question. What is the probability that he gets 8 or more correct answers on the test? Show your work using Pascal's Triangle. **See margin.**
22. **REGULARITY:** Use Pascal's Triangle to find the fourth term in the expansion of $(2x + 3)^5$. Why is 4 the same as the fourth term in the expansion of $(2 + 3x)^5$? **See margin.**
23. **USE A SOURCE:** Research the number of judges on the Supreme Court. For most rulings, a majority is needed. How many combinations of votes are possible for a majority to be reached? **See margin.**
24. **STRUCTURE:** Find the term in $(x + a)^{12}$ where the exponent of a is 5. $792a^5x^8$
25. **PRECISION:** Use the first four terms of the binomial expansion of $(1 + 0.02)^{10}$ to approximate $(1.02)^{10}$. Evaluate $(1.02)^{10}$ using a calculator and compare the value to your approximation. 1.21896 , $(1.02)^{10} = 1.21899442$; the approximation differs from the value given by a calculator by 0.00000442 .
- Higher-Order Thinking Skill**
26. **PERSEVERE:** Find the sixth term of the expansion of $(x + y)^{10}$. $792x^4y^6$
27. **ANALYZE:** Explain how the terms of $(x + y)^n$ and $(x - y)^n$ are the same and how they are different. **See margin.**
28. **REGULARITY:** Each row of Pascal's triangle is like a palindrome. That is, the numbers read the same left to right as they do right to left. Explain why this is the case. **See margin.**
29. **CREATE:** Write a power of a binomial for which the second term of the expansion is $6x^3y$. **See margin.**
30. **WRITE:** Explain how to write out the terms of Pascal's triangle. **See margin.**

112 Module 2 • Polynomials and Polynomial Functions

19. If C represents a correct circuit board and N represents an incorrect circuit board, then the number of ways for the robot to produce 5 of 7 circuit boards accurately is given by the coefficient of C^5N in the expansion of $(C + N)^7$. Using the Binomial Theorem, there are 21 ways for the robot to produce a correct circuit board out of 128 possibilities. So, the probability that 5 of 7 are correct is $\frac{21}{128}$ or about 16%.
21. If c represents a correct answer and w represents a wrong answer, then the coefficients expansion of $(c + w)^8$ can be used to represent situation. Using Pascal's triangle, there are 45 ways to get 8 questions correct, 10 ways to get 9 questions correct, and 1 way to get all correct. So there are 56 ways to get 8 or more correct. By adding all of the values in this row of Pascal's triangle, I found that there are 1024 different ways he could answer the questions in the quiz. Matthew has a $\frac{56}{1024}$ or about a 5.5% chance of getting 8 or more correct.
22. Sample answer: Using Pascal's triangle, the fourth term of the expansion $(a + b)^5$ is 20. If $a = 2x$ and $b = 7$, then the fourth term is $20a^3b$, or $280x^3$. If $a = 7$ and $b = 2x$, then the fourth term is also $20a^3b$, or $280x^3$.
23. There are 9 judges on the Supreme Court. The majority could be 5, 6, 7, 8, or 9 votes. So, there are $\frac{C}{95} + \frac{C}{95} + \frac{C}{95} + \frac{C}{95} + \frac{C}{95} = 256$
27. Sample answer: While they have the same terms, the signs for $(x + y)^n$ will all be positive, while the signs for $(x - y)^n$ will alternate.
28. Sample answer: If you switch x and y in the Binomial Theorem, the coefficients reverse their order, yet $(x + y) = (y + x)$ because $x + y = y + x$. Therefore, the coefficients must read the same forward and back.
29. Sample answer: $(x + \frac{6}{5}y)^5$
30. Sample answer: The first row is a 1. The second row is two 1s. Each new row begins and ends with 1. Each coefficient is the sum of the two coefficients above it in the previous row.

Module 2 • Polynomials and Polynomial Functions

Review

Rate Yourself!

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have them respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- How does a power function's graph change as the degree of the function increases?
- Are polynomials closed under addition, subtraction, and multiplication?
- How do you know you have your remainder and that you are done dividing?

Then have them write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

 A completed Foldable for this module should include the key concepts related to polynomials and polynomial functions.

 **LearnSmart** Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Polynomial, Rational, and Radical Relationships** and **Modeling with Functions**.

- Polynomial Expressions
- Graphing Rational, Radical, and Polynomial Functions
- Creating Function Models
- Interpreting Function Models

Module 2 • Polynomials and Polynomial Functions

Review

**Essential Question**

How does an understanding of polynomials and polynomial functions help us understand and interpret real-world events?

Polynomials and polynomial functions can be used to model situations where quantities increase or decrease in a nonlinear pattern.

Module Summary

Lessons 2-1 and 2-2

Polynomial Functions and Graphs

- A power function is any function of the form $f(x) = a x^n$, where a and n are nonzero real numbers. The leading coefficient is a and the degree is n .
- Odd-degree functions will always have at least one real zero.
- Even-degree functions may have any number of real zeros or no real zeros at all.
- A polynomial function is a continuous function that can be described by a polynomial equation in one variable.
- The degree of a polynomial function tells the maximum number of times that the graph of a polynomial function intersects the x -axis.
- If the value of $f(x)$ changes signs from one value of x to the next, then there is a zero between those two x -values.
- Extrema occur at relative maxima or minima of the function.

Lessons 2-3 and 2-4

Operations with Polynomials

- Polynomials can be added or subtracted by performing the operations indicated and combining like terms.
- To subtract a polynomial, add its additive inverse.
- Polynomials can be multiplied by using the Distributive Property to multiply each term in one polynomial by each term in the other.

- The set of polynomials is closed under the operations of addition, subtraction, and multiplication.
- To multiply two binomials, you can use a shortcut called the FOIL method.
- You can divide a polynomial by a polynomial with more than one term by using a process similar to long division of real numbers.
- Synthetic division is an alternate method used to divide a polynomial by a binomial of degree 1.

Lesson 2-5

Powers of Binomials

- Pascal's triangle is a triangle of numbers in which a row represents the coefficients of an expanded binomial $(a + b)^n$. Each row begins and ends with 1. Each coefficient can be found by adding the two coefficients above it in the previous row.
- You can also use the Binomial Theorem to expand a binomial. If n is a natural number, then $(a + b)^n = C_n^0 a^n b^0 + C_n^1 a^{n-1} b^1 + C_n^2 a^{n-2} b^2 + \dots + C_n^n a^0 b^n$.

Study Organizer

 **Foldables**

Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.



Module 2 Review • Polynomials and Polynomial Functions 113

Test Practice

1. MULTIPLE CHOICE The weight of an ideal cut round diamond can be modeled by $f(d) = 0.0075d^3 - 0.090d^2 + 0.48d$, where d is the diameter of the diamond. Find the domain of the function in the context of the situation. (Lesson 2-3)

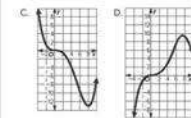
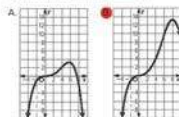
- A. The domain is all real numbers.
☒ B. The domain is $\{d | d \geq 0\}$.
 C. The domain is $\{d | d \leq 0\}$.
 D. The domain is $\{d | d \geq 0.48\}$.

2. OPEN RESPONSE Use the function $f(x) = 13 - 2x^2 + 6x - 9x^3$ to answer the following questions. (Lesson 2-3)



- a) What is the degree?
 b) What is the leading coefficient?
 c) Describe the end behavior using the leading coefficient and degree.
 d) $\lim_{x \rightarrow -\infty} f(x) = -\infty$
 e) $\lim_{x \rightarrow \infty} f(x) = \infty$
 f) Because the leading coefficient is negative and the degree is odd, the end behavior is that as $x \rightarrow -\infty$, $f(x) \rightarrow \infty$ and as $x \rightarrow \infty$, $f(x) \rightarrow -\infty$.

3. MULTIPLE CHOICE The revenue of a certain business can be modeled using $f(x) = -0.04x^2 - 11x^3 + 4x^2 - 5x + 7$, where x is the number of years since the business was started and $f(x)$ is the revenue in hundred thousands of dollars. Which graph represents the function? (Lesson 2-3)



- 4. MULTI-SELECT** Select all intervals in which a real zero is located for the function $f(x) = x^3 - 2x^2 + 3x - 5$. (Lesson 2-3)
- A. $x = -2$ and $x = -1$
☒ B. $x = -1$ and $x = 0$
 C. $x = 0$ and $x = 1$
☒ D. $x = 1$ and $x = 2$
 E. $x = 2$ and $x = 3$
 F. $x = 3$ and $x = 4$

5. OPEN RESPONSE Describe the end behavior for $g(x) = -2x^4 - 6x^3 + 11x - 10$ as $x \rightarrow \infty$. (Lesson 2-3)

$$g(x) \rightarrow -\infty$$

Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Put It All Together: Lessons 2-1 through 2-5

Vocabulary Activity

Module Review

Assessment Resources

Vocabulary Test

AL Module Test Form B

OL Module Test Form A

BL Module Test Form C

Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

Test Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1-16 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Multiple Choice	Students select one correct answer.	1, 3, 7, 10, 11, 12, 14, 15
Multi-Select	Multiple answers may be correct. Students must select all correct answers.	4, 16
Open Response	Students construct their own response.	2, 5, 6, 8, 9, 13

To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
A.CED.2	2-2	7
A.APR.1	2-3, 2-4	8-12
A.APR.6	2-4, 2-5	13-16
F.IF.4	2-1, 2-2	2, 4-6
F.IF.5	2-1	1
F.IF.7c	2-1	3

6. **OPEN RESPONSE** Marshall claims that there is only one real zero in the function $f(x) = 4x^2 + 7x - 3$. Based on the table provided, determine whether you agree with Marshall. Then name the interval(s) in which the zero(s) is/are located. (Lesson 2-3)

x	f(x)
-3	-27
-2	9
-1	11
0	3
1	9
2	53

Sample answer: I agree with Marshall. There is one zero located between $x = -3$ and $x = -2$. I know this because the y-values change from a negative to a positive.

7. **MULTIPLE CHOICE** Helen started a business several years ago. The table shows her profits, in millions of dollars, for the first 7 years. Select the polynomial function of best fit that could be used to model Helen's profits. (Lesson 2-2)

x	f(x)
1	1.425
2	1.46
3	1.5
4	1.53
5	1.56
6	1.58
7	1.58

- A. $f(x) = 0.000(-3.27x^4 + 53.53x + 137)$
 B. $f(x) = 0.000(-6.944x^4 + 50.6x^3 + 250.4x + 13.957)$
 C. $f(x) = 0.0000(9.47x^4 + 82.07x^3 - 312.5x^2 + 420.3x + 138.500)$
 D. $f(x) = 0.000(-x^4 + 12x^3 - 77x^2 + 600x + 13.650)$

8. **OPEN RESPONSE** What is the difference? $(3x^2 - 5x^2 + 5x^2 + 8x - 17) - (3x^2 - 6x^2 - 4x^2 + 12x + 4)$ (Lesson 2-3)

$$6x^2 + 5x^2 + 9x^2 - 4x - 15$$

9. **OPEN RESPONSE** What is the sum? $(3x^2 - 5x^2 + 8x^2 - 4x) + (-2x^2 + 7x^2 - x^2 + 6x)$ (Lesson 2-3)

$$x^2 + 7x^2 - 6x^2 + 8x^2 + 2x$$

10. **MULTIPLE CHOICE** Enrique is designing a flag for a new school club. A smaller sloped square is placed as part of the design and the rest of the flag will have chevrons. (Lesson 2-3)



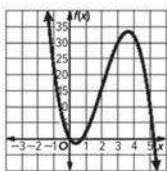
Which expression can be used to represent the area of the flag that is not sloped?

- A. $16x^2 + 57x + 46$
 B. $8x^2 + 35x + 16$
 C. $8x^2 + 17x + 4$
 D. $8x^2 + 17x - 4$

Lesson 2-2

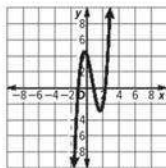
5. relative minimum between $x = 0$ and $x = 1$; relative maximum near $x = 4$

x	$f(x)$
-1	22
0	0
2	16
3	30
4	32
5	10
6	-48
7	-154



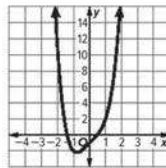
6. relative minimum near $x = 2$; relative maximum near $x = 0$;

x	$f(x)$
-4	-176
-3	-77
-2	-22
-1	1
0	4
1	-1
2	-2
3	13
4	56



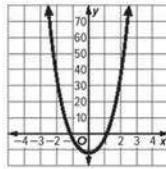
7. relative minimum near $x = -1$; no relative maximum

x	$f(x)$
-4	247
-3	74
-2	11
-1	-2
0	-1
1	2
2	19
3	86
4	263



8. relative minimum near $x = 0$; no relative maximum

x	$f(x)$
-4	372
-3	141
-2	36
-1	-3
0	-12
1	-3
2	36
3	141
4	372



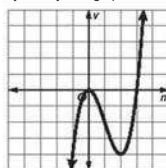
9. Domain and Range: The domain and range of the function is all real numbers. Because the function models years, the relevant domain and range are $\{n | n \geq 0\}$ and $\{y | y \geq 0\}$.

Extrema: There is a relative minimum at $n = 2$ in the relevant domain.

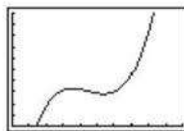
End Behavior: As $x \rightarrow \infty$, $f(x) \rightarrow \infty$.

Intercepts: In the relevant domain, the y -intercept is at $(0, 0)$. There is a n -intercept, or zero, is at about $(3, 0)$.

Symmetry: The graph of the function has symmetry at $x = 2$.



- 10.



$[0, 10]$ scl: 1 $[0, 100]$ scl: 10

Domain and Range: The domain and range of the function is all real numbers. Because the function models weeks of growth, the relevant domain and range are $\{x | x \geq 0\}$ and $\{y | y \geq 0\}$.

Extrema: There is a relative maximum between $x = 3$ and $x = 4$ and a relative minimum between $x = 5$ and $x = 6$ in the relevant domain.

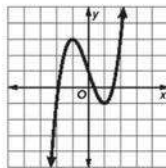
End Behavior: As $x \rightarrow \infty$, $f(x) \rightarrow \infty$.

Intercepts: In the relevant domain, there is no y -intercept. There is an x -intercept, or zero, at about $(1.4, 0)$.

Symmetry: The graph does not have symmetry in the relevant domain.

- 16.

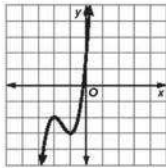
x	$f(x)$
-3	-17
-2	1
-1	3
0	1
1	-1
2	3
3	19



zeros between $x = -2$ and $x = -1$, between $x = 0$ and $x = 1$, and between $x = 1$ and $x = 2$; rel. max. at $x = -1$, rel. min. at $x = 1$;
D = all real numbers; R = all real numbers

- 17.

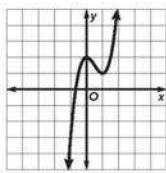
x	$f(x)$
-3	-7
-2	-2
-1	-3
0	2
1	25



zero between $x = -1$ and $x = 0$; rel. max. at $x = -2$, rel. min. at $x = -1$;
D = all real numbers; R = all real numbers

18.

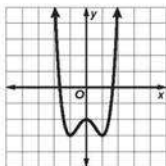
x	$f(x)$
-1	-3
0	2
1	1
2	6
3	29



zero between $x = -1$ and $x = 0$; rel. max. at $x = 0$, rel. min. at $x = 1$;
 $D = \text{all real numbers}$; $R = \text{all real numbers}$

19.

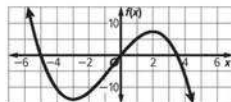
x	$f(x)$
-3	61
-2	6
-1	-3
0	-2
1	-3
2	6
3	61



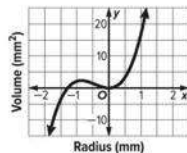
zeros between $x = -2$ and $x = -1$, and between $x = 1$ and $x = 2$;
 rel. max. at $x = 0$, min. at $x = -1$, and $x = 1$; $D = \text{all real numbers}$;
 $R = \{f(x) \mid f(x) \geq -3\}$

20. Domain and Range: $D = \text{all real numbers}$; $R = \{y \mid y > 0\}$ Extrema: rel. min at $(-4, 16)$; rel. min near $x = 0$ End Behavior: as $x \rightarrow \infty$, $y \rightarrow \infty$ and as $x \rightarrow -\infty$, $y \rightarrow \infty$ Intercepts: no x -intercept; y -intercept: 5

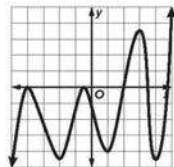
Symmetry: none

25. Sample answer: $f(x) \rightarrow -\infty$ as $x \rightarrow \infty$ and $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$; the leading coefficient is negative.

26.



31. Sample answer:



11. **MULTIPLE CHOICE** Determine the quotient.
 $(5x^2 + 12x^2 - 64x^2 - 95x + 132) \div (x - 3)$
Lesson 2-6

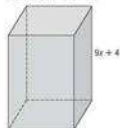
A. $3x^2 + 25x^2 + x - 31$
 B. $3x^2 - 25x^2 - x + 31$
☒ C. $5x^2 + 27x^2 + 17x - 44$
 D. $5x^2 - 27x^2 - 17x + 44$

12. **MULTIPLE CHOICE** Determine the quotient.
Lesson 2-6

$$\frac{6x^2 - 7x^2 + 199x + 190}{3x + 2}$$

A. $2x^2 - 25x + 63 + \frac{8}{3x+2}$
 B. $2x^2 - 25x + 63 + \frac{3}{3x+2}$
 C. $2x^2 - 25x + 63 + \frac{5}{3x+2}$
☒ D. $2x^2 - 25x + 63 + \frac{4}{3x+2}$

13. **OPEN RESPONSE** The volume of the rectangular prism shown is $45x^3 + 83x^2 + x - 12$.



What is the area of the base? Lesson 2-6

☒ $5x^2 + 7x - 3$

14. **MULTIPLE CHOICE** Which of the following is the expansion of $(2h + 7f)^2$? Lesson 2-5

A. $2h^2 + 4hf + 6h^2f + 49f^2 + f^2$
 B. $16h^2 + 32hf + 24hf^2 + 32hf^2 + 16f^2$
 C. $16h^2 + 32hf + 32hf^2 + 8hf^2 + f^2$
☒ D. $16h^2 + 32hf + 24hf^2 + 8hf^2 + f^2$

15. **MULTIPLE CHOICE** The first shelf on Hannah's bookshelf holds an equal number of fiction and nonfiction books. If Hannah selects 5 books randomly, what is the probability that 4 of the books will be fiction and 1 will be nonfiction?

Round your answer to the nearest tenth of a percent. Lesson 2-5

A. 31.3%
☒ B. 15.6%
 C. 12.5%
 D. 3.8%

16. **MULTI-SELECT** Select all of the following that would be a coefficient of a term in the binomial expansion of $(x + y)^7$. Lesson 2-5

☒ A. 1
☐ B. 3
☒ C. 7
☐ D. 14
☒ E. 21
☐ F. 28
☐ G. 30
☒ H. 35

Polynomial Equations

Module Goals

- Students solve polynomial equations by graphing, factoring, and the Fundamental Theorem of Algebra.
- Students prove polynomial identities.
- Students evaluate and factor equations using the Factor and Remainder Theorem.

Focus

Domain: Algebra

Standards for Mathematical Content:

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

A.APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Also addresses *N.CN.9*, *A.APR.2*, *A.APR.4*, *A.REI.11*, and *F.IF.7c*.

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this module.

Coherence

Vertical Alignment

Previous

Students graphed polynomial functions and performed operations with polynomial expressions. **A.APR.1, F.IF.7c**

Now

Students solve polynomial equations, factor polynomial expressions, and identify the zeros of polynomial functions.

A.CED.1, A.APR.3, F.IF.7c

Next

Students will solve rational equations and identify key features of rational functions.

F.IF.4, A.REI.2

Rigor

The Three Pillars of Rigor

To help students meet standards, they need to illustrate their ability to use the three pillars of rigor. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. Once they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Independent Practice.

1 CONCEPTUAL UNDERSTANDING 2 FLUENCY 3 APPLICATION

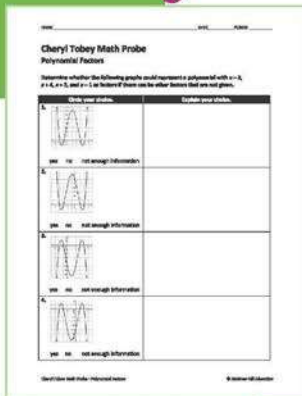
EXPLORE

LEARN

EXAMPLE & PRACTICE

Suggested Pacing

Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
3-1 Solving Polynomial Equations by Graphing	A.CED.1, A.REI.11	1	0.5
3-2 Solving Polynomial Equations Algebraically	A.CED.1	2	1
3-3 Proving Polynomial Identities	A.APR.4	1	0.5
3-4 The Remainder and Factor Theorems	A.APR.2	1	0.5
3-5 Roots and Zeros	N.CN.9, A.APR.3, F.IF.7c	2	1
Module Review		1	0.5
Module Assessment		1	0.5
Total Days		10	5



Cheryl Tobey Math Probe
Polynomial Factors

Determine whether the following graphs could represent a polynomial with $x = 2$, $x = 4$, $x = 3$, and $x = 1$ as factors. If there are no other factors that are not given.

Check your graphs	Explain your checks
1. 	
2. 	
3. 	
4. 	

Cheryl Tobey Math Probe: Polynomial Factors © 2008 Cheryl Tobey

**Correct Answers: 1. no; 2. yes;
3. no; 4. yes**

Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will connect polynomial factors with graphs of polynomial functions.

Targeted Concepts Understand how different representations of polynomial functions can give you information about the solutions of their related equations.

Targeted Misconceptions

- Students may not set factors equal to zero to find the roots of polynomial equations.
- Students may not consider that various polynomial equations can have the same roots.

Use the Probe after Lesson 3-1.

Collect and Assess Student Answers

If the student selects these responses...

- yes
- no
- yes
- no

Then the student likely...

does not take the additive inverse of the factors to find the roots.

Example: The zero for the factor $x - 2$ is found by setting $x - 2 = 0$ and then solving to get $x = 2$.

does not consider the polynomial with the additive inverse of each term.

Example: The polynomial $x^4 + 4x^3 - 7x^2 - 22x + 24$ has the same roots as $-(x^4 + 4x^3 - 7x^2 - 22x + 24)$.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

-  **ALEKS** Real Zeros of Polynomial Functions
- Lesson 3-1, Learn, all Examples

Revisit the Probe at the end of the module to be sure that your students no longer carry these misconceptions.



The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

What methods are useful for solving polynomial equations and finding zeros of polynomial functions? Sample answer: You can use graphing or factoring to solve polynomial equations or find roots of polynomial functions. The Remainder and Factor Theorems can also be useful.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

Focus Students write notes about new terms and concepts as they are presented in each lesson of this module.

Teach Have students construct their Foldable as illustrated. Have students write an explanation of each term or concept on the appropriate section of their Foldable while working through each lesson. Encourage students to record examples of each term or concept on the back of each flap.

When to Use It Encourage students to add to their Foldable as they work through the module, and to use it to review for the module test.

Launch the Module

For this module, the Launch the Module video uses exploring Yellowstone National Park to describe modeling real-world situations with polynomial functions. Students learn about polynomial models in population growth and decay.

Module 3 Polynomial Equations

Essential Question

What methods are useful for solving polynomial equations and finding zeros of polynomial functions?

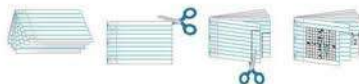
What Will You Learn?

How much do you already know about each topic **before** starting this module?

KEY	Before	After
👉 I don't know	👉 I've heard of it	👉 I know it!
solve polynomial equations by graphing		
solve polynomial equations by factoring		
solve polynomial equations in quadratic form		
prove polynomial identities		
apply the Remainder Theorem		
use the Factor Theorem to determine whether a binomial is a factor of a polynomial		
use the Fundamental Theorem of Algebra		
find zeros of polynomial functions		

Foldable Make this Foldable to help you organize your notes about polynomial equations. Begin with three sheets of notebook paper.

1. Fold each sheet of paper in half from top to bottom.
2. Cut along the fold. Staple the six half-sheets together to form a booklet.
3. Cut tabs into the margin. The top tab is 2 lines deep, the next tab is 6 lines deep, and so on.
4. Label each tab except the first with a lesson number. Use the first tab as a cover page decorating it with a graph from lesson 1.



Module 3 • Polynomial Equations 117

Interactive Presentation



What Vocabulary Will You Learn?

- depressed polynomial
- polynomial identity
- synthetic substitution
- identity
- prime polynomial
- quadratic form
- multiplicity

Are You Ready?

Complete the Quick Review to see if you are ready to start this module. Then complete the Quick Check.

Quick Review	
Example 1 Use the Distributive Property to multiply $(x^2 - 2x - 4)(x + 5)$. $(x^2 - 2x - 4)(x + 5)$ $= x^2(x + 5) - 2x(x + 5) - 4(x + 5)$ Distributive Property $= x^3(x) + x^2(5) + (-2x)(x) + (-2x)(5) + (-4)(x) + (-4)(5)$ Distributive Property $= x^3 + 5x^2 - 2x^2 - 10x - 4x - 20$ Multiply $= x^3 + 3x^2 - 14x - 20$ Combine like terms.	Example 2 Solve $2x^2 + 8x + 1 = 0$. $x = \frac{-8 \pm \sqrt{8^2 - 4(2)(1)}}{2(2)}$ Quadratic Formula $= \frac{-8 \pm \sqrt{64 - 8}}{4}$ $= \frac{-8 \pm \sqrt{56}}{4}$ Simplify $= -2 \pm \frac{\sqrt{14}}{2}$ The exact solutions are $-2 + \frac{\sqrt{14}}{2}$ and $-2 - \frac{\sqrt{14}}{2}$. The approximate solutions are -0.13 and -3.87.
Quick Check Use the Distributive Property to multiply each set of polynomials. 1. $(5x^2 - x + 2)(4x + 2)$ $24x^2 + 8x + 4$ 2. $(x^2 - 2x + 7)(x - 3)$ $7x^3 - 17x^2 + 55x - 21$ 3. $(7x^3 - 6x - 6)(2x - 4)$ $14x^4 - 40x^3 + 12x + 24$ 4. $(x^2 + 6x - 4)(2x - 4)$ $2x^3 + 8x^2 - 32x + 16$ 5. $x^2 + 2x - 8 = 0$ $-4, 2$ 6. $2x^2 + 7x + 3 = 0$ $-3, -\frac{1}{2}$ 7. $6x^2 + 5x - 4 = 0$ $-\frac{4}{3}, \frac{3}{2}$ 8. $4x^2 - 2x - 1 = 0$ $\frac{1 \pm \sqrt{5}}{4}$	
How Did You Do? Which exercises did you answer correctly in the Quick Check?	

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define The quadratic form is a form of polynomial equation, $ax^2 + bx + c$, where u is an algebraic expression in x .

Example $-(2x - 1) + 3(2x - 1) + 5$

Ask Is the expression in quadratic form? Explain. **Yes**, the expression is in the form $ax^2 + bx + c$ with $a = -1$, $b = 3$, and $c = 5$ while $u = 2x - 1$.

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- finding x-intercepts
- solving simple quadratic equations
- identifying equations with no solution vs. identities
- performing synthetic division
- finding zeros of functions



ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You can use the ALEKS pie report to see which students know the topics in the **Quadratic and Polynomial Functions** module—who is ready to learn these topics and who isn't quite ready to learn them yet—in order to adjust your instruction as appropriate.

Mindset Matters

Promote Growth Over Speed

Learning requires time and effort—time to think, reason, make mistakes, and learn from your mistakes and the mistakes of others. Ultimately, it is about the deep connections students make in their thinking and reasoning that matter more than the speed at which a problem is solved.

How Can I Apply It?

Have students complete the **Rate Yourself** chart before starting the module, discuss their mistakes and progress as you work through each lesson, and then reflect on their growth at the end of the module.

Solving Polynomial Equations by Graphing

LESSON GOAL

Students solve polynomial equations by graphing.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Explore:** Solutions of Polynomial Equations

 **Develop:**

Solving Polynomial Functions by Graphing

- Solve a Polynomial Equation by Graphing
- Solve a Polynomial Equation by Using a System

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

 Formative Assessment Math Probe

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources

Remediation: Multiplying Polynomials

Extension: Complex Roots of Polynomial Equations

AL	LB	EL
●	●	●
●	●	●

Language Development Handbook

Assign page 14 of the *Language Development Handbook* to help your students build mathematical language related to solving polynomial equations by graphing.

 You can use the tips and suggestions on page T14 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra

Standards for Mathematical Content:

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

A.REI.11 Explain why the x -coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g. using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

5 Use appropriate tools strategically.

Coherence

Vertical Alignment

Previous

Students analyzed the graphs of polynomial functions by identifying key features. **F.IF.4, F.IF.7c**

Now

Students solve polynomial equations by graphing. **A.CED.1, A.REI.11**

Next

Students will determine the numbers and types of roots of polynomial equations, find zeros, and use zeros to graph polynomial functions. **A.APR.3, F.IF.7c, N.CN.9**

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

 **Conceptual Bridge** In this lesson, students expand on their understanding of graphing polynomial functions by using the graphs to solve related polynomial equations. They build fluency by finding solutions by graphing, and they apply their understanding by using graphs to solve real-world problems.



Interactive Presentation

Warm Up

Name the x -intercept of each graph. If an exact coordinate cannot be found, name the two integers between which the x -intercept lies.

1.

2.

3.

4.

[Show Answers](#)

Warm Up

Launch the Lesson

Watch this video to see how to use a polynomial equation to solve a real-world situation.

Launch

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- finding x -intercepts

Answers:

- 1
- 3
- 2, 4
- between -2 and -1, between 0 and 1, 1

Launch the Lesson



Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the dimensions, volume, and graph in the Launch the Lesson.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Mathematical Background

Tables of values can be used to explore two types of changes in the values of a polynomial function. A change of signs in the value of $f(x)$ from one value of x to the next indicates that the graph of the function crosses the x -axis between the two x -values. A change between increasing values and decreasing values indicates that the graph is turning for that interval. A turning point on a graph is a relative maximum or minimum.



Explore Solutions of Polynomial Equations

Objective

Students use a sketch to explore solving polynomial equations by graphing a related function.

MP Teaching the Mathematical Practices

7 Use Structure Help students to explore how the structure of the factored form of a polynomial equation relates to the zeros of the related function.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

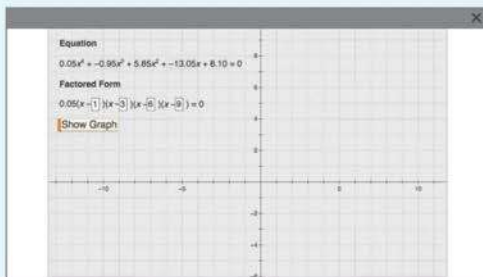
Students will complete guiding exercises throughout the Explore activity. They will learn that they can use related functions to solve polynomial equations. They will be presented with different polynomial equations, and will use the sketch to graph the related function. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation



Explore



Explore

WEB SKETCHPAD



Students will use the sketch to graph the related function of a polynomial equation.

TYPE



Students answer questions pertaining to solving polynomial equations by graphing related functions.



Interactive Presentation



Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Solutions of Polynomial Equations
(continued)**Questions**

Have students complete the Explore activity.

Ask:

- What would be the solution(s) of the polynomial equation $(2x + 1)(x - 3)(4x - 3) = 0$? $x = -\frac{1}{2}, 3, \frac{3}{4}$
- When graphing the related function of a polynomial equation, where are the solutions to the equation located on the graph?
the x -intercepts

Inquiry

How can you solve a polynomial equation by using the graph of a related polynomial function? **Sample answer:** I can find the zeros of the graph of the related polynomial function. The zeros will be the same as the solutions of the equation.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Solving Polynomial Equations by Graphing

Objective

Students solve polynomial equations by examining graphs of the related functions and by graphing systems of equations.

Teaching the Mathematical Practices

- 1 Explain Correspondences** Help students to see the relationship between the solutions of a polynomial equation and the zeros of the related function.

Example 1 Solve a Polynomial Equation by Graphing

Teaching the Mathematical Practices

- 5 Decide When to Use Tools** Mathematically proficient students make sound decisions about when to use mathematical tools such as graphing calculators. Help them see why using a graphing calculator will help solve the polynomial equation and what the limitations are of using the tool.

Questions for Mathematical Discourse

- A1** Why do you set the equation equal to zero before you graph the related function? **Sample answer:** This makes the x -intercepts the solutions to the original equation because they are where the related function is equal to zero.
- OL** When finding the first zero using the TI-84 family calculator, the y -value was $1E-12$. What does this value represent? **Sample answer:** $1E-12$ is scientific notation for 0.000000000001, which is effectively zero.
- BL** Suppose all of the x -intercepts do not show up when graphing the related function. What might you need to do on the calculator? **Sample answer:** I may need to change the viewing window because the other x -intercepts might be greater than 10 or less than -10 .

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Solving Polynomial Equations by Graphing

Lesson 3-1

Explore Solutions of Polynomial Equations

Online Activity Use graphing technology to complete the Explore.

INQUIRY How can you solve a polynomial equation by using the graph of a related polynomial function?

Learn Solving Polynomial Equations by Graphing

A related function is found by rewriting the equation with 0 on one side, and then replacing 0 with $f(x)$. The values of x for which $f(x) = 0$ are the real zeros of the function and the x -intercepts of its graph.



$$\begin{aligned} x^3 + 2x^2 - 4x - 6 &= 0 \\ x^3 + 2x^2 - 4x - 6 &= 0 \\ x^3 + 2x^2 - 4x - 6 &= 0 \\ x^3 + 2x^2 - 4x - 6 &= 0 \end{aligned}$$

Example 1 Solve a Polynomial Equation by Graphing

Use a graphing calculator to solve $x^4 + 3x^2 - 5 = -4x^2$ by graphing.

Step 1 Find a related function. Write the equation with 0 on the right.

$$\begin{aligned} x^4 + 3x^2 - 5 &= -4x^2 \\ x^4 + 3x^2 + 4x^2 - 5 &= -4x^2 + 4x^2 \\ x^4 + 7x^2 - 5 &= 0 \end{aligned}$$

A related function is $f(x) = x^4 + 7x^2 - 5$.

Step 2 Graph the related function.

Enter the equation in the $Y=$ list and graph the function.



Step 3 Find the zeros. Use the zero feature from the CALC menu. The real zeros are about -3.22 and 0.84 .

Check

Use a graphing calculator to solve $4x^2 + x = \frac{1}{4}x^3 + 1$ by graphing. Round to the nearest hundredth, if necessary.

$$x = \text{ ? } \quad -2.64, -0.66, 0.39, 2.91$$

Today's Goals

- Solve polynomial equations by graphing.

Think About It!

How can you use the structure of the related function to determine the number of real solutions of the equation?

Sample answer: The degree of the related polynomial function is the maximum number of real solutions. When repeated roots are counted as distinct roots, if the degree is odd, there will be an odd number of real solutions, and if the degree is even, there will be an even number of real solutions or no real solutions.

Talk About It!

Explain how you could use the table feature to more accurately estimate the zeros of the related function. What are the limitations of the table feature?

Sample answer: By changing the table settings to use a smaller interval, I can find more accurate intervals in which the zeros lie. Sometimes there is no interval small enough to identify a zero exactly using the table feature. Also, using a very small interval may require a lot of scrolling to find the zeros.

Lesson 3-1 • Solving Polynomial Equations by Graphing 119

Interactive Presentation

Solving Polynomial Equations by Graphing

The real solutions of a polynomial equation can be found by graphing. One method for solving a polynomial equation is to rewrite the equation as $f(x) = 0$, where $f(x)$ is a polynomial function. A related function is found by rewriting the equation as $f(x) = 0$ with $f(x)$.

Equation	Related Function
$x^4 + 3x^2 - 5 = -4x^2$	$f(x) = x^4 + 7x^2 - 5$
$4x^2 + x = \frac{1}{4}x^3 + 1$	$f(x) = \frac{1}{4}x^3 - 4x^2 - x + 1$

Learn

TYPE



Students answer a question to determine whether they understand solving polynomial functions by graphing.



Example 2 Solve a Polynomial Equation by Using a System



Teaching the Mathematical Practices

2 Represent a Situation Symbolically Guide students to define variables to solve the problem in Example 2. Then work with them to find the other relationships in the problem.

Questions for Mathematical Discourse

AL Why is the total volume of the pool 1085 ft³ rather than 217 ft³?

Sample answer: 217 ft³ is the volume per penguin and there are 5 penguins in the pool.

OL Could you have found the solutions by finding the zeros of a related function? Explain. **Yes; sample answer:** You could subtract 1085 from each side and then solve for x by finding the zeros of the related function.

BL To graph a system of equations, must the polynomial equation be simplified first? Explain. **No; sample answer:** The polynomial equation does not have to be simplified because the two forms are equivalent and produce the same graph.

Common Error

Many students will distribute the monomial term to both binomials before multiplying the binomials. This is incorrect as the monomial term can only be multiplied once. Encourage students to multiply binomial terms together first, then distribute any monomial terms.

Exit Ticket

Recommended Use

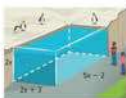
At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Example 2 Solve a Polynomial Equation by Using a System

ANIMALS For an exhibit with six or fewer Emperor penguins, the pool must have a depth of at least 4 feet and a volume of at least 1020 gallons, or about 217 ft³, per bird. If a zoo has five Emperor penguins, what should the dimensions of the pool shown at the right be to meet the minimum requirements?



Part A Write a polynomial equation.

Use the formula for the volume of a rectangular prism, $V = lwh$, to write a polynomial equation that represents the volume of the pool. Let h represent the depth of the pool.

Since the minimum required volume for the pool is 217 ft³ per penguin, or 217 $\cdot$ 5 = 1085 ft³, the equation that represents the volume of the pool is $(2x + 3)(5x - 2)(7) = 1085$. Simplify the equation.

$$\begin{aligned} (2x + 3)(5x - 2)(7) &= 1085 && \text{Volume of pool} \\ 7(2x + 3)(5x - 2) &= 1085 && \text{FOIL} \\ 7(10x^2 - 4x + 15x - 6) &= 1085 && \text{Simplify} \\ 7(10x^2 + 11x - 6) &= 1085 && \text{Combine like terms} \\ 70x^2 + 77x - 42 &= 1085 && \text{Distributive Property} \\ 70x^2 + 77x - 112x &= 1085 - 42 && \end{aligned}$$

So, the volume of the pool is $70x^2 + 77x - 112x = 1085$.

Part B Write and solve a system of equations.

Set each side equal to y to create a system of equations.

$$\begin{aligned} y &= 70x^2 + 77x - 112x && \text{First equation} \\ y &= 1085 && \text{Second equation} \end{aligned}$$

Enter the equations in the $Y =$ list and graph.

Use the **intersect** feature on the **CALC** menu to find the coordinates of the point of intersection.

The real solution is the x -coordinate of the intersection, which is 3.5.

Part C Find the dimensions.

Substitute 3.5 feet for x in the length, width, and depth of the pool.

$$\begin{aligned} \text{Length: } 2x + 3 &= 10 \text{ ft} && \text{Width: } 5x - 2 = 15.5 \text{ ft} \\ \text{Depth: } 7 \text{ ft} &&& \end{aligned}$$



Think About It! Is your solution reasonable? Justify your conclusion.

Yes; sample answer: When $x = 3.5$, all of the dimensions are positive and the depth of 7 feet meets the minimum required depth of 4 feet.

Go Online You can complete an Exit Ticket online.

Interactive Presentation

Example 2

TAP



Students tap to see the steps to write and solve a polynomial equation.

TYPE



Students complete the calculations to determine the dimensions of the pool.

CHECK



Students complete the Check online to determine whether they are ready to move on.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–9
2	exercises that use a variety of skills from this lesson	10–15
2	exercises that extend concepts learned in this lesson to new contexts	16
3	exercises that emphasize higher-order and critical-thinking skills	17–21

ASSESS AND DIFFERENTIATE

 Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign:

- Practice Exercises 1–17 odd, 19–23
- Extension: Complex Roots of Polynomial Equations
-  **ALEKS** Real Zeros of Polynomial Functions

IF students score 66%–89% on the Checks, **THEN** assign:

- Practice Exercises 1–23 odd
- Remediation, Review Resources: Multiplying Polynomials
- Personal Tutors
- Extra Examples 1–2
-  **ALEKS** Finding x-intercepts

IF students score 65% or less on the Checks, **THEN** assign:

- Practice, Exercises 1–9 odd
- Remediation, Review Resources: Multiplying Polynomials
- Quick Review Math Handbook*: Solving Polynomial Equations
-  **ALEKS** Finding x-intercepts



A.CED.1, A.REI.11

Practice

 No Online You can complete your homework online.

Example 1

Use a graphing calculator to solve each equation by graphing. If necessary, round to the nearest hundredth.

- $x^2 + x^2 - 5x = -9$
- $x^2 - 9x + 27x = 20$
- $x^2 + 1 = 4x^2$
- $x^6 - 15 = 5x^4 - x^2$
- $\frac{1}{2}x^2 = \frac{1}{2}x^2 - 2$
- $x^4 = -x^2 + 3$

Example 2

7. **SHIPPING** A shipping company will ship a package for \$750 when the volume is no more than 15,000 cm^3 . Grace needs to ship a package that is $3x - 5$ cm long, $2x$ cm wide, and $x + 20$ cm tall.

- Write a polynomial equation to represent the situation. $6x^3 + 110x^2 - 200x = 15,000$
- Write and solve a system of equations. $y = 6x^3 + 110x^2 - 200x$, $y = 15,000$; $x = 10$ cm
- What should the dimensions of the package be to have the maximum volume? 25 cm by 20 cm by 30 cm

- Write a polynomial equation to represent the situation. $4x^2 + 56x + 192 = 285$
- Write and solve a system of equations. $y = 4x^2 + 56x + 192$, $y = 285$; $x = 1.3$ ft
- What are the dimensions of the garden plus border? 15 ft by 19 ft

9. **PACKAGING** A juice manufacturer is creating new cylindrical packaging. The height of the cylinder is to be 3 inches longer than the radius of the can. The cylinder is to have a volume of 628 cubic inches. Use 3.14 for π .

- Write a polynomial equation to support the model. $\pi r^2 \cdot 3 = 628$
- Write and solve a system of equations. $y = \pi r^2 + 3\pi r^2$, $y = 628$; $r = 5$
- What are the radius and height of the new packaging? radius = 5 in., height = 8 in.

Mixed Exercises

Solve each equation, if necessary, round to the nearest hundredth.

- $x^4 + 2x^2 = 7$
- $x^4 - 15x^2 = -24$
- $x^3 - 6x^2 + 4x = -6$
- $x^4 - 15x^2 + x + 65 = 0$

14. **BALLOON** Treyvon is standing 9 yards from the base of a hill that has a slope of $\frac{3}{4}$. He throws a water balloon from a height of 2 yards. Its path is modeled by $f(x) = -0.1x^2 + 0.8x + 2$, where h is the height of the balloon in yards and x is the distance the balloon travels in yards.

- Write a polynomial equation to represent the situation. $-0.1x^2 + 0.8x + 2 = \frac{3}{4}(x - 9)$
- How far from Treyvon will the balloon hit the hill? about 9.6 yd

Lesson 3.1 • Solving Polynomial Equations by Graphing 121



15. **USE TOOLS** A company models its revenue in dollars using the function $f(x) = 70,000(x - x^2)$ on the domain $(0, 1)$ where x is the price at which they sell their product in dollars. Use a graphing calculator to sketch a graph and find the price at which their product should be sold to make revenue of \$20,000. Describe your solution process. *See margin.*

16. **ROLLER COASTERS** On a racing roller coaster, two trains start at the same time and race to see which returns to the station first. On one coaster, the height of a train on the blue track can be modeled by $f(x) = \frac{1}{20}(x^3 - 60x^2 + 900x)$ and the height of a train on the green track can be modeled by $g(x) = \frac{1}{200}(x^5 - 144x^4 + 7384x^3 - 158,400x^2 + 1,210,000x)$ where x is time in seconds for the first 35 seconds of the ride.



- a. What equation would determine the times when the blue and green trains are at the same height? $\frac{1}{20}(x^3 - 60x^2 + 900x) = \frac{1}{200}(x^5 - 144x^4 + 7384x^3 - 158,400x^2 + 1,210,000x)$
- b. Use a graphing calculator to sketch a graph of $f(x)$ and $g(x)$ and solve the equation from part a. Interpret the solution in the context of the situation. *See margin.*
- c. Write an equation to determine the times for which the blue train modeled by $f(x)$ is at a height of 150 feet. Use a graphing calculator to solve the equation. Interpret the solution in the context of the situation. *See margin.*

Higher-Order Thinking Skills

17. **WRITE** Use a graph to explain why a function with an even degree can have zero real solutions, but a function with an odd degree must have at least one real solution. *See margin.*

18. **CREATE** Write a polynomial equation and solve it by graphing a related function and finding its zeros. Sample answer: $x^2 - 4x^2 + 12 = x - 22$; $x = -2.39$

19. **ANALYZE** Determine whether the following statement is sometimes, always, or never true. Justify your argument.
If a system of equations has more than one solution, then the positive solution is the only viable solution. *See margin.*

20. **PERSEVERE** During practice, a player kicks a ball from the ground with an initial velocity of 32 feet per second. The polynomial $f(x) = -16x^2 + 32x$ models the height of the ball, where x represents time in seconds. At the same time, another player heads a ball at some distance c feet off the ground with an initial velocity of 27 feet per second. The polynomial $g(x) = -16x^2 + 27x + c$ models the height of the ball.

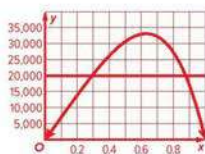
- a. If the balls are at the same height after 1.2 seconds, from what height did the second player head the ball? **6 ft**
- b. If $c > 0$, is it possible that the soccer balls are never at the same height? Is it reasonable in the context of the situation? Explain your reasoning.
Sample answer: Only if $c > 10$ will the balls never be at the same height before they strike the ground. Because it is unreasonable for a player to head the ball more than 10 feet above the ground, it is not reasonable that the soccer balls will never be at the same height.

21. **WHICH ONE DOESN'T BELONG?** Which polynomial doesn't belong? Justify your conclusion. $5x^3 = -2x - 11$; It is the only one that has no real solutions.

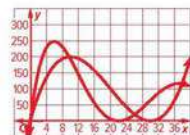
$$x - 17 = 18x^2 + 3x^2 \quad x^2 = 4x^4 + 3x^2 - 8 \quad 5x^2 = -2x - 11 \quad -4 = 2x^3 - x^2$$

Answers

15. \$0.29, 0.88; sample answer: graph $f(x) = 70,000(x - x^2)$ and $f(x) = 20,000$ and find the x -values of the points of intersections.



- 16b. The graphs of the functions intersect at $x = 0$ and $x \approx 9.6$ and $x \approx 26.0$. The passenger cars are at the same height at 0 seconds and approximately 9.6 and 26.0 seconds.



- 16c. $\frac{1}{20}(x^3 - 60x^2 + 900x) = 150$, which has solutions at $x \approx 16.5$ and $x \approx 4.7$. The passenger car is at 150 feet at approximately 4.7 seconds and 16.5 seconds.

17. Sample answer: A function with an even degree reverses direction, so both ends extend in the same direction. That means that if the function opens up and the vertex is above the x -axis or if the function opens down and the vertex is below the x -axis, there are no real solutions. A function with an odd degree does not reverse direction, so the ends extend in opposite directions. Therefore, it must cross the x -axis at least once.

19. Sometimes; sample answer: The positive solution is often correct when a negative value doesn't make sense, such as for distance or time; however, sometimes a negative solution is reasonable, such as for temperature or position problems. Also, sometimes there are two positive solutions and one may be unreasonable.

Solving Polynomial Equations Algebraically

LESSON GOAL

Students solve polynomial equations by factoring and by writing them in quadratic form.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Develop:**

Solving Polynomial Equations by Factoring

- Factor Sums and Differences of Cubes
- Factor by Grouping
- Combine Cubes and Squares
- Solve a Polynomial Equation by Factoring
- Write and Solve a Polynomial Equation by Factoring

Solving Polynomial Equations in Quadratic Form

- Write Expressions in Quadratic Form
- Solve Equations in Quadratic Form

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources

Remediation: Solving Quadratic Equations by Factoring

Extension: History of Quadratic Equations

AR	LB	EL
●	●	●
	●	●

Language Development Handbook

Assign page 15 of the *Language Development Handbook* to help your students build mathematical language related to solving polynomial equations by factoring.

ELL You can use the tips and suggestions on page T15 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Algebra

Standards for Mathematical Content:

A.CED.1 Create equations and inequalities in one variable and use them to solve problems.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous

Students solved quadratic equations by factoring.

N.CN.7, N.CN.8, F.IF.8a

Now

Students solve polynomial equations by factoring and by writing them in quadratic form.

A.CED.1

Next

Students will evaluate and factor functions by using the Remainder and Factor Theorems.

A.APR.2

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

 **Conceptual Bridge** In this lesson, students draw on their understanding of factoring to solve polynomial equations. They build fluency by factoring to find solutions, and they apply their understanding of factoring of solving real-world problems.

Mathematical Background

If a polynomial can be written as a product of linear and quadratic factors, its zeros can be found. Set each of the factors equal to 0 and solve the resulting equations.



Interactive Presentation

Warm Up

Solve each equation.

- $x(x - 4) = 0$
- $(x + 1)(x - 2) = 0$
- $x^2 - 7x = 0$
- $x^2 - 3 = 6$
- $x^2 - 80 = 20$

Show Answers

Warm Up

Launch the Lesson

Round-cut diamonds are the most popular shape of diamond available, making up approximately 75% of total diamond sales. The ideal round-cut diamond was first described by Marcel Tolkowsky in 1919. Using theories of light behavior and mathematical calculations, he was able to determine the proportions and angles that would maximize the brilliance of a diamond. A polynomial equation can be used to model the weight, in carats, of an ideal round-cut diamond as it relates to its diameter in millimeters. You could solve this equation algebraically to find the diameter of an ideal round-cut diamond given its weight.



Launch the Lesson

Vocabulary

Extend All Collapse All

▼ prime polynomial

A polynomial that cannot be written as a product of two polynomials with integral coefficients.

▼ quadratic form

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- solving simple quadratic equations

Answers:

- 0, 4
- 1, 2
- 0, 7
- 2, 3
- 10, 10

Launch the Lesson



Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students apply polynomial equations to a real-world problem.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.



Learn Solving Polynomial Equations by Factoring

Objective

Students solve polynomial equations by factoring.

MP Teaching the Mathematical Practices

1 Understand the Approaches of Others Mathematically proficient students can explain the methods to solve a problem. The Think About It! feature asks students to justify the reasoning of Mateo.

Example 1 Factor Sums and Differences of Cubes

MP Teaching the Mathematical Practices

7 Use Structure Guide students to use the structure of each polynomial expression in Example 1 to determine which method of factoring to apply.

Questions for Mathematical Discourse

AL What is a greatest common factor? It is the greatest term that divides evenly from all terms of the algebraic expression.

OL When raising a power to another power, what operation do you perform on the exponents? **multiply**

BL Is the resulting trinomial in the sum or difference of cubes factorable? Explain. **No; sample answer: The resulting trinomial will never be factorable since there are no two factors that can multiply to $a^2 \pm ab + b^2$.**

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 3-2

Solving Polynomial Equations Algebraically

Learn Solving Polynomial Equations by Factoring

Like quadratics, some polynomials of higher degrees can be factored. A polynomial that cannot be written as a product of two polynomials with integral coefficients is called a **prime polynomial**. Like a prime real number, the only factors of a prime polynomial are 1 and itself.

Similar to quadratics, some cubic polynomials can be factored by using polynomial identities.

Key Concept: Sum and Difference of Cubes

Factoring Technique	General Case
Sum of Two Cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of Two Cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Polynomials can be factored by using a variety of methods, the most common of which are summarized in the table below. When factoring a polynomial, always look for a common factor first to simplify the expression. Then, determine whether the resulting polynomial factors can be factored using one or more methods.

Concept Summary: Factoring Techniques

Number of Terms	Factoring Technique	General Case
any	Greatest Common Factor (GCF)	$2a^2b^3 + 6ab = 2ab(a^2b^2 + 3)$
any number	Difference of Two Squares	$a^2 - b^2 = (a + b)(a - b)$
two	Sum of Two Cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
	Difference of Two Cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
three	Perfect Square Trinomial	$a^2 + 2ab + b^2 = (a + b)^2$
	General Trinomial	$ax^2 + bx + c = (ax + d)(x + e)$
four or more	Grouping	$ax + bx + cy + dy = (a + b)(x + y)$

Today's Goals

- Solve polynomial equations by factoring.
- Solve polynomial equations by writing them in quadratic form and factoring.

Today's Vocabulary
prime polynomial
quadratic form

Think About It!
Mateo says that you could use the sum of two cubes to factor $x^3 + y^3$. Is he correct? Why or why not?

Yes, sample answer: Because $(x^3)^{1/3} = x$ and because $(y^3)^{1/3} = y$, you can rewrite the expression as $a^3 + b^3$ where $a = x$ and $b = y$.

Think About It!
How can you check that an expression has been factored correctly?

Sample answer: Multiply the factors to check that the product is the same as the original expression.

Interactive Presentation



Learn

TYPE



Students explain whether a polynomial expression can be factored using the sum of two cubes.

**Example 1** Factor Sums and Differences of Cubes

Factor each polynomial. If the polynomial cannot be factored, write prime.

a. $8x^3 + 125y^3$

The GCF of the terms is 1, but $8x^3$ and $125y^3$ are both perfect cubes. Factor the sum of two cubes.

$$\begin{aligned} 8x^3 + 125y^3 &= (2x)^3 + (5y)^3 \\ &= (2x + 5y)(2x^2 - 2x(5y) + (5y)^2) \\ &= (2x + 5y)(2x^2 - 10xy + 25y^2) \end{aligned}$$

Original expression

$$(2x)^3 + (5y)^3 = 8x^3 + 125y^3$$

Sum of two cubes

Simplify

b. $54x^3 - 128y^3$

$54x^3 - 128y^3$

$$\begin{aligned} 54x^3 - 128y^3 &= 2x^3(27x^2 - 64y^3) \\ &= 2x^3(3x - 4y)(3x^2 + 3x(4y) + (4y)^2) \\ &= 2x^3(3x - 4y)(3x^2 + 12xy + 16y^2) \end{aligned}$$

Original expression

Factor out the GCF

$$(2x)^3 - 2(64y)^3 = 54x^3 - 128y^3$$

Difference of two cubes

Simplify

Study Tip

Grouping When grouping 6 or more terms, group the terms that have the most common values.

Think About It!

When factoring by grouping, what must be true about the expressions inside parentheses after factoring out a GCF from each group?

Sample answer: They must be the same.

Example 2 Factor by Grouping

Factor $14x^2 - 16y + 20cy + 28bx^2 - 35cx^2 - 8cy$. If the polynomial cannot be factored, write prime.

$$\begin{aligned} 14x^2 - 16y + 20cy + 28bx^2 - 35cx^2 - 8cy &= (14x^2 + 28bx^2 - 35cx^2) + (-16y - 8cy + 20cy) \\ &= 7x^2(2x + 4b - 5c) - 4y(2x + 4b - 5c) \\ &= (7x^2 - 4y)(2x + 4b - 5c) \end{aligned}$$

Original expression

Group to find a GCF

Factor out the GCF

Distributive Property

Example 3 Combine Cubes and Squares

Factor $64x^3 - y^3$. If the polynomial cannot be factored, write prime.

This polynomial could be considered the difference of two squares or the difference of two cubes. The difference of two squares should always be done before the difference of two cubes for easy factoring.

$$\begin{aligned} 64x^3 - y^3 &= (8x)^3 - (y)^3 \\ &= (8x^2 + y^2)(8x^2 - y^2) \\ &= (8x^2 + y^2)(2x - y)(2x + y) \\ &= (2x + y)(8x^2 - 2xy + y^2)(2x - y) \\ &= (2x + y)(4x^2 + 2xy + y^2)(2x - y) \end{aligned}$$

Original expression

Difference of cubes

Factor out the GCF

Sum and difference of cubes

Go Online You can complete an Extra Example online.

Example 2 Factor by Grouping**MP Teaching the Mathematical Practices**

1 Seek Information Students must transform the polynomial expression in order to apply the method of factoring by grouping.

Questions for Mathematical Discourse

- AL** How should an original polynomial be grouped so that it can be factored? **Sample answer:** so that each group has a GCF which results in two expressions that are the same after factoring out the GCF.
- OL** Why is the GCF for the second grouping negative? **Sample answer:** to make the signs match the first grouping.
- BL** Is there another way to group the polynomial expression into two groups and factor? Explain. **No; sample answer:** Even though there may be a GCF from a new grouping, the resulting polynomials will not be the same. Thus it cannot be factored by grouping.

Example 3 Combine Cubes and Squares**MP Teaching the Mathematical Practices**

1 Analyze Givens and Constraints In Example 3, guide students through planning the solution pathway. Help students to see why the difference of squares should be factored first.

Questions for Mathematical Discourse

- AL** Why is the expression a difference of squares and a difference of cubes? **The expression can be written as $(8x^2 - y^2)(2x - y)$ or $(4x^2)^2 - (y^2)^2$.**
- OL** What must be true about the coefficients and exponents of an expression that is both a difference of cubes and a difference of squares? **Sample answer:** The coefficients must be both perfect squares and perfect cubes, and the exponents must be a multiple of both 2 and 3.
- BL** Why is it easier to factor by difference of squares before using the sum or difference of cubes? **Sample answer:** If the polynomial expression is factored using the sum or difference of cubes first, the resulting trinomial must be factored. Applying the difference of squares first, results in a simpler expression to factor.

Interactive Presentation**Factor by Grouping**

To factor polynomials with four or more terms, the easy method out there is factoring by grouping.

Factor $14x^2 - 16y + 20cy + 28bx^2 - 35cx^2 - 8cy$. If the polynomial cannot be factored, write prime.

$$\begin{aligned} 14x^2 - 16y + 20cy + 28bx^2 - 35cx^2 - 8cy &= (14x^2 + 28bx^2 - 35cx^2) + (-16y - 8cy + 20cy) \\ &= 7x^2(2x + 4b - 5c) - 4y(2x + 4b - 5c) \\ &= (7x^2 - 4y)(2x + 4b - 5c) \end{aligned}$$

Original expression

Group to find a GCF

Factor out the GCF

Distributive Property

Example 2

TYPE**a**

Students answer a question to determine whether they understand factoring by grouping.



Example 4 Solve a Polynomial Equation by Factoring

MP Teaching the Mathematical Practices

7 Use the Distributive Property The Distributive Property is one of the most-used properties in algebra. Students should be able to use the Distributive Property to factor by grouping.

Questions for Mathematical Discourse

- AL** How did you know to start the factoring process by using grouping? **There were 4 terms.**
- OL** Describe another way to factor the original polynomial equation by grouping. **Sample answer:** I could group $(4x^2 - 9x) + (12x^2 - 27)$. This would factor to $x(4x - 9) + 3(4x^2 - 9)$, which yields the same factors of $(x + 3)(4x - 9)$.
- BL** What pattern do you see in the original equation that indicates a factor of $(x + 3)$? **Sample answer:** The coefficient of the second term is 3 times the coefficient of the first term, and the coefficient of the fourth term is 3 times the coefficient of the third term.

Example 5 Write and Solve a Polynomial Equation by Factoring

MP Teaching the Mathematical Practices

4 Use Tools Point out that to solve the problem in Example 5, students will need to use the diagram to write and solve a polynomial equation.

Questions for Mathematical Discourse

- AL** Why do you subtract the volume of the small cube from the volume of the large cube? **You are given the volume of the figure, which is the large cube with the small cube removed.**
- OL** Why are the solutions you found when solving $x^2 + 3x + 9 = 0$ not viable solutions? **Sample answer:** The solutions are imaginary and because x represents a length, it must be a real number.
- BL** How many and what type of cube roots does a real number have? **one real cube root and a complex conjugate pair of cube roots**

Learn Solving Polynomial Equations in Quadratic Form

Objective

Students solve polynomial equations by writing them in quadratic form and factoring.

MP Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. Point out that to factor some polynomial expressions, it may be helpful to rewrite the expression in quadratic form.

Example 4 Solve a Polynomial Equation by Factoring

Solve $4x^2 + 12x^2 - 9x - 27 = 0$.

$$\begin{aligned} 4x^2 + 12x^2 - 9x - 27 &= 0 && \text{Original equation} \\ (4x^2 + 12x^2) + (-9x - 27) &= 0 && \text{Group to find a GCF} \\ 4x^2(3x + 3) - 9(x + 3) &= 0 && \text{Factor out the GCFs} \\ (4x^2 - 9x) + 3(-x + 3) &= 0 && \text{Distributive Property} \\ (2x + 3)(2x - 3) + 3(-x + 3) &= 0 && \text{Difference of squares} \\ (2x + 3)(2x - 3) - 3(x - 3) &= 0 && \text{Zero Product Property} \\ (2x + 3)(2x - 3) - 3(x - 3) &= 0 && \text{Zero Product Property} \end{aligned}$$

The solutions of the equation are -3 , $-\frac{3}{2}$, and $\frac{3}{2}$.

Check:

$$\begin{aligned} \text{Solve } x^2 + 4x^2 - 25x + 100 &= 0 \\ x = -7, -\frac{7}{2}, \text{ and } \frac{7}{2} \end{aligned}$$

Example 5 Write and Solve a Polynomial Equation by Factoring

GEOMETRY In the figure, the small cube is one fourth the length of the larger cube. If the volume of the figure is 1703 cubic centimeters, what are the dimensions of the cubes?

$$\begin{aligned} (4x)^3 - x^3 &= 1703 && \text{Volume of figure} \\ 64x^3 - x^3 &= 1703 && (4x)^3 = 64x^3 \\ 63x^3 &= 1703 && \text{Subtract} \\ x^3 &= 27 && \text{Divide each side by 63} \\ x^3 - 27 &= 0 && \text{Subtract 27 from each side} \\ (x - 3)(x^2 + 3x + 9) &= 0 && \text{Difference of cubes} \\ x = 3 \text{ or } x = -3 \pm 3i && \text{Solve} \end{aligned}$$

Since 3 is the only real solution, the lengths of the cubes are 3 cm and 12 cm.

Learn Solving Polynomial Equations in Quadratic Form

Some polynomials in x can be rewritten in **quadratic form**.

Key Concept - Quadratic Form

An expression in quadratic form can be written as $ax^2 + bx + c$ for any numbers a , b , and c , or 0 if a is some expression in x . The expression $ax^2 + bx + c$ is called the quadratic form of the original expression.

Now Do This You can complete an Extra Example online.

Lesson 3-2 • Solving Polynomial Equations Algebraically 125

Sample answer: The exponent of one of the variable terms is twice the exponent of the other variable terms.

Think About It!

The following expressions can be written in quadratic form. What do you notice about the terms with variables in the original expressions?

$$\begin{aligned} 2x^2 + x^2 + 6 \\ 12x^2 - 20x^3 + 9 \\ 15x^4 + 9x^2 - 1 \end{aligned}$$

Interactive Presentation

Solve a Polynomial Equation by Factoring

You can solve a polynomial equation by factoring and then using the Zero Product Property to set each factor equal to zero.

Solve $4x^2 + 12x^2 - 9x - 27 = 0$.

$$\begin{aligned} 4x^2 + 12x^2 - 9x - 27 &= 0 && \text{Original Equation} \\ (4x^2 + 12x^2) + (-9x - 27) &= 0 && \text{Group to find a GCF} \\ 4x^2(x + 3) - 9(x + 3) &= 0 && \text{Factor out the GCFs} \\ (4x^2 - 9)(x + 3) &= 0 && \text{Distributive Property} \end{aligned}$$

Example 4

CHECK



Students complete the Check online to determine whether they are ready to move on.

**Talk About It**

Describe how the exponent of the expression equal to u relates to the exponents of the terms with variables.

Sample answer: The exponent of the expression equal to u is the same as the lesser exponent of the two terms with variables.

Example 6 Write Expressions in Quadratic Form

Write each expression in quadratic form, if possible.

a. $4x^{10} + 6x^6 + 15$

Examine the terms with variables to choose the expression equal to u .
 $4x^{10} + 6x^6 + 15 = (2x^5)^2 + 3(2x^6) + 15$ $(2x^5)^2 = 4x^{10}$

b. $18x^4 + 180x^2 - 28$

If the polynomial is not already in standard form, rewrite it. Then examine the terms with variables to choose the expression equal to u .
 $18x^4 + 180x^2 - 28 = 18(6x^4) + 18x^2 - 28$ Standard form of a polynomial
 $(6x^2)^2 = 36x^4$
 $= 5(6x^2)^2 + 3(6x^2) - 28$

c. $9x^6 - 4x^2 - 12$

Because $x^6 \neq (x^2)^2$, the expression cannot be written in quadratic form.

Check

What is the quadratic form of $10x^4 + 100x^2 - 9$?

$4(5x^2)^2 + 2(5x^2) - 9$

Example 7 Solve Equations in Quadratic Form

Solve $8x^4 + 10x^2 - 12 = 0$.

$8x^4 + 10x^2 - 12 = 0$

Original equation

$2(2x^2)^2 + 5(2x^2) - 12 = 0$

$2(2x^2)^2 = 8x^4$

$2u^2 + 5u - 12 = 0$

Let $u = 2x^2$

$(2u - 3)(u + 4) = 0$

Factor

$u = \frac{3}{2}$ or $u = -4$

Zero Product Property

$2x^2 = \frac{3}{2}$ $2x^2 = -4$

Replace u with $2x^2$

$x^2 = \frac{3}{4}$ $x^2 = -2$

Divide each side by 2

$x = \pm\sqrt{\frac{3}{4}}$ $x = \pm\sqrt{-2}$

Take the square root of each side

The solutions are $\pm\frac{\sqrt{3}}{2}$, $\pm\frac{\sqrt{3}}{2}i$, $\pm\sqrt{2}i$, and $\pm\sqrt{2}$.

Check

What are the solutions of $16x^4 + 24x^2 - 40 = 0$?

$x = \pm\frac{\sqrt{5}}{2}$, $\pm\frac{\sqrt{5}}{2}i$, $\pm\sqrt{2}$, and $\pm\sqrt{2}i$

Go Online You can complete an Extra Example online.

Interactive Presentation

Write Expressions in Quadratic Form

Write each expression in quadratic form, if possible.

a. $4x^{10} + 6x^6 + 15$

Examine the terms with variables to choose the expression equal to u .

$4x^{10} + 6x^6 + 15 = (2x^5)^2 + 3(2x^6) + 15$ $(2x^5)^2 = 4x^{10}$

Example 6

TYPE

Students answer a question to determine whether they understand how to write polynomials in quadratic form.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 6 Write Expressions in Quadratic Form**MP Teaching the Mathematical Practices**

7 Use Structure Help students to use the structure of the expressions to rewrite them in quadratic form.

Questions for Mathematical Discourse

- AL** How do you square a variable and its coefficient? **Multiply the exponent of the variable by two and square the coefficient.**
- OL** How do you determine the expression equal to u ? **Sample answer:** The coefficient of u is the principal square root of the original leading coefficient, and the exponent is half the greatest exponent.
- BL** What must be true about a polynomial expression to write it in quadratic form? **Sample answer:** The exponent of the leading term must be twice the exponent of the middle term, and the third term must be a constant.

Example 7 Solve Equations in Quadratic Form**MP Teaching the Mathematical Practices**

6 State the Meaning of Symbols Guide students to define u in the problem. Point out that it is important to clarify the meaning of the variable.

Questions for Mathematical Discourse

- AL** Why are $\frac{3}{2}$ and -4 not the solutions of the equation? **Sample answer:** $\frac{3}{2}$ and -4 are the values of u , not x . So, you must set each value of u equal to $2x^2$ to find the solutions of the equation.
- OL** Why are there four solutions to the polynomial equation when there were only two factors? **Sample answer:** Each factor generated an equation involving x . When the square root of each side is taken, the plus or minus sign must be used. This creates two solutions per factor.
- BL** What is another first step you could take when solving this equation? **Sample answer:** You could factor 2 out of the equation. The solutions will remain the same.

Exit Ticket**Recommended Use**

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.



3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–27
2	exercises that use a variety of skills from this lesson	28–67
2	exercises that extend concepts learned in this lesson to new contexts	68–71
3	exercises that emphasize higher-order and critical-thinking skills	72–76

ASSESS AND DIFFERENTIATE

 Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, THEN assign:

- Practice Exercises 1–85 odd, 87–91
- Extension: Complex Roots of Polynomial Equations
-  ALEKS® Real Zeros of Polynomial Functions

BL

IF students score 66%–89% on the Checks, THEN assign:

- Practice Exercises 1–91 odd
- Remediation, Review Resources: Solving Quadratic Equations by Factoring
- Personal Tutors
- Extra Examples 1–7
-  ALEKS® Solving Quadratic Equations by Factoring

OL

IF students score 65% or less on the Checks, THEN assign:

- Practice Exercises 1–27 odd
- Remediation, Review Resources: Solving Quadratic Equations by Factoring
- Quick Review Math Handbook: Solving Polynomial Equations
-  ALEKS® Solving Quadratic Equations by Factoring

AL

Practice

Go Online You can complete your homework online.

Examples 1–3

Factor completely. If the polynomial is not factorable, write prime.

$$1. 8z^2 - 27z^3 \quad 2. 64x^4 + x^3$$

$$(2z - 3)(4z^2 + 6z + 9z^3) \quad 4(4x + x^2)(4x^2 - 4x + x^2)$$

$$3. w^2 - w^2z^2 \quad 4. x^2y^2 + y^2$$

$$w^2(1 - z^2) = w^2(1 - z)(1 + z) \quad y^2(x^2 + 1) = y^2(x^2 + 1)$$

$$5. 18x^2 + 5x^3 \quad 6. x^2 - 2x^3$$

$$\text{prime} \quad \text{prime}$$

$$7. 9x^2 - 36x^2 - 63x^2 - 9x^2 + 66x^2 + 36x^2 \quad 8. 12xy^2 - 20xy^2 - 18xy^2 - 10xy^2 + 15xy^2 + 24xy^2$$

$$(x + y)(x + y)(x + y + 3) \quad (5x - 3y)(2x - 2y + 4)$$

$$9. x^2y^2 - 16x^2y + 64y^2 - x^2y^2 + 16x^2y - 64y^2$$

$$(x - 4y)(x + 4y - 8)$$

$$10. 8x^3 - 25x^3 + 30x^3 - x^3 + 20x^3 - 10x^3$$

$$(2x - y)(4x^2 + 2xy + y^2) + 5x^3$$

Example 4

Solve each equation.

$$11. x^2 - 9x + 14z = 0$$

$$0, 7, 2$$

$$12. x^2 = 3x^2$$

$$0, 3$$

$$13. x^2 - 3x - 40z = 0$$

$$0, -5, 8$$

$$14. x^2 - 8x + 15z = 0$$

$$0, 4$$

Example 5

15. **FLUENCY** A modern table is constructed with four legs made of concrete cubes with cube-shaped notches.

a. Define one or more variables and write an expression that represents the volume of one table leg.

Let x represent the length of a side of the leg and y represent the length of the notch. $x^3 - y^3$.

b. Maribel determined that she will use 12,636 cubic inches of concrete to construct the four table legs. If the sides of the notches are 45% of the sides of the legs, how long are the sides of the legs? 15 in.



Lesson 3.2 • Solving Polynomial Equations Algebraically 127

Example 6

Write each expression in quadratic form, if possible.

$$16. x^4 - 12x^2 + 8 \quad 17. -16x^4 + 18x^2 - 4$$

$$(x^2)^2 - 12(x^2) + 8 \quad -16(x^2)^2 + 18(x^2) - 4$$

$$18. 3x^4 + 5x^2 + 7 \quad 19. 5x^4 - 3x^2 + 8$$

$$3(x^2)^2 + 5(x^2) + 7 \quad \text{not possible}$$

$$20. 3x^4 - 2x^2 + 12 \quad 21. 16x^2 + 2x^2 + 6$$

$$(3x^2)^2 - 2(x^2) + 12 \quad 4(4x^2 + x^2 + 1.5)$$

Example 7

Solve each equation.

$$22. x^4 + 5x^2 + 5 = 0$$

$$\pm i\sqrt{5}, \pm i$$

$$23. x^4 - 3x^2 - 10 = 0$$

$$\pm\sqrt{5}, \pm\sqrt{10}$$

$$24. 4x^4 - 16x^2 + 12 = 0$$

$$\pm\sqrt{2}, \pm\sqrt{\frac{3}{2}}$$

$$25. 9x^4 - 37x^2 + 20 = 0$$

$$\pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{4}{3}}$$

$$26. 4x^4 - 5x^2 - 6 = 0$$

$$\pm\sqrt{2}, \pm\sqrt{\frac{3}{2}}$$

$$27. 24x^4 + 16x^2 - 3 = 0$$

$$\pm\sqrt{\frac{1}{3}}, \pm\sqrt{\frac{2}{3}}$$

Mixed Exercises

Factor completely. If the polynomial is not factorable, write prime.

$$28. x^4 - 625 \quad 29. x^4 - 64$$

$$(x^2 + 25)(x^2 - 5) \quad (x + 2)(x - 2 + 4i)(x - 2 - 4i)(x^2 + 2x + 4)$$

$$30. x^4 - 16x \quad 31. 8x^3y^2 - 27x^2y^3$$

$$x(x^3 - 16) = x(x - 2)(x^2 + 2x + 4) \quad x^2y^2(2x - 3y)(4x^2 + 6xy + 9y^2)$$

$$32. x^4 - 4x^3 - 8x^2 + 32x - 16 \quad 33. y^3 - y^2 - 2y^2 + 2y^2 + y^2 - 1$$

$$(x + 2)(x - 2)^2 \quad y(y - 1)(y + 1)(y + 1)$$

14. **DECODE** Each box in a set of decorative storage boxes is a cube.a. The sides of the smallest box are 30% of the length of the sides of the largest box. The volume of the largest box is 7784 cubic inches more than the volume of the smallest box. Define one or more variables, and write an equation that represents the situation. Let x represent the length of the side of the largest box. $x^3 - (0.3x)^3 = 7784$

b. What are the lengths of the sides of the largest and smallest boxes? 20 in., 6 in.

c. Two of the other boxes have a total volume of 2457 cubic inches. The length of the sides of the smaller of these boxes is 75% of those of the larger box. Write and solve an equation to find the lengths of the sides of the boxes. $y^3 + (0.75y)^3 = 2457$; 12 in., 9 in.

128 Module 3 • Polynomial Equations



Solve each equation.

35. $x^4 + x^2 - 30 = 0$
 $\pm\sqrt{1}, \pm\sqrt{10}$

37. $x^4 - 7x^2 + 44 = 0$
 $\pm\sqrt{1}, \pm 2i$

39. $x^4 + 29x^2 = 0$
 $-6, 3 \pm 3\sqrt{3}$

41. $8x^4 + 10x^2 - 3 = 0$
 $\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}$

43. $20x^4 - 53x^2 + 18 = 0$
 $\pm\frac{3}{2}, \pm\frac{3}{2}$

45. $8x^4 - 18x^2 + 4 = 0$
 $\pm\frac{1}{2}, \pm\sqrt{2}$

47. $x^4 + 7x^2 - 8 = 0$
 $1, -2, -\frac{1+\sqrt{17}}{2}, -\frac{1-\sqrt{17}}{2}$

49. $8x^4 + 999x^2 = 125$
 $-\frac{1}{2}, -\frac{1}{2}, \frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}$

51. $x^4 - 5x^2 + 9 = 0$
 $\pm 3, \pm 1, \pm i$

36. $x^4 - 10x^2 + 20 = 0$
 $6, -6, 2\sqrt{10}$

38. $x^4 + 6x^2 - 91 = 0$
 $\pm\sqrt{7}, \pm i\sqrt{13}$

40. $64x^4 + 1 = 0$
 $-\frac{1}{4}, \frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}$

42. $6x^4 - 8x^2 - 8 = 0$
 $\pm\frac{\sqrt{3}}{2}, \pm\frac{\sqrt{3}}{2}, \pm\frac{\sqrt{3}}{2}, \pm\frac{\sqrt{3}}{2}$

44. $18x^4 + 43x^2 - 5 = 0$
 $\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}$

46. $3x^4 - 22x^2 + 45 = 0$
 $3, -3, \pm\frac{\sqrt{3}}{2}, \pm\frac{\sqrt{3}}{2}$

48. $x^4 - 26x^2 - 27 = 0$
 $-1, 3, -\frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}$

50. $8x^4 - 4x^2 - x^2 + 1 = 0$
 $-1, 1, \pm\frac{1}{2}, \pm\frac{1}{2}$

52. $x^4 + 8x^2 + 15 = 0$
 $\pm i\sqrt{5}, \pm i\sqrt{3}$

53. STRUCTURE Consider the equation $x^4 - 8x^2 + 15 = 0$.

a. How are the exponents in the equation related?

Sample answer: $\frac{1}{2}$ and 1 b. How could you define u so that you could rewrite the equation as a quadratic equation in terms of u ? Write the quadratic equation. $u = x^2$, $u^2 - 8u + 15 = 0$

c. Solve the original equation.

31, 625

Lesson 9.2 • Solving Polynomial Equations Algebraically 129

Factor completely. If the polynomial is not factorable, write prime.

54. $2x^4 - 18x^2y + 24xy^2$
 $3x(2x^3 - 6xy + 8y^2)$

56. $x^4 + 7x - 6$
 $(x + 5)(x - 2)$

58. $x^4 + 8x + 7$
 $(x + 7)(x + 1)$

60. $4x^4 - 64$
 $4x^2(x^2 - 16)$

62. $2x^4 + 25$
prime

64. $7x^4 - 14x$
 $7x(x^3 - 2)$

66. $n^4 - 125$
 $(n - 5)(n^3 + 5n + 25)$

55. $5x^4 - 4x^3 - 7$
prime

57. $20x^4 - 6x^2 + 4 - 3$
 $(2x + 1)(x - 3)$

59. $x^4 - 8x - 10$
prime

61. $x^4 - 12x + 36$
 $(x - 6)^2$

63. $x^4 + 25x + 81$
 $(x + 9)^2$

65. $10x^4 - 38x^3$
 $18x^3(x - 2)$

67. $m^4 - 1$
 $(m^2 + 1)(m - 1)(m + 1)$

68. REASONING A rectangular box has dimensions of x inches, $(x + 5)$ inches, and $(x - 2)$ inches. The volume of the box is 30x cubic inches. Find the dimensions of the box. Explain your reasoning. See margin.

69. GEOMETRY The combined volume of a cube and a cylinder is 1000 cubic inches. If the height of the cylinder is twice the radius and the side of the cube is four times the radius, find the radius of the cylinder to the nearest tenth of an inch. 2.4 in.

Higher-Order Thinking Skills

70. ANALYZE Find the solutions of $(x + 3)^4 - 25x + 3x^2 - 8 = 0$. Show your work. See margin.71. WRITE If the equation $ax^2 + bx + c = 0$ has solutions $x = m$ and $x = n$, what are the solutions to $ax^2 + bx^2 + c = 0$? Explain your reasoning. See margin.72. PERSISTENCE Factor $36x^{20} + 12x^4 + 1$. $(6x^4 + 1)^2$ 73. PERSISTENCE Solve $6x - 11\sqrt{3x} + 12 = 0$. $\frac{11}{6}, \frac{1}{6}$ 74. ANALYZE Find a counterexample to the statement $d^2 + d^2 = (d + d)^2$.
Sample answer: $d = 1, d = -1$ 75. CREATE The cubic form of an equation is $ax^3 + bx^2 + cx + d = 0$. Write an equation with degree 4 that can be written in cubic form. Sample answer: $13x^4 + 8x^3 + 8x^2 + 4 = 13x^3 + 40x^2 + 80x + 4$

76. WRITE Explain how the graph of a polynomial function can help you factor the polynomial. See margin.

130 Module 3 • Polynomial Equations

Answers

68. 5 inches by 10 inches by 3 inches; sample answer: Volume equals the product of length, width, and height. An equation to represent the volume is $x^3 + 3x^2 - 10x = 30x$. The solutions of the equation are $x = -8, 0, 5$. Therefore, $x = 5$ and the dimensions of the box are 5 inches, 10 inches, and 3 inches.70. $a = -1, -5, -3 + i\sqrt{2}, -3 - i\sqrt{2}$; sample answer: Use the substitution $u = (a + 3)^2$ and factor to $(u - 4)(u + 2) = 0$. Substitute for u to get $(a + 3)^2 = 4$ and $(a + 3)^2 = -2$.71. $x = \pm \sqrt{m}$ and $x = \pm \sqrt{n}$; sample answer: Because both equations have the same coefficients, take the solutions for the quadratic equation and substitute x^2 for x .76. Sample answer: The factors can be determined by the x -intercepts of the graph. An x -intercept of 5 represents a factor of $(x - 5)$.

Proving Polynomial Identities

LESSON GOAL

Students prove polynomial identities and use them to describe numerical relationships.

1 LAUNCH



Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP



Explore: Polynomial Identities



Develop:

Proving Polynomial Identities

- Transform One Side
- Use Polynomial Identities



You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE



Exit Ticket



Practice

DIFFERENTIATE



View reports of student progress on the **Checks** after each example.

Resources

Remediation: Using the Distributive Property

Extension: Generating Pythagorean Triples

AL	1B		ELL
●	●		●
	●	●	●

Language Development Handbook

Assign page 16 of the *Language Development Handbook* to help your students build mathematical language related to proving polynomial identities.

ELL You can use the tips and suggestions on page T16 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra

Standards for Mathematical Content:

A.APR.4 Prove polynomial identities and use them to describe numerical relationships.

Standards for Mathematical Practice:

6 Attend to precision.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous

Students added, subtracted, and multiplied polynomials. **A.APR.1**

Now

Students prove polynomial identities and use them to describe numerical relationships. **A.APR.4**

Next

Students will evaluate and factor functions by using the Remainder and Factor Theorems. **A.APR.2**

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION



Conceptual Bridge In this lesson, students develop an understanding of polynomial identities, and they build fluency by proving the identities and using them to describe numerical relationships. They apply their understanding by solving real-world problems.

Mathematical Background

An identity is an equation that is satisfied by any numbers that replace the variables. Thus, a polynomial identity is a polynomial equation that is true for any values that are substituted for the variables.



Interactive Presentation

Warm Up

Answer each question for both categories.

Identities	Equations with No Solution
1. Is it an equation?	Is it an equation?
2. Can you swap one side for the other?	Can you swap one side for the other?
3. What type of solution does it have?	What type of solution does it have?
4. What is the actual solution?	What is the actual solution?
5. What is the difference between an equation that is an identity and an equation with no solution?	

Warm Up

Launch the Lesson

A mathematical identity is a mathematical relationship that is always true. Engineers use identities every day. One example they use is the identity that describes the relationship between the sides and hypotenuse of a triangle, the Pythagorean Theorem. Identities allow engineers to look at the different relationships between variables and measurements, because they know that the relationship will be true no matter the specifics of what they are doing.



Launch the Lesson

Vocabulary

Expand All Collapse All

identity

An equation that is true for every value of the variable.

polynomial identity

A polynomial equation that is true for any values that are substituted for the variables.

• If an equation is an identity, what must be true about the expressions on each side of the equation?

• What is an example of a factoring technique that is also a polynomial identity?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- comparing equations with no solution and identities

Answers:

Identities:

- yes
- yes
- a statement that is always true
- all values

Equations with No Solution:

- yes
- no
- a statement that is never true
- Empty
- An identity is true for every value of the variable; an equation with no solution is never true.

Launch the Lesson



Teaching the Mathematical Practices

4 Apply Mathematics Students will apply polynomial identities to a real-world situation. Ask students to identify instances in which engineers may use the Pythagorean Theorem.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Polynomial Identities

Objective

Students use a sketch to explore polynomial identities.

MP Teaching the Mathematical Practices

8 Look for a Pattern Help students to see the pattern in which expressions result in the same value for any value of x and y .

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. They will be presented with different polynomial equations, and will use the sketch to determine which ones generate polynomial identities. Then, students will answer the Inquiry question.

(continued on the next page)

Interactive Presentation

Polynomial Identities

INQUIRY How can you prove that two polynomial expressions form a polynomial identity?

An identity is an equation that is satisfied by any numbers that replace the variables. Use the sketch to complete the table by dragging Point A across the grid and recording the values.

Explore

An identity is an equation that is satisfied by any numbers that replace the variables. Use the sketch to complete the table by dragging Point A across the grid and recording the values.

Explore

WEB SKETCHPAD



Students will use the sketch to explore polynomial identities.

TYPE



Students complete the table by finding the value of each polynomial expression for various values of x and y .

DRAW & DROP



Students drag and drop the corresponding expression to complete the polynomial identities.



Interactive Presentation

INQUIRY How can you prove that two polynomial expressions form a polynomial identity?

Submit

Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

Explore Polynomial Identities (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- To be considered an identity, can two expressions be equal for only certain values of x and y ? Explain. **No; sample answer: They must be equal for all values of x and y , not just certain pairs.**
- Test your coordinates in the expression $x^2 - 2xy + y^2$. Does this appear to make an identity with another expression? If so, which one? **yes; $(x - y)^2$**

Inquiry

How can you prove that two polynomial expressions form a polynomial identity? **Sample answer: I can take one side of the equation and simplify it until it is identical to the other side of the equation.**



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Proving Polynomial Identities

Objective

Students prove polynomial identities and use them to describe numerical relationships.

MP Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. To verify polynomial identities, students must transform one side of the equation until it is identical to the other side.

About the Key Concept

A polynomial identity is a polynomial equation that is true for any values that are substituted for the variables. To verify identities by transforming one side, first simplify one side of the equation until the two sides are the same. Transform that expression into the form of the simpler side.

Example 1 Transform One Side

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. Point out that they should use clear mathematical language when they discuss their answer to the Talk About It! feature.

Questions for Mathematical Discourse

- AL** How do you multiply terms with the same variables? **Sample answer:** Multiply the coefficients and then add the exponents of each like variable.
- OL** How would the polynomial identity have changed if we were given a sum of cubes? **Sample answer:** The identity to prove would now be $(x^3 + y^3)(x - xy + y)$.
- BL** Could you have factored the difference of cubes to show the two sides are equal? Explain. **No; sample answer:** Factoring relies upon the identity and does not prove anything. You must multiply the factors together to prove that it equals the difference of cubes.

Common Error

While most students understand the process of simplifying one side of the polynomial equation, many may make an algebraic mistake. Encourage students to work slowly and check back over their work periodically. If the two sides are not equal then they know they have made a mistake and can try again.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 3-3

Proving Polynomial Identities

Explore: Polynomial Identities

Online Activity Use graphing technology to complete the Explore.

INQUIRY How can you prove that two polynomial expressions form a polynomial identity?

Learn: Proving Polynomial Identities

An **identity** is an equation that is satisfied by any numbers that replace the variables. Thus, a **polynomial identity** is a polynomial equation that is true for any values that are substituted for the variables.

Unlike solving an equation, do not begin by assuming that an identity is true. You cannot perform the same operation to both sides and assume that equality is maintained.

Key Concept Verifying Identities by Transforming One Side:

- Simplify one side of an equation until the two sides of the equation are the same. It is often easier to transform the more complicated expression into the form of the simpler side.
- Factor or multiply expressions as necessary. Simplify by combining like terms.

Example 1 Transform One Side

Prove that $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

Original equation:
 $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Distributive Property:
 $x^3 - y^3 = x(x^2 + xy + y^2) - y(x^2 + xy + y^2)$

Simplify:
 $x^3 - y^3 = x^3 + x^2y + xy^2 - yx^2 - xy^2 - y^3$

Commutative Property:
 $x^3 - y^3 = x^3 - y^3$

Thus:
 Because the expression on the right can be simplified to be the same as the expression on the left, this proves the polynomial identity.

Go Online: You can complete an Extra Example online.

Today's Goal

- Prove polynomial identities and use them to describe numerical relationships.

Today's Vocabulary

identity
polynomial identity

Study Tip

Transforming One Side
 It is often easier to work with the more complicated side of an equation. Look at each side and determine which requires more steps to be simplified. For example, it is often easier to work on the side that involves the square or cube of an algebraic expression.

Talk About It!

If you multiplied each side of the equation by a variable z , would the result still be a polynomial identity? Explain your reasoning.

Yes; sample answer: It would still be an identity because both sides of the equation would be multiplied by the same quantity z . It would be a polynomial identity because it would still consist of multiple algebraic terms.

Lesson 3-3 • Proving Polynomial Identities 131

Interactive Presentation

Interactive Presentation

1. Identity: is an equation that is satisfied by any numbers that replace the variables. Thus, a polynomial identity is a polynomial equation that is true for any values that are substituted for the variables.

KEY CONCEPT: VERIFYING IDENTITIES BY TRANSFORMING ONE SIDE

Step 1: Simplify one side of an equation until the two sides of the equation are the same.

Step 2: Factor or multiply expressions as necessary to simplify the side.

Learn

TAP



Students tap to see a Study Tip.



Math History Minute

Former quarterback Frank Ryan (1936–) earned his Ph.D. in mathematics about six months after he led the Cleveland Browns to the NFL championship game of 1964, where they won 27–0. During part of his academic career, Ryan studied prime numbers, including Oppenheimer's Conjecture that there is a prime number between n^2 and $n^2 + n$, where n is an integer. This work could eventually lead to a polynomial identity that could be used to identify prime numbers.

Use a Source

Research an application of prime numbers. How could a polynomial identity for identifying prime numbers impact the application?

Sample answer: Prime numbers are often used in cryptography. A polynomial identity for identifying prime numbers could make encoding using prime numbers obsolete.

Example 2 Use Polynomial Identities

TRIANGLES Pedro claims that you can always create three lengths that form a right triangle by using the following method: take two positive integers x and y where $x > y$. Two legs of a right triangle are defined as $x^2 - y^2$ and $2xy$. The hypotenuse is defined as $x^2 + y^2$. Is Pedro correct? Explain your reasoning in the context of polynomial identities.

To determine whether Pedro is correct, we can use information about right triangles and the expressions involving x and y to try to construct a polynomial identity. If $x^2 - y^2$ and $2xy$ are the legs of the triangle, and $x^2 + y^2$ is the hypotenuse, then it should be true that $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$.

If this is an identity, you can simplify the expressions for the sides to be the same expression.



$$(x^2 - y^2)^2 + (2xy)^2 \stackrel{?}{=} (x^2 + y^2)^2$$

Original equation

$$x^4 - 2x^2y^2 + y^4 + 4x^2y^2 \stackrel{?}{=} x^4 + 2x^2y^2 + y^4$$

Simplify each term

$$x^4 + 2x^2y^2 + y^4 \stackrel{?}{=} x^4 + 2x^2y^2 + y^4$$

True

Because the identity is true, this proves that Pedro is correct. His process for creating the sides of a right triangle will always work.

Check

Write in the missing explanations to prove that $x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)$.

$$x^4 - y^4 \stackrel{?}{=} (x - y)(x + y)(x^2 + y^2)$$

Original equation

$$x^4 - y^4 \stackrel{?}{=} (x^2 - y^2)(x^2 + y^2)$$

$$x^4 - y^4 \stackrel{?}{=} (x^2 - y^2)(x^2 + y^2)$$

FOIL

$$x^4 - y^4 \stackrel{?}{=} x^4 + x^2y^2 - x^2y^2 - y^4$$

FOIL

$$x^4 - y^4 \stackrel{?}{=} x^4 - y^4$$

Product of a square and a difference

$$x^4 - y^4 \stackrel{?}{=} x^4 - y^4$$

Subtract

Interactive Presentation



Example 2

CHECK



Students complete the Check online to determine whether they are ready to move on.

Example 2 Use Polynomial Identities



Teaching the Mathematical Practices

1 Understand the Approaches of Others Mathematically proficient students can explain the solution methods of others.

Example 2 asks students to justify the reasoning of Pedro by verifying a polynomial identity.

Questions for Mathematical Discourse

AL What does the Pythagorean Theorem state? **Sample answer:** $a^2 + b^2 = c^2$

OL For Pedro's method to work, why must $x > y$? **Sample answer:** If x is not greater than y , then $x^2 - y^2$ could give a side length of a non-positive number.

BL If x and y are not restricted to positive numbers, what would need to be true for this method to work? **Sample answer:** x and y would need to have the same sign, and $|x| > |y|$.

Common Error

Students may try to "distribute" the square over the plus or minus sign of a binomial, meaning they will only square the two terms. Remind students what squaring means and then have students write out the binomial twice to FOIL.

DIFFERENTIATE

Reteaching Activity **AL** **ELL**

IF students are struggling to prove polynomial identities, THEN have students write each side of the equation on two different note cards. This breaks the equation into two expressions. Students simplify the more complicated expression to see if it matches the expression on the other notecard.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–10
2	exercises that use a variety of skills from this lesson	11–15
2	exercises that extend concepts learned in this lesson to new contexts	16–20
3	exercises that emphasize higher-order and critical thinking skills	21–25

ASSESS AND DIFFERENTIATE

Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign:

- Practice Exercises 1–21 odd, 22–26
- Extension: Generating Pythagorean Triples

IF students score 66%–89% on the Checks, **THEN** assign:

- Practice Exercises 1–26 odd
- Remediation, Review Resources: Using the Distributive Property
- Personal Tutors
- Extra Examples 1–2
- ALEKS** Equations With No Solutions vs. Identities

IF students score 65% or less on the Checks, **THEN** assign:

- Practice Exercises 1–6 odd
- Remediation, Review Resources: Using the Distributive Property
- Quick Review Math Handbook*: Polynomial Identities
- ALEKS** Equations With No Solutions vs. Identities

Practice

Go Online You can complete your homework online.

Example 1

Prove each polynomial identity. 1–8: See margin.

- $(x - y)^2 = x^2 - 2xy + y^2$
- $(x + 5)^2 = x^2 + 10x + 25$
- $4(x - 7)^2 = 4x^2 - 56x + 196$
- $(2x^2 + y^2)^2 = (2x^2 - y^2)^2 + (2xy)^2$
- $a^2 - b^2 = (a + b)(a - b)$
- $x^4 + y^4 = (x + y)(x^3 - xy^2 + y^3)$
- $a^4 - a^2 = (a - a^2)(a + a^2)$
- $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$

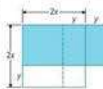
Example 2

9. SQUARES Apurvi claims that you can find the area of a square, using the following method: take two positive integers x and y . The side length of the square is defined by the expression $3x + y$. The area of the square is defined by the expression $(3x + y)^2 = 6xy + y^2$. Is Apurvi correct? Explain your reasoning in the context of polynomial identities.

See margin.

10. USE A MODEL Julio claims that you can find the area of a rectangle using the following method: take two positive integers x and y , where $x > y$. The side lengths of the rectangle are defined by the expressions $2x + y$ and $2x - y$. The area of the rectangle is defined by the expression $4x^2 - y^2$. Is Julio correct? Explain your reasoning in the context of polynomial identities.

See margin.



Mixed Exercises

Determine whether each equation is an identity.

- $(x + 3)^2(x^2 + 3x^2 + 3x + 1) = (x^2 + 6x + 9)(x^2 + 3x + 1)$ **identity**
- $(x + 2)(x + 1)^2 = (x^2 + 3x + 2)(x + 1)$ **identity**
- $(x + 3)(x - 1)^2 = (x^2 - 2x - 3)(x - 1)$ **not an identity**
- $(x + 2)^2(x^3 - 3x^2 + 3x - 1) = (x^2 + 4x + 4)(x - 1)^2$ **identity**
- $(a + b)^2 = a^2 - 2ab + b^2$ **not an identity**



16. **USE TOOLS** Consider the following equation.

$$(x - 2)^2(x^2 + 9x^2 + 27x + 27) = (x^2 - 4x + 4)(x + 3)$$

a. Evaluate the expressions for each value. Copy and complete the table.

x	$(x - 2)^2(x^2 + 9x^2 + 27x + 27)$	$(x^2 - 4x + 4)(x + 3)$
0	508	108
1	64	64
2	0	0
3	216	216
4	1372	1372

- b. What conclusion can you make about the equation, based on the results in your table? Explain. The equation may be a polynomial identity because the left side of the equation equals the right side of the equation for each x -value in the table.
- c. How can you prove your conclusion from part b? **Sample answer:** Transform one side of the equation to determine if it is a polynomial identity.

USE TOOLS Use a computer algebra system (CAS) to prove each identity.

17. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ **See Mod. 3 Answer Appendix.**

18. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ **See Mod. 3 Answer Appendix.**

19. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ **See Mod. 3 Answer Appendix.**

20. $(a + b)^2(x - y)^2 = (a^2 - 3x - 4b^2 - 3x^2 + 8x + 36)$ **See Mod. 3 Answer Appendix.**

Higher-Order Thinking Skills

21. **WRITE** Explain the meaning of polynomial identity and summarize the method for proving an equation is a polynomial identity. **See Mod. 3 Answer Appendix.**

22. **CREATE** Write and solve a system of equations using the identity $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$ to find the values of x and y that make a 3, 4, 5 Pythagorean triple. **See Mod. 3 Answer Appendix.**

23. **ANALYZE** Refer to Example 2. Notice that Pedro says x and y must be positive integers and x must be greater than y . Explain why these restrictions are necessary. **See Mod. 3 Answer Appendix.**

24. **PERSEVERE** Rebecca has a square garden with side length a that she wants to transform into a rectangle. Rebecca speculates that if she subtracts the same length b from one dimension of the garden and adds it to the other dimension the new rectangle's area will be smaller than the original garden in the amount of b^2 . Draw a diagram and show algebraically that Rebecca is correct. **See Mod. 3 Answer Appendix.**

25. **FIND THE ERROR** George is proving the identity $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ by simplifying the right side. His work is shown. Is George correct? If not, identify and correct his error. **See Mod. 3 Answer Appendix.**

$$\begin{aligned} (a + b)(a^2 - ab + b^2) &= a^3 - a^2b + ab^2 - ab^2 + ab^2 + b^3 \\ &= a^3 - 2a^2b + b^3 \end{aligned}$$

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Answers

$$\begin{aligned} 1. (x - y)^2 &\stackrel{?}{=} x^2 - 2xy + y^2 \\ x^2 - 2xy + y^2 &\stackrel{?}{=} x^2 - 2xy + y^2 \end{aligned}$$

Original equation
Distributive Property

$$\begin{aligned} 2. (x + 5)^2 &\stackrel{?}{=} x^2 + 10x + 25 \\ x^2 + 10x + 25 &= x^2 + 10x + 25 \end{aligned}$$

Original equation
Distributive Property

$$\begin{aligned} 3. 4(x - 7)^2 &\stackrel{?}{=} 4x^2 - 56x + 196 \\ 4(x^2 - 14x + 49) &\stackrel{?}{=} 4x^2 - 56x + 196 \\ 4x^2 - 56x + 196 &= 4x^2 - 56x + 196 \end{aligned}$$

Original equation
Distributive Property
Distributive Property

$$\begin{aligned} 4. (2x^2 + y)^2 &\stackrel{?}{=} (2x^2 + y)(2x^2 + y) \\ (2x^2 + y)^2 &\stackrel{?}{=} (2x^2 + y)(2x^2 + y) \\ (2x^2 + y)^2 &\stackrel{?}{=} 4x^4 + 4xy^2 + 8x^2y + 4xy^2 + y^2 \\ (2x^2 + y)^2 &\stackrel{?}{=} 4x^4 + 8x^2y + 8xy^2 + y^2 \end{aligned}$$

Original equation
Evaluate the exponent
Distributive Property
Combine like terms.
Factor.

$$\begin{aligned} 5. a^2 - b^2 &\stackrel{?}{=} (a + b)(a - b) \\ a^2 - b^2 &\stackrel{?}{=} a^2 - ab + ab - b^2 \\ a^2 - b^2 &\stackrel{?}{=} a^2 - b^2 \end{aligned}$$

Original equation
FOIL
Simplify.

$$\begin{aligned} 6. x^2 + y^2 &\stackrel{?}{=} (x + y)(x - xy + y) \\ x^2 + y^2 &\stackrel{?}{=} x^2 - xy^2 + xy^2 + y^2 - xy + y^2 \\ x^2 + y^2 &\stackrel{?}{=} x^2 + y^2 \end{aligned}$$

Original equation
Distributive Property
Simplify

$$\begin{aligned} 7. p^4 - q^4 &\stackrel{?}{=} (p - q)(p + q)(p^2 + q^2) \\ p^4 - q^4 &\stackrel{?}{=} (p - q)(p^2 + q^2) \end{aligned}$$

Original equation
Difference of two squares

$$p^4 - q^4 = (p^2 - q^2)(p^2 + q^2)$$

Difference of two squares

$$\begin{aligned} 8. a^5 - b^5 &\stackrel{?}{=} (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4) \\ a^5 - b^5 &\stackrel{?}{=} a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 - ab^4 - a^4b - a^3b^2 - a^2b^3 - ab^4 - b^5 \\ a^5 - b^5 &\stackrel{?}{=} a^5 - b^5 \end{aligned}$$

Distributive Property
Simplify.

$$\begin{aligned} 9. (3x + y)^2 &\stackrel{?}{=} 9x^2 + 6xy + y^2 \\ 9x^2 + 6xy + y^2 &\stackrel{?}{=} 9x^2 + 6xy + y^2 \\ 9x^2 + 6xy + y^2 &\stackrel{?}{=} 9x^2 + 6xy + y^2 \end{aligned}$$

Original equation
Square the left side.
True

Sample answer: Because the identity is true, this proves that Aponi is correct. Her process for finding the area of a square will always work.

$$\begin{aligned} 10. (2x + y)(2x - y) &\stackrel{?}{=} 4x^2 - y^2 \\ 4x^2 - 2xy + 2xy - y^2 &\stackrel{?}{=} 4x^2 - y^2 \\ 4x^2 - y^2 &\stackrel{?}{=} 4x^2 - y^2 \end{aligned}$$

Original equation
Use the FOIL Method.
True

Sample answer: Because the identity is true, this proves that Julio is correct. His process for finding the area of a rectangle figure will always work.

The Remainder and Factor Theorems

LESSON GOAL

Students evaluate and factor functions by using the Remainder and Factor Theorems.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Explore:** Remainders

 **Develop:**

The Remainder Theorem

- Synthetic Substitution
- Apply the Remainder Theorem

The Factor Theorem

- Use the Factor Theorem

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources

Remediation: Dividing Polynomials

Extension: Radical Notation

ALC	BL	EL
●	●	●
	●	●

Language Development Handbook

Assign page 17 of the *Language Development Handbook* to help your students build mathematical language related to evaluating and factoring functions by using the Remainder and Factor Theorems.

 You can use the tips and suggestions on page T17 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	0.5 day
45 min	1 day

Focus

Domain: Algebra

Standards for Mathematical Content:

A.APR.2 Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a , the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

Standards for Mathematical Practice:

- 3 Construct viable arguments and critique the reasoning of others.
- 4 Model with mathematics.

Coherence

Vertical Alignment

Previous

Students divided polynomials by using long division and synthetic division.
A.APR.6

Now

Students evaluate and factor functions by using the Remainder and Factor Theorems.
A.APR.2

Next

Students will determine the number and types of roots, find zeros, and use zeros to graph polynomial equations.
N.CN.9, A.APR.3, F.IF.7c

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

 **Conceptual Bridge** In this lesson, students build on their understanding of polynomial division. They build fluency by using the Remainder and Factor Theorems, and they apply their understanding by solving real-world problems.

Mathematical Background

The Remainder Theorem says that the value of $f(a)$ is the same as the remainder when the polynomial is divided by $x - a$. The Factor Theorem is a special case of the Remainder Theorem. It says: If $f(a)$ has a value of 0, then $x - a$ is a factor of the polynomial.



Interactive Presentation

Warm Up

Use synthetic division to find each quotient.

- $2x^3 + 7x^2 + 4x^3 - x^2 + 6x + 6; x + 1$
- $x^3 - 4x^2 - 7x^2 + 3x + 6; x - 2$

Show Answers

Warm Up

Launch the Lesson

As smartphones and other mobile devices continue to grow in popularity, Americans are buying more products using these devices. In 2015, nearly \$50 billion in transactions were made using mobile devices. The growth in this sector can be modeled by a cubic function. You can use the Remainder Theorem to estimate the amount that Americans spent during any given year.



Launch the lesson

Vocabulary

Expand All Collapse All

synthetic substitution

The process of using synthetic division to find a value of a polynomial function.

depressed polynomial

A polynomial resulting from division with a degree one less than the original polynomial.

- What two methods can you use to find $f(2)$ of a polynomial function?
- How can you evaluate and determine factors of polynomials using synthetic substitution?
- How does synthetic division relate to synthetic substitution?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- using synthetic division

Answers:

$$1. 2x^4 - 5x^3 + x^2 + 6$$

$$2. x^3 - 2x^2 - 11x + 19 - \frac{32}{x-2}$$

Launch the Lesson



Teaching the Mathematical Practices

4 Apply Mathematics Encourage students to consider how smartphone and mobile device growth can be modeled by a cubic function, and how the function can be used to estimate values.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Remainders

Objective

Students use a sketch to explore the divisor and quotient of a polynomial when the remainder is zero.

Teaching the Mathematical Practices

3 Construct Arguments In the Explore, students will use previously established results to make a conjecture about the divisor and quotient of a polynomial when the remainder is 0.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will move through a series of exercises using the sketch to determine how the quotient, divisor and remainder are related. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Inquiry How are the divisor and quotient of a polynomial related to its factors when the remainder is zero?

Move the sliders to create different divisors and dividends, and then complete the exercises.

Explore

divisor

dividend

$$\begin{array}{r}
 x + (2) + R 7 \\
 x + (-3) \overline{) x^2 + (-1)x + (1)} \\
 \underline{(-) x^2 + (-3)x} \\
 (2)x + (1) \\
 \underline{(-) (2)x + (-6)} \\
 7
 \end{array}$$

Explore

WEB SKETCHPAD



Students use the sketch to explore polynomial division.

TYPE



Students answer questions about polynomial division.



Interactive Presentation

INQUIRY How are the divisor and quotient of a polynomial related to its factors when the remainder is zero?

Submit

Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

Explore Remainders (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- If $x^2 - 9$ were divided by $x - 3$, would the remainder be zero? Explain.
Yes; sample answer: $x - 3$ is a factor of $x^2 - 9$.
- Robert says he divided $x^2 - 7x + 12$ by a binomial and got a remainder of zero. By what binomial could Robert have divided? $x - 4$ or $x - 3$



Inquiry

How are the divisor and quotient of a polynomial related to its factors when the remainder is zero? Sample answer: When the remainder is zero, the divisor and quotient are factors of the polynomial.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn The Remainder Theorem

Objective

Students evaluate functions by using synthetic substitution.

MP Teaching the Mathematical Practices

- 1 Explain Correspondences** Guide students to analyze the relationship between the polynomial division and evaluating a function for a specific value.

Important to Know

The Remainder Theorem says that polynomial division can be used to find a function value. For any polynomial, the function value of the polynomial at $x = a$ is the value of the remainder when the polynomial is divided by $x - a$. This can be found using synthetic substitution.

Common Misconception

Some students may think that the remainder must be zero. Not all polynomials divide evenly, just like all numbers do not divide evenly. Remind students that the remainder can be any real number.

DIFFERENTIATE

Enrichment Activity

The remainder when $x^4 - 2x^2 - kx + 3$ is divided by $x + 3$ is 36. What is the value of k ?

$$k = 10$$

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 3-4

The Remainder and Factor Theorems

Explore: Remainders

Online Activity Use the interactive tool to complete the Explore.

INQUIRY How are the divisor and quotient of a polynomial related to its factors when the remainder is zero?

Learn: The Remainder Theorem

Polynomial division can be used to find the value of a function. From the Division Algorithm, we know that $\frac{p(x)}{q(x)} = q(x) + \frac{r(x)}{q(x)}$ and that $p(x) = q(x) \cdot q(x) + r(x)$, where q and r are unique and the degree of r is less than the degree of q . Suppose we were to call the dividend $p(x)$ and the divisor $x - a$. Then the Division Algorithm would be $\frac{p(x)}{x - a} = q(x) + \frac{r(x)}{x - a}$ and $p(x) = q(x) \cdot (x - a) + r$ where q is a constant and r is the remainder. Since any polynomial can be written in this form, evaluating $p(x)$ at a gives the following:

$$\begin{aligned} p(a) &= q(a) \cdot (a - a) + r \\ p(a) &= q(a) \cdot 0 + r \\ p(a) &= q(a) \cdot 0 + r \\ p(a) &= r \end{aligned}$$

Polynomial function $p(x)$

Substitute a for x .

$0 - 0 = 0$

$q(a) \cdot 0 = 0$

This shows how the Remainder Theorem can be used to evaluate a polynomial at $p(a)$.

Key Concept • Remainder Theorem

Words: For a polynomial $p(x)$ and a number a , the remainder upon division by $x - a$ is $p(a)$.

Example: Evaluate $p(x) = x^2 - 4x + 7$ when $x = 5$.

Synthetic division

$$\begin{array}{r|rrrr} 5 & 1 & -4 & 7 & \\ & & 5 & 5 & \\ \hline & 1 & 1 & 12 & \\ p(5) = 12 \end{array}$$

Direct substitution

$$\begin{aligned} p(x) &= x^2 - 4x + 7 \\ p(5) &= 5^2 - 4(5) + 7 \\ p(5) &= 12 \end{aligned}$$

Interactive Presentation

The Remainder Theorem

Polynomial division can be used to find the value of a function. From the Division Algorithm, we know that $\frac{p(x)}{q(x)} = q(x) + \frac{r(x)}{q(x)}$ and that $p(x) = q(x) \cdot q(x) + r(x)$, where q and r are unique and the degree of r is less than the degree of q . Suppose we were to call the dividend $p(x)$ and the divisor $x - a$. Then the Division Algorithm would be $\frac{p(x)}{x - a} = q(x) + \frac{r(x)}{x - a}$ and $p(x) = q(x) \cdot (x - a) + r$ where q is a constant and r is the remainder. Since any polynomial can be written in this form, evaluating $p(x)$ at a gives the following:

$$\begin{aligned} p(a) &= q(a) \cdot (a - a) + r \\ p(a) &= q(a) \cdot 0 + r \\ p(a) &= q(a) \cdot 0 + r \\ p(a) &= r \end{aligned}$$

Polynomial function $p(x)$
Substitute a for x .
 $0 - 0 = 0$
 $q(a) \cdot 0 = 0$

Learn



Learn The Factor Theorem

Objective

Students determine factors of polynomials by using the Factor Theorem.

MP Teaching the Mathematical Practices

3 Analyze Cases Guide students to examine how the Factor Theorem is related to the Factor Theorem for the case where $p(a) = 0$.

Important to Know

When a binomial divides evenly into a polynomial, the binomial is a factor. A depressed polynomial is the quotient of the division and has a degree one less than the original polynomial.

Essential Question Follow-Up

Students have begun learning about the Remainder and Factor Theorems.

Ask:

Why are the Remainder and Factor Theorems important when solving polynomial equations? **Sample answer:** They tell you whether a binomial is a factor of a polynomial, which helps you find the roots.

Example 3 Use the Factor Theorem

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write and explain their solution methods. The Talk About It! feature asks students to explain the steps they would take to solve a problem.

Questions for Mathematical Discourse

- A** What is a depressed polynomial? the quotient when a binomial evenly divides a polynomial
- O** Without using synthetic division, how could you determine that $x + 8$ is a factor of $2x^3 + 15x^2 - 9x - 24$? **Sample answer:** Substitute -8 for x and see if the result is zero.
- B** When might you want to use the Factor Theorem? **Sample answer:** When you are given a potential factor and need to quickly test several possible factors, or are only interested in whether a specific binomial is a factor. You can use synthetic division to verify the factor and simultaneously determine the depressed polynomial. You can then potentially factor the depressed polynomial to find the other factors of the original expression.

Check

KITTENS The ideal weight of a kitten in pounds is modeled by the function $f(x) = 0.009x^2 + 0.127x + 0.377$, where x is the age of the kitten in weeks. Determine the ideal weight of a 9-week-old kitten. Round to the nearest tenth.

T pounds
2.3

Learn The Factor Theorem

When a binomial evenly divides a polynomial, the binomial is a factor of the polynomial. The quotient of this division is called a depressed polynomial. The **depressed polynomial** has a degree that is one less than the original polynomial.

A special case of the Remainder Theorem is called the Factor Theorem.

Key Concept - Factor Theorem

Words: The binomial $x - a$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0$.

Example:

dividend	quotient	divisor	remainder
$x^3 - x^2 - 30x + 72$	$(x^2 - 7x + 12)$	$(x + 6)$	0
$x + 6$ is a factor of $x^3 - x^2 - 30x + 72$.			

Example 3 Use the Factor Theorem

Show that $x + 8$ is a factor of $2x^3 + 15x^2 - 11x - 24$. Then find the remaining factors of the polynomial.

-8	2	15	-11	-24
		-16	8	24
	2	-1	-3	0

Because the remainder is 0, $x + 8$ is a factor of the polynomial by the Factor Theorem. So $2x^3 + 15x^2 - 11x - 24$ can be factored as $(x + 8)(2x^2 - x - 3)$. The depressed polynomial is $2x^2 - x - 3$. Check to see if this polynomial can be factored.

$$2x^2 - x - 3 = (2x - 3)(x + 1) \quad \text{Factor the trinomial.}$$

$$\text{Therefore, } 2x^3 + 15x^2 - 11x - 24 = (x + 8)(2x - 3)(x + 1).$$

Lesson 3-4 • The Remainder and Factor Theorems 137

Interactive Presentation

The Factor Theorem

When a binomial evenly divides a polynomial, the binomial is a factor of the polynomial. The quotient of this division is called a depressed polynomial. The **depressed polynomial** has a degree that is one less than the original polynomial.

A special case of the Remainder Theorem is called the Factor Theorem.

Key Concept: Factor Theorem

Words: The binomial $x - a$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0$.

Example	dividend	quotient	divisor	remainder
	$x^3 - x^2 - 30x + 72$	$(x^2 - 7x + 12)$	$(x + 6)$	0
$x + 6$ is a factor of $x^3 - x^2 - 30x + 72$.				

Learn

MULTIPLE CHOICE



Students select features of polynomials based on specific quotients and remainders.

**Talk About It**

Suppose you were asked to determine whether $3x + 4$ is a factor of $3x^3 - 2x^2 - 8x$. Describe the steps necessary to find a solution.

Sample answer: First, divide the factor and the polynomial by 3 so that the x -term of the factor is in the form $x - a$. In this case, $a = -\frac{4}{3}$. Then, use synthetic division to determine the remainder. If the remainder is 0, then $3x + 4$ is a factor.

Check

Select all of the factors of $3x^3 + 10x^2 - 27x - 10$. **A, B, E**

- A. $x - 2$
 B. $x + 5$
 C. $x + 9$
 D. $x - 10$
 E. $3x + 1$
 F. $3x - 10$

Pause and Reflect

Did you struggle with anything in this lesson? If so, how did you deal with it?

See students' observations.

Practice**Example 1**

Use synthetic substitution to find $f(-5)$ and $f(2)$ for each function.

1. $f(x) = x^3 + 2x^2 - 3x + 1$ -59; 11	2. $f(x) = x^3 - 8x + 6$ 71; -6
3. $f(x) = 3x^4 + x^3 - 2x^2 + x + 12$ 1707; 62	4. $f(x) = 2x^3 - 8x^2 - 2x + 5$ -435; -15
5. $f(x) = x^3 - 5x + 2$ -98; 0	6. $f(x) = x^3 + 8x^2 + 2x - 15$ -4150; 85
7. $f(x) = x^4 - 4x^3 + 3x^2 - 10$ 13,190; 2	8. $f(x) = x^3 - 6x - 8$ 647; -4

See Online You can't complete an Extra Example online.

138 Module 3 • Polynomial Equations

Interactive Presentation**Question 1**

Select all of the factors of $3x^3 + 10x^2 - 27x - 10$.

- ☐ A) $x - 2$
☐ B) $x + 5$
☐ C) $x + 9$
☐ D) $x - 10$
☐ E) $3x + 1$
☐ F) $3x - 10$

Example 3

CHECK

Students complete the Check online to determine whether they are ready to move on.

1 CONCEPTUAL UNDERSTANDING**2 FLUENCY****3 APPLICATION****Common Error**

When students are given a potential factor of a polynomial, they often forget to use the opposite sign when performing synthetic division. Remind students that factors are in the form $x - a$ so the sign of a is opposite of the factor.

DIFFERENTIATE**Language Development Activity**

Intermediate Use an interactive whiteboard to project an example and mark text during reading to reinforce steps and concepts. Use different colors to represent the different steps in the process.

Advanced Have students write a paragraph summarizing the sequence of steps to solving a problem using synthetic substitution. They could create a numbered flowchart to help organize their paragraphs.

Advanced High Instruct collaborative groups to create a pamphlet illustrating the sequence of steps. Have students share their pamphlets with the class.

Exit Ticket**Recommended Use**

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–44
2	exercises that use a variety of skills from this lesson	45–60
3	exercises that emphasize higher-order and critical-thinking skills	61–67

ASSESS AND DIFFERENTIATE

 Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, THEN assign:

- Practice Exercises 1–65 odd, 67–75
- Extension: Radical Notation
-  **ALEKS** Polynomial Division; Remainder and Factor Theorems

IF students score 66%–89% on the Checks, THEN assign:

- Practice Exercises 1–75 odd
- Remediation, Review Resources: Dividing Polynomials
- Personal Tutors
- Extra Examples 1–3
-  **ALEKS** Synthetic Division

IF students score 65% or less on the Checks, THEN assign:

- Practice Exercises 1–19 odd
- Remediation, Review Resources: Dividing Polynomials
- Quick Review Math Handbook: The Remainder and Factor Theorems*
-  **ALEKS** Synthetic Division

Use synthetic substitution to find $f(2)$ and $f(-1)$ for each function.

- $f(x) = x^2 + 6x + 5$
21, 0
- $f(x) = x^2 - x + 1$
3, 3
- $f(x) = x^2 - 2x - 2$
-2, 1
- $f(x) = x^2 - x^2 - 2x + 3$
3, 3
- $f(x) = x^2 - 3x^2 + x - 2$
-4, -7
- $f(x) = x^2 + 2x^2 - 9$
15, -6
- $f(x) = x^2 - 3x^2 - 4x + 10$
-22, 20
- $f(x) = x^2 - x^2 - x + 1$
3, 3
- $f(x) = x^2 + 6x^2 + x - 4$
30, 0
- $f(x) = x^2 - 3x^2 + x + 6$
-8, 1
- $f(x) = x^2 - 3x^2 + 2x^2 - 2x + 6$
2, 14
- $f(x) = x^2 - 2x^2 + x^2 + x^2 - 3x^2 - 20$
-32, -26

Example 2

21. **BUSINESS** Advertising online generates billions of dollars for global businesses each year. The revenue from online advertising in the United States from 2000 to 2015 can be modeled by $y = 0.01x^3 + 0.02x^2 + x + 6$, where x is the number of years since 2000 and y is the revenue in billions of U.S. dollars.

- Estimate the revenue from online advertising in 2008.
- Predict the revenue from online advertising in 2022.

22. **PROFIT** The profit, in thousands, of Cypko's Corporation can be modeled by $P(x) = x^3 - 6x^2 + 2x^2 + 10x - 200$, where x is the number of years after the business was started. Predict the profit of Cypko's Corporation after 10 years.

Example 2

Given a polynomial and one of its factors, find the remaining factors of the polynomial.

- $x^3 - 3x^2 + 2x + 2$
 $(x - 1)^2$
- $x^3 + 2x^2 - 8x - 16$
 $x - 2, x^2 + 2x + 4$
- $x^3 - x^2 - 10x - 8$
 $x - 4, x + 1$
- $x^3 - x^2 - 5x - 6$
 $(x + 1)^2$
- $2x^3 + 17x^2 + 23x - 42$
 $x + 6, 2x + 3$
- $2x^3 + 7x^2 - 53x - 28$
 $x + 7, 2x + 1$
- $x^3 + 2x^2 + 2x - 3$
 $x + 1, 2x + 3$
- $x^3 + 2x^2 - 3x - 2$
 $x - 1, x + 1$

Lesson 3-4 • The Remainder and Factor Theorems 139

- $x^3 + 2x^2 - x - 2$
 $x - 1, x + 2$
- $x^3 + x^2 - 5x + 3$
 $x - 1, x + 3$
- $x^3 + 3x^2 - 4x - 12$
 $x - 2, x + 2$
- $x^3 - 6x^2 + 11x - 6$
 $x - 1, x - 2$
- $x^3 + 2x^2 - 33x - 90$
 $x + 3, x - 6$
- $x^3 - 6x^2 + 32x - 4$
 $x - 4, x + 2$
- $x^3 - x^2 - 10x - 8$
 $x + 1, x - 4$
- $x^3 - 19x + 30$
 $x + 5, x - 3$
- $2x^3 + x^2 - 2x - 1$
 $x - 1, 2x - 1$
- $2x^3 + x^2 - 5x + 2$
 $x - 1, 2x - 1$
- $3x^3 + 4x^2 - 5x - 2$
 $x - 1, x + 3$
- $3x^3 + x^2 + x - 2$
 $x^2 + x + 1$
- $16x^3 - 32x^2 - 81x + 162$
 $x - 2, 2x + 3, 4x + 9$

Mixed Exercises

45. **REASONING** Jessica evaluates the polynomial $p(x) = x^3 - 5x^2 + 3x + 5$ for a factor using synthetic substitution. Some of her work is shown below. Find the values of a and b .

a	1	-5	3	5
	11	66	759	
	1	6	69	b

46. **STATE YOUR ASSUMPTION** The revenue from streaming music services in the United States from 2005 to 2016 can be modeled by $y = 0.26x^3 - 7.48x^2 + 79.20x + 333.33x^3 + 481.68x + 99.13$, where x is the number of years since 2005 and y is the revenue in millions of U.S. dollars.

- Estimate the revenue from streaming music services in 2015.
- What might the revenue from streaming music services be in 2020? What assumption did you make to make your prediction?

47. **NATURAL EXPONENTIAL FUNCTION** The natural exponential function $y = e^x$ is a special function that is applied in many fields such as physics, biology, and economics. It is not a polynomial function, however, for small values of x , the value of e^x is very closely approximated by the polynomial function $f(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1$. Use synthetic substitution to determine $f(0.9)$. See margin.

Find values of k so that each remainder is 3.

48. $(k^2 - x + k) \div (x - 1)$

49. $(k^2 + kx - 17) \div (x - 2)$

50. $(k^2 + 5x + 7) \div (x + k)$

51. $(k^2 + 4x^2 + x + k) \div (x + 2)$

52. If $(k-8) = 0$ and $f(x) = x^3 - x^2 - 5x + 112$, find all the zeros of $f(x)$ and use them to graph the function. See margin.53. REASONING If $P(x) = 0$ and $P(x) = 10x^3 + kx^2 - 18x + 3$, find all the factors of $P(x)$ and use them to graph the function. Explain your reasoning. See margin.54. GEOMETRY The volume of a box with a square base is $V(x) = 2x^3 + 15x^2 + 36x + 27$ if the height of the box is $(2x + 3)$ units, what are the measures of the sides of the base in terms of x ? See margin.55. SPORTS The average value of a function in the National Football League from 2000 to 2008 can be modeled by $y = -0.032x^3 + 1.608x^2 - 24.904x + 145.100x^2 - 207.594x + 482.098$, where x is the number of years since 2000 and y is the value in millions of U.S. dollars.

a. Copy and complete the table of estimated values. Round to the nearest million.

Year	2002	2003	2004	2005	2006	2007	2008
Estimated Average Franchise Value (millions \$)	621	1997	1740	15,255			

b. What assumption did you make to make your predictions? Do you think the assumption is valid? Explain. The model still represents the situation after 25 years; no, the average value is unlikely to fall so quickly.

56. CONSTRUCT ARGUMENTS Divide the polynomial function $f(x) = 4x^3 - 10x^2 + 8$ by the factor $(x + 5)$. Then state and confirm the Remainder Theorem for this particular polynomial function and factor. See margin.57. REGULARITY The polynomial function $P(x)$ is symmetric in the y -axis and contains the point $(2, -5)$. What is the remainder when $P(x)$ is divided by $(x + 2)$? Explain your reasoning. See margin.58. STRUCTURE Verify the Remainder Theorem for the polynomial $x^3 + 3x + 5$ and the factor $(x - 5)$ by first using synthetic division and then evaluating for $x = 5$. See Mod. 3 Answer Appendix.

Lesson 3-4 • The Remainder and Factor Theorems 541

59. STRUCTURE If $(x + 6)$ is a factor of $kx^2 + 15x^2 + 12x - 30$, determine the value of k . Factor the polynomial and confirm the result graphically. See Mod. 3 Answer Appendix.60. REGULARITY Polynomial $f(x)$ is divided by $x - c$. What can you conclude if:

- the remainder is 0? $x - c$ is a factor of $f(x)$.
- the remainder is 1? $x - c$ is not a factor of $f(x)$.
- the quotient is 1 and the remainder is 0? $f(x) = x - c$.

Higher-Order Thinking Skills

61. CREATE Write a polynomial function that has a double zero of 1 and a double zero of -5 . Graph the function. Sample answer: $f(x) = (x-1)^2(x+5)^2$. See Mod. 3 Answer Appendix for graph.62. PERSISTENCE For a cubic function $P(x)$, $P(2) = -90$, $P(-8) = 0$, and $P(5) = 0$.a. Write two possible equations for $P(x)$. Explain your answer. See Mod. 3 Answer Appendix.

b. Graph your equations from part a. What three points do these graphs have in common? See Mod. 3 Answer Appendix.

c. If $P(4) = 60$, write the equation for $P(x)$. See Mod. 3 Answer Appendix.

63. ANALYZE Review the definition for the Factor Theorem. Provide a proof of the theorem. See Mod. 3 Answer Appendix.

64. CREATE Write a cubic function that has a remainder of 8 for $f(x)$ and a remainder of -5 for $g(x)$. Sample answer: $f(x) = -x^3 + x^2 + x + 30$.65. PERSISTENCE Show that the quartic function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ will always have a rational zero when the numbers 1, -2 , 3, 4, and -6 are randomly assigned to replace a through e , and all of the numbers are used. See Mod. 3 Answer Appendix.

66. WRITE Explain how the zeros of a function can be located by using the Remainder Theorem and making a table of values for different input values and then comparing the remainders. See Mod. 3 Answer Appendix.

67. FIND THE ERROR The table shows x -values and their corresponding $P(x)$ values for a polynomial function. Tyrone and Nia used the Factor Theorem to find factors of $P(x)$. Is either of them correct? Explain your reasoning. See Mod. 3 Answer Appendix.

x	-3	-1	0	1	2	4
$P(x)$	-18	0	6	2	0	122
Tyrone ($x + 1$) and ($x - 2$)						
Nia ($x + 1$)						

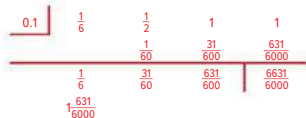
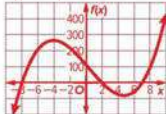
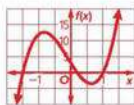
1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Answers

47.

52. $-8, 7, 2$; sample answer: By the Factor Theorem, $(x + 8)$ is a factor.When you use synthetic division with $x = -8$, the depressed polynomial is $x^2 - 9x + 14$, which factors to $(x - 7)(x - 2)$. Therefore, $x^3 - x^2 - 58x + 112 = (x + 8)(x - 7)(x - 2)$.53. $-1.5, 0.2, 1$; sample answer: By the Factor Theorem, $(x - 1)$ is a factor.Use synthetic division with $x = 1$. The remainder is $k - 3$. For $(x - 1)$ to be a factor, $k - 3 = 0$, so $k = 3$. The quotient is $10x^2 + 13x - 3$, which factors as $(5x - 1)(2x + 3)$. $P(x) = (x - 1)(5x - 1)(2x + 3)$. The cubic has a positive leading coefficient and zeros at $-1.5, 0.2$, and 1 .54. $x + 3$; sample answer: Since $(2x + 3)$ is a factor of $V(x)$, use synthetic division with $x = -\frac{3}{2}$. The remainder is zero and the other factor is $2(x^2 + 6x + 9)$, or $2(x + 3)^2$. The sides of the base are $(x + 3)$.56. For $f(x) = 4x^3 - 10x + 8$, the remainder on division by $(x + 5)$ is -442 . This means that $f(-5) = -442$; $4(-5)^3 - 10(-5) + 8 = -500 + 50 + 8 = -442$.57. -5 ; sample answer: If $P(x)$ is symmetric to the y -axis, it must also contain the point $(-2, -5)$. According to the Remainder Theorem, the remainder when polynomial $P(x)$ is divided by $(x - n)$ is $P(n)$. Since $P(-2) = -5$, the remainder when $P(x)$ is divided by $(x + 2)$ is -5 .

LESSON GOAL

Students determine the numbers and types of roots of polynomial equations, find zeros, and use zeros to graph polynomial functions.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Explore:** Roots of Quadratic Polynomials

 **Develop:**

Fundamental Theorem of Algebra

- Determine Number and Type of Roots
- Find Numbers of Positive and Negative Zeros

Finding Zeros of Polynomial Functions

- Use Synthetic Substitution to Find Zeros
- Use a Graph to Write a Polynomial Function
- Use Zeros to Graph a Polynomial Function

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources

Remediation: Intercepts of Graphs

Extension: The Rational Zero Theorem

AL	LB	EL
●	●	●
●	●	●

Language Development Handbook

Assign page 18 of the *Language Development Handbook* to help your students build mathematical language related to determining the number and types of roots of polynomial equations.

 You can use the tips and suggestions on page T18 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Number and Quantity, Algebra, Functions

Standards for Mathematical Content:

N.CN.9 Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

A.APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

F.IF.7c Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous

Students evaluated and factored functions by using the Remainder and Factor Theorems.

A.APR.2

Now

Students determine the numbers and types of roots, find zeros, and use zeros to graph polynomial equations.

N.CN.9, A.APR.3, F.IF.7c

Next

Students will graph and analyze reciprocal functions

F.BF.3, F.IF.5

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

In this lesson, students solidify their understanding of the relationships among roots, zeros, and factors. They build fluency by identifying zeros when factorizations are available, and they apply their understanding by solving real-world problems.

Mathematical Background

The real zeros of a polynomial function $f(x)$ are the x -intercepts of the graph $f(x)$. They are also the real solutions of the polynomial equation $f(x) = 0$. A polynomial function cannot have more zeros than its degree.



Interactive Presentation

Warm Up

State the zeros of the functions.

- $f(x) = (x - 3)^2(x + 2)(x + 4)$
- $f(x) = (x - 3)^3(x + 2)$
- $f(x) = (x - 3)^4$

4. How do you think the double roots will appear on the graphs?

5. Using a graphing calculator to graph each function. How do you think the number of turns in a graph is related to the degree of the polynomial function?

Show Answers

Warm Up

Launch the Lesson

Banks determine how much money to borrow from the Federal Reserve based on the interest they will have to pay back. If a bank knows the period over which they plan to pay back the loan and how much they want to pay back each month, they can use the zeros of a polynomial function to determine the desired interest rate. This can help them determine whether the loan would be profitable for them to take.



Launch the Lesson

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- finding zeros of functions

Answers:

1. $-4, -2, 3, 3$
2. $-2, 3, 3, 3$
3. $3, 3, 3, 3$
4. The graph will touch the x -axis at the double roots.
5. Sample answer: If the degree is n , then the number of turns is at most $n - 1$.

Launch the Lesson



Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will learn how the process of banks borrowing money from the Federal Reserve relates to finding the zeros of a polynomial function.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.



Explore Roots of Quadratic Polynomials

Objective

Students use a sketch to prove the Fundamental Theorem of Algebra for quadratic polynomials.

Teaching the Mathematical Practices

3 Analyze Cases Throughout the Explore, work with students to examine the case where the discriminant of a quadratic equation is negative to prove that the Fundamental Theorem of Algebra holds true for quadratic equations.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students use the sketch to determine how the discriminant of a quadratic equation is related to the number of zeros. Then, students will answer the Inquiry Question.

(continued on the next page)

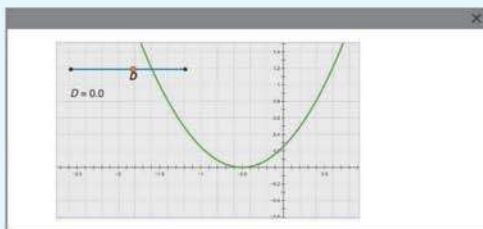
Interactive Presentation

Roots of Quadratic Polynomials

INQUIRY Is the Fundamental Theorem of Algebra true for quadratic polynomials?

The Fundamental Theorem of Algebra states that every polynomial equation with degree greater than zero has at least one root in the set of complex numbers. Consider this theorem in terms of quadratic polynomials.

Explore



Explore

WEB SKETCHPAD



Students use the sketch to explore the relationship between the number of zeros a quadratic has and the discriminant.

TYPE



Students answer a question about determining the number of real roots a quadratic polynomial has given its graph.



Interactive Presentation

INQUIRY Is the Fundamental Theorem of Algebra true for quadratic polynomials? Explain.

Submit

Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Roots of Quadratic Polynomials

(continued)

Questions

Have students complete the Explore activity.

Ask:

- If a quadratic equation has a discriminant of 12, would the function have 2 real roots, 1 real root, or 2 complex roots? **2 real roots**
- What type of roots would the quadratic equation $4x^2 - 12x + 9 = 0$ have? **1 real root**

Inquiry

Is the Fundamental Theorem of Algebra true for quadratic polynomials?

Yes; sample answer: A quadratic polynomial can have 2 real roots, 1 real root, or at least 1 complex root.



Go Online to find additional teaching notes and sample answers for the guiding exercises.

Learn Fundamental Theorem of Algebra

Objective

Students determine the numbers and types of roots of polynomial equations by using the Fundamental Theorem of Algebra.

Teaching the Mathematical Practices

- 1 Explain Correspondences** Encourage students to explain the relationships between the zeros, factors, roots, and intercepts of polynomial functions and the relationship between the degree of a polynomial and the number of roots of the function.

About the Key Concept

The zero of a function $f(x)$ is any value c such that $f(c) = 0$. The real zeros are the x -intercepts of the graph of the function. For a polynomial function $p(x)$, if c is a zero of $p(x)$, then c is a root or solution of $p(x) = 0$, and $x - c$ is a factor of the polynomial. Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers. The degree n of a polynomial equation gives the exact number of roots in the set of complex numbers, including repeating roots. Descartes' Rule of Signs identifies the number of positive and negative zeros for a polynomial function.

Essential Question Follow-Up

Students have begun learning about the Fundamental Theorem of Algebra.

Ask:

Why is the Fundamental Theorem of Algebra important as we use polynomial equations to model and solve real-world situations?

Sample answer: Many real-world situations involve multiple factors, which generate polynomial functions. Because only certain values make sense in real-world situations, the Fundamental Theorem of Algebra makes it possible to find the number and type of roots for a polynomial equation.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Roots and Zeros

Lesson 3-5

Explore: Roots of Quadratic Polynomials

Online Activity Use graphing technology to complete the Explore.

INQUIRY Is the Fundamental Theorem of Algebra true for quadratic polynomials?

Learn: Fundamental Theorem of Algebra

The zero of a function $f(x)$ is any value c such that $f(c) = 0$.

Key Concept - Zeros, Factors, Roots, and Intercepts

Words: Let $P(x) = a_n x^n + \dots + a_1 x + a_0$ be a polynomial function. Then the following statements are equivalent:

- c is a zero of $P(x)$.
- c is a root or solution of $P(x) = 0$.
- $x - c$ is a factor of $P(x)$.
- $(c, 0)$ is an x -intercept of the graph of $P(x)$.

Example: Consider the polynomial function $P(x) = x^2 + 3x - 18$.

The zeros of $P(x) = x^2 + 3x - 18$ are -6 and 3 .

The roots of $x^2 + 3x - 18 = 0$ are -6 and 3 .

The factors of $x^2 + 3x - 18$ are $(x + 6)$ and $(x - 3)$.

The x -intercepts of $P(x) = x^2 + 3x - 18$ are $(-6, 0)$ and $(3, 0)$.

Key Concept - Fundamental Theorem of Algebra

Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.

Key Concept - Corollary to the Fundamental Theorem of Algebra

Words: A polynomial equation of degree n has exactly n roots in the set of complex numbers, including repeated roots.

Examples:

$$2x^3 - 5x = 2 \quad \rightarrow x^3 + 2x^2 = 2x$$

$$3 \text{ roots} \quad 4 \text{ roots}$$

$$x^5 - 6x^3 + x^2 - 5$$

$$5 \text{ roots}$$

Repeated roots can also be called roots of multiplicity m where m is an integer greater than 1. **Multiplicity** is the number of times a number is a zero for a given polynomial. For example, $5x^3 = x \cdot x \cdot x$ has a zero at $x = 0$ with multiplicity 3, because x is a factor three times. However, the graph of the function still only intersects the x -axis once at the origin.

Today's Goals

- Use the Fundamental Theorem of Algebra to determine the numbers and types of roots of polynomial equations.
- Determine the numbers and types of roots of polynomial equations, find zeros, and use zeros to graph polynomial functions.

Today's Vocabulary
multiplicity

Think About It!

What is the multiplicity of the zero at $x = 1$ for $p(x) = 5x - 5x^2$? Explain your reasoning.

Sample answer: The zero at $x = 1$ has a multiplicity of 5, because $(x - 1)^5$ has $(x - 1)$ as a factor five times.

Lesson 3-5 • Roots and Zeros 143

Interactive Presentation

FUNDAMENTAL THEOREM OF ALGEBRA

Every non-constant polynomial $P(x)$ has at least one zero in the complex number system.

KEY CONCEPT: ZEROS, FACTORS, ROOTS, AND INTERCEPTS

Words: Let $P(x) = a_n x^n + \dots + a_1 x + a_0$ be a polynomial function. Then the following statements are equivalent:

- c is a zero of $P(x)$.
- c is a root or solution of $P(x) = 0$.
- $x - c$ is a factor of $P(x)$.
- $(c, 0)$ is an x -intercept of the graph of $P(x)$.

Learn

TYPE

a|

Students answer a question about multiplicity of a zero of a polynomial function.



Example 1 Determine the Number and Type of Roots

MP Teaching the Mathematical Practices

7 Use the Distributive Property Point out that the Distributive Property is one of the most-used properties. Students will use the Distributive Property to factor and determine the roots of a polynomial equation.

Questions for Mathematical Discourse

AL Why does $x^2 = 0$ yield a double root? **Sample answer:** x has two factors, x and x .

OL What is the value of the discriminant of the quadratic factor? What does this indicate about the solutions of the quadratic factor? **—196; sample answer:** It will have 2 imaginary roots.

BL Will the graph of the related function cross the x -axis? Explain. **No; sample answer:** Because $x = 0$ is the only real root, the graph will touch the x -axis at $(0, 0)$, but will not cross the axis.

Example 2 Find the Number of Positive and Negative Zeros

MP Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of the polynomial function to determine the possible number of roots of the function.

Questions for Mathematical Discourse

AL What is the smallest possible number of positive roots if a polynomial equation has 3 sign changes? **1**

OL Is it possible for an odd-degree polynomial function to have no real roots? Explain in the context of Descartes' Rule of Signs. **No; sample answer:** An odd-degree polynomial function that does not have any sign changes and has a constant term would have sign changes for $f(-x)$. If the odd function does not have a constant term, the polynomial would have a common factor of x and result in a root of zero.

BL Why are the numbers of possible real zeros in increments of 2? **Complex roots come in conjugate pairs.**

Common Error

To check for possible negative zeros, students must substitute in $-x$, which may result in sign errors when simplifying. Encourage students to consider each term and the effect of a negative value to limit the number of sign errors.

Key Concept • Descartes' Rule of Signs

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ be a polynomial function with real coefficients, and $a_n \neq 0$. Then the number of positive real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of $P(x)$, or is less than this by an even number, and the number of negative real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this by an even number.

Example 1 Determine the Number and Type of Roots

Solve $x^4 + 49x^2 = 0$. State the number and type of roots.

$x^4 + 49x^2 = 0$	Original equation
$x^2(x^2 + 49) = 0$	Factor
$x^2 = 0$ or $x^2 + 49 = 0$	Zero Product Property
$x = 0$ or $x^2 = -49$	Subtract 49 from each side
$x = \pm \sqrt{-49}$	Square Root Property
$x = \pm 7i$	Simplify

The polynomial has degree 4, so there are four roots in the set of complex numbers. Because x^2 is a factor, $x = 0$ is a root with multiplicity 2, also called a double root. The equation has one real repeated root, 0, and two imaginary roots, $7i$ and $-7i$.

Example 2 Find the Number of Positive and Negative Zeros

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $f(x) = x^5 - 2x^4 - x^3 + 6x^2 - 5x + 10$.

Because $f(x)$ has degree 5, it has five roots, either real or imaginary. Use Descartes' Rule of Signs to determine the possible number and types of real zeros.

Part A Find the possible number of positive real zeros.

Count the number of changes in sign for the coefficients of $f(x)$.

$$f(x) = x^5 - 2x^4 - x^3 + 6x^2 - 5x + 10$$

$$+ \quad - \quad - \quad + \quad - \quad +$$

There are 4 sign changes, so there are 4, 2, or 0 positive real zeros.

Part B Find the possible number of negative real zeros.

Count the number of changes in sign for the coefficients of $f(-x)$.

$$f(-x) = (-x)^5 - 2(-x)^4 - (-x)^3 + 6(-x)^2 - 5(-x) + 10$$

$$= -x^5 - 2x^4 + x^3 + 6x^2 + 5x + 10$$

$$- \quad - \quad + \quad + \quad + \quad +$$

There is 1 sign change, so there is 1 negative real zero.

See Online: You can complete an Explore Example online.

Interactive Presentation

Determine the Number and Type of Roots

Solve $x^4 + 49x^2 = 0$. State the number and type of roots.

$x^4 + 49x^2 = 0$	Original equation
$x^2(x^2 + 49) = 0$	Factor
$x = 0$ or $x^2 + 49 = 0$	Zero Product Property
$x^2 = -49$	Subtract 49 from each side
$x = \pm \sqrt{-49}$	Square Root Property
$x = \pm 7i$	Simplify

Example 1

TAP



Students tap to see a Study Tip and Common Error.



1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Learn Finding Zeros of Polynomial Functions

Objective

Students find zeros of polynomial functions and graph polynomial functions by using zeros.

MP Teaching the Mathematical Practices

1 Explain Correspondences Guide students to see the relationship between the zeros of a polynomial function and its factors.

Important to Know

The Complex Conjugates Theorem states that if a polynomial function with real coefficients has an imaginary number as a root, then its conjugate is also a root.

DIFFERENTIATE

Enrichment Activity

A polynomial function has the roots $1 - \sqrt{5}$ and $4i$. What is the equation of the polynomial function in standard form?

$$x^4 - 2x^3 + 12x^2 - 32x - 64 = 0$$

Example 3 Use Synthetic Substitution to Find Zeros

MP Teaching the Mathematical Practices

7 Use Structure Students will use the structure of a polynomial function to determine the number and type of zeros in order to sketch a graph of the function.

(continued on the next page)

Part C Find the possible number of imaginary zeros.

Positive Real Zeros	Negative Real Zeros	Imaginary Zeros	Total Zeros
4	1	0	$4 + 1 + 0 = 5$
2	1	2	$2 + 1 + 2 = 5$
0	1	4	$0 + 1 + 4 = 5$

Check

Copy and complete the table. State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 3x^2 + 15x + 1$. Write the rows in ascending order of positive real zeros.

Number of Positive Real Zeros	Number of Negative Real Zeros	Number of Imaginary Zeros
0	0	6
0	2	4
2	0	4
2	2	2
4	0	2
6	2	0

Learn Finding Zeros of Polynomial Functions

Key Concept • Complex Conjugates Theorem

Words: Let a and b be real numbers, and $b \neq 0$. If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.

Example: If $1 + 2i$ is a zero of $f(x) = x^3 - x^2 + 3x + 5$, then $1 - 2i$ is also a zero of the function.

When you are given all of the zeros of a polynomial function and asked to determine the function, use the zeros to write the factors and multiply them together. The result will be the polynomial function.

Example 3 Use Synthetic Substitution to Find Zeros

Find all of the zeros of $f(x) = x^3 + x^2 - 7x - 15$ and use them to sketch a rough graph.

Part A Find all of the zeros.

Step 1 Determine the total number of zeros.

Since $f(x)$ has degree 3, the function has 3 zeros.

(continued on the next page)

Lesson 3-5 • Roots and Zeros 145

Interactive Presentation

Finding Zeros of Polynomial Functions

For example, consider the cubic function $f(x) = x^3 - x^2 + 3x + 5$. If $1 + 2i$ is a zero of $f(x)$, then $1 - 2i$ is also a zero of $f(x)$.

Key Concept: Complex Conjugates Theorem

Words: Let a and b be real numbers, and $b \neq 0$. If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.

Example: If $1 + 2i$ is a zero of $f(x) = x^3 - x^2 + 3x + 5$, then $1 - 2i$ is also a zero of the function.

Learn

CHECK



Students complete the Check online to determine whether they are ready to move on.



Questions for Mathematical Discourse

AL What is shown in the table of values in **Step 3**? **Sample answer:**

Each value in the x -column represents a value tested with synthetic substitution. The corresponding columns show the values that result from the synthetic substitution (i.e. the bottom row when performing synthetic substitution). The values in the column under -15 are the remainders.

OL How can synthetic substitution and the Quadratic Formula be used to find all the roots of a polynomial of degree 4? **Sample answer:** Use synthetic substitution to find the depressed polynomial of degree 3, and then use synthetic substitution one more time to produce a quadratic polynomial. Then the Quadratic Formula can be used to find the remaining 2 roots.

BL Can you determine certain values that make sense to test as possible zeros? Explain. **Yes; sample answer:** You know from the Distributive Property that the constant of the binomial factor multiplied by the constant of the depressed polynomial must equal the constant of the original function. Thus, test values that are the negative of factors of the constant term.

Common Error

Many students incorrectly substitute values into the Quadratic Formula, especially the $-b$ portion. Encourage students to label a , b , and c in the quadratic function, if necessary, before using the Quadratic Formula.

DIFFERENTIATE

Reteaching Activity **AL** **ELL**

IF students are having difficulties finding the number, type, and signs of the roots of a polynomial equation,

THEN have students check each problem by graphing the polynomial. They can verify the number and sign of real roots.

Common Misconception

A common misconception some students may have is that only the given zeros are the zeros of a polynomial function. Remind students of what the Complex Conjugates Theorem states.

Step 2 Determine the type of zeros.

Examine the number of sign changes for $f(x)$ and $f(-x)$.

$$f(x) = x^3 + x^2 - 7x - 15 \quad f(-x) = -x^3 + x^2 + 7x - 15$$

Because there is 1 sign change for the coefficients of $f(x)$, the function has 1 positive real zero. Because there are 2 sign changes for the coefficients of $f(-x)$, $f(x)$ has 2 or 0 negative real zeros. Thus, $f(x)$ has 3 real zeros, or 1 real zero and 2 imaginary zeros.

Step 3 Determine the real zeros.

List some possible values, and then use synthetic substitution to evaluate $f(x)$ for real values of x .

x	1	1	-7	-15
-3	1	-2	-1	-12
-2	1	-1	-5	-5
-1	1	0	-7	-8
0	1	1	-7	-15
1	1	2	-5	-20
2	1	3	-1	-17
3	1	4	5	0
4	1	5	13	37

3 is a zero of the function, and the depressed polynomial is $x^2 + 4x + 5$. Since it is quadratic, use the Quadratic Formula. The zeros of $f(x) = x^2 + 4x + 5$ are $-2 - i$ and $-2 + i$.

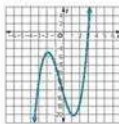
The function has zeros at 3, $-2 - i$ and $-2 + i$.

Part B Sketch a rough graph.

The function has one real zero at $x = 3$, so the function goes through $(3, 0)$ and does not cross the x -axis at any other place.

Because the degree is odd and the leading coefficient is positive, the end behavior is that as $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$ and as $x \rightarrow \infty$, $f(x) \rightarrow \infty$.

Use this information and points with coordinates found in the table above to sketch the graph.



See Online You can complete an Extra Example online.

146 Module 3 • Polynomial Equations

Interactive Presentation

Use Synthetic Substitution to Find Zeros

Find all of the zeros of $f(x) = x^3 + x^2 - 7x - 15$ and use them to sketch a rough graph.

Find all of the zeros.

Find a rough graph.

Example 3

TAP



Students tap to see the steps to find roots of a polynomial function and sketch its graph.

Example 4 Use a Graph to Write a Polynomial Function

MP Teaching the Mathematical Practices

1 Explain Correspondences Students must be able to explain the relationship between the graph of a polynomial function and the equation of the polynomial function.

Questions for Mathematical Discourse

AL What are two features of the polynomial function that are consistent with the graph? **Sample answer:** When $x = 0$, $y = -8$, and the y -intercept of the graph is -8 . The leading coefficient of the polynomial is positive, and the end behavior of the graph indicates a positive leading coefficient.

OL Why does the example say *a polynomial that could be represented by the graph*, rather than asking to find the exact polynomial represented by the graph? **Sample answer:** The graph only shows real roots, but it could represent a polynomial with pairs of imaginary roots. Thus, the polynomial of degree 3 is a possibility rather than an exact answer.

BI Does the function you wrote have any imaginary roots? Explain. **No;** sample answer: Because the polynomial has degree 3, it has exactly 3 roots. All 3 of those roots are real and represented on the graph.

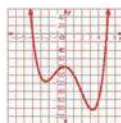
Common Error

Many students will simply add the zero to x to create a factor instead. Remind students that if c is a root, then $x - c$ is the factor.

Check

Determine all of the zeros of $f(x) = x^3 - x^2 - 16x^2 - 4x - 80$, and use them to sketch a rough graph.

Real Zeros: $-4, -5, -7$
Imaginary Zeros: $-2i, 2i$



Example 4 Use a Graph to Write a Polynomial Function

Write a polynomial function that could be represented by the graph.

The graph crosses the x -axis 3 times, so the function is at least of degree 3. It crosses the x -axis at $x = -4$, $x = -2$, and $x = 1$, so its factors are $x + 4$, $x + 2$, and $x - 1$.



To determine a polynomial, find the product of the factors.

$$y = (x + 4)(x + 2)(x - 1)$$

Set the product of the factors equal to y .

$$= (x^2 + 6x + 8)(x - 1)$$

FOIL

$$= x^3 + 5x^2 + 2x - 8$$

Multiply

A polynomial that could be represented by the graph is

$$y = x^3 + 5x^2 + 2x - 8$$

Check

Write a polynomial that could be represented by the graph.

A. $y = x^3 - 6x^2 - 24x + 64$

B. $y = x^3 + 4x - 32$

C. $y = x^3 + 6x^2 - 24x - 64$

D. $y = x^3 - 64$



Go Online You can complete an Extra Example online.

Lesson 3-5 • Roots and Zeros 147

Interactive Presentation

Use a Graph to Write a Polynomial Function

Write a polynomial that could be represented by the graph.

The graph crosses the x -axis 3 times, so it is at least of degree 3. It crosses the x -axis at $x = -4$, $x = -2$, and $x = 1$, so its factors are $x + 4$, $x + 2$, and $x - 1$.



Example 4

TYPE



Students complete the statement to write a polynomial function for the graph.



Apply Example 5 Use Zeros to Graph a Polynomial Function



Teaching the Mathematical Practices

1 Make Sense of Problems and Persevere in Solving Them,

4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- What factors must a polynomial have in order for it to equal zero when $x = 5$ and $x = 40$?
- Which direction must the parabola open and what must be true about the coefficients in order for that to happen?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Apply Example 5 Use Zeros to Graph a Polynomial Function

PROFIT MARGIN A book publisher wants to release a special hardcover version of several Charles Dickens books. They know that if they charge \$5 or \$40, their profit will be \$0. Graph a polynomial function that could represent the company's profit in thousands of dollars given the price they charge for the book.

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: Let x represent the price that the publisher charges and let y represent the profit. I need to write and graph a polynomial function that relates x and y .

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: I know 5 and 40 are zeros of the function, I can use them to write factors to write an equation of the function.

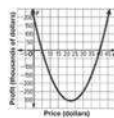
3 What is your solution?

Use your strategy to solve the problem.

What is a function that represents the given information?

$$y = x^2 - 45x + 200$$

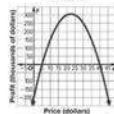
Graph the function.



Does this function make sense in the context of the situation? If not, explain why not and write and graph a more reasonable function.

No; the graph passed through the zeros, but did not show reasonable book prices that would result in profit.

$$y = -x^2 + 45x - 200$$



4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: With multiplying the function by -1 , the new function shows that the profit is negative when charging less than \$5 per book. This makes sense in the context of the situation.

Problem-Solving Tip
Logical Reasoning
When solving a problem it is important to use logical reasoning skills to analyze the problem.

Go Online
You can complete an Extra Example online.

148 Module 3 • Polynomial Equations

Interactive Presentation



Example 5

TAP



Students tap to see the steps to graphing a polynomial using the zeros.

TYPE



Students answer a question to determine the reasonableness of their solution.

CHECK



Students complete the Check online to determine whether they are ready to move on.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–45
2	exercises that use a variety of skills from this lesson	46–57
3	exercises that emphasize higher-order and critical-thinking skills	58–66

ASSESS AND DIFFERENTIATE

Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, THEN assign:

- Practice Exercises 1–65 odd, 67–72
- Extension: The Rational Root Theorem
- ALEKS** Real Zeros of Polynomial Functions

IF students score 66%–89% on the Checks, THEN assign:

- Practice Exercises 1–71 odd
- Remediation, Review Resources: Intercepts of Graphs
- Personal Tutors
- Extra Examples 1–5
- ALEKS** Zeros of Functions

IF students score 65% or less on the Checks, THEN assign:

- Practice Exercises 1–23 odd
- Remediation, Review Resources: Intercepts of Graphs
- Quick Review Math Handbook: Roots and Zeros
- ALEKS** Zeros of Functions

Practice

Example 1

Solve each equation. State the number and type of roots.

- $5x + 12 = 0$
 $-\frac{12}{5}$, 1 real
- $x^2 - 4x + 40 = 0$
 $2 \pm 6i$, 2 imaginary
- $x^3 + 4x^2 = 0$
 $0, 0, -4$, 3 real, 2 imaginary
- $x^4 - 625 = 0$
 $5, 5i, -5i, -5$, 2 real, 2 imaginary
- $4x^3 - 4x - 1 = 0$
 $\frac{1}{4}, \frac{1}{4}i, \frac{1}{4}i$, 2 real
- $x^3 - 85x = 0$
 $0, -3, -3i, 3i$, 3 real, 2 imaginary
- $2x^2 + x - 6 = 0$
 $-2, \frac{3}{2}$, 2 real
- $4x^2 + 1 = 0$
 $\frac{1}{2}i, \frac{1}{2}i$, 2 imaginary
- $x^2 + 1 = 0$
 $i, -i$, 2 imaginary
- $2x^2 - 5x + 14 = 0$
 $\frac{5}{4} \pm \frac{3}{4}i$, 2 imaginary
- $3x^3 - 27 = 0$
 $3, -3, -3i$, 1 real, 2 imaginary
- $x^2 - 6x^2 + 7x = 0$
 $0, 3 \pm \sqrt{2}$, 3 real
- $x^3 + 2x^2 + x = 0$
 $0, -1, -1, 1$, 1 real, 4 imaginary

Example 2

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

- $f(x) = 3x^4 - 4x^3 - 17x + 6$
2 or 0; 1, 2 or 0
- $f(x) = x^3 - x^2 + 2x - 4$
3 or 1; 0, 2 or 0
- $f(x) = x^4 + 7x^2 + 9$
1, 3
- $f(x) = x^4 - 5x^3 + 2x^2 + 5x + 7$
0 or 2; 0 or 2, 0, 2, or 4
- $f(x) = -3x^4 + 5x^3 + 4x^2 - 8$
0 or 2; 0, 2, or 4
- $f(x) = x^4 - 5x^3 - x^2 + 24$
0 or 2; 0 or 2, 4, or 6
- $f(x) = x^4 - 4x^3 - 12x^2 + 13x - 10$
2, 2 or 1, 2 or 0
- $f(x) = x^4 - 2x^3 - 7x^2 - 2x + 12$
0 or 2; 0, 0, 2, or 4
- $f(x) = x^4 - 3x^3 - 5x^2 + 5x + 19$
0 or 2; 0 or 2, 0, 2, or 4
- $f(x) = x^4 - 16x^3 + 18x^2 - 36$
0 or 2; 2, 2, or 4

Lesson 3-5 • Roots and Zeros 149

Example 3

Find all of the zeros of each function and use them to sketch a rough graph.

- $f(x) = x^3 - 5x^2 + 3x + 3$
 $1, 1 \pm \sqrt{2}$, 1 or 2
- $f(x) = x^3 - 4x^2 + 3x - 6$
 $2, -1, -4$
- $f(x) = x^4 - 3x^3 - 5x^2 + 3x + 4$
 $-1, -1, 1, 4$
- $f(x) = x^4 - x^3 - 17x^2 + 17x + 10$
 $-4, 4, -\sqrt{5}, \sqrt{5}$
- $f(x) = x^4 + 7x^3 + 4x - 12$
 $-6, -2, 1$
- $f(x) = x^4 + 6x^3 + 73x^2 + 384x + 576$
 $-3, -3, -6i, 6i$
- $f(x) = x^4 - 8x^3 + 20x^2 - 32x + 64$
 $4, 4, -3i, 3i$
- $f(x) = x^4 - 16x^3 + 18x^2 - 36$
 $-3, 0, 3, -1, 1$

Example 4

Write a polynomial that could be represented by each graph.

- $y = x^2 + x - 6$
- $y = x^3 - 2x^2 - 5x + 6$
- $y = x^3 - 6x^2 + 7x + 8$

Example 5

44. **FISH** Some fish jump out of the water. When a fish is out of the water, its location is above sea level. When a fish dives back into the water, its location is below sea level. A biologist can use polynomial functions to model the location of fish compared to sea level. A biologist noticed that a fish is at sea level at $-3, -2, -1, 1, 2$, and 3 minutes from noon. Graph a polynomial function that could represent the location of the fish compared to sea level p , in centimeters, x seconds from noon. See Mod. 3 Answer Appendix.

45. **BUSINESS** After introducing a new product, a company's profit is modeled by a polynomial function. In 2012 and 2013, the company's profit on the product was \$40. Graph a polynomial function that could represent the amount of profit $p(x)$, in thousands of dollars, x years since 2010. See Mod. 3 Answer Appendix.



Mixed Exercises

Write a polynomial function of least degree with integral coefficients that has the given zeros. 48–51. Sample answers given.

48. 5, -2, -1 $f(x) = x^3 - 2x^2 - 13x + 10$ 49. -4, -3, 5 $f(x) = x^3 + 2x^2 - 23x - 60$

50. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ $f(x) = 6x^3 + 5x^2 + 8x + 4$ 51. -2, -3, -5 $f(x) = x^3 + 2x^2 + 6x^2 + 8x + 27$

52. $f(x) = x^3 - 20x^2 + 50x$ 53. $f(x) = x^3 - 3x^2 - 6x + 10$

54. $f(x) = x^4 - 8x^3 + 7x^2 + 5x - 8$ 55. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

56. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 57. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

58. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 59. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

60. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 61. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

62. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 63. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

64. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 65. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

66. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 67. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

68. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 69. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

70. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 71. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

72. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 73. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

74. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 75. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

76. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 77. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

78. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 79. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

80. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 81. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

82. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 83. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

84. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 85. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

86. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 87. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

88. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 89. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

90. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 91. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

92. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 93. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

94. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 95. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

96. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 97. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

98. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 99. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

100. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 101. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

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106. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 107. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

108. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 109. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

110. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 111. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

112. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 113. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

114. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 115. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

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138. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 139. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

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150. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 151. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

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154. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 155. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

156. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 157. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

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166. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 167. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

168. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 169. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

170. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 171. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

172. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 173. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

174. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 175. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

176. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 177. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

178. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 179. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

180. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 181. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

182. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 183. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

184. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 185. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

186. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 187. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

188. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 189. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

190. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 191. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

192. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 193. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

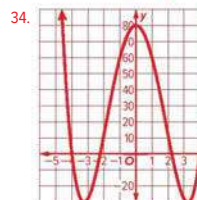
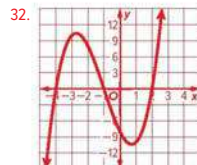
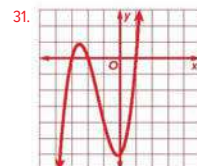
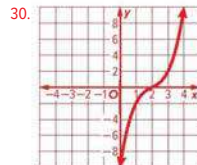
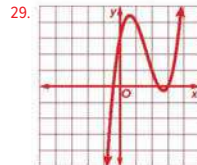
194. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 195. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

196. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 197. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

198. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 199. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

200. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ 201. $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

Answers



Higher-Order Thinking Skills

58. CREATE Consider two polynomial functions, $f(x)$ and $g(x)$. $a < c$. See Mod. 3 Answer Appendix.

a. Write a polynomial function $f(x)$ of least degree with integral coefficients and zeros that include -1 , -4 , and $\frac{5}{2}$. Explain how you found the function.

b. Write another polynomial function $g(x)$ with integral coefficients that has the same degree and zeros. How did you find this function?

c. Are you able to sketch the graphs of $f(x)$ and $g(x)$ based on the zeros? Explain your reasoning. Then sketch the graphs of $f(x)$ and $g(x)$.

59. ANALYZE Use the zeros to draw the graph of $f(x) = x^3 - 7x^2 + 7x + 15$ by hand. Discuss the accuracy of your graph, and what could be done to improve the accuracy. See Mod. 3 Answer Appendix.

60. PERSEVERE Let the polynomial function $f(x)$ have real coefficients, be of degree 5, and have zeros $4 + 3i$, $2 - 7i$, and $6 + 8i$, where i is a real number.

a. What can be determined about f ? Explain your reasoning. By the Complex Conjugates Theorem $4 - 3i$, $2 + 7i$, and $6 - 8i$ are also zeros of $f(x)$. Because $f(x)$ has degree 5, it can have at most 5 zeros. This means that 0 must be zero.

b. Write a possible equation for $f(x)$. Sample answer: $f(x) = x^5 - 18x^4 + 182x^3 - 1184x^2 + 4468x - 7950$

61. CREATE Sketch the graph of a polynomial function with: See Mod. 3 Answer Appendix.

a. 3 real, 2 imaginary zeros b. 4 real zeros c. 2 imaginary zeros

62. PERSEVERE Write an equation in factored form of a polynomial function of degree 5 with 2 imaginary zeros, 1 real nonrational zero, and 2 irrational zeros. Explain. Sample answer: $f(x) = (x + 2)(x - 2)(x + 5)(x + 1)(x - \sqrt{2})$. Use conjugates for the imaginary and irrational values.

63. WHICH ONE DOESN'T BELONG? Determine which equation is not like the others. Justify your conclusion. Sample answer: $x^4 + 1 = 0$. The equation has imaginary solutions and all of the others have real solutions.

$x^4 + 1 = 0$ $x^2 + 1 = 0$ $x^2 - 1 = 0$ $x^2 - 8 = 0$

64. ANALYZE Provide a counterexample for each statement.

a. All polynomial functions of degree greater than 2 have at least 1 negative real root. Sample answer: $f(x) = x^3 + 4x^2 + 4$

b. All polynomial functions of degree greater than 2 have at least 1 positive real root. Sample answer: $f(x) = x^3 + 6x^2 + 9x$

65. WRITE Explain to a friend how you would use Descartes' Rule of Signs to determine the number of possible positive real roots and the number of possible negative real roots of the polynomial function $f(x) = x^4 - 2x^3 + 6x^2 + 5x - 12$. See Mod. 3 Answer Appendix.

66. FIND THE ERROR The graph shows a polynomial function. Brannon says the function is a 4th degree polynomial. Anita says the function is a 2nd degree polynomial. Is either of them correct? Explain your reasoning. See Mod. 3 Answer Appendix.



Review

Rate Yourself!

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have them respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- Why are the Remainder and Factor Theorems important when solving polynomial equations?
- Why is the Fundamental Theorem of Algebra important as we use polynomial equations to model and solve real-world situations?

Then have them write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

A completed Foldable for this module should include the key concepts related to roots and zeros of polynomial equations.

LearnSmart Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Polynomial, Rational, and Radical Relationships**.

- Solving Equations Graphically

Review

Essential Question

What methods are useful for solving polynomial equations and finding zeros of polynomial functions?

You can use graphing or factoring to solve polynomial equations or find roots of polynomial functions. The Remainder and Factor Theorems can also be useful.

Module Summary

Lessons 3.1 and 3.2

Solving Polynomial Equations

- Polynomial equations can be solved by graphing or can be solved algebraically.
- Use patterns such as the sum or difference of two cubes to factor.
 - $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
 - $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- When factoring a polynomial, look for a common factor to simplify the expression.
- An expression in quadratic form can be written as $ax^2 + bx + c$ for any numbers a , b , and c , $a \neq 0$, where u is some expression in x .

Lesson 3.3

Polynomial Identities

- An identity is an equation that is satisfied by any numbers that replace the variables.
- A polynomial identity is a polynomial equation that is true for any values that are substituted for the variables.
- To verify an identity, simplify one side of an equation until the two sides of the equation are the same.

Lesson 3.4

The Remainder and Factor Theorems

- Remainder Theorem: For a polynomial $p(x)$ and a number a , the remainder upon division by $x - a$ is $p(a)$.
- Factor Theorem: The binomial $x - a$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0$.

Lesson 3.5

Roots and Zeros

- Let $P(x) = a_nx^n + \dots + a_1x + a_0$ be a polynomial function. Then the following statements are equivalent.
 - c is a zero of $P(x)$.
 - $x - c$ is a root or solution of $P(x) = 0$.
 - $x - c$ is a factor of $a_nx^n + \dots + a_1x + a_0$.
- If c is a real number, then $(c, 0)$ is an x -intercept of the graph of $P(x)$.
- A polynomial equation of degree n has exactly n roots in the set of complex numbers, including repeated roots.
- The number of positive real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and the number of negative real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this by an even number.
- If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.

Study Organizer

Foldables

Use your Foldable to review this module. Working with a partner can be helpful. Ask for clarification of concepts as needed.



Test Practice

1. **MULTIPLE CHOICE** Which function can be used to solve $x^3 - x = 2x^2 - 2$ by graphing?

Lesson 3-5

- ☒ A. $f(x) = x^3 - 2x^2 - x + 2$
☐ B. $f(x) = x - 2$
☐ C. $f(x) = x^3 + 2x^2 - x - 2$
☐ D. $f(x) = 2x^3 - 4x^2 + 2x$

2. **OPEN RESPONSE** The graph of $f(x) = x^3 - 4x^2 + x + 1$ is shown.



How many real solutions does the function have? (Lesson 3-5)

A.

3. **MULTI-SELECT** Use a graphing calculator to solve $x^3 - 10x + 4 = 4 - x$. Select all of the solutions. (Lesson 3-5)

- ☒ A. -3
☒ B. 0
☐ C. 1
☒ D. 3
☐ E. 4
☐ F. 7

4. **MULTIPLE CHOICE** A jewelry box is 3 inches by 4 inches by 2 inches. If increasing the length of each edge by x inches doubles the volume of the jewelry box, what is the value of x ? Round your answer to the nearest hundredth if necessary. (Lesson 3-5)

- A. 0.40
 B. 0.69
☒ C. 0.73
 D. 1.24

5. **OPEN RESPONSE** The volume of a figure is $x^3 - 9x$. The surface area of another figure is $8x^2$. Disregarding the units, the volume of the first figure equals the surface area of the second figure. What are the possible values of x ? Explain your reasoning. (Lesson 3-5)

Set the expression for the volume of the first figure equal to the expression for the surface area of the second figure. Solve the equation $x^3 - 9x = 8x^2$. Solving the equation gives the solutions -1 , 0 , and 9 . The value of x cannot be 0 because when substituting 0 for x in the original expressions, the volume of the first figure is $(0)^3 - 9(0) = 0 - 0 = 0$ and surface area of the second figure is $8(0)^2 = 8(0) = 0$. Volume and surface area cannot be 0 , so the value of x is -1 or 9 .

6. **MULTI-SELECT** Find the solutions of $x^4 - x^2 - 2 = 0$. Select all that apply. (Lesson 3-2)

- ☒ A. $\pm\sqrt{2}$
☒ B. $\pm i$
☐ C. ± 1
☐ D. ± 2
☐ E. $\pm\sqrt{2}i$

Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Vocabulary Activity

Module Review

Assessment Resources

Vocabulary Test

AL Module Test Form B

OL Module Test Form A

BL Module Test Form C

Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

Test Practice

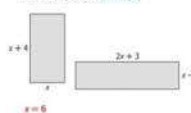
You can use these pages to help your students review module content and prepare for online assessments. Exercises 1-18 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Multiple Choice	Students select one correct answer.	4, 9, 10, 11, 13, 18
Multi-Select	Multiple answers may be correct. Students must select all correct answers.	3, 6, 8, 12
Table Item	Students complete a table by entering in the correct values.	15, 16
Open Response	Students construct their own response.	2, 5, 7, 14

To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
A.CED.1	3-1, 3-2, 3-4	3, 4, 5, 6, 7
A.CED.2	3-1	1, 2
A.REI.11	3-1	1, 2, 3
A.APR.2	3-4	11, 12, 13, 14
A.APR.3	3-1, 3-5	15, 17, 18
A.APR.4	3-3	8, 9, 10
N.CN.9	3-5	15, 16, 18
F.IF.4	3-5	17
F.IF.7c	3-1, 3-5	17

7. **OPEN RESPONSE** Evelyn is making two rectangular table tops. The dimensions of both are shown. If both designs have the same area, what is the value of x ? (Lesson 3-2)



8. **MULTI-SELECT** Select all of the choices that are steps in the proof that $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$. (Lesson 3-3)

- ☒ A. $x^3 + y^3 = x(x^2) + y(y^2) + xy(x^2) + y(xy^2) + y(xy^2) + y(y^2)$
- ☒ B. $x^3 + y^3 = x^3 - x^2y + x^2y + xy^3 - x^2y + y^3$
- ☒ C. $x^3 + y^3 = x^3 - (x^2y + xy^2) + (x^2y - xy^2) + y^3$
- ☒ D. $x^3 + y^3 = x^3 - x^2y + xy^2 + xy^2 - xy^2 + y^3$
- ☒ E. $x^3 + y^3 = x^3 - x^2y + x^2y + y^3$
- ☒ F. $x^3 + y^3 = x^3 + y^3$

9. **MULTIPLE CHOICE** Which of the following is an equivalent expression to $(x + y)^2$? (Lesson 3-3)

- ☐ A. $x^2 + y^2 + 3xy(x - y)$
- ☒ B. $x^2 + y^2 + 3xy(x + y)$
- ☐ C. $x^2 + y^2 + xy(x + y)$
- ☐ D. $x^2 + y^2 + xy(x - y)$

10. **MULTIPLE CHOICE** Which of the following is an equivalent expression to $4xy$? (Lesson 3-2)

- ☐ A. $(x + y)^2 - (x - y)^2$
- ☒ B. $(x + y)^2 - (x - y)^2$
- ☐ C. $2(x + y)^2 - (x - y)^2$
- ☐ D. $(x + y)^2 - 2(x - y)^2$

11. **MULTIPLE CHOICE** What is the remainder when $6x$ is $x^4 + x^3 - 2x^2 + 5x - 4$ is divided by $x + 2$? (Lesson 3-4)

- ☐ A. -25
- ☐ B. -17
- ☒ C. 17
- ☐ D. 25

12. **MULTI-SELECT** If $x - 1$ is a factor of $x^3 - 6x^2 + 5x - 6$, find the remaining factors of the polynomial. Select all that apply. (Lesson 3-4)

- ☐ A. $x^2 - 5x + 6$
- ☒ B. $x - 3$
- ☐ C. $x + 2$
- ☒ D. $x - 2$

13. **MULTIPLE CHOICE** The average price of gasoline, in dollars, from 2010 to 2016 can be modeled by the function $f(x) = 0.03x^2 - 0.4x + 118x + 2.75$, where x represents the years since 2010. What is the estimated price of gasoline in 2011 in dollars? (Lesson 3-6)

A. \$4.70
☒ B. \$3.56
 C. \$2.42
 D. \$1.34

14. **OPEN RESPONSE** Use synthetic substitution to find $f(-6)$ if $f(x) = -3x^3 - 20x^2 + 80x + 12$. (Lesson 3-6)

$f(-6) = -36$

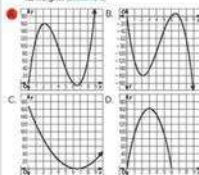
15. **MULTI-SELECT** Which value of x is a real root in the equation $x^3 + 3x^2 - 4 = 0$? Select all that apply. (Lesson 3-5)

A. -2
☒ B. -1
☒ C. 1
 D. 2
 E. $-2i$
 F. $-i$
 G. i
 H. $2i$

16. **MULTI-SELECT** What are the possible numbers of positive real roots and negative real roots of $p(x) = x^5 - 2x^4 + 3x^3 - 4x^2 + 1$? Select all that apply. (Lesson 3-5)

☒ A. 0
 B. 1
☒ C. 2
 D. 3
☒ E. 4
 F. 5

17. **MULTIPLE CHOICE** A template for a shipping box is made by cutting a square with side length x inches from each corner of a rectangular piece of cardboard that is 12 inches wide and 14 inches long. Which graph could represent the relationship between the volume of a shipping box y and its height? (Lesson 3-5)



18. **MULTIPLE CHOICE** Consider the function $p(x) = x^4 - 3x^2 + 3x^2 - x^2$. Which of the following is a root of the function with multiplicity 3? (Lesson 3-5)

A. -1
 B. 0
☒ C. 1
 D. 2

Lesson 3-3

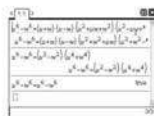
17. Sample answer:



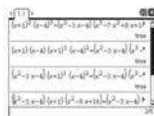
18. Sample answer:



19. Sample answer:



20. Sample answer:

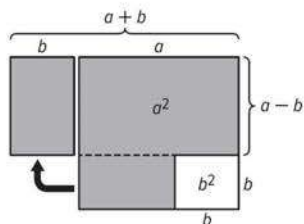


21. Sample answer: A polynomial identity is a polynomial equation that is satisfied for any values that are substituted for the variables. To prove that a polynomial equation is an identity you begin with the more complicated side of the equation and use algebra properties to transform that side of the equation until it is simplified to look like the other side.

22. $x - 2y = 3$, $2xy = 4$, $x + y = 5$; $x = 2$, $y = 1$

23. Sample answer: All Pythagorean triples must be made from positive integers. Therefore, x must be greater than y to prevent a negative side length and x cannot be equal to y to prevent a zero side length.

24. Sample answer:



$$a^2 - (a+b)(a-b) = b^2 \quad \text{Original equation}$$

$$a^2 - (a^2 - ab + ab - b^2) = b^2 \quad \text{Use the FOIL Method.}$$

$$a^2 - a^2 + ab - ab + b^2 = b^2 \quad \text{Distribute -1.}$$

$$b^2 = b^2 \quad \text{Combine like terms.}$$

$$b^2 = b^2 \quad \text{True}$$

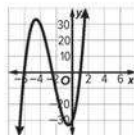
25. Sample answer: When George multiplied b and a in the second line he mistakenly made the term negative so it did not cancel when he simplified.

Lesson 3-4

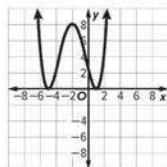
58.

$\sqrt{3}$	1	3	5
	$\sqrt{3}$	$\sqrt{3}$	$3\sqrt{3} + 3$
	1	$3 + \sqrt{3}$	$3\sqrt{3} + 8$
$(\sqrt{3})2 + 3\sqrt{3} + 5 = 3 + 3\sqrt{3} + 5 = 3\sqrt{3} + 8$			

59. Use synthetic substitution with $a = -6$. The remainder must be zero, so $432 - 216k = 0$, or $k = 2$. The depressed polynomial is $2x^2 + 3x - 5$, which factors to $(2x + 5)(x - 1)$. Therefore, we can write $kx^3 + 15x^2 + 13x - 30 = (x + 6)(2x + 5)(x - 1)$.

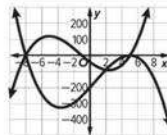


61.



62a. Sample answer: By the Factor Theorem, $P(x) = (x + 8)(x - 5)(ax + b)$ where $a \neq 0$. Substitute for x using $P(2) = -90$ to get $-90 = (10)(-3)(2a + b)$ or $3 = 2a + b$. Two solutions are $a = -1$, $b = 5$ and $a = 1$, $b = 1$. Two possible equations for $P(x)$ are $P(x) = (x + 8)(x - 5)(-x + 5)$ and $P(x) = (x + 8)(x - 5)(x + 1)$.

62b. $(2, -90)$, $(-8, 0)$, and $(5, 0)$.



62c. $P(x) = (x + 8)(x - 5)(-4x + 11)$; sample answer: Because $3 = 2a + b$, then $b = 3 - 2a$ so $P(x) = (x + 8)(x - 5)(ax - 2a + 3)$. Substitute for x by using $P(4) = 60$ to get $60 = (12)(-1)(2a + 3)$ or $-5 = 2a + 3$ so $a = -4$ and $b = 11$.

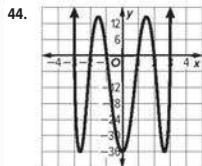
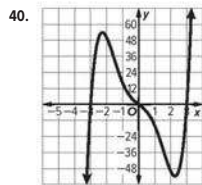
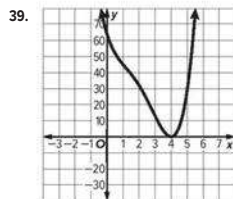
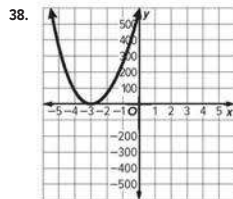
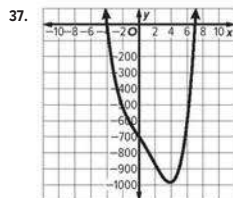
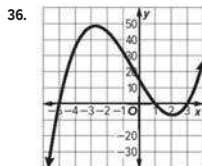
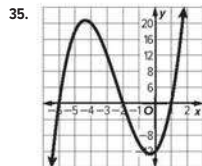
63. Sample answer: If $x - a$ is a factor of $f(x)$, then $f(a)$ has a factor of $(a - a)$ or 0. Since a factor of $f(a)$ is 0, $f(a) = 0$. Now assume that $f(a) = 0$. If $f(a) = 0$, then the Remainder Theorem states that the remainder is 0 when $f(x)$ is divided by $x - a$. This means that $x - a$ is a factor of $f(x)$. This proves the Factor Theorem.

65. Sample answer: When $x = 1$, $f(1)$ is the sum of all of the coefficients and constants in $f(x)$, in this case, a, b, c, d , and e . The sum of a, b, c, d , and e is 0, so however the coefficients are arranged, $f(1)$ will always equal 0, and $f(x)$ will have a rational root.

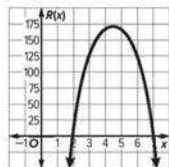
66. Sample answer: A zero can be located using the Remainder Theorem and a table of values by determining when the output, or remainder, is equal to zero. For instance, if $f(6)$ leaves a remainder of 2 and $f(7)$ leaves a remainder of -1, then you know that there is a zero between $x = 6$ and $x = 7$.

67. Tyrone; sample answer: By the Factor Theorem, $(x - r)$ is a factor when $P(r) = 0$. $P(r) = 0$ when $x = -1$ and $x = 2$.

Lesson 3-5



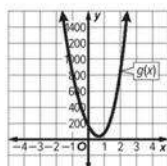
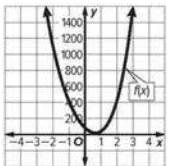
45. Sample answer:



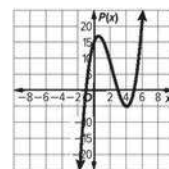
58a. $f(x) = (x^2 - 2x + 17)(9x^2 - 12x + 5)$, or $f(x) = 9x^4 + 6x^3 + 134x^2 - 194x + 85$; sample answer: If $-1 - 4i$ is a zero, $-1 + 4i$ is also a zero, and the quadratic factor is $x^2 + 2x + 17$. If $\frac{2}{3} + \frac{1}{3}i$ is a zero, then $\frac{2}{3} - \frac{1}{3}i$ is also a zero, and the quadratic factor is $x^2 - \frac{4}{3}x + \frac{5}{9}$, or $9x^2 - 12x + 5$.

58b. Sample answer: $g(x) = 2(x^2 - 2x + 17)(9x^2 - 12x + 5)$; another polynomial function with the same zeros and degree can be obtained by multiplying $f(x)$ by any nonzero whole number. For instance, $g(x) = 2(x^2 - 2x + 17)(9x^2 - 12x + 5)$.

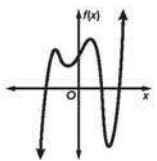
58c. No; sample answer: The graphs cannot be sketched from only imaginary zeros because they do not show on a graph.



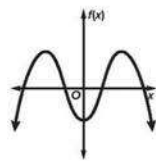
59. Sample answer: The graph is accurate as far as the location of the zeros, but does not consider any vertical dilation. It could be improved by finding more points between the roots.



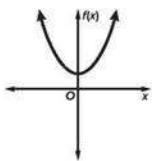
61a. Sample answer:



61b. Sample answer:



61c. Sample answer:



65. Sample answer: To determine the number of positive real roots, determine how many times the signs change in the polynomial as you move from left to right. In this function there are 3 changes in sign. Therefore, there may be 3 or 1 positive real roots. To determine the number of negative real roots, I would first evaluate the polynomial for $-x$. All of the terms with an odd-degree variable would change signs. Then I would again count the number of sign changes as I move from left to right. There would be only one change. Therefore there may be 1 negative root.

66. Both Brianne and Amrita could be correct; sample answer: The graph crosses the x -axis twice, so there are two real zeros, one positive and one negative. This means Amrita could be correct. There could be two other zeros of the function that are imaginary. This means Brianne could also be correct.

Inverse and Radical Functions

Module Goals

- Students graph and verify inverses.
- Students simplify expressions with radical and rational exponents.
- Students graph, analyze, and solve radical functions and equations.

Focus

Domain: Algebra, Functions

Standards for Mathematical Content:

A.SSE.2 Use the structure of an expression to identify ways to rewrite it.

F.IF.7b Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Also addresses A.REI.2, F.IF.5, F.BF.1b, and F.BF.4a.

Standards for Mathematical Practice:

All Standards for Mathematical Practice will be addressed in this module.

Coherence

Vertical Alignment

Previous

Students simplified numerical square root expressions.

N.RN.2

Now

Students simplify radical expressions and solve radical equations. Students write and graph the inverse of a function and analyze the relationship between a function and its inverse.

A.SSE.2, F.IF.5, F.BF.4a

Next

Students will analyze the relationship between a function and its inverse (exponential and logarithmic).

A.SSE.2, F.IF.7e

Rigor

The Three Pillars of Rigor

To help students meet standards, they need to illustrate their ability to use the three pillars of rigor. Students gain conceptual understanding as they move from the Explore to Learn sections within a lesson. Once they understand the concept, they practice procedural skills and fluency and apply their mathematical knowledge as they go through the Examples and Independent Practice.

1 CONCEPTUAL UNDERSTANDING 2 FLUENCY 3 APPLICATION

EXPLORE

LEARN

EXAMPLE & PRACTICE

Suggested Pacing

Lessons	Standards	45-min classes	90-min classes
Module Pretest and Launch the Module Video		1	0.5
4-1 Operations on Functions	F.BF.1b	2	1
4-2 Inverse Relations and Functions	F.IF.5, F.BF.4a	2	1
4-3 n th Roots and Rational Exponents	A.SSE.2	2	1
4-4 Graphing Radical Functions	F.IF.7b, F.BF.3	2	1
4-5 Operations with Radical Expressions	A.SSE.2	2	1
4-6 Solving Radical Equations	A.REI.2	2	1
Module Review		1	0.5
Module Assessment		1	0.5
Total Days		15	7.5

Cheryl Tobey Math Probe
Combining Functions

Use the following functions to determine whether each statement is true.

$f(x) = 4x + 3$ $g(x) = \sqrt{x+3}$ $h(x) = 8$

Circle yes, no, or not enough information.	Explain your answer.
1. $\frac{f(x)}{g(x)} = \frac{f}{g}$	
yes no not enough information	
2. $h(g(h(x))) = x$	
yes no not enough information	
3. $f^{-1}(f(x)) = \frac{1}{f(x)}$	
yes no not enough information	
4. $g(x) \times (x) = \sqrt{x+3} \times x$	
yes no not enough information	
5. $h^{-1}(g(x)) = \sqrt{x+3} \times 8$	
yes no not enough information	
6. $h^{-1}(f(x)) = 8$	
yes no not enough information	
7. $g^{-1}(f(x)) = x^2 + 3$	
yes no not enough information	

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Correct Answers:

1. no 2. yes 3. no 4. no
5. no 6. yes 7. yes

Analyze the Probe

Review the probe prior to assigning it to your students.

In this probe, students will combine functions by finding function values, finding inverses, finding compositions, and using operations with functions.

Targeted Concepts Understand processes to evaluate functions, find compositions of functions, and find inverse functions.

Targeted Misconceptions

- Students may treat the f , g , h , and so on in function notation as variables and incorrectly use properties of operations with them.
- Students may have difficulty evaluating constant functions and applying one function to the results of another (composition of functions).
- Students may have difficulty evaluating a function for a value involving a variable such as, evaluating $f(x)$ for $f(x + h)$.
- Students may misinterpret the -1 in inverse function notation as “the reciprocal of” instead of the inverse function and/or they may have difficulty recognizing that with inverse functions, they have to be able to “undo” operations in a specific order.

Use the Probe after Lesson 4-2.

Collect and Assess Student Answers

If the student selects these responses...

1, 3. yes or not enough information

2, 6. no or not enough information

5. yes or not enough information

4. yes or not enough information

7. no or not enough information

Then the student likely...

is having difficulty with function notation. In Item 1, the student is considering the letter f as a variable and believes they can be “canceled.” For Item 3, the -1 in the inverse function notation is being used as the reciprocal of the function rather than the inverse.

does not understand how to evaluate constant functions.

does not understand how to evaluate functions for variable values.

is having difficulty with the process of “undoing” operations in the correct order.

Take Action

After the Probe Design a plan to address any possible misconceptions. You may wish to assign the following resources.

-  **ALEKS** Combining Functions, Composite Functions, and/or Inverse Functions
- Lessons 4-1 and 4-2, all Learns, all Examples



The Ignite! activities, created by Dr. Raj Shah, cultivate curiosity and engage and challenge students. Use these open-ended, collaborative activities, located online in the module Launch section, to encourage your students to develop a growth mindset towards mathematics and problem solving. Use the teacher notes for implementation suggestions and support for encouraging productive struggle.

Essential Question

At the end of this module, students should be able to answer the Essential Question.

How can the inverse of a function be used to help interpret a real-world event or solve a problem? Sample answer: Similar to how it is necessary to undo operations when solving equations in algebra, an inverse of a function can be used to undo a function to solve a problem.

What Will You Learn?

Prior to beginning this module, have your students rate their knowledge of each item listed. Then, at the end of the module, you will be reminded to have your students return to these pages to rate their knowledge again. They should see that their knowledge and skills have increased.

DINAH ZIKE FOLDABLES

Focus Students write notes as they study inverses and radical functions and relations in the lessons of this module.

Teach Have students make and label their Foldables as illustrated. Have students use the appropriate tab as they cover each lesson in this module. Encourage them to apply what they have learned by writing examples as well.

When to Use It Encourage students to add to their Foldables as they work through the module and use them to review for the module test.

Launch the Module

For this module, the Launch the Module video uses world travel to introduce students to inverse and radical functions. Students learn about foreign exchange rates and Fahrenheit and Celsius temperature conversions.

Module 4 Inverse and Radical Functions

Essential Question

How can the inverse of a function be used to help interpret a real-world event or solve a problem?

What Will You Learn?

How much do you already know about each topic **before** starting this module?

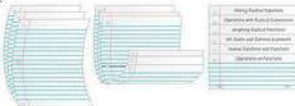
KEY:	Before	After
👉 I don't know	👉 I've heard of it	👉 I know it!
perform operations on functions		
combine functions using a composition of functions		
verify mathematically two relations or functions are inverses		
write expressions with rational exponents		
graph square root functions		
graph cube root functions		
perform operations with radical expressions		
solve radical equations by graphing		
solve radical equations algebraically		

Foldables. Make this Foldable to help you organize your notes about inverse and radical functions. Begin with three sheets of notebook paper.

1. **Stack** three sheets of notebook paper so that each is one inch higher than the previous.

2. **Align** the bottom of all the sheets.

3. **Fold** the papers and crease well. Open the papers and staple them. Label the pages with lesson titles.



Module 4 • Inverse and Radical Functions 157

Interactive Presentation



What Vocabulary Will You Learn?

- composition of functions
- conjugates
- cube root function
- index
- inverse functions
- inverse relations
- like radical expressions
- nth root
- principal root
- radical equation
- radical function
- radicand
- rational exponent
- square root function

Are You Ready?

Complete the Quick Review to see if you are ready to start this module.
Then complete the Quick Check.

Quick Review

Example 1

Use the related graph of $0 = 3x^2 - 4x + 1$ to determine its roots.

If exact roots cannot be found, state the consecutive integers between which the roots are located.



The roots are the x -coordinates where the graph crosses the x -axis.

The graph crosses the x -axis between 0 and 1 and at 1.

Example 2

Simplify $(3x^2 + 4x^2 + x^2 + 9x - 6) \div (x + 2)$ by using synthetic division.

$x - 2 \mid x^2 + 2, \text{ so } r = -2$

-2	3	4	1	9	-6
	6	-8	4	-10	2
	3	-2	5	-1	-4

The result is $3x^2 - 2x + 5x - 1 = \frac{3}{x+2}$.

Quick Check

Use the related graph of each equation to determine its roots. If exact roots cannot be found, state the consecutive integers between which the roots are located.

1. $x^2 - 4x + 1 = 0$ between 0 and 1, and between 2 and 4

2. $2x^2 + x - 6 = 0$ $\frac{3}{2}$ and -2

3. $3x^2 - 3x - 1 = 0$ between -1 and 0, and between 1 and 2

4. $2x^2 - 9x + 4 = 0$ $\frac{3}{2}$ and 4

Simplify each expression by using synthetic division.

5. $(5x^2 - 22x - 15) \div (x - 5)$ $5x + 3$

6. $(3x^2 + 14x - 12) \div (x + 4)$ $3x + 2 - \frac{20}{x+4}$

7. $(2x^2 - 7x^2 - 36x + 36) \div (x - 6)$ $2x^2 + 5x - 6$

8. $(3x^3 - 13x^2 + 17x^2 - 16x + 15) \div (x - 3)$ $3x^2 - 4x^2 + 5x - 3 + \frac{5}{x-3}$

How Did You Do?

Which exercises did you answer correctly in the Quick Check?

What Vocabulary Will You Learn?

ELL As you proceed through the module, introduce the key vocabulary by using the following routine.

Define A radical equation is an equation that has a variable in the radicand.

Example $\sqrt{4x + 8} - 1 = 15$

Ask Is $7x - 11 = \sqrt{9}$ a radical equation? Explain. **No; sample answer:** because there is no variable in the radicand.

Are You Ready?

Students may need to review the following prerequisite skills to succeed in this module.

- finding function values
- identifying relations as functions
- understanding powers of rational numbers
- simplifying square roots
- simplifying radicals
- simplifying radical expressions



ALEKS is an adaptive, personalized learning environment that identifies precisely what each student knows and is ready to learn, ensuring student success at all levels.

You may want to use the Radicals and Advanced Functions section to ensure student success in this module.



Mindset Matters

Model Constructive Feedback

In order for students to grow, they need to receive timely, constructive feedback that references a specific skill or area. You can also model what appropriate feedback looks and sounds like so that students can collaborate and give one another constructive feedback in a way that is positive and helpful.

How Can I Apply It?

Use the **Leveled Discussion Questions** in the Teacher Edition to ask students questions and share feedback on their thinking. This is a great opportunity to model feedback for the class so that students can give one another feedback during collaborative activities.

Operations on Functions

LESSON GOAL

Students simplify and graph operations on functions, and determine compositions of functions.

1 LAUNCH



Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP



Explore: Adding Functions



Develop:

Operations on Functions

- Add and Subtract Functions
- Multiply and Divide Functions
- Graphs of Combined Functions (online)

Compositions of Functions

- Compose Functions by Using Ordered Pairs
- Compose Functions
- Use Composition of Functions



You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE



Exit Ticket



Practice

DIFFERENTIATE



View reports of student progress on the **Checks** after each example.

Resources

Remediation: Combining Functions

Extension: Iteration

AL	LB	EL
●	●	
	●	●

Language Development Handbook

Assign page 19 of the *Language Development Handbook* to help your students build mathematical language related to simplifying and graphing operations on functions.

You can use the tips and suggestions on page T19 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min

1 day

45 min

2 days

Focus

Domain: Functions

Standards for Mathematical Content:

F.BF.1b Combine standard function types using arithmetic operations.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

4 Model with mathematics.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous

Students performed operations on polynomials.

A.CED.1

Now

Students find the sum, difference, product, and quotient of functions. Students compose functions.

F.BF.1b

Next

Students will describe and analyze the relationship between a function and its inverse.

F.IF.5, F.BF.4a

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING **2 FLUENCY** **3 APPLICATION**



Conceptual Bridge In this lesson, students develop an understanding of function operations. They build fluency by using arithmetic operations to combine functions, and they apply their understanding by solving real-world problems.

Mathematical Background

Addition, subtraction, multiplication, and division can be used to combine functions. Composition of functions is a method of combining functions, and is fundamentally different from arithmetic operations on functions. In the composition $g \circ f$, the output $f(x)$ is used as input for g .



Interactive Presentation

Warm Up

Given that $f(x) = -4x + 3$ and $g(x) = \frac{1}{2}x - 3$, find each value.

- $f(7)$
- $g(-8)$
- $f\left(\frac{1}{2}\right)$
- $g\left(-\frac{1}{2}\right)$
- $f(6) - g(-4)$

Show Answers

Warm Up

Launch the Lesson

Accurately predicting the weather involves complex mathematical models based on data. Data is collected at and above the Earth's surface using weather balloons, ships, buoys, aircraft, radar, weather balloons, and satellites and transmitted to centers throughout the world for analysis. Sometimes analysis requires performing more than one function on the data to find many weather indicators. Combining functions by performing one after another is called **composition of functions**.

Launch the Lesson

Vocabulary

Expand All Collapse All

composition of functions

An operation that uses the results of one function to evaluate a second function.

- How do you think you could find the sum, difference, or product of two functions?
- One definition of composition is "the act of combining parts or elements to form a whole." How can that help you remember what a composition of functions is?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- finding function values

Answers:

- 25
- 7
- 1
- $-3\frac{1}{4}$
- 16

Launch the Lesson



Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will learn how predicting the weather requires combining functions to analyze data.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then discuss the questions below with the class.



Explore Adding Functions

Objective

Students analyze the slopes and y-intercepts of the sums of linear functions.

MP Teaching the Mathematical Practices

3 Make Conjectures Students will make conjectures about the graph of the sum of two linear functions and then build a logical progression of statements to validate the conjectures. Once students have made their conjectures, guide the students to validate them.

5 Decide When to Use Tools Mathematically proficient students can make sound decisions about when to use mathematical tools such as a sketch. Help them see why using a sketch to analyze the graphs will help them draw conclusions.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will be presented with a sketch with three linear functions, $f(x)$, $g(x)$, and $f(x) + g(x)$, and will compare the slopes of these functions. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

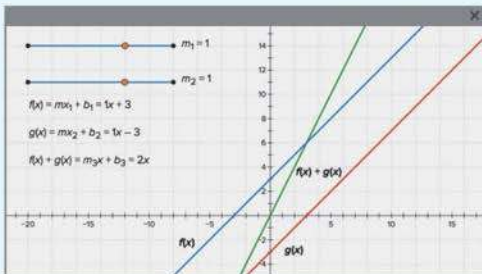
Adding Functions

INQUIRY Do you think that the graph of $f(x) + g(x)$ will be more or less steep than the graphs of $f(x)$ and $g(x)$?

Consider two linear functions $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. What would it mean to find the sum of the functions, that is, $f(x) + g(x)$?

You can use the sketch to compare the slope m_1 of $f(x)$ to m_2 and $m_1 + m_2$ and then complete the exercises.

Explore



Explore

WEB SKETCHPAD



Students use the sketch to explore adding functions.



Interactive Presentation

Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Adding Functions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What is the relationship between m_1 , m , and m_2 ? **Sample answer:** m is the slope of the sum equation, so $m = m_1 + m_2$.
- How could you find the relationship for $f(x) - g(x)$? **Sample answer:** Experiment with different functions and discover how the slopes are related.

Inquiry

Do you think that the graph of $f(x) + g(x)$ will be more or less steep than the graphs of $f(x)$ and $g(x)$? Explain your reasoning. **Sample answer:** The slope of $f(x) + g(x)$ would be more steep than either of them if both $f(x)$ and $g(x)$ have a positive slope; it would be less steep than either of them if $m_1 < 0 < m_2$ and $m_1 + m_2 < m_1$ or m_2 . It could also be equally as steep as one of the lines if m_1 or m_2 is 0.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Operations on Functions

Objective

Students find sums, differences, products, and quotients of functions.

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between how sums and differences of functions are found algebraically and how they are found by graphing.

What Students Are Learning

If $h(x)$ is a linear function, it can be written in the form $h(x) = mx + b$, where m is the slope and b is the y -intercept. If two linear functions are parallel, the functions have the same slope, but different y -intercepts. Consider this when defining $(f + g)(x)$ and comparing it to $f(x)$ and $g(x)$.

Example 1 Add and Subtract Functions

MP Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. Use the Study Tip to guide students to see what information they can gather about the sum or difference of two functions just by looking at the functions.

Questions for Mathematical Discourse

- AL** Why are parentheses used when the expressions are substituted into the function operations? **Sample answer:** the parentheses around each expression help you keep track of how addition or subtraction are distributed to the terms of each expression.
- OL** When adding two functions, is it possible for the sum to have less terms than either of the original functions? **Yes; Sample answer:** Two like terms could result in zero when performing the operation on the functions.
- BL** If the domain of $f(x)$ is $(-\infty, 3) \cup (3, \infty)$ and the domain of $g(x)$ is $[-5, 7]$, what is the domain of $(f + g)(x)$? **$[-5, 3) \cup (3, 7]$**

Common Error

When subtracting functions, remind students to distribute the negative throughout the function.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Operations on Functions

Explore: Adding Functions

Online Activity Use graphing technology to complete the Explore.

INQUIRY Do you think that the graph of $(f + g)(x)$ will be more or less steep than the graphs of $f(x)$ and $g(x)$?

Today's Goals

- Find sums, differences, products, and quotients of functions.
- Find compositions of functions.

Today's Vocabulary
composition of functions

Learn: Operations on Functions

Key Concept • Operations on Functions

Operation	Definition	Example: Let $f(x) = 3x$ and $g(x) = 2x - 4$.
Addition	$(f + g)(x) = f(x) + g(x)$	$(f + g)(x) = 3x + (2x - 4)$ $= 5x - 4$
Subtraction	$(f - g)(x) = f(x) - g(x)$	$(f - g)(x) = 3x - (2x - 4)$ $= x + 4$
Multiplication	$(f \cdot g)(x) = f(x) \cdot g(x)$	$(f \cdot g)(x) = 3x(2x - 4)$ $= 6x^2 - 12x$
Division	$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ $g(x) \neq 0$	$(\frac{f}{g})(x) = \frac{3x}{2x - 4}$, $x \neq 2$

To graph the sum or difference of functions, graph each function separately. Then add or subtract the corresponding function values.

Example 1: Add and Subtract Functions

Given $f(x) = -x^2 + 3x + 1$ and $g(x) = 2x^2 - 5$, find each function.

a. $(f + g)(x)$

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ &= (-x^2 + 3x + 1) + (2x^2 - 5) \quad \text{Addition of functions} \\ &= -x^2 + 3x + 1 + 2x^2 - 5 \\ &= x^2 + 3x - 4 \end{aligned}$$

$f(x) = -x^2 + 3x + 1$

and $g(x) = 2x^2 - 5$

Add:

Simplify:

(continued on the next page)

Go Online You can complete an Extra Example online.

Study Tip

Degree If the degree of $f(x)$ is m and the degree of $g(x)$ is n , then the degrees of $(f + g)(x)$ and $(f - g)(x)$ can be at most m or n , whichever is greater.

Interactive Presentation

Operations on Functions

The following table summarizes the operations on functions. Use the table to find the sum, difference, product, and quotient of two functions.

KEY CONCEPT: OPERATIONS ON FUNCTIONS

Operation	Definition	Example: Let $f(x) = 3x$ and $g(x) = 2x - 4$.
Addition	$(f + g)(x) = f(x) + g(x)$	$(f + g)(x) = 3x + (2x - 4)$ $= 5x - 4$
Subtraction	$(f - g)(x) = f(x) - g(x)$	$(f - g)(x) = 3x - (2x - 4)$ $= x + 4$
Multiplication	$(f \cdot g)(x) = f(x) \cdot g(x)$	$(f \cdot g)(x) = 3x(2x - 4)$ $= 6x^2 - 12x$
Division	$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ $g(x) \neq 0$	$(\frac{f}{g})(x) = \frac{3x}{2x - 4}$, $x \neq 2$

Learn

TAP



Students can tap on a Study Tip to learn more about operations on linear function.



b. $(f - g)(x)$

$$(f - g)(x) = f(x) - g(x)$$

$$= (-x^2 + 3x + 5) - (2x^2 - 5)$$

$$= -x^2 + 3x + 5 - 2x^2 + 5$$

$$= -3x^2 + 3x + 10$$

Subtraction of functions

$$f(x) = -x^2 + 3x + 5 \text{ and } g(x) = 2x^2 - 5$$

Subtract

Simplify

Example 2. Multiply and Divide FunctionsGiven $f(x) = 4x^2 - 2x + 3$ and $g(x) = -x + 5$, find each function.

a. $(f \cdot g)(x)$

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$= (4x^2 - 2x + 3)(-x + 5)$$

$$= -4x^3 + 20x^2 + 2x^3 - 10x + 3x + 15$$

$$= -4x^3 + 22x^2 - 13x + 15$$

Multiplication of functions

$$f(x) = 4x^2 - 2x + 3 \text{ and } g(x) = -x + 5$$

Distributive Property

Simplify

b. $\left(\frac{f}{g}\right)(x)$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$= \frac{4x^2 - 2x + 3}{-x + 5}, x \neq 5$$

Division of functions

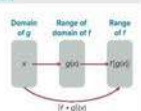
$$f(x) = 4x^2 - 2x + 3 \text{ and } g(x) = -x + 5$$

Simplify

CheckGiven $f(x) = -x^2 + 1$ and $g(x) = x + 1$, find each function.

$$(f \cdot g)(x) = \frac{f(x)}{g(x)}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Learn Compositions of Functions**Key Concept • Composition of Functions**Suppose f and g are functions such that the range of g is a subset of the domain of f . Then the composition function $f \circ g$ can be described by $(f \circ g)(x) = f(g(x))$.

Go Online You can complete an Extra Example online.

Example 2 Multiply and Divide Functions**Teaching the Mathematical Practices**

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. In Example 2, students must transform the expressions to find the product or quotient of two functions.

Questions for Mathematical Discourse

- A1** In part b, why must you exclude any domain values that cause the denominator to be 0? **You cannot divide by 0.**
- O1** If the denominator was a factor of the numerator, would the domain restriction remain? Explain. **Yes; the domain must exclude any values that are undefined for the quotient.**
- B1** What would the graph of $\left(\frac{f}{g}\right)(x)$ look like near $x = 5$? As the denominator approaches zero, the function value increases more rapidly, so as x approaches 5, $\left(\frac{f}{g}\right)(x)$ would approach ∞ or $-\infty$.

Learn Compositions of Functions**Objective**

Students find compositions of functions.

**Teaching the Mathematical Practices**

7 Use Structure Help students to explore how to use the structure of functions to compose them.

About the Key Concept

Addition, subtraction, multiplication, and division can be used to combine functions. Composition of functions is a method of combining functions, and is fundamentally different from arithmetic operations on functions. In the composition $g \circ f$, the output $f(x)$ is used as input for g .

Interactive Presentation

Example 2

TAP

Students move through the steps to multiply and divide functions.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Example 4 Compose Functions by Using Ordered Pairs

MP Teaching the Mathematical Practices

1 Monitor and Evaluate Point out that in Example 4, students should stop and evaluate their progress and change course if necessary. Students should evaluate whether the values they have calculated for $g(x)$ make sense before evaluating $[f \circ g](x)$.

Questions for Mathematical Discourse

- AL** For $f[g(x)]$, which function do you evaluate first? $g(x)$ Is $[f \circ g](x)$ the same as $f(x) \cdot g(x)$? **No** Does $f[g(x)] = g[f(x)]$? **No**
- OL** Why does the domain of $[g \circ f](x)$ have fewer values than $[f \circ g](x)$? **Not all values of $f(x)$ are defined for $g(x)$.**
- BL** What needs to be true for the composition of functions to have the same number of ordered pairs as the original functions? **The range of the inner function must equal the domain of the outer function.**

Common Error

Remind students to not confuse composition of functions, $[f \circ g](x)$, with $(f \cdot g)(x)$.

Example 5 Compose Functions

MP Teaching the Mathematical Practices

1 Check Answers Mathematically proficient students continually ask themselves, “Does this make sense?” Use the Study Tip to encourage students to graph the composition functions to confirm the domain and range.

Questions for Mathematical Discourse

- AL** In what order do you perform substitution when composing functions? **Substitute into the inside function first, then substitute into the outside function.**
- OL** What kind of function will be the result of the composition of two linear functions? **A linear function**
- BL** What would the degree of $f(g(x))$ be if $f(x)$ is a degree 5 polynomial and $g(x)$ is a degree 3 polynomial? **15**

Example 4 Compose Functions by Using Ordered Pairs

Given f and g , find $[f \circ g](x)$ and $[g \circ f](x)$. State the domain and range for each.

$$f = \{(1, 12), (0, 11), (2, 13), (9, 7)\}$$

$$g = \{(4, 1), (5, 0), (3, 9), (12, 10)\}$$

Part A Find $[f \circ g](x)$ and $[g \circ f](x)$.

To find $f \circ g$, evaluate $g(x)$ first then use the range to evaluate $f(x)$.

$$f(g(4)) = f(1) \text{ or } 12$$

$$g(f(1)) = 1$$

$$f(g(0)) = f(11) \text{ or } 13$$

$$g(f(11)) = 10$$

$$f(g(2)) = f(9) \text{ or } 7$$

$$g(f(9)) = 11$$

$$f(g(9)) = f(0) \text{ or } 11$$

$$g(f(0)) = 9$$

$$f(g(12)) = f(10) \text{ or } 11$$

$$g(f(12)) = 7$$

Because 11 and 7 are not in the domain of g , $g \circ f$ is undefined for $x = 11$ and $x = 7$. So, $g \circ f = \{(1, 10), (0, 9)\}$.

Part B State the domain and range.

$[f \circ g](x)$ The domain is the x -coordinates of the composed function, so $D = \{4, 5, 12, 13\}$. The range is the y -coordinates of the composed function, so $R = \{7, 11, 12, 13\}$.

$[g \circ f](x)$ The domain is the x -coordinates of the composed function, so $D = \{0, 1\}$. The range is the y -coordinates of the composed function, so $R = \{9, 10\}$.

Example 5 Compose Functions

Given $f(x) = 2x - 5$ and $g(x) = 3x$, find $[f \circ g](x)$ and $[g \circ f](x)$. State the domain and range for each.

Part A Find $[f \circ g](x)$ and $[g \circ f](x)$.

$$[f \circ g](x) = f(g(x))$$

$$[g \circ f](x) = g(f(x))$$

$$= f(3x)$$

$$= g(2x - 5)$$

$$= 2(3x) - 5$$

$$= 3(2x - 5)$$

$$= 6x - 5$$

$$= 6x - 15$$

Part B State the domain and range.

Because $[f \circ g](x)$ and $[g \circ f](x)$ are both linear functions with nonzero slopes, $D = \{\text{all real numbers}\}$ and $R = \{\text{all real numbers}\}$ for both functions.

Gas Online You can complete an Extra Example online.

Study Tip
Domain and Range To ensure you have the right domain and range, it can help to graph $[f \circ g](x)$ and $[g \circ f](x)$.

Interactive Presentation

Compose Functions

Find $[f \circ g](x) = f(g(x))$ and $[g \circ f](x) = g(f(x))$. State the domain and range for each.

Find $[f \circ g](x) = f(g(x))$:

Drop the expressions and justifications for $f(g(x))$ and $g(f(x))$.

Compose **Simplify** **Check Answer**

$[f \circ g](x) = f(g(x))$

$= f(3x)$

$= 2(3x) - 5$

$= 6x - 5$

Composition of Functions $[g \circ f](x) = g(f(x))$

Compose $= g(2x - 5)$

$= 3(2x - 5)$

$= 6x - 15$

Example 5

DRAG & DROP



Students drag the expressions and justifications to compose functions.



Apply Example 6 Use Composition of Functions

Teaching the Mathematical Practices

- 1 Make Sense of Problems and Persevere in Solving Them,**
4 Model with Mathematics Students will be presented with a task. They will first seek to understand the task, and then determine possible entry points to solving it. As students come up with their own strategies, they may propose mathematical models to aid them. As they work to solve the problem, encourage them to evaluate their model and/or progress, and change direction, if necessary.

Check

Given $f(x) = -x + 1$ and $g(x) = 2x^2 - x$, find $f \circ g(x)$ and $g \circ f(x)$. State the domain and range for each.

$f \circ g(x) = -2x^2 + x + 1$
 Domain of $f \circ g(x)$: all real numbers
 Range of $f \circ g(x)$: all real numbers

$g \circ f(x) = -2x^2 + 6x^2 - 5x + 1$
 Domain of $g \circ f(x)$: all real numbers
 Range of $g \circ f(x)$: all real numbers

Apply Example 6 Use Composition of Functions

BOX OFFICE A movie theater charges \$8.50 for each of the x tickets sold. The manager wants to determine how much the movie theater gets to keep of the ticket sales if they have to give the studios 75% of the money earned on ticket sales $f(x)$. If the amount they keep of each ticket sale is $h(x)$, which composition represents the total amount of money the theater gets to keep?

1 What is the task?

Describe the task in your own words. Then list any questions that you may have. How can you find answers to your questions?

Sample answer: I need to write functions for $f(x)$ and $h(x)$ and use them to create a composition that represents the money that the theater keeps. If the studios get 75%, what does the theater get to keep? Should the composition be $h \circ f(x)$ or $f \circ h(x)$?

2 How will you approach the task? What have you learned that you can use to help you complete the task?

Sample answer: First, I will determine functions for $f(x)$ and $h(x)$. Then, I will determine the order of the composition and simplify it. I will apply what I have learned in previous examples to complete the task.

3 What is your solution?

Use your strategy to solve the problem.

What function represents the money earned on ticket sales,

$$f(x)? f(x) = 8.50x$$

What function represents the amount of money the theater keeps from each ticket sale, $h(x)$? $h(x) = 0.25x$

Because the theater uses the total earnings to determine the amount they keep from the ticket sales, what composition should be used to represent the situation? $h \circ f(x) = 2.125x$

4 How can you know that your solution is reasonable?

Write About It! Write an argument that can be used to defend your solution.

Sample answer: If the theater sells 1000 tickets in a weekend, they will earn \$8500, or \$8500. The theater will keep 25% of \$8500, which is \$2125. This is the same value as $h \circ f(1000)$.

Watch Out!

Order Remember that, for two functions $f(x)$ and $g(x)$, $f \circ g(x)$ is not always equal to $g \circ f(x)$. Given that the studios take their cut after the tickets have been sold, consider how that affects the order of $f(x)$ and $h(x)$.

Go Online

You can complete an Extra Example online.

Interactive Presentation

Unit Composition of Functions

BOX OFFICE A movie theater charges \$8.50 for each of the x tickets sold. The manager wants to determine how much the movie theater gets to keep of the ticket sales if they have to give the studios 75% of the money earned on ticket sales $f(x)$. If the amount they keep of each ticket sale is $h(x)$, which composition represents the total amount of money the theater gets to keep?

Apply Example 6

CHECK



Students complete the Check online to determine if students are ready to move on.

Recommended Use

Have students work in pairs or small groups. You may wish to present the task, or have a volunteer read it aloud. Then allow students the time to make sure they understand the task, think of possible strategies, and work to solve the problem.

Encourage Productive Struggle

As students work, monitor their progress. Instead of instructing them on a particular strategy, encourage them to use their own strategies to solve the problem and to evaluate their progress along the way. They may or may not find that they need to change direction or try out several strategies.

Signs of Non-Productive Struggle

If students show signs of non-productive struggle, such as feeling overwhelmed, frustrated, or disengaged, intervene to encourage them to think of alternate approaches to the problem. Some sample questions are shown.

- How much money would the theater keep if they sold 1 ticket?
- If the theater has to give the studio 75% of the money earned on a ticket sale, what percentage of a ticket sale do they keep?

Write About It!

Have students share their responses with another pair/group of students or the entire class. Have them clearly state or describe the mathematical reasoning they can use to defend their solution.

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–18
2	exercises that use a variety of skills from this lesson	19–35
2	exercises that extend concepts learned in this lesson to new contexts	36–39
3	exercises that emphasize higher-order and critical-thinking skills	40–44

ASSESS AND DIFFERENTIATE

Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, THEN assign:

- Practice Exercises 1–39 odd, 40–44
- Extension: Iteration
- ALEKS** Function Operations Inverse Functions

IF students score 66%–89% on the Checks, THEN assign:

- Practice Exercises 1–43 odd
- Remediation, Review Resources: Combining Functions
- Personal Tutors
- Extra Examples 1–6
- ALEKS** Applications of Linear Equations with Two Variables

IF students score 65% or less on the Checks, THEN assign:

- Practice Exercises 1–17 odd
- Remediation, Review Resources: Combining Functions
- Quick Review Math Handbook:** Operations on Functions
- ALEKS** Applications of Linear Equations with Two Variables

Practice

Examples 1 and 2

Find $f \circ g(x)$, $f \circ g(x)$, $f \circ g(x)$, and $(\frac{f}{g})(x)$ for each $f(x)$ and $g(x)$. See margin.

1. $f(x) = 2x$

$g(x) = -4x + 5$

3. $f(x) = x - 2$

$g(x) = 2x - 7$

5. $f(x) = -x^2 + 6$

$g(x) = 2x^2 + 3x - 5$

2. $f(x) = x - 1$

$g(x) = 5x - 2$

4. $f(x) = x^2$

$g(x) = x - 5$

6. $f(x) = 3x^2 - 4$

$g(x) = x^2 - 8x + 4$

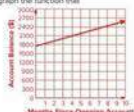
Example 3

7. FINANCE Theron opens a checking account that he only uses to pay fixed bills, which are expenses that are the same each month, such as car loans or rent. The checking account has an initial balance of \$1750 and Theron deposits \$925 each month. The balance of the account can be modeled by $b(x) = 1750 + 925x$, where x is the number of months since the account was opened. The total of Theron's fixed bills is modeled by $b(x) = 846x$. Define and graph the function that represents the account balance after he pays his bills.

a. Identify and write a new function to represent the account balance.

$b(x) - b(x) = b(x) = 846x + 1750$

b. Graph the combined function.



8. BASEBALL A coach is ordering custom practice T-shirts and game jerseys for each of the team members. The coach orders T-shirts from a local shop that charges \$1.50 for each, plus a \$35 initial printer fee. The cost of the T-shirts is modeled by $t(x) = 1.5x + 35$, where x is the number of team members. He orders jersey pants, which cost \$18 each with \$20 shipping. The cost of the jerseys is modeled by $j(x) = 18x + 20$. Define and graph the function that represents the total cost of the T-shirts and jerseys.

a. Identify and write a new function to represent total cost.

$t(x) + j(x) = t(x) = 1.5x + 55$

b. Graph the combined function.



Lesson 4-1 • Operations on Functions 163

Example 4

For each pair of functions, find $f \circ g$ and $g \circ f$ if they exist. State the domain and range for each. 9–12. See margin.

9. $f(x) = -8x - 4$, $g(x) = 2x - 6$, $h(x) = 12x - 3$

$g(x) = -4x - 1$, $h(x) = -4x - 5$

10. $f(x) = -7x - 3$, $g(x) = 12x - 3$, $h(x) = 12x - 3$

$g(x) = 12x - 3$, $h(x) = 12x - 3$, $h(x) = 12x - 3$

11. $f(x) = 15x - 4$, $g(x) = -2x - 8$, $h(x) = 12x - 3$

$g(x) = -2x - 8$, $h(x) = 12x - 3$, $h(x) = 12x - 3$

12. $f(x) = -8x - 4$, $g(x) = -4x - 5$, $h(x) = 12x - 3$

$g(x) = -4x - 5$, $h(x) = 12x - 3$, $h(x) = 12x - 3$

13. $f(x) = 2x$

$g(x) = x + 5$

14. $f(x) = -3x$

$g(x) = -x + 8$

15. $f(x) = x^2 - 6x - 2$

$g(x) = x - 6$

16. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$

17. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$

18. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$

19. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$

20. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$

21. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$

22. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$

23. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$

24. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$

25. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$

26. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$

27. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$

28. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$

29. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$

30. $f(x) = 2x^2 - x + 1$

$g(x) = 4x + 3$



If $f(x) = 3x$, $g(x) = x + 4$, and $h(x) = x^2 - 1$, find each value.

21. $f(g(1))$ 15

22. $g(h(5))$ 3

23. $g(h(-1))$ 1

24. $h(f(5))$ 224

25. $g(h(-3))$ 12

26. $h(g(0))$ 889

27. $f(h(5))$ 189

28. $f(h + g)(5)$ 72

29. $f(h + g)(-2)$ 21

30. $h(f(-6))$ 323

31. $f(h(0))$ -3

32. $f(g(7))$ 33

33. $f(h(-2))$ 9

34. $g(h + f)(-3)$ 4

35. $h(f + g)(3)$ 440

36. **AREA** Valerie wants to know the area of a figure made by joining an equilateral triangle and square along an edge. The function $f(x) = \frac{\sqrt{3}}{4}x^2$ gives the area of an equilateral triangle with side x . The function $g(x) = x^2$ gives the area of a square with side x . Write a function $h(x)$ that gives the area of the figure as a function of its side length x .

$$h(x) = f(x) + g(x) = \left(\frac{\sqrt{3}}{4} + 1\right)x^2$$



37. **USE A MODEL** The volume V of a weather balloon with radius r is given by $V(r) = \frac{4}{3}\pi r^3$. The balloon is being inflated so that the radius increases at a constant rate $\frac{dr}{dt} = 2$, where r is in inches and t is the number of seconds since inflation began.

a. Determine the function that represents the volume of the weather balloon in terms of time.

$$V(t) = \frac{4}{3}\pi (2t)^3 = 2\pi t^3 = 8\pi t^3 = \frac{8\pi}{3}t^3$$

b. Find the volume of the balloon 12 seconds after inflation begins. Round your answer to the nearest cubic meter. 2545 m³

Lesson 4 • Operations on Functions 165

38. **REASONING** The National Center for Education Statistics reports data showing that since 2006, college enrollment for men in thousands can be modeled by $f(x) = 389x + 7920$, where x represents the number of years since 2006. Similarly, enrollment for women can be modeled by $g(x) = 480x + 10,075$. Write a function for $(f + g)(x)$ and interpret what it represents. See margin.

39. **STRUCTURE** The table shows various values of functions $f(x)$, $g(x)$, and $h(x)$.

x	-1	0	1	2	3	4
$f(x)$	7	-2	0	2	4	1
$g(x)$	-3	-4	-5	0	1	1
$h(x)$	0	4	1	1	5	5

Use the table to find the following values:

a. $(f + g)(-4)$

b. $(f - g)(5)$

c. $(f \cdot h)(4)$

d. $\left(\frac{f}{g}\right)(3)$ 4

e. $\left(\frac{f}{h}\right)(3)$ 9

f. $\left(\frac{f}{h}\right)(h)$ undefined

Higher-Order Thinking Skills

40. **PERSEVERE** If $f(x) = g(3) = 5$ and $(f \cdot g)(3) = 6$, find $f(3)$ and $g(3)$. Explain. See Mod. 4 Answer Appendix.

41. **CREATE** Write two functions $f(x)$ and $g(x)$ such that $(f + g)(x) = x^2$. Sample answer: $f(x) = x$, $g(x) = x^2 - x$

42. **FIND THE ERROR** Chris and Tobias are finding $(f \cdot g)(x)$, where $f(x) = x^2 - 8$ and $g(x) = x^2 + 8$. Is either of them correct? Explain your reasoning. Tobias' sample answer: Chris did not substitute $g(x)$ for every x in $f(x)$.

Chris	Tobias
$(f \cdot g)(x) = \frac{f(x)}{g(x)}$	$(f \cdot g)(x) = \frac{f(x)}{g(x)}$
$= \frac{x^2 - 8}{x^2 + 8} = \frac{x^2 - 8}{x^2 + 8}$	$= \frac{(x^2 - 8)(x^2 + 8)}{(x^2 + 8)(x^2 + 8)}$
$= \frac{x^4 - 64}{x^4 + 16x^2 + 64}$	$= \frac{x^4 - 64}{x^4 + 16x^2 + 64}$
$= \frac{x^4 - 64}{x^4 + 16x^2 + 64}$	$= \frac{x^4 - 64}{x^4 + 16x^2 + 64}$

43. **PERSEVERE** Given $f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{x-1}$, determine each domain.

a. $g(x) \cdot g(x)$ $D = \{\text{all real numbers}\}$

b. $f(x) \cdot g(x)$ $D = \{x | x \geq 0\}$

44. **ANALYZE** State whether the following statement is sometimes, always, or never true. Justify your argument. See Mod. 4 Answer Appendix.

The domain of two functions $f(x)$ and $g(x)$ that are composed $g(f(x))$ is restricted by the domain of $g(x)$.

Answers

1. $(f + g)(x) = -2x + 5$; $(f - g)(x) = 6x - 5$; $(f \cdot g)(x) = -8x^2 + 10x$;

$$\left(\frac{f}{g}\right)(x) = \frac{2x}{-4x + 5}, x \neq \frac{5}{4}$$

2. $(f + g)(x) = 6x - 3$; $(f - g)(x) = -4x + 1$; $(f \cdot g)(x) = 5x^2 - 7x + 2$;

$$\left(\frac{f}{g}\right)(x) = \frac{x-1}{5x-2}, x \neq \frac{2}{5}$$

3. $(f + g)(x) = 3x - 9$; $(f - g)(x) = -x + 5$; $(f \cdot g)(x) = 2x^2 - 11x + 14$;

$$\left(\frac{f}{g}\right)(x) = \frac{x-2}{2x-7}, x \neq \frac{7}{2}$$

4. $(f + g)(x) = x^2 + x - 5$; $(f - g)(x) = x^2 + x + 5$; $(f \cdot g)(x) = x^3 - 5x$;

$$\left(\frac{f}{g}\right)(x) = \frac{x^2}{x-5}, x \neq 5$$

5. $(f + g)(x) = x^2 + 3x + 1$; $(f - g)(x) = -3x^2 - 3x + 11$; $(f \cdot g)(x) = -2x^4 - 3x^3 + 17x^2 + 18x - 30$;

$$\left(\frac{f}{g}\right)(x) = \frac{-x^2 + 6}{2x^2 + 3x - 5}, x \neq 1 \text{ or } -\frac{5}{2}$$

6. $(f + g)(x) = 4x^2 - 8x$; $(f - g)(x) = 2x + 8x - 8$; $(f \cdot g)(x) = 3x^4 - 24x^3 + 8x^2 + 32x - 16$;

$$\left(\frac{f}{g}\right)(x) = \frac{3x^2 - 4}{x^2 - 8x + 4}, x \neq 4 \pm 2\sqrt{3}$$

9. $f \circ g = \{(-4, 4)\}$, $D = \{-4\}$, $R = \{4\}$; $g \circ f = \{(-8, 0), (0, -4), (2, -5), (-6, -1)\}$, $D = \{-8, -6, 0, 2\}$, $R = \{-5, -4, -1, 0\}$

10. $f \circ g = \{(6, 12)\}$, $D = \{6\}$, $R = \{12\}$; $g \circ f = \{(-7, 5), (4, 1), (-3, 8)\}$, $D = \{-7, -3, 4\}$, $R = \{1, 5, 8\}$

11. $f \circ g$ is undefined, $D = \emptyset$, $R = \emptyset$; $g \circ f$ is undefined, $D = \emptyset$, $R = \emptyset$

12. $f \circ g$ is undefined, $D = \emptyset$, $R = \emptyset$; $g \circ f = \{(-4, 9), (0, 1), (-6, 13), (2, -3)\}$, $D = \{-6, -4, 0, 2\}$, $R = \{-3, 1, 9, 13\}$

13. $(f \circ g)(x) = 2x + 10$, $D = \{\text{all real numbers}\}$, $R = \{\text{all real numbers}\}$; $(g \circ f)(x) = 2x + 5$, $D = \{\text{all real numbers}\}$, $R = \{\text{all real numbers}\}$

14. $(f \circ g)(x) = 3x - 24$; $(g \circ f)(x) = 3x + 8$, $D = \{\text{all real numbers}\}$, $R = \{\text{all real numbers}\}$

15. $(f \circ g)(x) = x^2 - 6x - 2$, $R = \{y | y \geq -11\}$; $(g \circ f)(x) = x^2 - 6x - 8$, $R = \{y | y \geq -17\}$

16. $(f \circ g)(x) = 32x^2 + 44x + 16$, $R = \{y | y \geq 0.875\}$; $(g \circ f)(x) = 8x^2 - 4x + 7$, $R = \{y | y \geq 6.5\}$

38. $(f + g)(x) = 869x + 17,575$. This function represents the enrollment of both men and women in college since 2006.

40. Let $a = f(3)$ and $b = g(3)$. Since $(f + g)(3) = 5$, then $a + b = 5$. Solving for a , we get $a = 5 - b$. Since $(f \cdot g)(3) = 6$, $ab = 6$. Substituting a , we get $(5 - b)(b) = 6$. Distributing the left hand side, we get $5b - b^2 = 6$, or $b^2 - 5b + 6 = 0$. Factoring, we get $(b - 2)(b - 3) = 0$, so $b = 2$ or $b = 3$. If $b = 2$, then $a = 3$. If $b = 3$, then $a = 2$. So the two solutions are $f(3) = 2$, $g(3) = 3$, and $f(3) = 3$, $g(3) = 2$.

44a. Always, since the range is dependent on the domain, the domain of $g(f(x))$ is restricted by the domain of $f(x)$.

44b. Sometimes; when $f(x) = 4x$ and $g(x) = \sqrt{x}$, $g(f(x)) = \sqrt{4x}$, $x \geq 0$. The domain of $g(x)$ restricts the domain of $g(f(x))$. When $f(x) = 4x$ and $g(x) = -\sqrt{x}$, $g(f(x)) = -\sqrt{4x}$. In this case, the domain of $g(x)$ does not restrict the domain of $g(f(x))$.

Inverse Relations and Functions

LESSON GOAL

Students graph and verify inverse functions.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Explore:** Inverse Functions

 **Develop:**

Inverse Relations and Functions

- Find an Inverse Relation
- Inverse Functions
- Inverses with Restricted Domains
- Interpret Inverse Functions

Verifying Inverses

- Use Compositions to Verify Inverses
- Verify Inverse Functions

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

 Formative Assessment Math Probe

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources	AL	LB	ELL
Remediation: Functions	●	●	●
Extension: Group Theory		●	●

Language Development Handbook

Assign page 20 of the *Language Development Handbook* to help your students build mathematical language related to graphing inverse functions.

 You can use the tips and suggestions on page T20 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

F.BF.4a Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

Standards for Mathematical Practice:

2 Reason abstractly and quantitatively.

6 Attend to precision.

Coherence

Vertical Alignment

Previous

Students found the sum, difference, product, and quotient of functions. Students composed functions.

F.BF.1b

Now

Students find the inverse of a function or relation. Students determine whether two functions or relations are inverses.

F.IF.5, F.BF.4a

Next

Students will simplify expressions involving radicals and rational exponents.

A.SSE.2

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

 **Conceptual Bridge** In this lesson, students develop an understanding of inverse relations and functions. They build fluency by finding inverses of relations and functions, and they apply their understanding by solving real-world problems.

Mathematical Background

To test whether two functions $f(x)$ and $g(x)$ are inverses, check that each of the two compositions $[f \circ g](x)$ and $[g \circ f](x)$ has the value x . When the inverse of a function is a function, then the original function is said to be one-to-one.



Interactive Presentation

Warm Up

Determine whether each relation is a function.

- $\{(-1, -5), (3, 5), (5, 7), (7, 11)\}$
- $\{(0, 8), (4, 7), (9, 9), (3, 0)\}$

Write a rule for each function. State the domain and range.

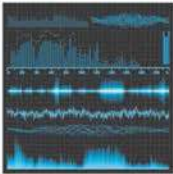
- $E = \{(-3, -15), (-1, -5), (0, 0), (4, 30)\}$
- $G = \{(-4, -3), (-3, 4), (1, 8), (9, 16)\}$

Show Answers

Warm Up

Launch the Lesson

The sound you hear is a result of particle-to-particle interaction. The speed of the sound varies with the temperature and many of the medium through which it is traveling. The formula $S = 108\sqrt{\frac{T}{m}}$ can be used to find the speed of sound in some mediums, where S represents the speed of sound, T represents the temperature, and m represents the average molecular mass. You can use inverse functions to rewrite the formula for the temperature in terms of speed.



Launch the Lesson

Today's Vocabulary

Expand All Collapse All

Inverse relations

Two relations, one of which contains points of the form (a, b) while the other contains points of the form (b, a) .

Inverse functions

Two functions for which both of their compositions are the identity function.

- What is an inverse of a function?
- How do you graph the inverse of a function?
- How can you find the inverse of a relation or function?
- How can you determine whether the inverse of a function is a function?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- determining whether relations are functions

Answers:

1. yes

2. no

3. $y = 5x$; $D = \{-3, -1, 0, 4\}$, $R = \{-15, -5, 0, 20\}$ 4. $y = x + 7$; $D = \{-4, -3, 0, 9\}$, $R = \{3, 4, 8, 16\}$

Launch the Lesson



Teaching the Mathematical Practices

2 Attend to Quantities Point out that it is important to note the meaning of the quantities used in the formula for the speed of sound. Once students have completed the lesson, encourage students to use inverse functions to rewrite the formula and ask students to describe the meaning of the formula.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.



Explore Inverse Functions

Objective

Students use a sketch to explore inverse functions.

MP Teaching the Mathematical Practices

5 Use Mathematical Tools Point out that to complete the Explore, students will need to use the sketches. Work with students to explore and deepen their understanding of inverse functions.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will be presented with a sketch of a function and its inverse. They will use the sketch to explore different points on each function and determine their relationship. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

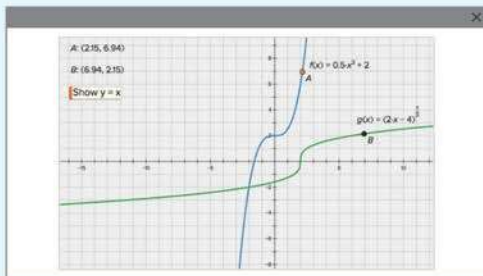
Inverse Functions

INQUIRY For what values of x and y is $(x, y) = x^2$ have an inverse that is also a function?

Graphs of Inverse Functions

You can use the sketch to explore the graph of a function and its inverse. Then complete the exercises.

Explore



Explore

WEB SKETCHPAD



Students use the sketch to complete an activity in which they explore graphs of inverse functions.



Interactive Presentation



Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

Explore Inverse Functions (*continued*)

Teaching the Mathematical Practices

3 Construct Arguments Throughout the Explore, students will use previously established results to construct arguments about graphs of functions and their inverses.

Questions

Have students complete the Explore activity.

Ask:

- What can the reflection of the graph in $y = x$ tell you about the graph of the inverse? **Sample answer:** You can look at the shape of the graph to make the reflection in $y = x$. Then, you can see whether the inverse would pass the vertical line test.
- Describe a function of degree 3 with an inverse that is not a function. **Sample answer:** If a function with degree 3 has a relative minimum and relative maximum, it will not pass the horizontal line test. For example, if it had more than one root, it would be crossing the x -axis more than once. Therefore, the inverse would not be a function.



Inquiry

For what values of n will $f(x) = x^n$ have an inverse that is also a function?

Sample answer: If n is an odd whole number, the inverse of $f(x)$ will also be a function.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Inverse Relations and Functions

Objective

Students find inverses of relations.



Teaching the Mathematical Practices

7 Use Structure Help students to explore the structure of inverse function. Guide them to see the pattern in the input and output values of a function and its inverse.

Important to Know

You can use a graphing calculator to compare a function and its inverse using tables and graphs.



Essential Question Follow-Up

Students have explored inverse functions.

Ask:

Why would you use the inverse of a function to model a real-world situation? **Sample answer:** You could use a function or the inverse of the function to model a relationship so that either quantity could be the independent variable. Then, you could use the functions to solve problems involving either quantity.

Example 1 Find an Inverse Relation



Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the graph of the function and its inverse to answer the question in the Think About It! feature.

Questions for Mathematical Discourse



A If you had a table of ordered pairs to graph a relation how could you create a table to graph the inverse? **switch the x and y columns**



OL For a relation of ordered pairs to be the same as its inverse, must $x = y$ for each pair? **No, a relation could have points (a, b) and (b, a) and its inverse would have the same points.**



BL What would be true about a relation that intersects the line $x = y$? **The relation would intersect its inverse.**



Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Inverse Relations and Functions

Lesson 4-2

Explore Inverse Functions

Online Activity Use graphing technology to complete the Explore.

INQUIRY For what values of n will $f(x) = x^n$ have an inverse that is also a function?

Learn Inverse Relations and Functions

Two relations are **inverse relations** if one relation contains elements of the form (a, b) when the other relation contains the elements of the form (b, a) .

Two functions f and g are **inverse functions** if and only if both of their compositions are the identity function.

Key Concepts • Inverse Functions

Words If f and f^{-1} are inverses, then $f(b) = a$ if and only if $f^{-1}(a) = b$.

Example Let $f(x) = x + 5$ and represent its inverse as $f^{-1}(x) = x + 5$.

Evaluate $f(7)$.

$f(7) = 7 + 5$

$f(7) = 7 + 5$ or 12 .

Evaluate $f^{-1}(12)$.

$f^{-1}(12) = 12 + 5$

$f^{-1}(12) = 2 + 5$ or 7 .

Not all functions have an inverse function. If a function fails the horizontal line test, you can restrict the domain of the function to make the inverse a function. Choose a portion of the domain on which the function is one-to-one. There may be more than one possible domain.

Example 1 Find an Inverse Relation

GEOMETRY The vertices of $\triangle ABC$ can be represented by the relation $\{(2, 4), (-3, 2), (4, 1)\}$. Find the inverse of the relation. Graph both the original relation and its inverse.

Step 1 Graph the relation.

Graph the ordered pairs and connect the points to form a triangle.



(continued on the next page)

Today's Goals

- Find inverses of relations.
- Verify that two relations are inverses by using compositions.

Today's Vocabulary
inverse relations
inverse functions

Think About It!

Write a function that does not pass the horizontal line test.

Sample answer: $f(x) = x^2$

Go Online You can complete an Extra Example online.

Lesson 4-2 • Inverse Relations and Functions 167

Interactive Presentation

Inverse Relations and Functions

Recall that a relation is a set of ordered pairs. Two relations are **inverse relations** if one relation contains elements of the form (a, b) when the other relation contains the elements of the form (b, a) . The domain of a relation becomes the range of its inverse, and the range of the relation becomes the domain of its inverse. For example, if $A = \{(2, 4), (-3, 2), (4, 1)\}$ and $B = \{(4, 2), (2, -3), (1, 4)\}$, then A and B are inverse relations.

All sets involving the ordered pairs of inverse functions are also related. The inverse of a function can be found by exchanging the domain and range. Two functions f and g are **inverse functions** if and only if both of their compositions are the identity function.

Key Concepts • Inverse Functions

Words If f and f^{-1} are inverses, then $f(b) = a$ if and only if $f^{-1}(a) = b$.

Example Let $f(x) = x + 5$ and represent its inverse as $f^{-1}(x) = x + 5$.

Evaluate $f(7)$.

$f(7) = 7 + 5$

$f(7) = 7 + 5$ or 12 .

Evaluate $f^{-1}(12)$.

$f^{-1}(12) = 12 + 5$

$f^{-1}(12) = 2 + 5$ or 7 .

Learn



Example 2 Inverse Functions



Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. In Example 2, students transform the function to find its inverse.

Questions for Mathematical Discourse

A1. What are the slopes of $f(x)$ and $f^{-1}(x)$? The slope of $f(x)$ is 3 and the slope of $f^{-1}(x)$ is $\frac{1}{3}$.

A2. Can a linear equation with a positive slope have an inverse with a negative slope? Explain. **No; sample answer:** the slope of the inverse of a linear equation will have the same sign because the inverse is a reflection in the line $y = x$.

B1. Are any quadratic functions one-to-one? Why or why not? **No; sample answer:** the graph of a quadratic function is a parabola which would never pass the horizontal line test.

Common Error

Remind students that f^{-1} is read *f inverse* or *the inverse of f*. Note that -1 is not an exponent.

Step 2 Find the inverse.

To find the inverse, exchange the coordinates of the ordered pairs. $\{(4, 2), (2, -3), (1, 4)\}$

Step 3 Graph the inverse.



Think About It
Describe the graph of the inverse relation.

Sample answer: It is a reflection in the line $y = x$.

Study Tip
Inverses If $f^{-1}(x)$ is the inverse of $f(x)$, the graph of $f^{-1}(x)$ will be a reflection of the graph of $f(x)$ in the line $y = x$.

Go Online
You can learn how to graph a relation and its inverse on a graphing calculator by watching the video online.

Example 2 Inverse Functions

Find the inverse of $f(x) = 3x + 2$. Then graph the function and its inverse.

Step 1 Rewrite the function.

Rewrite the function as an equation relating x and y .

$$f(x) = 3x + 2 \rightarrow y = 3x + 2$$

Step 2 Exchange x and y .

Exchange x and y in the equation.

$$x = 3y + 2$$

Step 3 Solve for y .

$$x = 3y + 2$$

$$x - 2 = 3y$$

$$\frac{x - 2}{3} = y$$

Step 4 Replace y with $f^{-1}(x)$.

Replace y with $f^{-1}(x)$ in the equation.

$$y = \frac{x - 2}{3} \rightarrow f^{-1}(x) = \frac{x - 2}{3}$$

The inverse of $f(x) = 3x + 2$ is

$$f^{-1}(x) = \frac{x - 2}{3}$$

Step 5 Graph $f(x)$ and $f^{-1}(x)$.



Check

Examine $f(x) = 3x + 2$.

Part A Find the inverse of $f(x) = 3x + 2$.

$$f^{-1}(x) = \frac{x - 2}{3}$$

Part B Graph $f(x) = 3x + 2$ and its inverse.



Go Online You can complete an Extra Example online.

Interactive Presentation

Inverse Functions

Find the inverse of $f(x) = 3x + 2$. Then graph the function and its inverse.

Step 1

Rewrite the function.

Rewrite the function as an equation relating x and y .

$f(x) = 3x + 2 \rightarrow y = 3x + 2$

Step 2

Exchange x and y .

Exchange x and y in the equation.

$x = 3y + 2$

Step 3

Solve for y .

$x - 2 = 3y$

$\frac{x - 2}{3} = y$

Step 4

Replace y with $f^{-1}(x)$.

Replace y with $f^{-1}(x)$ in the equation.

$y = \frac{x - 2}{3} \rightarrow f^{-1}(x) = \frac{x - 2}{3}$

The inverse of $f(x) = 3x + 2$ is

$f^{-1}(x) = \frac{x - 2}{3}$

Step 5

Graph $f(x)$ and $f^{-1}(x)$.

Example 2

TAP



Students move through the steps to find and graph the inverse of a function.



Example 3 Inverses with Restricted Domains

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the inverse, its graph, and the domains that make the inverse a function.

Questions for Mathematical Discourse

- AL** Why is the domain of $f^{-1}(x)$ restricted? $f^{-1}(x)$ isn't a function.
- OL** How does the axis of symmetry of $f(x)$ relate to the restricted domain of $f^{-1}(x)$? **Sample answer:** the axis of symmetry divides the function into two pieces that are each one-to-one. It divides the x -axis into the pieces that can be the restricted domain.
- BL** List the two possible restricted domains for $f(x) = ax^2 + bx + c$?
 $(-\infty, -\frac{b}{2a}]$, $[\frac{b}{2a}, \infty)$

Example 4 Interpret Inverse Functions

MP Teaching the Mathematical Practices

2 Attend to Quantities Point out that it is important to note the meaning of the inverse function in Example 4. Students should understand what the inverse function represents and how it can be used.

Questions for Mathematical Discourse

- AL** Let F equal the temperature in degrees Fahrenheit and C equal the temperature in degrees Celsius. How do F and C relate to each other through $T(x)$ and $T^{-1}(x)$? $C = T(F)$ and $F = T(C)$
- OL** How do you know that $T^{-1}(x)$ will be a function? **The graph of $T(x)$ passes the horizontal line test.**
- BL** Will there be a point where $T(x)$ and $T^{-1}(x)$ intersect? Explain. **Yes; sample answer:** they must intersect because they are linear functions with different slopes. What does the intersection represent? **It is the value for which the temperature in Celsius and in Fahrenheit are the same.** What else do you need to consider when determining whether the functions intersect? **Sample answer:** You need to consider whether the intersection occurs in the valid domain of the functions. In this case it does, at -40 . The domain is limited by absolute zero, which is 273.15 in Celsius and 459.67 in Fahrenheit.

Example 3 Inverses with Restricted Domains

Examine $f(x) = x^2 + 2x + 4$.

Part A. Find the inverse of $f(x)$.

$f(x) = x^2 + 2x + 4$	Original function
$y = x^2 + 2x + 4$	Replace $f(x)$ with y .
$x = y^2 + 2y + 4$	Exchange x and y .
$x - 4 = y^2 + 2y$	Subtract 4 from each side.
$x - 4 + 1 = y^2 + 2y + 1$	Complete the square.
$x - 3 = (y + 1)^2$	Simplify.
$\pm\sqrt{x-3} = y + 1$	Take the square root of each side.
$-1 \pm \sqrt{x-3} = y$	Subtract 1 from each side.
$f^{-1}(x) = -1 \pm \sqrt{x-3}$	Replace y with $f^{-1}(x)$.

Part B. If necessary, restrict the domain of the inverse so that it is a function.

Because $f(x)$ fails the horizontal line test, $f^{-1}(x)$ is not a function. Find the restricted domain of $f(x)$ so that $f^{-1}(x)$ will be a function. Look for a portion of the graph that is one-to-one. If the domain of $f(x)$ is restricted to $[-1, \infty)$, then the inverse is $f^{-1}(x) = -1 + \sqrt{x-3}$.

If the domain of $f(x)$ is restricted to $(-\infty, -1]$, then the inverse is $f^{-1}(x) = -1 - \sqrt{x-3}$.



Watch Out!
Inverse Functions f^{-1} is read f inverse or the inverse of f. Note that -1 is not an exponent.

Example 4 Interpret Inverse Functions

TEMPERATURE A formula for converting a temperature in degrees Fahrenheit to degrees Celsius is $T(x) = \frac{5}{9}(x - 32)$.

Find the inverse of $T(x)$, and describe its meaning.

$T(x) = \frac{5}{9}(x - 32)$	Original function
$y = \frac{5}{9}(x - 32)$	Replace $T(x)$ with y .
$x = \frac{9}{5}(y - 32)$	Exchange x and y .
$\frac{5}{9}x = y - 32$	Multiply each side by $\frac{9}{5}$.
$\frac{5}{9}x + 32 = y$	Add 32 to each side.
$T^{-1}(x) = \frac{5}{9}x + 32$	Replace y with $T^{-1}(x)$.

$T^{-1}(x)$ can be used to convert a temperature in degrees Celsius to degrees Fahrenheit.

Go Online You can compute an Extra Example online.

Go Online to see Part B of the example on using the graph of $T(x)$ and $T^{-1}(x)$.

Think About It! Find the domain of $T(x)$ and its inverse. Explain your reasoning.

Sample answer: Algebraically, the domain of both $T(x)$ and $T^{-1}(x)$ are all real numbers because temperatures can be positive and negative and do not have to be integer values.

Interactive Presentation

Inverses with Restricted Domains

Examine $f(x) = x^2 + 2x + 4$.

Part A.
Find the inverse of $f(x)$.

$f(x) = x^2 + 2x + 4$	Original function
$y = x^2 + 2x + 4$	Replace $f(x)$ with y .
$x = y^2 + 2y + 4$	Exchange x and y .
$x - 4 = y^2 + 2y$	Subtract 4 from each side.
$x - 4 + 1 = y^2 + 2y + 1$	Complete the square.

Example 3

TAP



Students move through the steps to graph the inverse of a function with a restricted domain.

CHECK



Students complete the Check online to determine whether they are ready to move on.



Learn Verifying Inverses

Objective

Students verify that two relations are inverses by using compositions.

MP Teaching the Mathematical Practices

6 Use Definitions To answer the question in the Think About It! feature, students will use definition of inverse functions.

Common Misconception

Remind students to check both $[f \circ g](x)$ and $[g \circ f](x)$ to verify that functions are inverses. By definition, both compositions must be the identity function.

Example 5 Use Compositions to Verify Inverses

MP Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of the functions to determine whether they are inverses.

Questions for Mathematical Discourse

- AL** Why is it unnecessary to check $k[h(x)]$? **For the functions to be inverses, both $h[k(x)]$ and $k[h(x)]$ must be the identity function. Once one is determined to not be an identity function, you know they cannot be inverses.**
- OL** What change would make $h(x)$ the inverse of $k(x)$? **if + 13 was outside the radical**
- BL** What is an example of a function that is its own inverse? **Sample answer: $y = \frac{1}{x}$**

Example 6 Verify Inverse Functions

MP Teaching the Mathematical Practices

2 Attend to Quantities Students must consider the meaning of the domain of the inverse function to answer the question in the Talk About It! feature.

Questions for Mathematical Discourse

- AL** Why is it useful to find inverses of real-world functions?
Sample answer: You can select a variable to be the dependent variable based on the problem you are solving. In this example you have functions representing how V depends on r and for how r depends on V .
- OL** What is another way you can tell the functions are inverses? **When you solve V for r , you get the same result as the given function r , likewise if you solve r for V .**
- BL** Would the functions be inverses without the real-world domain restriction? **Explain. No, V does not pass the horizontal line test.**

Think About It!
If $f(x)$ and $g(x)$ are inverses, find $(f \circ g)(x)$.

x

Watch Out!
Compositions of Functions Be sure to check both $(f \circ g)(x)$ and $(g \circ f)(x)$ to verify that functions are inverses. By definition, both compositions must result in the identity function.

Go Online: to see another example on verifying inverse functions.

Talk About It!

Find the domain of the inverse, and describe its meaning in the context of the situation.

Sample answer: The inverse represents the radius of a cylinder with a height of 5 inches in terms of its volume. The domain is all positive real numbers because the radius cannot be negative.

Learn Verifying Inverses

Key Concept • Verifying Inverse Functions:

Words: Two functions f and g are inverse functions if and only if both of their compositions are the identity function.

Symbols: $f(x)$ and $g(x)$ are inverses if and only if $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

Example 5 Use Compositions to Verify Inverses

Determine whether $h(x) = \sqrt{x + 13}$ and $k(x) = (x - 13)^2$ are inverse functions.

First $(h \circ k)(x)$:

$$\begin{aligned} (h \circ k)(x) &= h(k(x)) \\ &= h((x - 13)^2) \\ &= \sqrt{(x - 13)^2 + 13} \\ &= \sqrt{x^2 - 26x + 169 + 13} \\ &= \sqrt{x^2 - 26x + 182} \end{aligned}$$

Composition of functions

Substitute

Simplify

Distribute

Simplify

Because $(h \circ k)(x)$ is not the identity function, $h(x)$ and $k(x)$ are not inverses.

Check

Determine whether $f(x) = \frac{x}{5} + \frac{3}{5}$ and $g(x) = 5x + 12$ are inverses.

Explain your reasoning. **No, $(f \circ g)(x) = x + \frac{3}{5}$ and $(g \circ f)(x) = x + 24$.**

Example 6 Verify Inverse Functions

GEOMETRY The formula for the volume of a cylinder with a height of 5 inches is $V = 5\pi r^2$. Determine whether $r = \sqrt{\frac{V}{5\pi}}$ is the inverse of the original function.

Find $V \circ r$:

$$V \circ r = 5\pi r^2$$

$$= 5\pi \left(\sqrt{\frac{V}{5\pi}} \right)^2$$

$$= 5\pi \left(\frac{V}{5\pi} \right)$$

$$= V$$

$$r = \sqrt{\frac{V}{5\pi}}$$

$$r = \sqrt{\frac{V}{5\pi}}$$

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$$r = \sqrt{\frac{V}{5\pi}}$$

Go Online: You can complete an Extra Example online.

170 Module 4 • Inverse and Radical Functions

Interactive Presentation

Use Compositions to Verify Inverses

Determine whether the functions are inverse functions.

$f(x) = 7x + 3$ and $g(x) = \frac{x-3}{7}$

To determine whether the functions are inverses, verify that the compositions of f and g result in the identity function.

First $(f \circ g)(x)$:

$(f \circ g)(x) = f(g(x))$

$= f\left(\frac{x-3}{7}\right)$

$= 7\left(\frac{x-3}{7}\right) + 3$

$= x - 3 + 3$

$= x$

Composition of functions

Substitute

Simplify

Substitute again

Distribute

Simplify

First $(g \circ f)(x)$:

$(g \circ f)(x) = g(f(x))$

$= g(7x + 3)$

$= \frac{7x + 3 - 3}{7}$

$= \frac{7x}{7}$

$= x$

Example 5

TYPE

a

Students answer a question to determine whether they understand how to write the inverse of a function.

CHECK

100

Students complete the Check online to determine whether they are ready to move on.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION



F.IF.5, F.BF.4a

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–24
2	exercises that use a variety of skills from this lesson	25–46
2	exercises that extend concepts learned in this lesson to new contexts	47, 48
3	exercises that emphasize higher-order and critical-thinking skills	49–53

ASSESS AND DIFFERENTIATE

Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, THEN assign:

- Practice Exercises 1–47 odd, 49–53
- Extension: Group Theory
- ALEKS Function Operations; Inverse Functions

IF students score 66%–89% on the Checks, THEN assign:

- Practice Exercises 1–53 odd
- Remediation, Review Resources: Functions
- Personal Tutors
- Extra Examples 1–6
- ALEKS Introduction to Functions

IF students score 65% or less on the Checks, THEN assign:

- Practice Exercises 1–23 odd
- Remediation, Review Resources: Functions
- Quick Review Math Handbook: Inverse Functions and Relations
- ALEKS Introduction to Functions

Practice

Go Online You can complete your homework online.

Example 1

For each polygon, find the inverse of the relation. Then, graph both the original relation and its inverse.

1. $\triangle AMP$ with vertices at $(-8, 6)$, $(5, -2)$, $(4, -6)$. See margin.

2. $\triangle XYZ$ with vertices at $(7, 7)$, $(4, 9)$, $(3, -7)$. See margin.

3. $\triangle RST$ with vertices at $(8, -5)$, $(-6, -1)$, $(-2, -8)$. See margin.

4. quadrilateral $PQWU$ with vertices at $(4, 3)$, $(-4, -3)$, $(-5, 5)$, $(5, 2)$. See margin.

Examples 2 and 3

Find the inverse of each function. Then graph the function and its inverse. If necessary, restrict the domain of the inverse so that it is a function.

5. $f(x) = x + 2$

$f^{-1}(x) = x - 2$

6. $g(y) = 5y$

$g^{-1}(y) = \frac{1}{5}y$

7. $f(x) = -2x + 1$

$f^{-1}(x) = -\frac{1}{2}(x - 1)$

8. $h(x) = \frac{3}{4}x - 2$

$h^{-1}(x) = \frac{4}{3}(x + 2)$

9. $f(x) = \frac{1}{3}x - 8$

$f^{-1}(x) = 3(x + 8)$

10. $g(x) = x + 4$

$g^{-1}(x) = x - 4$

11. $f(x) = 4x$

$f^{-1}(x) = \frac{1}{4}x$

12. $f(x) = -8x + 9$

$f^{-1}(x) = -\frac{1}{8}(x - 9)$

13. $f(x) = 5x^2$

If the domain of $f(x)$ is restricted to $x \geq 0$, then the inverse is $f^{-1}(x) = \sqrt{\frac{x}{5}}$.

If the domain of $f(x)$ is restricted to $x \leq 0$, then the inverse is $f^{-1}(x) = -\sqrt{\frac{x}{5}}$.

14. $h(x) = x^2 + 4$

If the domain of $h(x)$ is restricted to $x \geq 0$, then the inverse is $h^{-1}(x) = \sqrt{x - 4}$.

If the domain of $h(x)$ is restricted to $x \leq 0$, then the inverse is $h^{-1}(x) = -\sqrt{x - 4}$.

Example 4

15. **WEIGHT** The formula to convert weight in pounds to stones is $p(x) = \frac{1}{16}x$, where x is the weight in pounds.

a. Find the inverse of $p(x)$, and describe its meaning. **Sample answer:** The inverse converts stones to pounds, where x is the weight in stones.

b. Graph $p(x)$ and $p^{-1}(x)$. Use the graph to find the weight in pounds of a dog that weighs about 2.5 stones. **About 25 pounds; see Mod. 4 Answer Appendix for graph.**

Lesson 4.2 • Inverse Relations and Functions 171

16. **CRYPTOGRAPHY** DeAndre is designing a code to send secret messages. He assigns each letter of the alphabet to a number, where $A = 1$, $B = 2$, $C = 3$, and so on. Then he uses $c(x) = 4x - 9$ to create the secret code.

a. Find the inverse of $c(x)$, and describe its meaning. **Sample answer:** The inverse can be used to convert the secret code to the original message.

b. Make tables of $c(x)$ and $c^{-1}(x)$. Use the table to decipher the message: 75, 75, 41, 3, 71, 27, 51, 47, 67. **message: FUNCTIONS; See Mod. 4 Answer Appendix for table.**

Example 5

Determine whether each pair of functions are inverse functions. Write yes or no.

17. $f(x) = x + 1$

$g(x) = 2x + 3$

$g(f(x)) = 2(x + 1) + 3 = 2x + 5$ **no**

18. $f(x) = 5x - 5$

$g(x) = \frac{1}{5}x + 1$ **yes**

19. $f(x) = 2x$

$g(x) = \frac{1}{2}x$ **yes**

20. $f(x) = 6x - 2$

$g(x) = \frac{1}{6}x + 3$ **no**

21. $f(x) = 8x - 10$

$g(x) = \frac{1}{8}x + \frac{5}{4}$ **yes**

Example 6

22. **GEOMETRY** The formula for the volume of a right circular cone with a height of 2 feet is $V = \frac{1}{3}\pi r^2$. Determine whether $r = \sqrt{\frac{3V}{\pi}}$ is the inverse of the original function. **yes**

23. **GEOMETRY** The formula for the area of a trapezoid is $A = \frac{1}{2}(b_1 + b_2)h$. Determine whether $b_2 = \frac{2A}{h} - b_1$ is the inverse of the original function. **no**

24. **GEOMETRY** The formula for the area of a trapezoid is $A = \frac{1}{2}(b_1 + b_2)h$. Determine whether $b_1 = \frac{2A}{h} - b_2$ is the inverse of the original function. **no**

25. **Mixed Exercises**

Find the inverse of each function. Then graph the function and its inverse. If necessary, restrict the domain of the inverse so that it is a function. See Mod. 4 Answer Appendix for graphs and restrictions.

26. $f(x) = 3x$

$f^{-1}(x) = \frac{1}{3}x$

27. $g(x) = 2x + 1$

$g^{-1}(x) = \frac{1}{2}(x - 1)$

28. $h(x) = \frac{1}{2}x - 1$

$h^{-1}(x) = 2(x + 1)$

29. $f(x) = \frac{1}{2}x^2 - 1$

$f^{-1}(x) = \pm\sqrt{2(x + 1)}$

If the domain of $f(x)$ is restricted to $x \geq 0$, then the inverse is $f^{-1}(x) = \sqrt{2(x + 1)}$.

If the domain of $f(x)$ is restricted to $x \leq 0$, then the inverse is $f^{-1}(x) = -\sqrt{2(x + 1)}$.

30. $g(x) = x + 2$

$g^{-1}(x) = x - 2$

31. $h(x) = x^2 + 4$

$h^{-1}(x) = \pm\sqrt{x - 4}$

If the domain of $h(x)$ is restricted to $x \geq 0$, then the inverse is $h^{-1}(x) = \sqrt{x - 4}$.

If the domain of $h(x)$ is restricted to $x \leq 0$, then the inverse is $h^{-1}(x) = -\sqrt{x - 4}$.

32. **Module 4 • Inverse Relations and Functions**



Determine whether each pair of functions are inverse functions. Write yes or no.

31. $f(x) = 4x^2$

$g(x) = \frac{1}{2}\sqrt{x}$ yes

32. $f(x) = \frac{1}{3}x^2 + 1$

$g(x) = \sqrt{3x-3}$ yes

33. $f(x) = x^2 - 9$

$g(x) = x + 3$ no

34. $f(x) = \frac{1}{3}x^2$

$g(x) = \sqrt{\frac{1}{3}x}$ no

35. $f(x) = (x + 6)^2$

$g(x) = -\sqrt{x} - 6$ yes

36. $f(x) = 2\sqrt{x-5}$

$g(x) = \frac{1}{2}x^2 - 5$ no

Restrict the domain of $f(x)$ so that its inverse is also a function. State the restricted domain of $f(x)$ and the domain of $f^{-1}(x)$.

37. $f(x) = x^2 + 5$

$\{x \geq 0\}$ or $\{x \leq 0\}$; $\{x \geq 5\}$

38. $f(x) = 3x^2$

$\{x \geq 0\}$ or $\{x \leq 0\}$; $\{x \geq 0\}$

39. $f(x) = \sqrt{x+6}$

$\{x \geq -6\}$; $\{x \geq 0\}$

40. $f(x) = \sqrt{x+3}$

$\{x \geq -3\}$; $\{x \geq 0\}$

Copy each graph. Sketch a graph of the inverse of each function. Then state whether the inverse is a function.



The inverse is not a function.



The inverse is a function.

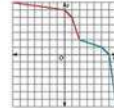


The inverse is a function.



The inverse is not a function.

45. **FITNESS** Alejandro is a personal trainer. For his clients to gain the maximum benefit from their exercise, Alejandro calculates their maximum target heart rate using the function $f(x) = 0.85(220 - x)$, where x represents the age of the client. Find the inverse of this function and interpret its meaning in the context of the situation.

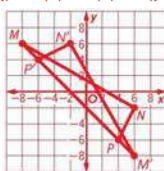


46. Copy the previous function shown. Graph the inverse of the previous function shown.

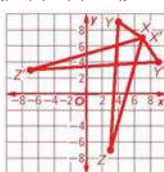
Lesson 4.2 • Inverse Relations and Functions 173

Answers

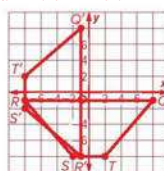
1. $\{(6, -8), (-2, 6), (-6, 4)\}$



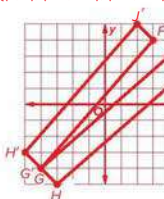
2. $\{(7, 7), (9, 4), (-7, 3)\}$



3. $\{(-1, 8), (-1, -8), (-8, -2), (-8, 2)\}$



4. $\{(3, 4), (-4, -4), (-5, -3), (2, 5)\}$



47. **USE A MODEL.** The diagram shows a sheet of metal with squares of side length x removed from each corner which can be used to form an open box.



- a. Write and graph the function $V(x)$ which gives the volume of the open box. Explain how the domain of the function must be restricted within the context of this scenario. See Mod. 4 Answer Appendix.
- b. Does restricting the domain in part a allow for the inverse to also be a function? Explain your reasoning. See Mod. 4 Answer Appendix.

48. **STRUCTURE** Use the table to find the relationship between $f + g^{-1}(x)$ and $f^{-1}(g(x))$.

x	0	1	2	3
$f(x)$	0	3	1	4
$g(x)$	1	0	4	3

- a. Suppose that functions $f(x)$, $g(x)$, and $f^{-1}(g(x))$ all have inverse functions defined on the domain $[0, 3]$. Calculate the following values.
- i. $f^{-1}(g(x) + g^{-1}(x))$ ii. $f^{-1}(g(x) + g^{-1}(x))$
- b. Use the value of $f + g(x)$ to find $f + g^{-1}(x)$. Use the value of $f + g(x)$ to find $f + g^{-1}(x)$. See Mod. 4 Answer Appendix.
- c. Joyce claims that $f + g^{-1}(x) = f^{-1}(g(x) + g^{-1}(x))$ and $f^{-1}(g(x) + g^{-1}(x)) = f + g^{-1}(x)$. Determine whether she is correct. Explain your reasoning. Appendix.
- d. Consider the functions $f(x) = 2x + 1$ and $g(x) = 2x - 1$. Compare $f + g^{-1}(x)$ and $f^{-1}(g(x) + g^{-1}(x))$. See Mod. 4 Answer Appendix.

Higher-Order Thinking Skills

49. **ANALYZE** If a relation is not a function, then its inverse is sometimes, always, or never a function. Justify your argument. Sample answer: Sometimes $y = \pm\sqrt{x}$ is an example of a relation that is not a function, with an inverse being a function. A circle is an example of a relation that is not a function with an inverse not being a function.
50. **CREATE** Give an example of a function and its inverse. Verify that the two functions are inverses. Sample answer: $f(x) = 2x$, $f^{-1}(x) = 0.5x$; $f^{-1}(f(x)) = f^{-1}(2x) = x$
51. **PERSERVE** Give an example of a function that is its own inverse. Sample answer: $f(x) = x$ and $f^{-1}(x) = x$ or $f(x) = -x$ and $f^{-1}(x) = -x$
52. **ANALYZE** Can the graphs of a linear function with a slope $\neq 0$ and its inverse ever be perpendicular? Justify your answer. See Mod. 4 Answer Appendix.
53. **WRITE** Suppose you have a composition of two functions that are inverses. When you put in a value of 5 for x , why is the result always 5? Sample answer: One of the functions carries out an operation on 5. Then the second function that is an inverse of the first function reverses the operation on 5. Thus, the result is 5.

n th Roots and Rational Exponents

LESSON GOAL

Students simplify expressions involving radicals and rational exponents.

1 LAUNCH

Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

Explore: Inverses of Rational Functions

Develop:

n th roots

- Find Roots
- Simplify Using Absolute Value

Rational Exponents

- Radical and Exponential Forms
- Use Rational Exponents
- Evaluate Expressions with Rational Exponents
- Simplify Expressions with Rational Exponents

You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

Exit Ticket

Practice

DIFFERENTIATE

View reports of student progress on the **Checks** after each example.

Resources

Remediation: Negative Exponents

Extension: Approximating Square Roots

AL	LB	ET
●	●	●
●	●	●

Language Development Handbook

Assign page 21 of the *Language Development Handbook* to help your students build mathematical language related to simplifying expressions involving radicals and rational exponents.

You can use the tips and suggestions on page T21 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Algebra

Standards for Mathematical Content:

A.SSE.2 Use the structure of an expression to identify ways to rewrite it.

Standards for Mathematical Practice:

- Reason abstractly and quantitatively.
- Attend to precision.
- Look for and make use of structure.

Coherence

Vertical Alignment

Previous

Students found the inverse of a function or relation. Students determined whether two functions or relations are inverses.

F.IF.5, F.BF.4a

Now

Students simplify expressions involving radicals and rational exponents.

A.SSE.2

Next

Students will graph and analyze square and cube root functions.

F.IF.7b, F.BF.3

Rigor

The Three Pillars of Rigor

Lessons with Explore(s)

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

Conceptual Bridge In this lesson, students use what they know about square roots to develop an understanding of n th roots. They build fluency by simplifying n th root expressions, and they apply their understanding by solving real-world problems related to n th roots.

Mathematical Background

In general, if n is an integer greater than or equal to 2 and a is a real number, then $\sqrt[n]{a^n} = a$ if n is odd and $|\sqrt[n]{a^n}| = |a|$ if n is even.



Interactive Presentation

Warm Up

- Which are squares of a rational number?
9 -9 0.9 0.81 0.0081
- Which are cubes of a rational number?
8 -8 0.8 0.08 0.008
- Which are fourth powers of a rational number?
16 -16 $\frac{1}{16}$ 0.0016 -625

[Show Answers](#)

Warm Up

Launch the Lesson

Astronomers are working to discover planets that could contain liquid water. The distance a planet is from its star impacts its temperature. In order for a planet to have liquid water, it must be warm enough for the water to change from a solid to a liquid, but not too so hot that the water turns to a gas. The formula $T = 0.67\left(\frac{r}{d}\right)^2$ can be used to find the temperature of a planet, where d is the distance from the planet to its star and r is the radius of the star. You can use the properties of exponents to simplify this equation and write it in radical form.



Launch the Lesson

Vocabulary

[Expand All](#) [Collapse All](#)

- > ***n*th root**
- > **index**
- > **radicand**
- > **principal root**
- > **exponential form**
- > **radical form**
- > **rational exponent**

- What is a rational exponent?
- What is the connection between rational exponents and radicals?
- To simplify the *n*th root of an expression, what must be true about the expression?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- evaluating powers of rational numbers

Answers:

- 9, 0.81, 0.0081
- 8, -8, 0.008
- 16, $\frac{1}{16}$, 0.0016

Launch the Lesson



Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will learn how rational exponents are used in the formula for the temperature of a planet.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.

Explore Inverses of Power Functions

Objective

Students explore n th roots and exponents for odd positive values of n .



Teaching the Mathematical Practices

3 Construct Arguments At the end of the Explore, students will use previously established results to make a conjecture about the inverses of rational functions.

7 Look for a Pattern Help students to see the pattern in the domain and range values for each pair of functions.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will work through exercises to compare two functions, $f(x)$ and $g(x)$, for different values of n to explore how the functions are related. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

Inverses of Rational Functions

INQUIRY What conjecture can you make about $f(x) = x^n$ and $g(x) = \sqrt[n]{x}$ for all odd positive values of n ?

Use a calculator to complete the exercises for $f(x) = x^n$ and $g(x) = \sqrt[n]{x}$.

Get It?

The $\sqrt[n]{x}$ Function On TI-84 Plus family calculators, access the $\sqrt[n]{x}$ function from the **MATH** menu. On a TI-84 Plus, use **2nd** **[$\sqrt[n]{x}$]**.

Explore

Answer 1

Complete the table for $f(x) = x^n$ and $g(x) = \sqrt[n]{x}$ where $n = 3$.

x	$f(x)$	x	$g(x)$
1		1	
-1		-1	
8		8	
-8		-8	
27		27	
-27		-27	
64		64	
-64		-64	
125		125	
-125		-125	

Check Answer Check Answer Check Answer Check Answer

Explore

TYPE



Students complete exercises that lead them to make a power function and its inverse.



Interactive Presentation

INQUIRY What conjectures can you make about $f(x) = x^n$ and $g(x) = \sqrt[n]{x}$ for all odd positive values of n ?

Done

Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Inverses of Power Functions (continued)

Questions

Have students complete the Explore activity.

Ask:

- Why does $\sqrt[3]{8}$ have only one answer, while $\sqrt{9}$ has two? **Sample answer:** For a cube root, there is only a positive root for a positive radicand and a negative root for a negative radicand. A positive number multiplied by itself three times is still positive, but a negative number cubed would be negative.
- Does the same conjecture hold true for x when n is even? **Sample answer:** The conjecture is not true when n is even because no even powered function will pass the horizontal line test. So the domain and range of the inverse cannot be the same as the domain and range of the original function.

Inquiry

What conjectures can you make about $f(x) = x^n$ and $g(x) = \sqrt[n]{x}$ for all odd positive values of n ? **Sample answer:** For all positive odd values of n , the domain and range of $f(x)$ and $g(x)$ are all real numbers and x^n and $\sqrt[n]{x}$ are inverses of each other.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn n th Roots

Objective

Students simplify expressions involving radicals and rational exponents by using the properties of exponents.

MP Teaching the Mathematical Practices

3 Analyze Cases Work with students to analyze the number of real roots when $a > 0$, $a < 0$, and $a = 0$ for the case when n is even and when n is odd. Encourage students to familiarize themselves with all of the cases.

Things to Remember

In general, if n is an integer greater than or equal to 2 and a is a real number, then $\sqrt[n]{a} = a$ if n is odd and $\sqrt[n]{a} = |a|$ if n is even.

DIFFERENTIATE

Reteaching Activity AL

Some students tend to think that x must represent a positive number and $-x$ must represent a negative number. Reading $-x$ as “the opposite of x ” should help them understand that $-x$ is 9 if $x = -9$. Also, explain that -9 has no square root that is a real number; that is, no real number can be squared to give -9 . Remind students that -9 is $3i$, an imaginary number.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

n th Roots and Rational Exponents

Lesson 4-3

Explore Inverses of Power Functions

Online Activity Use a calculator to complete the Explore.

INQUIRY What conjectures can you make about $f(x) = x^n$ and $g(x) = \sqrt[n]{x}$ for all odd positive values of n ?

Learn n th Roots

Finding the square root of a number and squaring a number are inverse operations. To find the square root of a , you must find a number with a square of a . The inverse of raising a number to the n th power is finding the n th root of a number. The symbol $\sqrt[n]{}$ indicates an n th root.

For any real numbers a and b and any positive integer n , if $a^n = b$, then a is an n th root of b . For example, because $(-2)^3 = -8$, -2 is a sixth root of -64 and 2 is a principal root.

An example of an n th root is $\sqrt[3]{36}$,

which is read as the n th root of 36. In this example, n is the **index** and 36 is the **radicand**, or the expression under the radical symbol.



Some numbers have more than one real n th root. For example, 16 has two square roots, 4 and -4 , because 4^2 and $(-4)^2$ both equal 16. When there is more than one real root and n is even, the nonnegative root is called the **principal root**.

Key Concept • Real n th Roots

Suppose n is an integer greater than 1, a is a real number, and a is an n th root of b .

a	n is even	n is odd
$a > 0$	1 unique positive and 1 unique negative real root: $\pm\sqrt[n]{a}$	1 unique positive and 0 negative real root: $\sqrt[n]{a}$
$a < 0$	0 real roots	0 positive and 1 negative real root: $\sqrt[n]{a}$
$a = 0$	1 real root: $\sqrt[n]{0} = 0$	1 real root: $\sqrt[n]{0} = 0$

A radical expression is simplified when the radicand contains no fractions and no radicals appear in the denominator.

Today's Goals

- Simplify expressions involving radicals and rational exponents.
- Simplify expressions in exponential or radical form.

Today's Vocabulary

- n th root
- index
- radicand
- principal root
- rational exponent

Think About It!

Lorena says she can tell that $\sqrt[3]{-64}$ will have a real root without graphing. Do you agree or disagree? Explain your reasoning.

Agree; sample answer: Because n is odd and a is less than 0, I know that the expression will have one negative real root.

Lesson 4-3 • n th Roots and Rational Exponents 175

Interactive Presentation

n th Roots

Finding the square root of a number and squaring a number are inverse operations. To find the square root of a , you must find a number with a square of a . The inverse of raising a number to the n th power is finding the n th root of a number. The symbol $\sqrt[n]{}$ indicates an n th root.

For any real numbers a and b and any positive integer n , if $a^n = b$, then a is an n th root of b . For example, because $(-2)^3 = -8$, -2 is a sixth root of -64 and 2 is a principal root.

An example of an n th root is $\sqrt[3]{36}$, which is read as the n th root of 36. In this example, n is the **index** and 36 is the **radicand**, or the expression under the radical symbol.

Learn

TYPE



Students answer a question to determine whether a radical expression has a real root.



Watch Out!

Principal Roots
Because $5^2 = 25$, 25 has two square roots, 5 and -5 . However, the value of $\sqrt{25}$ is 5 only. To indicate both square roots and not just the principal square root, the expression must be written as $\pm\sqrt{25}$.

Talk About It!

Compare the simplified expressions in the previous example with the ones in this example. Explain why the simplified expressions in this example require absolute value bars when the simplified expressions in the previous example did not.

Sample answer: None of the expressions in the previous example resulted in an odd power when taking an even root of an even power. However, both expressions in this example did result in an odd power when taking an even root of an even power.

Example 1 Find Roots

Simplify.

a. $\pm\sqrt{25x^4}$

$$\pm\sqrt{25x^4} = \pm\sqrt{5^2x^4}$$

$$= \pm 5x^2$$

b. $-\sqrt{9x^2 + 7y^2}$

$$-\sqrt{9x^2 + 7y^2} = -\sqrt{9x^2 + 7y^2}$$

$$= -(x^2 + y^2)$$

c. $\sqrt[3]{343a^3b^3}$

$$\sqrt[3]{343a^3b^3} = \sqrt[3]{7^3a^3b^3}$$

$$= 7ab$$

d. $\sqrt{-289c^2d^2}$

There are no real roots of $\sqrt{-289}$. However, there are two imaginary roots, $17i$ and $-17i$. Because we are only finding the principal square root, use $17i$.

$$\sqrt{-289c^2d^2} = \sqrt{-1 \cdot 17^2 \cdot c^2d^2}$$

$$= i \cdot \sqrt{289c^2d^2}$$

$$= 17icd$$

Check:

Write the simplified form of each expression.

a. $\pm\sqrt{196x^4} = \pm\sqrt{14^2x^4}$

b. $-\sqrt{196x^4} = -14x^2$

c. $\sqrt[3]{125x^3} = 5x$

Example 2 Simplify Using Absolute Value

Simplify.

a. $\sqrt[3]{8x^3}$

$$\sqrt[3]{8x^3} = \sqrt[3]{2^3x^3}$$

$$= 2|x|$$

Because x could be negative, you must use the absolute value of x to ensure that the principal square root is nonnegative.

b. $\sqrt[3]{256y^3 - 27z^3}$

$$\sqrt[3]{256y^3 - 27z^3} = \sqrt[3]{256 \cdot y^3 - 27z^3}$$

$$= 2|y^3 - z^3|$$

Because $(y^3 - z^3)$ could be negative, you must use the absolute value of $(y^3 - z^3)$ to ensure that the principal square root is nonnegative.

Example 1 Find Roots



Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of the expressions in Example 1 to find the n th roots.

Questions for Mathematical Discourse

AL What should you look for when simplifying n th roots? **Sample answer:** I should check to see whether elements of the radicand can be rewritten as factors to the n th power.

OL Why is an odd root of a negative number a real number? **When a negative number is raised to an odd power, the answer is negative.**

BL What is the cube root of x^3 ? **What is the square root of x^2 ? $|x|$**

Example 2 Simplify Using Absolute Value



Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solutions methods. Students should use clear definitions and mathematical language when they discuss their answer to the question in the Talk About It! feature.

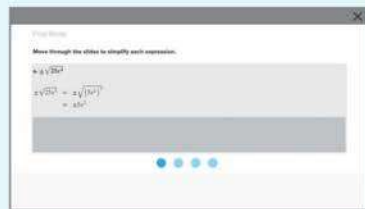
Questions for Mathematical Discourse

AL In part b, why isn't the answer $2|y^3 - z^3|$? **because $|2| = 2$**

OL Why would the absolute value be unnecessary when taking odd roots? **There is only one root when taking odd roots.**

BL Write a rule that explains how you find the exponent on a factor after taking the n th root. **Sample answer: divide the exponent in the radicand by the root.**

Interactive Presentation



Example 1

SWIPE



Students move through the slides to simplify expressions.

TYPE



Students answer a question about principle roots.

CHECK



Students complete the Check online to determine whether they are ready to move on.



Learn Rational Exponents

Objective

Students simplify expressions in exponential or radical form.

MP Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. Encourage students to familiarize themselves with the conditions that must be met for an expression with rational exponents to be in simplest form. Point out that when all conditions are met, they are done transforming the expression.

Example 3 Radical and Exponential Forms

MP Teaching the Mathematical Practices

7 Use Structure Help students to use the structure of the expressions to write them in radical or exponential form.

Questions for Mathematical Discourse

AL What operation do you perform with the exponents when an exponent is raised to another exponent? **Y** ou multiply them.

OL Why is $b^{\frac{1}{2}} = \sqrt[2]{b}$? because $(b^{\frac{1}{2}})^2 = b$

BL What is an advantage of using exponential form?

Sample answer: Exponential form can make it easier to identify how to simply an expression using the rules of exponents.

Learn Rational Exponents

You can use the properties of exponents to translate expressions from exponential form to radical form or from radical form to exponential form. An expression is in **exponential form** if it is in the form x^n , where n is an exponent. An expression is in **radical form** if it contains a radical symbol.

For any real number b and a positive integer n , $b^{\frac{1}{n}} = \sqrt[n]{b}$, except where $b < 0$ and n is even. When $b < 0$ and n is even, a complex root may exist.

Examples: $125^{\frac{1}{3}} = \sqrt[3]{125}$ or 5 $(-49)^{\frac{1}{2}} = \sqrt{-49}$ or 7i

The expression $b^{\frac{1}{n}}$ has a **rational exponent**. The rules for exponents also apply to rational exponents.

Key Concept • Rational Exponents

For any nonzero number b and any integers x and y , with $y > 0$, $b^x = \sqrt[y]{b^x}$, except when $b < 0$ and y is even. When $b < 0$ and y is even, a complex root may exist.

Examples: $125^{\frac{2}{3}} = (\sqrt[3]{125})^2 = 5^2$ or 25 $(-49)^{\frac{3}{2}} = (\sqrt{-49})^3 = (7i)^3$ or $-343i$

Key Concept • Simplest Form of Expressions with Rational Exponents

An expression with rational exponents is in simplest form when all of the following conditions are met.

- It has no negative exponents.
- It has no exponents that are not positive integers in the denominator.
- It is not a complex fraction.
- The index of any remaining radical is the least number possible.

Example 3 Radical and Exponential Forms

Simplify.

a. Write $x^{\frac{1}{2}}$ in radical form.

$$x^{\frac{1}{2}} = \sqrt[2]{x}$$

b. Write $\sqrt[3]{x^2}$ in exponential form.

$$\sqrt[3]{x^2} = x^{\frac{2}{3}}$$

Go Online You can complete an Extra Example online.



Math History Minute:

Christoff Rudolf (1498–1553) wrote the first German algebra textbook. It is believed that he introduced the radical symbol $\sqrt{}$ in 1525 in his book *Die Clavis*. Some feel that this symbol was used because it resembled a small r , the first letter in the Latin word *radix* or root.

Think About It!

Write two equivalent expressions, one in radical form and one in exponential form.

Sample answer: $\sqrt[3]{x^2}$, $x^{\frac{2}{3}}$

Interactive Presentation

Rational Exponents

Recall that raising a number to a power means to multiply the number by itself. For example, $2^3 = 2 \cdot 2 \cdot 2 = 8$. A power of a number is called a **power**. The number being multiplied is called the **base**. The number of times the base is multiplied is called the **exponent**. The expression 2^3 is in **exponential form**. It is in the form a^b , where a is the base, b is the exponent, and a^b is the result. The expression 2^3 is also in **radical form**. It is in the form $\sqrt[n]{a^b}$, where a is the base, b is the exponent, and $\sqrt[n]{a^b}$ is the result. The expression 2^3 is also in **radical form**. It is in the form $\sqrt[n]{a^b}$, where a is the base, b is the exponent, and $\sqrt[n]{a^b}$ is the result.

The expression 2^3 is in **exponential form**. The base is 2, the exponent is 3, and the result is 8.

The expression 2^3 is in **radical form**. The base is 2, the exponent is 3, and the result is 8.

The expression 2^3 is in **radical form**. The base is 2, the exponent is 3, and the result is 8.

$125^{\frac{2}{3}} = (\sqrt[3]{125})^2 = 5^2 = 25$ $(-49)^{\frac{3}{2}} = (\sqrt{-49})^3 = (7i)^3 = -343i$

Learn

TYPE



Students explain how to solve an equation using the rules of exponents.



Think About It!
Why did you set t equal to $\frac{1}{2}$?

Sample answer: t is the time in years, and 3 months is equal to $\frac{1}{4}$ of a year.

Think About It!
How can you tell if $x^{\frac{1}{2}}$ will simplify to an integer?

Sample answer: If you can rewrite x as an integer to the power of 2, it will simplify to an integer.

Go Online
to see more examples of evaluating expressions with rational exponents.

Watch Out!
Exponents Recall that when you multiply powers, the exponents are added, and when you raise a power to a power, the exponents are multiplied.

Think About It!
How would the expression in part c change if the exponents were $\frac{1}{3}$ and $-\frac{1}{3}$?

Sample answer: The expression would not have a positive exponent, so I would need to follow the process used in part b.

Example 4 Use Rational Exponents

FINANCIAL LITERACY The expression $ci + rf$ can be used to estimate the future cost of an item due to inflation, where c represents the current cost of the item, r represents the annual rate of inflation, and t represents the time in years. Write the expression in radical form for the future cost of an item 3 months from now if the annual rate of inflation is 4.7%.

$$ci + rf = ci + 0.047t \quad r = 0.047 \quad t = \frac{1}{4}$$

$$= ci(0.047) \quad \text{Add.}$$

$$= ci\sqrt[4]{0.047} \quad \text{Simplify.}$$

Example 5 Evaluate Expressions with Rational Exponents

Evaluate $32^{-\frac{1}{5}}$.

$$32^{-\frac{1}{5}} = \frac{1}{32^{\frac{1}{5}}}$$

$$= \frac{1}{\sqrt[5]{32}}$$

$$= \frac{1}{2^5}$$

$$= \frac{1}{32} \quad \text{or} \quad 2^{-5}$$

Power of a Power
Multiply the exponents.

Example 6 Simplify Expressions with Rational Exponents

Simplify each expression.

a. $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}}$

$$x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{2} + \frac{1}{2}}$$

$$= x^1$$

$$= x$$

Add powers.
 $\frac{1}{2} + \frac{1}{2} = 1$
Add the exponents.

b. $y^{-\frac{1}{3}} \cdot y^{\frac{1}{3}}$

$$y^{-\frac{1}{3}} \cdot y^{\frac{1}{3}} = y^{-\frac{1}{3} + \frac{1}{3}}$$

$$= y^0$$

$$= 1$$

Add powers.
 $-\frac{1}{3} + \frac{1}{3} = 0$
Add the exponents.

c. $x^{-\frac{1}{2}} \cdot x^{\frac{1}{2}}$

$$x^{-\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{-\frac{1}{2} + \frac{1}{2}}$$

$$= x^0$$

$$= 1$$

Add powers.
 $-\frac{1}{2} + \frac{1}{2} = 0$
Add the exponents.

Go Online You can complete an Entry Example online.

178 Module 4 • Inverse and Radical Functions

Interactive Presentation

Use Rational Exponents

FINANCIAL LITERACY The expression $ci + rf$ can be used to estimate the future cost of an item due to inflation, where c represents the current cost of the item, r represents the annual rate of inflation, and t represents the time in years. Write the expression in radical form for the future cost of an item 3 months from now if the annual rate of inflation is 4.7%.

Example 4

TYPE

a

Students answer a question about units used in the example.

CHECK

||||

Students complete the Check online to determine whether they are ready to move on.

Example 4 Use Rational Exponents

MP Teaching the Mathematical Practices

2 Consider Units Students must note the units involved in this problem to help them substitute the correct value for t in the expression. The Think About It! features asks students to explain how the units affect the value of t .

Questions for Mathematical Discourse

AL What would be the value of t if we were considering the value of the item 8 months from now? $\frac{2}{3}$

OL When does this equation have integer exponents? **The exponent is an integer when the time is a whole number of years.**

BL What equation could you solve to find when the cost would double? $2 = (1 + r)^t$

Example 5 Evaluate Expressions with Rational Exponents

MP Teaching the Mathematical Practices

8 Look for a Pattern Help student to see the pattern in how expressions with rational exponents are evaluated to generalize how you can tell if the expressions will simplify to an integer.

Questions for Mathematical Discourse

AL When evaluating a rational exponent, which part of the exponent is the index of the root? **the denominator.**

OL Why does $32^{-\frac{1}{5}} = \frac{1}{32^{\frac{1}{5}}}$? Explain your answer.
Sample answer: Multiplying either form by $32^{\frac{1}{5}}$ yields 1.

BL What is the multiplicative inverse of $a^{\frac{1}{2}}$? Explain. $a^{-\frac{1}{2}}$ because $a^{\frac{1}{2}} \cdot a^{-\frac{1}{2}} = a^0 = 1$.

Example 6 Simplify Expressions with Rational Exponents

MP Teaching the Mathematical Practices

7 Use Structure Guide students to use the structure of the expressions in Example 6 to write them in simplest form.

Questions for Mathematical Discourse

AL What do you need to do before adding or subtracting fractions? **Ensure they have the same denominator.**

OL In part b, why do we multiply the numerator and denominator by $y^{\frac{1}{3}}$? **to rationalize the denominator so that the final form meets the rules for simplest form.**

BL How could you solve $x^{\frac{a}{b}} \cdot x^{\frac{c}{d}} = x$ for a ? **Write an equation for the exponents and solve for a .** $\frac{a}{b} + \frac{c}{d} = 1$

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION



A.SSE.2

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–30
2	exercises that use a variety of skills from this lesson	31–56
2	exercises that extend concepts learned in this lesson to new contexts	57–60
3	exercises that emphasize higher-order and critical-thinking skills	61–68

ASSESS AND DIFFERENTIATE



Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, THEN assign:

- Practice Exercises 1–59 odd, 61–68
- Extension: Approximating Square Roots
- ALEKS** Simplifying Expressions, Rational Exponents

BL

IF students score 66%–89% or more on the Checks, THEN assign:

- Practice Exercises 1–67 odd
- Remediation, Review Resources: Negative Exponents
- Personal Tutors
- Extra Examples 1–6
- ALEKS** Negative Exponents

OL

IF students score 65% or less on the Checks, THEN assign:

- Practice Exercises 1–29 odd
- Remediation, Review Resources: Negative Exponents
- Quick Review Math Handbook: *n*th Roots
- ALEKS** Negative Exponents

AL

Practice

Examples 1 and 2

Simplify.

1. $\pm\sqrt[3]{125x^3y^3} \pm 10x^3y^3$

2. $\pm\sqrt[3]{250x^3y^3} \pm 10x^3y^3$

3. $\pm\sqrt[3]{49x^3} \pm 7x$

4. $-\sqrt[3]{8x^3y^3} - 6x^3y^3$

5. $-\sqrt[3]{8x^3y^3} - 6x^3y^3$

6. $-\sqrt[3]{400x^3y^3} - 20x^3y^3$

7. $\sqrt[3]{90x^3} - 3x^3$

8. $\sqrt[3]{x^3y^3} x^3y^3$

9. $\sqrt[3]{81x^3} - 4x^3$

10. $\sqrt[3]{x^3} x^3$

11. $\sqrt[3]{x^3} x^3$

12. $\sqrt[3]{x^3} x^3$

Example 3

Write each expression in radical form, or write each radical in exponential form.

13. $x^{\frac{2}{3}}$

14. $x^{\frac{1}{6}}$

15. $x^{\frac{1}{3}} \sqrt[3]{x^3}$

16. $\sqrt[3]{x^3}$

17. $\sqrt[3]{50x^3} \sqrt[3]{x^3}$

18. $\sqrt[3]{25x^3} \sqrt[3]{5x^3}$

Example 4

19. **GEOMETRY** The distance in millions of miles a planet is from the Sun in terms of t , the number of Earth days it takes for the planet to orbit the Sun, can be modeled by the expression $16t^2$. Write the expression in exponential form. $64t^2$

20. **DEPRECIATION** The depreciation rate is calculated by the expression $1 - \left(\frac{P}{O}\right)^n$, where n is the age of the item in years, P is the resale price in dollars, and O is the original price in dollars. Write the expression in radical form for an 8-year-old car that was originally purchased for \$52,425. $1 - \sqrt[8]{\frac{52,425}{O}}$

Example 5

Evaluate each expression.

21. $27^{\frac{1}{3}}$

22. $256^{\frac{1}{4}}$

23. $16^{-\frac{1}{4}}$

24. $81^{-\frac{1}{4}}$

25. $1024^{\frac{1}{5}}$

26. $16^{-\frac{1}{5}}$

Lesson 4.3 • *n*th Roots and Rational Exponents 179

Example 6

Simplify each expression.

27. $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} x^{\frac{1}{2}}$

28. $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} x^{\frac{1}{2}}$

29. $16^{-\frac{1}{4}}$

30. $y^{-\frac{1}{2}}$

Mixed Exercises

Simplify.

31. $\sqrt[3]{27x^3y^3} \sqrt[3]{30x^3y^3}$

32. $-\sqrt[3]{2x^3y^3} - \sqrt[3]{2x^3y^3}$

33. $\sqrt[3]{81x^3} + 4x^3$

34. $\sqrt[3]{64x^3} - 4x^3$

35. $\sqrt[3]{9x^3} + 9x^3$

36. $\sqrt[3]{250x^3} - 2x^3$

37. $\sqrt[3]{100x^3y^3} \sqrt[3]{16x^3y^3}$

38. $\sqrt[3]{64x^3y^3} \sqrt[3]{8x^3y^3}$

39. $\sqrt[3]{-27x^3y^3} - 3x^3y^3$

40. $\sqrt[3]{-16x^3y^3} - 2x^3y^3$

41. $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} x^{\frac{1}{2}}$

42. $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} x^{\frac{1}{2}}$

43. $(x^{\frac{1}{2}})^{\frac{1}{2}}$

44. $(x^{\frac{1}{2}})^{\frac{1}{2}}$

45. $x^{-\frac{1}{2}}$

46. $x^{-\frac{1}{2}}$

47. **GEOMETRY** The volume V of a regular octahedron with edge length ℓ is given by $V = \frac{\sqrt{2}}{3} \ell^3$. Write the volume in simplest form for octahedron with the given edge lengths.

a. $\sqrt{15}$ cm

b. $\sqrt[3]{24}$ cm

c. $3\sqrt[3]{8}$ cm

180 Module 4 • Inverse and Radical Functions



Answers

57. There are four cases to consider. If $m > 0$ and b is even, then $\sqrt[b]{m^{3b}} = m^3$. If $m < 0$ and b is even, then $\sqrt[b]{m^{3b}} = -m^3$. If $m > 0$ and b is odd, then $\sqrt[b]{m^{3b}} = m^3$. If $m < 0$ and b is odd, then $\sqrt[b]{m^{3b}} = m^3$.
60. $f(x)$ and $r(x)$ are equivalent. They simplify to x^3 . $g(x)$ is different and simplifies to $|x^3|$. $s(x)$ is almost the same as x^3 , but the function is not defined for negative values of x .
61. Sample answer: It may be easier to simplify an expression when it has rational exponents because all the properties of exponents apply. We do not have as many properties dealing directly with radicals. However, we can convert all radicals to rational exponents, and then use the properties of exponents to simplify.
66. Sample answer: They are needed to ensure that the answer is not a negative number. When we take any odd root of a number, we find that there is just one answer. If the number is positive, the root is positive. If the number is negative, the root is negative. Absolute value symbols are never needed when finding odd roots. Every positive real number has two n th roots when n is even; one of these roots is positive and one is negative. Negative real numbers do not have n th roots when n is even. When finding even n th roots, absolute value symbols are sometimes necessary to ensure the principal root is the result if it is possible the root could be negative.

Simplify.

48. $\frac{1}{x^2} \cdot \frac{1}{x^3}$

49. $\frac{1}{x^2} \cdot \frac{1}{x^3}$

50. $\frac{1}{x^2} \cdot \frac{1}{x^3}$

51. $\frac{1}{x^2} \cdot \frac{1}{x^3}$

52. $\frac{1}{x^2} \cdot \frac{1}{x^3}$

53. $\frac{1}{x^2} \cdot \frac{1}{x^3}$

54. **SPORTS** A volleyball has a volume of $854\pi \text{ cm}^3$. A tennis ball has a volume of $32\pi \text{ cm}^3$. By how much does the radius of the volleyball exceed that of the tennis ball? Write your answer using rational exponents.

$\frac{1}{3}\sqrt[3]{17}$ cm

55. **WATER TOWER** One of the largest sphere water towers in the country is located in Edmond, Oklahoma. It is 218 feet tall and holds 900,000 gallons, or about 68,840 cubic feet of water. Another town is planning to build a similar water tower. However, the new water tower will hold $\frac{1}{8}$ as much water as the tower in Edmond. Determine the radius of the new water tower to the nearest foot.

21 feet



56. **CELLS** The number of cells in a cell culture grows exponentially. The number of cells in the culture as a function of time is given by the expression $A\left(\frac{1}{2}\right)^{\frac{t}{T}}$, where t is measured in hours and A is the initial size of the culture. Write following expressions in radical form.

a. the number of cells after 20 minutes with N initial cells: $N \cdot \left(\frac{1}{2}\right)^{\frac{1}{3}}$

b. the number of cells after 44 minutes with N initial cells: $N \cdot \left(\frac{1}{2}\right)^{\frac{11}{15}}$

c. the number of cells after 1 hour and 15 minutes with 4000 initial cells: $4000 \cdot \left(\frac{1}{2}\right)^{\frac{5}{4}}$

Lesson 4.3 • nth Roots and Rational Exponents 181

57. **REASONING** Simplify $\sqrt[n]{m^{3n}}$, where $b > 0$. Explain your reasoning. See margin.

58. **REGULARITY** There are no real n th roots of a number w . What can you conclude about the index and the number w ? The index n is even and w is negative.

59. **CONSTRUCT ARGUMENTS** Determine the values of x for which $\sqrt[n]{x^2} \neq x$. Explain your answer. $x < 0$; $0 < x < 1$; then $\sqrt[n]{x^2} = -x$.

60. **STRUCTURE** Which of the following functions are equivalent? Justify your answer. See margin.

a. $f(x) = \sqrt[3]{x^3}$

b. $g(x) = \sqrt{x^2}$

c. $h(x) = \sqrt[3]{x^3}$

d. $k(x) = \sqrt{x^2}$

e. $l(x) = \sqrt[3]{x^3}$

Higher Order Thinking Skills

61. **WRITE** Explain how it might be easier to simplify an expression using rational exponents rather than using radicals. See margin.

62. **FIND THE ERROR** Destiny and Kim are simplifying $\sqrt[3]{8x^3y^3}$. Is either of them correct? Explain your reasoning.

Kim's sample answer: Destiny's error was keeping the y^3 inside the absolute value symbol.

Destiny	Kim
$\sqrt[3]{8x^3y^3} = \sqrt[3]{8x^3} \cdot \sqrt[3]{y^3} = 2x \cdot y = 2xy$	$\sqrt[3]{8x^3y^3} = \sqrt[3]{8x^3} \cdot \sqrt[3]{y^3} = 2x \cdot y = 2xy$

63. **PERSEVERE** Under what conditions is $\sqrt{x^2 + y^2} = x + y$ true? when x or y are 0 and the other variable is ≥ 0 .

64. **ANALYZE** Determine whether the statement $\sqrt[n]{x^2} = x$ is sometimes, always, or never true. Justify your argument.

Sometimes; sample answer: When $x = -3$, $\sqrt[n]{x^2} = \sqrt[n]{9} = 3$ or -3 . When $x = 3$, $\sqrt[n]{x^2} = 3$ or -3 .

65. **PERSEVERE** For what real values of x is $\sqrt{x} > x$? $0 < x < 1$, $x < -1$.

66. **WRITE** Explain when and why absolute value symbols are needed when taking an n th root. See margin.

67. **PERSEVERE** Write an equivalent expression for $\sqrt[3]{2x} \cdot \sqrt[3]{16y}$. Simplify the radical. $2\sqrt[3]{2xy}$

68. **CREATE** Find two different expressions that equal 2 in the form $a^{\frac{1}{b}}$. Sample answer: $4^{\frac{1}{2}}$ and $16^{\frac{1}{4}}$

Graphing Radical Functions

LESSON GOAL

Students graph and analyze square and cube root functions.

1 LAUNCH

 Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP

 **Explore:** Using Technology to Analyze Square Root Functions

 **Develop:**

Graphing Square Root Functions

- Identify Domain and Range Algebraically
- Graph a Transformed Square Root Function
- Analyze the Graph of a Square Root Function
- Graph a Square Root Inequality

Graphing Cube Root Functions

- Graph Cube Root Functions
- Compare Radical Functions
- Write a Radical Function

 You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE

 Exit Ticket

 Practice

DIFFERENTIATE

 View reports of student progress on the **Checks** after each example.

Resources	AL	OE	EL	
Remediation: Rational Exponents	●	●		●
Extension: Relativistic Mass		●	●	●

Language Development Handbook

Assign page 22 of the *Language Development Handbook* to help your students build mathematical language related to graphing and analyzing square and cube root functions.

 You can use the tips and suggestions on page T22 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min	1 day
45 min	2 days

Focus

Domain: Functions

Standards for Mathematical Content:

F.IF.7b Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Standards for Mathematical Practice:

1 Make sense of problems and persevere in solving them.

3 Construct viable arguments and critique the reasoning of others.

5 Use appropriate tools strategically.

Coherence

Vertical Alignment

Previous

Students simplified expressions involving radicals and rational exponents.
A.SSE.2

Now

Students graph and analyze square and cube root functions.
F.IF.7b, F.BF.3

Next

Students will simplify and perform operations with radical expressions.
A.SSE.2

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION

 **Conceptual Bridge** In this lesson, students develop an understanding of rational functions and build fluency by graphing them. They apply their understanding of graphing rational functions by solving real-world problems.

Mathematical Background

A function that contains a variable inside a square root symbol is called a square root function. The domains and ranges of these functions are limited to values for which the function is defined.



Interactive Presentation

Warm Up

Simplify:

- $\sqrt{60}$
- $\sqrt{16} + \sqrt{99}$
- $\sqrt{3} - \sqrt{12}$
- $\frac{3}{\sqrt{5}} + \frac{2\sqrt{5}}{3}$

Show Answers

Warm Up

Launch the Lesson

For large mammals, including humans, their maximum walking speed is related to the length of their legs from hip to foot. The maximum speed in feet per second is given by the square root of the product of 32 and the length of the animal's leg.

Launch the Lesson

Vocabulary

Expand All Collapse All

- > radical function
- > square root function
- > square root inequality
- > cube root function
- * How do you graph radical functions?
- * How are the graphs of square root functions and cube root functions similar and different?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- simplifying square roots

Answers:

- $2\sqrt{10}$
- 11
- 6
- 1

Launch the Lesson

MP Teaching the Mathematical Practices

2 Create Representations Guide students to define variables and write an equation that models the situation in the Launch the Lesson. Then ask students to use the equation to find the maximum walking speed of an animal with a given leg length.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.



Explore Using Technology to Analyze Square Root Functions

Objective

Students use a graphing calculator to explore transformations and key features of square root functions.



Teaching the Mathematical Practices

5 Analyze Graphs Throughout the Explore, students will analyze the graphs they have created using graphing calculators to draw conclusions about transformations of square root functions.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

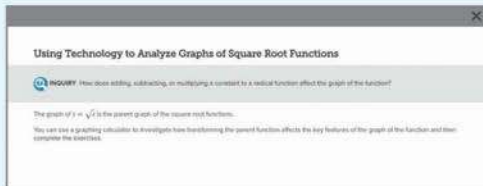
What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

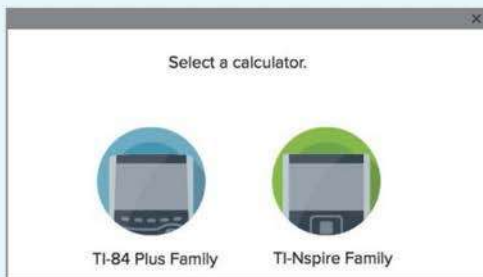
Students will complete guiding exercises throughout the Explore activity. Students will use a graphing calculator to investigate how transforming a parent function affects the key features of the graph of the function. Key features in the exercises will include x - and y -intercepts, symmetry, domain, range and end behavior. Then, students will answer the Inquiry Question.

(continued on the next page)

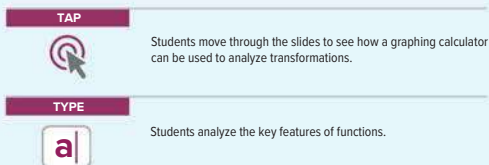
Interactive Presentation



Explore



Explore





Interactive Presentation



Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Using Technology to Analyze Square Root Functions (*continued*)

Questions

Have students complete the Explore activity.

Ask:

- What is an easy way to determine if a shift will be horizontal or vertical?
Sample answer: A horizontal shift happens when a constant is added to or subtracted from x , so look under the radical. Otherwise, a constant added or subtracted will be a vertical shift.
- Does multiplying by a positive number ever change the end behavior of a function? Why or why not? No; sample answer: it may change the steepness of the function, but it will not change the end behavior. As the x approaches positive or negative infinity, the function is still going to have either the same or different behavior.

Inquiry

How does adding, subtracting, or multiplying a constant to a radical function affect the graph of the function? Sample answer: Adding or subtracting a constant causes the graph to be shifted horizontally or vertically. Multiplying by a constant stretches or compresses the graph. Multiplying by a negative constant causes the graph to be reflected in the x -axis.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Graphing Square Root Functions

Objective

Students graph and analyze square root functions.

MP Teaching the Mathematical Practices

3 Construct Arguments The Think About It! feature asks students to use stated assumptions, definitions, and previously established results to construct an argument about the domain of the parent square root function.

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. Guide students to see what information they can gather about square root functions just from looking at the constants in the functions.

Essential Question Follow-Up

Students have explored square root functions.

Ask:

Why would you choose a square root function to model a set of data instead of a polynomial function? **Sample answer:** The end behavior of a square root function might fit the data better than a polynomial function. Also, the domain of the square root function is $x \geq 0$, which may be applicable to many real-world situations that involve quantities such as time and distance.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Lesson 4-4

Graphing Radical Functions

Explore Using Technology to Analyze Square Root Functions

Online Activity Use graphing technology to complete the Explore.

INQUIRY How does adding, subtracting, or multiplying a constant to a function affect the graph of the function?

Learn Graphing Square Root Functions

A radical function is a function that contains radicals with variables in the radicand. One type of radical function is a **square root function**, which is a function that contains the square root of a variable expression.

Key Concept • Parent Function of Square Root Functions

The parent function of the square root functions is $f(x) = \sqrt{x}$.

Domain: $\{x \mid x \geq 0\}$

Range: $\{f(x) \mid f(x) \geq 0\}$

Intercepts: $x = 0, f(x) = 0$

End behavior: As $x \rightarrow 0, f(x) \rightarrow 0$, and as $x \rightarrow \infty, f(x) \rightarrow \infty$.

Increasing/decreasing: increasing when $x \geq 0$

Positive/negative: positive for $x \geq 0$

Symmetry: no symmetry



A square root function can be written in the form $g(x) = a\sqrt{x-h} + k$. Each constant in the equation affects the parent graph.

- The value of a stretches or compresses (tilts) the parent graph.
- When the value of a is negative, the graph is reflected in the x -axis.
- The value of h shifts (translates) the parent graph left or right.
- The value of k shifts (translates) the parent graph up or down.

Today's Goals

- Graph and analyze square root functions.
- Graph and analyze cube root functions.

Today's Vocabulary

radical function
square root function
cube root function

Think About It!

Why is the domain limited to $x \geq 0$ for the parent of the square root function?

Sample answer: The square root of a negative value of x is not a real number and is not defined in the domain.

Go Online

You may want to complete the Concept Check to check your understanding.

Interactive Presentation

Graphing Radical Functions

Radical Function A function that contains radicals with variables in the radicand. One type of radical function is a **square root function**, which is a function that contains the square root of a variable expression.

KEY CONCEPT: PARENT FUNCTION OF SQUARE ROOT FUNCTIONS

The parent function of the square root functions is $f(x) = \sqrt{x}$.

Domain: $\{x \mid x \geq 0\}$

Range: $\{f(x) \mid f(x) \geq 0\}$

Intercepts: $x = 0, f(x) = 0$

End Behavior: As $x \rightarrow 0, f(x) \rightarrow 0$, and as $x \rightarrow \infty, f(x) \rightarrow \infty$.

Increasing/decreasing: increasing for $x \geq 0$

Positive/negative: positive for $x \geq 0$

Symmetry: no symmetry

Learn

TYPE



Students answer a question to determine whether they understand the limitations on the domain for a square root function.

SELECT



Students select how adding, subtracting, or multiplying the function by a number affects the graph.

**Example 1** Identify Domain and Range AlgebraicallyIdentify the domain and range of $f(x) = \sqrt{2x-6} + 1$.

The domain is restricted to values for which the radicand is nonnegative.

$$2x - 6 \geq 0$$

Write an inequality using the radicand.

$$2x \geq 6$$

Add 6 to each side.

$$x \geq 3$$

Divide each side by 2.

The domain is $\{x \mid x \geq 3\}$.Find $f(3)$ to determine the lower limit of the range.

$$f(3) = \sqrt{2(3) - 6} + 1 \text{ or } 1$$

The range is $\{y \mid f(y) \geq 1\}$.**Example 2** Graph a Transformed Square Root FunctionGraph $g(x) = -3\sqrt{x+1} + 2$, and identify the domain and range.

Then describe how it is related to the graph of the parent function.

Step 1 Determine the minimum domain value.

$$x + 1 \geq 0$$

Write an inequality using the radicand.

$$x \geq -1$$

Simplify.

Step 2 Make a table and graph.Use x -values determined from Step 1 to make a table.

x	$g(x)$
-1	2
0	-1
1	-2.2
2	-3.2
3	4

The domain is $\{x \mid x \geq -1\}$ and the range is $\{y \mid y \leq 2\}$.**Step 3** Compare $g(x)$ to the parent function.The maximum is $(-1, 2)$.Because $f(x) = \sqrt{x}$, $g(x) = a\sqrt{x-h} + k$ where $a = -3$, $h = -1$, and $k = 2$. $a < 0$ and $|a| > 1$, so the graph of $f(x) = \sqrt{x}$ is reflected in the x -axis and stretched vertically by a factor of 3, or 3. $h < 0$, so the graph is then translated left $|h|$ units, or 1 unit. $k > 0$, so the graph is then translated up k units, or 2 units.**Go Online**

You can learn how to graph radical functions by watching the video online.

Think About It!How are the values of h and k related to the domain and range in this example?**Sample answer:**In this example, the domain is $\{x \mid x \geq -1\}$ or $\{x \mid x \geq h\}$ and the range is $\{y \mid y \leq 2\}$ or $\{y \mid y \leq k\}$.**Go Online** You can complete an Extra Example online.**Interactive Presentation**

Identify Domain and Range Algebraically

Identify the domain and range of $f(x) = \sqrt{2x-6} + 1$.

The domain is restricted to values for which the radicand is nonnegative.

$$2x - 6 \geq 0$$

Write an inequality using the radicand.

$$2x \geq 6$$

Add 6 to each side.

$$x \geq 3$$

Divide each side by 2.

The domain is $\{x \mid x \geq 3\}$. Find $f(3)$ to determine the lower limit of the range.

$$f(3) = \sqrt{2(3) - 6} + 1 \text{ or } 1$$

The range is $\{y \mid f(y) \geq 1\}$.

Example 1

EXPAND

Students can tap to see how to avoid a common error when identifying domain and range of a square root function.

Example 1 Identify Domain and Range Algebraically**MP Teaching the Mathematical Practices****7 Interpret Complicated Expressions** Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. Guide students to use the radicand of the function to determine the domain of the function.**Questions for Mathematical Discourse****AL** Why do we set the radicand of a square root function greater than or equal to 0 to find the domain? **The domain and range can only contain real numbers, and a negative radicand yields imaginary numbers.****OL** What is another way to identify the lower limit of the range?**Sample answer:** The lower limit of the radicand is zero, so you can find the value of the function when the radical is zero.**BL** How would the domain and range change if there were absolute value symbols around the radicand? **The domain would be all real numbers, and the range would still be $\{f(x) \mid f(x) \geq 1\}$.****Example 2** Graph a Transformed Square Root Function**MP Teaching the Mathematical Practices****1 Explain Correspondences** In Example 2, students must explain the relationships between the transformed function, graph, and parent function.**Questions for Mathematical Discourse****AL** Why are negative values of x part of the domain? **x can be negative as long as the radicand is not negative; all real numbers greater than or equal to -1 result in a nonnegative radicand.****OL** What would the equation of $g(x) = -3\sqrt{x} + 2$ reflected across the x -axis be? **$g(x) = 3\sqrt{x} + 2$** **BL** How would you change $g(x)$ to reflect it in the y -axis? **Distribute a negative to all terms in the radicand.**



Example 3 Analyze the Graph of a Square Root Function

MP Teaching the Mathematical Practices

4 Use Tools In Example 3, students will need to identify important quantities in the problem and map their relationships using a function, table, and graph.

Questions for Mathematical Discourse

AL What does multiplying by 104.23 do to the parent function? This stretches the function vertically.

OL Do the designations of the independent and dependent variable make sense in the context? Explain. **Yes; sample answer:** While the force generated depends on the RPM of the machine, in this example we are determining what RPM value is needed to achieve a given force. As the function is one-to-one, it can be considered either way.

BL Describe the end behavior in the context of the problem. As the gravitational force increases, the RPMs of the centrifuge must increase.

Example 3 Analyze the Graph of a Square Root Function

BLOOD DONATION When blood is donated, medical professionals use a centrifuge to separate it. The centrifuge spins the blood, causing it to separate into three components, which are red cells, platelets, and plasma. In order to efficiently separate the blood, the centrifuge must spin at a specified rate, measured in rotations per minute (RPM), for the required gravitational force, or g force, exerted on the blood. For a centrifuge with a radius of 78 centimeters, the RPM setting of the centrifuge is determined by the product of 104.23 and the square root of the g -force required.

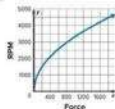
Part A Write and graph the function.

Complete the table to write the function.

Words	The RPM setting of the centrifuge	is	the product of 104.23	and the square root of the g -force required.
Variables	Let g represent the force and r represent the RPM setting.			
Function	r	=	$104.23 \cdot \sqrt{g}$	

Make a table to graph the function.

g	r
0	0
400	2085
800	2948
1200	3501
1600	4169



Part B Describe key features of the function.

Domain: $\{g \mid g \geq 0\}$

Range: $\{r \mid r \geq 0\}$

x -intercept: 0

y -intercept: 0

Increasing/Decreasing: increasing as $g \rightarrow \infty$

Positive/Negative: positive for $g > 0$

End Behavior: As $g \rightarrow \infty$, $r \rightarrow \infty$

Go Online You can compute an Extra Example online.

Think About It
What do the domain and range mean in the context of the situation?

Sample answer: In this situation, the domain and range indicate that both the force and RPM must be greater than or equal to zero.

Interactive Presentation

Analyze the Graph of a Square Root Function

BLOOD DONATION When blood is donated, medical professionals use a centrifuge to separate it. The centrifuge spins the blood, causing it to separate into three components, which are red cells, platelets, and plasma. In order to efficiently separate the blood, the centrifuge must spin at a specified rate, measured in rotations per minute (RPM), for the required gravitational force, or g force, exerted on the blood. For a centrifuge with a radius of 78 centimeters, the RPM setting of the centrifuge is determined by the product of 104.23 and the square root of the g -force required.

Example 3

SELECT



Students select variables and operations to write an equation to represent the situation.

TYPE



Students answer a question about interpreting the domain and range of a square root function in context.



Example 4 Graph a Square Root Inequality



Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the inequality and graph in Example 4.

Questions for Mathematical Discourse

AL How do you know where to stop shading on the left side of the graph? **The radical expression does not output real values less than $x = -2.5$, so the shading stops along this boundary.**

OL What points $(2.5, y)$ satisfy the inequality? **$y < 0$**

BL What points should not be used as test points for this inequality? **Points on the boundary line and points in which $x < -2.5$ should not be used.**

Learn Graphing Cube Root Functions

Objective

Students graph and analyze cube root functions.



Teaching the Mathematical Practices

3 Analyze Cases The Talk About It! feature asks students to examine the domain and range of radical function for cases when the index is even and when it is odd. Encourage students to familiarize themselves with both cases.

Example 5 Graph Cube Root Functions



Teaching the Mathematical Practices

7 Use Structure Guide students to use the structure of the cube root functions to identify transformations and graph the functions.

Questions for Mathematical Discourse

AL Is a cube root an odd function or an even function? **odd**

OL Where would you use a negative to reflect a cube root function across the x -axis? **The negative would be outside the radical.**
Where would you use a negative to reflect a cube root function across the y -axis? **The negative would be inside the radical, distributed to all terms in the radicand.**

BL Compare transformations of radical functions and transformations of polynomial functions. **Sample answer: The transformations of both function types follow the same form. A difference is needing to take domain restrictions of even roots into account along with the transformations.**

Example 4 Graph a Square Root Inequality

Graph $y < \sqrt{2x + 5}$.

Step 1 Graph the related function.
Graph the boundary $y = \sqrt{2x + 5}$, using a dashed line because the inequality is $<$.

Step 2 Shade.
The domain is $\{x | x \geq -2.5\}$. Because the inequality is less than, shade the region below the boundary and within the domain. Select a test point in the shaded region to verify the solution.

Watch Out!
Test Point When selecting a test point, make sure the point is within the domain of the related function.

Talk About It!
Describe how the domain and range differ for a radical function with an odd index and a radical function with an even index.

Sample answer: The domain and range of a radical function with an odd index are unrestricted. For a radical function with an even index, the domain is restricted so that the radicand is greater than or equal to 0. The restriction of the domain causes the range to be restricted.

Go Online
You can learn how to graph radical functions on a graphing calculator by watching the video online.

Learn Graphing Cube Root Functions

A **cube root function** is a radical function that contains the cube root of a variable expression.

Key Concept • Parent Function of Cube Root Functions
The parent function of the cube root functions is $f(x) = \sqrt[3]{x}$.

Domain: all real numbers
Range: all real numbers
Intercepts: $x = 0, f(x) = 0$
End behavior: As $x \rightarrow -\infty, f(x) \rightarrow -\infty$; and as $x \rightarrow \infty, f(x) \rightarrow \infty$.
Increasing/decreasing: increasing as $x \rightarrow \infty$
Positive/negative: positive for $x > 0$; negative for $x < 0$
Symmetry: symmetric about the origin

A cube root function can be written in the form $g(x) = a\sqrt[3]{b(x - h)} + k$.

Example 5 Graph Cube Root Functions
Graph each function. State the domain and range.

a. $g(x) = \frac{1}{3}\sqrt[3]{2x}$

In $g(x) = a\sqrt[3]{b(x - h)} + k$, $a = \frac{1}{3}$, $b = 2$, $h = 0$, and $k = 0$. So the function is centered at the origin and vertically compressed.

x	$g(x)$
-2	≈ -0.42
-1	≈ -0.33
0	0
1	≈ 0.33
2	≈ 0.42

The domain is all real numbers, and the range is all real numbers.

Interactive Presentation

Graph a Square Root Inequality

Graph $y < \sqrt{2x + 5}$.

Step 1 Graph the related function.
Graph the boundary $y = \sqrt{2x + 5}$, using a dashed line because the inequality is $<$.

Preview **Next**

Example 4

TAP



Students move through the steps to see how to graph a square root inequality.

CHECK



Students complete the Check online to determine whether they are ready to move on.



DIFFERENTIATE

Language Development Activity **AL ELL**

IF students continue to struggle with graphing square root inequalities, **THEN** have students work in pairs or small groups to discuss how to graph square root inequalities. Students should discuss how to determine the domain and range of an inequality, how to determine whether the boundary is solid or dashed, and how to determine where to shade the inequality.

Example 6 Compare Radical Functions

MP Teaching the Mathematical Practices

1 Explain Correspondences Encourage students to explain the relationships between the key features of the function and graph in Example 6.

Questions for Mathematical Discourse

A What kind of function does $q(x)$ appear to be? **Sample answer:** square root function

OL Assuming that $q(x)$ can be written in the general form for transformed functions, what is the value of h for each function?
 $h = 6$ for both functions.

BL If $q(x)$ is a square root function, is $|a|$ greater than or less than 1?
greater than

$$b. p(x) = \sqrt{x+5}$$

In $p(x) = a\sqrt{x-h} + k$, $a = 1$, $h = -5$, and $k = 0$. So the function is translated 5 units left from the parent graph. The domain is all real numbers, and the range is all real numbers.

x	$p(x)$
-2	≈ -1.26
-6	-1
-5	0
-4	1
-3	≈ 1.26



$$c. q(x) = \sqrt[3]{4-x} + 1$$

The function can be written as $q(x) = \sqrt[3]{-(x-4)} + 1$. So the function is reflected and translated 4 units right and 1 unit up from the parent graph.

x	$q(x)$
2	≈ 2.26
3	2
4	1
5	0
6	≈ -0.26



The domain is all real numbers, and the range is all real numbers.

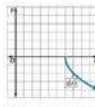
Example 6 Compare Radical Functions

Examine $p(x) = -2\sqrt{x-6}$ and $q(x)$ shown in the graph.

Part A Graph $p(x)$.

Make a table of values for $p(x)$. Then, graph the function.

x	$p(x)$
6	≈ 3.63
7	≈ 3.17
8	≈ 2.52
9	0
10	≈ -2.52
11	≈ -3.18



(continued on the next page)

Study Tip

Tables When making a table of values for a radical function, first determine h . Then make a table using values that are greater than and less than h .

Think About It!

Describe the end behavior of the function in part c.
 $y = \sqrt[3]{4-x} + 1$.

Sample answer: As x approaches negative infinity, y approaches negative infinity. As x approaches positive infinity, y approaches positive infinity.

Think About It!

Determine whether the indexes of $p(x)$ and $q(x)$ are even or odd. Justify your response.

Sample answer: $p(x)$ has an odd index because the domain and range are unrestricted. $q(x)$ has an even index because the domain is restricted.

Interactive Presentation

Compare Radical Functions

Examine $p(x) = -2\sqrt{x-6}$ and $q(x)$ shown in the graph.

Part A

Make a table of values for $p(x)$. Then, graph the function.

Example 6

EXPAND

Students move through the steps to see how to compare the graphs of two radical functions.



Part B: Compare key features.

	$p(x)$	$q(x)$
Domain and Range	D: all real numbers, R: all real numbers	D: $\{x \mid x \geq 5\}$, R: $\{y \mid y \geq 0\}$
Intercepts	x-intercept: 6, y-intercept: 3.63	x-intercept: none, y-intercept: none
Increasing/Decreasing	decreasing as $x \rightarrow \infty$	decreasing as $x \rightarrow \infty$
Positive/Negative	positive for $x < 6$; negative for $x > 6$	negative for $x > 5$
End Behavior	As $x \rightarrow -\infty$, $p(x) \rightarrow \infty$, and as $x \rightarrow \infty$, $p(x) \rightarrow -\infty$.	As $x \rightarrow \infty$, $q(x) \rightarrow \infty$, and as $x \rightarrow 5$, $q(x) \rightarrow -\infty$.

Example 7 Write a Radical FunctionWrite a radical function for the graph of $g(x)$.**Step 1** Identify the index.

Because the domain and range for $g(x)$ is all real numbers, the index is odd. This function can be represented by $g(x) = a\sqrt[n]{x-h} + k$.

**Step 2** Identify any transformations.

The function has been translated 3 units left and 4 units down. Therefore, $h = -3$ and $k = -4$. To find the value of a , use a point as well as the values of h and k .

$$\begin{aligned} g(x) &= a\sqrt[n]{x-h} + k \\ -2 &= a\sqrt[n]{-2 - (-3)} + (-4) && \text{Cube root function} \\ -2 &= a\sqrt[3]{-2 - (-3)} + (-4) && \text{Simplify.} \\ -2 &= a\sqrt[3]{-1} - 4 && \text{Simplify.} \\ a &= 2 \end{aligned}$$

Step 3 Write the function.

Substitute the values of a , h , and k to write the function. The graph is represented by $g(x) = 2\sqrt[3]{x+3} - 4$.

CheckWrite a radical function for the graph of $g(x)$.

$$g(x) = 3\sqrt[3]{x-1} + 2$$



Go Online You can complete an Extra Example online.

Study Tip:

Index While one reasonable guess is that the index is 3, the index could be any odd positive integer.

Think About It!

Write a cube root function for a graph that has been reflected in the x -axis and translated 7 units right and 4 units down from the parent function.

$$y = -\sqrt[3]{x-7} - 4$$

Example 7 Write a Radical Function

Teaching the Mathematical Practices

2 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. In Example 7, students should justify their conclusions as they solve the problem.

Questions for Mathematical Discourse

- A1** Why do we need to use a point on the graph to determine the function? **We need to isolate a and determine its value. We know h and k from the translation, so if we substitute in a point, the only remaining variable is a .**
- A2** How do you know that the index is not an even integer? **Sample answer: The domain is not restricted.**
- B1** How can you check the function to make sure you identified the correct radical n value? **Sample answer: Test more points on the graph to see whether they work in the function that was written, or graph the function using a graphing calculator and compare the result to the given graph.**

Exit Ticket

Recommended Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Interactive Presentation

Write a radical function for the graph of $g(x)$.

Identify the index.

Identify whether the function has been translated.

Use the domain and range to determine the index. Because the domain and range for $g(x)$ is all real numbers, the index is odd. This function can be represented by $g(x) = a\sqrt[n]{x-h} + k$.

Example 7

EXPAND

Students move through the steps to see how to write a radical function.

CHECK

Students complete the Check online to determine whether they are ready to move on.



Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–28
2	exercises that use a variety of skills from this lesson	29–37
2	exercises that extend concepts learned in this lesson to new contexts	38–42
3	exercises that emphasize higher-order and critical-thinking skills	43–47

ASSESS AND DIFFERENTIATE

Use the data from the Checks to determine whether to provide resources for extension, remediation, or intervention.

If students score 90% or more on the Checks, THEN assign:

- Practice Exercises 1–41 odd, 43–47
- Extension: Relativistic Mass
- ALEKS® Radical Functions

If students score 66%–89% on the Checks, THEN assign:

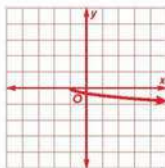
- Practice Exercises 1–47 odd
- Remediation, Review Resources: Rational Exponents
- Personal Tutors
- Extra Examples 1–6
- ALEKS® Rational Exponents

If students score 65% or less on the Checks, THEN assign:

- Practice, Exercises 1–27 odd
- Remediation, Review Resources: Rational Exponents
- Quick Review Math Handbook: Square Root Functions and Inequalities
- ALEKS® Rational Exponents

Answer

7. reflected in the x -axis, compressed vertically, and translated left 1 unit



Practice

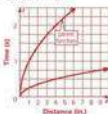
Example 1
Identify the domain and range of each function.

- $y = \sqrt{x-9}$
 $D = \{x | x \geq 9\}$; $R = \{y | y \geq 0\}$
- $y = \sqrt{x+7}$
 $D = \{x | x \geq -7\}$; $R = \{y | y \geq 0\}$
- $y = -\sqrt{x}$
 $D = \{x | x \geq 0\}$; $R = \{y | y \leq 0\}$
- $y = 5\sqrt{x+2} - 1$
 $D = \{x | x \geq -2\}$; $R = \{y | y \geq -1\}$
- $y = \sqrt{x-4}$
 $D = \{x | x \geq 4\}$; $R = \{y | y \geq 0\}$
- $y = -\sqrt{x-2} + 3$
 $D = \{x | x \geq 2\}$; $R = \{y | y \leq 3\}$

Example 2
Graph each function. State the domain and range of each function. Then describe how it is related to the graph of the parent function. 7–12. See margin for graphs and descriptions.

- $y = -\frac{1}{2}\sqrt{x+1}$
 $D = \{x | x \geq -1\}$; $R = \{y | y \leq 0\}$
- $y = -\sqrt{x-2} + 3$
 $D = \{x | x \geq 2\}$; $R = \{y | y \leq 3\}$
- $y = 3\sqrt{x-5}$
 $D = \{x | x \geq 5\}$; $R = \{y | y \geq 0\}$
- $y = \sqrt{x+4} - 2$
 $D = \{x | x \geq -4\}$; $R = \{y | y \geq -2\}$

- Example 3
13. **HELPER** Rigoberto and Ashley are testing one another's reflexes. Rigoberto drops a ruler from a given height so that it falls between Ashley's thumb and index finger. Ashley tries to catch the ruler before it falls through her hand. The time, in seconds, required to catch the ruler is given by the product of $\frac{1}{2}$ and the square root of the distance the ruler falls in inches. b , c . See margin.
- Write and graph a function, where d is the distance in inches, and t is the time, in seconds. $t = \frac{1}{2}\sqrt{d}$
 - Describe the key features of the function.
 - Graph the parent function on the same coordinate grid. How does the function you wrote in part a compare to the parent function?



Lesson 4-4 • Graphing Radical Functions 189

Example 4
Graph each inequality. 14–19. See Mod. 4 Answer Appendix.

- $y < \sqrt{x-5}$
- $y \geq -\sqrt{x+6}$
- $y \geq -4\sqrt{x+3}$
- $y \leq -2\sqrt{x-6}$
- $y > 2\sqrt{x+7} - 5$
- $y \geq 4\sqrt{x-2} - 12$

Example 5
Graph each function. State the domain and range of each function. 20–23. See Mod. 4 Answer Appendix for graphs.

- $f(x) = \sqrt{x+1}$
 $D = [-1, \infty)$; $R = [0, \infty)$
- $f(x) = 3\sqrt{x-2}$
 $D = [2, \infty)$; $R = [0, \infty)$
- $g(x) = \sqrt{x+7} - 1$
 $D = [-7, \infty)$; $R = [-1, \infty)$
- $g(x) = -\sqrt{x-2} + 9$
 $D = [2, \infty)$; $R = (-\infty, 9]$

Example 6
24. Examine $p(x) = -3\sqrt{x+2}$ and $q(x)$ shown in the graph.

- Graph $p(x)$.
- Compare the key features of the functions.
a, b. See Mod. 4 Answer Appendix.



25. Examine $p(x)$, which is 3 less than the cube root of x , and $q(x)$ shown in the graph.

- Graph $p(x)$.
- Compare the key features of the functions.
a, b. See Mod. 4 Answer Appendix.



26. Examine $p(x) = \sqrt{x-2} + 5$ and $q(x)$ shown in the graph.

- Graph $p(x)$.
- Compare the key features of the functions.
a, b. See Mod. 4 Answer Appendix.





Example 7

Write a radical function for each graph.



Mixed Exercises

Graph each function and state the domain and range. Then describe how it is related to the graph of the parent function. 29–34. See Mod. 4 Answer Appendix for graphs and descriptions.

29. $f(x) = 2\sqrt{x-5} - 6$

30. $f(x) = \frac{1}{2}\sqrt{x+12} + 3$

31. $f(x) = -\frac{1}{2}\sqrt{x-1} - 4$

32. $f(x) = -3\sqrt{x-7} + 9$

33. $f(x) = -\frac{1}{2}\sqrt{x+2} - 3$

34. $f(x) = -\frac{1}{2}\sqrt{2x-1} + 3$

35. $y \leq 6 - 3\sqrt{x-4}$

36. $y \geq \sqrt{x-12} + 8$

37. $f(x) = -\sqrt{x+4} - 2$

38. $f(x) = \frac{1}{2}\sqrt{x+1} - 2$

39. **STRUCTURE** Consider the function $f(x) = -\sqrt{x+3} + \frac{1}{2}$ and the function $g(x)$ shown in the graph.

a. Determine which function has the greater maximum value. Explain your reasoning.
b. Compare the domains of the two functions.
c. Compare the average rates of change of the two functions over the interval $[6, 12]$.

40. $y \leq 6 - 3\sqrt{x-4}$

41. $y \geq \sqrt{x-12} + 8$

42. $f(x) = -\sqrt{x+4} - 2$

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190. $f(x) = -\sqrt{x+4} - 2$

191. $f(x) = \frac{1}{2}\sqrt{x+1} - 2$

192. $f(x) = -\sqrt{x+4} - 2$

193. $f(x) = \frac{1}{2}\sqrt{x+1} - 2$

194. $f(x) = -\sqrt{x+4} - 2$

195. $f(x) = \frac{1}{2}\sqrt{x+1} - 2$

196. $f(x) = -\sqrt{x+4} - 2$

197. $f(x) = \frac{1}{2}\sqrt{x+1} - 2$

198. $f(x) = -\sqrt{x+4} - 2$

199. $f(x) = \frac{1}{2}\sqrt{x+1} - 2$

200. $f(x) = -\sqrt{x+4} - 2$

201. $f(x) = \frac{1}{2}\sqrt{x+1} - 2$

202. $f(x) = -\sqrt{x+4} - 2$

203. $f(x) = \frac{1}{2}\sqrt{x+1} - 2$

204. $f(x) = -\sqrt{x+4} - 2$

205. $f(x) = \frac{1}{2}\sqrt{x+1} - 2$

206. $f(x) = -\sqrt{x+4} - 2$

Operations with Radical Expressions

LESSON GOAL

Students simplify and perform operations with radical expressions.

1 LAUNCH



Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP



Develop:

Simplifying Radical Expressions

- Simplify Expressions with the Product Property
- Simplify Expressions with the Quotient Property

Operations with Radical Expressions

- Add and Subtract Radicals
- Multiply Radicals
- Use the Distributive Property to Multiply Radicals

Rationalizing the Denominator

- Rationalize the Denominator
- Use Conjugates to Rationalize the Denominator



You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE



Exit Ticket



Practice

DIFFERENTIATE



View reports of student progress on the **Checks** after each example.

Resources

Remediation: Simplify Radical Expressions

Extension: Special Products with Radicals

AL	OE	EL		
●	●			●
		●	●	●

Language Development Handbook

Assign page 23 of the *Language Development Handbook* to help your students build mathematical language related to simplifying and performing operations with radical functions.

ELL You can use the tips and suggestions on page T23 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min

1 day

45 min

2 days

Focus

Domain: Algebra

Standards for Mathematical Content:

A.SSE.2 Use the structure of an expression to identify ways to rewrite it.

Standards for Mathematical Practice:

4 Model with mathematics.

7 Look for and make use of structure.

Coherence

Vertical Alignment

Previous

Students graphed and analyzed square and cube root functions.

F.IF.7b, F.BF.3

Now

Students simplify and perform operations with radical expressions.

A.SSE.2

Next

Students will solve radical equations algebraically and by graphing.

A.REI.2

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION



Conceptual Bridge In this lesson, students expand on their understanding of radical expressions to build fluency by adding, subtracting, and multiplying radical expressions. They apply their understanding by performing operations with radical expressions to solve real-world problems.

Mathematical Background

The Product Property of Radicals states that $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

This statement is true for all real numbers a and b when n is an odd integer greater than 1. If n is even, then a and b must be nonnegative real numbers.



Interactive Presentation

Warm Up

Simplify each radical.

- $\sqrt[3]{27}$
- $\sqrt{32}$
- $\sqrt{125x^4}$
- $\sqrt{32x^3}$

Show Answers

Warm Up

Launch the Lesson

Golden rectangles, which have side lengths with a ratio of $\frac{2}{\sqrt{5}-1}$, have long been used in art and architecture to create beautiful designs. Rectangles with sides in this ratio are considered the most pleasing to the eye and have been found in some of the world's most historic sites such as the Parthenon, the Great Pyramids of Giza, and Notre Dame Cathedral. Recently, golden rectangles were discovered in the Sun Temple at Mesa Verde National Park in Colorado. You can simplify the ratio $\frac{2}{\sqrt{5}-1}$ by using properties of radicals and by performing operations with radicals.



Launch the Lesson

Today's Vocabulary

Expand All Collapse All

like radical expressions

Radicals in which both the index and the radicand are the same.

conjugates

Expressions of the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$, where a, b, c , and d are rational numbers.

- How do you simplify expressions with radicals?
- What are two radical expressions that are conjugates?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- simplifying radicals

Answers:

- $\sqrt{3}$
- 2
- $5x\sqrt[3]{x}$
- $2x^2\sqrt[4]{2}$

Launch the Lesson



Teaching the Mathematical Practices

4 Apply Mathematics In the Launch the Lesson, students will learn how the ratio of the side lengths of a golden rectangle can be simplified by performing operations with radical expressions.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet this standard?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using these vocabulary terms in this lesson. You can expand each row if you wish to share the definitions. Then discuss the questions below with the class.



Learn Properties of Radicals

Objective

Students simplify radical expressions by applying the properties of radicals.

MP Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. Encourage students to familiarize themselves with the conditions that must be met for radical expression to be in simplest form. Point out that when all conditions are met, they are done transforming the expression.

7 Use Structure Help students to understand how to use the structure of the Product Property and Quotient Property of Radicals to simplify radical expressions.

Example 1 Simplify Expressions with the Product Property

MP Teaching the Mathematical Practices

6 Communicate Precisely Encourage students to routinely write or explain their solution methods. The Think About It! feature asks students to explain why the solution to **part b** does not include absolute value symbols.

Questions for Mathematical Discourse

AI In **part a**, why is b^{14} written in two factors? **Sample answer:** we want to find perfect cubes that will simplify when applying the cube root. Though b^{14} is not a perfect cube, b^9 , so b^9 is factored into the cube part and the remaining part.

OL How do you factor a coefficient in a radicand to simplify an n th root? **Sample answer:** First check to see if the coefficient is a perfect n th power. If not, determine the greatest factor of the coefficient that is a perfect n th power, and separate the coefficient into this factor and the remaining factor.

BL How can you use the product property of radicals to simplify $\sqrt[3]{18a^3} \cdot \sqrt[3]{2a}$? **Multiply the terms together. Then take the square root, resulting in $\sqrt[3]{36a^4} = \sqrt[3]{6^2 a^4}$.**

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Operations with Radical Expressions

Learn Properties of Radicals

The properties used to simplify radical expressions involving square roots can be extended to radical expressions involving n th roots.

Key Concept - Product Property of Radicals

Words: For any real numbers a and b and any integer $n > 1$,

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}, \text{ if } n \text{ is even and } a, b \geq 0, \text{ or if } n \text{ is odd.}$$

Examples: $\sqrt[3]{12} \cdot \sqrt[3]{3} = \sqrt[3]{36}$ or $\sqrt[4]{4} \cdot \sqrt[4]{4} = \sqrt[4]{16}$ or 4

Key Concept - Quotient Property of Radicals

Words: For any real numbers a and b and any integer $n > 1$,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, \text{ if all roots are defined.}$$

Examples: $\sqrt[3]{\frac{27}{125}} = \frac{\sqrt[3]{27}}{\sqrt[3]{125}} = \frac{3}{5}$ or $\sqrt[4]{\frac{81}{625}} = \frac{\sqrt[4]{81}}{\sqrt[4]{625}} = \frac{3}{5}$

The properties of radicals and process of rationalizing the denominator can be used to write radical expressions in simplest form.

Key Concept - Simplest Form of Radical Expressions

A radical expression is in simplest form when the following conditions are met.

- The index n is as small as possible.
- The radicand contains no factors (other than 1) that are n th powers of an integer or polynomial.
- The radicand contains no fractions.
- No radicals appear in the denominator.

Example 1 Simplify Expressions with the Product Property

Simplify each expression

a. $\sqrt[3]{-27a^9b^{14}}$

$$= \sqrt[3]{(-3)^3 \cdot (a^3)^3 \cdot (b^4)^3 \cdot b^2}$$

$$= \sqrt[3]{(-3)^3} \cdot \sqrt[3]{(a^3)^3} \cdot \sqrt[3]{(b^4)^3} \cdot \sqrt[3]{b^2}$$

$$= -3a^3b^4\sqrt[3]{b^2}$$

Factor into cubes.

Product Property of Radicals

Simplify.

b. $\sqrt[3]{75a^3b^3}$

$$= \sqrt[3]{3^3 \cdot (a^3)^3 \cdot (b^3)^3}$$

$$= \sqrt[3]{3^3} \cdot \sqrt[3]{(a^3)^3} \cdot \sqrt[3]{(b^3)^3}$$

$$= 3a^3b^3\sqrt[3]{1}$$

$$= 3a^3b^3$$

Factor into squares.

Product Property of Radicals

Simplify.

Today's Goals

- Simplify radical expressions.
- Add, subtract, and multiply radicals.
- Divide and simplify radical expressions by rationalizing the denominator.

Today's Vocabulary
like radical expressions
conjugates

Think About It!

Why are absolute value symbols not necessary around y^2 in part b even though it is an odd power and the result of finding the even root of an even power?

Sample answer: Because y is raised to an odd power, in order for $\sqrt[3]{75a^3b^3}$ to be defined, y must be nonnegative.

Interactive Presentation

Properties of Radicals

The properties used to simplify radical expressions involving square roots can be extended to radical expressions involving n th roots.

KEY CONCEPT: PROPERTIES OF RADICALS

Product Property of Radicals

Words: For any real numbers a and b and any integer $n > 1$,
 $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$, if n is even and $a, b \geq 0$, or if n is odd.

Examples: $\sqrt[3]{12} \cdot \sqrt[3]{3} = \sqrt[3]{36}$ or $\sqrt[4]{4} \cdot \sqrt[4]{4} = \sqrt[4]{16}$ or 4

Learn

TYPE



Students answer a question applying the properties of radicals.

SELECT



Students select whether each expression is in simplest form.



Example 2 Simplify Expressions with the Quotient Property

MP Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. In Example 2, guide students to simplify the numerator and denominator separately.

Questions for Mathematical Discourse

- AL** In **part b**, why do you need absolute value symbols around y ?
Sample answer: This ensures that the even root of an even power is a positive number and thus is the principal root.
- OL** In **part b**, what is another approach you could take to simplify the coefficients? **Sample answer:** You could simplify the fraction $\frac{80}{256}$ to $\frac{5}{16}$ first.
- BL** How can you check your answer? **Sample answer:** Raise your answer to the exponent equal to the root you took. You should get the radicand of the original expression.

Learn Adding and Subtracting Radical Expressions

Objective

Students add, subtract, and multiply radicals by using the Distributive Property and combining like terms.

MP Teaching the Mathematical Practices

- 3 Find the Error** The Think About It! feature requires students to read Jon's argument, decide whether it makes sense, and explain why or why not.
- 8 Look for a Pattern** Help students to see the pattern in the product when multiplying radical expressions with coefficients. Encourage them to write a general rule for finding the product.

About the Key Concept

Exact roots occur when the powers of the constants and variables are all identical to or multiples of the index. For example, $\sqrt[3]{2 \cdot \sqrt[3]{2^2}} = \sqrt[3]{2^3}$ or 2.

Check

Simplify each expression.

a. $\sqrt[3]{4x^3y^{20}}$

$\sqrt[3]{4x^3y^{20}}$

b. $\sqrt[3]{60x^3y^{12}}$

$20x^3y^{12}$

Example 2 Simplify Expressions with the Quotient Property

Simplify each expression.

a. $\sqrt[3]{\frac{24x^3}{125}}$

$\sqrt[3]{\frac{24x^3}{125}}$

$\sqrt[3]{\frac{24x^3}{125}}$

$\sqrt[3]{\frac{24x^3}{125}}$

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Think About It!
 Jon says that you cannot combine $\sqrt[3]{40}$ and $6\sqrt[3]{5}$ because they do not have the same radicand. Is he correct? Why or why not?

No; sample answer: $\sqrt[3]{40}$ can be simplified to $2\sqrt[3]{5}$, which has the same index and radicand as $6\sqrt[3]{5}$.

Learn Adding and Subtracting Radical Expressions

Radicals can be added and subtracted in the same manner as monomials. In order to add or subtract, the radicals must be like terms. Radicals are like radical expressions if both the index and the radicand are the same.

Go Online You can complete an Extra Example online.

Interactive Presentation

Example 2

TAP



Students move through the steps to simplify an expression.

CHECK



Students complete the Check online to determine whether they are ready to move on.

**Example 3** Add and Subtract Radicals**MP Teaching the Mathematical Practices**

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. In Example 3, students must transform the terms in the expression to simplify it.

Questions for Mathematical Discourse

AL Why is the first step simplifying the radicals? **Simplifying the radicals results in like terms that can be combined.**

OL How does the distributive property relate to being able to combine radicals with like index and radicand? **Sample answer:** An expression with like radicals could be factored into the radical multiplied by the expression of the coefficient terms:
 $a\sqrt{x} + b\sqrt{x} = (a + b)\sqrt{x}$.

BL What values of x would yield an expression with only one term?

Sample answer: $\left(\frac{3}{5}\right)^n$, where n is any perfect square, i.e. any value that makes the radicand simplify to $\sqrt{3}$.

Example 4 Multiply Radicals**MP Teaching the Mathematical Practices**

7 Use Structure In Example 4, help students see how the radicand can be broken into several objects.

Questions for Mathematical Discourse

AL What must be true about the indices of the two roots you are multiplying together? **They must be the same.**

OL What is another approach you could take to simplify the expression? **Sample answer:** Simplify each root first and then find the product.

BL What is a potential advantage of multiplying radicands before simplifying? **Multiplying first may make it possible to find factors to the n th power. If you simplify first, you would still need to multiply and simplify again.**

DIFFERENTIATE**Reteaching Activity AL**

Have students work in small groups. This strategy allows every student to have an opportunity to speak several times. Ask a question or give a prompt about radical expressions such as "Name one thing to keep in mind as you simplify radical expressions?" Then pass a stick or other object to the student. The student speaks, everyone listens, and then passes the object to the next person. The next student speaks, everyone listens, then the student passes the object on until everyone has one or two turns.

Although two radicals, such as $\sqrt{18}$ and $\sqrt{32}$, may not appear to be like radicals, if you simplify each radical, you can see that $\sqrt{18} = 3\sqrt{2}$ and $\sqrt{32} = 4\sqrt{2}$. These simplified expressions are like radical expressions and can be combined.

Radicals with the same index can be multiplied by using the Product Property of Radicals. If the radicals have coefficients before the radical symbol, multiply the coefficients. Then, multiply the radicands of each expression. To multiply radical expressions with more than one term, you can use the Distributive Property or FOIL method.

$$\left(\frac{1}{6} + 1\right)(\sqrt{2} - \sqrt{7}) = \frac{1}{6}\sqrt{2} - \frac{1}{6}\sqrt{7} + \sqrt{2} - \sqrt{7}$$

Think About It!

Complete the statement to write a general method for multiplying radicals with coefficients.

For any real numbers a , b , c , and d and any integer $n \geq 1$,
 $c\sqrt[n]{a} + d\sqrt[n]{b} = \frac{c+d}{cd\sqrt[n]{ab}}$

Example 3 Add and Subtract Radicals

$$\text{Simplify } 6\sqrt{45x} + \sqrt{12} - 3\sqrt{20x}.$$

$$\begin{aligned} &= 6\sqrt{9 \cdot 5x} + \sqrt{4 \cdot 3} - 3\sqrt{4 \cdot 5x} && \text{Factor using squares.} \\ &= 6(\sqrt{9} \cdot \sqrt{5x}) + (\sqrt{4} \cdot \sqrt{3}) - 3(\sqrt{4} \cdot \sqrt{5x}) && \text{Product Property} \\ &= 6(3\sqrt{5x}) + 2\sqrt{3} - 3(2\sqrt{5x}) && \text{Simplify radicals.} \\ &= 18\sqrt{5x} + 2\sqrt{3} - 6\sqrt{5x} && \text{Multiply.} \\ &= 12\sqrt{5x} + 2\sqrt{3} && \text{Simplify.} \end{aligned}$$

Check

Write the simplified form of $\sqrt{18x} - 5\sqrt{28} - 3\sqrt{98x} + 3\sqrt{7x}$.

$$3\sqrt{7x} - 18\sqrt{2x} - 10\sqrt{7}$$

Example 4 Multiply Radicals

$$\text{Simplify } 4\sqrt{-10x^3y^2} \cdot 3\sqrt{16x^2y^2}.$$

$$\begin{aligned} &= 4 \cdot 3 \cdot \sqrt{-10x^3y^2} \cdot \sqrt{16x^2y^2} && \text{Product Property of Radicals} \\ &= 12 \cdot \sqrt{-1 \cdot 2 \cdot 5 \cdot x^3 \cdot y^2} \cdot \sqrt{2^4 \cdot x^2 \cdot y^2} && \text{Factor the constants.} \\ &= 12 \cdot \sqrt{-1 \cdot 2^2 \cdot 5 \cdot x^2 \cdot x \cdot y^2} \cdot \sqrt{2^2 \cdot x \cdot y^2} && \text{Group into powers of 5.} \\ &= 12 \cdot (-1 \cdot 2 \cdot x \cdot y) \cdot \sqrt{5 \cdot x \cdot y^2} && \text{Product Property of Radicals} \\ &= 12 \cdot (-2xy) \cdot \sqrt{5xy} && \text{Simplify.} \\ &= -24xy^2\sqrt{5xy} && \text{Multiply.} \end{aligned}$$

Check

Write the simplified form of $5\sqrt{-9x^3y^2} \cdot \sqrt{27x^2y^2}$.

$$-45xy^2\sqrt{3x}$$

Go Online You can complete an Extra Example online.

Lesson 4-5 • Operations with Radical Expressions 195

Interactive Presentation

Add and Subtract Radicals

Simplify $6\sqrt{45x} + \sqrt{12} - 3\sqrt{20x}$.

$$\begin{aligned} &6\sqrt{45x} + \sqrt{12} - 3\sqrt{20x} \\ &= 6\sqrt{9 \cdot 5x} + \sqrt{4 \cdot 3} - 3\sqrt{4 \cdot 5x} && \text{Factor using squares.} \\ &= 6(\sqrt{9} \cdot \sqrt{5x}) + (\sqrt{4} \cdot \sqrt{3}) - 3(\sqrt{4} \cdot \sqrt{5x}) && \text{Product Property} \\ &= 6(3\sqrt{5x}) + 2\sqrt{3} - 3(2\sqrt{5x}) && \text{Simplify radicals.} \\ &= 18\sqrt{5x} + 2\sqrt{3} - 6\sqrt{5x} && \text{Multiply.} \\ &= 12\sqrt{5x} + 2\sqrt{3} && (18 - 6)\sqrt{5x} = 12\sqrt{5x} \end{aligned}$$

Example 4

TYPE

Students answer a question about even and odd roots.

**Example 5** Use the Distributive Property to Multiply Radicals**SPORTS** A sports pennant has the dimensions shown. Find the area, in square inches.

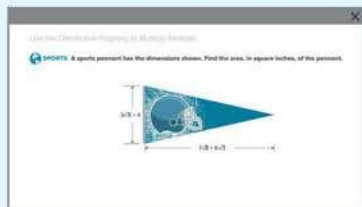
$$\begin{aligned}
 \text{Area} &= \frac{1}{2} \cdot \text{base} \cdot \text{height}, \text{ so the area is } \frac{1}{2} (3\sqrt{8} + 4)(7\sqrt{8} + 6\sqrt{3}) \\
 &= \frac{1}{2} [3\sqrt{8} \cdot 7\sqrt{8} + 3\sqrt{8} \cdot 6\sqrt{3} + 4 \cdot 7\sqrt{8} + 4 \cdot 6\sqrt{3}] \\
 &= \frac{1}{2} [21\sqrt{64} + 18\sqrt{24} + 28\sqrt{8} + 24\sqrt{3}] \\
 &= \frac{1}{2} [21(8) + 18\sqrt{2^3 \cdot 3} + 28\sqrt{2^3} \cdot 2 + 24\sqrt{3}] \\
 &= \frac{1}{2} [21(8) + 18 \cdot \sqrt{2^2} \cdot \sqrt{6} + 28 \cdot \sqrt{2^2} \cdot 2 + 24\sqrt{3}] \\
 &= \frac{1}{2} [168 + 36\sqrt{6} + 56\sqrt{2} + 24\sqrt{3}] \\
 &= 84 + 18\sqrt{6} + 28\sqrt{2} + 12\sqrt{3}
 \end{aligned}$$

The area of the pennant is $84 + 18\sqrt{6} + 28\sqrt{2} + 12\sqrt{3}$ in², or about 188.5 in².**Check****POOLS** A rectangular pool safety cover has a length of $7\sqrt{10} - 4$ feet and a width of $6\sqrt{10} + 8\sqrt{5}$ feet. Which expression represents the area of the pool cover in simplest form? **C**

- A. $420 + 280\sqrt{2} + 24\sqrt{10} + 32\sqrt{5}$ ft²
 B. $42\sqrt{100} + 280\sqrt{2} - 24\sqrt{10} - 32\sqrt{5}$ ft²
 C. $420 + 280\sqrt{2} - 24\sqrt{10} - 32\sqrt{5}$ ft²
 D. $420 + 56\sqrt{50} - 24\sqrt{10} - 32\sqrt{5}$ ft²

Learn Rationalizing the Denominator

If a radical expression contains a radical in the denominator, you can rationalize the denominator to simplify the expression. Recall that to rationalize a denominator, you should multiply the numerator and denominator by a quantity so that the radicand has an exact root.

Watch Out!**Simplest Form** Do not forget to check that your result is in simplest form. Make sure that none of the individual radicals can be further simplified or combined.**Interactive Presentation**

Example 5

TAP

Students can tap to see a tip about writing results in simplest form.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 5 Use the Distributive Property to Multiply Radicals**Teaching the Mathematical Practices****4 Apply Mathematics** In Example 5, students will apply what they have learned about adding, subtracting, and multiplying radical expressions to solve a real-world problem.**Questions for Mathematical Discourse**

- A1.** Why does the exact solution have four terms? **None of the terms have the same radicand, so they cannot be combined. 188.5 is an approximation.**
- O1.** How would you multiply a radical expression with three terms by a radical expression with two terms? **Use the Distributive Property and multiply each term of the first expression by each term in the second expression.**
- B1.** What is the maximum number of terms that will result from multiplying two radical expressions? **The maximum number of terms will be the number of terms of the first expression times the number of terms of the second expression.**

DIFFERENTIATE**Reteaching Activity****IF** when presented with a radical expression such as $11 + 6\sqrt{3}$, some students persist in trying to add 11 and 6,**THEN** to help them understand why this cannot be done, compare the radical expression $11 + 6\sqrt{3}$ to the expression $11 + 6x$. Stress that the radical $6\sqrt{3}$ is a multiplication expression just like $6x$. Remind students that the order of operations requires that multiplication be performed before addition.



Learn Rationalizing the Denominator

Objective

Students divide and simplify radical expressions by rationalizing the denominator.

Teaching the Mathematical Practices

3 Analyze Cases Guide students to examine the cases of rationalizing the denominator. Encourage students to familiarize themselves with each case and know when to apply each.

7 Use Structure Help students to understand how the structure of conjugates is used to rationalize the denominator.

Things to Remember

The product of conjugates is always a rational number.

Example 6 Rationalize the Denominator

Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. In Example 6, students must use properties of radicals and the process of rationalizing the denominator to transform and simplify the expression.

Questions for Mathematical Discourse

A When rationalizing the denominator, why are the numerator and denominator both multiplied by $\sqrt[3]{7^3d}$? **Sample answer:** this is done so that there will no longer be a radical in the denominator. The term must be multiplied to both the numerator and denominator so that the entire expression is only being multiplied by 1.

OL Why is the radicand of the expression used to rationalize the denominator $\sqrt[3]{7^3d}$? **To simplify the radical from the denominator, the resulting radicand after the multiplication must be cubed factors.**

BL What is another way that you could have begun to simplify the expression? **cancel a from numerator and denominator**

If the denominator is:	Multiply the numerator and denominator by:	Examples
$\sqrt{b}$	$\sqrt{b}$	$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \cdot \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$
$\sqrt[n]{b^n}$	$\sqrt[n]{b^n}$	$\frac{a}{\sqrt[n]{b^n}} = \frac{a}{\sqrt[n]{b^n}} \cdot \frac{\sqrt[n]{b^n}}{\sqrt[n]{b^n}} = \frac{a\sqrt[n]{b^n}}{b}$

Binomials of the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$, where $a, b, c,$ and d are rational numbers, are called **conjugates** of each other. Multiplying the numerator and denominator by the conjugate of the denominator will eliminate the radical from the denominator of the expression.

Example 6 Rationalize the Denominator

Simplify	$\frac{\sqrt[3]{250a^3}}{\sqrt[3]{7^3d}}$	Quotient Property of Radicals
	$= \frac{\sqrt[3]{2 \cdot 125 \cdot a^3}}{\sqrt[3]{7^3d}}$	Factor into cubes.
	$= \frac{\sqrt[3]{2 \cdot 5^3 \cdot a^3}}{\sqrt[3]{7^3d}}$	Product Property of Radicals
	$= \frac{5a\sqrt[3]{2}}{\sqrt[3]{7^3d}}$	Simplify.
	$= \frac{5a\sqrt[3]{2}}{\sqrt[3]{7^3} \cdot \sqrt[3]{d}}$	Rationalize the denominator.
	$= \frac{5a\sqrt[3]{2} \cdot \sqrt[3]{7^3d}}{\sqrt[3]{7^3d} \cdot \sqrt[3]{d}}$	Product Property of Radicals
	$= \frac{5a\sqrt[3]{2} \cdot \sqrt[3]{7^3d}}{\sqrt[3]{7^3d^4}}$	Multiply.
	$= \frac{5a\sqrt[3]{2} \cdot \sqrt[3]{7^3d}}{\sqrt[3]{7^3d^4}}$	Simplify.

Check

Write the simplified form of $\frac{\sqrt[3]{250a^3}}{\sqrt[3]{7^3d}}$.

$$\frac{5a\sqrt[3]{2}}{\sqrt[3]{7^3d}}$$

Go Online You can complete an Extra Example online.

Lesson 4-4 • Operations with Radical Expressions 197

Think About It!
Why is the product of $\sqrt{a}$ and $\sqrt{b}$ an exact root?

Sample answer: Using the Product Property of Radicals, $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$. Then, by the Product of Powers, $\sqrt{a^n} \cdot \sqrt{b^n} = \sqrt{a^n \cdot b^n} = \sqrt{(ab)^n} = \sqrt{ab}^n$ if both a and b are even.

Think About It!
How does multiplying conjugates relate to the difference of squares identity that can be used when multiplying binomials?

Sample answer: The difference of squares says that $(a + b)(a - b) = a^2 - b^2$. Similarly, when two conjugates are multiplied, the products of the outer and inner terms are subtracted and you are left with $(a\sqrt{b})^2 - (c\sqrt{d})^2$, or $a^2b - c^2d$.

Watch Out!
Rationalizing the Denominator When determining the quantity to multiply by when rationalizing the denominator, make sure you raise the entire term under the radical to the power of $n - 1$.

Interactive Presentation

Rationalizing the Denominator

If a radical expression contains a radical in the denominator, you can rationalize the denominator to simplify the expression. Recall that to rationalize a denominator, you should multiply the numerator and denominator by a quantity so that the radical will be an integer.

If the denominator is:	Multiply the numerator and denominator by:	Examples
$\sqrt{b}$	$\sqrt{b}$	$\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \cdot \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$
$\sqrt[n]{b^n}$	$\sqrt[n]{b^n}$	$\frac{a}{\sqrt[n]{b^n}} = \frac{a}{\sqrt[n]{b^n}} \cdot \frac{\sqrt[n]{b^n}}{\sqrt[n]{b^n}} = \frac{a\sqrt[n]{b^n}}{b}$

Learn

TYPE



Students use the properties of radicals to simplify an expression.

**Example 7** Use Conjugates to Rationalize the DenominatorSimplify $\frac{4x}{2\sqrt{7}-5}$.To rationalize the denominator, multiply the numerator and denominator by the conjugate of $2\sqrt{7}-5$.

$$\begin{aligned}\frac{4x}{2\sqrt{7}-5} &= \frac{4x}{2\sqrt{7}-5} \cdot \frac{2\sqrt{7}+5}{2\sqrt{7}+5} && 2\sqrt{7}+5 \text{ is the conjugate of } 2\sqrt{7}-5. \\ &= \frac{4x(2\sqrt{7}+5)}{(2\sqrt{7}-5)(2\sqrt{7}+5)} && \text{Multiply.} \\ &= \frac{8x\sqrt{7}+20x}{28-25} && \text{Simplify.} \\ &= \frac{8x\sqrt{7}+20x}{3} && \text{Subtract.}\end{aligned}$$

CheckWrite the simplified form of $\frac{4x}{2\sqrt{7}-5}$.
 $\frac{8x\sqrt{7}+20x}{3}$ **Pause and Reflect**

Did you struggle with anything in this lesson? If so, how did you deal with it?

See students' observations.

Go Online You can complete an Exit Example online.

198 Module 4 • Inverse and Radical Functions

Example 7 Use Conjugates to Rationalize the Denominator**Teaching the Mathematical Practices****7 Use Structure** Help students to use the structure of the denominator to determine what to multiply the expression by to rationalize the denominator and simplify the expression.**Questions for Mathematical Discourse**

- A1** Why is a conjugate used to rationalize the denominator? **When the denominator is multiplied with the conjugate, the radical terms cancel, and the resulting denominator is rational.**
- O1** Why are both the numerator and denominator multiplied by the conjugate? **Sample answer: the whole expression is being multiplied by 1, so it does not change the expression.**
- B1** Does the order of the terms matter for determining the conjugate? **Explain. No; sample answer: as long as one term keeps the same sign while the other term changes sign, and the numerator and denominator are multiplied by the same expression, the result will be the same.**

Common Error

Some students may try to multiply the numerator and denominator by the denominator. Remind them to multiply the numerator and denominator by the conjugate of the denominator.

Exit Ticket**Recommended Use**

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

Use Conjugates to Rationalize the Denominator

Simplify $\frac{4x}{2\sqrt{7}-5}$.

To rationalize the denominator, multiply the numerator and denominator by the conjugate of $2\sqrt{7}-5$.

$$\frac{4x}{2\sqrt{7}-5} = \frac{4x}{2\sqrt{7}-5} \cdot \frac{2\sqrt{7}+5}{2\sqrt{7}+5} \quad 2\sqrt{7}+5 \text{ is the conjugate of } 2\sqrt{7}-5.$$

$$= \frac{4x(2\sqrt{7}+5)}{(2\sqrt{7}-5)(2\sqrt{7}+5)}$$

$$= \frac{8x\sqrt{7}+20x}{28-25}$$

$$= \frac{8x\sqrt{7}+20x}{3}$$

Pause

Final

Answer

Example 7

TYPE

Students compare and contrast the method used in Example 7 with an alternative method.

CHECK

Students complete the Check online to determine whether they are ready to move on.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2 exercises that mirror the examples		1–38
2	exercises that use a variety of skills from this lesson	39–51
2	exercises that extend concepts learned in this lesson to new contexts	52–55
3	exercises that emphasize higher-order and critical-thinking skills	56–60

ASSESS AND DIFFERENTIATE

 Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign:

- Practice Exercises 1–55 odd, 56–60
- Extension: Special Products with Radicals
-  **ALEKS** Simplifying Expressions

IF students score 66%–89% or more on the Checks, **THEN** assign:

- Practice Exercises 1–59 odd
- Remediation, Review Resources: Simplifying Radical Expressions
- Personal Tutors
- Extra Examples 1–7
-  **ALEKS** Other Topics Available: Radicals

IF students score 65% or less on the Checks, **THEN** assign:

- Practice Exercises 1–37 odd
- Remediation, Review Resources: Simplifying Radical Expressions
- Quick Review Math Handbook*: Operations with Radical Expressions
-  **ALEKS** Other Topics Available: Radicals

Practice

Examples 1 and 2

Simplify.

$$1. \sqrt{72} \sqrt{3} \quad 6. 9\sqrt{2} \sqrt{3}$$

$$2. \sqrt{50} \sqrt{2} \quad 3. 2\sqrt{2} \sqrt{3}$$

$$3. \sqrt{24} \sqrt{3} \quad 2. 9\sqrt{2} \sqrt{3}$$

$$4. \sqrt{18} \sqrt{3} \quad 3. 9\sqrt{2} \sqrt{3}$$

$$5. \sqrt{64} \sqrt{3} \quad 2. 9\sqrt{2} \sqrt{3}$$

$$6. \sqrt{18} \sqrt{3} \quad 3. 9\sqrt{2} \sqrt{3}$$

$$7. \sqrt{18} \sqrt{3} \quad 2. 9\sqrt{2} \sqrt{3}$$

$$8. \sqrt{18} \sqrt{3} \quad 3. 9\sqrt{2} \sqrt{3}$$

$$9. \sqrt{18} \sqrt{3} \quad 2. 9\sqrt{2} \sqrt{3}$$

$$10. \sqrt{18} \sqrt{3} \quad 3. 9\sqrt{2} \sqrt{3}$$

Example 3

Simplify.

$$11. \sqrt{2} + \sqrt{8} + \sqrt{50} \quad 8\sqrt{2}$$

$$12. \sqrt{2} - 2\sqrt{3} + \sqrt{50} \quad 6\sqrt{3}$$

$$13. 8\sqrt{5} - 4\sqrt{5} - \sqrt{80} \quad 4\sqrt{5}$$

$$14. 2\sqrt{48} - \sqrt{75} - \sqrt{12} \quad 3\sqrt{3}$$

$$15. \sqrt{28} - \sqrt{4} + \sqrt{63} \quad 5\sqrt{7} - 4\sqrt{14} \quad 16. \sqrt{35} + 5\sqrt{50} - 3\sqrt{60} \quad 5\sqrt{10} - 3\sqrt{15}$$

Example 4

Simplify.

$$17. 3\sqrt{5} - 8\sqrt{50} \quad 100\sqrt{2}$$

$$18. 2\sqrt{32} \sqrt{3} + \sqrt{80} \sqrt{2} \quad 32\sqrt{6} \sqrt{3}$$

$$19. 6\sqrt{32} + 4\sqrt{240} \quad 144\sqrt{2}$$

$$20. 5\sqrt{2} \sqrt{2} - 5\sqrt{2} \sqrt{2} \quad 25\sqrt{2} \sqrt{2}$$

$$21. 5\sqrt{2} - 3\sqrt{2} \sqrt{2} \quad 15\sqrt{2} \sqrt{2}$$

$$22. 3\sqrt{2} \sqrt{2} - 2\sqrt{50} \sqrt{2} \quad 6\sqrt{2} \sqrt{2}$$

Example 5

23. TRAMPOLINE There are two trampoline runways at a gymnastics practice facility. Both runways are $\sqrt{3}$ meters wide. One is $6\sqrt{3}$ meters long and the other is $5\sqrt{2}$ meters long. What is the total area of the trampoline runways? $38 + 5\sqrt{6}$ m²

24. DISTANCE Jayla walks 5 blocks north, then 8 blocks east to get to the library. Each block is $5\sqrt{10}$ yards long. If Jayla could walk in a straight line to the library instead, how far would the walk be, in yards? $5\sqrt{10}$ yd

Lesson 4.5 • Operations with Radical Expressions 199

Simplify.

$$25. (7\sqrt{2} - 3\sqrt{3})(4\sqrt{6} + 3\sqrt{10})$$

$$56\sqrt{3} + 42\sqrt{6} - 36\sqrt{2} - 54$$

$$27. (2\sqrt{10} - 6\sqrt{5})(2\sqrt{10} + 6\sqrt{5})$$

$$28. (4\sqrt{3} + 5\sqrt{2})(\sqrt{6} + 3\sqrt{3})$$

$$(16)$$

$$36\sqrt{2} + 36\sqrt{3} + 20\sqrt{3} + 60$$

Examples 6 and 7

Simplify.

$$29. \frac{\sqrt{50}}{3} - \frac{\sqrt{18}}{3}$$

$$30. \frac{\sqrt{10}}{\sqrt{2}} - \frac{\sqrt{6}}{\sqrt{3}}$$

$$31. \frac{\sqrt{10}}{3\sqrt{2}} - \frac{\sqrt{6}}{3\sqrt{3}}$$

$$32. \frac{\sqrt{10}}{3\sqrt{2}} - \frac{\sqrt{6}}{3\sqrt{3}}$$

$$33. \frac{8}{\sqrt{3} - \sqrt{2}} - 4\sqrt{3} + 6\sqrt{2}$$

$$34. \frac{\sqrt{2} + 2\sqrt{5}}{\sqrt{5} - \sqrt{2}} - \frac{\sqrt{10} + \sqrt{15}}{2}$$

$$35. \frac{9 - 2\sqrt{3}}{\sqrt{3} + 6} - \frac{2\sqrt{3} - 3}{6}$$

$$36. \frac{2\sqrt{2} + 2\sqrt{5}}{\sqrt{5} + \sqrt{2}} - 2$$

$$37. \frac{3\sqrt{2}}{\sqrt{3} - \sqrt{2}} - \frac{3\sqrt{2} + 3\sqrt{6}}{4}$$

$$38. \frac{3}{1 - \sqrt{2}} - 3x + x\sqrt{2}$$

Mixed Exercises

Simplify.

$$39. \sqrt{50} \sqrt{2} \quad 2\sqrt{2} \sqrt{2}$$

$$40. \sqrt{54} \sqrt{3} - 3\sqrt{2} \sqrt{2}$$

$$41. \frac{\sqrt{3} - 1}{2} - \frac{\sqrt{3} + 1}{2}$$

$$42. \frac{\sqrt{3} - 1}{2} - \frac{\sqrt{3} + 1}{2}$$

$$43. 3\sqrt{24} - 2\sqrt{54} + \sqrt{48} \quad 4\sqrt{3}$$

$$44. 5\sqrt{16} + 3\sqrt{25} + 6\sqrt{7} \quad 33\sqrt{2} + 12\sqrt{10}$$

$$45. 10\sqrt{175} - 4\sqrt{100} - 2\sqrt{630}$$

$$46. 7\sqrt{204} + 4\sqrt{456} - 8\sqrt{140}$$

$$26\sqrt{15} - 10\sqrt{35}$$

47. VOLUME McKenna has a rectangular prism with dimensions 20 inches by 35 inches by 40 inches. She would like to replace it with a cube with the same volume. What should the length of a side of the cube be? Express your answer as a radical expression in simplest form. $10\sqrt{28}$ in.

48. MUSIC Traditionally, musical instruments are tuned so that the note A has a frequency of 440 Hertz. With each note higher on the instrument, the frequency of the pitch is multiplied by a factor of $\sqrt{2}$. What is the ratio of the frequencies of two notes that are 6 steps apart on the instrument? What is the ratio of the frequencies of two notes that are 9 steps apart on the instrument? Express your answers in simplest form. $\sqrt{2}, \sqrt{8}$

200 Module 4 • Inverse and Radical Functions



1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Answers

56. Twyla; Brandon's mistakes were multiplying the 4 by 16 instead of 4 and multiplying the 6 by 9 instead of 3.

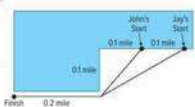
$$\begin{aligned} 58. \quad \left(\frac{-1-i\sqrt{3}}{2} \right)^3 &= \left(\frac{-1-i\sqrt{3}}{2} \right) \left(\frac{-1-i\sqrt{3}}{2} \right) \left(\frac{-1-i\sqrt{3}}{2} \right) \\ &= \frac{(-1-i\sqrt{3})(-1-i\sqrt{3})(-1-i\sqrt{3})}{8} \\ &= \frac{(1+i\sqrt{3}+i\sqrt{3}+3i)(-1-i\sqrt{3})}{8} \\ &= \frac{(2i\sqrt{3}-2)(-1-i\sqrt{3})}{8} \\ &= \frac{-2i\sqrt{3}-6i^2+2+2i\sqrt{3}}{8} \\ &= \frac{-6i^2+2}{8} = \frac{8}{8} \text{ or } 1 \end{aligned}$$

59. Sample answer: It is only necessary to use absolute values when it is possible that n could be odd or even and still be defined. It is when the radicand must be nonnegative in order for the root to be defined that the absolute values are not necessary.

49. **PHYSICS** The speed of a wave traveling over a string is given by $\frac{\sqrt{T}}{\sqrt{\rho}}$, where T is the tension of the string and ρ is the density. Simplify the expression.

- 50. LIGHT** Suppose a light has a brightness intensity of I_1 when it is at a distance of d_1 and a brightness intensity of I_2 when it is at a distance of d_2 . These quantities are related by the equation $\frac{I_1}{d_1^2} = \frac{I_2}{d_2^2}$. If $I_1 = 50$ units and $I_2 = 24$ units, find $\frac{d_1}{d_2}$. Express your answer in simplest form. **1-7**

- 51. WRITING** Jay likes to race his younger brother while running. To make the race fair, John and Jay start at different locations, but finish at the same point. The diagram shows their running paths. Both of them finished the race in exactly 4 minutes.



8. If John and Jay continued at their average paces during the race, exactly how many minutes would it take them each to run a mile? Express your answer as a radical expression in simplest form.

$$\text{Josh: } \frac{0.8 - 4\sqrt{0.01}}{0.01} \text{ or } 40 - 20\sqrt{2} \text{ min; Jay: } \frac{0.8 - 4\sqrt{0.04}}{0.01} \text{ or } 40\sqrt{5} - 80 \text{ min}$$

- b. Exactly how many times faster is Jay compared to John? Express your answer as a radical expression in simplest form. $2 - \sqrt{2} + \sqrt{5} - \frac{1}{2}$

- 82. STRUCTURE** Write the ratio of the side lengths of the two cubes described in simplest form.

- a. The volumes of the two cubes are $270x$ cubic inches and $32x^2$ cubic inches. $\frac{270x^3}{4}$

- b. The surface areas of the two cubes are $6x^2$ square feet and $6(x + 5)$ square feet.

53. **REGULARITY** Rewrite each of the following expressions as a single expression in the form $a \cdot 10^m$ for appropriate choices of a and m . Show your work.

- a. $\sqrt{x}(\sqrt{x} + \sqrt{4x})$ $3x: (\sqrt{x} + \sqrt{x}) + (\sqrt{x} + \sqrt{4x}) = (\sqrt{x^2} + \sqrt{4x^2}) = x + 2x = 3x$

- b. $\sqrt{x}(\sqrt{x} + \sqrt{4x} + \sqrt{9x})$ $6x(\sqrt{x} + \sqrt{x}) + (\sqrt{x} + \sqrt{4x}) + (\sqrt{x} + \sqrt{9x}) = \sqrt{x^3} + \sqrt{4x^3} + \sqrt{9x^3}$
 $= x + 2x + 3x = 6x$

- $$c. \quad \sqrt{x}(\sqrt{x} + \sqrt{4x} + \sqrt{9x} + \sqrt{16x}) \quad 10c. (\sqrt{x} \cdot \sqrt{x}) + (\sqrt{x} \cdot \sqrt{4x}) + (\sqrt{x} \cdot \sqrt{9x}) + (\sqrt{x} \cdot \sqrt{16x})$$

- d. More generally, simplify $\sqrt{x}(\sqrt{x} + \sqrt{4x} + \cdots + \sqrt{n^2x})$ for any positive integer n .

Use the fact that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$:

$$(\sqrt{x} \cdot \sqrt{x}) + (\sqrt{x} \cdot \sqrt{4x}) + \cdots + (\sqrt{x} \cdot \sqrt{n^2x}) = x + 2x + \cdots + nx = [1 + 2 + \cdots + n]x = \left(\frac{n(n+1)}{2}\right)x$$

Lesson 4-5 • Operations with Radical Expressions 201

- 54. USE A MODEL** If the area of the trapezoid shown is 200 square feet, what is the height h of the trapezoid? $\frac{1400 \cdot 2}{h} = \frac{1000 \cdot 2}{h}$



- 95. CONSTRUCT ARGUMENTS:** A spherical paperweight with a volume of 72π cubic centimeters is to be packaged in a gift box that is a cube. There must be at least 2 centimeters of packing material around the paperweight to protect it during shipping. The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

- a. Write an expression for the minimum length of a side of the gift box. Show your work.
 $6\sqrt{2} + 4; 7 = \frac{5\sqrt{3}}{2} \cdot 72\pi = 154 = 3\sqrt{2}; s = 2 \cdot 3\sqrt{2} + 2(2)$

5. The shipper wants to use a box with a volume of 384 cubic centimeters that they already have in inventory. Is this box suitable? Justify your argument. **Yes; sample answer:** A box with a volume of 384 cm³ has a side that is $\sqrt[3]{384}$, or about 7.27 cm; $s = 6\sqrt{2} + 4 \approx 11.55$ is the least possible value for a side of the gift box.

 Higher-Order Thinking

- 56. FIND THE ERROR** Twyla and Brandon are simplifying $4\sqrt{32} + 6\sqrt{18}$. Is either of them correct? Explain your reasoning. **See margin.**

$$\begin{aligned} & 4\sqrt{32} + 6\sqrt{18} \\ &= 4\sqrt{16 \cdot 2} + 6\sqrt{9 \cdot 2} \\ &= 16\sqrt{2} + 18\sqrt{2} \\ &= 34\sqrt{2} \end{aligned}$$

Brandon

$$\begin{aligned}4\sqrt{32} + 6\sqrt{18} \\&= 4\sqrt{16 \cdot 2} + 6\sqrt{9 \cdot 2} \\&= 64\sqrt{2} + 54\sqrt{2} \\&= 108\sqrt{2}\end{aligned}$$

57. **PERSEVERE** Find four combinations of whole numbers that satisfy $\sqrt[4]{256} = a$
 $a = 1, b = 256$; $a = 2, b = 16$; $a = 4, b = 4$; $a = 8, b = 2$

58. **PERSISTENT** Show that $\frac{-1 + \sqrt{3}i}{2}$ is a cube root of 1. **See margin.**

59. **WRITE** Explain why absolute values may be unnecessary when an n th root of an even power results in an odd power. **See margin.**

- 60. WHICH ONE DOESN'T BELONG?** Determine which of the radical expressions does not belong. Justify your answer.

$\sqrt{256g^4h^{18}}$; Sample answer: The other two simplify to $2g^2\sqrt{h}$.

$\sqrt[3]{250g^3n^6}$ $\sqrt[3]{50g^3n^2}$ $\sqrt[3]{64g^{12}n^3}$

Solving Radical Equations

LESSON GOAL

Students solve radical equations algebraically and by graphing.

1 LAUNCH



Launch the lesson with a **Warm Up** and an introduction.

2 EXPLORE AND DEVELOP



Explore: Solutions of Radical Equations



Develop:

Solving Radical Equations Algebraically

- Solve a Square Root Equation
- Solve a Cube Root Equation
- Identify Extraneous Solutions
- Solve a Radical Equation

Solving Radical Equations by Graphing

- Solve a Radical Equation by Graphing
- Solve a Radical Equation by Using a System
- Confirm Solutions by Using Technology



You may want your students to complete the **Checks** online.

3 REFLECT AND PRACTICE



Exit Ticket



Practice

DIFFERENTIATE



View reports of student progress on the **Checks** after each example.

Resources

Remediation: Operations with Radical Expressions

Extension: Lesser-Known Geometric Formulas

AL	DE	EL	U
●	●		●
	●	●	●

Language Development Handbook

Assign page 24 of the *Language Development Handbook* to help your students build mathematical language related to solving radical equations.

ELL You can use the tips and suggestions on page T24 of the handbook to support students who are building English proficiency.



Suggested Pacing

90 min

1 day

45 min

2 days

Focus

Domain: Algebra

Standards for Mathematical Content:

A.REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Standards for Mathematical Practice:

3 Construct viable arguments and critique the reasoning of others.

5 Use appropriate tools strategically.

8 Look for and express regularity in repeated reasoning.

Coherence

Vertical Alignment

Previous

Students simplified and performed operations with radical expressions.
A.SSE.2

Now

Students solve radical equations algebraically and by graphing.
A.REI.2

Next

Students will graph exponential growth and decay functions.
F.IF.4, F.IF.7e

Rigor

The Three Pillars of Rigor

1 CONCEPTUAL UNDERSTANDING	2 FLUENCY	3 APPLICATION
Conceptual Bridge In this lesson, students expand on their understanding of radical functions to build fluency by solving radical equations. They apply their understanding of radical equations by solving real-world problems.		

Mathematical Background

It is always important to check the solution to an equation in the original equation, but it is especially important when both sides of an equation are raised to a power. If a solution is an approximation, it is sometimes difficult to determine whether a discrepancy is due to rounding or if it is an incorrect solution. Students should check exact solutions whenever possible.



Interactive Presentation

Warm Up

Simplify each expression.

- $(\sqrt{3x-1})^2$
- $(\sqrt[3]{2x+5})^3$
- $(7\sqrt{x+6})^2$
- $(\sqrt{x-5}+3)^2$

Show Answers

Warm Up



Launch the Lesson

Vocabulary

Expand All Collapse All

☒ radical equation

An equation with a variable in a radicand.

- What values must be eliminated from the solution set of a radical equation?
- How do you solve a radical equation in one variable?

Today's Vocabulary

Warm Up

Prerequisite Skills

The Warm Up exercises address the following prerequisite skill for this lesson:

- simplifying radical expressions

Answers:

- $3x - 1$
- $2x + 5$
- $49x + 294$
- $x + 4 + 6\sqrt{x-5}$

Launch the Lesson



Teaching the Mathematical Practices

4 Analyze Relationships Mathematically In the Launch the Lesson, encourage students to analyze the mathematical relationships in the formula for the speed of a car.



Go Online to find additional teaching notes and questions to promote classroom discourse.

Today's Standards

Tell students that they will be addressing these content and practice standards in this lesson. You may wish to have a student volunteer read aloud *How can I meet these standards?* and *How can I use these practices?*, and connect these to the standards.

See the Interactive Presentation for I Can statements that align with the standards covered in this lesson.

Today's Vocabulary

Tell students that they will be using this vocabulary term in this lesson. You can expand the row if you wish to share the definition. Then discuss the questions below with the class.



Explore Solutions of Radical Equations

Objective

Students use a sketch to explore solutions of radical equations.



Teaching the Mathematical Practices

3 Reason Inductively Throughout the Explore, students will use inductive reasoning to analyze data and make arguments about when radical equations will have a solution.

5 Use Mathematical Tools Point out that to complete the Explore, students will need to use the sketch. Work with students to deepen their understanding of solutions of radical equations.

Ideas for Use

Recommended Use Present the Inquiry Question, or have a student volunteer read it aloud. Have students work in pairs to complete the Explore activity on their devices. Pairs should discuss each of the questions. Monitor student progress during the activity. Upon completion of the Explore activity, have student volunteers share their responses to the Inquiry Question.

What if my students don't have devices? You may choose to project the activity on a whiteboard. A printable worksheet for each Explore is available online. You may choose to print the worksheet so that individuals or pairs of students can use it to record their observations.

Summary of the Activity

Students will complete guiding exercises throughout the Explore activity. Students will use a sketch to explore when radical equations will have a solution and when they will not have a solution. Then, students will answer the Inquiry Question.

(continued on the next page)

Interactive Presentation

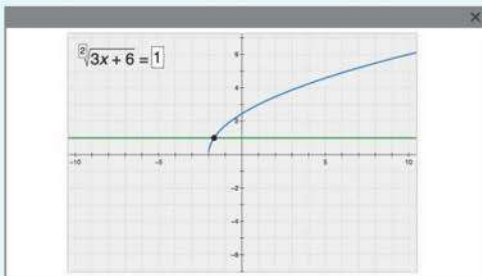
Solutions of Radical Equations

INQUIRY When will a radical equation $\sqrt{ax + b} = c$ have a solution? When will it have no solution?

You can solve equations, including those involving radical expressions, by graphing functions on each side of the equation and finding where the graphs intersect. The graphs may intersect at one or more points or not at all.

You can use the sketch to explore when radical equations of the form $\sqrt{ax + b} = c$ will have a solution.

Explore



Explore

WEB SKETCHPAD



Students use the sketch to complete an activity in which they explore the solutions of radical equations.



Interactive Presentation



Explore

TYPE



Students respond to the Inquiry Question and can view a sample answer.

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Explore Solutions of Radical Equations (continued)

Questions

Have students complete the Explore activity.

Ask:

- What must be true about the radicand of $\sqrt{ax+b} = c$ in order to have a real solution? **Sample answer:** The radicand of a square root must be greater than or equal to zero in order to have a real root.
- How many solutions will $\sqrt[4]{3x+7} = -2$ have? Why? **Sample answer:** There will be no real solution because the index is even, and the value of c is negative

Inquiry

When will a radical equation $\sqrt[n]{ax+b} = c$ have a solution? When will it have no solution? **Sample answer:** If n is odd, there will always be a solution. If n is even, there will be a solution if $c > 0$, and there will be no solution if $c < 0$.



Go Online to find additional teaching notes and sample answers for the guiding exercises.



Learn Solving Radical Equations Algebraically

Objective

Students solve radical equations in one variable and identify extraneous solutions.

MP Teaching the Mathematical Practices

8 Look for a Pattern Guide students to see the pattern in how the radical is eliminated when solving a radical equation. Ask students to explain the relationship between the index and the power to which the radical should be raised.

Example 1 Solve a Square Root Equation

MP Teaching the Mathematical Practices

7 Interpret Complicated Expressions Mathematically proficient students can see complicated expressions as single objects or as being composed of several objects. In Example 1, guide students to see the radical expression as a single object that should be isolated.

Questions for Mathematical Discourse

AL Why are both sides of the equation squared in the second step? **Sample answer:** the same operation must be performed on both sides of the equation to maintain equality.

OL What is the domain of the related function for this equation? Explain. $\{x \mid x \geq \frac{5}{3}\}$; the radicand must be positive.

BL What is the y -intercept of the related function for this equation? There is no y -intercept because $x = 0$ yields a negative radicand.

Example 2 Solve a Cube Root Equation

MP Teaching the Mathematical Practices

1 Check Answers Point out that when solving radical equation, extraneous solutions may be introduced. In Example 2, students need to check their answer.

Questions for Mathematical Discourse

AL If you wrote the equation as a radical equation, what would be the radicand, and what would be the index? **The radicand would be $2x + 6$, and the index would be 3.**

OL What rule of exponents is used when cubing $(2x + 6)^{\frac{1}{3}}$? **Power of a Power Property;** to raise an exponent to an exponent, multiply the exponents: $\frac{1}{3} \cdot 3 = 1$.

BL To what exponent would you raise both sides if the exponent of the expression was $\frac{a}{b}$? **$\frac{b}{a}$**

Lesson 4-6

Solving Radical Equations

Explore Solutions of Radical Equations

Online Activity Use graphing technology to complete the Explore.

INQUIRY When will a radical equation have a solution? When will it have no solution?

Learn Solving Radical Equations Algebraically

A radical equation has a variable in a radicand. When solving a radical equation, the result may be an extraneous solution.

Key Concept Solving Radical Equations

- Step 1** Isolate the radical on one side of the equation.
- Step 2** To eliminate the radical, raise each side of the equation to a power equal to the index of the radical.
- Step 3** Solve the resulting polynomial equation. Check your results.

Example 1 Solve a Square Root Equation

Solve $\sqrt{3x - 5} + 2 = 6$.

$$\sqrt{3x - 5} + 2 = 6$$

$$\sqrt{3x - 5} = 4$$

$$3x - 5 = 16$$

$$x = 7$$

Check that the result satisfies the original equation.

Original equation

Subtract 2 from each side.

Square each side to eliminate the radical.

Simplify.

Example 2 Solve a Cube Root Equation

Solve $4(2x + 6)^{\frac{1}{3}} - 9 = 3$.

To remove the $\frac{1}{3}$ power, or cube root, you must first isolate it and then raise each side of the equation to the third power.

$$4(2x + 6)^{\frac{1}{3}} - 9 = 3$$

$$4(2x + 6)^{\frac{1}{3}} = 12$$

$$(2x + 6)^{\frac{1}{3}} = 3$$

$$2x + 6 = 27$$

$$2x = 21$$

$$x = \frac{21}{2}$$

Original equation

Add 9 to each side.

Divide each side by 4.

Cube each side.

Subtract 6 from each side.

Divide each side by 2.

(continued on the next page)

Lesson 4-6 • Solving Radical Equations 203

Interactive Presentation



Learn

TYPE



Students determine the power to raise each expression to eliminate the radical.



Example 3 Identify Extraneous Solutions

MP Teaching the Mathematical Practices

3 Justify Conclusions Mathematically proficient students can explain the conclusions drawn when solving a problem. In Example 3 and the Talk About It! feature, students must justify their conclusions about the solution of the equation.

Questions for Mathematical Discourse

AL Why does the right side of the equation have three terms after the first step of squaring each side? **Sample answer:** The right side of the original equation has two terms. Squaring the expression is the same as multiplying the expression by itself, which requires using the Distributive Property.

OL What is an extraneous solution, and what makes one occur in this case? **An extraneous solution** is a result that does not satisfy the original equation. In this case it is the result of an even radical being equal to a negative value, when the even radical is defined to yield the nonnegative root.

BL Can an odd radical equation have an extraneous solution? Explain. **No; sample answer:** odd radicals yield one root that can be positive or negative.

Example 4 Solve a Radical Equation

MP Teaching the Mathematical Practices

1 Seek Information Mathematically proficient students must be able to transform algebraic expressions to reach solutions. In Example 4, students must transform the radical equation to reach a solution.

Questions for Mathematical Discourse

AL Why should the radical be isolated before raising both sides to the index? **Sample answer:** It simplifies the process to raise single terms to the power of the index.

OL What would happen to the solution if the right side of the original equation was -10 ? **The solution would be extraneous.**

BL What would happen to the related function if the right side of the original equation was changed to -10 ? **The graph would be translated up such that it would not intersect the x-axis.**

Common Error

Have a discussion with students about which operations may introduce extraneous solutions when solving an equation that contains a radical. Remind students that the square root sign in an equation means the principal root.

Go Online

- Find additional teaching notes.
- View performance reports of the Checks.
- Assign or present an Extra Example.

Talk About It! In Example 3, could you tell that 4 was an extraneous solution before checking the result? Explain your reasoning.

Yes; sample answer: Because a square root is never negative, once the equation simplified to $-x = -2$, I could tell that there was no solution.

Check

$$4(2x + 6) - 9 = 3$$

$$4(2 \cdot \frac{1}{2} + 6) - 9 \neq 3$$

$$4(1 + 6) - 9 \neq 3$$

$$4(7) - 9 \neq 3$$

$$28 - 9 \neq 3$$

$$19 \neq 3$$

Original equation
Replace x with $\frac{1}{2}$
Simplify
The cube root of 27 is 3
True

Example 3 Identify Extraneous Solutions

Solve $\sqrt{x+21} = 3 - \sqrt{x}$.

$$\sqrt{x+21} = 3 - \sqrt{x}$$

$$x+21 = 9 - 6\sqrt{x} + x$$

$$12 = -6\sqrt{x}$$

$$-2 = -\sqrt{x}$$

$$4 = x$$

Original equation
Square each side
Isolate the radical
Divide each side by -6
Square each side

Check

$$\sqrt{x+21} = 3 - \sqrt{x}$$

$$\sqrt{4+21} \neq 3 - \sqrt{4}$$

$$\sqrt{25} \neq 3 - 2$$

$$5 \neq 1$$

The result does not satisfy the original equation, so it is an **extraneous** solution. Therefore, there is **no real solution**.

Example 4 Solve a Radical Equation

Solve $\frac{1}{2}(11x + 14) + 8 = 10$.

$$\frac{1}{2}(11x + 14) + 8 = 10$$

$$\frac{1}{2}(11x + 14) = 2$$

$$(11x + 14) = 4$$

$$11x + 14 = 2$$

$$11x = -12$$

$$x = -\frac{12}{11}$$

Original equation
Subtract 8 from each side
Multiply each side by $\frac{1}{2}$
Rearranging each side to the sixth power
Subtract 14 from each side
Divide each side by 11

The value of $-\frac{12}{11}$ does not make the equation true.

Go Online You can complete an Extra Example online.

Interactive Presentation

Identify Extraneous Solutions

Solve $\sqrt{x+21} = 3 - \sqrt{x}$.

$$\sqrt{x+21} = 3 - \sqrt{x}$$

$$x+21 = 9 - 6\sqrt{x} + x$$

$$12 = -6\sqrt{x}$$

$$-2 = -\sqrt{x}$$

$$4 = x$$

Check

$$\sqrt{x+21} = 3 - \sqrt{x}$$

$$\sqrt{4+21} \neq 3 - \sqrt{4}$$

Original equation
Square each side
Isolate the radical
Divide each side by -6
Square each side

Example 3

TYPE



Students complete the steps of solving a radical equation.

CHECK



Students complete the Check online to determine whether they are ready to move on.



Learn Solving Radical Equations by Graphing

Objective

Students solve radical equations by examining graphs of related functions and graphing systems of equations.

MP Teaching the Mathematical Practices

1 Understand Different Approaches Work with students to look at both methods shown. Ask students to compare and contrast the methods.

Important to Know

Students should know the constraints on the values of the variables in a radical equation so that the solutions are real numbers.

Example 5 Solve a Radical Equation by Graphing

MP Teaching the Mathematical Practices

5 Decide When to Use Tools Mathematically proficient students can make sound decision about when to use mathematical tools such as graphing calculators. Help them see what using these tools will help to solve radical equations and what the limitations are of using the tools.

Questions for Mathematical Discourse

- A1** Why did we rewrite the equation with 0 on one side? **Sample answer:** This allows the solution(s) of the equation to correspond to the zeros of the related function.
- OL** Do we need to check if our solution is an extraneous solution while using a graphing method to find a solution? Explain. **No; sample answer:** only real solutions can be represented as a coordinate of a zero.
- BL** If you graphed an even root with an extraneous solution, how could you use the graph to determine the extraneous solution? **Sample answer:** The extraneous solution occurs at the x -value where the reflection of the function across $x = k$ intersects the x -axis, where k is the vertical translation from the parent function. Thus, find the x -value for which $y = 2k$.

Learn Solving Radical Equations by Graphing

To solve a radical equation using the graph of a related function, rewrite the equation with 0 on one side and then replace 0 with $f(x)$.

$$\text{Equation: } \sqrt{2x+5} + 1 = 4$$

$$\text{Related Function: } f(x) = \sqrt{2x+5} - 3 \text{ or } y = \sqrt{2x+5} - 3$$

The values of x for which $f(x) = 0$ are the zeros of the function and occur at the x -intercepts of its graph. The solutions or roots of an equation are the zeros or x -intercepts of its related function.

You can also solve a radical equation by writing and solving a system of equations based on the equation. Set the expressions on each side of the equation equal to y to create the system of equations.

$$\text{Equation: } \sqrt{2x+5} + 1 = 4$$

$$\text{System of Equations: } y = \sqrt{2x+5} + 1 \quad y = 4$$

The x -coordinate of the intersection of the system of equations is the value of x where the two equations are equal. Thus, the x -coordinate of the point of intersection is the solution of the radical equation.

Example 5 Solve a Radical Equation by Graphing

Use a graphing calculator to solve $2\sqrt{3x-4} + 10 = 9$ by graphing.

Step 1 Find a related function. Rewrite equation with 0 on right side.

$$2\sqrt{3x-4} + 10 = 9$$

Original equation

$$2\sqrt{3x-4} + 10 = 0$$

Subtract 10 from each side.

$$\text{Replacing 0 with } f(x) \text{ gives the related function } f(x) = 2\sqrt{3x-4} + 1.$$

Step 2 Graph the related function.

Use the $Y=$ list to graph.

Step 3 Use a table.

You can use the **TABLE**

feature to find the interval

where the zero lies.

The function changes sign

between $x = 1$ and $x = 2$

which indicates that there is

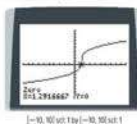
a zero between 1 and 2.

Step 4 Find the zero.

Use the **zero** feature from the **CALC** menu to find the zero of the function.

The zero is about 1.29. This is between 1 and 2, which is

consistent with the interval we found using the table.



Go Online You can complete an Extra Example online.

Lesson 4-6 • Solving Radical Equations 205

Interactive Presentation

Solving Radical Equations by Graphing

You can use the graph of a related function to solve a radical equation. An extraneous solution is found by solving the equation. The 0 can then be replaced by $f(x)$.

Equation	Related Function
$\sqrt{2x+5} + 1 = 4$	$y = \sqrt{2x+5} - 3$ $y = \sqrt{2x+5} - 3$

Learn

TYPE



Students consider the graph of a radical equation with no solution.

Think About It

What would the graph of a radical equation with no solution look like?

Sample answer: The graph would not intersect the x -axis.

Watch Out!

Misleading Graphs

Although the TI-84 may show what appears to be a discontinuity in the graphs of radical functions with odd roots, these functions are in fact continuous for all real numbers.

**Think About It!**

How can you use the table feature on your calculator to find the intersection?

Sample answer: I can scroll through the table to find the value of x when the two equations have the same y -value.

Example 6 Solve a Radical Equation by Using a System

Use a graphing calculator to solve $\sqrt{x+6} - 5 = -\sqrt{2x} + 1$ by using a system of equations.

Step 1 Write a system. Set each side of $\sqrt{x+6} - 5 = -\sqrt{2x} + 1$ equal to y to create a system of equations.

$$y = \sqrt{x+6} - 5$$

First equation

$$y = -\sqrt{2x} + 1$$

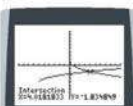
Second equation

Step 2 Graph the system. Enter the equations in the $Y=$ list and graph in the standard viewing window.

Step 3 Find the intersection.

Use the intersect feature from the **CALC** menu to find the coordinates of the point of intersection.

The solution is the x -coordinate of the intersection, which is about 4.02.



$(-9.10, 10)$ and $(4.02, -10)$

Check.

Use a graphing calculator to solve $-\sqrt{4(x-2)} = \sqrt{x} - 3 - 6$ by using a system of equations. Round to the nearest hundredth if necessary.

$$x \approx 7$$

$$3.96$$

Example 7 Confirm Solutions by Using Technology

SPACE The square of the time it takes a planet to orbit the Sun T is proportional to the cube of the planet's mean distance from the Sun a . This relationship can also be written as $T = \sqrt[3]{a^3}$, where T is measured in years and a is measured in astronomical units (AU). If it takes Mars 1.88 years to orbit the Sun, use a graphing calculator to find the mean distance from Mars to the Sun.

$$1.88 = \sqrt[3]{a^3}$$

$$T = 1.88$$

$$3.5344 = a^3$$

$$1.5233 \approx a$$

Separate each side.

Take the cube root of each side.

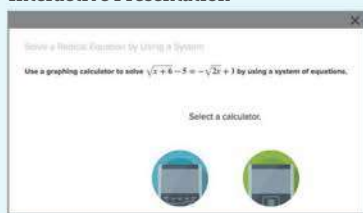
So, the mean distance from Mars to the Sun is about 1.5233 AU. Use a graphing calculator to confirm this solution by graphing.



$(-9.10, 10)$ and $(1.52, -10)$

Go Online: You can complete an Extra Example online.

206 Module 4 • Inverse and Radical Functions

Interactive Presentation

Example 6

TAP

Students move through the slides to see how to use a graphing calculator to solve a radical equation by using a system.

TYPE

Students consider how the table feature on a graphing calculator could be used to find the intersection.

CHECK

Students complete the Check online to determine whether they are ready to move on.

Example 6 Solve a Radical Equation by Using a System**Teaching the Mathematical Practices**

5 Analyze Graphs Help students analyze the graph they have generated using graphing calculators. Point out that to see the entire graph, students may need to adjust the viewing window.

Questions for Mathematical Discourse

AL Why does graphing the two sides of the equation and finding the intersection yield the value for x ? **Sample answer:** The intersection shows the value of x where the two functions are equal, and thus makes the two sides of the original equation equal.

OL How can you graph the two sides without a graphing calculator? **Sample answer:** The equations can be graphed using transformations from the parent function and a table of values.

BL Would a reflection of either of the equations in $x = k$ result in a system with a solution? Explain. **Sample answer:** Reflecting $y = \sqrt{x+6} - 5$ across $x = -5$ would also have a solution. Both functions would be decreasing, but $y = -\sqrt{2x} + 1$, which has a higher starting point, decreases faster due to the coefficient of 2.

Example 7 Confirm Solutions by Using Technology**Teaching the Mathematical Practices**

5 Use a Source Guide students to find external information about the time it takes other planets to orbit the Sun to answer the questions posed in the Use a Source feature.

Questions for Mathematical Discourse

AL How do you confirm the solution by graphing? **Either rewrite the equation equal to zero and find the x -intercept of the related function, or use each side to create a system and find the intersection.**

OL What would be the exponent if this equation was written in exponential form? $\frac{3}{2}$

BL What would you input into a calculator to find the approximate solution in one step instead of two? $1.88^{\frac{2}{3}}$

Exit Ticket**Recommended Use**

At the end of class, go online to display the Exit Ticket prompt and ask students to respond using a separate piece of paper. Have students hand you their responses as they leave the room.

Alternate Use

At the end of class, go online to display the Exit Ticket prompt and ask students to respond verbally or by using a mini-whiteboard. Have students hold up their whiteboards so that you can see all student responses. Tap to reveal the answer when most or all students have completed the Exit Ticket.

3 REFLECT AND PRACTICE

1 CONCEPTUAL UNDERSTANDING

2 FLUENCY

3 APPLICATION

Practice and Homework

Suggested Assignments

Use the table below to select appropriate exercises.

DOK	Topic	Exercises
1, 2	exercises that mirror the examples	1–30
2	exercises that use a variety of skills from this lesson	31–47
2	exercises that extend concepts learned in this lesson to new contexts	48–50
3	exercises that emphasize higher-order and critical-thinking skills	51–56

ASSESS AND DIFFERENTIATE

 Use the data from the **Checks** to determine whether to provide resources for extension, remediation, or intervention.

IF students score 90% or more on the Checks, **THEN** assign:

- Practice Exercises 1–49 odd, 51–56
- Extension: Lesser-Known Geometric Formulas
-  **ALEKS** Radical Equations and Applications

BL

IF students score 66%–89% or more on the Checks, **THEN** assign:

- Practice Exercises 1–55 odd
- Remediation, Review Resources: Operations with Radical Expressions
- Personal Tutors
- Extra Examples 1–7
-  **ALEKS** Other Topics Available: Radicals

OL

IF students score 65% or less on the Checks, **THEN** assign:

- Practice, Exercises 1–29 odd
- Remediation, Review Resources: Operations with Radical Expressions
- Quick Review Math Handbook: Solving Radical Equations and Inequalities*
-  **ALEKS** Other Topics Available: Radicals

AL

Practice

Example 1
Solve each equation.

1. $5x^2 = 1$ $\frac{1}{5}$

2. $\sqrt{0} - 5 = 4$ 21

3. $\sqrt{3x} + 1 = 5$ 8

4. $2 + \sqrt{3x+7} = 6$ 3

5. $\sqrt{x-4} + 1 = 5$ 40

6. $5 + \sqrt{2x-7} = 15$

Example 2
Solve each equation.

7. $\sqrt{3x-6} = 3$ 11

8. $(2x+3) = 2$ $\frac{5}{2}$

9. $q - 3 = 2$ 11

10. $4 - (q - 7) = 0$ -9

11. $\sqrt{2x-7} = -2$ $-\frac{3}{2}$

12. $4(5x-1) = 0$ $\frac{1}{4}$

Examples 3 and 4

Solve each equation. Identify any extraneous solutions.

13. $\sqrt{x-15} = 3 - \sqrt{x}$

no real solution; 3 is an extraneous solution.

14. $(5x+3)^2 + 7 = 5$

no real solution; 15 is an extraneous solution.

15. $(3x+7)^2 - 2 = 1$ 83

16. $(3x-2)^2 + 5 = 0$ 1

17. $(4x-9)^2 - 1 = 2$ 61

18. $\sqrt{x-10} = 3 - \sqrt{x}$

no real solution; 12 is an extraneous solution.

19. $\sqrt[3]{y^2+2} + 9 = 14$ 15,622

20. $(2x-9)^2 - 2 = 1$ 41

Example 5

Use a graphing calculator to solve each equation by graphing.

21. $\sqrt{x-7} + 2 = 8$ 43

22. $5 + \sqrt{3x+9} = 10$ 5, 3

23. $\sqrt[3]{50-4x} = 2$ 2, 4

24. $\sqrt[3]{2x+3} = -2$ -5, 5

Lesson 4.6 • Solving Radical Equations 207

Example 6

Use a graphing calculator to solve each equation by using a system of equations.

25. $2(\sqrt{2x-3}) = (3x+4)$ 2

26. $3(\sqrt{y-5}) = (5\sqrt{y+2})$ 5, 32

27. $\sqrt{2x+7} + 5 = \sqrt{3x}$ 1, 23

28. $\sqrt{x+8} = \sqrt{4x-9}$ 5, 67

Example 7

29. GEOMETRY Heron's Formula states that the area of a triangle whose sides have lengths a , b , and c is $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$. If the area of the triangle is 270 cm^2 , $a = 45$ cm, $b = 15$ cm, and $c = 39$ cm, what is the length of side a ? 36 cm

30. TITLES The largest ball of twine was started in 1953 by Frank Stoeber, in 4 years, the ball had a volume of approximately 268 ft³. What was the radius of the ball of twine at that time? Round your answer to the nearest tenth. 4.0 feet

Mixed Exercises

Solve each equation. Identify any extraneous solutions.

31. $6 + \sqrt{4x} = 9$ $\frac{9}{4}$

32. $\sqrt{7x-2} = \sqrt{0+3}$ $\frac{5}{4}$

33. $\sqrt{x+3} - \sqrt{x} = -2$ $\frac{5}{4}$

34. $\sqrt{0-6} + \sqrt{5} = 3$ $\frac{7}{2}$

35. $2(x-10)^2 + 4 = 0$ 2

36. $3x + 5 = 5 - 0$ 3

37. $\frac{1}{2}(4x) = 1$ 24, 5

38. $\frac{1}{2}(2x) = 1$ 2

39. $\sqrt{x-3} = 3 + x$

40. $\sqrt{x-2} = 2x + x$

3, 4 (extraneous solution)

16, 27 (extraneous solution)

41. $\sqrt{x+30} = x$

42. $\sqrt{x+22} = x + 2$

6, -5 (extraneous solution)

3, -4 (extraneous solution)

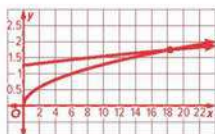
43. GEOMETRY The lateral area of a cone with base radius r and height h is given by the formula $L = \pi r(\sqrt{r^2 + h^2})$. A cone has a lateral area of 60 π square units and a base radius of 5 units. What is the height of the cone? 12 units



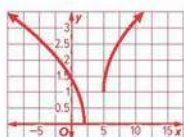


Answers

48. Isolate the radicals and write each side of the equation as a function. Then graph the functions and find their intersection(s). The intersections appear to be at $x = -1$ and $x = 7$. To verify algebraically, raise each side of the equation to the 4th power. The result is $(x + 2)^2 = 10x + 11$, or $(x - 7)(x + 1) = 0$. The solutions are $x = -1$ and $x = 7$.
49. Graph functions $f(x) = \sqrt{\frac{x}{6}}$ and $g(x) = \sqrt{\frac{x+20}{4\pi}}$, where x is the surface area of the cube. The graphs intersect at $x \approx 18.27$. The surface areas of the cube and sphere are approximately 18.27 cm^2 and 38.27 cm^2 .



50. The domain of the left-hand side is $x \geq 5$. The domain of the right-hand side is $x \leq 2$. These two domains do not overlap, so no real x -value could simultaneously satisfy both sides of the equation. The graph confirms this.



53. never;

$$\sqrt{\frac{(x^2)^2}{-x}} = x$$

$$\frac{x^2}{-x} = x$$

$$x^2 = x(-x)$$

$$x^2 \neq -x^2$$

44. **TECHNICAL** A tether of length y secures a telephone pole at 25 feet off the ground. The distance from the tether to the pole along the ground is represented by x . By the Pythagorean Theorem, the length of the tether is given by $y = \sqrt{x^2 + 25^2}$. If $x = y = 50$ feet, what is the measure of x ? **18.25 ft**



45. **SPACE** NASA's Near-Earth Asteroid Tracking project tracked more than 300 asteroids. An asteroid is passing near Earth. If Earth is located at the origin of a coordinate plane, the path that the asteroid will trace out is given by $y = -x^2 + 2$, where unit corresponds to one million miles. One asteroid will be visible by telescope when it is within $\frac{1}{10}$ million miles of Earth.
- a. Write an expression that gives the distance of the asteroid from Earth as a function of x . $\sqrt{x^2 + \frac{1}{100}}$
- b. For what values of x will the asteroid be in range of a telescope? $-\frac{1}{10} \leq x \leq \frac{1}{10}$

46. **DRIVING** To determine the speed of a car when it begins to skid to a stop, the formula $s = \sqrt{30fd}$ can be used, where s is the speed of the car, f is the coefficient of friction, and d is the length of the skid marks in feet. If the speed limit is 25 mph and the coefficient of friction is 0.6, what is the length of the skid marks if the driver is obeying the speed limit? **34.7 feet**

47. **CHEMISTRY** The nuclear radius of an element can be approximated by $r = (1.2 \times 10^{-16})A^{1/3}$, where r is the length of the radius in meters and A is the molecular mass of the element.
- a. The nuclear radius of neon is about 3.267×10^{-16} meters. Find its molecular mass. **20.2**
- b. Which element has a molecular radius of approximately 5.793×10^{-16} meter? Justify your conclusion. **Silver; $5.793 \times 10^{-16} = (1.2 \times 10^{-16})A^{1/3}$, $4.261 = A^{1/3}$, $101.917 = A$**

Element	Molecular Mass
copper (Cu)	63.5
gold (Au)	197.0
magnesium (Mg)	24.3
neon (Ne)	20.2
silver (Ag)	107.9
titanium (Ti)	47.9

48. **USE A MODEL** Explain how to find the solutions to $\sqrt{10x+11} = \sqrt{x+2} + 0$ graphically and confirm your results algebraically. What are the solutions? Construct the graph. See margin.

Lesson 4.4 • Solving Radical Equations 209

49. **USE TOOLS** The surface area of a sphere is 20 cm^2 greater than the surface area of a cube. First, functions to represent the radius of the sphere and the side length of the cube, in terms of the surface area of the cube. If the radius of the sphere equals the side length of the cube, describe how to use a graphing calculator to find the surface area of the cube and sphere. Sketch the graph. Find the surface area of the cube and the sphere. See margin.

50. **CONSTRUCT ARGUMENTS** Explain how we know that the equation $\sqrt{x-5} + 1 = \sqrt{2-x}$ has no solutions without having to actually solve it. Confirm this by graphing the two sides of the equation. See margin.

Higher-Order Thinking Skills

51. **WHICH ONE DOESN'T BELONG?** Which equation does not have a solution? Justify your conclusion.

$$\sqrt{x-1} + 3 = 4$$

$$\sqrt{x+1} + 3 = 4$$

$$\sqrt{x-2} + 7 = 10$$

$$\sqrt{x+2} - 7 = -10$$

$\sqrt{x+2} - 7 = -10$ Sample answer: This equation does not have a real solution while the other three equations do.

52. **PERSEVERE** Haruko is working to solve $(x+5)^2 = -4$. He said that he could tell there was no real solution without even working the problem. Is Haruko correct? Explain your reasoning.
Yes; sample answer: Because $(x+5)^2 \geq 0$, the left side of the equation is nonnegative. Therefore, the left side of the equation cannot equal -4 . Thus the equation has no solution.

53. **ANALYZE** Determine whether $\sqrt{-x} = -x$ is sometimes, always, or never true when x is a real number. Justify your argument. See margin.

54. **CREATE** Write an equation that can be solved by raising each side of the equation to the given power.

a. $\frac{2}{3}$ power

b. $\frac{3}{2}$ power

c. $\frac{1}{4}$ power

Sample answer: $0 = 6x - 5$ Sample answer: $0 = x^2 - 9$ Sample answer: $16x^2 = -1$

- ANALYZE** Determine whether the following statements are sometimes, always, or never true for $x \in \mathbb{R}$. Explain your reasoning.

55. If n is odd, there will be extraneous solutions.

Never; sample answer: The radicand can be negative.

56. If n is even, there will be extraneous solutions.

Sometimes; sample answer: when the radicand is negative, then there will be extraneous roots.

Review

Rate Yourself!

Have students return to the Module Opener to rate their understanding of the concepts presented in this module. They should see that their knowledge and skills have increased. After completing the chart, have them respond to the prompts in their *Student Edition* and share their responses with a partner.

Answering the Essential Question

Before answering the Essential Question, have students review their answers to the Essential Question Follow-Up questions found throughout the module.

- Why would you use the inverse of a function to model a real-world situation?
- Why would you choose a square root function to model a set of data instead of a polynomial function?

Then have them write their answer to the Essential Question.

DINAH ZIKE FOLDABLES

ELL A completed Foldable for this module should include the key concepts related to inverses and radical functions.

LE **LearnSmart** Use LearnSmart as part of your test preparation plan to measure student topic retention. You can create a student assignment in LearnSmart for additional practice on these topics for **Polynomial, Rational, and Radical Relationships**.

- Solving Rational and Radical Equations
- Graphing Rational, Radical, and Polynomials Functions

Review

Essential Question

How can the inverse of a function be used to help interpret a real-world event or solve a problem? Much like it is necessary to undo operations when solving equations in algebra, an inverse of a function can be used to undo a function to solve a problem.

Module Summary

Lesson 4-1

Operations on Functions

- $(f + g)(x) = f(x) + g(x)$
- $(f - g)(x) = f(x) - g(x)$
- $(f \cdot g)(x) = f(x) \cdot g(x)$
- $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$
- $(f \circ g)(x) = f(g(x))$

Lesson 4-2

Inverse Relations and Functions

- If f and f^{-1} are inverses, then $f(x) = b$ if and only if $f^{-1}(b) = a$.
- $f(x)$ and $g(x)$ are inverses if and only if $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

Lesson 4-3

nth Roots and Rational Exponents

- For any real numbers a and b and any positive integer n , if $a^n = b$, then a is an n th root of b .
- When there is more than one real root and n is even, the nonnegative root is called the principal root.
- An expression with rational exponents is in simplest form when it has no negative exponents, it has no exponents that are not positive integers in the denominator, it is not a complex fraction, and the index of any remaining radical is the least number possible.

Lesson 4-4

Graphing Radical Functions

- The parent function of the square root functions is $f(x) = \sqrt{x}$. The parent function of the cube root functions is $f(x) = \sqrt[3]{x}$.
- A square root function can be written in the form $g(x) = a\sqrt{x - h} + k$. A cube root function can be written in the form $g(x) = a\sqrt[3]{x - h} + k$.

Lessons 4-5 and 4-6

Radical Expressions and Equations

- For any real numbers a and b and any integer $n > 1$, $\sqrt[n]{a} = \sqrt[n]{b}$ if n is even and $a, b \geq 0$ or if n is odd.
- For any real numbers a and $b \neq 0$ and any integer $n > 1$, $\sqrt[n]{a} = \sqrt[n]{b}$ if all roots are defined.
- $a\sqrt[n]{b} + c\sqrt[n]{b}$ and $a\sqrt[n]{b} - c\sqrt[n]{b}$ are conjugates of each other.
- To solve a radical equation, isolate the radical on one side of the equation. Raise each side of the equation to a power equal to the index of the radical. Solve the resulting polynomial equation. Check your results.

Study Organizer

Foldables

Use your Foldable

to review this

module. Working

with a partner can

be helpful. Ask

for clarification

of concepts as

needed.

Using Radical Equations
Operations on Radical Expressions
Graphing Radical Functions
nth Roots and Rational Exponents
Inverse Relations and Functions
Operations on Functions

Test Practice

1. **MULTIPLE CHOICE** Given $f(x) = 4x^2 - 5x + 8$ and $g(x) = 2x^2 - 9x - 7$, find $f - g(x)$. (Lesson 4-3)

- A. $f - g(x) = 2x^2 - 14x + x$
 B. $f - g(x) = 2x^2 + 4x + 15x$
 C. $f - g(x) = 2x^2 - 14x^2 - 7x + 8$
☒ D. $f - g(x) = 2x^2 + 4x^2 + 7x + 8$

2. **OPEN RESPONSE** Given $f(x) = 2x^2 + 7x - 7$ and $g(x) = x - 2$, find $\left(\frac{f}{g}\right)(x)$. (Lesson 4-3)

$2x^2 + 15x + 15$

3. **MULTIPLE CHOICE** Given $f(x) = 4x^2 - 2x^3$ and $g(x) = -x^2 + 3x - 2$, find $f \cdot g(x)$. (Lesson 4-3)

- A. $f \cdot g(x) = 4x^3 - x^2 - 3x + 2$
 B. $f \cdot g(x) = 4x^3 - x^2 + 3x - 2$
☒ C. $f \cdot g(x) = -4x^3 + 14x^2 - 14x^3 + 6x^2$
 D. $f \cdot g(x) = 4x^3 - 14x^2 + 14x^3 - 4x^2$

4. **MULTIPLE CHOICE** Given $f(x) = 14x^2 - x^2 + x + 5$ and $g(x) = 7x^2 + 4x^2 - 2x - 1$, find $f - g(x)$. (Lesson 4-3)

- A. $f - g(x) = 7x^2 + 3x^2 + x + 4$
☒ B. $f - g(x) = 7x^2 - 5x^2 + 3x + 6$
 C. $f - g(x) = 7x^2 + 3x^2 + 3x + 6$
 D. $f - g(x) = 7x^2 - 5x^2 + x + 4$

5. **OPEN RESPONSE** Given $f(x) = 3x - 7$ and $g(x) = 4x + 5$, find $f \circ g(x)$. (Lesson 4-3)

$f \circ g(x) = 12x + 8$

6. **MULTIPLE CHOICE** The graph shows $f(x)$. Which of the following represents $f^{-1}(x)$? (Lesson 4-2)



- ☒ A. $f^{-1}(x) = 2x + 3$
 B. $f^{-1}(x) = 3x + 2$
 C. $f^{-1}(x) = x + 4$
 D. $f^{-1}(x) = 4x$

7. **OPEN RESPONSE** Find the inverse of $f(x) = x^2 + 4x + 3$. Restrict the domain, if necessary. (Lesson 4-2)

If the domain of $f(x)$ is restricted to $(-\infty, -2]$, then the inverse is $f^{-1}(x) = -2 - \sqrt{x+1}$. If the domain of $f(x)$ is restricted to $[-2, \infty)$, then the inverse is $f^{-1}(x) = -2 + \sqrt{x+1}$.

Review and Assessment Options

The following online review and assessment resources are available for you to assign to your students. These resources include technology-enhanced questions that are auto-scored, as well as essay questions.

Review Resources

Vocabulary Activity
 Module Review

Assessment Resources

Vocabulary Test

AL Module Test Form B

OL Module Test Form A

BL Module Test Form C

Performance Task*

*The module-level performance task is available online as a printable document. A scoring rubric is included.

Test Practice

You can use these pages to help your students review module content and prepare for online assessments. Exercises 1–19 mirror the types of questions your students will see on online assessments.

Question Type	Description	Exercise(s)
Multiple Choice	Students select one correct answer.	1, 3, 4, 6, 8, 9, 10, 13, 14, 16–20
Open Response	Students construct their own response.	2, 5, 7, 10–12, 15

To ensure that students understand the standards, check students' success on individual exercises.

Standard(s)	Lesson(s)	Exercise(s)
F.BF.1b	4-1	1–5
F.IF.7b	4-4	11, 13, 14
F.BF.4a	4-2	6, 7
A.SSE.2	4-3, 4-5	8–10, 12, 15–17
A.REI.2	4-6	18–20

8. **MULTIPLE CHOICE** Evaluate $256i^4 + 100,000i^{-1}$. (Lesson 4-3)

A. 79
B. 799
C. $8.0i^4$
D. 81

9. **MULTIPLE CHOICE** Which of the following is the simplified form of $\sqrt[3]{729x^3 - 77^3}$? (Lesson 4-3)

A. $27(x - 7)^3$
B. $27(x - 7)^2$
C. $9(x - 7)^3$
D. $9(x - 7)^2$

10. **MULTIPLE CHOICE** The lung volume for mammals V can be modeled using the expression $170x\sqrt{V}$, where x is the mass of the mammal. How can this expression be rewritten using radicals? (Lesson 4-3)

A. $170\sqrt[3]{V^2}$
B. $170\sqrt[3]{V^2}$
C. $\sqrt[3]{170x^2}$
D. $\sqrt[3]{170x^3}$

11. **OPEN RESPONSE** Describe how the graph of $g(x) = -2(x + 5) - 3$ is related to the graph of the parent function. (Lesson 4-4)

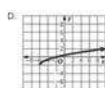
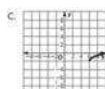
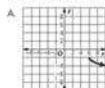
The parent function is $f(x) = x^2$ and $g(x) = a(x - h) + k$ where $a = -2$, $h = -5$, and $k = -3$. Because $a < 0$ and $|a| > 1$, the graph of $g(x)$ is the graph of the parent function reflected across the y -axis and stretched vertically by a factor of $|a|$, or 2. Also, because $h < 0$, the graph of $g(x)$ is the graph of the parent function translated left $|h|$ units, or 5 units. Finally, because $k < 0$, the graph of $g(x)$ is the graph of the parent function translated down $|k|$ units, or 3 units.

12. **OPEN RESPONSE** Determine the values of a and b . (Lesson 4-3)

$$x^2 + x^2 = a^2$$

$$a = 5, b = 12$$

13. **MULTIPLE CHOICE** Which of the following is the graph of $g(x) = -\sqrt{x + 6} - 2$? (Lesson 4-4)



14. **MULTIPLE CHOICE** Which function is shown on the graph? (Lesson 4-4)



- A. $g(x) = (x + 4)^2$
 B. $g(x) = (x - 4)^2$
☒ C. $g(x) = x^2 - 4$
 D. $g(x) = x^2 + 4$

15. **OPEN RESPONSE** Determine the sum.
 $7\sqrt{12} + 2\sqrt{48} - 4\sqrt{75}$ (Lesson 4-5)

$2\sqrt{3}$

16. **MULTIPLE CHOICE** Heather drew a rectangle and labeled the length as $3\sqrt{16}2\sqrt{y}$ and the width as $2\sqrt{40}3\sqrt{y}$. Which is the simplified form of the area of the rectangle? (Lesson 4-5)

- A. $36\sqrt{5x^2y^{11}}$
☒ B. $36xy^2\sqrt{5x^2y^{11}}$
 C. $36xy^2\sqrt{5x^2y^2}$
 D. $36\sqrt{5y^{11}}$

17. **MULTIPLE CHOICE** Letasia wants to rationalize the denominator of $\frac{2\sqrt{y}}{\sqrt{12}}$. What should she multiply the numerator and denominator by? (Lesson 4-5)

- A. $\sqrt{y}$
☒ B. $\sqrt{12}$
 C. x^2
 D. x^3

18. **MULTIPLE CHOICE** Solve for x .
 $3(6x - 8) + 3 = 9$ (Lesson 4-6)

- A. $x = 2$
☒ B. $x = 4$
 C. $x = 6$
 D. $x = 8$

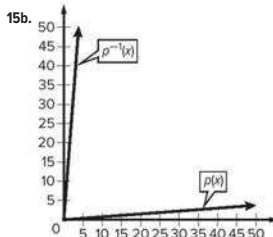
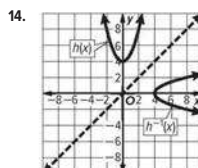
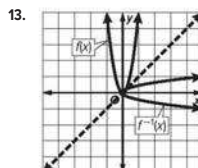
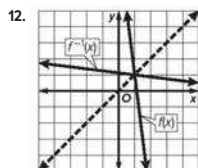
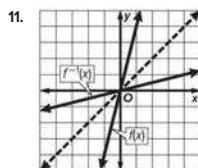
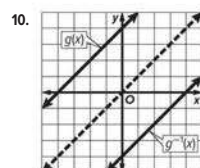
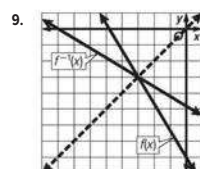
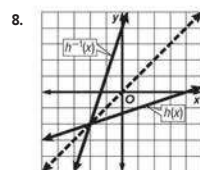
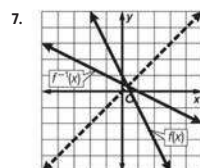
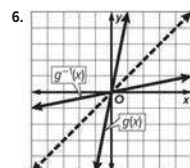
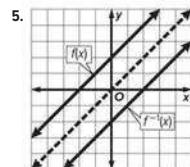
19. **MULTIPLE CHOICE** Select the solution(s) of the equation $\sqrt{3x + 7} = x - 1$. (Lesson 4-6)

- ☒ A. 6
 B. 3
 C. 2 and 3
 D. 6 and -1

20. **MULTIPLE CHOICE** The distance in miles, d , a pilot can see to the horizon is 123% of the square root of the altitude in feet above sea level, a , of the plane. How many miles to the horizon can a pilot flying a plane at an altitude of 30,000 feet see? Round your answer to the nearest mile. (Lesson 4-6)

- ☒ A. 213 miles
 B. 369 miles
 C. 21,304 miles
 D. 36,900 miles

Lesson 4-2

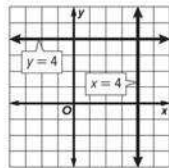


16b.

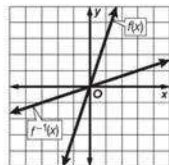
Letter	$c(x)$	Letter	$c(x)$	Letter	$c(x)$
A = 1	-5	J = 10	31	S = 19	67
B = 2	-1	K = 11	35	T = 20	71
C = 3	3	L = 12	39	U = 21	75
D = 4	7	M = 13	43	V = 22	79
E = 5	11	N = 14	47	W = 23	83
F = 6	15	O = 15	51	X = 24	87
G = 7	19	P = 16	55	Y = 25	91
H = 8	23	Q = 17	59	Z = 26	95
I = 9	27	R = 18	63		

$c(x)$	Letter	$c^{-1}(x)$	Letter	$c^{-1}(x)$	Letter
-5	A = 1	31	J = 10	67	S = 19
-1	B = 2	35	K = 11	71	T = 20
3	C = 3	39	L = 12	75	U = 21
7	D = 4	43	M = 13	79	V = 22
11	E = 5	47	N = 14	83	W = 23
15	F = 6	51	O = 15	87	X = 24
19	G = 7	55	P = 16	91	Y = 25
23	H = 8	59	Q = 17	95	Z = 26
27	I = 9	63	R = 18		

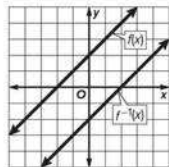
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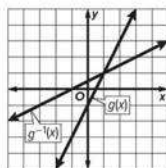
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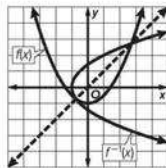
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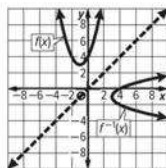
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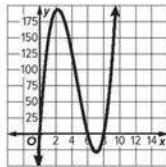
29. If the domain of $f(x)$ is restricted to $(-\infty, 0]$, then the inverse $f^{-1} = \sqrt{2x} + 2$. If the domain of $f(x)$ is restricted to $[0, \infty)$, then the inverse is $f^{-1} = -\sqrt{2x} + 2$.



30. If the domain of $f(x)$ is restricted to $(-\infty, -1]$, then the inverse $f^{-1} = -1 + \sqrt{x - 3}$. If the domain of $f(x)$ is restricted to $[-1, \infty)$, then the inverse is $f^{-1} = -1 - \sqrt{x - 3}$.



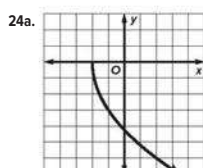
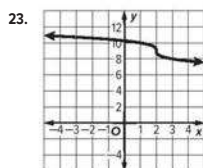
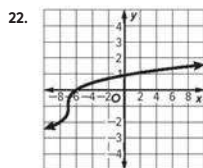
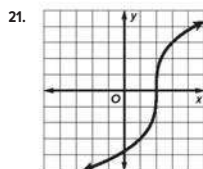
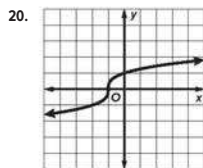
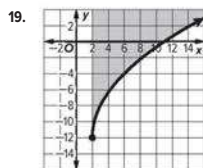
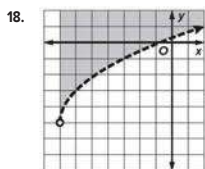
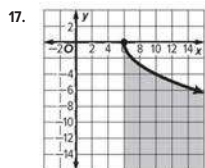
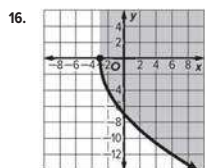
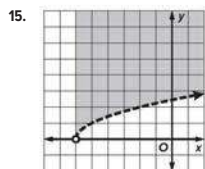
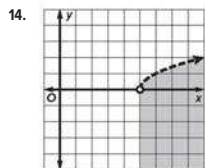
- 47a. $V(x) = x(12 - 2x)(16 - 2x)$; the value of x must be between 0 and 6 since the length x must be more than 0, but the length of the side of the box $(12 - 2x)$ cannot be less than 0.



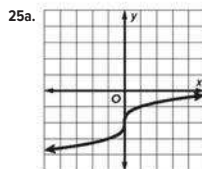
- 47b. No; sample answer: even with the restriction, the function does not pass the horizontal line test, so its inverse will not be a function.
- 48b. $(f + g)(1) = f(1) + g(1) = 3 + 0 = 3$. Since $(f + g)(1) = 3$, $(f + g)(3) = 1$. $(f + g)(0) = f(0) + g(0) = 0 + 1 = 1$. Since $(f + g)(0) = 1$, $(f + g)(1) = 0$.
- 48c. Part b provides two counterexamples to this property. $(f + g)(3) = 1$, but $f(3) + g(3) = 4$, $(f + g)(1) = 0$, but $f(1) + g(1) = 2$.
- 48d. $f(x) = \frac{1}{2}x - \frac{1}{2}$, and $g(x) = \frac{1}{2}x + \frac{1}{2}$. So $f(x) + g(x) = x$. However, since $(f + g)(x) = 4x$, $(f + g)(x) = \frac{1}{4}x$. The two are not the same.

52. No; sample answer: If $f(x) = mx + b$ then $f^{-1} = \frac{1}{m}x - \frac{b}{m}$. Perpendicular lines have slopes that are opposite reciprocals. Because $m \cdot \frac{1}{m} \neq -1$, a function with slope $\neq 0$ is not perpendicular to its inverse.

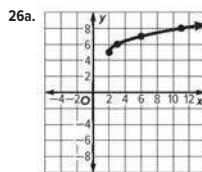
Lesson 4-4



24b.	$p(x) = -3\sqrt{x+2}$	$q(x)$
Domain	$\{x x \geq -2\}$	all real numbers
Range	$\{y y \leq 0\}$	all real numbers
Intercepts	x-int: -2; y-int: -4.24	x-int: 2; y-int: -1.25
Increasing/Decreasing	decreasing as $x \rightarrow \infty$	increasing as $x \rightarrow \infty$
Positive/Negative	negative for $x > -2$	negative for $x < 2$; positive for $x > 2$
End Behavior	as $x \rightarrow \infty$, $p(x) \rightarrow -\infty$	as $x \rightarrow -\infty$, $q(x) \rightarrow -\infty$ as $x \rightarrow \infty$, $q(x) \rightarrow \infty$

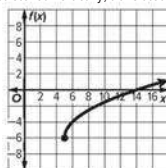


25b.	$p(x) = \sqrt[3]{x} - 2$	$q(x)$
Domain	all real numbers	$\{x x \geq 0\}$
Range	all real numbers	$\{y y \geq -2\}$
Intercepts	x-int: 8; y-int: -2	x-int: 4; y-int: -2
Increasing/Decreasing	increasing as $x \rightarrow \infty$	increasing as $x \rightarrow \infty$
Positive/Negative	negative for $x < 8$; positive for $x > 8$	negative for $x < 4$; positive for $x > 4$
End Behavior	as $x \rightarrow -\infty$, $p(x) \rightarrow -\infty$ as $x \rightarrow \infty$, $p(x) \rightarrow \infty$	as $x \rightarrow \infty$, $q(x) \rightarrow \infty$

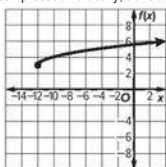


26b.	$p(x) = \sqrt{x-2} + 5$	$q(x)$
Domain	$\{x x \geq 2\}$	$\{x x \leq 2\}$
Range	$\{y y \geq 5\}$	$\{y y \geq 5\}$
Intercepts	x-int: none; y-int: none	x-int: none; y-int: about 6.5
Increasing/Decreasing	increasing as $x \rightarrow \infty$	decreasing as $x \rightarrow 2$
Positive/Negative	positive for $x \geq 2$	positive for $x \leq 2$
End Behavior	as $x \rightarrow \infty$, $p(x) \rightarrow \infty$	as $x \rightarrow -\infty$, $q(x) \rightarrow \infty$

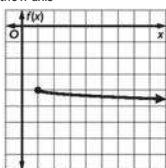
29. stretched vertically, translated 5 units right and 6 units down



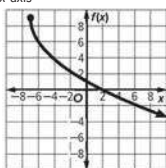
30. compressed vertically, translated 12 units left and 3 units up



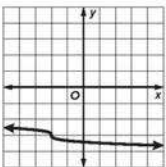
31. compressed vertically, translated 1 unit right and 4 units down, reflected in the x-axis



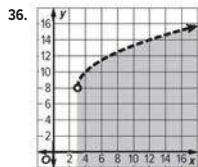
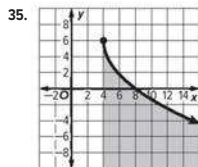
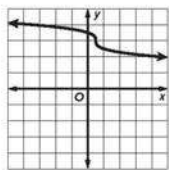
32. stretched vertically, translated 7 units left and 9 units up, reflected in the x-axis



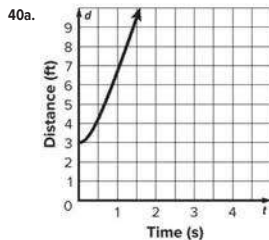
33. compressed vertically, translated 2 units left and 3 units down, reflected in the x-axis



34. compressed vertically and horizontally, translated $\frac{1}{2}$ unit right and 3 units up, reflected in the x -axis

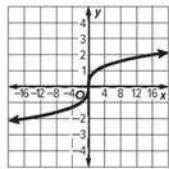


- 39a. $f(x)$ has the greater maximum value because its maximum, $\frac{13}{2}$, is greater than 6, the maximum value of $g(x)$.
- 39b. The domain of $f(x)$ is $x \geq -3$, since any values greater than 3 produce a negative value under the radical. The domain of $g(x)$ is $x \geq \frac{5}{2}$.
- 39c. The average rate of change over the interval is $-\frac{1}{7}$ for $f(x)$ and about $-\frac{1}{5}$ for $g(x)$. Both have a negative rate of change, but $f(x)$ is slightly steeper over the interval.

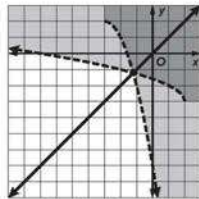


- 40b. $D = \{t \mid t \geq 0\}$; $R = \{d \mid d \geq 3\}$; increasing as $x \rightarrow \infty$; positive for $x > 0$;
As $x \rightarrow \infty$, $y \rightarrow \infty$

41. $D = \{V \mid -\infty < V < \infty\}$ or $(-\infty, \infty)$; $R = \{r(V) \mid -\infty < r(V) < \infty\}$ or $(-\infty, \infty)$; End behavior: The values of r increase as the values of V increase.



42. The inverse inequality is $f(x) > -(x + 3) + 2$. The domain of the inverse is restricted to $x > -3$.



45. Sample answer: The domain is limited because square roots of negative numbers are imaginary. The range is limited due to the limitation of the domain.
46. To be a function, for every x -value there must be exactly one y -value. For every x in this equation there are two y -values, one that is negative and one that is positive. Also, the graph of $y = \pm\sqrt{x}$ does not pass the vertical line test.
- 47a. Sample answer: The original is $y = x^2 + 2$ and the inverse is $y = \pm\sqrt{x-2}$.
- 47b. Sample answer: The original is $y = \pm\sqrt{x} + 4$ and the inverse is $y = (x-4)^2$.

Selected Answers

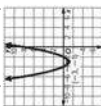
Module 1

Quick Check

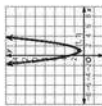
1. 22 3. -21 5. $(x-3)(x-7)$
7. $2(x+3)(x-9)$

Lesson 1-1

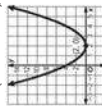
1. D = (all real numbers), $R = \{y | y \geq -1\}$



3. D = (all real numbers), $R = \{y | y \geq 1\}$



5. D = (all real numbers), $R = \{y | y \geq 0\}$



7. $g(x)$ Sample answer: Its vertex is a maximum point at (1, 3), which is 4 units above the vertex of $f(x)$ which is (1, -1).

9. $f(x)$ Sample answer: Its vertex is a minimum point at (5, -20), which is 10 units below the vertex of $g(x)$ which is (5, -10).

- 11a. For x = the number of price increases and y = revenue, $y = (80 + 4x)(80 - 16x)$
11b. \$100 per permit

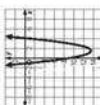
- 11c. D = $\{x | 0 \leq x \leq 30\}$, $R = \{y | 0 \leq y \leq 40,000\}$

13. -10 15. -3 17. 0 19. 2 21. 3 23. 0

- 25a. $\frac{1}{5}$, $\frac{2}{5}$, or $2\frac{3}{5}$; $\frac{29,397}{5} = 2,476$
25b. Sample answer: From 2011–2016, the number of people in the U.S. who consumed between 8 and 11 bags of potato chips annually increased by 2,476 million people per year.

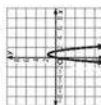
- 27a. $y = x^2 - 1$; axis of symmetry: $x = 1$; x-intercepts at (0, 0) and (2, 0); y-intercept at (1, -1)

x	$f(x)$
0	9
1	-13
1.5	-13.5
2	-13
3	-9



- 29a. $y = x^2 - 1$; axis of symmetry: $x = 1$; x-coordinate of vertex = 1

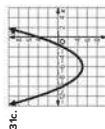
x	$f(x)$
$-\frac{\pi}{2}$	-6
$-\frac{\pi}{4}$	1
0	1.5625
$\frac{\pi}{4}$	1
$\frac{\pi}{2}$	-6



31a. $y = (x + 4)^2$; axis of symmetry: $x = -4$;

31b. $y = (x + 4)^2$; vertex: $(-4, 0)$

x	$f(x)$
-10	-1
-8	-4
-6	-5
-4	-4
-2	-1



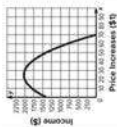
31c. $\max \frac{1}{2}, 23$

32. $\max, 14, 3.8$ 37. $\max = -4.11$

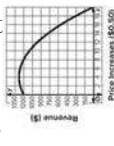
39a. Let x represent the number of \$1 price increases. Let y represent the income. Then $y = (70 - x)(20 + x)$.

x	y
5	1025
10	1800
15	1925
20	2000
25	2025
30	2000
35	1925
40	1800
45	1025
50	1000

39b. (25, 2025); The club should increase the price by \$4.35 to make the profit of \$2025. Since the current price is \$20 by \$25 is more than twice the current price, so this may be unreasonable. The club may want to revisit their assumptions.



41a. If x = number of price increases, the revenue is $R(x) = -5x^2 + 40x + 1200$. Because the price changes and the revenue will not be negative, the domain is $\{x \mid 0 \leq x \leq 20\}$.



41b. The graph shows that the maximum revenue value occurs at $x = 4$, which corresponds to a ticket price of \$6.00 + 4(\$0.50) or \$8.00. The maximum revenue is $R(4) = \$1280.00$.

41c. As price increases continue, the demand for tickets will decrease. The x-intercept indicates a price that is too and no one is willing to buy tickets for the cinema.

42. 20 ft

43. 20 ft

44. Median; sample answer: $f(x)$ has a

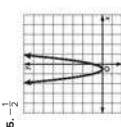
maximum at $x = -2$, $g(x)$ has a minimum at $x = 1$.

47. Sample answer: The function is quadratic if it has no other terms than a quadratic term, linear term, and constant term. The function has a maximum if the coefficient of the quadratic term is negative and has a minimum if the coefficient of the quadratic term is positive.

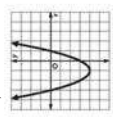
Lesson 1-2

1. no real solution 3. -4

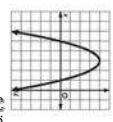
5. $-\frac{1}{2}$



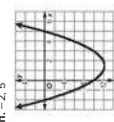
7. -3, 1



9. 1, 5

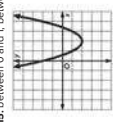


11. -2, 5

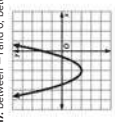


13. 6 and -4

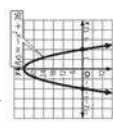
15. between 0 and 1; between 3 and 4



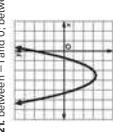
17. between -1 and 0; between -4 and -3



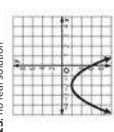
19. -6, 6



21. between -1 and 0; between -5 and -4



23. no real solution

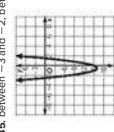


25. between 0 and 1; between 2 and 3

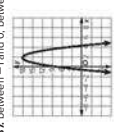
27. -1, 1 29. $\approx -1.45, \approx 3.45$

31. $\approx 3.24, \approx -1.24$ 33. 21 seconds

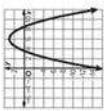
35. between -3 and -2; between 1 and 2



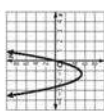
37. between -1 and 0; between 4 and 5



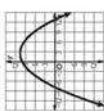
39. between 3 and 4, between 8 and 9



41. between -3 and -2 , 0



43. between -8 and -7 ; between 12 and 13



45. no real solution

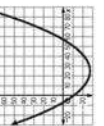
47. -2

49. between -3 and -2 , between 4 and 5

51. about -5 and 17

53. 11 and -19

55. 54 m



57. 5 seconds

59. No, sample answer: Haseem is right about the location of one of the roots, but his reason is not accurate. The roots are located where $f(x)$ changes signs.

SA4 Selected Answers

61. 5; Sample answer: The intercepts are equidistant from the axis of symmetry.

63. Sample answer: Graph the function using the axis of symmetry. Determine where the graph intersects the x-axis. The x-coordinates of those points are solutions of the quadratic equation.

Lesson 1-3

$$1.4\sqrt{3} \quad 3.6\sqrt{2} \quad 5.2\sqrt{21} \quad 7.-23\sqrt{2}$$

$$9.-5\sqrt{2} \quad 11.-1 \quad 13.23i \quad 15.3i \quad 17.5\sqrt{5}$$

$$19.3, 3 \quad 21.-\frac{1}{2}, -3 \quad 23.4, -3 \quad 25.10 - 4i$$

$$27.2 + i \quad 29.10 - 5i \quad 31.7 + i \quad 33.-10i$$

$$35.\frac{3}{2} - \frac{1}{2}i \quad 37.-\frac{3}{2} - 2i \quad 39.\frac{5}{6} - \frac{10}{9}i$$

$$41.(3 + i)x + (-2 + i)y - 8 + 7$$

$$43.10 + 9i \text{ volts} \quad 45.8 - 2i \text{ ohms}$$

$$47.Zoe: $i^2 = -1$, not -1 .$$

49. Always. Sample answer: the value of 5 can be represented by $5 + 0i$, and the value of $3i$ can be represented by $0 + 3i$.

51. Some quadratic equations have complex solutions and cannot be solved using only the real numbers.

Lesson 1-4

$$1.0, \frac{1}{3}, 3, 0, -\frac{5}{4}, 5, -1i, -3, 7.7i \text{ by } 9 \text{ ft}$$

$$9.\frac{3}{5}, -1, 1i, \frac{3}{5}, -6, 13, 8, \frac{5}{2}$$

$$15.-8, 8, 17, -17, 19, -13, 13, 21, \frac{7}{2}$$

$$23.\frac{3}{4}, 25, -\frac{8}{9}, 27, 5i, -5i$$

$$29.25i, -25i, 3i, \frac{5}{6}i, -\frac{5}{6}i, 33, -3, \frac{1}{2}$$

$$35.9i, -9i, 37, \frac{5}{4}i, -\frac{5}{4}i$$

$$39.13 \text{ inches by } 16 \text{ inches}$$

$$41.24, 26$$

43. Neither; both students made a mistake in Step 3.

45. Sample answer: 3 and $6 \rightarrow x^2 - 9x + 18 = 0$, -3 and $-6 \rightarrow x^2 + 9x + 18 = 0$. The linear term changes sign.

Selected Answers

71a. w = width, $V(w)$ = the volume;

$$V(w) = 6w^2 - 32w$$

71b. 207 by 621 in.

73. Alonso; Alka did not add 16 to each side; she added it only to the left side.

75a. 2; rational; 16 is a perfect square, so $x + 2$ and x are rational.

75b. 2; rational; 16 is a perfect square, so $x - 2$ and x are rational.

75c. 2; complex; if the opposite of a square is not a square, then the opposite of a square root of a negative number is complex.

75d. 2; real; the square must equal 20. Since that is positive but not a perfect square, the solutions will be real but not rational.

75e. 1; rational; the expression must be equal to 0 and only -2 makes the expression equal to 0.

75f. 1; rational; the expressions $(x + 4)$ and $(x + 6)$ must either be equal or opposites. No value makes them equal, -5 makes them opposites, so only -5 is a solution.

77. Sample answer: Rewrite the equation in the form of a quadratic equation. Once in this form, the equation can be solved by using the Square Root Property.

Lesson 1-6

$$1.-3,-5 \quad 3.\frac{2}{3}, \frac{2}{3} \quad 5.-4 + \sqrt{11}$$

$$7.2.9 \pm$$

$$9.\frac{1}{3}, -5 \quad 11.-2, \frac{1}{3} \quad 13.-\frac{5 \pm \sqrt{97}}{2}$$

$$15.\frac{3 \pm \sqrt{33}}{4} \quad 17.\pm 5i \quad 19.\frac{1 \pm i}{4}$$

$$21.7 \pm 2i \quad 23.\frac{3 \pm \sqrt{6}}{2}$$

$$25.225; 2 \text{ rational roots}$$

$$27.289; 2 \text{ rational roots}$$

$$29.24; 2 \text{ rational roots}$$

$$31.21; 2 \text{ rational roots}$$

$$33.-196; 2 \text{ complex roots}$$

$$35.-7; 2 \text{ complex roots}$$

$$37.b^2 - 4ac > 0;$$

$$2 \text{ real rational or irrational roots}$$

Selected Answers SA5

Selected Answers

39. 2 complex roots; $\frac{1 \pm 4i}{2}$

41. 2 rational roots; 0, $\frac{9}{2}$

43. 2 rational roots; 3, 8

45. 2 rational roots; 4, $\frac{4}{3}$

47. 2 irrational roots; $-\frac{5 \pm \sqrt{3}}{2}$

49. about 2.9 seconds

51. x is the length of a side of the base, $x^2 + 4x + 4 = x^2 + 4x + 4 = (x + 2)^2$, 5 in. by 5 in.

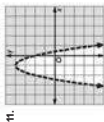
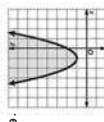
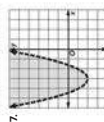
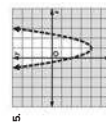
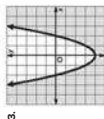
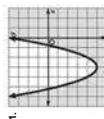
53. $7x^2 + 6x + 2 = 0$ is different from the other 3 equations because it has 2 complex roots, where the other 3 equations each have 2 rational roots.

55a. Simple answer: Always; when a and c are opposite signs, then ac will always be negative and $-4ac$ will always be positive. Because b^2 will always be positive, the discriminant will always be the sum of two positive values, which will never be negative. Hence, the discriminant can never be negative and the solutions can never be imaginary.

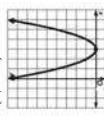
55b. Simple answer: Sometimes; the roots will only be irrational if $b^2 - 4ac$ is not a perfect square.

57. -0.75

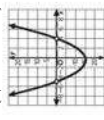
Lesson 1-7



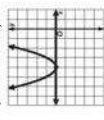
13. $\{x | x = 3\}$



15. $\{x | x < -5 \text{ or } x > 4\}$



17. $\{x | x \geq 1\}$



19. 30 ft to 60 ft

21. $\{x | x < -106 \text{ or } x > 706\}$

23. $\{x | x \leq -275 \text{ or } x \geq 1\}$

25. $\{x | x < 0.61 \text{ or } x > 272\}$

27. $\{x | x \geq 272\}$

29. $\{x | x \leq -924 \text{ or } x \geq -076\}$

31. $\{x | -5 \leq x \leq -3\}$

33. $\{x | x \geq -\frac{1}{3}\}$

35. $\{x | -3 \leq x \leq -\frac{4}{3}\}$

37. $x^2 > 4x - 6$

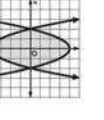
39. $x^2 > -0.25x^2 - 4x + 2$

41. No; the graphs of the inequalities intersect the x-axis at the same points.

43a. Simple answer: $x^2 + 2x + 1 \geq 0$

43b. Simple answer: $x^2 - 4x + 6 < 0$

45.



Lesson 1-8

1. $(2, -3)$, $(-1, -4)$

3. $(-1, 2)$, $(1.5, 4.5)$

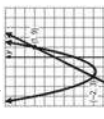
5. $(2, -2)$, $(\frac{1}{2}, 1)$

7. $(1, 2)$

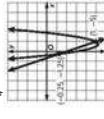
9. $(-\frac{1+\sqrt{3}}{2}, -\frac{1+\sqrt{3}}{2})$, $(-\frac{1-\sqrt{3}}{2}, -\frac{1-\sqrt{3}}{2})$

Selected Answers

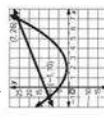
11. $-\frac{1}{2}, 1$



13. $\frac{1}{2}, 1$



15. $-1, 7$

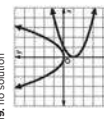


17a. $P = 2x^2 + 30x$ and $P = 50,000$

17b. $x \approx -165.75$, $x \approx 150.79$

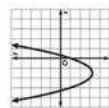
17c. Simple answer: Because the selling price cannot be negative, $(150.79, 50,000)$ is the only solution. This means that the business can earn a \$50,000 profit when the selling price is about \$150.79 per item.

19. no solution



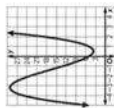
Lesson 2-2

1. zeros between $x = -4$ and $x = -3$, and between $x = 0$ and $x = 1$



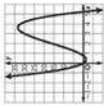
x	$f(x)$
-4	247
-3	74
-2	11
-1	-2
0	-1
1	2
2	19
3	86
4	263

- 3a. zero between $x = -6$ and $x = -5$



x	$f(x)$
-6	37
-5	5
-4	25
-3	29
-2	23
-1	13
0	5
1	5
2	19
3	53

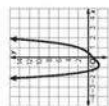
5. relative minimum between $x = 0$ and $x = 1$; relative maximum near $x = 4$



x	$f(x)$
-1	22
0	0
1	2
2	16
3	30
4	42
5	10
6	-48
7	-151

7. relative minimum near $x = -1$, no relative maximum

x	$f(x)$
-4	247
-3	74
-2	11
-1	-2
0	-1
1	2
2	19
3	86
4	263



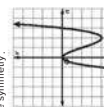
9. Domain and Range: The domain and range of the function models years. Because the relevant domain is $[n \geq 0]$ and the relevant range is all real numbers.

Extrema: There is a relative minimum at $n = 2$ in the relevant domain.

End Behavior: As $x \rightarrow \infty$, $y \rightarrow \infty$.

Intercepts: In the relevant domain, the y-intercept is at $(0, 0)$. The x-intercept, or zero, is at about $(5, 0)$.

Symmetry: In the relevant domain, the graph does not have symmetry.



11. Curve of best fit: $y = -0.012x^4 + 0.217x^3 - 1.162x^2 + 1.488x + 7552$; car sales in 2007: 999 million



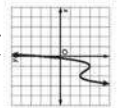
13. Curve of best fit: $y = -0.122x^4 + 1.565x^3 - 6.235x^2 + 30.2x + 33.125$; the number of months since January 4 average volunteer hours for September 44



13. See the graph on page 19.

15. average rate of change: 4500; From 2002 to 2007, the average rate of change in salary was an increase of \$4500 per year.

x	$f(x)$
-2	-7
-1	-3
0	2
1	25

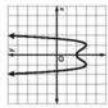


zero between $x = -1$ and $x = 0$;

rel. max. at $x = -2$, rel. min. at $x = -1$;

$D =$ all real numbers; $R =$ all real numbers

x	$f(x)$
-3	61
-2	6
-1	-3
0	-2
1	-3
2	6
3	61



zero between $x = -2$ and $x = -1$, and between $x = 1$ and $x = 2$;

rel. max. at $x = 0$, min. at $x = -1$ and $x = 2$;

$D =$ all real numbers; $R = \{f(x) | f(x) \geq -3\}$

21. rel. max. $x \approx -2.73$; rel. min. $x = 0.73$

23. max. $x \approx 134$; no rel. min

13. See the graph on page 19.

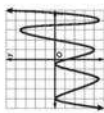
25. Sample answer: $f(x) \rightarrow -\infty$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$; the leading coefficient is negative.



27. The type of polynomial function that should be used to model the graph is a function with an even degree with a negative leading coefficient. This is based on the fact that the graph is reflected in the x-axis and as $x \rightarrow -\infty$, $f(x) \rightarrow \infty$ and as $x \rightarrow \infty$, $f(x) \rightarrow -\infty$.

29. As the x-values approach large positive or large negative values, the degree of the polynomial becomes more and more dominant in determining the value of $f(x)$.

31. Sample answer:



33. Sample answer: No. $f(x) = x^2 + x$ is an even degree, but $f(1) \neq f(-1)$.

Lesson 2-3

1. yes; 2. 3. no. 5. $2a^3 - a - 2$. 7. $3a + 12$

9. $3x^2 + 4x - 5$. 11. $-3x + 3$. 13. $3ap^3 - 3p$

15. $-10c^2 + 5d^2$. 17. $c^2 - 10a + 25$

19. $x^2 + xy - y^2$. 21. $2x^3 + xy^2 - 2xy^2 - y^3$. 23. $t^2 - 4t^2$

25. $2x^3 - 7x^2 + 5x^2 - 4x^2 - x + 2$

27. $s^4 + 16 + 15$. 29a. $4x^2 + 8x + 3$

- 29b. $8x^2 + 16x + 6$

31. $12a^2b + 8a^2b - 15ab^3 + 4b^3$

51. $(3 - 2\sqrt{3})(3 + 2\sqrt{3})$

Lesson 2-4

1. $9x^2 + 2y + 1$ 3. $2x - 3$ 5. $n + 2$
 2. $\frac{7}{2} + \frac{6}{5}$ 4. $9x - 3$ 6. $3x + 5 + \frac{10}{\sqrt{-4}}$
 3. $y^2 - 2y + 4 - \frac{y^2}{2}$
 4. $\frac{3}{5}y^2 + \frac{6}{5} + \frac{29}{5\sqrt{2}}$
 5. $m - 3 + \frac{m}{\pi} + 4$ 7. $2x^2 - 3x + 1$
 8. $2x^2 - 2x + 4$ 9. $2x^2 - \frac{3}{2}$

25. $n^2 - n - 1$

27. $3x^2 - x^2 + 2x^2 - 4x + 9 - \frac{13}{13}$

29. A is $x^2 B$ is $6x$. C is 1 . $31. \pi(k^2 - 8x + 6)$

33. Yes. Because $3x$ times the divisor is $9x^2 + 3x$, the divisor must be $3x + 1$. The second and third terms of the dividend must be $9x^2 + 3x$ because the first difference is $-3x + 1$.

35. $\frac{8x^2 - 8x - 2x}{8x^2 - 8x - 2x} = \frac{8x^2 - 10x - 2}{8x^2 - 10x - 2}$
 37. No. The degree of dx and dx^2 is $2x^2 + x + 14x - 6 + (-4) = 8x^2 - 8x^2 - 2x - 10$.

39. No. The degree of dx and dx^2 may be equal. For example, $\frac{3x^2 + 6}{3x^2 + 6} = 3 + \frac{6}{3x^2 + 6}$.

37a. Yes; if the degree of dx is greater than or equal to the degree of dx^2 , then the expression $\frac{dx}{dx^2}$ may be simplified by division. For example, $\frac{1}{\frac{dx}{dx^2}} = \frac{1}{\frac{1}{8x^2 + 1}}$, then $8x^2 + 1$ may be divided by x to get $8x + \frac{1}{x}$.

37c. Yes; because $\frac{dx}{dx^2} = \frac{dx}{dx^2}$, the degree of $\frac{dx}{dx^2}$ must equal the degree of dx^2 . The degree of dx^2 is less than the degree of dx , so the degree of dx^2 equals the degree of dx . This means the degree of $\frac{dx}{dx^2}$ equals the degree of dx , and the degree of $\frac{dx}{dx^2}$ is the degree of dx minus the degree of dx^2 .

For example, $\frac{2x^2 - 1}{x^2 + 3} = 2x + \frac{-6x - 1}{x^2 + 3}$, the degree of dx^2 is the degree of dx minus the degree of dx^2 .

39. The binomial is a factor of the polynomial.

33. $2x^2 - 3y + 6xy^2$

35. $2x^2 - 14x^2 + 4x^2 - 28$

37. $64x^2 - 240x^2 + 300x^2 - 125$

39. $7x - 2y$

41a. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41b. $h(x)f(x) = (-x^2 - x)(3x^2 - 1) = -3x^4 + x^3 - x^2 - x$, 4th degree

41c. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41d. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41e. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41f. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41g. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41h. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41i. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41j. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41k. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41l. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41m. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41n. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41o. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41p. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41q. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41r. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41s. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41t. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41u. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41v. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41w. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41x. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41y. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41z. $f(x)g(x) = (3x^2 - 1)(x + 2) = 3x^3 + 6x^2 - x - 2$, 3rd degree

41. Simple answer: $x^2 + 5x + 9$

43a. $B^2 + 1$

43b. $B + \frac{1}{B + 8 + 1}$

Lesson 2-5

1. $x^2 - 3xy + 3xy - y^2$
 3. $9x^2 - 49y^2 + 6xy^2 - 49y^2 + y^2$
 5. $y^2 - 21y + 14y - 343$
 7. 38
 9. $243x^4 + 1620x^3y + 4320x^2y^2 + 5760xy^3 + 3840y^4 + 1024y^5$
 11. $4096x^4 - 6448x^3y + 3456x^2y^2 - 864xy^3 + 81y^4$
 13. $x^4 + \frac{8}{3}x^3 + \frac{8}{3}x^2 + \frac{8}{3}x + \frac{8}{3}$
 15. $32b^2 + 20b^2 + 5b^2 + \frac{8}{3}b^2 + \frac{1}{128}b^2 + \frac{1}{128}$
 17. 23%

19. If C represents a correct circuit board and N represents an incorrect circuit board, then there are 128 possible ways for the robot to produce a correct circuit board out of 128 possibilities. So, the probability that 5 of 7 are correct is $\frac{5}{128}$ or about 16%.

21. If C represents a correct answer and N represents a wrong answer, then the

coefficients expansion of $(x + y)^n$ can be used to represent situation. Using Pascal's triangle, there are 45 ways to get 8 questions correct, 10 ways to get 9 questions correct, and 1 way to get all correct. So there are 46 ways to get 8 or more questions correct. In this row of Pascal's triangle, I found that there are 1024 different ways he could answer the questions in the quiz. Matthew has a $\frac{46}{1024}$ or about a 5.5% chance of getting 8 or more correct.

23. There are 9 judges on the Supreme Court. The majority could be 5, 6, 7, 8, or 9 votes. So, the number of possible combinations is $\binom{9}{5} + \binom{9}{6} + \binom{9}{7} + \binom{9}{8} + \binom{9}{9} = 250$

25. 1.28986; $(102)^9 \approx 1.2898642$; the approximation differs from the value given by a calculator by 0.0000342.

27. Simple answer: While they have the same terms, the signs for $(x + y)^n$ will all be positive, while the signs for $(x - y)^n$ will alternate.

29. Sample answer: $(x + \frac{5}{3}y)^3$

Module 2 Review

1. B 3. B 5. $g(x) = -\infty$ 7. C

9. $x^4 + 7x^3 - 6x^2 + 8x^2 + 2x$

11. C 13. $5x^2 + 7x - 3$ 15. B



6c.

Module 3 Review

1. A. 3, A, B, D

5. Set the expression for the volume of the first figure equal to the expression for the surface area of the second figure. Solve the equation $x^3 - 9x = 8x^2$. Solving the equation gives the solutions -1 , 0 , and 9 . The value of x cannot be 0 , so the solutions are -1 and 9 . For the original expressions, the volume of the first figure is $(3 - 9)(-90) = 0 - 0 = 0$ and surface area of the second figure is $8(9)^2 = 8(81) = 648$. Volume and surface area cannot be 0 , so the value of x is -1 or 9 .

7. $x = 6$

9. B. 11. C. 13. B

15. B, C

17. A

63. $r^4 + 1 = 0$: Simple answer: The equation has imaginary solutions and all of the others have real solutions.
65. Simple answer: To determine the number of positive real roots, determine how many times the graph crosses the x -axis. You move from left to right. In this function there are 3 changes in sign. Therefore, there may be 3 or 1 positive real roots. To determine the number of negative real roots, I would first evaluate the polynomial for $-x$. All of the terms change sign. Therefore, there are 3 changes in sign. Then I would again count the number of sign changes as I move from left to right. There would be only one change. Therefore there may be 1 negative root.

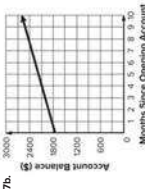
Module 4

Quick Check

1. between 0 and 1, and between 3 and 4
2. between -1 and 0, and between 1 and 2
5. $3x + 3$ 7. $2x^2 + 5x - 6$

Lesson 4-1

1. $f + g)(x) = -2x + 5$; $(f - g)(x) = 6x - 5$;
 $(f \cdot g)(x) = -8x^2 + 10x(\frac{1}{2}) - \frac{2x}{-4x \cdot 5} \cdot x \neq \frac{5}{2}$
3. $(f + g)(x) = 2x^2 - 11x + 14$; $(f - g)(x) = -x + 5$;
 $(f \cdot g)(x) = x^4 + 3x + 1$; $(\frac{1}{2}g)(x) = -3x^2 - 3x + 1$; $(f - g)(x) = -2x^2 - 3x^2 + 17x^2 + 18x - 30$; $(\frac{1}{2}g)(x) = \frac{2x^2 + 3x}{2} \cdot 5 \neq 1$ or $-\frac{3}{2}$
7a. $(a - b)(x) = 85x + 1750$
7b.



9. $f \circ g = \{(-4, 4), (-2, -4), (-6, -4), (-8, 0), (0, -4), (2, -5), (-5, -6), (-1), D = \{-6, 0, 2\}$; $R = \{-5, -4, -1, 0\}$
11. $f \circ g$ is undefined; $D = \{0\}$; $R = \{0, g \circ f\}$ is undefined; $D = \{0\}$; $R = \{0, f\}$ is undefined.

13. $(f \circ g)(x) = 2x + 10$; $D = \{\text{all real numbers}\}$; $R = \{\text{all real numbers}\}$; $(g \circ f)(x) = 2x + 5$; $D = \{\text{all real numbers}\}$; $R = \{\text{all odd numbers}\}$; $D = \{\text{all real numbers}\}$; $(f \circ g)(x) = x^2 - 6x - 2$; $D = \{\text{all real numbers}\}$; $R = \{y | y \geq -1\}$; $(g \circ f)(x) = x^2 + 6x - 8$; $D = \{\text{all real numbers}\}$; $R = \{y | y \geq -17\}$
17. $f(x) = 0.85x$; $f(x) = 1.065x$; $f(x) = 1.065(0.85x)$; $\$1222.09$

19. Simple answer: The order of the discounts does not matter. Either composition results in a final cost of $\$54.91$.

21. 15, 23, 1, 25, 12, 27, 89, 29, 21

31. -3 33. 0 35. 440

37a. $V(t) = \frac{\pi}{6} + 2\pi t^2 + 8\pi t + \frac{32}{3}\pi$

37b. 2145 m^3

39a. $(f + g)(t) = 4$

39b. $(f - g)(t) = 8$

39c. $(f \cdot g)(t) = 5$

39d. $(\frac{1}{2}g) = 4$

39e. $(\frac{1}{2}g) = 0$

39f. $(\frac{1}{2}g) = 0$

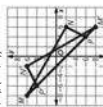
41. Simple answer: $f(x) = x - 9$; $g(x) = x + 5$

43a. $D = \{\text{all real numbers}\}$

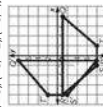
43b. $D = \{x | x \geq 0\}$

Lesson 4-2

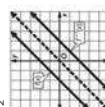
1. $\{(6, -8), (-2, 6), (-3, 7)\}$



3. $\{(-1, 8), (-1, -8), (-8, -2), (-8, 2)\}$

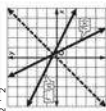


5. $f^{-1}(x) = x - 2$

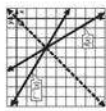


15b. About 35 pounds

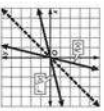
7. $f^{-1}(x) = -\frac{1}{2}(x + 3)$



9. $f^{-1}(x) = -\frac{2}{3}(x + 8)$



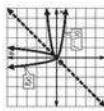
11. $f^{-1}(x) = \frac{1}{3}x$



13. $f^{-1}(x) = \pm \frac{x\sqrt{5}}{5}$

If the domain of $f(x)$ is restricted to $(-\infty, 0]$, then the inverse is $f^{-1}(x) = \frac{x\sqrt{5}}{5}$.

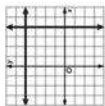
If the domain of $f(x)$ is restricted to $[0, \infty)$, then the inverse is $f^{-1}(x) = \frac{x\sqrt{5}}{5}$.



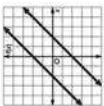
15a. $g^{-1}(y) = 14y$: Sample answer: The inverse converts stones to pounds, where x is the weight in stones.



17. no 19. yes 21. no 23. yes
25. $x = 4$

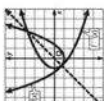


27. $f^{-1}(x) = x - 2$



29. $f^{-1}(x) = \pm \sqrt{2x + 2}$

If the domain of $f(x)$ is restricted to $(-\infty, 0]$, then the inverse is $f^{-1}(x) = \sqrt{2x + 2}$. If the domain of $f(x)$ is restricted to $[0, \infty)$, then the inverse is $f^{-1}(x) = -\sqrt{2x + 2}$.

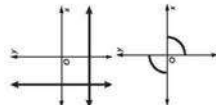


31. yes 33. no 35. yes

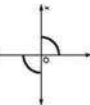
37. $\{x | x \geq 0\}$ or $\{x | x \leq 0\}$; $\{x | x \geq 5\}$

39. $\{x | x \geq -6\}$; $\{x | x \geq 0\}$

41.

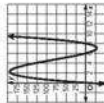


43.



45. $f^{-1} = 220 - \frac{x}{0.85}$. Sample answer: The inverse function (not the functional age of the clients based on their heart rate).

47a. $V(x) = x(12 - 2x)(6 - 2x)$. Sample answer: The value of x must be between 0 and 6, but the length of the side of the box $(12 - 2x)$ cannot be less than 0.



47b. No, sample answer: even with the restriction, the function does not pass the horizontal line test, so its inverse will not be a function.

49. Sample answer: Sometimes, $y = \pm\sqrt{x}$ is an example of a relation that is not a function, with an inverse being a function. A circle is an example of a relation that is not a function with an inverse not being a function.

51. Sample answer: $f(x) = x$ and $f^{-1}(x) = x$ or $f(x) = x$ and $f^{-1}(x) = -x$.

53. Sample answer: One of the functions carries out an operation on 5. Then the second function that is an inverse of the first function reverses the operation on 5. Thus, the result is 5.

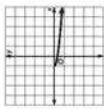
Selected Answers

Lesson 4-3

1. $\pm 1\sqrt{3}y$ 3. $\pm 2x^2$ 5. $-9\sqrt{b}b^3c$ 7. $2(x - 3)^3$
9. $3(x - 4)$ 11. $|a|$ 13. $\sqrt[3]{8}$ 15. $\sqrt[3]{x^6}$
17. $5\sqrt{x}y^3$ 19. $6\sqrt[3]{a^3}$ 21. 3 23. $\frac{64}{64}$
25. 64 27. $x^{\frac{3}{4}}$ 29. $\frac{5}{5}$ 31. $3b^2c^4$
33. $3[(k + 4)]$ 35. $(y + 5)^8$
37. $14|c|^{10}$ 39. $-3a^2b^3$ 41. c^2
43. b^4 45. $\frac{a^2}{a^2}$
- 47a. $5\sqrt[3]{50} \text{ cm}^3$
- 47b. $32\sqrt[3]{3} \text{ cm}^3$
- 47c. 288 cm^3
49. c^3 51. $6\sqrt[3]{h^3}$ 53. $\frac{xy^2z^3}{x^2y^2z^3}$ 55. 21 feet
57. There are four cases to consider. If $m > 0$ and b is even, then $\sqrt[m]{m^m} = m^1$. If $m < 0$ and b is even, then $\sqrt[m]{m^m} = -m^1$. If $m > 0$ and b is odd, then $\sqrt[m]{m^m} = m^1$. If $m < 0$ and b is odd, then $\sqrt[m]{m^m} = m^1$.
59. $x < 0$; if $x < 0$, then $\sqrt{x^2} = -x$.
61. Sample answer: It may be easier to simplify an expression when it has rational exponents because all the properties of exponents apply. We do not have as many properties dealing directly with radicals. However, we can convert all radicals to rational exponents, and then use all the properties of exponents to simplify.
63. When x or $y = 0$ and the other variable is ≥ 0 .
65. $0 < x < 1$ or $x < -1$ 67. $2\sqrt[3]{2xy}$

Lesson 4-4

1. $D = \{x | x \geq 9\}$; $R = \{y | y \geq 0\}$
3. $D = \{x | x \geq 0\}$; $R = \{y | y \leq 0\}$
5. $D = \{x | x \geq \frac{3}{2}\}$; $R = \{y | y \geq 0\}$
7. $D = \{x | x \geq -1\}$; $R = \{y | y \leq 0\}$; reflected in the x -axis, compressed vertically, and translated left 1 unit

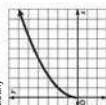


Selected Answers SA21

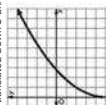
SA20 Selected Answers

Selected Answers

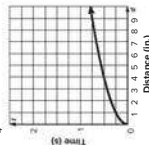
9. $D = \{x \mid x \geq 0\}$; $R = \{y \mid y \geq 0\}$
stretched vertically



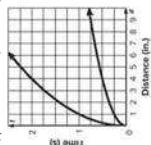
11. $D = \{x \mid x \geq 0\}$; $R = \{y \mid y \geq -5\}$; stretched vertically and translated down 5 units



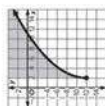
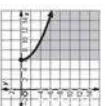
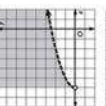
13a. $t = \frac{1}{3}\sqrt{d}$



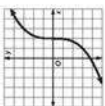
13b. $D = \{d \mid d \geq 0\}$; $R = \{t \mid t \geq 0\}$; increasing as $x \rightarrow \infty$; positive for $x > 0$; As $x \rightarrow \infty$, $y \rightarrow \infty$.



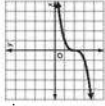
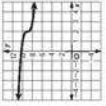
15. compressed vertically



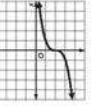
21. $D = (-\infty, \infty)$; $R = (-\infty, \infty)$



23. $D = (-\infty, \infty)$; $R = (-\infty, \infty)$



25a.

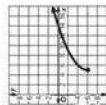


25b.

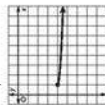
	$f(x) = \sqrt{x} - 2$	$g(x)$
Domain	all real numbers $\{x \mid x \geq 0\}$	
Range	all real numbers $\{y \mid y \leq -3\}$	
Intercepts	x-int: 8; y-int: -2	x-int: 9; y-int: -3
Increasing/Decreasing	increasing as x increases	increasing as x increases
Positive/Negative	negative for $x < 8$; positive for $x > 8$	negative for $x < 9$; positive for $x > 9$
End Behavior	as $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$ as $x \rightarrow \infty$, $f(x) \rightarrow \infty$	as $x \rightarrow -\infty$, $g(x) \rightarrow -\infty$ as $x \rightarrow \infty$, $g(x) \rightarrow \infty$

27. $f(x) = \sqrt[3]{x} - 2 + 1$

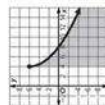
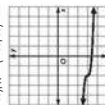
29. $D = \{x \mid x \geq 5\}$; $R = \{f(x) \mid f(x) \geq -6\}$
stretched vertically translated 5 units right and 6 units down



31. $D = \{x \mid x \geq 1\}$; $R = \{f(x) \mid f(x) \leq -4\}$
compressed vertically, translated 1 unit right and 4 units down, reflected across the x-axis



33. $D = (-\infty, \infty)$; $R = (-\infty, \infty)$



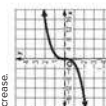
37. $f(x) = -\sqrt{x} + 4 - 2$

39a. $f(x)$ has the greatest maximum value of 2, which is greater than 6, the maximum value of $g(x)$.

39b. The domain of $f(x)$ is $x \geq -3$, since any values less than -3 produce a negative value under the radical. The domain of $g(x)$ is $x \geq \frac{1}{2}$.

39c. The average rate of change over the interval is $-\frac{1}{2}$ for $f(x)$. It appears that the rate of change for $g(x)$ is the same.

41. $D = \{V \mid -\infty < V < \infty\}$ or $(-\infty, \infty)$; $R = \{f(V) \mid -\infty < f(V) < \infty\}$ or $(-\infty, \infty)$; End behavior: $f(V)$ increases or decreases as the values of V increase



43. Simple answer: $y = -\sqrt{x} + 4 + 6$

45. Simple answer: The domain is limited because square roots of negative numbers are imaginary. The range is limited due to the limitation of the domain.

47a. Simple answer: The original is $y = x^2 + 2$. The inverse is $y = \sqrt{x} + 2$.

47b. Simple answer: The original is $y = \pm\sqrt{x} + 4$ and inverse is $y = (x - 4)^2$.

Lesson 4-5

1. $6\sqrt{b} - \sqrt{2b}$ 3. $2\sqrt{b} - \sqrt{6b}$ 5. $2\sqrt{ab}/\sqrt{4}$
7. $\frac{3}{4}\sqrt{14}$ 9. $\frac{4\sqrt{30}}{3}$ 11. $8\sqrt{2}$ 13. $\sqrt{5}$
15. $5\sqrt{x} - \sqrt{14}$ 17. $20\sqrt{22}$ 19. $144\sqrt{b}/2$

Lesson 4-6

21. $15\sqrt{y} + 15\sqrt{y}$
 23. $18 + 5\sqrt{6}n^2$
 25. $56\sqrt{3} + 42\sqrt{6} - 36\sqrt{2} - 54$ 27. 1260
 29. $\frac{b^2\sqrt{5b}}{5b^2}$ 31. $\frac{3\sqrt{5}}{5}$
 33. $6\sqrt{5} + 6\sqrt{2}$ 35. $\frac{20 - 2\sqrt{3}}{4}$ 37. $\frac{3\sqrt{7} + 3\sqrt{35}}{4}$
 39. $9\sqrt{2}\sqrt{27}$
 41. $\frac{1}{x-1} \sqrt{x^2-9} = \frac{x^2-9}{x^2-1} = \frac{x+3}{x-1}$
 43. $4\sqrt{3}$ 45. $28\sqrt{2}$ 47. $10\sqrt{28}$ in. 49. $\frac{\sqrt{12}}{2}$
 51a. John: $0.8 - \frac{4}{1000}$ or 40 - 200/2 min;
 Jay: $\frac{0.8}{1000}$ or 40 $\sqrt{5} - 80$ min
 51b. $2 - \sqrt{2} + \sqrt{5} + \sqrt{10}$
 53a. $3\sqrt{x^2-9} + (\sqrt{x^2-9}) = \sqrt{x^2-9} + \sqrt{x^2-9} = 2\sqrt{x^2-9}$
 53b. $6x(\sqrt{x^2-9}) + (\sqrt{x^2-9}) + (\sqrt{x^2-9}) = (\sqrt{x^2-9}) + \sqrt{x^2-9} = x + 2x + 3x = 6x$
 53c. $10x(\sqrt{x^2-9}) + (\sqrt{x^2-9}) + (\sqrt{x^2-9}) = (\sqrt{x^2-9}) + \sqrt{x^2-9} + \sqrt{x^2-9} = x + 2x + 3x + 4x = 10x$
 53d. $(\sqrt{x^2-9}) + (\sqrt{x^2-9}) + \dots + (\sqrt{x^2-9}) = x + 2x + \dots + nx = 1 + 2 + \dots + n$
 $n(x) = \frac{(n+1)}{2}x$
 55a. $6\sqrt{2} + 1 = \sqrt{\frac{3}{2}} \cdot 72\pi = \frac{3}{2}\sqrt{2} = 3\sqrt{2}$
 $S = 3\sqrt{2} + 2(2)$
 55b. No; sample answer: a box with a volume of 384 cm³ has a side that is $\sqrt[3]{384}$, or about 7.27 cm; $s = 6\sqrt{2} + 4 \approx 11.55$ is the least possible value for a side of the gift box.
 57. $a = 1, b = 256, a = 2, b = 16; a = 4, b = 4; a = 8, b = 2$
 59. Sample answer: It is only necessary to use absolute values when it is possible that n could be odd or even and still be defined. It is when n is even that the root is defined and the root to be defined that the absolute values are not necessary.

SA24 Selected Answers

Module 4 Review

1. D
 3. C
 5. $f \circ g(x) = 12x + 8$
 7. If the domain of $f(x)$ is restricted to $(-\infty, -2]$, then the inverse is $f^{-1}(x) = -2 - \sqrt{x+1}$. If the domain of $f(x)$ is restricted to $[-2, \infty)$, then the inverse is $f^{-1}(x) = -2 + \sqrt{x+1}$.
 9. D
 11. The parent function is $f(x) = \sqrt[3]{x}$ and $g(x) = \alpha\sqrt[3]{x} + h + k$ where $\alpha = -2, h = -5$, and

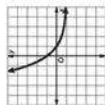
$k = -3$. Because $\alpha < 0$ and $|a| > 1$, the graph of $g(x)$ is the graph of the parent function reflected across the x -axis and stretched vertically by a factor of $|a|$, or 2. Also, because $h < 0$, the graph of $g(x)$ is the graph of the parent function translated $|h|$ units, or 5 units, down. Finally, because $k < 0$, the graph of $g(x)$ is the graph of the parent function translated $|k|$ units, or 3 units, down.

13. B 15. $2\sqrt{3}$ 17. B 19. A

Selected Answers

Selected Answers SA25

45. Reflect $f(x)$ in the y -axis and translate the result one unit up to obtain the graph of $g(x)$. The y -intercept of $g(x)$ is 2, which was translated up 1 unit from the y -intercept of $f(x)$. $g(x)$ is decreasing on its entire domain. $g(x)$ is increasing on its entire domain. The asymptote of $g(x)$ is $y = 1$, which was translated up one unit from the asymptote of $f(x)$. $y = 0$. For $g(x)$ the end behavior is as $x \rightarrow -\infty$, $g(x) \rightarrow \infty$; as $x \rightarrow \infty$, $g(x) \rightarrow 1$. The end behavior of $f(x)$ is as $x \rightarrow -\infty$, $f(x) \rightarrow 0$; as $x \rightarrow \infty$, $f(x) \rightarrow \infty$.



- 47a. A is the amount in grams, t is time in years.
 $A(t) = 27.3(0.9)^t$
 47b. rate of change for $(0, 2) = -2.5935$; rate of change for $(3, 9) = -1.3891$; The parent of the compound decreases quickly at first and then more slowly as time passes.
 48a. Always; sample answer: The domain of exponential functions is all real numbers, so $(0, y)$ always exists.
 48b. Sometimes; sample answer: The graph of an exponential function crosses the x -axis when $k < 0$.
 49c. Sometimes; sample answer: The function is not exponential if $a = 1$ or -1 .
 51. about 251 mg

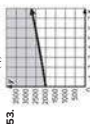
53. Sample answer: The parent function, $g(x) = b^x$, is stretched if b is greater than 1 or compressed if b is less than 1. The parent function is translated up k units if k is positive and down $|k|$ units if k is negative. The parent function is stretched if k is positive and compressed if k is negative. The parent function is stretched if k is positive and compressed if k is negative.

Lesson 5-2

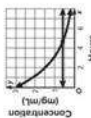
1. 4
 5. 2
 7b. 33.0 years
 9a. $y = 5000(0.5)^x$
 9b. $y = 23(0.995)^x$

- 9b. 85 years
 11. \$1476.79
 13. \$5309.08
 17. $x > -73$
 21. 0
 23. $a < -4$
 25. 5
 27. $-\frac{2}{3}$
 29. $c > 4$
 31. -7
 33. -6
 35. 16
 37. 1
 39. 12

41. $z < -1$
 43a. Ingrid: $y = 2(250)^x$; Alberto: $y = 321(38)^x$
 43b. Ingrid; 427; Alberto; 162
 43c. Sample answer: Not a business cannot open a bank account.
 45. $y = -12x$
 47. $y = -\frac{3}{2}y$
 49. $y = -6\left(\frac{1}{2}\right)^x$
 51. $y = (x)$



55. No. A function to model the concentration is $h(t) = 3(0.5)^t$. The intersection with $g(t) = 0.6$ is at $t = 1.6$. The concentration of the drops below the effective level before 3 P.M.



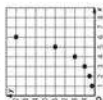
- 57a. Sample answer: According to the US Census Bureau and the American Association of University Professors, the average salary for a professor in 2017 was \$100,000. After 15 years, the professor's annual salary would be $100,000(0.02)^n = \$134,721.42$.
 57b. Sample answer: It depends on how long the professor intends to work. For years 1–16, the average salary with a 2% raise earns more. After year 17, the lower starting salary with a 3% raise earns more. The professor should consider the cumulative salary over the time he or she intends to work to determine which is better.

Selected Answers

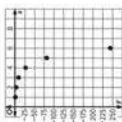
35. Final amount corresponds to final population. Initial amount corresponds to initial population. Interest rate corresponds to growth rate, and time of growth is the same for both equations.
 37. Sample answer: Since e is irrational, evaluating any expression with e will produce an approximation. To minimize the error, do as much work as possible with the exact value (e) before evaluating.
 39. $g(x) = 2e^x + 9$ because it has a range of all real numbers greater than 9 and the other two functions have a range of all real numbers greater than 0.

Lesson 5-4

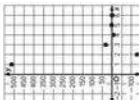
1. not geometric
 3. geometric
 5. not geometric
 7. 1, 2, 4, 8, 16, 32



9. -1, -3, -9, -27, -81, -243



11. 517, -128, 32, -8, 2, -0.5



13. 32

15. $a_1 = 39r^{n-1}$

19. $a_1 = 2(-3)^{n-1}$

21. $a_1 = 30(6)^{n-1}$

23. $a_1 = -\left(\frac{1}{2}\right)^{n-1}$

27. 198

29a. The heights in feet after the first 6 seconds

are 1000, 2000, 3000, 4050, 4001, 5010, 00501,

29b. Yes, the height after 60 minutes is about

81,669 ft.

29c. 6.34×10^8

31. $\frac{1}{3}$

33. -2

35. 255

37. 0

39. 4374

41. -5120

43. -0.2

45. geometric; $a_1 = 25(2)^{n-1}$

47. geometric; $d = -15$; -29^{n-1}

49. 519,375

51. 9707.869

53. 3, -6, 12, -24, 48

55. 18,351

57. $a_1 = 60\left(\frac{3}{2}\right)^{n-1}$

59. $\pm \frac{3}{16}$, $\frac{1}{4}$, $\pm \frac{1}{8}$

61. -8, 4

63. Sample answer: The second option,

receiving 20 annual payments represents a

geometric series, and $S = 4,000,000 - (1.02)^{20}$

approximately \$17 million more than by

choosing the single payout.

65a. \$347,514.00

65b. \$206,000

67. Sample answer: $256 + 192 + 144 + 108 +$

$81 + \frac{243}{8}$

69. Sample answer:

Let a_n = the n th term of the sequence and

r = the common ratio.

Definition of the second term

of a geometric sequence

Substitution

of a geometric sequence

$a_1 = a_1 \cdot r \cdot r$

Substitution

$a_1 = a_1 \cdot r^2$

Associative Property of

Multiplication

$a_1 = a_1 \cdot r^{2-1}$

$3 - 1 = 2$

$a_1 = a_1 \cdot r^{2-1}$

$71. x \cdot y$

73. Sample answer: A series is arithmetic

if every pair of consecutive terms shares a

common difference. A series is geometric

if every pair of consecutive terms shares

a common ratio. If the series displays both

quizzes, then it is both arithmetic and

geometric. If the series displays neither quality,

then it is neither geometric nor arithmetic.

Lesson 5-5

1.

linear

3.

quadratic

5a.

linear; $y = 6.4x + 45.85$

5b. ≈ 2.2 hours

7.

linear

9. strong, positive

11. weak, positive

13.

linear

1990, 2020 set 1 by 00, 80 set 5

The scatter plot is linear, with a positive

direction, and a strong correlation.

15. Enter the x-values in the table in L1 in a

calculator. Perform linear, quadratic, and

exponential regressions. Then compare the

coefficients of determination, r^2 . The function

with a coefficient of determination closest to 1

will fit the data best.

17. quadratic; $SA = 3\pi r^2$

Module 5 Review

1. A

3. A, B, F

5. D

7. Sample answer: The x-coordinate of

the intersection of the graphs will give the

approximate solution of the equation.

9. C

11. 18, 21

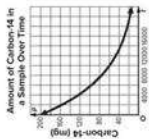
13. $a_1 = 10\left(\frac{1}{2}\right)^{n-1}$

15. C

17. A

Lesson 6-4

1. $\ln x = 15$ 3. $-5x = \ln 0.2$ 5. $x = \ln 3$
 7. $\ln(x-3) = 4$ 9. $5 = \ln 10x$ 11. $e^x = 36$
 13. $x = e^3$ 15. $e^x = 0.0002$ 17. $e^x = 15$
 19. $\ln 3$ 21. $\ln 16x^2$ or $2 \ln 4x$ 23. $\ln 5$
 25. $-8 \ln \frac{1}{2}$ 27. $\ln \frac{e^2}{625}$ 29. 1.0986
 31. 175.79 33. 2.7081 35. 18.0855
 37. 357.4953 39. 2.4630
 41. 3813.28 43. about 36.3 years
 45. $y = 53.90 \ln x$; about 624 grams
 47. $y = 75.85 \ln x$; about 228,940 visitors
 49. 13,900 51. about 24.75 years
 53. $\ln(5 \cdot 5 + 8) = \ln 5 + \ln 5 = \ln 8 = 1394$
 55a. $P = P_0 e^{kt}$ 0.5P = $P_0 e^{0.05t}$ 0.5 = $e^{0.05t}$
 $\ln(0.5) = 5730k \ln e$; $k = \frac{\ln 0.5}{5730}$
 55b. $P = 200e^{\frac{\ln(0.5)}{5730}t}$



- 55c. Sample answer: about 7500 years.
 Solve for t : $80 \leq 200e^{\frac{\ln(0.5)}{5730}t}$
 $80 \leq 200e^{\frac{\ln(0.5)}{5730}t}$ $0.4 \leq e^{\frac{\ln(0.5)}{5730}t}$ $\ln(0.4) \leq \frac{\ln(0.5)}{5730}t$
 $\ln(0.4) \geq \frac{\ln(0.5)}{5730}t$ $\ln(0.4) \leq \frac{\ln(0.5)}{5730}t$
 $\ln(0.4) \geq \frac{\ln(0.5)}{5730}t$ $\ln(0.4) \leq \frac{\ln(0.5)}{5730}t$

57. Disregard; $e^x = x$ can be rewritten as $e^{\ln(x)} = \ln x$, which can be written as $e^x = \ln x$.
 The functions $y = e^x$ and $y = \ln x$ are inverse functions, so there are no values of x for which $e^x = x$.

59. Let $p = \ln a$ and $q = \ln b$. That means that $e^p = a$ and $e^q = b$.
 $ab = e^p \times e^q$
 $ab = e^{p+q}$
 $\ln(ab) = \ln(e^{p+q})$
 $\ln(ab) = p + q$
 $\ln(ab) = \ln a + \ln b$

SA34 Selected Answers

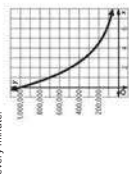
Lesson 6-5

- 1a. $y = 6.124e^{0.0004x}$ 1b. 2016
 3a. about 9.0256 or about 2.66%;
 $y = 6e^{-0.0004x}$
 3b. about 0.57 5. about 12.618 years old
 7a. $k = 0.071$ 7b. about 60.8 min 7c. about 30.85 min 9. about 6.9 yr
 11. $y = 100e^{0.0004x}$ 13. $P = 8e^{0.0004x}$ about 4.4 years
 15. Sample answer: Based on the data in the table, the decay constant is about -0.000124 , which is approximately equal to the given constant. $f(t) = 1000e^{-0.000124t}$
 the exponential model: $f(t) = 1000e^{-0.000124t}$
 1000 $e^{-0.000124(999.628)} = e^{-0.12395628} = \ln e^{-0.12395628} = \ln e^{-1}$
 in the model, $f(t) = 1000e^{-0.000124t}$. Verify:
 $f(2) = 1000e^{-0.000124(2)} = 999.752$

7b.

t (min)	Surviving Cells (Approximates)	$f(t)$
0	Initial Amount	(0, 1000.000)
1	(0.700, 000.000) = 700,000 survive	(1, 700.000)
2	(0.700, 000.000) = 490,000	(2, 490.000)
3	(0.700, 990.000) = 343,000	(3, 343.000)

Simple answer: An exponential model best describes the points because the number of cells decreases by the same percentage every minute.



- 7b. In the model $f(t) = 6e^{kt}$, $f(t)$ represents the number of cells remaining after t minutes, a represents the number of cells at the start of the experiment, t represents the number of minutes since the experiment began, and k is the growth or decay constant. $f(t) = 1000.000e^{kt}$.

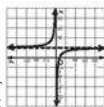
Module 6 Review

- 1a. $\log_4 1296 = 4$ 1b. $\log_{24} 27 = \frac{3}{2}$
 3. $\log_4 16 = -2$

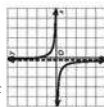
 5. Sample answer: If I divide 15,309 by 7, the result is 2,187 or 37. So, the expression that needs to be approximated can be written as $\log_7 37$. Then, using the Product Property of Logarithms, the expression can be rewritten again as $\log_7 7 + \log_7 37$. I know that $\log_7 7 = 1$, so $\log_7 37 = 1 + \log_7 37$. I know that $\log_7 37$ is about 1.7712, so $\log_7 37$ is about 1.7712 + 1 or 2.7712.
 7 D 9 C E 11 B 13a. $\ln 8x = 7$
 13b. $\ln 17 = 2x$ 15 B 17 A 19. 365 days

Selected Answers SA35

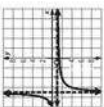
19. $a = 9, h = -3, k = 6; g(h) = \frac{9}{-3+3} + 6$
 21. $a = -6, h = -4, k = -2; g(h) = \frac{-6}{-4+2} - 2$
 23. $\frac{5}{3}, -1, 3$
 27. $-3, 5$
 29. $(x|x \neq -1)$ and $(f(x)|f(x) \neq 0)$
 31. $(x|x \neq 2, f(x) = 0, D = \{x|x \neq 2\}; R = \{f(x)|f(x) \neq 0\})$



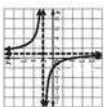
33. asymptotes: $x = \frac{1}{3}, h = 0$;
 $D = \{x|x \neq -\frac{1}{3}\}; R = \{f(x)|f(x) \neq 0\}$



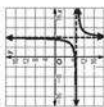
35. asymptotes: $x = -7, h = -1$;
 $D = \{x|x \neq -7\}; R = \{f(x)|f(x) \neq -1\}$



37. asymptotes: $x = 1, h = 2, D = \{x|x \neq 1\};$
 $R = \{f(x)|f(x) \neq 2\}$

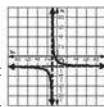


39. asymptotes: $x = 8, h = -9$;
 $D = \{x|x \neq 8\}; R = \{f(x)|f(x) \neq -9\}$



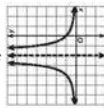
41. $(h) = \frac{2}{3}, 43, C = \frac{59}{3} + 0.3$
 45a. $C = \frac{59}{3} + 0.3$
 45b. They cannot travel zero miles or a negative number of miles.

47. The graph of $g(h)$ is a reflection in the x-axis, a translation 2 units left, and a translation 1 unit up of the graph of $f(h)$.

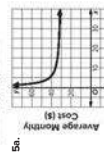
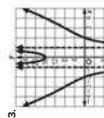
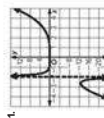


49. The graph of $h(x)$ is a reflection in the y-axis, a translation 2 units left, and a translation 2 units up of the graph of $f(x)$, so $h(x) = -f(x + 2) + 2 = -\frac{1}{x+2} + 2$.

51. Sample answer: $h(x) = \frac{x+4}{x+6}$
 53. Sample answer: $g(h)$; all other choices have unknowns only in the denominator.
 55. asymptotes: $x = -2, y = 0$



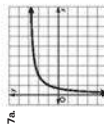
Lesson 7-4



Months of Service

- 5b. x-intercept: -2 ; y-intercept: none; end behavior: As $x \rightarrow -\infty, f(x) \rightarrow 30$ and as $x \rightarrow \infty, f(x) \rightarrow 30$.

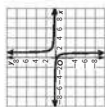
5c. \$37.50



- 7b. x-intercept: $\frac{2}{3}$; y-intercept: none; end behavior: As $x \rightarrow -\infty, f(x) \rightarrow 3$ and as $x \rightarrow \infty, f(x) \rightarrow 3$.

7c. = 2.7 cents

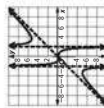
9a.



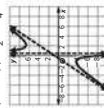
- 9b. $h(x)$, since $-\frac{1}{x} > -1$
 9c. Vertical asymptotes $h(x): x = \frac{1}{4}, f(x): x = -3$
 Horizontal asymptotes $h(x): y = \frac{1}{4}, f(x): y = -2$
 11. zero: $x = 4$; vertical asymptote: $x = -2$;
 oblique asymptote: $f(x) = x - 10$



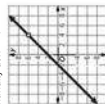
13. zero: $x = -1$; vertical asymptote: $x = -2$
 and $x = 2$; oblique asymptote: $f(x) = x$



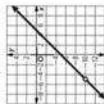
15. zero: none; vertical asymptote: $x = \frac{1}{2}$;
 oblique asymptote: $f(x) = \frac{3}{2}x + \frac{3}{2}$



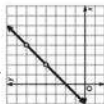
17. point discontinuity at $x = 4$



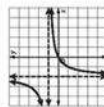
19. point discontinuity at $x = -5$



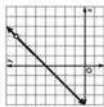
21. point discontinuity at $x = 2$ and $x = 4$



23.



25.

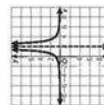


31b. $f(x)$

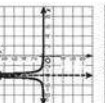
31c. The graph of $f(x)$ has a hole at $x = -2$, but no vertical asymptote. Its graph is a straight line with a positive slope, passing through (0, 0) and (-2, 2). The function $f(x)$ also has a hole at $x = -2$, but has a vertical asymptote at $x = 0$. Its graph is not a straight line, but two curves having a hole in the graph at $(-2, \frac{3}{2})$.

33. $x = 0$

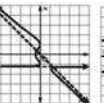
35.



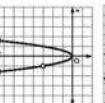
37.



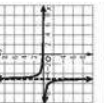
39.



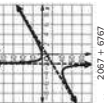
41.



43.



45.



47a. $y = \frac{2087 + 9367}{9367 + x}$

47b. $y = \frac{2087 + 9367}{9367 + x}$

47c. As the number of bits increases, the number of hits grows closer and closer to zero, since the number of hits remains constant.

48. Sample answer: The domain is $\{x \mid x \neq -2 \text{ and } x \neq 2\}$. The graph has zeros at $x = 1$ and $x = 4$ and a y-intercept at $(0, 1)$. The graph has

vertical asymptotes at $x = -2$ and $x = 2$ and a horizontal asymptote at $f(x) = -1$ for $2 \leq x$ or $x \leq -2$. A function with these features is

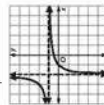
$$f(x) = -\frac{(x+2)(x-2)}{x^2-4}$$

51. Sample answer:



53. Sample answer: If the degrees of the numerator and denominator are equal, there is a horizontal asymptote determined by the coefficients of the numerator and denominator. If the degree of the numerator is less than the degree of the denominator, there is no horizontal asymptote. If the degree of the numerator is exactly 1 greater than the degree of the denominator, there is an oblique asymptote. If the degree of the denominator is greater, the horizontal asymptote is the line $y = 0$.

55. Sample Answer:



57. The graph of $g(x)$ has a hole at $x = -3$.

59. Sample answer: By factoring the denominator, we are exposed to a rational function that has a hole at the value that causes each factor to equal zero. You can find the asymptotes or discontinuity of the function. If the denominator has a factor $x - c$ that does not appear in the numerator, then there is a vertical asymptote at $x = c$. If the numerator and denominator both have a factor of $x - c$, then there is point continuity at $x = c$.

Lesson 7-5

1.15 3. -56 5. 55 7. -48 9. 72

11. 50 13. -22.5 15. 38

conduct a survey to find what gifts or activities might motivate the students.

21. convenience; survey

23. stratified; experiment

25. observational study

27. Sample answer: She should use an observational study to determine the average miles per gallon that identical cars are able to go on each type of fuel. She can then divide the current price of each fuel by the number of miles to find the average cost per mile. The group with the lower cost per mile would be the one that is more cost-efficient. She might also want to check out a website or a magazine club or magazine stating the average gas mileage for cars similar to her new car.

29. No; sample answer: the difference between the means of the experimental group and control group is only 0.1 ounce. Both groups had cats that gained significant weight and others that gained none. To compare the two groups, she should have used cats that were similar in age, starting weight, and health. I also assumed that each cat was given the same amount of food.

31. Simple answer: This method of selecting a sample is valid. Each student has an equally likely chance of being selected for the sample. However, if my best friends would not select that one grade, they would feel strongly about the dress code than another.

33. False; sample answer: A sample statistic is used to estimate a population parameter.

35. Sample answer: objective: Determine the average amount of time that students spend studying at the library. population: All students that study at the library. sample: 25 randomly selected students studying at the library during a given week.

37. observational study; Sample answer: Teenagers are more willing than adults to try a new sports drink.

39. control group: students who do not receive the gift cards; experimental group: students receiving gift cards; Sample answer: The cards they may not have any interest in the gift cards they are receiving. The principal should

divide the T-shirts by team before the sample is selected.

3. Sample: the customers who submit suggestions; population: all customers; self-selected: the customers voluntarily submit suggestions.

5. unbiased; Sample answer: the students are not influenced by the fact that they were selected at band camp does not influence the response of the selected sample.

7. biased; Sample answer: The question is asking about two issues: whether the workout facility needs a new treadmill and whether the workout facility needs a new racquetball court.

9. unbiased; Sample answer: The question does not influence participants.

11. survey; sample: customers that take the online survey; population: all customers

13. experimental; the 50 adults participating in the study; all adults

15. objective: to determine the number of juniors and seniors planning to attend prom; population: all juniors and seniors at the school; sample survey questions: What grade are you in? Do you plan on attending prom?

17. observational study; Sample answer: Teenagers are more willing than adults to try a new sports drink.

19. control group: students who do not receive the gift cards; experimental group: students receiving gift cards; Sample answer: The cards they may not have any interest in the gift cards they are receiving. The principal should

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9. unbiased; Sample answer: The question does not influence participants.

11. survey; sample: customers that take the online survey; population: all customers

13. experimental; the 50 adults participating in the study; all adults

Module 8

Module 8 Opener

1. 89 customers, 88 customers, no mode

3. 77, 5, 0

5. 4

7. 1

Lesson 8-1

1. Sample: the T-shirts; Bertion selects; population: all of Bertion's sports T-shirts; stratified; Bertion divides the T-shirts by team before the sample is selected.

3. Sample: the customers who submit suggestions; population: all customers; self-selected: the customers voluntarily submit suggestions.

5. unbiased; Sample answer: the students are not influenced by the fact that they were selected at band camp does not influence the response of the selected sample.

7. biased; Sample answer: The question is asking about two issues: whether the workout facility needs a new treadmill and whether the workout facility needs a new racquetball court.

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19. control group: students who do not receive the gift cards; experimental group: students receiving gift cards; Sample answer: The cards they may not have any interest in the gift cards they are receiving. The principal should

divide the T-shirts by team before the sample is selected.

39b. $x > 1050$; the company must produce and sell at least 1050 speakers in order to ensure that the revenue from each one is greater than the average cost of producing each one.

41. $x = 2$; this method works because the x-coordinate of the point of intersection will make both sides of the equation equal to the same value, and no extraneous roots are shown.

43. at least 40 students

45. Sample answer: I do not agree with the solution set. Multiplying both sides of the inequality by -4 will change the direction of the inequality to 0 . Only -4 is a solution.

47. Sample answer: Multiplying both sides of the inequality by -4 will change the direction of the inequality if x is negative. The work is only valid for positive x -values. The student could have graphed the two equations $f(x) = \frac{1}{3}$ and $g(x) = 2$ and looked at x -values for which the graph of $f(x)$ is below the graph of $g(x)$.

49. all real numbers except 5 , -5 , 0

51. Sample answer: Multiplying each side of a rational equation or inequality by the LCD can introduce extraneous solutions. The student should check all solutions to make sure that they satisfy the original equation or inequality.

53. $d = 0.009$; **63a.** Joint variation; the amount deducted varies directly as the product of the number of hours worked and the number of hours worked. **63c.** Substitute $r = \$19.50$ and $h = 36$ in the formula for part a. The amount deducted was \$70.20.

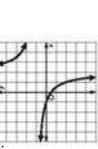
65. a and g are directly related.

67. Sample answer: The force of an object varies jointly as its mass and acceleration.

Module 7 Review

1. $\frac{2x^2}{3x}$ **3.** $\frac{2x^2}{3x}$ **5.** $\frac{2x^2}{3x}$ **7.** D

9. $\frac{2x^2}{3x}$



11a. x-intercept: undefined; y-intercept: -1

11b. horizontal $y = -2$; vertical $x = -2$

13. D **45a.** 25 mph **45b.** 30 mph **47.** D

17. 19 **19.** -10 **21.** about 0.7 **22.** 0.00079311

25. 1336 am **27.** inverse; 4 **29.** inverse; 15

31. direct; 5280 **33.** inverse; -25

35. inverse; 8 **37.** direct; 21 **39.** inverse; -2

41. combined; 10 **43.** direct; 4 **45.** direct; -2

47. inverse; 7 **49.** joint; 20 **51.** 53 **53.** 7

55. $y = 30k$; theaters **57.** inverse

59a. 11 x ; $f = \frac{11}{12}w$; 12 w ; $f = \frac{11}{12}w$; $w = 16$; $f = \frac{11}{12}w$; $w = 18$; $f = \frac{11}{12}w$; $w = 24$; $f = \frac{11}{12}w$; $w = 30$; $f = \frac{11}{12}w$; $w = 36$; $f = \frac{11}{12}w$; $w = 42$; $f = \frac{11}{12}w$; $w = 48$; $f = \frac{11}{12}w$; $w = 54$; $f = \frac{11}{12}w$; $w = 60$; $f = \frac{11}{12}w$; $w = 66$; $f = \frac{11}{12}w$; $w = 72$; $f = \frac{11}{12}w$; $w = 78$; $f = \frac{11}{12}w$; $w = 84$; $f = \frac{11}{12}w$; $w = 90$; $f = \frac{11}{12}w$; $w = 96$; $f = \frac{11}{12}w$; $w = 102$; $f = \frac{11}{12}w$; $w = 108$; $f = \frac{11}{12}w$; $w = 114$; $f = \frac{11}{12}w$; $w = 120$; $f = \frac{11}{12}w$; $w = 126$; $f = \frac{11}{12}w$; $w = 132$; $f = \frac{11}{12}w$; $w = 138$; $f = \frac{11}{12}w$; $w = 144$; $f = \frac{11}{12}w$; $w = 150$; $f = \frac{11}{12}w$; $w = 156$; $f = \frac{11}{12}w$; $w = 162$; $f = \frac{11}{12}w$; $w = 168$; $f = \frac{11}{12}w$; $w = 174$; $f = \frac{11}{12}w$; $w = 180$; $f = \frac{11}{12}w$; $w = 186$; $f = \frac{11}{12}w$; $w = 192$; $f = \frac{11}{12}w$; $w = 198$; $f = \frac{11}{12}w$; $w = 204$; $f = \frac{11}{12}w$; $w = 210$; $f = \frac{11}{12}w$; $w = 216$; $f = \frac{11}{12}w$; $w = 222$; $f = \frac{11}{12}w$; $w = 228$; $f = \frac{11}{12}w$; $w = 234$; $f = \frac{11}{12}w$; $w = 240$; $f = \frac{11}{12}w$; $w = 246$; $f = \frac{11}{12}w$; $w = 252$; $f = \frac{11}{12}w$; $w = 258$; $f = \frac{11}{12}w$; $w = 264$; $f = \frac{11}{12}w$; $w = 270$; $f = \frac{11}{12}w$; $w = 276$; $f = \frac{11}{12}w$; $w = 282$; $f = \frac{11}{12}w$; $w = 288$; $f = \frac{11}{12}w$; $w = 294$; $f = \frac{11}{12}w$; $w = 300$; $f = \frac{11}{12}w$; $w = 306$; $f = \frac{11}{12}w$; $w = 312$; $f = \frac{11}{12}w$; $w = 318$; $f = \frac{11}{12}w$; $w = 324$; $f = \frac{11}{12}w$; $w = 330$; $f = \frac{11}{12}w$; $w = 336$; $f = \frac{11}{12}w$; $w = 342$; $f = \frac{11}{12}w$; $w = 348$; $f = \frac{11}{12}w$; $w = 354$; $f = \frac{11}{12}w$; $w = 360$; $f = \frac{11}{12}w$; $w = 366$; $f = \frac{11}{12}w$; $w = 372$; $f = \frac{11}{12}w$; $w = 378$; $f = \frac{11}{12}w$; $w = 384$; $f = \frac{11}{12}w$; $w = 390$; $f = \frac{11}{12}w$; $w = 396$; $f = \frac{11}{12}w$; $w = 402$; $f = \frac{11}{12}w$; $w = 408$; $f = \frac{11}{12}w$; $w = 414$; $f = \frac{11}{12}w$; $w = 420$; $f = \frac{11}{12}w$; $w = 426$; $f = \frac{11}{12}w$; $w = 432$; $f = \frac{11}{12}w$; $w = 438$; $f = \frac{11}{12}w$; $w = 444$; $f = \frac{11}{12}w$; $w = 450$; $f = \frac{11}{12}w$; $w = 456$; $f = \frac{11}{12}w$; $w = 462$; $f = \frac{11}{12}w$; $w = 468$; $f = \frac{11}{12}w$; $w = 474$; $f = \frac{11}{12}w$; $w = 480$; $f = \frac{11}{12}w$; $w = 486$; $f = \frac{11}{12}w$; $w = 492$; $f = \frac{11}{12}w$; $w = 498$; $f = \frac{11}{12}w$; $w = 504$; $f = \frac{11}{12}w$; $w = 510$; $f = \frac{11}{12}w$; $w = 516$; $f = \frac{11}{12}w$; $w = 522$; $f = \frac{11}{12}w$; $w = 528$; $f = \frac{11}{12}w$; $w = 534$; $f = \frac{11}{12}w$; $w = 540$; $f = \frac{11}{12}w$; $w = 546$; $f = \frac{11}{12}w$; $w = 552$; $f = \frac{11}{12}w$; $w = 558$; $f = \frac{11}{12}w$; $w = 564$; $f = \frac{11}{12}w$; $w = 570$; $f = \frac{11}{12}w$; $w = 576$; $f = \frac{11}{12}w$; $w = 582$; $f = \frac{11}{12}w$; $w = 588$; $f = \frac{11}{12}w$; $w = 594$; $f = \frac{11}{12}w$; $w = 600$; $f = \frac{11}{12}w$; $w = 606$; $f = \frac{11}{12}w$; $w = 612$; $f = \frac{11}{12}w$; $w = 618$; $f = \frac{11}{12}w$; $w = 624$; $f = \frac{11}{12}w$; $w = 630$; $f = \frac{11}{12}w$; $w = 636$; $f = \frac{11}{12}w$; $w = 642$; $f = \frac{11}{12}w$; $w = 648$; $f = \frac{11}{12}w$; $w = 654$; $f = \frac{11}{12}w$; $w = 660$; $f = \frac{11}{12}w$; $w = 666$; $f = \frac{11}{12}w$; $w = 672$; $f = \frac{11}{12}w$; $w = 678$; $f = \frac{11}{12}w$; $w = 684$; $f = \frac{11}{12}w$; $w = 690$; $f = \frac{11}{12}w$; $w = 696$; $f = \frac{11}{12}w$; $w = 702$; $f = \frac{11}{12}w$; $w = 708$; $f = \frac{11}{12}w$; $w = 714$; $f = \frac{11}{12}w$; $w = 720$; $f = \frac{11}{12}w$; $w = 726$; $f = \frac{11}{12}w$; $w = 732$; $f = \frac{11}{12}w$; $w = 738$; $f = \frac{11}{12}w$; $w = 744$; $f = \frac{11}{12}w$; $w = 750$; $f = \frac{11}{12}w$; $w = 756$; $f = \frac{11}{12}w$; $w = 762$; $f = \frac{11}{12}w$; $w = 768$; $f = \frac{11}{12}w$; $w = 774$; $f = \frac{11}{12}w$; $w = 780$; $f = \frac{11}{12}w$; $w = 786$; $f = \frac{11}{12}w$; $w = 792$; $f = \frac{11}{12}w$; $w = 798$; $f = \frac{11}{12}w$; $w = 804$; $f = \frac{11}{12}w$; $w = 810$; $f = \frac{11}{12}w$; $w = 816$; $f = \frac{11}{12}w$; $w = 822$; $f = \frac{11}{12}w$; $w = 828$; $f = \frac{11}{12}w$; $w = 834$; $f = \frac{11}{12}w$; $w = 840$; $f = \frac{11}{12}w$; $w = 846$; $f = \frac{11}{12}w$; $w = 852$; $f = \frac{11}{12}w$; $w = 858$; $f = \frac{11}{12}w$; $w = 864$; $f = \frac{11}{12}w$; $w = 870$; $f = \frac{11}{12}w$; $w = 876$; $f = \frac{11}{12}w$; $w = 882$; $f = \frac{11}{12}w$; $w = 888$; $f = \frac{11}{12}w$; $w = 894$; $f = \frac{11}{12}w$; $w = 900$; $f = \frac{11}{12}w$; $w = 906$; $f = \frac{11}{12}w$; $w = 912$; $f = \frac{11}{12}w$; $w = 918$; $f = \frac{11}{12}w$; $w = 924$; $f = \frac{11}{12}w$; $w = 930$; $f = \frac{11}{12}w$; $w = 936$; $f = \frac{11}{12}w$; $w = 942$; $f = \frac{11}{12}w$; $w = 948$; $f = \frac{11}{12}w$; $w = 954$; $f = \frac{11}{12}w$; $w = 960$; $f = \frac{11}{12}w$; $w = 966$; $f = \frac{11}{12}w$; $w = 972$; $f = \frac{11}{12}w$; $w = 978$; $f = \frac{11}{12}w$; $w = 984$; $f = \frac{11}{12}w$; $w = 990$; $f = \frac{11}{12}w$; $w = 996$; $f = \frac{11}{12}w$; $w = 1000$; $f = \frac{11}{12}w$; $w = 1006$; $f = \frac{11}{12}w$; $w = 1012$; $f = \frac{11}{12}w$; $w = 1018$; $f = \frac{11}{12}w$; $w = 1024$; $f = \frac{11}{12}w$; $w = 1030$; $f = \frac{11}{12}w$; $w = 1036$; $f = \frac{11}{12}w$; $w = 1042$; $f = \frac{11}{12}w$; $w = 1048$; $f = \frac{11}{12}w$; $w = 1054$; $f = \frac{11}{12}w$; $w = 1060$; $f = \frac{11}{12}w$; $w = 1066$; $f = \frac{11}{12}w$; $w = 1072$; $f = \frac{11}{12}w$; $w = 1078$; $f = \frac{11}{12}w$; $w = 1084$; $f = \frac{11}{12}w$; $w = 1090$; $f = \frac{11}{12}w$; $w = 1096$; $f = \frac{11}{12}w$; $w = 1102$; $f = \frac{11}{12}w$; $w = 1108$; $f = \frac{11}{12}w$; $w = 1114$; $f = \frac{11}{12}w$; $w = 1120$; $f = \frac{11}{12}w$; $w = 1126$; $f = \frac{11}{12}w$; $w = 1132$; $f = \frac{11}{12}w$; $w = 1138$; $f = \frac{11}{12}w$; $w = 1144$; $f = \frac{11}{12}w$; $w = 1150$; $f = \frac{11}{12}w$; $w = 1156$; $f = \frac{11}{12}w$; $w = 1162$; $f = \frac{11}{12}w$; $w = 1168$; $f = \frac{11}{12}w$; $w = 1174$; $f = \frac{11}{12}w$; $w = 1180$; $f = \frac{11}{12}w$; $w = 1186$; $f = \frac{11}{12}w$; $w = 1192$; $f = \frac{11}{12}w$; $w = 1198$; $f = \frac{11}{12}w$; $w = 1200$; $f = \frac{11}{12}w$; $w = 1206$; $f = \frac{11}{12}w$; $w = 1212$; $f = \frac{11}{12}w$; $w = 1218$; $f = \frac{11}{12}w$; $w = 1224$; $f = \frac{11}{12}w$; $w = 1230$; $f = \frac{11}{12}w$; $w = 1236$; $f = \frac{11}{12}w$; $w = 1242$; $f = \frac{11}{12}w$; $w = 1248$; $f = \frac{11}{12}w$; $w = 1254$; $f = \frac{11}{12}w$; $w = 1260$; $f = \frac{11}{12}w$; $w = 1266$; $f = \frac{11}{12}w$; $w = 1272$; $f = \frac{11}{12}w$; $w = 1278$; $f = \frac{11}{12}w$; $w = 1284$; $f = \frac{11}{12}w$; $w = 1290$; $f = \frac{11}{12}w$; $w = 1296$; $f = \frac{11}{12}w$; $w = 1300$; $f = \frac{11}{12}w$; $w = 1306$; $f = \frac{11}{12}w$; $w = 1312$; $f = \frac{11}{12}w$; $w = 1318$; $f = \frac{11}{12}w$; $w = 1324$; $f = \frac{11}{12}w$; $w = 1330$; $f = \frac{11}{12}w$; $w = 1336$; $f = \frac{11}{12}w$; $w = 1342$; $f = \frac{11}{12}w$; $w = 1348$; $f = \frac{11}{12}w$; $w = 1354$; $f = \frac{11}{12}w$; $w = 1360$; $f = \frac{11}{12}w$; $w = 1366$; $f = \frac{11}{12}w$; $w = 1372$; $f = \frac{11}{12}w$; $w = 1378$; $f = \frac{11}{12}w$; $w = 1384$; $f = \frac{11}{12}w$; $w = 1390$; $f = \frac{11}{12}w$; $w = 1396$; f

Lesson 8-2

1a. 25%

1b. 38%

3a. 15.7%

3b. 18%

5. Sample answer: The theoretical probability of making a goal is $\frac{8}{20}$, and the theoretical probability of missing the goal is $\frac{12}{20}$. I can use a spinner to simulate this situation. For numbers 1–20, let 1–7 represent a goal and 8–20 represent a miss. Based on 100 trials, the experimental probability of making a goal is 41%. Because Priola's success rate was only about 13%, it is likely that there is an error in the programming.

7. spinner with 2 regions, one central angle 216°

9. Sample answer: Use a random number generator where 0, 1, 2, 3, and 4 represent brown hair, 5, 6, 7, and 8 represent blonde hair, and 9 represents red hair.

11. Jaymie's model

13a. 20%, 13b. 13c. Sample answer: I assumed that each car color was equally likely.

15a. Sample answer: Use a random number generator to generate a set of three numbers from 1 to 42. Numbers 1–15 will represent the basketball players, and numbers 16–42 will represent the other students. Discard any trial that repeats any of the same number. Run the trial fifty times.

15b.

35, 13, 36	15, 32, 9	6, 14, 12	16, 26, 27	4, 34, 20	16, 32, 24	12, 9, 18	10, 17, 26
15, 13, 23	18, 15, 41	13, 9, 20	38, 30, 34	40, 35	37, 18, 36	22, 16, 36	23, 5, 28
29, 7, 2	21, 22, 1	33, 31, 22	30, 39, 8	31, 7, 17	35, 10, 36	35, 9, 29	32, 20, 20
21, 3, 23	16, 36, 5	30, 36, 11	28, 35, 10	5, 42, 21	11, 18, 40	26, 10, 2	28, 35, 41
13, 11, 23	12, 7, 24	5, 3, 7	15, 39, 2	30, 21, 24	32, 25, 7	18, 9, 34	29, 20, 2
18, 26, 6	32, 6, 39	17, 3, 21	31, 18, 27	24, 29, 31	25, 37, 21	32, 23, 13	21, 5, 28
3, 34, 15	12, 2, 20						

SA44 Selected Answers

Selected Answers

Most of the exam sessions will have between 71 and 107 attendees.

13. No; sample answer: adding \$1 to each employee's hourly wage has no effect on the standard deviation of the data. The standard deviation both before and after the wage change is 0.07.

14. Sample answer: Assume that each data value falls in the center of each bar of the histogram. Using technology, the mean is 17 and the standard deviation is about 7.4.

17. Find the mean of the set of data. Then, find the square of the difference between each data value and the mean. Next, find the sum of all of the differences. Then, divide the sum by the number of data values. Finally, take the square root of the quotient, or variance.

Lesson 8-4

1. tests; discrete; The number of tests can only be a whole number.

3. height of plant; continuous; The height of a plant may be any positive value.

5. Sample answer: The scale of the y-axis misleadingly shows the differences in probabilities.

7a. $397 < X < 3331$ 7b. 16% 7c. $X > 282.2$

9. –219 11. 0.281 13. 0.9861

15. 2.5% 17. 6200 hours

19. Hiciko; Sample answer: Monica's solution would work with a uniform distribution.

21. Sample answer: The z-value represents the position of a value X in a normal distribution. The area under a curve to the left of a z-value represents the probability that a value from the distribution will be less than the given value X .

Lesson 8-5

4.E = 016

3.E = 006

5. At a 99% confidence interval, the population mean is $5.35 \leq \mu \leq 5.65$. Therefore, we are 99% confident that the rating of the airline is between 5.35 and 5.65.

Selected Answers SA45

19. Sample answer: Jevon's concern about the fairness of the coin should be dependent on the number of times he tossed it. If he completed only 2, or even 20, trials, then the results may not be enough to cause a major concern. However, if he completed many trials, then he should be concerned since the experimental probability should be closer to the theoretical probability of $\frac{1}{2}$, or 50%.

21. True; sample answer: The event will always occur. When the theoretical probability is 1, therefore, the experimental probability can never be 0. For example, the theoretical probability of rolling a 1 on a six-sided die is $\frac{1}{6}$, or 6.67%. Since I will always show one of these six numbers, so when a die is rolled, the result will be one of these six numbers, making the experimental probability always greater than 0.

23. Yes; sample answer: If the spinner were going to be divided equally into three outcomes, each sector would measure 120°. But the red sector is larger than the blue sector. If outcome C, you can record spins that end in the red area as a success, or the occurrence of outcome C, and spins that end in the blue area as a failure, or an outcome of A or B.

Lesson 8-3

1. 38.22; Sample answer: because the standard deviation is great compared to the mean of \$7817, the data are spread farther from the mean. Based on the standard deviation, most of the cars cost between \$34.41 and \$10.65.

3. 2.26; 13

5. 6.5; 17

7. Before the season the mean time was 9.2 minutes. After the season the mean time was 7.0 minutes. After the season the mean time was 7.0 minutes. This means that before the season the mile times were higher on average, but that the mile times were generally spread the same before and after the season.

9. 0.375; 0.123

11. B.4; Sample answer: The standard deviation is small compared to the mean of 89.

Selected Answers

7. At a 90% confidence interval, the population proportion is estimated to be between 18.8 hours; therefore, we are 90% confident that the proportion of high school students who agree that the proportion of students who agree that high school students should have a part-time job is between 73.7% and 81.3%.

Hours of Television Watched Each Week

12	8	9	27	20	18
0	33	25	44	9	19
23	6	20	26	30	29

9. At a 99% confidence interval, the population proportion is $0.326 \leq p \leq 0.506$. Therefore, we are 99% confident that the proportion of students who agree that the principal's plan is between 32.6% and 50.6%.

11. Instead of stating that the population proportion is estimated at $\frac{50}{100} = 0.48 = 48\%$ with a margin of error of 3%, the claim uses the highest possible estimate for the population proportion to give the impression that the proportion of people who want parks to be built is greater than 50%.

13. $p = \frac{0.056}{0.056 + 0.050} \approx 0.050$, so $CI = 15.6\% \pm 5.0\%$.

- Sample answer: With 95% confidence, the mean proportion of discredits for the population of all pieces fired in her skin is between 10.6% and 20.6%. The probability that she will buy a new skin on the basis that the "discredit" rate is most likely higher than 10%.

15. As the sample size increases, the maximum error decreases.

Module 8 Review

- 1.D 3.A 5.A

- 7.A,D 9.B,C 11. $x < 55.9$ and $x > 70.9$

13. $E = 196 \sqrt{\left(\frac{0.60}{360}\right)\left(\frac{0.40}{360}\right)}$

$E \approx 0.0508$

$p = \frac{279}{360} \approx 0.7750$

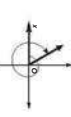
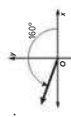
$q = \frac{81}{360} \approx 0.2250$

- Sample answer: At a 95% confidence level, I am confident that the proportion of patrons who are 18 years of age or younger who are under 18 is between 70.75% and 80.9%.

Module 9 Opener

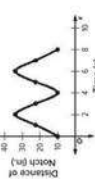
1. 117 3. 205 5. $x = 6$, $y = 9/2$
7. $x = 12$, $y = 12\sqrt{3}$

Lesson 9-1



17. 285° , -435° , 19 , 405° , -315°
21. 10° , -350° , 23 , 60° , -300°
25. 435° , -295° , 27 , 470° , -260°
29. $\frac{7\pi}{6}$, $-\frac{7\pi}{6}$, 31 , $\frac{8\pi}{3}$, $-\frac{4\pi}{3}$, $\frac{3\pi}{4}$, $-\frac{3\pi}{4}$
33. $-\frac{9\pi}{9}$, 37 , $-\frac{9\pi}{9}$, 38 , 4π , 41 , $\frac{\pi}{2}$
35. -108° , 45 , 50° , 47 , 120° , 49 , -135°
39. 45° , 53 , $-\frac{3\pi}{4}$, or approximately 20.9 in.
43. 55 , 30 , 57 , $-\frac{\pi}{9}$, 59 , 450°
47. -105° , 63 , 8.4 in.
51. 67 , 3309° , 58 radians
55. 90° , $\frac{\pi}{2}$ radians; 3π inches
59. $73a$, $\frac{\pi}{3}$. Because a complete rotation takes 30 minutes, 5 minutes represents $\frac{5}{30}$ or $\frac{1}{6}$ of a complete rotation, and $\frac{1}{6} \cdot 2\pi = \frac{\pi}{3}$.
63. The radius r of the wheel is 67.5 m. The distance the passenger traveled is the arc length, so $s = r\theta = 67.5\left(\frac{\pi}{3}\right) \approx 71$ m.
67. The circumference is $2\pi r$, which is approximately $3.14(2 \cdot 71) = 3.14(142) \approx 443.9$ m and $\frac{1}{6}(443.9) \approx 73.98$, which is close to 71, so the answer is reasonable.

19. period = 4 seconds



21. $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $-\frac{3}{2}$, $-\frac{5}{2}$, $\frac{7}{2}$, $\frac{9}{2}$

23. $\frac{5\sqrt{3}}{2}$, $\frac{3\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}$, $-\frac{3\sqrt{3}}{2}$, $-\frac{5\sqrt{3}}{2}$

25. $\frac{5\sqrt{3}}{2}$, $\frac{3\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}$, $-\frac{3\sqrt{3}}{2}$, $-\frac{5\sqrt{3}}{2}$

35a. Sometimes; the cosine function can only result in values between -1 and 1, inclusive.

35b. Always; the sine function has a period of 2π .

35c. Sometimes; $\cos \theta = 1$ when n is even and $\cos \theta = -1$ when n is odd.

35d. Never; the y-coordinate of the corresponding point on the unit circle is negative.

37a. $\frac{3\pi}{4}$ meters; because the radian measure of the angle is the length of the arc on the unit circle subtended by the angle.



37c. The period is 2π because the values of the function repeat every 2π units, as shown in the graph when the shape of the curve from 0 to 2π is repeated from 2π to 4π .

37d. The heights of point P above or below the surface of the water repeat once as the wheel makes a complete rotation (2π radians).

37e. The x-intercepts are whole-number multiples of π ; these are the angles of rotation for which the wheel is at the water surface.

37f. Sample answer: $(\frac{3\pi}{4}, \frac{\sqrt{2}}{2})$ as the wheel rotates through an angle of $\frac{3\pi}{4}$ radians to $\frac{3\pi}{2}$ radians, P is at the same distance from the equilibrium point every 2 seconds.

37g. 72° F

37h. 39c. 12° F

39d. September

41. Benoit France incorrectly wrote $\cos(\frac{3\pi}{2}) = -\cos \frac{3\pi}{2}$.

43. $\cos \frac{3\pi}{2} = 0$, which is not a multiple of π .

45. The period of a periodic function is the horizontal distance of the part of the graph that is nonrepeating. Each nonrepeating part of the graph is one cycle.

Lesson 9-4

1. 2π ; $y = 0$; 360°

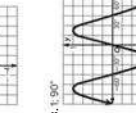
3. 1 ; $y = 0$; 120°

5. 2 ; 360°

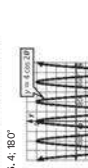
7. 1 ; 720°

9. $\frac{1}{2}$; 180°

11. 3 ; 360°



13. 1 ; 90°



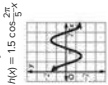
15. 4 ; 180°



17a. period: 2; frequency: $\frac{1}{2}$ The object completes $\frac{1}{2}$ of a cycle per second, and it will take 2 seconds to return to the equilibrium point every 2 seconds.

7b. Domain: all real numbers; because time cannot be negative, the relevant domain is $[-25, 25]$.

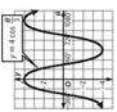
19. $h(t) = 15 \cos \frac{2\pi}{3}t$



21. 5, 540°



23. 4; 1080°



25. 180°; $y = 5 \sin 2\theta$

27. Sample answer: The sine function has the same shape as an ocean wave. Additionally, the sine function and cosine waves share similar behaviors, as they both repeat themselves over a constant interval of time and distance.

29a. 12; a calendar year

29b. 85° F; July 15

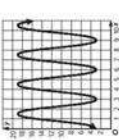
31a. 200 nm, 520 nm

31b. yellow

31c. violet; red

33a. amplitude: 4.5; period: 2 seconds

33b. $h = -4.5 \cos \pi t + 7.5$



35. Sample answer: $y = 2 \sin 6\theta$; the amplitude is $|a| = 2$. The period is $\frac{2\pi}{b} = \frac{2\pi}{6} = \frac{\pi}{3}$, so there are 3 complete cycles between $\theta = 0$ and $\theta = \pi$.

37. The graph of $y = \frac{1}{2} \sin \theta$ has an amplitude of $\frac{1}{2}$ and a period of 360° . The graph of $y = \sin \frac{1}{2}\theta$ has an amplitude of 1 and a period of 720° .

39. Sample answer: Determine the amplitude and period of the function; find and graph and x-intercepts, extrema, and asymptotes; use the parent function to sketch the graph.

Lesson 9-5

1. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

3. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

5. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

7. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

9. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

11. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

13. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

15. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

17. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

19. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

21. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

23. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

25. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

27. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

29. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

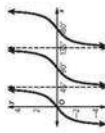
31. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

33. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

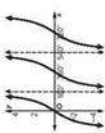
35. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

37. period = 36° ; asymptotes: $(36n + 18)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 36^\circ, 72^\circ, 108^\circ, \dots$; midline: $y = 0$; Because $a = 1$, the function is not reflected horizontally.

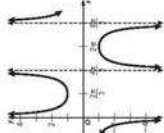
3. period = 90° ; asymptotes: $(90n + 45)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 90^\circ, 180^\circ, 270^\circ, \dots$; midline: $y = 0$. Because $a = 1$, the function is not vertically dilated in relation to the parent function. Because $b = 2$, the function is stretched horizontally in relation to the parent function.



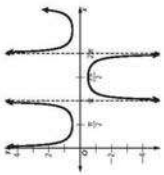
5. period = 360° ; asymptotes: $(360n + 180)^\circ$, where n is an odd integer; x-intercepts: $0^\circ, 360^\circ, 720^\circ, 1080^\circ, \dots$; midline: $y = 0$. Because $a = 2$, the function is stretched vertically in relation to the parent function. Because $b = \frac{1}{2}$, the function is stretched horizontally in relation to the parent function.



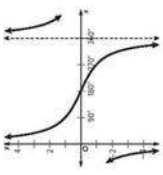
7. period = $\frac{6\pi}{5}$; asymptotes: $x = \frac{4\pi}{5} + \frac{4\pi n}{5}$, where n is an integer; relative minima at $x = \frac{\pi}{5}, x = \frac{3\pi}{5}, \dots$ or $x = \frac{\pi}{5} + 2\pi n$, where n is an integer; relative maxima at $x = 2\pi, x = \frac{7\pi}{5}, \dots$ or $x = 2\pi + \frac{2\pi n}{5}$, where n is an integer.



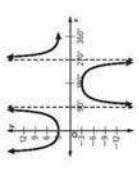
9. period = 2π ; asymptotes: $x = 2\pi + 2\pi n$, where n is an integer; relative minima at $x = \frac{\pi}{2}, x = \frac{3\pi}{2}, \dots$ or $x = \frac{\pi}{2} + 2\pi n$, where n is an integer; relative maxima at $x = \frac{5\pi}{2}, x = \frac{7\pi}{2}, \dots$ or $x = \frac{5\pi}{2} + 2\pi n$, where n is an integer.



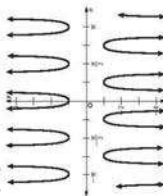
11. period = 360° ; asymptotes: $x = 27\pi n$, where n is an integer; x-intercepts: $180^\circ, 540^\circ, \dots$ (odd multiples of 180°).



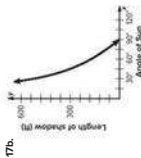
13. period = 360° ; asymptotes: $x = 270^\circ + 360n$, where n is an integer



15. period = $\frac{\pi}{2}$; asymptotes: $x = \frac{2\pi}{3} + \frac{\pi}{2}n$, where n is an integer



- 17a. $y = 300 \cot x$



- 17c. Sample answer: In the context of the problem, the unit of x is irrelevant, so focus on analyzing that portion of the graph.

- domain: $\{0 < x < 90\}$

- range: $\{y \mid -\infty < y < \infty\}$

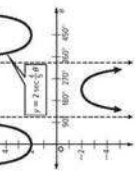
- x-intercept: 90

- y-intercept: none

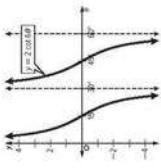
- relative minimum: none

- relative maximum: none

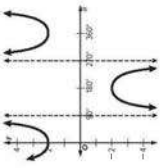
- The period of $y = 300 \cot x$ is 180° .



21. period: 30°



23. period: 360°



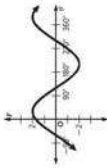
25. $d = 2000 \csc x$

27. Tyler: It appears that Lazretta gave the period of the function $y = \frac{\pi}{2}$ rad rather than $\frac{\pi}{2}$ sec.

29. Sample answer: Determine the amplitude and period of the function, then find and graph any x-intercepts, extrema, and asymptotes. Sketch the parent function and transform the graph to graph the given function.

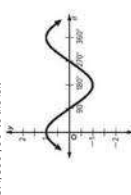
Lesson 9-6

1. 1.2π ; π to the left

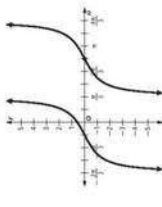


- domain: $-\infty < y < \infty$; range: $-2 \leq f(\theta) \leq 2$

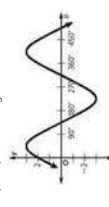
3. 1, 360°; 90° to the left



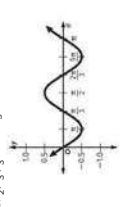
domain: $-\infty < \theta < \infty$; range: $-1 \leq f(\theta) \leq 1$
5. no amplitude; $\pi/6$ to the left



domain: $m\pi < \theta < \frac{\pi}{3} + m\pi$, where n is an integer; range: $-\infty < f(\theta) < \infty$
7. 3, 360°; 45° to the right

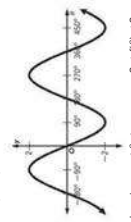


domain: $-\infty < \theta < \infty$; range: $-3 \leq f(\theta) \leq 3$
9. $\frac{3}{2}, \frac{2\pi}{3}, \frac{\pi}{3}$ to the right

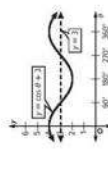


domain: $-\infty < \theta < \infty$; range: $-0.5 \leq f(\theta) \leq 0.5$

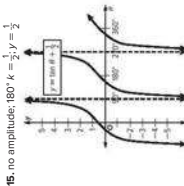
11. 2, 360°; 90° to the left



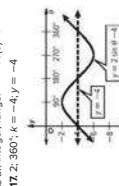
domain: $-\infty < \theta < \infty$; range: $-2 \leq f(\theta) \leq 2$
13. 1, 360°; $k = 3$; $y = 3$



domain: $-\infty < \theta < \infty$; range: $2 \leq f(\theta) \leq 4$
15. no amplitude; 180°; $k = \frac{1}{2}$; $y = \frac{1}{2}$



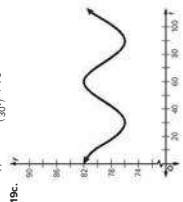
domain: $90 + 180n < \theta < 180 + 180n$, where n is an integer; range: $-\infty < f(\theta) < \infty$
17. 2, 360°; $k = -4$; $y = -4$



domain: $-\infty < \theta < \infty$; range: $-6 \leq f(\theta) \leq -2$

19a. The amplitude is 3, the midline is $y = 7/5$, the period is 60 minutes.

19b. $h(t) = 3 \cos(\frac{\pi}{30}t) + 7/5$

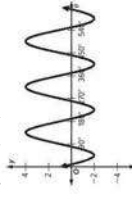


19d. 2 lines; these are the lines when the minute hand is on the 6 (i.e., 1230 and 1350).

21a. $k = -3$, so the vertical shift is 3 units down; amplitude = 1 21b. none 21c. π

21d. $y = \cos 2\theta - 3$ 21e. $y = \sin(k - 4) + 3$

21f. $y = \tan(k - \pi) + 2.5$ 21g. 180°; -45°; 1



29. Sample answer: sine. Because as a Ferris wheel turns, the distance a rider is above the ground varies sinusoidally. The sine function can also be modeled with a cosine function using the appropriate horizontal shift and/or reflection about the horizontal axis.

31a. 120, in 2.5 years 31b. 125, in 7 years

33a. $(\frac{\pi}{2}, 2)$ 35. no maximum values

37. The graphs are reflections of each other over the x-axis.

39. The graphs are identical.

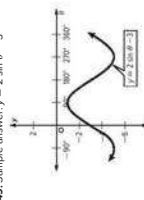
41. 360°; Sample answer: $y = \sin \theta - 5$

43. 360°; Sample answer: $y = 4 \cos \theta + 1$

45. Sample answer: $f(\theta) = 1.5 \cos 2\theta - 2$

47. The graph of $y = 3 \sin 2\theta + 1$ has an amplitude of 3 rather than an amplitude of 1, it is shifted up 1 unit from the parent graph and is compressed so that it has a period of 180°.

49. Sample answer: $y = 2 \sin \theta - 3$



51. Sample answer: $y = \sin(\theta + \frac{\pi}{3}) - 3$ is equivalent to $y = \cos(\theta)$. When the sine graph is shifted $\frac{\pi}{3}$ units to the left, it is equivalent to the cosine graph.

Lesson 9-7

1. 30°, $\frac{\pi}{6}$ 3. 120°, $\frac{2\pi}{3}$ 5. 135°, $\frac{3\pi}{4}$ 7. 45°, $\frac{\pi}{4}$

9. 90°, $\frac{\pi}{2}$ 11. -102 13. 0.71 15. 0.95

17. 0.5, 19, 53.1° 21. 60° 23. -5.24

25. 59.67° 27. 20.49° 29. 75.52°

31a. 48.2°

31b. Sample answer: I assumed that the angle at which the doors would be opened would be equal in measure.

33. $\frac{\pi}{3}$ 35. 0, 2π 37. $\frac{\pi}{4}, \frac{5\pi}{4}$

41. Sample answer: Neither, cosine is not positive in the second quadrant.

43. Sample answer: $y = \tan^{-1}x$ is a relation that has a domain of all real numbers and a range of all real numbers except odd multiples of $\frac{\pi}{2}$. The relation is not a function: $y = \tan^{-1}x$ is a function, but the domain is not all real numbers and a range of $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

45. Sample answer: $y = \cos^{-1}(\theta + 1)$; this function is a horizontal stretch by a factor of 2 of the parent function $y = \cos^{-1}\theta$, so the domain is $[-2, 2]$. It is also a vertical translation 1 unit up, so the range is $[1, \pi + 1]$.

Module 9 Review

- 1a. $\frac{2}{3}$ 1b. $\frac{5}{12}$ 1c. $\frac{8}{15}$
 3. 157 meters 5. $\frac{21}{24}$ 7. $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ 9a. 30
 9b. 20 11. A 13. A 15. A, D, E 17. C

Glossary

English

30°-60°-90° triangle A right triangle with two acute angles that measure 30° and 60°.

45°-45°-90° triangle A right triangle with two acute angles that measure 45°.

absolute value The distance a number is from zero on the number line.

absolute value function A function written as

$$f(x) = |x|, \text{ in which } f(x) \geq 0 \text{ for all values of } x.$$

accuracy The nearness of a measurement to the true value of the measure.

additive identity Because the sum of any number a and 0 is equal to a , 0 is the additive identity.

additive inverses Two numbers with a sum of 0.

adjacent angles Two angles that lie in the same plane and have a common vertex and a common side but have no common interior points.

adjoint arcs Arcs in a circle that have exactly one point in common.

algebraic expression A mathematical expression that contains at least one variable.

algebraic notation Mathematical notation that describes a set by using algebraic expressions.

alternate exterior angles When two lines are cut by a transversal, nonadjacent exterior angles that lie on opposite sides of the transversal.

alternate interior angles When two lines are cut by a transversal, nonadjacent interior angles that lie on opposite sides of the transversal.

altitude of a parallelogram A perpendicular segment between any two parallel bases.

Español

triángulo 30°-60°-90° Un triángulo rectángulo con dos ángulos agudos que miden 30° y 60°.

triángulo 45°-45°-90° Un triángulo rectángulo con dos ángulos agudos que miden 45°.

valor absoluto La distancia que un número es de cero en la línea numérica.

función del valor absoluto Una función que se escribe

$$f(x) = |x|, \text{ donde } f(x) \geq 0, \text{ para todos los valores de } x.$$

exactitud La proximidad de una medida al valor verdadero de la medida.

identidad aditiva Debido a que la suma de cualquier número y 0 es igual a 0, 0 es la identidad aditiva.

inversos aditivos Dos números con una suma de 0.

ángulos adyacentes Dos ángulos que se encuentran en el mismo plano y tienen un vértice común y un lado común, pero no tienen puntos comunes en el interior.

arcos adyacentes Arcos en un círculo que tienen un solo punto en común.

expresión algebraica Una expresión matemática que contiene al menos una variable.

notación algebraica Notación matemática que describe un conjunto usando expresiones algebraicas.

ángulos alternos exteriores Cuando dos líneas son cortadas por un ángulo transversal, no adyacente exterior que se encuentran en lados opuestos de la transversal.

ángulos alternos interiores Cuando dos líneas son cortadas por un ángulo transversal, no adyacente interior que se encuentran en lados opuestos de la transversal.

altura de un paralelogramo Un segmento perpendicular entre dos bases paralelas.

Glossary G1

Glossary - Glosario

altitude of a prism or cylinder A segment perpendicular to the bases that joins the planes of the bases.	area The number of square units needed to cover a surface.	area El número de unidades cuadradas para cubrir una superficie.
altitude of a pyramid or cone A segment perpendicular to the base that lies the vertex as one endpoint and a point in the plane of the base as the other endpoint.	arithmetic sequence A pattern in which each term after the first is found by adding a constant, the common difference d , to the previous term.	secuencia aritmética Un patrón en el cual cada término después del primero se encuentra añadiendo una constante, la diferencia común d , al término anterior.
altitude of a triangle A segment from a vertex of the triangle to the line containing the opposite side and perpendicular to that side.	asymptote A line that a graph approaches.	asintota Una línea que se aproxima a un gráfico.
amblyopias When two different triangles could be created or described using the given information.	auxiliary line An extra line or segment drawn in a figure to help analyze geometric relationships.	línea auxiliar Una línea o segmento extra dibujado en una figura para ayudar a analizar las relaciones geométricas.
amplitude For functions of the form $y = a \sin bx$ or $y = a \cos bx$, the amplitude is $ a $.	average rate of change The change in the value of the dependent variable divided by the change in the value of the independent variable.	tasa media de cambio El cambio en el valor de la variable dependiente dividido por el cambio en el valor de la variable independiente.
analytic geometry The study of geometry that uses the coordinate system.	axiom A statement that is accepted as true without proof.	axioma Una declaración que se acepta como verdadera sin prueba.
angle The intersection of two noncollinear rays at a common endpoint.	axiomatic system A set of axioms from which theorems can be derived.	sistema axiomático Un conjunto de axiomas de los cuales se pueden derivar teoremas.
angle bisector A ray or segment that divides an angle into two congruent angles.	axis of symmetry A line about which a graph is symmetric.	eje de simetría Una línea sobre la cual se grafica es simétrica.
angle of depression The angle formed by a horizontal line and an observer's line of sight to an object below the horizontal line.	axis symmetry If a figure can be mapped onto itself by a rotation between 0° and 360° in a line.	eje simetría Si una figura puede ser asignada sobre sí misma por una rotación entre 0° y 360° en una línea.
angle of elevation The angle formed by a horizontal line and an observer's line of sight to an object above the horizontal line.	bar graph A graphical display that compares categories of data using bars of different heights.	gráfico de barras Una pantalla gráfica que compara las categorías de datos usando barras de diferentes alturas.
angle of rotation The angle through which a figure rotates.	base In a power, the number being multiplied by itself.	base En un poder, el número se multiplica por sí mismo.
apothem A perpendicular segment between the center and the midpoint of a side of the polygon or the length of that line segment.	base angles of a trapezoid The two angles formed by the bases and legs of a trapezoid.	ángulos de base de un trapecio Los dos ángulos formados por las bases y patas de un trapecio.
approximate error The positive difference between an actual measurement and an approximate or estimated measurement.	base angles of an isosceles triangle The two angles formed by the base and the congruent sides of an isosceles triangle.	ángulo de la base de un triángulo isósceles Los dos ángulos formados por la base y los lados congruentes de un triángulo isósceles.
arc Part of a circle that is defined by two endpoints.	base edge The intersection of a lateral face and a base in a solid figure.	arista de la base La intersección de una cara lateral y una base en una figura sólida.
arc length The distance between the endpoints of an arc measured along the arc in linear units.	base of a parallelogram Any side of a parallelogram.	base de un paralelogramo Cualquier lado de un paralelogramo.
	base of a pyramid or cone The face of the solid opposite the vertex of the solid.	base de una pirámide o cono La cara del sólido opuesta al vértice del sólido.

Glossary · Glosario	
bases de un prisma o cilindro The two parallel congruent faces of the solid. bases de un trapecio The parallel sides in a trapezoid. best-fit line The line that most closely approximates the data in a scatter plot. betweenness of points Point C is between A and B if and only if A, B, and C are collinear and $AC + CB = AB$. bias An error that results in a misrepresentation of a population. biconditional statement The conjunction of a conditional and its converse. binomial The sum of two monomials. bisect To separate a line segment into two congruent segments. bivariate data Data that consists of pairs of values. boundary The edge of the graph of an inequality that separates the coordinate plane into regions. bounded When the graph of a system of constraints is a polygonal region. box plot A graphical representation of the five-number summary of a data set.	bases de un prisma o cilindro Las dos caras congruentes paralelas de la figura sólida. bases de un trapecio Los lados paralelos en un trapecio. línea de ajuste óptimo La línea que más se aproxima a los datos en un diagrama de dispersión. intermedios de puntos El punto C está entre A y B si y sólo si A, B, y C son colineales y $AC + CB = AB$. s sesgo Un error que resulta en una tergiversación de una población. declaración bicondicional La conjunción de un condicional y su inverso. binomio La suma de dos monomios. bisecar Separar un segmento de línea en dos segmentos congruentes. datos bivariables Datos que constan de pares de valores. frontera El borde de la gráfica de una desigualdad que separa el plano de coordenadas en regiones. acotada Cuando la gráfica de un sistema de restricciones es una región poligonal. diagrama de caja Una representación gráfica del resumen de cinco números de un conjunto de datos.
categorical data Data that can be organized into different categories. causation When a change in one variable produces a change in another variable. center of a circle The point from which all points on a circle are the same distance. center of a regular polygon The center of the circle circumscribed about a regular polygon. center of dilation The center point from which dilations are performed.	datos categóricos Datos que pueden organizarse en diferentes categorías. causalidad Cuando un cambio en una variable produce un cambio en otra variable. centro de un círculo El punto desde el cual todos los puntos de un círculo están a la misma distancia. centro de un polígono regular El centro del círculo circunscrito alrededor de un polígono regular. centro de dilatación Punto fijo en torno al cual se realizan las homotecias.
center of rotation The fixed point about which a figure rotates. center of symmetry A point in which a figure can be rotated onto itself. central angle of a circle An angle with a vertex at the center of a circle and sides that are radii. central angle of a regular polygon An angle with its vertex at the center of a regular polygon and sides that pass through consecutive vertices of the polygon. centroid The point of concurrency of the medians of a triangle. chord of a circle or sphere A segment with endpoints on the circle or sphere. circle The set of all points in a plane that are the same distance from a given point called the center. circular function A function that describes a point on a circle as the function of an angle defined in radians. circumcenter The point of concurrency of the perpendicular bisectors of the sides of a triangle. circumference The distance around a circle. circumscribed angle An angle with sides that are tangent to a circle. circumscribed polygon A polygon with vertices outside the circle and sides that are tangent to the circle. closed If for any members in a set, the result of an operation is also in the set. closed half-plane The solution of a linear inequality that includes the boundary line. codomain The set of all the y-values that could possibly result from the evaluation of the function. coefficient The numerical factor of a term. coefficient of determination An indicator of how well a function fits a set of data.	centro de rotación El punto fijo sobre el que gira una figura. centro de la simetría Un punto en el que una figura se puede girar sobre sí misma. ángulo central de un círculo Un ángulo con un vértice en el centro de un círculo y los lados que son radios. ángulo central de un polígono regular Un ángulo con su vértice en el centro de un polígono regular y lados que pasan a través de vértices consecutivos del polígono. baricentro El punto de intersección de las medianas de un triángulo. cuerda de un círculo o esfera Un segmento con extremos en el círculo o esfera. círculo El conjunto de todos los puntos en un plano que están a la misma distancia de un punto dado llamado centro. función circular Función que describe un punto en un círculo como la función de un ángulo definido en radianes. circuncentro El punto de concurrencia de las bisectrices perpendiculares de los lados de un triángulo. circunferencia La distancia alrededor de un círculo. ángulo circunscrito Un ángulo con lados que son tangentes a un círculo. polígono circunscrito Un polígono con vértices fuera del círculo y lados que son tangentes al círculo. cerrado Si para cualquier número en el conjunto, el resultado de la operación es también en el conjunto. semi-plano cerrado La solución de una desigualdad lineal que incluye la línea de límite. codominio El conjunto de todos los valores y que podrían resultar de la evaluación de la función. coeficiente El factor numérico de un término. coeficiente de determinación Un indicador de lo bien que una función se ajusta a un conjunto de datos.

Glossary · Glosario

figura compuesta Una figura que se puede separar en regiones que son figuras básicas, tales como triángulos, rectángulos, trapecios, y círculos.

figura tridimensional Una figura tridimensional que se compone de figuras más simples.

función de transformación Operación que utiliza los resultados de una función para evaluar una segunda función.

evento compuesto Dos o más eventos simples.

desigualdad compuesta Dos o más desigualdades que están unidas por las palabras *y* o *e*.

interés compuesto Intereses calculados sobre el principal y sobre el interés acumulado de períodos anteriores.

enunciado compuesto Dos o más declaraciones unidas por la palabra *y* o *e*.

polígono cóncavo Un polígono con uno o más ángulos interiores con medidas superiores a 180° .

círculos concéntricos Círculos coplanarios que tienen el mismo centro.

conclusión La declaración que inmediatamente sigue la palabra *entonces* en un condicional.

líneas concurrentes Tres o más líneas que se intersecan en un punto común.

probabilidad condicional La probabilidad de que un evento ocurra dado que otro evento ya ha ocurrido.

frecuencia relativa condicional La relación entre la frecuencia de la afirmación y la frecuencia marginal.

enunciado condicional Una declaración compuesta que consiste en una premisa, o hipótesis, y una conclusión, que es falsa solo cuando su premisa es verdadera y su conclusión es falsa.

región compuesta A figure that can be separated into regions that are basic figures, such as triangles, rectangles, trapezoids, and circles.

figura tridimensional A three-dimensional figure that is composed of simpler solids.

función de transformación An operation that uses the results of one function to evaluate a second function.

composición de transformaciones When a transformation is applied to a figure and then another transformation is applied to its image.

evento compuesto Two or more simple events.

desigualdad compuesta Two or more inequalities that are connected by the words *and* or *or*.

interés compuesto Interest calculated on the principal and on the accumulated interest from previous periods.

enunciado compuesto Two or more statements joined by the word *and* or *or*.

polígono cóncavo A polygon with one or more interior angles with measures greater than 180° .

círculos concéntricos Coplanar circles that have the same center.

conclusión The statement that immediately follows the word *then* in a conditional.

líneas concurrentes Three or more lines that intersect at a common point.

probabilidad condicional The probability that an event will occur given that another event has already occurred.

frecuencia relativa condicional The ratio of the joint frequency to the marginal frequency.

enunciado condicional A compound statement that consists of a premise, or hypothesis, and a conclusion, which is false only when its premise is true and its conclusion is false.

identidades de adición Identidades que muestran las relaciones entre seno y coseno, tangente y cotangente, y secante y cosecante.

colinear Acostado en la misma línea.

combinación Una selección de objetos en los que el orden no es importante.

variación combinada Cuando una cantidad varía directamente y/o inversamente como dos o más cantidades.

diferencia común La diferencia entre términos consecutivos de una secuencia aritmética.

logaritmos comunes Logaritmos de base 10.

razón común El razón de términos consecutivos de una secuencia geométrica.

tangente común Una línea o segmento que es tangente a dos círculos en el mismo punto.

complemento de A Todos los resultados en el espacio muestral que no se incluyen como resultados del evento A.

ángulos complementarios Dos ángulos con medidas que tienen una suma de 90° .

completar el cuadrado Un proceso usado para hacer una expresión cuadrática en un trinomio cuadrado perfecto.

conjugados complejos Dos números complejos de la forma $a + bi$ y $a - bi$.

fracción compuesta Una expresión racional con un numerador y/o denominador que también es una expresión racional.

número complejo Cualquier número que se puede escribir en la forma $a + bi$, donde a y b son números reales y i es la unidad imaginaria.

forma de componente Un vector escrito como $\langle x, y \rangle$, que describe el vector en términos de su componente horizontal x y componente vertical y .

colocación de identidades Identidades que muestran las relaciones entre seno y coseno, tangente y cotangente, y secante y cosecante.

colinear Lying on the same line.

combinación A selection of objects in which order is not important.

variación combinada When one quantity varies directly and/or inversely as two or more other quantities.

común diferencia The difference between consecutive terms in an arithmetic sequence.

común logaritmo Logarithms of base 10.

común ratio The ratio of consecutive terms of a geometric sequence.

común tangente A line or segment that is tangent to two circles in the same point.

complemento de A All of the outcomes in the sample space that are not included as outcomes of event A.

complementarios ángulos Two angles with measures that have a sum of 90° .

completar el cuadrado A process used to make a quadratic expression into a perfect square trinomial.

conjugados complejos Two complex numbers of the form $a + bi$ and $a - bi$.

fracción compuesta A rational expression with a numerator and/or denominator that is also a rational expression.

número complejo Any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit.

forma de componente A vector written as $\langle x, y \rangle$, which describes the vector in terms of its horizontal component x and vertical component y .

Glossary · Glosario

coarse The ratio of the length of the leg adjacent to an angle to the length of the hypotenuse.	coarse Relación entre la longitud de la pata adyacente a un ángulo y la longitud de la hipotenusa.
contangent The ratio of the length of the leg adjacent to an angle to the length of the leg opposite the angle.	contangente La relación entre la longitud de la pata adyacente a un ángulo y la longitud de la pata opuesta a un ángulo.
conteminal angles Angles in standard position that have the same terminal side.	ángulos coterminales Ángulos en posición estándar que tienen el mismo lado terminal.
countereample An example that contradicts the conjecture showing that the conjecture is not always true.	contrajemplo Un ejemplo que contradice la conjetura que muestra que la conjetura no siempre es cierta.
critical values The x-values corresponding to the most common degrees of certainty.	valores críticos Los valores x correspondientes a los grados de certeza más comunes.
cross section The intersection of a solid and a plane.	sección transversal Intersección de un sólido con un plano.
cube root One of three equal factors of a number.	raíz cúbica Uno de los tres factores iguales de un número.
cube root function A radical function that contains the cube root of a variable expression.	función de la raíz del cubo Función radical que contiene la raíz cúbica de una expresión variable.
curve fitting Finding a regression equation for a set of data that is approximated by a function.	ajuste de curvas Encontrar una ecuación de regresión para un conjunto de datos que se aproxima por una función.
cycle One complete pattern of a periodic function.	ciclo Un patrón completo de una función periódica.
cylinder A solid figure with two congruent and parallel circular bases connected by a curved surface.	cilindro Una figura sólida con dos bases circulares congruentes y paralelas conectadas por una superficie curva.
decay factor The base of an exponential expression, $0 < 1 - r$.	factor de decaimiento La base de una expresión exponencial, $0 < 1 - r$.
decomposition Separating a figure into two or more nonoverlapping parts.	descomposición Separar una figura en dos o más partes que no se solapan.
decreasing When the graph of a function goes down when viewed from left to right.	decreciente Cuando la gráfica de una función disminuye cuando se ve de izquierda a derecha.
deductive argument An argument that guarantees the truth of the conclusion provided that its premises are true.	argumento deductivo Un argumento que garantiza la verdad de la conclusión siempre que sus premisas sean verdaderas.
deductive reasoning The process of reaching a specific conclusion based on general facts, rules, definitions, or properties.	razonamiento deductivo El proceso de alcanzar una conclusión específica basado en hechos generales, reglas, definiciones, o propiedades.
define a variable To choose a variable to represent an unknown value.	definir una variable Para elegir una variable que represente un valor desconocido.
defined term A term that has a definition and can be explained.	término definido Un término que tiene una definición y se puede explicar.
definitions An explanation that assigns properties to a mathematical object.	definiciones Una explicación que asigna propiedades a un objeto matemático.
degree The value of the exponent in a power function; 360° of the circular rotation about a point.	grado Valor del exponente en una función de potencia; 360° de la rotación circular alrededor de un punto.
degree of a monomial The sum of the exponents of all its variables.	grado de un monomio La suma de los exponentes de todas sus variables.
degree of a polynomial The greatest degree of any term in the polynomial.	grado de un polinomio El grado mayor de cualquier término del polinomio.
density A measure of the quantity of some physical property per unit of length, area, or volume.	densidad Una medida de la cantidad de alguna propiedad física por unidad de longitud, área o volumen.
dependent A consistent system of equations with an infinite number of solutions.	dependiente Un sistema consistente de ecuaciones con un número infinito de soluciones.
dependent events Two or more events in which the outcome of one event affects the outcome of the other events.	eventos dependientes Dos o más eventos en que el resultado de un evento afecta el resultado de los otros eventos.
dependent variable The variable in a relation, usually y , with values that depend on x .	variable dependiente La variable de una relación, generalmente y , con los valores que dependen de x .
depressed polynomial A polynomial resulting from division with a degree one less than the original polynomial.	polinomio reducido Un polinomio resultante de la división con un grado uno menos que el polinomio original.
descriptive modeling A way to mathematically describe real-world situations and the factors that cause them.	modelado descriptivo Una forma de describir matemáticamente las situaciones del mundo real y los factores que las causan.
descriptive statistics The branch of statistics that focuses on collecting, summarizing, and displaying data.	estadística descriptiva Rama de la estadística cuyo enfoque es la recopilación, resumen y demostración de los datos.
diagonal A segment that connects any two nonconsecutive vertices within a polygon.	diagonal Un segmento que conecta cualquier dos vértices no consecutivos dentro de un polígono.

diameter of a circle or sphere A chord that passes through the center of a circle or sphere.	diámetro de un círculo o esfera Un acorde que pasa por el centro de un círculo o esfera.
difference of squares A binomial in which the first and last terms are perfect squares.	diferencia de cuadrados Un binomio en el que los términos primero y último son cuadrados perfectos.
difference of two squares The square of one quantity minus the square of another quantity.	diferencia de dos cuadrados El cuadrado de una cantidad menos el cuadrado de otra cantidad.
dilation A worded definition that enlarges or reduces a geometric figure. A transformation that stretches or compresses the graph of a function.	dilatación Un enunciado no lógico que agranda o reduce una figura geométrica. Una transformación que estira o comprime el gráfico de una función.
dimensional analysis The process of performing operations with units.	análisis dimensional El proceso de realizar operaciones con unidades.
direct variation When one quantity is equal to a constant times another quantity.	variación directa Cuando una cantidad es igual a una constante multiplicada por otra cantidad.
discrete line segment A line segment with an initial endpoint and a terminal endpoint.	segmento de línea discreto Un segmento de línea con un punto final inicial y un punto final terminal.
discrete An exterior line perpendicular to the line containing the foci of a curve.	discreto Una línea exterior perpendicular a la línea que contiene los focos de una curva.
discontinuous function A function that is not continuous.	función discontinua Una función que no es continua.
discrete function A function in which the points on the graph are not connected.	función discreta Una función en la que los puntos del gráfico no están conectados.
discrete random variable The numerical outcome of a random event that is finite and can be counted.	variable aleatoria discreta El resultado numérico de un evento aleatorio que es finito y puede ser contado.
discriminant In the Quadratic Formula, the expression $b^2 - 4ac$ that determines the nature of the information about the roots of the quadratic equation.	discriminante En la Fórmula cuadrática, la expresión $b^2 - 4ac$ que determina la información sobre las raíces de la ecuación cuadrática.
disjunction A compound statement using the word or.	disyunción Una declaración compuesta usando la palabra o.
distance The length of the line segment between two points.	distancia La longitud del segmento de línea entre dos puntos.
distribution A graph or table that shows the theoretical frequency of each possible data value.	distribución Un gráfico o una tabla que muestra la frecuencia teórica de cada valor de dato posible.
domain The set of the first numbers of the ordered pairs in a relation. The set of x -values to be evaluated by a function.	dominio El conjunto de los primeros números de los pares ordenados en una relación. El conjunto de valores x para ser evaluados por una función.

dot plot A diagram that shows the frequency of data on a number line.	gráfica de puntos Una diagrama que muestra la frecuencia de los datos en una línea numérica.
double root They roots of a quadratic equation that are the same number.	raíces dobles Dos raíces de una función cuadrática que son el mismo número.
edge of a polygon A line segment where the faces of the polyhedron intersect.	arista de un poliedro Un segmento de línea donde las caras del poliedro se cruzan.
elimination A method that involves eliminating a variable by combining the individual equations within a system of equations.	eliminación Un método que consiste en eliminar una variable combinando las ecuaciones individuales dentro de un sistema de ecuaciones.
empty set The set that contains no elements, symbolized by $\{\}$ or $\emptyset$.	conjunto vacío El conjunto que no contiene elementos, simbolizado por $\{\}$ o $\emptyset$.
end behavior The behavior of a graph at the positive and negative extremes in its domain.	comportamiento extremo El comportamiento de un gráfico en los extremos positivo y negativo en su dominio.
enlargement A dilation with a scale factor greater than 1.	ampliación Una dilatación con un factor de escala mayor que 1.
equation A mathematical statement that contains two expressions and an equal sign, $=$.	ecuación Un enunciado matemático que contiene dos expresiones y un signo igual, $=$.
equiangular polygon A polygon with all angles congruent.	polígono equiangular Un polígono con todos los ángulos congruentes.
equidistant A point is equidistant from other points if it is the same distance from them.	equidistante Un punto es equidistante de otros puntos si está a la misma distancia de ellos.
equidistant lines Two lines for which the distance between the two lines, measured along a perpendicular line or segment to the two lines, is always the same.	líneas equidistantes Dos líneas para las cuales la distancia entre las dos líneas, medida a lo largo de una línea o segmento perpendicular a las dos líneas, es siempre la misma.
equilateral polygon A polygon with all sides congruent.	polígono equilátero Un polígono con todos los lados congruentes.
equivalent equations Two equations with the same solution.	ecuaciones equivalentes Dos ecuaciones con la misma solución.
equivalent expressions Expressions that represent the same value.	expresiones equivalentes Expresiones que representan el mismo valor.
evaluate To find the value of an expression.	evaluar Calcular el valor de una expresión.

Glossary - Glosario

even functions Functions that are symmetric in the y -axis.	funciones pares Funciones que son simétricas en el eje y .	desigualdad exponencial Una desigualdad en la que la variable independiente es un exponente.
event A subset of the sample space.	evento Un subconjunto del espacio de muestra.	ángulo exterior de un triángulo Un ángulo formado por un lado del triángulo y la extensión de un lado adyacente.
excluded values Values for which a function is not defined.	valores excluidos Valores para los que no se ha definido una función.	ángulos externos Cuando dos líneas son cortadas por una transversal, cualquiera de los cuatro ángulos que se encuentran fuera de la región entre las dos líneas interiores.
experimental group A sample is divided into two groups. The experimental group undergoes a change, while there is no change to the control group. The effects on the groups are then compared. A situation involving chance.	experimento Una muestra se divide en dos grupos. El grupo experimental experimenta un cambio, mientras que no hay cambio en el grupo de control. A continuación se comparan los efectos sobre los grupos. Una situación de riesgo.	exterior de un ángulo El área fuera de los dos rayos de un ángulo.
experimental probability Probability calculated by using data from an actual experiment.	probabilidad experimental Probabilidad calculada utilizando datos de un experimento real.	solución extrínseca Una solución de una forma simplificada de una ecuación que no satisface la ecuación original.
exponent When n is a positive integer is the expression x^n indicates the number of times x is multiplied by itself.	exponente Cuando n es un entero positivo en la expresión x^n , n indica el número de veces que x se multiplica por sí mismo.	extremo Puntos que son las subsecciones de valores de función relativamente alta o baja.
exponential decay Change that occurs when an initial amount decreases by the same percent over a given period of time.	deintegración exponencial Cambio que ocurre cuando una cantidad inicial disminuye en el mismo porcentaje durante un período de tiempo dado.	valores extremos Los valores mínimo y máximo en un conjunto de datos.
exponential decay function A function in which the independent variable is an exponent, where $a > 0$ and $0 < b < 1$.	función exponencial de decaimiento Una ecuación en la que la variable independiente es un exponente, donde $a > 0$ y $0 < b < 1$.	cara de un poliedro Superficie plana de un poliedro.
explicit formula A formula that allows you to find any term a_n of a sequence by using a formula written in terms of n .	fórmula explícita Una fórmula que le permite encontrar cualquier término a_n de una secuencia usando una fórmula escrita en términos de n .	forma factorizada Una forma de ecuación cuadrática, $0 = ax^2 + bx + c$, donde $a \neq 0$, en la que p y q son las intersecciones x de la gráfica de la función relacionada.
exponential equation An equation in which the independent variable is an exponent.	ecuación exponencial Una ecuación en la que la variable independiente es un exponente.	factorial de n El producto de los enteros positivos inferiores o iguales a n .
exponential form When an expression is in the form x^n .	forma exponencial Cuando una expresión es la forma x^n .	factorización por agrupamiento Utilizando la propiedad distributiva para factorizar polinomios que poseen cuatro o más términos.
exponential function A function in which the independent variable is an exponent.	función exponencial Una función en la que la variable independiente es el exponente.	factorización El proceso de expresar un polinomio como el producto de monomios y polinomios.
exponential growth Change that occurs when an initial amount increases by the same percent over a given period of time.	crecimiento exponencial Cambio que ocurre cuando una cantidad inicial aumenta por el mismo porcentaje durante un período de tiempo dado.	familia de gráficas Gráficas y ecuaciones de gráficas que tienen al menos una característica común.
exponential growth function A function in which the independent variable is an exponent, where $a > 0$ and $b > 1$.	función de crecimiento exponencial Una función en la que la variable independiente es el exponente, donde $a > 0$ y $b > 1$.	región factible La intersección de los gráficos en un sistema de restricciones.
		espacio de muestra finito Un espacio de muestra que contiene un número contable de resultados.

Glossary 015

014 Glossary

reflexión del deslizamiento La composición de una traslación seguida de una reflexión en un eje es paralela al vector de traslación.

función entera más grande Una función del tipo $f(x) = \lfloor x \rfloor$ es el número más grande menor que o igual a x .

factor de crecimiento La base de una expresión exponencial, a o $1 + r$.

H

semi-plano Una región de la gráfica de una desigualdad en un lado de un límite.

altura de un paralelogramo La longitud de la altura del paralelogramo.

altura de un sólido La longitud de la altura de una figura sólida.

altura de un triángulo La distancia perpendicular entre las bases de un triángulo.

histograma Una exhibición gráfica que utiliza barras paralelas para mostrar los datos que se han organizado en intervalos iguales.

asíntota horizontal Una línea horizontal que se aproxima a un gráfico.

hipérbola La gráfica de una función recíproca.

hipótesis La declaración que sigue inmediatamente a la palabra *if* en un condicional.

I

identidad Una ecuación que es verdad para cada valor de la variable.

función identidad La función $f(x) = x$.

enunciado si-entonces Enunciado compuesto de la forma *if p, entonces q*, donde *p* y *q* son enunciados.

imagen La nueva figura en una transformación.

unidad imaginaria i La raíz cuadrada principal de -1 .

intersección El punto de intersección de las bisectrices interiores de un triángulo.

glide reflection The composition of a translation followed by a reflection in a line parallel to the translation vector.

greatest integer function A step function f which $f(x)$ is the greatest integer less than or equal to x .

growth factor The base of an exponential expression, a or $1 + r$.

half-plane A region of the graph of an inequality on one side of a boundary.

height of a parallelogram The length of an altitude of the parallelogram.

height of a solid The length of the altitude of a solid figure.

height of a triangle The perpendicular distance between the bases of a triangle.

histogram A graphical display that uses bars to display numerical data that have been organized in equal intervals.

horizontal asymptote A horizontal line that a graph approaches.

hyperbola The graph of a reciprocal function.

hypothesis The statement that immediately follows the word *if* in a conditional.

identity An equation that is true for every value of the variable.

identity function The function $f(x) = x$.

if-then statement A compound statement of the form *if p, then q*, where *p* and *q* are statements.

image The new figure in a transformation.

imaginary unit i The principal square root of -1 .

incenter The point of concurrency of the angle bisectors of a triangle.

secuencia finita Una secuencia que contiene un número limitado de términos.

resumen de cinco números El mínimo, cuartiles y máximo de un conjunto de datos.

demostración de flujo Una prueba que usa cajas y flechas para mostrar la progresión lógica de un argumento.

foco Un punto dentro de una parábola que tiene la propiedad de que las distancias desde cualquier punto de la parábola a ellos y a una línea fija tienen una relación constante para cualquier punto de la parábola.

fórmula Una ecuación que expresa una relación entre ciertas cantidades.

distancia fraccional Un punto intermedio de alguna fracción de la longitud de un segmento de línea.

frecuencia El número de ciclos en una unidad del tiempo dado.

función Una relación en la que a cada elemento del dominio de correspondencia le corresponde un único elemento del rango.

notación funcional Una forma de escribir una ecuación para que $y = f(x)$.

medios geométricos Los términos entre dos términos no consecutivos de una secuencia geométrica. La n -ésima raíz, donde n es el número de elementos de un conjunto de números, del producto de los números.

modelo geométrico Una figura geométrica que representa un objeto de la vida real.

probabilidad geométrica Probabilidad que implica una medida geométrica como longitud o área.

secuencia geométrica Un patrón de números que comienza con un término distinto de cero y cada término de aquí se encuentra multiplicando el término anterior por una constante no nula r .

series geométricas La suma indicada de los términos en una secuencia geométrica.

finite sequence A sequence that contains a limited number of terms.

five-number summary The minimum, quartiles, and maximum of a data set.

flow proof A proof that uses boxes and arrows to show the logical progression of an argument.

focus A point inside a parabola having the property that the distance from any point on the parabola to them and to a fixed line have a constant ratio for any point on the parabola.

formula An equation that expresses a relationship between certain quantities.

fractional distance An intermediate point some fraction of the length of a line segment.

frequency The number of cycles in a given unit of time.

function A relation in which each element of the domain is paired with exactly one element of the range.

function notation A way of writing an equation so that $y = f(x)$.

geometric means The terms between two nonconsecutive terms of a geometric sequence; the n th root, where n is the number of elements in a set of numbers, of the product of the numbers.

geometric model A geometric figure that represents a real-life object.

geometric probability Probability that involves a geometric measure such as length or area.

geometric sequence A pattern of numbers that begins with a nonzero term and each term after is found by multiplying the previous term by a nonzero constant r .

geometric series The indicated sum of the terms in a geometric sequence.

Glossary - Glosario		Glossary - Glosario	
included angle The interior angle formed by two adjacent sides of a triangle.	ángulo incluido El ángulo interior formado por dos lados adyacentes de un triángulo.	inferential proof A paragraph that explains why the conjecture for a given situation is true.	prueba infernal Un párrafo que explica por qué la conjetura para una situación dada es verdadera.
included side The side of a triangle between two angles.	lado incluido El lado de un triángulo entre dos ángulos.	initial side The part of an angle that is fixed on the x-axis.	lado inicial La parte de un ángulo que se fija en el eje x.
inconsistent A system of equations with no ordered pair that satisfies both equations.	inconsistente Un sistema de ecuaciones para el cual no existe par ordenado alguno que satisfaga ambas ecuaciones.	inscribed angle An angle with its vertex on a circle and sides that contain chords of the circle.	ángulo inscrito Un ángulo con su vértice en un círculo y lados que contienen arcos del círculo.
increasing Where the graph of a function goes up when viewed from left to right.	creciente Donde la gráfica de una función sube cuando se ve de izquierda a derecha.	inscribed polygon A polygon inside a circle in which all of the vertices of the polygon lie on the circle.	polygono inscrito Un polígono dentro de un círculo en el que todos los vértices del polígono se encuentran en el círculo.
independent A consistent system of equations with exactly one solution.	independiente Un sistema consistente de ecuaciones con exactamente una solución.	intercept A point at which the graph of a function intersects an axis.	interceptar Un punto en el que la gráfica de una función corta un eje.
independent events Two or more events in which the outcome of one event does not affect the outcome of the other events.	eventos independientes Dos o más eventos en los que el resultado de un evento no afecta el resultado de los otros eventos.	intercepted arc The part of a circle that lies between the two lines intersecting it.	arco interceptado La parte de un círculo que se encuentra entre las dos líneas que se cruzan.
independent variable The variable in a relation, usually x, with a value that is subject to choice.	variable independiente La variable de una relación, generalmente x, con el valor que sujeta a elección.	interior angle of a triangle An angle at the vertex of a triangle.	ángulo interior de un triángulo Un ángulo en el vértice de un triángulo.
index In r/n roots, the value that indicates to what root the value under the radical is being taken.	índice En radicandos raíces, el valor que indica a qué raíz está el valor bajo la raíz cuadrada.	interior angles When two lines are cut by a transversal, any of the four angles that lie inside the region between the two intersected lines.	ángulos interiores Cuando dos líneas son cortadas por una transversal, cualquiera de los cuatro ángulos que se encuentran dentro de la región entre las dos líneas intersectadas.
indirect measurement Using similar figures and proportions to measure an object.	medición indirecta Usando figuras y proporciones similares para medir un objeto.	interior of an angle The area between the two rays of an angle.	interior de un ángulo El área entre los dos rayos de un ángulo.
indirect proof One assumes that the statement to be proved is false and then shows that this leads to a contradiction or a statement that contradicts a postulate, theorem, or one of the assumptions.	demonstración indirecta Se supone que la afirmación que se quiere demostrar es falsa y se muestra que esto conduce a una contradicción o a una afirmación contraria a un postulado, teorema o uno de los supuestos.	interquartile range The difference between the upper and lower quartiles of a data set.	 rango intercuartil La diferencia entre el cuartil superior y el cuartil inferior de un conjunto de datos.
indirect reasoning Reasoning that eliminates all possible conclusions but one so that the one remaining conclusion must be true.	razonamiento indirecto Razonamiento que elimina todas las posibles conclusiones, pero una de ellas, de manera que la conclusión que queda debe ser verdadera.	inductive reasoning The process of reaching a conclusion based on a pattern of examples.	inferencia Un conjunto de puntos comunes a dos o más figuras geométricas: inferencia. La gráfica de una desigualdad compuesta que contiene la palabra y.
inequality A mathematical sentence that contains $<$, $>$, $\geq$, $\leq$, or $\neq$.	desigualdad Una expresión matemática que contiene uno o más de $<$, $>$, $\geq$, $\leq$, o $\neq$.	intersection of A and B The set of all outcomes in the sample space of event A that are also in the sample space of event B.	intersección de A y B El conjunto de todos los resultados en el espacio muestral del evento A que también se encuentran en el espacio muestral del evento B.
inferential statistics When the data from a sample is used to make inferences about the corresponding population.	estadísticas inferenciales Cuando los datos de una muestra se utilizan para hacer inferencias sobre la población correspondiente.	interval The distance between two numbers on the scale of a graph.	intervalo La distancia entre dos números en la escala de un gráfico.
infinite sample space A sample space with outcomes that cannot be counted.	espacio de muestra infinito Un espacio de muestra con resultados que no pueden ser contados.	interval notation Mathematical notation that describes a set by using endpoints with parentheses or brackets.	notación de intervalo Notación matemática que describe un conjunto utilizando puntos finales con paréntesis o corchetes.
infinite sequence A sequence that continues without end.	secuencia infinita Una secuencia que continúa sin fin.	inverse A statement formed by negating both the hypotheses and conclusion of a conditional statement.	inverso Una declaración formada negando tanto la hipótesis como la conclusión de la declaración condicional.

Glossary · Glosario	
inverse cosine The ratio of the length of the hypotenuse to the length of the leg adjacent to an angle. inverse functions Two functions, one of which contains points of the form (a, b) while the other contains points of the form (b, a). inverse relations Two relations, one of which contains points of the form (a, b) while the other contains points of the form (b, a). inverse sine The ratio of the length of the hypotenuse to the length of the leg opposite an angle. inverse tangent The ratio of the length of the leg adjacent to an angle to the length of the leg opposite the angle. inverse trigonometric functions Arcsine, Arccosine, and Arctangent. inverse variation When the product of two quantities is equal to a constant k. isosceles trapezoid A quadrilateral in which two sides are parallel and the legs are congruent. isosceles triangle A triangle with at least two sides congruent.	inverso del coseno Relación de la longitud de la hipotenusa con la longitud de la pierna adyacente a un ángulo. funciones inversas Dos funciones, una de las cuales contiene puntos de la forma (a, b) mientras que la otra contiene puntos de la forma (b, a). relaciones inversas Dos relaciones, una de las cuales contiene puntos de la forma (a, b) mientras que la otra contiene puntos de la forma (b, a). inverso del seno Relación de la longitud de la hipotenusa con la longitud de la pierna opuesta a un ángulo. inverso del tangente Relación de la longitud de la pierna adyacente a un ángulo con la longitud de la pierna opuesta a un ángulo. funciones trigonométricas inversas Arcsine, Arccosine y Arctangent. variación inversa Cuando el producto de dos cantidades es igual a una constante k. trapecio isósceles Un cuadrilátero en el que dos lados son paralelos y sus patas son congruentes. triángulo isósceles Un triángulo con al menos dos lados congruentes.
joint frequencies Entries in the body of a two-way frequency table. In a two-way frequency table, the frequencies in the interior of the table. joint variation When one quantity varies directly as the product of two or more other quantities. kite A convex quadrilateral with exactly two distinct pairs of adjacent congruent sides. lateral area The sum of the areas of the lateral faces of the figure.	frecuencias articulares Entradas en el cuerpo de una tabla de frecuencias de dos vías. En una tabla de frecuencia bidimensional, las frecuencias en el interior de la tabla. variación conjunta Cuando una cantidad varía directamente como el producto de dos o más cantidades. cometa Un cuadrilátero convexo con exactamente dos pares de lados adyacentes congruentes. área lateral La suma de las áreas de las caras laterales de la figura.
lateral edges The intersection of two lateral faces. lateral faces The face that joins the bases of a solid. lateral surface of a cone The curved surface that joins the base of a cone to the vertex. lateral surface of a cylinder The curved surface that joins the bases of a cylinder. leading coefficient The coefficient of the first term when a polynomial is in standard form. legs of a trapezoid The nonparallel sides in a trapezoid. legs of an isosceles triangle The two congruent sides of an isosceles triangle. like radical expressions Radicals in which both the index and the radicand are the same. like terms Terms with the same variables, with corresponding variables having the same exponent. line A line is made up of points, but so thick lines or width, and extends indefinitely in both directions. line of fit A line used to describe the trend of the data in a scatter plot. line of reflection A line midway between a preimage and an image. The line in which a reflection flips the graph of a function. line of symmetry An imaginary line that separates a figure into two congruent parts. line segment A measurable part of a line that consists of two points, called endpoints, and all of the points between them. line symmetry A graph has line symmetry if it can be reflected in a vertical line so that each half of the graph maps exactly to the other half. linear equation An equation that can be written in the form $Ax + By = C$ with a graph that is a straight line.	aristas laterales La intersección de dos caras laterales. caras laterales Las caras que unen las bases de un sólido. superficie lateral de un cono La superficie curva que une la base de un cono con el vértice. superficie lateral de un cilindro La superficie curva que une las bases de un cilindro. coeficiente líder El coeficiente del primer término cuando un polinomio está en forma estándar. patas de un trapecio Los lados no paralelos en un trapecio. patas de un triángulo isósceles Los dos lados congruentes de un triángulo isósceles. expresiones radicales semejantes Radicales en los que tanto el índice como el radicando son iguales. términos semejantes Términos con las mismas variables, con las variables correspondientes que tienen el mismo exponente. línea Una línea está formada por puntos, no tiene espesor ni anchura, y se extiende indefinidamente en ambas direcciones. línea de ajuste Una línea usada para describir la tendencia de los datos en un diagrama de dispersión. línea de reflexión Una línea a medio camino entre una preimagen y una imagen. La línea en la que una reflexión volteó la gráfica de una función. línea de simetría Una línea imaginaria que separa una figura en dos partes congruentes. segmento de línea Una parte medible de una línea que consta de dos puntos, llamados extremos, y todos los puntos entre ellos. simetría de línea Un gráfico tiene simetría de línea si puede reflejarse en una línea vertical, de modo que cada mitad del gráfico se asigna exactamente a la otra mitad. ecuación lineal Una ecuación que puede escribirse de la forma $Ax + By = C$ con un gráfico que es una línea recta.

Glossary · Glosario

magnitud de la simetría The smallest angle through which a figure can be rotated so that it maps onto itself. *magnitude of symmetry*

arco mayor An arc with a measure greater than 180° . *major arc*

emparejamiento An illustration that shows how each element of the domain is paired with an element in the range. *mapping*

frecuencias marginales In a table of frequencies of two variables, the frequencies in the totals of the rows and columns. *marginal frequencies*

mediana The point in the range of a function. *maximum*

error máximo de la estimación The difference between the estimate of the population mean and its actual value. *maximum error of the estimate*

medidas de datos Measures of what is average. *measures of center*

medidas de dispersión Measures of how spread out the data are. *measures of spread*

mediana The beginning of the second quartile that separates the data into upper and lower halves. *median*

mediana de un triángulo A line segment with endpoints at a vertex of a triangle and the midpoint of the side opposite the vertex. *median of a triangle*

métrico A rule for assigning a number to some characteristic or attribute. *metric*

línea media The line about which the graph of a function oscillates. *midline*

midpoint The point on a line segment halfway between the endpoints of the segment. *midpoint*

midsegment of a trapezoid The segment that connects the midpoints of the legs of a trapezoid. *midsegment of a trapezoid*

midsegment of a triangle The segment that connects the midpoints of the legs of a triangle. *midsegment of a triangle*

minimum The lowest point on the graph of a function. *minimum*

extrapolación lineal The use of a linear equation to predict values that are outside the range of data. *linear extrapolation*

función lineal A function in which the independent variable is raised to a power greater than 1. A function with a graph that is a line. *linear function*

línea desigualdad A half-plane with a boundary that is a straight line. *linear inequality*

línea de intersección The use of a linear equation to predict values that are inside the range of data. *linear intersection*

par lineal A pair of adjacent angles with noncommon sides that are opposite rays. *linear pair*

programación lineal The process of finding the maximum or minimum value of a function for a region defined by a system of inequalities. *linear programming*

regresión lineal An algorithm used to find a precise line of fit for a set of data. *linear regression*

transformación lineal One or more operations performed on a set of data that can be written as a linear function. *linear transformation*

ecuación literal A formula or equation with several variables. *literal equation*

logaritmo In $x = b^y$, y is called the logarithm, base b , of x . *logarithm*

ecuación logarítmica An equation that contains one or more logarithms. *logarithmic equation*

función logarítmica A function of the form $f(x) = \log_b x$, where $b > 0$ and $b \neq 1$. *logarithmic function*

lógicamente equivalentes Statements with the same truth value. *logically equivalent*

cuartil inferior The median of the lower half of a set of data. *lower quartile*

magnitud The length of a vector from the initial point to the terminal point. *magnitude*

línea de intersección The use of a linear equation to predict values that are inside the range of data. *linear intersection*

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cuartil inferior The median of the lower half of a set of data. *lower quartile*

magnitud The length of a vector from the initial point to the terminal point. *magnitude*

minor arc An arc with measure less than 180° .

mixture problems Problems that involve creating a mixture of two or more kinds of things, and then determining some quantity of the resulting mixture.

monomial A number, a variable, or a product of a number and one or more variables.

monomial function A function of the form $f(x) = ax^n$, for which a is a nonzero real number and n is a positive integer.

multi-step equation An equation that uses more than one operation to solve it.

multiplicative identity Because the product of any number a and 1 is equal to a , 1 is the multiplicative identity.

multiplicative inverses Two numbers with a product of 1.

multiplicity The number of times a number is a zero for a given polynomial.

mutually exclusive Events that cannot occur at the same time.

negatively skewed distribution A distribution that typically has a median greater than the mean and less data on the left side of the graph.

net A two-dimensional figure that forms the surfaces of a three-dimensional object when folded.

no correlation Bivariate data in which x and y are not related.

nonlinear function A function in which a set of points cannot all lie on the same line.

nonrigid motion A transformation that changes the dimensions of a given figure.

normal distribution A continuous, symmetric, bell-shaped distribution of a random variable.

n th root If $a^n = b$ for a positive integer n , then a is the n th root of b .

n th term of an arithmetic sequence The n th term of an arithmetic sequence with first term a and common difference d is given by $a_n = a_1 + (n - 1)d$, where n is a positive integer.

numerical expression A mathematical phrase involving only numbers and mathematical operations.

distribución negativamente sesgada Una distribución que típicamente tiene una mediana a mayor que la media y menos datos en el lado izquierdo del gráfico.

red Una figura bidimensional que forma las superficies de un objeto tridimensional cuando se dobla.

sin correlación Datos bivariados en los que x e y no están relacionados.

función no lineal Una función en la que un conjunto de puntos no puede estar en la misma línea.

movimiento no rígido Una transformación que cambia las dimensiones de una figura dada.

distribución normal Distribución con forma de campana, simétrica y continua de una variable aleatoria.

raíz enésima Si $a^n = b$ para cualquier entero positivo n , entonces a se llama una raíz enésima de b .

enésimo término de una secuencia aritmética El enésimo término de una secuencia aritmética con el primer término a_1 y la diferencia común d viene dado por $a_n = a_1 + (n - 1)d$, donde n es un número entero positivo.

expresión numérica Una frase matemática que implica sólo números y operaciones matemáticas.

N

natural base exponential function An exponential function with base e , written as $y = e^x$.

natural logarithm The inverse of the natural base exponential function, most often abbreviated as $\ln x$.

negation A statement that has the opposite meaning, as well as the opposite truth value, of an original statement.

negative Where the graph of a function lies below the x -axis.

negative correlation Bivariate data in which y decreases as x increases.

negative exponent An exponent that is a negative number.

oblique asymptote An asymptote that is neither horizontal nor vertical.

observational study Members of a sample are observed or observed without being affected by the study.

octant One of the eight divisions of three-dimensional space.

odd functions Functions that are symmetric in the origin.

one-to-one function A function for which each element of the range is paired with exactly one element of the domain.

onto function A function for which the codomain is the same as the range.

oblique asintota Una asintota que no es ni horizontal ni vertical.

estudio de observación Los miembros de una muestra son observados o observados sin ser afectados por el estudio.

octante Una de las ocho divisiones del espacio tridimensional.

funciones extrañas Funciones que son simétricas en el origen.

función biunívoca Función para la cual cada elemento del rango está emparejado con exactamente un elemento del dominio.

sobre la función Función para la cual el codominio es el mismo que el rango.

Glossary · Glosario

open half plane The solution of a linear inequality that does not include the boundary line.	parámetro Una medida que describe una característica de una población. Un valor en la ecuación de una función que se puede variar para producir una familia de funciones.
opposite rays Two collinear rays with a common endpoint.	función básica La función más fundamental de un familia de funciones.
optimization The process of seeking the optimal value of a function subject to given constraints.	triángulo de Pascal Un triángulo de números en el que una fila representa los coeficientes de un binomio expandido $(x + y)^n$.
order of symmetry The number of lines a figure maps onto itself.	por ciento tasa de cambio El porcentaje de aumento por período de tiempo.
ordered triple Three numbers given in a specific order used to locate points in space.	percentil Una medida que indica qué porcentaje de las puntuaciones totales estaban por debajo de una puntuación determinada.
orthocenter The point of concurrency of the altitudes of a triangle.	cubo perfecto Un número racional con un raíz cúbica que es un número racional.
orthographic drawing The two-dimensional views of the top, left, front, and right sides of an object.	cuadrado perfecto Un número racional con un raíz cuadrada que es un número racional.
oxidation How much the graph of a function varies from the values as it approaches positive or negative infinity.	trinomio cuadrado perfecto Cuadrados de los binomios.
outcome The result of a single event; the result of a single performance or trial of an experiment.	perímetro La suma de las longitudes de los lados de un polígono.
outlier A value that is more than 1.5 times the interquartile range above the third quartile or below the first quartile.	período La longitud horizontal de un ciclo.
	función periódica Una función con y -valores aquella repetición con regularidad.
parabola A curved shape that results when a cone is cut at an angle by a plane that intersects the base. The graph of a quadratic function.	permutación Un arreglo de objetos en el que el orden es importante.
paragraph proof A paragraph that explains why the conjecture for a given situation is true.	perpendicular Intersección en ángulo recto.
parallel lines Coplanar lines that do not intersect. Nonvertical lines in the same plane that have the same slope.	mediatriz Cualquier línea, segmento o rayo que pasa por el punto medio de un segmento y es perpendicular a ese segmento.
parallel planes Planes that do not intersect.	líneas perpendiculares Líneas no verticales en el mismo plano para las que el producto de las pendientes es -1 .
parallelogram A quadrilateral with both pairs of opposite sides parallel.	cambio de fase Una traducción horizontal de la gráfica de una función trigonométrica.

Glossary · Glosario	
piecewise-defined function A function defined by at least two subfunctions, each of which is defined differently depending on the interval of the domain. piecewise linear function A function defined by at least two linear subfunctions, each of which is defined differently depending on the interval of the domain. plane A flat surface made up of points that has no depth and extends infinitely in all directions. plane symmetry When a plane intersects a three-dimensional figure so one half is the reflected image of the other half. platonic solid One of five regular polyhedra. point A location with no size, only position. point discontinuity An area that appears to be a hole in a graph. point of concurrency The point of intersection of concurrent lines. point of symmetry The point about which a figure is rotated. point of tangency For a line that intersects a circle in one point, the point at which they intersect. point symmetry A figure or graph has this when a figure is rotated 180° about a point and maps exactly onto the other part. polygon A closed plane figure with at least three straight sides. polyhedron A closed three-dimensional figure made up of flat polygonal regions. polynomial A monomial or the sum of two or more monomials. polynomial function A continuous function that can be described by a polynomial equation in one variable.	función definida por partes Una función definida por al menos dos subfunciones, cada una de las cuales se define de manera diferente dependiendo del intervalo del dominio. función lineal por partes Una función definida por al menos dos subfunciones lineales, cada una de las cuales se define de manera diferente dependiendo del intervalo del dominio. plano Una superficie plana compuesta de puntos que no tiene profundidad y se extiende infinitamente en todas las direcciones. simetría plana Cuando un plano cruza una figura tridimensional, una mitad es la imagen reflejada de la otra mitad. sólido platónico Uno de cinco poliedros regulares. punto Una ubicación sin tamaño, sólo posición. discontinuidad de punto Un área que parece ser un agujero en un gráfico. punto de concurrencia El punto de intersección de líneas concurrentes. punto de simetría El punto sobre el que se gira una figura. punto de tangencia Para una línea que cruza un círculo en un punto, el punto en el que se cruzan. simetría de punto Una figura o gráfica tiene esto cuando una figura se gira 180° alrededor de un punto y se superpone exactamente sobre la otra parte. pólígono Una figura plana cerrada con al menos tres lados rectos. poliedro Una figura tridimensional cerrada formada por regiones poligonales planas. polinomio Un monomio o la suma de dos o más monomios. función polinómica Función continua que puede describirse mediante una ecuación polinómica en una variable.
polynomial identity A polynomial equation that is true for any values that are substituted for the variables. population All of the members of a group of interest about which data will be collected. population proportion The number of members in the population with a particular characteristic, divided by the total number of members in the population. positive Where the graph of a function lies above the x-axis. positive correlation Bivariate data in which y increases as x increases. positively skewed distribution A distribution that typically has a mean greater than the median. postulate A statement that is accepted as true without proof. power function A function of the form $f(x) = ax^n$, where a and n are nonzero real numbers. precision The repeatability, or reproducibility, of a measurement. preimage The original figure in a transformation. prime polynomial A polynomial that cannot be written as a product of two polynomials with integer coefficients. principal root The nonnegative root of a number. principal square root The nonnegative square root of a number. principal values The values in the restricted domains of trigonometric functions. principle of superposition Two figures are congruent if and only if there is a rigid motion or series of rigid motions that maps one figure exactly onto the other. prism A polyhedron with two parallel congruent bases connected by parallelogram faces.	identidad polinomial Una ecuación polinómica que es verdadera para cualquier valor que se sustituya por las variables. población Todos los miembros de un grupo de interés sobre el cual los datos serán recopilados. proporción de la población El número de miembros en la población con una característica particular, dividida por el número total de miembros en la población. positiva Donde la gráfica de una función se encuentra por encima del eje x. correlación positiva Datos bivariable en el cual y aumenta a x disminuye. distribución positivamente sesgada Una distribución que típicamente tiene una media mayor que la mediana. postulado Una declaración que se acepta como verdadera sin prueba. función de potencia Una ecuación polinomial que es verdadera para una función de la forma $f(x) = ax^n$, donde a y n son números reales no nulos. precisión La repetibilidad, o reproducibilidad, de una medida. preimagen La figura original en una transformación. polinomio primo Un polinomio que no puede escribirse como producto de dos polinomios con coeficientes enteros. raíz principal La raíz no negativa de un número. raíz cuadrada principal La raíz cuadrada no negativa de un número. valores principales Valores de los dominios restringidos de las funciones trigonométricas. principio de superposición Dos figuras son congruentes si y sólo si hay un movimiento rígido o una serie de movimientos rígidos que traza una figura exactamente sobre la otra. prisma Un poliedro con dos bases congruentes paralelas conectadas por caras de paralelogramo.

probability The number of outcomes in which a specified event occurred to the total number of trials.

probability distribution A function that maps the sample space to the probabilities of the outcomes in this sample space for a particular random variable.

probability model A mathematical representation of a random event that consists of the sample space and the probability of each outcome.

projectile motion problems Problems that involve objects being thrown or dropped.

proof A logical argument in which each statement is supported by a statement that is accepted as true.

proof by contradiction One assumes that the statement to be proven is false and then uses logical reasoning to deduce that a statement contradicts a postulate, theorem, or one of the assumptions.

proportion A statement that two ratios are equivalent.

pure imaginary number A number of the form bi , where i is a real number and i is the imaginary unit.

pyramid A polyhedron with a polygonal base and three or more triangular faces that meet at a common vertex.

Pythagorean identities Identifies that express the Pythagorean Theorem in terms of the trigonometric functions.

Pythagorean triple A set of three nonzero whole numbers that make the Pythagorean Theorem true.

quadrantal angle An angle in standard position with a terminal side that coincides with one of the axes.

quadratic equation An equation that includes a quadratic expression.

probabilidad El número de resultados en los que se produce un evento especificado al número total de ensayos.

distribución de probabilidad Una función que mapea el espacio de muestra a las probabilidades de los resultados en el espacio de muestra para una variable aleatoria particular.

modelo de probabilidad Una representación matemática de un evento aleatorio que consiste en el espacio muestral y la probabilidad de cada resultado.

problemas de movimiento del proyectil Problemas que involucran objetos que se lanzan o caen.

prueba Un argumento lógico en el que cada sentencia está respaldada por una sentencia aceptada como verdadera.

prueba por contradicción Se supone que la afirmación a ser probada es falsa y luego utiliza el razonamiento lógico para deducir que una afirmación contradice un postulado, teorema o uno de los supuestos.

proporción Una declaración de que dos proporciones son equivalentes.

número imaginario puro Un número de la forma bi , donde i es un número real e i es la unidad imaginaria.

pirámide Poliedro con una base poligonal y tres o más caras triangulares que se encuentran en un vértice común.

identidades pitagóricas Identidades que expresan el Teorema de Pitágoras en términos de las funciones trigonométricas.

tríples pitagóricas Un conjunto de tres números enteros positivos que hacen que el Teorema de Pitágoras sea verdadero.

ángulo de cuadrante Un ángulo en posición estándar con un lado terminal que coincide con uno de los ejes.

ecuación cuadrática Una ecuación que incluye una expresión cuadrática.

quadratic expression An expression in one variable with a degree of 2.

quadratic form A form of polynomial equation, $ax^2 + bx + c$, where x is an algebraic expression in x .

quadratic function A function with an equation of the form $y = ax^2 + bx + c$, where $a \neq 0$.

quadratic inequality An inequality that includes a quadratic expression.

quadratic relations Equations of parabolas with horizontal axes of symmetry that are not functions.

quartic function A fourth-degree function.

quartiles Measures of position that divide a data set arranged in ascending order into four groups, each containing about one fourth or 25% of the data.

quintic function A fifth-degree function.

radian A unit of angular measurement equal to $\frac{180^\circ}{\pi}$ or about 57.296° .

radical equation An equation with a variable in a radical.

radical expression An expression that contains a radical symbol, such as a square root.

radical form When an expression contains a radical symbol.

radical function A function that contains radicals with variables in the radicand.

radicand The expression under a radical sign.

radius of a circle or sphere A line segment from the center of a circle or sphere.

radius of a regular polygon The radius of the circle circumscribed about a regular polygon.

expresión cuadrática Una expresión en una variable con un grado de 2.

forma cuadrática Una forma de ecuación polinomial, $ax^2 + bx + c$, donde x es una expresión algebraica en x .

función cuadrática Una función con una ecuación de la forma $y = ax^2 + bx + c$, donde $a \neq 0$.

desigualdad cuadrática Una desigualdad que incluye una expresión cuadrática.

relaciones cuadráticas Ecuaciones de parábolas con ejes horizontales de simetría que no son funciones.

función cuártica Una función de cuarto grado.

cuartiles Medidas de posición que dividen un conjunto de datos dispuestos en orden ascendente en cuatro grupos, cada uno de los cuales contiene aproximadamente un cuarto o el 25% de los datos.

función quintica Una función de quinto grado.

radian Una unidad de medida angular igual a $\frac{180^\circ}{\pi}$ o alrededor de 57.296° .

ecuación radical Una ecuación con una variable en un radical.

expresión radical Una expresión que contiene un símbolo radical, tal como una raíz cuadrada.

forma radical Cuando una expresión contiene un símbolo radical.

función radical Función que contiene radicales con variables en el radicando.

radicando La expresión debajo del signo radical.

radio de un círculo o esfera Un segmento de línea desde el centro hasta un punto en un círculo o esfera.

radio de un polígono regular El radio del círculo circunscrito al exterior de un polígono regular.

range The difference between the greatest and least values in a set of data. The set of second numbers of the ordered pairs in a relation. The set of y -values that actually result from the evaluation of the function.

rate of change How a quantity is changing with respect to a change in another quantity.

rational equation An equation that contains at least one rational expression.

rational exponent An exponent that is expressed as a fraction.

rational expression A ratio of two polynomial expressions.

rational function An equation of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial expressions and $q(x) \neq 0$.

rational inequality An inequality that contains at least one rational expression.

rationalizing the denominator A method used to remove the denominator of a fraction or fractions from a radicand.

ray Part of a line that starts at a point and extends to infinity.

reciprocal function An equation of the form $f(x) = \frac{a}{x-h} + k$ where n is a real number and h and k are real numbers that cannot equal 0.

reciprocal trigonometric functions Trigonometric functions that are reciprocals of each other.

reciprocals Two numbers with a product of 1.

rectangle A parallelogram with four right angles.

recursive formula A formula that gives the value of the first term in the sequence and then defines the next term by using the preceding term.

reduction A division with a scale factor between 0 and 1.

reference angle The acute angle formed by the terminal side of an angle and the x -axis.

range La diferencia entre los valores de datos más grande o menor en un sistema de datos. El conjunto de los segundos números de los pares ordenados de una relación. El conjunto de valores que realmente resultan de la evaluación de la función.

tasas de cambio Cómo cambia una cantidad con respecto a un cambio en otra cantidad.

ecuación racional Una ecuación que contiene al menos una expresión racional.

exponente racional Un exponente que se expresa como una fracción.

expresión racional Una relación de dos expresiones polinomiales.

función racional Una ecuación de la forma $f(x) = \frac{p(x)}{q(x)}$ donde $p(x)$ y $q(x)$ son expresiones polinomiales y $q(x) \neq 0$.

desigualdad racional Una desigualdad que contiene al menos una expresión racional.

racionalizando el denominador Método utilizado para eliminar el denominador de una fracción o fracciones de una raíz cuadrada.

rayo Parte de una línea que comienza en un punto y se extiende hasta el infinito.

función recíproca Una ecuación de la forma $f(x) = \frac{a}{x-h} + k$ donde n es un número real y h y k son números reales que no pueden ser igual a 0.

funciones trigonométricas recíprocas Funciones trigonométricas que son recíprocas entre sí.

recíprocos Dos números con un producto de 1.

rectángulo Un paralelogramo con cuatro ángulos rectos.

formula recursiva Una fórmula que da el valor del primer término en la sucesión y luego define el siguiente término usando el término anterior.

reducción Una división con un factor de escala entre 0 y 1.

ángulo de referencia El ángulo agudo formado por el lado terminal de un ángulo en posición estándar y el eje x .

reflection A function in which the preimage is reflected in the line of reflection. A transformation in which a figure, line, or curve is flipped across a line.

regression function A function generated by an algorithm to find a line or curve that fits a set of data.

regular polygon A convex polygon that is both equilateral and equiangular.

regular polyhedron A polyhedron in which all of its faces are regular congruent polygons and all of the edges are congruent.

regular pyramid A pyramid with a base that is a regular polygon.

regular tessellation A tessellation formed by only one type of regular polygon.

relation A set of ordered pairs.

relative frequency In a two-way frequency table, the ratio of the number of observations in a category to the total number of observations. The ratio of the number of observations in a category to the total number of observations.

relative maximum A point on the graph of a function where no other nearby points have a greater y -coordinate.

relative minimum A point on the graph of a function where no other nearby points have a lesser y -coordinate.

remote interior angles Interior angles of a triangle that are not adjacent to an exterior angle.

residual The difference between an observed y -value and its predicted y -value on a regression line.

rhombus A parallelogram with all four sides congruent.

rigid motion A transformation that preserves distance and angle measure.

reflexión Función en la que la preimagen se refleja en la línea de reflexión. Una transformación en la que una figura, línea o curva se voltea a través de una línea.

función de regresión Función generada por un algoritmo para encontrar una línea o curva que se ajuste a un conjunto de datos.

polygono regular Un polígono convexo que es a la vez equilateral y equiangular.

poliedro regular Un poliedro en el que todas sus caras son polígonos congruentes y todos los bordes son congruentes.

pirámide regular Una pirámide con una base que es un polígono regular.

tessellado regular Un tessellado formado por un solo tipo de polígono regular.

relación Un conjunto de pares ordenados.

frecuencia relativa En una tabla de frecuencia bidimensional, las relaciones entre el número de observaciones en una categoría y el número total de observaciones. La relación entre el número de observaciones en una categoría y el número total de observaciones.

máximo relativo Un punto en la gráfica de una función donde ningún otro punto cercano tiene una coordenada y mayor.

mínimo relativo Un punto en la gráfica de una función donde ningún otro punto cercano tiene una coordenada y menor.

ángulos interiores no adyacentes Ángulos interiores de un triángulo que no estén adyacentes a un ángulo exterior.

residual La diferencia entre un valor y observado y su valor y predicho en una línea de regresión.

rombo Un paralelogramo con los cuatro lados congruentes.

movimiento rígido Una transformación que preserva la distancia y la medida del ángulo.

root A solution of an equation. rotation A function that moves every point of a preimage through a specified angle and direction about a fixed point. rotational symmetry A figure can be rotated less than 360° about a point so that the image and the preimage are indistinguishable.	raz Una solución de una ecuación. rotación Función que mueve cada punto de una preimagen a través de un ángulo y una dirección especificados alrededor de un punto fijo. simetría rotacional Una figura puede girar menos de 360° alrededor de un punto para que la imagen y la preimagen sean indistinguibles.	series The indicated sum of the terms in a sequence. set builder notation Mathematical notation that describes a set by stating the properties that its members must satisfy. sides of an angle The rays that form an angle. sigma notation A notation that uses the Greek uppercase letter Σ to indicate that a sum should be found. significant figures The digits of a number that are used to express a measure to an appropriate degree of accuracy. similar polygons Two figures are similar polygons if one can be obtained from the other by a dilation or a dilation with one or more rigid motions. similar solids Solid figures with the same shape but not necessarily the same size. similar triangles Triangles in which all of the corresponding angles are congruent and all of the corresponding sides are proportional. similarity ratio The scale factor between two similar polygons. sin binomial transformation A transformation composed of a dilation or a dilation and one or more rigid motions. simple random sample Each member of the population has an equal chance of being selected as part of the sample. simplest form An expression is in simplest form when it is replaced by an equivalent expression having no like terms or parentheses. simulation The use of a probability model to imitate a process or situation so it can be studied. sine The ratio of the length of the leg opposite an angle to the length of the hypotenuse.	serie La suma indicada de los términos en una secuencia. notación de construcción de conjuntos Notación matemática que describe un conjunto al declarar las propiedades que sus miembros deben satisfacer. lados de un ángulo Los rayos que forman un ángulo. notación de sigma Una notación que utiliza la letra mayúscula griega Σ para indicar que debe encontrarse una suma. dígitos significativos Los dígitos de un número que se utilizan para expresar una medida con un grado apropiado de precisión. polígonos similares Dos figuras son polígonos similares si uno puede ser obtenido del otro por una dilatación o una dilatación con uno o más movimientos rígidos. sólidos similares Figuras sólidas con la misma forma pero no necesariamente del mismo tamaño. triángulos similares Triángulos en los cuales todos los ángulos correspondientes son congruentes y todos los lados correspondientes son proporcionales. relación de similitud El factor de escala entre dos polígonos similares. transformación de sinusoidal Una transformación compuesta por una dilatación o una dilatación y uno o más movimientos rígidos. muestra aleatoria simple Cada miembro de la población tiene la misma posibilidad de ser seleccionado como parte de la muestra. forma reducida Una expresión está reducida cuando se puede sustituir por una expresión equivalente que no tiene ni términos semejantes ni paréntesis. simulación El uso de un modelo de probabilidad para imitar un proceso o situación para que pueda ser estudiado. seno La relación entre la longitud de la pierna opuesta a un ángulo y la longitud de la hipotenusa.
sample A subset of a population. sample space The set of all possible outcomes. sampling error The variation between samples taken from the same population. scale The distance between tick marks on the x and y-axes. scale factor of a dilation The ratio of a length on an image to a corresponding length on the preimage. scatter plot A graph of bivariate data that consists of ordered pairs on a coordinate plane. secant Any line or ray that intersects a circle in exactly two points; the ratio of the length of the hypotenuse to the length of the leg adjacent to the angle. sector A region of a circle bounded by a central angle and its intercepted arc. segment bisector Any segment, line, plane, or point that intersects a line segment at its midpoint. self-selected sample Members volunteer to be included in the sample. semicircle An arc that measures exactly 180°. semiregular tessellation A tessellation formed by two or more regular polygons. sequence A list of numbers in a specific order.	muestra Un subconjunto de una población. espacio muestral El conjunto de todos los resultados posibles. error de muestreo La variación entre muestras tomadas de la misma población. escala La distancia entre las marcas en los ejes x e y. factor de escala de una dilatación Relación de una longitud en una imagen con una longitud correspondiente en la preimagen. gráfica de dispersión Una gráfica de datos bivariados que consiste en pares ordenados en un plano de coordenadas. secante Cualquier línea o rayo que cruce un círculo en exactamente dos puntos; Relación entre la longitud de la hipotenusa y la longitud de la pierna adyacente al ángulo. sector Una región de un círculo delimitado por un ángulo central y su arco interceptado. bisectriz del segmento Cualquier segmento, línea, plano o punto que interseca un segmento de línea en su punto medio. muestra auto-seleccionada Los miembros se ofrecen como voluntarios para ser incluidos en la muestra. semicírculo Un arco que mide exactamente 180°. tesselado semi-regular Un tesselado formado por dos o más polígonos regulares. secuencia Una lista de números en un orden específico.	series The indicated sum of the terms in a sequence. set builder notation Mathematical notation that describes a set by stating the properties that its members must satisfy. sides of an angle The rays that form an angle. sigma notation A notation that uses the Greek uppercase letter Σ to indicate that a sum should be found. significant figures The digits of a number that are used to express a measure to an appropriate degree of accuracy. similar polygons Two figures are similar polygons if one can be obtained from the other by a dilation or a dilation with one or more rigid motions. similar solids Solid figures with the same shape but not necessarily the same size. similar triangles Triangles in which all of the corresponding angles are congruent and all of the corresponding sides are proportional. similarity ratio The scale factor between two similar polygons. sin binomial transformation A transformation composed of a dilation or a dilation and one or more rigid motions. simple random sample Each member of the population has an equal chance of being selected as part of the sample. simplest form An expression is in simplest form when it is replaced by an equivalent expression having no like terms or parentheses. simulation The use of a probability model to imitate a process or situation so it can be studied. sine The ratio of the length of the leg opposite an angle to the length of the hypotenuse.	serie La suma indicada de los términos en una secuencia. notación de construcción de conjuntos Notación matemática que describe un conjunto al declarar las propiedades que sus miembros deben satisfacer. lados de un ángulo Los rayos que forman un ángulo. notación de sigma Una notación que utiliza la letra mayúscula griega Σ para indicar que debe encontrarse una suma. dígitos significativos Los dígitos de un número que se utilizan para expresar una medida con un grado apropiado de precisión. polígonos similares Dos figuras son polígonos similares si uno puede ser obtenido del otro por una dilatación o una dilatación con uno o más movimientos rígidos. sólidos similares Figuras sólidas con la misma forma pero no necesariamente del mismo tamaño. triángulos similares Triángulos en los cuales todos los ángulos correspondientes son congruentes y todos los lados correspondientes son proporcionales. relación de similitud El factor de escala entre dos polígonos similares. transformación de sinusoidal Una transformación compuesta por una dilatación o una dilatación y uno o más movimientos rígidos. muestra aleatoria simple Cada miembro de la población tiene la misma posibilidad de ser seleccionado como parte de la muestra. forma reducida Una expresión está reducida cuando se puede sustituir por una expresión equivalente que no tiene ni términos semejantes ni paréntesis. simulación El uso de un modelo de probabilidad para imitar un proceso o situación para que pueda ser estudiado. seno La relación entre la longitud de la pierna opuesta a un ángulo y la longitud de la hipotenusa.

Glossary · Glosario	
standard error of the mean The standard deviation of the distribution of sample means taken from a population. standard form of a linear equation Any linear equation can be written in this form: $ax + by = c$, where $A \geq 0$, A and B are not both 0, and A, B, and C are integers with a greatest common factor of 1. standard form of a polynomial A polynomial that is written in terms of x in order from greatest degree to least degree. standard form of a quadratic equation A quadratic equation can be written in the form $ax^2 + bx + c = 0$, where $a \neq 0$ and a, b, and c are integers. standard normal distribution A normal distribution with a mean of 0 and a standard deviation of 1. standard position An angle positioned so that the origin and the initial side is on the positive x-axis. statement Any sentence that is either true or false, but not both. statistic A measure that describes a characteristic of a sample. statistics An area of mathematics that deals with collecting, analyzing, and interpreting data. step function A type of piecewise-linear function with a graph that is a series of horizontal line segments. straight angle An angle that measures 180°. stratified sample The population is first divided into similar, nonoverlapping groups. Then members are randomly selected from each group. substitution A process of solving a system of equations in which one equation is solved for one variable in terms of the other. supplementary angles Two angles with measures that have a sum of 180°.	error estándar de la media La desviación estándar de la distribución de los medios de muestra se toma de una población. forma estándar de una ecuación lineal Cualquier ecuación lineal se puede escribir de esta forma: $Ax + By = C$, donde $A \geq 0$, A y B no son ambos 0, y A, B, y C son enteros con el mayor factor común de 1. forma estándar de un polinomio Un polinomio que se escribe en términos de x en orden del grado más grande a menos grado. forma estándar de una ecuación cuadrática Una ecuación cuadrática puede escribirse en la forma $ax^2 + bx + c = 0$, donde $a \neq 0$, b, y c son enteros. distribución normal estándar Distribución normal con una media de 0 y una desviación estándar de 1. posición estándar Un ángulo colocado de manera que el origen y el lado inicial está en el eje x positivo. enunciado Cualquier oración que sea verdadera o falsa, pero no ambas. estadística Una medida que describe una característica de una muestra. estadísticas El proceso de recolección, análisis e interpretación de datos. función escalonada Un tipo de función lineal por piezas con un gráfico que es una serie de segmentos de línea horizontal. ángulo recto Un ángulo que mide 180°. muestra estratificada La población se divide primero en grupos similares, sin superposición. A continuación, los miembros se seleccionan aleatoriamente de cada grupo. substitución Un proceso de resolución de un sistema de ecuaciones en el que una ecuación se resuelve para una variable en términos de la otra. ángulos suplementarios Dos ángulos con medidas que tienen una suma de 180°.
sinusoidal function A function that can be produced by translating, reflecting, or dilating the sine function. slaw lines Noncoplanar lines that do not intersect. start height of a pyramid or right cone The length of a segment with one endpoint on the base edge of the figure and the other at the vertex. slope The rate of change in the y-coordinates (rise) to the corresponding change in the x-coordinates (run) for points on a line. slope criteria Outlines a method for proving the relationship between lines based on a comparison of the slopes of the lines. solid of revolution A solid figure obtained by rotating a shape around an axis. solution A value that makes an equation true. solve an equation The process of finding all values of the variable that make the equation a true statement. solving a triangle When you are given measurements to find the unknown angle and side measures of a triangle. space A boundless three-dimensional set of all points. sphere A set of all points in space equidistant from a given point called the center of the sphere. square A parallelogram with all four sides and all four angles congruent. square root One of two equal factors of a number. square root function A radical function that contains the square root of a variable expression. square root inequality An inequality that contains the square root of a variable expression. standard deviation A measure that shows how data deviates from the mean.	función sinusoidal Función que puede producirse traduciendo, reflejando o dilatando la función sinusoidal. líneas alabeadas Líneas no coplanares que no se cruzan. altura inclinada de una pirámide o cono derecho La longitud de un segmento con un punto final en el borde base de la figura y el otro en el vértice. pendiente La tasa de cambio en las coordenadas y (salida) al cambio correspondiente en las coordenadas x (corriente) para puntos en una línea. criterios de pendiente Describe un método para probar la relación entre líneas basadas en una comparación de las pendientes de las líneas. sólido de revolución Una figura sólida obtenida girando una forma alrededor de un eje. solución Un valor que hace que una ecuación sea verdadera. resolver una ecuación El proceso en que se hallan todos los valores de la variable que hacen verdadera la ecuación. resolver un triángulo Cuando se le dan mediciones para encontrar el ángulo desconocido y las medidas laterales de un triángulo. espacio Un conjunto tridimensional ilimitado de todos los puntos. esfera Un conjunto de todos los puntos del espacio equidistantes de un punto dado llamado centro de la esfera. cuadrado Un paralelogramo con los cuatro lados y los cuatro ángulos congruentes. raíz cuadrada Uno de dos factores iguales de un número. función raíz cuadrada Función radical que contiene la raíz cuadrada de una expresión variable. square root inequality Una desigualdad que contiene la raíz cuadrada de una expresión variable. desviación típica Una medida que muestra cómo los datos se desvían de la media.

Glossary · Glosario

surface area The sum of the areas of all faces and side surfaces of a three-dimensional figure.	área de superficie La suma de las áreas de todas las caras y superficies laterales de una figura tridimensional.	term of a sequence A number in a sequence.	término de una sucesión Un número en una sucesión.
survey Data are collected from responses (even by members of a group) regarding their characteristics, behaviors, or opinions.	encuesta Los datos se recogen de las respuestas dadas por los miembros de un grupo con respecto a sus características, comportamientos u opiniones.	terminal side The part of an angle that rotates about the center.	lado terminal La parte de un ángulo que gira alrededor de un centro.
symmetric distribution A distribution in which the mean and median are approximately equal.	distribución simétrica Una distribución en la que la media y la mediana son aproximadamente iguales.	tesellation A repeating pattern of one or more figures that covers a plane with no overlapping or empty spaces.	teselado Patrón repetitivo de una o más figuras que cubre un plano sin espacios superpuestos o vacíos.
symmetry A figure has this if there exists a rigid motion—reflection, translation, rotation, or glide reflection—that maps the figure onto itself.	simetría Una figura tiene esto si existe una reflexión, rotación, una traslación o una reflexión de deslizamiento rigurosa que mapea la figura sobre a sí misma.	theorem A statement that can be proven true using only facts, terms, definitions, and postulates.	teorema Una afirmación o conjetura que se puede probar usando únicamente términos, definiciones y postulados o axiomas.
synthetic division An alternate method used to divide a polynomial by a binomial of degree 1.	división sintética Un método alternativo utilizado para dividir un polinomio por un binomio de grado 1.	theoretical probability Probability based on what is expected to happen.	probabilidad teórica Probabilidad basada en lo que se espera que suceda.
synthetic geometry The study of geometric figures without the use of coordinates.	geometría sintética El estudio de figuras geométricas sin el uso de coordenadas.	transformation A function that takes points in the plane as inputs and gives other points as outputs. The movement of a graph on the coordinate plane.	transformación Función que toma puntos en el plano como entradas y da otros puntos como salidas. El movimiento de un gráfico en el plano de coordenadas.
synthetic substitution The process of using synthetic division to find a value of a polynomial function.	sustitución sintética El proceso de utilizar la división sintética para encontrar un valor de una función polinomial.	translation A function in which all of the points of a figure move the same distance in the same direction. A transformation in which a figure is slid from one position to another without being turned.	traslación Función en la que todos los puntos de una figura se mueven en la misma dirección. El movimiento de un gráfico en el plano de coordenadas.
system of equations A set of two or more equations with the same variables.	sistema de ecuaciones Un conjunto de dos o más ecuaciones con las mismas variables.	translation vector A directed line segment that describes both the magnitude and direction of the slide if the magnitude is the length of the vector from its initial point to its terminal point.	vector de traslación Un segmento de línea dirigida que describe tanto la magnitud como la dirección de la deslizamiento. La longitud es la longitud del vector desde su punto inicial hasta su punto terminal.
system of inequalities A set of two or more inequalities with the same variables.	sistema de desigualdades Un conjunto de dos o más desigualdades con las mismas variables.	transversal A line that intersects two or more lines in a plane at different points.	transversal Una línea que interseca dos o más líneas en un plano en diferentes puntos.
systematic sample Members are selected according to a specified interval from a random starting point.	muestra sistemática Los miembros se seleccionan de acuerdo con un intervalo especificado desde un punto de partida aleatorio.	trapezoid A quadrilateral with exactly one pair of parallel sides.	trapecio Un cuadrilátero con exactamente un par de lados paralelos.
tangent The ratio of the length of the leg opposite an angle to the length of the leg adjacent to the angle.	tangente La relación entre la longitud de la pata opuesta a un ángulo y la longitud de la pata adyacente al ángulo.	trend A general pattern in the data.	tendencia Un patrón general en los datos.
tangent to a circle A line or segment in the plane of a circle that intersects the circle in exactly one point and does not contain any points in the interior of the circle.	tangente a un círculo Una línea o segmento en el plano de un círculo que interseca el círculo en exactamente un punto y no contiene ningún punto en el interior del círculo.	trigonometric equation An equation that includes at least one trigonometric function.	ecuación trigonométrica Una ecuación que incluye al menos una función trigonométrica.
tangent to a sphere A line that intersects the sphere in exactly one point.	tangente a una esfera Una línea que interseca la esfera exactamente en un punto.	trigonometric function A function that relates the measure of one or more angles of a right triangle to the ratios of the lengths of any two sides of the triangle.	función trigonométrica Función que relaciona la medida de un ángulo no recto de un triángulo rectángulo con las relaciones de las longitudes de cualquiera de los lados del triángulo.
term A number, a variable, or a product or quotient of numbers and variables.	término Un número, una variable, o un producto o cociente de números y variables.	topographic identity An equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.	identidad topográfica Una ecuación que implica funciones trigonométricas que es verdadera para todos los valores para los cuales se define cada expresión en la ecuación.

cuartil superior La mediana de la mitad superior de un conjunto de datos.

V

argumento válido Un argumento es válido si es imposible que todas las premisas o argumentos de apoyo del argumento sean verdaderos y su conclusión sea falsa.

variable Una letra utilizada para representar un número o valor no especificado. Cualquier característica, número, o cantidad que pueda ser contada o medida.

termino variable Un término que contiene una variable.

varianza El cuadrado de la desviación estándar.

vértice El punto más bajo o el punto más alto en una función.

ángulo del vértice de un triángulo isósceles El ángulo entre los lados que son las patas de un triángulo isósceles.

forma de vértice Una función cuadrática se escribe de la forma $f(x) = ax^2 - bx + c$.

vértice de un polígono La intersección de tres bordes de un polígono.

vértice de un ángulo El punto final común de los dos rayos que forman un ángulo.

ángulos verticales Dos ángulos no adyacentes formados por dos líneas de intersección.

asintota vertical Una línea vertical que se aproxima a un gráfico.

cambio vertical Una traducción vertical de la gráfica de una función trigonométrica.

volumen La medida de la cantidad de espacio encerrada por una figura tridimensional.

W

problemas de trabajo Problemas que involucran a dos personas trabajando a diferentes ritmos que están tratando de completar un solo trabajo.

upper quartile The median of the upper half of a set of data.

valid argument An argument is valid if it is impossible for all of the premises, or supporting statements, of the argument to be true and its conclusion false.

variable A letter used to represent an unspecified number or value. Any characteristic, number, or quantity that can be counted or measured.

variable term A term that contains a variable.

variance The square of the standard deviation.

vertex Either the lowest point or the highest point of a function.

vertex angle of an isosceles triangle The angle between the sides that are the legs of an isosceles triangle.

vertex form A quadratic function written in the form $f(x) = ax^2 - bx + c$.

vertex of a polygon The intersection of three edges of a polygon.

vertex of an angle The common endpoint of the two rays that form an angle.

vertical angles Two nonadjacent angles formed by two intersecting lines.

vertical asymptote A vertical line that a graph approaches.

vertical shift A vertical translation of the graph of a trigonometric function.

volume The measure of the amount of space enclosed by a three-dimensional figure.

work problems Problems that involve two people working at different rates who are trying to complete a single job.

relación trigonométrica Una relación de las longitudes de los lados de un triángulo rectángulo.

trigonometría El estudio de las relaciones entre los lados y los ángulos de los triángulos.

trinomio La suma de tres monomios.

valor de verdad La verdad o la falsedad de una declaración.

prueba de dos columnas Una prueba que contiene declaraciones y razones organizadas en un formato de dos columnas.

tabla de frecuencia bidireccional Una tabla utilizada para mostrar las frecuencias de los datos clasificados de acuerdo con dos categorías, con las filas que indican una categoría y las columnas que indican la otra.

tabla de frecuencias relativas bidireccional Una tabla usada para mostrar las frecuencias de datos basadas en un porcentaje del número total de observaciones.

U

no acotado Cuando la gráfica de un sistema de restricciones está abierta.

terminos indefinidos Palabras que no se explican formalmente mediante palabras y conceptos más básicos.

problemas de movimiento uniforme Problemas que utilizan la fórmula $d = rt$, donde d es la distancia, r es la velocidad y t es el tiempo.

trésido uniforme Un tréside que contiene la misma disposición de formas y ángulos en cada vértice.

unión La gráfica de una desigualdad compuesta que contiene la palabra o.

unión de A y B El conjunto de todos los resultados en el espacio muestral del evento A combinado con todos los resultados en el espacio muestral del evento B.

círculo unitario Un círculo con un radio de 1 unidad centrado en el origen en el plano de coordenadas.

datos univariante Datos de medición en una variable.

trigonometric ratio A ratio of the lengths of two sides of a right triangle.

trigonometry The study of the relationships between the sides and angles of triangles.

trinominal The sum of three monomials.

truth value The truth or falsity of a statement.

two-column proof A proof that contains statements and reasons organized in a two-column format.

two-way frequency table A table used to show frequencies of data classified according to two categories, with the rows indicating one category and the columns indicating the other.

two-way relative frequency table A table used to show frequencies of data based on a percentage of the total number of observations.

unbounded When the graph of a system of constraints is open.

undefined terms Words that are not formally explained by means of more basic words and concepts.

uniform motion problems Problems that use the formula $d = rt$, where d is the distance, r is the rate, and t is the time.

uniform tessellation A tessellation that contains the same arrangement of shapes and angles at each vertex.

union The graph of a compound inequality containing or.

union of A and B The set of all outcomes in the sample space of event A combined with all outcomes in the sample space of event B.

unit circle A circle with a radius of 1 unit centered at the origin on the coordinate plane.

univariate data Measurement data in one variable.

x

x-intercept The x-coordinate of a point where a graph crosses the x-axis.

intersección x La coordenada x de un punto donde la gráfica corte al eje de x .

y

y-intercept The y-coordinate of a point where a graph crosses the y-axis.

intersección y La coordenada y de un punto donde la gráfica corte al eje de y .

z

z-value The number of standard deviations that a given data value is from the mean.

valor z El número de variaciones estándar que separa un valor dado de la media.

zero An x-intercept of the graph of a function; a value of x for which $f(x) = 0$.

cero Una intersección de la gráfica de una función; un punto x para los que $f(x) = 0$.

Index

A

algebra tiles, 34, 101
amplitudes, 445
Analyze See *Practice for the lessons*.

angles

central, 419
coterminal, 417
general, 427
quadrantal, 427
of rotation, 417
reference, 428
apply example, 96, 148, 162, 240,
272, 406, 438
arc length, 419
asymptotes, 217
horizontal, 327
oblique, 340
vertical, 327

B

bias, 371
binomial, 93
expansion, 109

C

closure
of polynomial expressions, 93
of rational expressions, 319
coefficient of determination, 249
common ratio, 239
completing the square, 35
complex conjugates, 23
complex fractions, 313
complex numbers, 23
arithmetic operations with, 24
properties of, 23
compound interest, 225
continuously compounded
interest, 233, 295
confidence intervals, 403
conjugates, 197
Create See *Practice for the lessons*.

critical values, 403
cubes
difference of two, 123
sum of two, 123
cycles, 437

D

decay factors, 219
degrees, 73
Descartes' Rule of Signs, 144
differences of squares, 29
discriminants, 45
distributions, 387
bimodal, 387
comparing, 390
negatively skewed, 387
normal, 396
positively skewed, 387
probability, 393
standard normal, 398
symmetric, 387
division of polynomials
algorithm, 101
long, 101
synthetic, 103

E

e, 233
The Empirical Rule, 396
equations
exponential, 225
logarithmic, 269
quadratic, 13
radical, 203
rational, 395
estimation, 89, 236, 282, 294
excluded values, 327
Expand, 244
expected values, 393
experiments, 372
exponential
continuous decay, 297
continuous growth, 295
decay, 219

equations, 225
form, 175
functions, 217
growth, 217
inequalities, 228

extrema

maximum, 3
minimum, 3

F

factored form, 27
factoring
by using the Distributive
Property, 27
differences of squares, 29
perfect square trinomials, 29
trinomials, 27
Zero Product Property, 27
Find the Error See *Practice for the lessons*.
FOIL method, 95
formulas
Change of Base, 279
explicit, 239
recursive, 239
frequencies, 449

functions

adding, 159
Arc cosine, 473
Arc sine, 473
Arc tangent, 473
circular, 435
composition of, 160
cosecant, 458
cosine, 445
cotangent, 458
cube root, 186
dividing, 159
exponential, 217
inverse, 167
inverse trigonometric, 473
logarithmic, 261
monomial, 73
multiplying, 159
natural base exponential, 285
periodic, 437
polynomial, 75
power, 73
quadratic, 3
quartic, 75

Index IN1

A

ALEKS

every Formative Assessment Math Probe
every What Will You Learn, second page
every Practice, first page

Assess and Differentiate, every Practice and
Homework, first page

Assessment, see *Review and Assessment Options, Formative Assessment Math Probe*

C

Cheryl Tobey, see *Formative Assessment Math Probe*

Common Misconception, 3, 33, 37, 60, 73, 84,
93, 101, 135, 146, 170, 239, 285, 295, 319, 330,
379, 417, 419, 425, 427, 429

Conceptual Bridge, see *Rigor*

D

Differentiate

Resources Chart, every Lesson Teacher
a page

Activities, 7, 21, 27, 35, 36, 43, 45, 51, 52, 73,
86, 88, 95, 101, 132, 135, 138, 145, 146, 175,
187, 195, 196, 217, 225, 233, 243, 259, 261,
269, 270, 277, 279, 285, 295, 312, 313, 319,
327, 330, 341, 347, 355, 371, 380, 393,
396, 403, 425, 435, 456, 458, 463, 465

Dinah Zike Foldables®, see *Foldables®*

E

ELL Support

every What Will You Learn, Teacher second
page
Language Development Support, every
Lesson Teacher a page
see *Differentiate, Activities*

Essential Question, 1, 71, 117, 157, 215, 257, 309, 369, 415

Essential Question Follow-Up, 13, 45, 67, 73, 93, 101, 103, 113, 137, 143, 153, 167, 183, 211, 217, 239, 253, 261, 280, 297, 305, 340, 348, 365, 380, 393, 405, 411, 426, 437, 473, 477

Explore, Explore and Develop, pages c-h

F

Foldables®, 1, 67, 71, 113, 117, 153, 157, 211, 215, 253, 257, 305, 309, 365, 369, 411, 415, 477

Formative Assessment Math Probe, 1b, 71b, 117b, 157b, 215b, 257b, 309b, 369b, 415b

G

Growth Mindset, *see Mindset Matters*

I

Inquiry, *see Explore*

M

Mathematical Background, 3b, 13b, 21a, 27b, 33b, 43a, 51a, 57a, 73a, 83a, 93a, 101a, 109a, 119a, 123a, 131a, 135a, 143a, 159a, 167a, 175a, 183a, 193a, 203a, 217a, 225a, 233a, 239a, 249a, 259a, 269a, 277b, 285a, 295a, 311a, 319a, 327b, 337a, 347b, 355b, 371a, 379a, 387a, 393a, 403a, 417a, 425a, 435a, 445a, 455a, 463a, 473a

Mindset Matters, 2, 72, 118, 158, 216, 258, 310, 370, 416

quintic, 75
radical, 183
rational, 337
reciprocal, 327
reciprocal trigonometric, 458
regression, 249
secant, 468
sine, 445
sinusoidal, 449
square root, 183
subtracting, 159
tangent, 455
trigonometric, 425

G

geometric means, 239
growth factors, 218

H

hyperbolas, 327

I

identities, 131
polynomial, 131
imaginary unit i , 21
indexes, 175

inequalities
exponential, 228
quadratic, 51
rational, 359
square root, 183
initial sides, 417

inverse
functions, 167
relations, 167

L

leading coefficients, 73
like radical expressions, 194
Location Principle, 83
logarithms, 259
common, 277
natural, 285

M

math history minutes, 29, 87, 132, 177, 220, 259, 340, 381, 428

maximum error of the estimate, 403

midlines, 445

Modeling See Practice for the lessons.
multiplicity, 143

N

notations
sigma, 243
 n th roots, 175

O

observational studies, 372
oscillation, 445
outcomes, 393
outliers, 387

P

parameters, 371
Pascal's triangle, 109
perfect square trinomials, 29
periods, 437
Persevere See Practice for the lessons.
phase shifts, 463
point discontinuities, 341
polynomials, 75, 93
adding, 93
degree of a, 75
depressed, 137
dividing, 101
factors of, 135
functions, 75
identities, 131
multiplying, 95
prime, 123
standard form of a, 75
subtracting, 93

populations, 371
population proportions, 405
principal roots, 175
principal values, 473
probability, 379
distributions, 393
experimental, 379
model, 380
theoretical, 379

problems
mixture, 357
projectile motion, 37
uniform motion, 358
work problems, 358

properties

Power Property of Logarithms, 270
Property of Equality for Exponential Equations, 225
Property of Equality for Logarithmic Equations, 269
Property of Inequality for Exponential Equations, 228
Product Property of Logarithms, 270
Product Property of Radicals, 193
Quotient Property of Logarithms, 270
Quotient Property of Radicals, 193
Square Root Property, 33
Zero Product Property, 27
pure imaginary numbers, 21

Q

quadratic
equations, 13
form, 125
functions, 3
inequalities, 51
relations, 60

Quadratic Formula, 43

R

radians, 419
radical
equations, 203
form, 177
functions, 183
radicands, 175
random sampling, 371
random variables
continuous, 393
discrete, 393
rates of change, 6
average, 6
percent, 218
rational
equations, 355
exponent, 177
expression, 311

IN2 index

functions, 337
inequalities, 358
rationalizing the denominator, 23
Reasoning See *Practice for the lessons*.

S

samples, 371
convenience, 371
self-selected, 371
simple random, 371
stratified, 371
systematic, 371
sample spaces, 393
sampling errors, 403
sequences, 239
as functions, 239
finite, 239
geometric, 239
infinite, 239
 n th term of a, 239
series, 241
geometric, 241
partial sum of a geometric, 241
simulations, 380
square root
functions, 183
inequalities, 183
standard deviations, 387
standard errors of the mean, 403
standard form
of a polynomial, 75
of a quadratic equation, 13
standard position, 417
statistic, 371
statistics
descriptive, 387
inferential, 403
surveys, 372
symmetry
axis of, 3
synthetic substitution, 136
systems
linear-nonlinear, 57

T

term of a sequence, 239
terminal sides, 417

theorems
Binomial Theorem, 109
Complex Conjugates Theorem, 145
Factor Theorem, 137
Fundamental Theorem of Algebra, 143
corollary, 143
Remainder Theorem, 135

trigonometric
functions, 425
ratios, 425
cosecant, 425
cosine, 425
cotangent, 425
secant, 425
sine, 425
tangent, 425
reciprocal functions, 458
sinusoidal functions, 449
trigonometry, 425
trinomials, 93
turning points, 83

U

unit circle, 435

V

variances, 387
variations
combined, 348
constant of, 347
direct, 347
inverse, 348
joint, 347
vertexes, 3
vertex form of a quadratic equation, 37
vertical shifts, 465

W

Write See *Practice for the lessons*.

Z

z-values, 398

P

Pacing, every Lesson Teacher a page

Prerequisite Skills

Are You Ready?, 2, 72, 118, 158, 216, 258, 310, 370, 416
Warm Up, every Lesson Teacher b page

R

Review, see *Review and Assessment Options*
Review and Assessment Options, 68, 114, 154, 212, 254, 306, 366, 412, 478
Rigor, every Lesson Teacher a page

S

Standardized Test Practice, see *Test Practice*

T

Teaching the Mathematical Practices
every Lesson Teacher b page
every Explore
every Explore and Develop

Test Practice, 68–70, 114–116, 154–156, 212–214, 254–256, 306–308, 366–368, 412–414, 478–480

V

Vertical Alignment, every Lesson Teacher a page a

Vocabulary, 2, 3b, 13b, 21b, 27b, 33b, 43b, 51b, 57b, 72, 73b, 93b, 101b, 109b, 118, 123b, 131b, 135b, 158, 159b, 167b, 175b, 183b, 193b, 203b, 216, 217b, 225b, 233b, 249b, 258, 259b, 269b, 277b, 285b, 310, 311b, 327b, 337b, 347b, 355b, 370, 371b, 379b, 387b, 393b, 403b, 416, 417b, 425b, 435b, 445b, 455b, 463b, 473b

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CONTENTS

- MODULE 1 QUADRATIC FUNCTIONS
- MODULE 2 POLYNOMIALS AND POLYNOMIAL FUNCTIONS
- MODULE 3 POLYNOMIAL EQUATIONS
- MODULE 4 INVERSES AND RADICAL FUNCTIONS
- MODULE 5 EXPONENTIAL FUNCTIONS
- MODULE 6 LOGARITHMIC FUNCTIONS
- MODULE 7 RATIONAL FUNCTIONS
- MODULE 8 INFERENCE STATISTICS
- MODULE 9 TRIGONOMETRIC FUNCTIONS

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